The Finite Embeddability Property for Topological Quasi-Boolean Algebra 5

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Abstract. In this paper we study some basic algebraic structures of rough algebras. We proved that the class of topological quasi-Boolean algebra 5s (tqBa5s) has the finite embeddability property (FEP). Further we also extend this result to some related classes of algebras.

1 Introduction

The algebraic structures called Pre-rough algebras [1,2] arose as a natural abstraction from the calculus of rough sets proposed by Pawlak in 1983 [3]. A Pre-rough algebra is a topological quasi-Boolean algebra (tqBa) which is quasi-Boolean algebra (see Definition 1) endowed with a topological (interior) operator. Although quasi Boolean algebra and topological Boolean algebra are presented in the similar work of Rasiowa [4], Topological quasi-Boolean algebra was first introduced in [1]. As a further step towards Pre-rough algebras, an axiom corresponding to modal logic axiom S5 has been added to tqBas resulting in tqBa5s [2]. Further studies with tqBas, tqBa5s and related algebras have been carried out in [1,2,5–7]. However, the finite embeddability property of these algebraic structures have not been investigated before.

On the study of the connection between logical systems and classes of algebras in general, one important and natural question is whether a given class of algebras has a decidable equational or even universal theory. The finite embeddability property (or FEP for short) i.e., every finite partial subalgebra of an algebra in the class is isomorphic to a subalgebra of a finite algebra in the class of algebras, entails the decidability of its universal theory if this class of algebras is finitely axiomatizable.

The study on FEP of classes of algebras dated back to Henkin [1956]. Henkin proves that the class of abelian groups has FEP. It is also well-known that the class HA of Heyting algebras has FEP. Block and Van Alten [8,9] show that various integral residuated lattices (groupoids) have FEP. Farulewski [10]

The work of the first author was supported by Chinese National Funding of Social Sciences (No. 17CZX048).

Md. A. Khan and A. Manuel (Eds.): ICLA 2019, LNCS 11600, pp. 30–41, 2019.
https://doi.org/10.1007/978-3-662-58771-3_4

shows that the integral condition is not necessary and proves FEP for residuated groupoids. Buszkowski [11] also proves that various lattice extensions of residuated groupoids (including Heything and Boolean extensions) have FEP. The first author of the current paper in [12] extends Buszkowski's results to various lattice extensions of residuated groupoids with modalities. In [13] the present authors investigate residuated Pre-rough algebras and show that residuated Pre-rough algebras have decidable quasiequational theories (see definition in Sect. 1), which entails that the pre-rough algebras have decidable quasiequational theories. Indeed residuated Pre-rough algebras have FEP. However FEP for Pre-rough algebras still remain open. In the present paper we study the class of the basic algebraic structures in Pre-rough algebras (the class of tqBa5s) and prove that it has FEP. This results can also be extended to Pre-rough and some other modal extensions of quasi-Boolean algebras.

The method we developed in the present paper is inspired by [11] and [12]. A sequent calculus which admits the interpolant lemma (see Lemma 1) is introduced and it plays a essential role in proof of FEP. Meanwhile a sequent calculus which does not admit the interpolant lemma for tqBa5 was earlier introduced in [14]. Our method can be regarded as an algebraic substitute of the filtration method for Kripke frames [15].

The paper is organized as below. In the next section we recall some basic algebraic definitions. Then in Sect. 3, we develop a sequent system for tqBa5s and prove the interpolant lemma. In Sect. 4, we present the main results and show FEP for the class of tqBa5s. In Sect. 5, we conclude our paper and make some simple extensions to some related classes of algebras. Hereafter, the class of all topological quasi-Boolean algebra 5 will be denoted by tqBa5 also.

2 Some Basic Definitions

Definition 1. A *quasi-Boolean algebra* (qBa) is an algebra $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and $\neg$ is an unary operation on A such that the following conditions hold for all $a, b \in A$:

$$\text{(DN)} \quad \neg\neg a = a, \quad \text{(DM)} \quad \neg(a \vee b) = \neg a \wedge \neg b$$

A *topological quasi-Boolean algebra* (tqBa) is an algebra $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1, \Box)$ where $(A, \wedge, \vee, \neg, 0, 1)$ is a quasi-Boolean algebra, and $\Box$ is an unary operation on A such that for all $a, b \in A$:

$$(\mathrm{K}_\Box) \quad \Box(a \wedge b) = \Box a \wedge \Box b, \quad (\mathrm{N}_\Box) \quad \Box\top = \top$$

$$(\mathrm{T}_\Box) \quad \Box a \leq a, \quad (4_\Box) \quad \Box a \leq \Box\Box a$$

A *topological quasi-Boolean algebra 5* (tqBa5) is a topological quasi-Boolean algebra $\mathbb{A} = (A, \wedge, \vee, \neg, \Box, 0, 1)$ such that for all $a \in A$:

$$(5) \quad \Diamond a \leq \Box\Diamond a,$$

where $\Diamond$ is an unary operation on A defined by $\Diamond a := \neg\Box\neg a$.

Proposition 1. *For any tqBa5* $\mathbb{A} = (A, \wedge, \vee, \neg, \Box, 0, 1)$ *and* $a, b \in A$*, the following hold:*

(1) $\neg 0 = 1$ *and* $\neg 1 = 0$*.*
(2) $\neg(a \wedge b) = \neg a \vee \neg b$*.*
(3) If $a \leq b$*, then* $\neg b \leq \neg a$*.*
(4) $\Diamond 0 = 0$ *and* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$*.*
(5) $\Box a = \Box\Box a$ *and* $\Diamond a = \Diamond\Diamond a$*.*
(6) $\Diamond a = \Box\Diamond a$ *and* $\Box a = \Diamond\Box a$*.*
(7) $\Diamond a \leq b$ *if and only if* $a \leq \Box b$*.*

The proof of Proposition 1 can be found in [6,7].

We now recall some concepts from universal algebra. *Equation* (identity) and *quasi-equation* (quasi-identity) are defined in standard manner (see Chap. 1 [16]). For any set of equations or quasi-equations Σ, let $\mathbb{A}(\Sigma)$ be the class of all algebras which validate all equations or quasi-equations in Σ. A class of algebras $\mathbb{K}$ is a *variety* if there is a set of equations Σ such that $\mathbb{K} = \mathbb{A}(\Sigma)$. A class of algebras $\mathbb{K}$ is a *quasi-variety* if there is a set of quasi-equations Θ such that $\mathbb{K} = \mathbb{A}(\Theta)$.

Theorem 1. *The class of tqBa5s is a variety.*

Due to the definition of tqBa5, the class of tqBa5s can be classified by a set of equations. Thus by Theorem 1.19, [16]. The class of tqBa5s is a variety.

Corollary 1. *The class of tqBa5s is a quasi-variety.*

Let $\mathbb{A} = \langle \mathrm{A}, \langle f_i^{\mathbb{A}} \rangle_{i \in I} \rangle$ be an algebra of fixed type and $\mathrm{B} \subseteq \mathrm{A}$. Then $\mathbb{B} = \langle \mathrm{B}, \langle f_i^{\mathbb{B}} \rangle_{i \in I} \rangle$ is a partial subalgebra of $\mathbb{A}$ where for every $n \in \mathrm{N}$, every n-ary function symbol $f_i^{\mathbb{A}}$ with $i \in I$, and for every $b_1, \ldots, b_n \in \mathrm{B}$, one defines $f_i^{\mathbb{B}}(b_1, \ldots, b_n) = f_i^{\mathbb{A}}(b_1, \ldots, b_n)$ if $f_i^{\mathbb{A}}(b_1, \ldots, b_n) \in \mathrm{B}$, otherwise, the value is not defined. If $\mathbb{A}$ is ordered, then $\leq^{\mathbb{B}} = \leq^{\mathbb{A}} |\mathrm{B}$, the restriction of $\leq^{\mathbb{A}}$ to $\mathbb{B}$. $f_i^{\mathbb{A}}$ denotes the operation interpreting the symbol f_i in the algebra $\mathbb{A}$. However we write f_i for $f_i^{\mathbb{A}}$, if it does not cause confusion.

By an embedding from a partial algebra $\mathbb{B}$ into an algebra $\mathbb{C}$, we mean an injection $h : \mathrm{B} \mapsto \mathrm{C}$ such that if $b_1, \ldots, b_n, f^{\mathbb{B}}(b_1, \ldots, b_n) \in \mathrm{B}$, then

$$h(f^{\mathbb{B}}(b_1, \ldots, b_n)) = f^{\mathbb{C}}(h(b_1), \ldots, h(b_n)).$$

If $\mathbb{B}$ and $\mathbb{C}$ are ordered, then h is required to be an order embedding i.e. $a \leq^{\mathbb{B}} b \Leftrightarrow h(a) \leq^{\mathbb{C}} h(b)$.

A class $\mathbb{K}$ of algebras has *the finite embeddability property* (FEP), if every finite partial subalgebra of a member of $\mathbb{K}$ can be embedded into a finite member of $\mathbb{K}$. FEP usually has some consequences on finite model property. FEP implies *the strong finite model property* (SFMP) i.e. every quasi-identity which fails to hold in a class $\mathbb{K}$ of algebras can be falsified in a finite member of $\mathbb{K}$. SFMP and FEP are equivalent in quasivarieties of finite type.

Lemma 1 (Lemma 6.40 [16]). *For any quasivariety $\mathbb{K}$ of finite type the following are equivalent:*

(1) $\mathbb{K}$ *has FEP*
(2) $\mathbb{K}$ *have SFMP*
(3) $\mathbb{K}$ *is generated as a quasivarieties by its finite members*

Remark 1. If a formal system S is strongly complete with respect to a class $\mathbb{K}$ of algebras, then it yields, actually, an axiomatization of the quasiequational theory of $\mathbb{K}$; hence SFMP for S with respect to $\mathbb{K}$ yields SFMP for $\mathbb{K}$. By SFMP for S, we mean that for any finite set of sequents Φ, if $\Phi \nvdash_S \Gamma \Rightarrow A$, then there exists a finite $\mathbb{A} \in \mathbb{K}$ and a valuation σ such that all sequents from Φ are true in $(\mathbb{A}\ \sigma)$, but $\Gamma \Rightarrow A$ is not.

3 Sequent Calculus of tqBa5

In this section we develop a sequent system G5 for tq$\mathbb{B}$a5 following the tradition [17]. The language of the logic of tqBa5 is defined as follows

$$\alpha ::= p \mid \bot \mid \top \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \neg\alpha \mid \Diamond\alpha \mid \Box\alpha,$$

where $p \in \mathbf{Prop}$, the set of propositional variables.

Formula structure are defined as follows with a unary structural operation $\langle\rangle$:

- α is a formula structure if α is a formula
- $\langle\Gamma\rangle^i$ is a formula structure if Γ is a formula structure

Hereafter we abbreviate $\underbrace{\langle\ldots\langle}_{n}\alpha\underbrace{\rangle\ldots\rangle}_{n}$ by $\langle\alpha\rangle^n$. Clearly if Γ is a formula structure, then it is of the form $\langle\alpha\rangle^i$ for some formula α and number $i \geq 0$. We use $\langle\alpha\rangle^{i_1}, \langle\beta\rangle^{i_2}, \ldots$ where $i_1, i_2 \geq 0$ to denote formula structures. A *sequent* is an expression of the form $\langle\alpha\rangle^i \Rightarrow \beta$ where $i \geq 0$ for some formulae α and β.

Definition 2. *The Gentzen sequent calculus* G5 *consists of the following axioms and inference rules:*

(1) *Axioms:*

$$(\mathrm{Id})\ \varphi \Rightarrow \varphi \qquad (\bot)\ \langle\bot\rangle^i \Rightarrow \varphi \qquad (\top)\ \langle\varphi\rangle^i \Rightarrow \top$$

$$(\mathrm{D})\ \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \qquad (\mathrm{DN})\ \varphi \Leftrightarrow \neg\neg\varphi$$

(2) *Connective rules:*

$$\frac{\langle\varphi\rangle^i \Rightarrow \chi}{\langle\varphi \wedge \psi\rangle^i \Rightarrow \chi}(\wedge L) \qquad \frac{\langle\chi\rangle^i \Rightarrow \varphi \quad \langle\chi\rangle^i \Rightarrow \psi}{\langle\chi\rangle^i \Rightarrow \varphi \wedge \psi}(\wedge\mathrm{R})$$

$$\frac{\langle\varphi\rangle^i \Rightarrow \chi \quad \langle\psi\rangle^i \Rightarrow \chi}{\langle\varphi \vee \psi\rangle^i \Rightarrow \chi}(\vee\mathrm{L}) \qquad \frac{\langle\chi\rangle^i \Rightarrow \psi}{\langle\chi\rangle^i \Rightarrow \psi \vee \varphi}(\vee\mathrm{R})$$

(3) *Modal rules*

$$\frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\Diamond\varphi\rangle^{i} \Rightarrow \psi}(\Diamond\mathrm{L}) \quad \frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\varphi\rangle^{i+1} \Rightarrow \Diamond\psi}(\Diamond\mathrm{R})$$

$$\frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\Box\varphi\rangle^{i+1} \Rightarrow \psi}(\Box\mathrm{L}) \quad \frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i} \Rightarrow \Box\psi}(\Box\mathrm{R})$$

$$\frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i} \Rightarrow \psi}(\mathrm{T}) \quad \frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i+2} \Rightarrow \psi}(4) \quad \frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\neg\psi\rangle^{i} \Rightarrow \neg\varphi}(\Diamond\Box)$$

(4) *Cut rule*

$$\frac{\langle\varphi\rangle^{i} \Rightarrow \chi \quad \langle\chi\rangle^{j} \Rightarrow \psi}{\langle\varphi\rangle^{i+j} \Rightarrow \psi}(\mathrm{Cut})$$

where $i, j \geq 0$. By $\vdash_{G5} \langle\alpha\rangle^i \Rightarrow \beta$, we mean the sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is provable in G5. A sequent is called *simple sequent* if it is of the form $\alpha \Rightarrow \beta$ for some formulae α and β. Let Φ be a finite set of simple sequents. By $\Phi \vdash_{G5} \langle\alpha\rangle^i \Rightarrow \beta$, we mean that sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is derivable from Φ in G5.

Proposition 2. *In G5, the following holds:*

- $\vdash_{G5} \Diamond(\alpha \vee \beta) \Rightarrow \Diamond\alpha \vee \Diamond\beta$
- $\vdash_{G5} \Box(\alpha \wedge \beta) \Rightarrow \Box\alpha \wedge \Box\beta$
- $\vdash_{G5} \Box\alpha \Rightarrow \Box\Box\alpha$
- $\vdash_{G5} \Box\alpha \Rightarrow \alpha$
- $\vdash_{G5} \Diamond\alpha \Rightarrow \Box\Diamond\alpha$
- $\vdash_{G5} \Diamond\alpha \Rightarrow \neg\Box\neg\alpha$
- $\vdash_{G5} \neg\Box\neg\alpha \Rightarrow \Diamond\alpha$

Let F be a finite set of formulae closed under subformulae. Define F^{qb} be the closure of F under $\wedge, \vee, \neg$. A set of formula T is called qb-closed if $T = F^{qb}$ for some finite set F which is closed under subformulae. A sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is called a T sequent if $\alpha, \beta \in T$. A derivation from Φ in G5 of a T-sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is called a T-derivation if all sequents appearing in the derivation are T-sequents, which is denoted by $\Phi \vdash_{G5} \langle\alpha\rangle^i \Rightarrow_T \beta$. Assume that $\Phi \vdash_{G5} \langle\varphi\rangle^{i+j} \Rightarrow_T \psi$. A formula γ is called a T interpolant of $\langle\varphi\rangle^i$ if $\gamma \in T$, $\Phi \vdash_{G5} \langle\varphi\rangle^i \Rightarrow_T \gamma$ and $\Phi \vdash_{G5} \langle\gamma\rangle^j \Rightarrow_T \psi$ and additionally $\Phi \vdash_{G5} \langle\gamma\rangle \Rightarrow_T \gamma$ if $i \geq 1$.

Lemma 2 (Interpolant). *If* $\Phi \vdash_{G5} \langle\varphi\rangle^{i+j} \Rightarrow_T \psi$*, then* $\langle\varphi\rangle^i$ *has a* T *interpolant.*

Proof. We proceed by induction on the length of derivation. Axiom is trivial. (Cut) is easy. Assume that the end sequent is obtained by a rule (R). If $i = 0$ then obviously φ is a required interpolant. Let $i \geq 1$ Here we consider three cases. Others can be treated similarly.

($\vee$L) Assume the premise are $\langle\delta\rangle^{i+j} \Rightarrow_T \psi$ and $\langle\chi\rangle^{i+j} \Rightarrow \psi$ and $\varphi = \delta \vee \chi$. Then by induction hypothesis, there are $\gamma_1, \gamma_2 \in T$ such that (1) $\Phi \vdash_{G5} \langle\delta\rangle^i \Rightarrow_T \gamma_1$, (2) $\Phi \vdash_{G5} \langle\chi\rangle^i \Rightarrow_T \gamma_2$, (3) $\Phi \vdash_{G5} \langle\gamma_1\rangle^j \Rightarrow_T \psi$, (4) $\Phi \vdash_{G5} \langle\gamma_2\rangle^j \Rightarrow_T \psi$, (5) $\Phi \vdash_{G5} \langle\gamma_1\rangle \Rightarrow \gamma_1$ and (6) $\Phi \vdash_{G5} \langle\gamma_2\rangle \Rightarrow \gamma_2$. By applying ($\vee R$) to (1) and (2), one gets (7) $\Phi \vdash_{G5} \langle\delta\rangle^i \Rightarrow_T \gamma_1 \vee \gamma_2$, (8) $\Phi \vdash_{G5} \langle\chi\rangle^i \Rightarrow_T \gamma_1 \vee \gamma_2$. Then by applying ($\vee L$) to (7) and (8), one obtains (9) $\Phi \vdash_{G5} \langle \delta \vee \chi \rangle^i \Rightarrow_T \gamma_1 \vee \gamma_2$. Further by applying ($\vee L$) to (3) and (4) one gets applying $\Phi \vdash_{G5} \langle \gamma_1 \vee \gamma_2 \rangle^i \Rightarrow_T \psi$. If $i \geq 1$, then applying ($\vee R$) and ($\vee L$) to (5) and (6), one gets $\Phi \vdash_{G5} \langle \gamma_1 \vee \gamma_2 \rangle \Rightarrow \gamma_1 \vee \gamma_2$. Thus $\gamma_1 \vee \gamma_2$ is a required interpolant.

($\Diamond\Box$) Assume that the premise is $\langle \varphi \rangle^{i+j} \Rightarrow_T \psi$. By induction hypothesis there is $\gamma \in T$ such that $\Phi \vdash_{G5} \langle \varphi \rangle^i \Rightarrow_T \gamma$, $\Phi \vdash_{G5} \langle \gamma \rangle^j \Rightarrow_T \psi$ and $\Phi \vdash_{G5} \langle \gamma \rangle \Rightarrow \gamma$. Then by rule ($\Diamond\Box$), one gets (1) $\Phi \vdash_{G5} \langle \neg\gamma \rangle^i \Rightarrow \neg\varphi$, (2) $\Phi \vdash_{G5} \langle \neg\psi \rangle^i \Rightarrow \neg\gamma$ and (3) $\Phi \vdash_{G5} \langle \neg\gamma \rangle \Rightarrow_T \neg\gamma$. Thus $\Phi \vdash_{G5} \langle \neg\gamma \rangle^k \Rightarrow_T \neg\gamma$ for any $k \geq 0$. Hence one gets (5) $\Phi \vdash_{G5} \langle \neg\gamma \rangle^i \Rightarrow_T \neg\gamma$ and (6) $\Phi \vdash_{G5} \langle \neg\gamma \rangle^j \Rightarrow_T \neg\gamma$. By applying (T) to (1) and (2) one gets (7) $\Phi \vdash_{G5} \neg\psi \Rightarrow_T \neg\gamma$ and (8) $\Phi \vdash_{G5} \neg\gamma \Rightarrow_T \neg\varphi$. Then by (Cut) to (7) and (5), one gets $\Phi \vdash_{G5} \langle \neg\psi \rangle^i \Rightarrow_T \neg\gamma$. By (Cut) to (8) and (6), one gets $\Phi \vdash_{G5} \langle \neg\gamma \rangle^j \Rightarrow_T \neg\varphi$. Obviously γ is a required interpolant. Hence $\neg\gamma \in T$ is a required interpolant.

(4) Assume that the premise is $\langle \varphi \rangle^{k+1} \Rightarrow_T \psi$ and the conclusion is $\langle \varphi \rangle^{k+2} \Rightarrow_T \psi$. Let $i + j = k + 2$. By induction hypothesis, there is $\gamma \in T$ such that $\Phi \vdash_{G5} \langle \varphi \rangle^i \Rightarrow_T \gamma$, $\Phi \vdash_{G5} \langle \gamma \rangle^{j-1} \Rightarrow_T \psi$ and $\Phi \vdash_{G5} \langle \gamma \rangle \Rightarrow_T \gamma$. Hence by (Cut) one gets $\Phi \vdash_{G5} \langle \gamma \rangle^j \Rightarrow_T \psi$. Hence γ is a required interpolant.

Notice that we did not assume that the set T is closed under any modal operations. Hence the above interpolant lemma with respect to this kind of T is based on the fact that in our sequent calculus we introduce modal structural operation and interpolate modal axioms by structural rules. Further the additional condition is required for the proof of case (4). Without the additional condition, one can not prove the case (4) when $k = 0$ and $i = 1$.

An algebraic model of G5 is a pair $(\mathbb{G}, \sigma)$ such that $\mathbb{G}$ is a **tqBa5**, and σ is a mapping from Prop into $\mathbb{G}$, called a *valuation*, which is extended to formulae and formula trees as follows:

$$\sigma(\Box\alpha) = \Box\sigma(\alpha), \sigma(\Diamond\alpha) = \Diamond\sigma(\alpha)$$
$$\sigma(\alpha \wedge \beta) = \sigma(\alpha) \wedge \sigma(\beta), \quad \sigma(\alpha \vee \beta) = \sigma(\alpha) \vee \sigma(\beta),$$
$$\sigma(\neg\alpha) = \neg\sigma(\alpha), \quad \sigma(\langle\alpha\rangle^{i+1}) = \Diamond\sigma(\langle\alpha\rangle^i).$$

A sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is said to be *true* in a model $(\mathbb{G}, \sigma)$ written $\mathbb{G}, \sigma \models \langle\alpha\rangle^i \Rightarrow \beta$, if $\sigma(\langle\alpha\rangle^i) \leq \sigma(\beta)$ (here $\leq$ is the lattice order in $\mathbb{G}$). It is *valid* in $\mathbb{G}$, if it is true in $(\mathbb{G}, \sigma)$, for any valuation σ. It is valid in a class of algebras $\mathbb{K}$, if it is valid in all algebras from $\mathbb{K}$. $\Phi \models \langle\alpha\rangle^i \Rightarrow \beta$ with respect to $\mathbb{K}$, if $\langle\alpha\rangle^i \Rightarrow \beta$ is true in all models $(\mathbb{G}, \sigma)$ such that $\mathbb{G} \in \mathbb{K}$ and all sequents from Φ are true in $(\mathbb{G}, \sigma)$.

Remark 2. G5 is *strongly complete* with respect to class tq$\mathbb{B}$a5: for any set of sequents Φ and any sequent $\langle\alpha\rangle^i \Rightarrow \beta$, $\Phi \vdash_{\mathrm{G5}} \langle\alpha\rangle^i \Rightarrow \beta$ if and only if $\Phi \models \langle\alpha\rangle^i \Rightarrow \beta$ with respect to tqBa5. So it follows that G5 is *weakly complete* with respect to tqBa5: the sequents provable in G5 are precisely the sequents valid in tqBa5. The proof of *strongly completeness* of G5 with respect to tqBa5 follows from the same proof of strong finite model property (definition see Sect. 4) of G5 in Sect. 4.

4 FEP for tqBa5

Given a qb-closed set of formula T and a set of simple T-sequents Φ, we define an order $\leq_T$ on formula structures. The set of T formula structures denote by T^s, consist of all formula structures whose formulae appearing in them belong to T. Let $\langle\alpha_1\rangle^i, \langle\alpha_2\rangle^j \in T^s$, we say $\langle\alpha_1\rangle^i \leq_T \langle\alpha_2\rangle^j$ if $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha_2\rangle^j\rangle^t \Rightarrow_T \beta$ implies $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha_1\rangle^i\rangle^t \Rightarrow_T \beta$ for any context $\langle\rangle^t$ where $t \geq 0$ and T formula β. Let $\langle\alpha_1\rangle^i \approx_T \langle\alpha_2\rangle^j$ if $\langle\alpha_1\rangle^i \leq_T \langle\alpha_2\rangle^j$ and $\langle\alpha_2\rangle^j \leq_T \langle\alpha_1\rangle^i$. Obviously $\approx_T$ is a equivalence relation on T formula structures.

We define

$$\{\langle\alpha\rangle^i\}^{\approxeq}_T = \{\langle\beta\rangle^j | \langle\beta\rangle^j \approx_T \langle\alpha\rangle^i\}(i,j \geq 0)$$

Obviously

$$\{\alpha\}^{\approxeq}_T = \{\langle\beta\rangle^j | \langle\beta\rangle^j \approx_T \alpha\}(j \geq 0)$$

Let $T^s/_{\approx_T}$ denote the set of all $\{\langle\alpha\rangle^i\}^{\approxeq}_T$ where $\langle\alpha\rangle^i \in T^s$ where $i \geq 0$ and $T/_{\approx_T}$ denote the set of all $\{\alpha\}^{\approxeq}_X$ where $\alpha \in T$. Define $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$ if $\langle\alpha\rangle^i \leq_T \langle\beta\rangle^j$. This is well defined since if $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$, $\langle\varphi\rangle^p \in \{\langle\alpha\rangle^i\}^{\approxeq}_T$ and $\langle\psi\rangle^q \in \{\langle\alpha\rangle^i\}^{\approxeq}_T$, then $\langle\varphi\rangle^p \leq_T \langle\psi\rangle^q$.

We define a closure operation C on $T^{\approx}$ as follows:

$$C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\underbrace{\bigwedge}_{1\leq j\leq n} \beta_j\}^{\approxeq}_T \quad \text{for any}\, \{\beta_j\}^{\approxeq}_T \in T^{\approx}\, \text{s.t.}\, \{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\beta_j\}^{\approxeq}_T$$

Remark 3. For any $\{\langle\alpha\rangle^i\}^{\approxeq}_T$, $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\top\}^{\approxeq}_T$. Thus $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$ always exists. Since T is closed under $\wedge$, for any $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$ there exists a $\{\beta\}^{\approxeq}_T$ such that $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\beta\}^{\approxeq}_T$. Let $C(T^s/_{\approx_T})$ denoted the sets of all $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$. Clearly $C(T^s/_{\approx_T}) \subseteq T/_{\approx_T}$. Since T is qb-closed, $T/_{\approx_T}$ is finite. Thus $C(T^s/_{\approx_T})$ is finite.

Lemma 3. *For any $\{\langle\alpha\rangle^i\}^{\approxeq}_T, \{\langle\beta\rangle^j\}^{\approxeq}_T \in T^{\approx}$, the following hold:*

(1) $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$.
(2) if $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$, *then* $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \preceq_T C(\{\langle\beta\rangle^j\}^{\approxeq}_T)$.
(3) $C(C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)) \subseteq C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$

Proof. (1) Let $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi_1 \wedge \ldots \wedge \varphi_n\}^{\approxeq}_T$ such that $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\varphi_j\}^{\approxeq}_T$ where $1 \leq j \leq n$. We suffice to show that $\langle\alpha\rangle^i \leq_T \varphi_1 \wedge \ldots \wedge \varphi_n$. Since $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\varphi_j\}^{\approxeq}_T$ where $1 \leq i \leq n$, $\langle\alpha\rangle^i \leq_T \varphi_j$ for all $1 \leq j \leq n$. Assume that $\Phi \vdash_{\mathrm{G5}} \langle\varphi_1 \wedge \ldots \wedge \varphi_n\rangle^t \Rightarrow_T \psi$. Further by $(\wedge R)$, one gets $\Phi \vdash_{\mathrm{G5}} \langle\alpha\rangle^i \Rightarrow_T \varphi_1 \wedge \ldots \wedge \varphi_n$. Then by (Cut) $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha\rangle^i\rangle^t \Rightarrow_T \psi$. Thus $\langle\alpha\rangle^i \leq_T \varphi_1 \wedge \ldots \wedge \varphi_n$. Consequently $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$.

(2) Let $C(\{\langle\beta\rangle^j\}^{\approxeq}_T) = \{\varphi\}^{\approxeq}_T$. By (1) $\{\langle\beta\rangle^j\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T$. Since $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$, $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T$. Hence by definition $C(\{\langle\alpha\rangle^i\}^{\approx}\} = \{\varphi\wedge\chi\}^{\approxeq}_T$ for some $\chi \in T$. We suffice to show that $\varphi\wedge\chi \leq_T \varphi$. Assume that $\Phi \vdash_{\mathrm{G5}} \langle\varphi\rangle^t \Rightarrow_T \psi$. Since $\Phi \vdash_{\mathrm{G5}} \varphi \wedge \chi \Rightarrow_T \varphi$, by (Cut), one gets $\Phi \vdash_{\mathrm{G5}} \langle\varphi \wedge \chi\rangle^t \Rightarrow_T \psi$. Thus $\varphi \wedge \chi \leq_T \varphi$. Consequently $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi \wedge \chi\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T = C(\{\langle\beta\rangle^j\}^{\approxeq}_T)$.

(3) First we show that $C(\{\varphi\}^{\approxeq}_T) \preceq_T \{\varphi\}^{\approxeq}_T$. Since $\{\varphi\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T$, by definition $C(\{\varphi\}^{\approxeq}_T) = \{\varphi \wedge \chi\}^{\approxeq}_T$ for some $\varphi \in T$. Obviously $\varphi \wedge \chi \leq_T \varphi$. Thus $\{\varphi \wedge \chi\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T$. Hence $C(\{\varphi\}^{\approxeq}_T) =\preceq_T \{\varphi\}^{\approxeq}_T$. Clearly for any $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$, there is a $\{\varphi\}^{\approxeq}_T$ where $\varphi \in T$ such that $C(\{\varphi\}^{\approxeq}_T) = \{\varphi\}^{\approxeq}_T$. Consequently $C(C(\{\varphi\}^{\approxeq}_T)) \subseteq C(\{\varphi\}^{\approxeq}_T)$.

We defined a interitor operation I on $T^{\approx}$ as follows:

$$I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{ \underbrace{\bigvee}_{1\leq j\leq n} \beta_j\}^{\approxeq}_T \quad \text{for any } \{\beta_j\}^{\approxeq}_T \text{ s.t. } \{\beta_j\}^{\approxeq}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T.$$

Remark 4. For any $\{\langle\alpha\rangle^i\}^{\approxeq}_T$, $\{\bot\}^{\approxeq}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$. Thus $C(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$ always exists. Since T is closed under $\vee$, for any $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$ there exists a $\{\beta\}^{\approxeq}_T$ such that $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\beta\}^{\approxeq}_T$. Let $I(T^s/_{\approx_T})$ denoted the sets of all $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$. Clearly $C(T^s/_{\approx_T}) \subseteq T/_{\approx_T}$. Since T is qb-closed, $T/_{\approx_T}$ is finite. Thus $I(T^s/_{\approx_T})$ is finite.

Lemma 4. *For any $\{\langle\alpha\rangle^i\}^{\approxeq}_T, \{\langle\beta\rangle^j\}^{\approxeq}_T \in T^{\approx}$, the following hold:*

(1) $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$.
(2) if $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$, then $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \preceq_T I(\{\langle\beta\rangle^j\}^{\approxeq}_T)$.
(3) $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \subseteq I(I(\{\langle\alpha\rangle^i\}^{\approxeq}_T))$

Proof. (1) Let $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi_1 \vee \ldots \vee \varphi_n\}^{\approxeq}_T$ such that $\{\varphi_j\}^{\approxeq}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$ where $1 \leq j \leq n$. We suffice to show that $\varphi_1 \vee \ldots \vee \varphi_n \leq_T \langle\alpha\rangle^i$. Since $\{\varphi_j\}^{\approxeq}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$ where $1 \leq j \leq n$, $\varphi_j \leq_T \langle\alpha\rangle^i$ for all $1 \leq j \leq n$. Assume that $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha\rangle^i\rangle^t \Rightarrow_T \psi$. Then $\Phi \vdash_{\mathrm{G5}} \langle\varphi_j\rangle^t \Rightarrow_T \psi$ for all $1 \leq j \leq n$. Further by $(\vee L)$, one gets $\Phi \vdash_{\mathrm{G5}} \langle\varphi_1 \vee \ldots \vee \varphi_n\rangle^t \Rightarrow_T \psi$. Thus $\varphi_1 \vee \ldots \vee \varphi_n \leq_T \langle\alpha\rangle^i$. Consequently $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$.

(2) Let $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi\}^{\approxeq}_T$. By (1) $\{\varphi\}^{\approxeq}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approxeq}_T$. Since $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$, $\{\varphi\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$. Hence by definition $I(\{\langle\beta\rangle^j\}^{\approx}\} = \{\varphi \vee \chi\}^{\approxeq}_T$ for some $\chi \in T$. We suffice to show that $\varphi \leq_T \varphi \vee \chi$. Assume that $\Phi \vdash_{\mathrm{G5}} \langle\varphi \vee \chi\rangle^t \Rightarrow_T \psi$. Since $\Phi \vdash_{\mathrm{G5}} \varphi \Rightarrow_T \varphi \vee \chi$, by (Cut), one gets $\Phi \vdash_{\mathrm{G5}} \langle\varphi\rangle^t \Rightarrow_T \psi$. Thus $\varphi \leq_T \varphi \vee \chi$. Consequent $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi\}^{\approxeq}_T \preceq_T \{\varphi \vee \chi\}^{\approxeq}_T = I(\{\langle\beta\rangle^j\}^{\approxeq}_T)$.

(3) First we show that $\{\varphi\}^{\approxeq}_T \preceq_T I(\{\varphi\}^{\approxeq}_T)$. Since $\{\varphi\}^{\approxeq}_T \preceq_T \{\varphi\}^{\approxeq}_T$, by definition $I(\{\varphi\}^{\approxeq}_T) = \{\varphi \vee \chi\}^{\approxeq}_T$ for some $\chi \in T$. Obviously $\varphi \leq_T \varphi \vee \chi$. Thus $\{\varphi \wedge \chi\}^{\approxeq}_T \preceq_T$. Hence $\{\varphi\}^{\approxeq}_T \preceq_T I(\{\varphi\}^{\approxeq}_T)T$. Clearly for any $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T)$, there is a $\{\varphi\}^{\approxeq}_T$ where $\varphi \in T$ such that $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) = \{\varphi\}^{\approxeq}_T$. Consequently $I(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \subseteq I(I(\{\langle\alpha\rangle^i\}^{\approxeq}_T))$.

We define a unary operations $\Diamond$ and $\Box$ on $T^{\approx}$:

$$\Diamond\{\langle\alpha\rangle^i\}^{\approxeq}_T = \{\langle\alpha\rangle^{i+1}\}^{\approxeq}_T$$
$$\Box\{\langle\alpha\rangle^i\}^{\approxeq}_T = \{\langle\varphi\rangle^j\}^{\approxeq}_T \quad \text{s.t} \quad (bc1) \quad \text{and} \quad (bc2) \quad \text{holds}$$

where (bc1): $\langle\varphi\rangle^{j+1} \leq_T \langle\alpha\rangle^i$ and (bc2): for any $\langle\delta\rangle^k \leq_T \langle\alpha\rangle^i$, $\langle\delta\rangle^k \leq_T \langle\varphi\rangle^{i+1}$ $(k \geq 1)$. Notice that such $\{\langle\varphi\rangle^j\}^{\approxeq}_T$ always exists. Since for any $\langle\delta_1\rangle^k_1 \leq_T \langle\alpha\rangle^i$ and $\langle\delta_2\rangle^k_2 \leq_T \langle\alpha\rangle^i$ where $k_1, k_2 \geq 1$, we have $\langle\delta_1\rangle^k_1 \leq \langle\delta_1 \vee \delta_2\rangle$, $\langle\delta_1\rangle^k_1 \leq_T \langle\delta_1 \vee \delta_2\rangle$ and $\langle\delta_1 \vee \delta_2\rangle \leq_T \langle\alpha\rangle^i$.

Lemma 5. *If* $\{\langle\varphi\rangle^i\}_T^{\approxeq} \preceq_T \{\langle\psi\rangle^j\}_T^{\approxeq}$ *where* $i, j \geq 0$, *then* $\Diamond\{\langle\varphi\rangle^i\}_T^{\approxeq} \preceq_T \Diamond\{\langle\psi\rangle^j\}_T^{\approxeq}$.

Proof. Assume that $\{\langle\varphi\rangle^i\}_T^{\approxeq} \preceq_T \{\langle\psi\rangle^j\}_T^{\approxeq}$. Then $\langle\varphi\rangle^i \leq_T \langle\psi\rangle^j$. Let $\vdash_{G5} \langle\langle\langle\psi\rangle^i\rangle\rangle^t \Rightarrow_T \chi$. We get $\vdash_{G5} \langle\langle\langle\varphi\rangle^j\rangle\rangle^t \Rightarrow_T \chi$ for any $\langle-\rangle^t$ and χ. Hence $\langle\varphi\rangle^{i+1} \leq_T \langle\psi\rangle^{j+1}$. Consequently $\Diamond\{\langle\varphi\rangle^i\}_T^{\approxeq} \preceq_T \Diamond\{\langle\psi\rangle^j\}_T^{\approxeq}$.

Lemma 6. *For any* $\{\langle\alpha\rangle^i\}_T^{\approxeq}, \{\langle\beta\rangle^j\}_T^{\approxeq} \in T^{\approx}$, $\Diamond C(\{\langle\alpha\rangle^i\}_T^{\approxeq}) \preceq_T C(\Diamond\{\langle\alpha\rangle^i\}_T^{\approxeq})$.

Proof. Let $C(\Diamond\{\langle\alpha\rangle^i\}_T^{\approxeq}) = \{\delta\}_T^{\approxeq}$. Then Lemma 3 (1) $\Diamond\{\langle\alpha\rangle^i\}_T^{\approxeq} \leq_T \{\delta\}_T^{\approxeq}$. Thus $\langle\alpha\rangle^{i+1} \leq_T \delta$. Hence $\vdash_{G5} \langle\alpha\rangle^{i+1} \Rightarrow_T \delta$. By Lemma 2 there is a γ such that $\vdash_{G5} \langle\alpha\rangle^i \Rightarrow_T \gamma$ and $\vdash_{G5} \langle\gamma\rangle \Rightarrow_T \delta$. Consequently $\{\langle\alpha\rangle^i\}_T^{\approxeq} \preceq_T \{\gamma\}_T^{\approxeq}$ and $\Diamond\{\gamma\}_T^{\approxeq} \preceq_T \{\delta\}_T^{\approxeq}$. By definition, $C(\{\langle\alpha\rangle^i\}_T^{\approxeq}) \preceq_T \{\gamma\}_T^{\approxeq}$. By Lemma 5 one gets $\Diamond C(\{\langle\alpha\rangle^i\}_T^{\approxeq}) \preceq_T \Diamond\{\gamma\}_T^{\approxeq}$. Hence $\Diamond C(\{\langle\alpha\rangle^i\}_T^{\approxeq}) \preceq_T \{\delta\}_T^{\approxeq}$. Therefore $\Diamond C(\{\langle\alpha\rangle^i\}_T^{\approxeq}) \preceq_T C(\Diamond\{\langle\alpha\rangle^i\}_T^{\approxeq})$.

We define two unary operation on $T/_{\approx_T}$ as follows:

$$\blacklozenge(\{\varphi\}_T^{\approxeq}) = C(\Diamond(\{\varphi\}_T^{\approxeq}))$$

$$\blacksquare(\{\varphi\}_T^{\approxeq}) = I(\Box(\{\varphi\}_T^{\approxeq}))$$

Lemma 7. *For any* $\{\varphi\}_T^{\approxeq} \in T/_{\approx_T}$, *the following hold:*

(1) $\blacklozenge\blacklozenge(\{\varphi\}_T^{\approxeq}) \preceq_T \blacklozenge(\{\varphi\}_T^{\approxeq})$
(2) $(\{\varphi\}_T^{\approxeq}) \preceq_T \blacklozenge(\{\varphi\}_T^{\approxeq})$
(3) If $\blacklozenge((\{\varphi\}_T^{\approxeq})) \preceq_T \{\psi\}_T^{\approxeq}$, *then* $\blacklozenge((\{\neg\psi\}_T^{\approxeq})) \preceq_T \{\neg\varphi\}_T^{\approxeq}$

Proof. (1) Let $\vdash_{G5} \langle\langle\varphi\rangle\rangle^t \Rightarrow_T \psi$ for some context $\langle-\rangle^t$ and formula $\psi \in T$. By rule (4), one gets $\vdash_{G5} \langle\langle\varphi\rangle^2\rangle^t \Rightarrow_T \psi$. Thus $\langle\varphi\rangle^2 \leq_T \langle\varphi\rangle$. Hence $\Diamond\Diamond\{\varphi\}_T^{\approxeq} \preceq \Diamond\{\varphi\}_T^{\approxeq}$. By Lemma 3 (2), one gets $C(\Diamond\Diamond\{\varphi\}_T^{\approxeq}) \preceq_T C(\Diamond\{\varphi\}_T^{\approxeq})$. By Lemma 6, one gets $\Diamond C(\Diamond\{\varphi\}_T^{\approxeq}) \preceq_T C(\Diamond\Diamond\{\varphi\}_T^{\approxeq})$. Therefore $\Diamond C(\Diamond\{\varphi\}_T^{\approxeq}) \preceq_T C(\Diamond\{\varphi\}_T^{\approxeq})$. By Lemma 3 (2) and (3), one gets $C(\Diamond C(\Diamond\{\varphi\}_T^{\approxeq})) \preceq_T C(\Diamond\{\varphi\}_T^{\approxeq})$. Hence $\blacklozenge\blacklozenge(\{\varphi\}_T^{\approxeq}) \preceq_T \blacklozenge(\{\varphi\}_T^{\approxeq})$.

(2) Let $\vdash_{G5} \langle\langle\varphi\rangle\rangle^t \Rightarrow_T \psi$ for some context $\langle-\rangle^t$ and formula $\psi \in T$. By rule (T), one gets $\vdash_{G5} \langle\varphi\rangle^t \Rightarrow_T \psi$. Thus $\varphi \leq_T \langle\varphi\rangle$. Hence $\{\varphi\}_T^{\approxeq} \preceq_T \Diamond\{\varphi\}_T^{\approxeq}$. By Lemma 3 (1), $\Diamond\{\varphi\}_T^{\approxeq} \preceq_T C(\Diamond\{\varphi\}_T^{\approxeq})$. Thus $\{\varphi\}_T^{\approxeq} \preceq_T C(\Diamond\{\varphi\}_T^{\approxeq})$.

(3) Assume that $\blacklozenge(\{\varphi\}_T^{\approxeq}) \preceq_T \{\psi\}_T^{\approxeq}$. By Lemma 6, one can get $\Diamond(\{\varphi\}_T^{\approxeq}) \preceq_T \blacklozenge(\{\varphi\}_T^{\approxeq})$. Thus $\Diamond(\{\varphi\}_T^{\approxeq}) \preceq_T \{\psi\}_T^{\approxeq}$. Hence $\langle\varphi\rangle \leq_T \psi$. Therefore $\vdash_{G5} \langle\varphi\rangle \Rightarrow_T \psi$. By rule $(\Diamond\Box)$, one gets $\vdash_{G5} \langle\neg\psi\rangle \Rightarrow_T \neg\varphi$. Hence $\Diamond(\{\neg\psi\}_T^{\approxeq}) \preceq_T \{\neg\varphi\}_T^{\approxeq}$. By definition $\blacklozenge((\{\neg\psi\}_T^{\approxeq})) \preceq_T \{\neg\varphi\}_T^{\approxeq}$

Lemma 8. *For any* $\{\varphi\}_T^{\approxeq}, \{\psi\}_T^{\approxeq} \in T/_{\approx_T}$, $\blacklozenge(\{\varphi\}_T^{\approxeq}) \preceq_T \{\psi\}_T^{\approxeq}$ *iff* $\{\varphi\}_T^{\approxeq} \preceq_T \blacksquare\{\psi\}_T^{\approxeq}$.

Proof. Assume that $\blacklozenge\{\varphi\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$. Then $C(\Diamond\{\varphi\}_T^{\approx}) \preceq_T \{\psi\}_T^{\approx}$. Thus $C(\{\langle\varphi\rangle\}_T^{\approx}) \preceq_T \{\psi\}_T^{\approx}$. By Lemma 3 (1), $\{\langle\varphi\rangle\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$. Hence $\langle\varphi\rangle \leq_T \psi$. So $\{\varphi\}_T^{\approx} \preceq_T \Box\{\psi\}_T^{\approx}$. By definition $\{\varphi\}_T^{\approx} \preceq_T I(\Box\{\psi\}_T^{\approx}) = \blacksquare\{\psi\}_T^{\approx}$. Conversely assume that $\{\varphi\}_T^{\approx} \preceq_T \blacksquare\{\psi\}_T^{\approx}$. Then $\{\varphi\}_T^{\approx} \preceq_T I(\Box\{\psi\}_T^{\approx}])$. By Lemma 4 (1), one gets $\{\varphi\}_T^{\approx} \preceq_T \Box\{\psi\}_T^{\approx}$. Let $\Box\{\psi\}_T^{\approx} = \{\langle\gamma\rangle^i\}_T^{\approx}$ such that $\langle\gamma\rangle^{i+1} \leq_T \psi$. Hence $\{\langle\gamma\rangle^{i+1}\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$. By Lemma 5 $\Diamond\{\varphi\}_T^{\approx} \preceq_T \Diamond\{\langle\gamma\rangle^i\}_T^{\approx}$. Thus $\{\langle\varphi\rangle\}_T^{\approx} \preceq_T \{\langle\gamma\rangle^{i+1}\}_T^{\approx}$. Consequently $\{\langle\varphi\rangle\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$. By definition, $C(\Diamond\{\varphi\}_T^{\approx}) = C(\{\langle\varphi\rangle\}_T^{\approx}) \preceq_T \{\psi\}_T^{\approx}$. Thus $\blacklozenge\{\varphi\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$.

Now we construct a finite tqBa5 on $T/_{\approx_T}$. For any $\{\varphi\}_T^{\approx}, \{\psi\}_T^{\approx} \in T/_{\approx_T}$ one defines:

- $\{\varphi\}_T^{\approx} \wedge \{\psi\}_T^{\approx} = \{\varphi \wedge \psi\}_T^{\approx}$,
- $\{\varphi\}_T^{\approx} \vee \{\psi\}_T^{\approx} = \{\varphi \vee \psi\}_T^{\approx}$
- $\neg\{\varphi\}_T^{\approx} = \{\neg\varphi\}_T^{\approx}$

Let $\blacklozenge$ and $\blacksquare$ be defined as above. Let $\mathbb{A}(T, \Phi) = (T/_{\approx_T}, \wedge, \vee, \neg, \blacklozenge, \blacksquare)$. Clearly $\mathbb{A}(T, \Phi)$ is a qBa. By Lemma 6 and 7, $\mathbb{A}(T, \Phi)$ is a tqBa5. Since $T/_{\approx_T}$ is finite, $\mathbf{A}(T, \Phi)$ is finite.

We define a assignment σ from T-formulae to $\mathbb{A}(T, \Phi)$ as follows: $\sigma(p) = \{p\}_T^{\approx}$ for any $p \in T$. σ can be extended to formulae and formula structures naturally. By induction on the complexity of formulae, one obtains the following fact

Proposition 3. $\sigma(\varphi) = \{\varphi\}_T^{\approx}$

Lemma 9. $\Diamond^i\{\varphi\}_T^{\approx} \preceq_T \Diamond_c^i\{\varphi\}_T^{\approx}$

Proof. By induction on the number i. If $i = 1$, then the claim holds by Lemma 5. Otherwise by induction hypothesis, one gets $\Diamond^{i-1}\{\varphi\}_T^{\approx} \preceq_T \Diamond_c^{i-1}\{\varphi\}_T^{\approx}$. Then by Lemma 4 and Lemma 3 (2), one gets $C(\Diamond\Diamond^{i-1}\{\varphi\}_T^{\approx}) \preceq_T \Diamond_c^i\{\varphi\}_T^{\approx}$. By Lemma 5, $\Diamond^i\{\varphi\}_T^{\approx} \preceq_T C(\Diamond\Diamond^{i-1}\{\varphi\}_T^{\approx})$. Hence $\Diamond^i\{\varphi\}_T^{\approx} \preceq_T \Diamond_c^i\{\varphi\}_T^{\approx}$.

Lemma 10. *If* $\Phi \nvdash_{G5} \langle\varphi\rangle^i \Rightarrow_T \psi$, *then* $\Phi \not\models_{\mathbb{A}(\mathbb{T},\Phi)} \sigma(\langle\varphi\rangle^i) \preceq_T \sigma(\psi)$

Proof. Let $\alpha \Rightarrow \beta \in \Phi$. Then $\Phi \vdash_{G5} \alpha \Rightarrow \beta$. Hence $\alpha \leq_T \beta$. Thus $\{\alpha\}_T^{\approx} \preceq \{\beta\}_T^{\approx}$. Hence $\models_{\mathbb{A}(\mathbb{T},\Phi)} \Phi$. Assume that $\models_{\mathbb{A}(\mathbb{T},\ \Phi)} \sigma(\langle\varphi\rangle^i) \preceq_T \sigma(\psi)$. Since $\sigma(\langle\varphi\rangle^i) = \Diamond_c^i\{\varphi\}_T^{\approx}$ and $\sigma(\psi) = \{\psi\}_T^{\approx}$. Hence $\Diamond_c^i\{\varphi\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$. Further by Lemma 8 $\Diamond^i\{\varphi\}_T^{\approx} \preceq_T \Diamond_c^i\{\varphi\}_T^{\approx}$. Thus $\Diamond^i\{\varphi\}_T^{\approx} \preceq_T \{\psi\}_T^{\approx}$ which yields $\langle\varphi\rangle^i \leq_T \psi$. Therefore $\Phi \vdash_{G5} \langle\varphi\rangle^i \Rightarrow \psi$. Contradiction.

Theorem 2. *If* $\Phi \nvdash_{G5} \langle\varphi\rangle^i \Rightarrow \psi$ *then there exists a model* $(\mathbb{G}, \sigma)$ *s.t.* $\mathbb{G}$ *is finite tqBa5 such that all sequents in* Φ *is true while* $\langle\varphi\rangle^i \Rightarrow \psi$ *is not.*

Proof. Let $\Phi \nvdash_{G5} \langle\varphi\rangle^i \Rightarrow \psi$. Then $\Phi \nvdash_{G5} \langle\varphi\rangle^i \Rightarrow_T \psi$. Therefore by Lemma 9, the claim holds.

Theorem 4 means that $G5$ has SFMP so from Remark 1, we get

Theorem 3. *The variety tqBa5 has SFMP*

Theorem 4. *The variety tqBa5 has FEP.*

5 Concluding Remarks

In this paper we have proved FEP for the class of tqBa5s. Indeed the class of Pre-rough algebras also has FEP.

A Pre-rough algebra is a tqBa5 enriched with the following conditions:

(IA1) $\Box a \vee \neg\Box a \leq 1$
(IA2) $\Box(a \vee b) \leq \Box a \vee \Box b$
(IA3) $\Diamond a \leq \Diamond b$ and $\Box a \leq \Box b$ implies $a \leq b$

Clearly in pre-rough one gets (i) $\Diamond(a \wedge b) = \Diamond a \wedge \Diamond b$ and (ii) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. Let T be a set of formula closed under $\neg, \wedge, \vee, \Diamond$. By the standard Lindenbaum Tarski method, one can construct a Pre-rough algebra from a sequent calculus for pre-rough with respect to a set of formulae T whose universe is the set of equivalence classes of formulae in T. Clearly by the De-morgan rules and (i), (ii), it is finite. Consequently the class of Pre-rough algebras has SFMP whence has FEP. By similar arguments the classes of intermediate pre-rough algebras those containing (IA2) have FEP. Since tqBa5 does not admit (IA2), the FEP for the class of tqBa5s can not be easily established by standard Lindenbaum Tarski method. The FEP for other classes of intermediate pre-rough algebras without (IA2) including IA1, IA3 remain open. The results in the present paper can also be extended to quasi Boolean algebra enriched with modal logic axioms (K) and (B) and its extensions. Further research can be finding more intermediate pre-rough algebras or modal quasi boolean algebras.

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