






Logics for Rough Concept Analysis

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Abstract. Taking an algebraic perspective on the basic structures of Rough Concept Analysis as the starting point, in this paper we introduce some varieties of lattices expanded with normal modal operators which can be regarded as the natural rough algebra counterparts of certain subclasses of rough formal contexts, and introduce proper display calculi for the logics associated with these varieties which are sound, complete, conservative and with uniform cut elimination and subformula property. These calculi modularly extend the multi-type calculi for rough algebras to a ‘nondistributive’ (i.e. general lattice-based) setting.

Keywords: Rough set theory · Formal Concept Analysis · Modal logic · Lattice-based logics · Algebras for rough sets · Proper display calculi

1 Introduction

This paper continues a line of investigation started in [10] and aimed at introducing sequent calculi for the logics of varieties of ‘rough algebras’, introduced and discussed in [1, 20]. The ‘rough algebras’ considered in the present paper are *nondistributive* (i.e. general lattice-based) generalizations of those of [20, 21]; specifically, they are varieties of lattices expanded with normal modal operators, natural examples of which arise in connection with (certain subclasses of) *rough formal contexts*, introduced by Kent in [15] as the basic notion of *Rough Concept Analysis* (RCA), a synthesis of Rough Set Theory [19] and Formal Concept Analysis [8]. The core idea of Kent’s approach is to use a given indiscernibility relation E on the objects of a formal context (A, X, I) to generate E -definable approximations R and S of the relation I such that $S \subseteq I \subseteq R$. The starting point of our approach is that R and S can be used to generate tuples of adjoint normal modal operators $\langle S \rangle \dashv [S]$ and $\langle R \rangle \dashv [R]$. We identify conditions under

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which $[S]$ and $\langle R \rangle$ are interior operators and $[R]$ and $\langle S \rangle$ are closure operators. This provides the basic algebraic framework, which we axiomatically extend so as to define ‘nondistributive’ counterparts of the varieties introduced in [21], whenever possible.

From an algebraic perspective, it is interesting to observe that, unlike $\langle S \rangle$ and $[S]$, the modal operators $\langle R \rangle$ and $[R]$ play the reverse roles they usually have in rough set theory: namely, $[R]$, being an *inflationary* map, plays naturally the role of the *closure* operator providing the upper lax approximation of a given formal concept, and similarly $\langle R \rangle$, being a *deflationary* map, plays the role of the *interior* operator, providing the lower lax approximation of a given formal concept.

From a proof-theoretic perspective, these properties make it possible to extend the *multi-type approach* (thanks to which, a modular family of analytic calculi was introduced in [10] for the logics of ‘rough algebras’) to varieties of ‘rough algebras’ on a ‘nondistributive’ propositional base. In particular, the calculi defined in Sect. 6 are all *proper display calculi* (cf. [24]), the cut elimination and subformula property of which can be straightforwardly verified by appealing to the meta-theorem of [5].¹ An interesting departure from the calculi of [10] concerns the counterparts of the IA3 condition, which in the present paper comes in two variants: the lower (strict), and the upper (lax). The inequality corresponding to the lower variant of IA3, which was analytic in the presence of distributivity, is not analytic inductive in the absence of distributivity (cf. [7, Definition 55]). However, the inequality corresponding to the upper variant of IA3 is analytic inductive, and hence can be captured by an analytic structural rule.

This paper contains the first algebraic and proof-theoretic contribution to a line of research aimed at integrating Rough Set Theory and Formal Concept Analysis, and at building the necessary logical machinery to support formal reasoning about categorization decisions under the assumption that categories and concepts can be *vague*. Future directions concern enriching this basic framework so as to formally account for the fact that the *dynamics* of categories also affect their becoming vaguer or sharper.

2 Preliminaries

The purpose of this section, which is based on [3, Appendix] and [2] and [18, Sects. 2.3 and 2.4], is to briefly recall the basic notions of the theory of *enriched formal contexts* (cf. Definition 2) while introducing the notation which will be used throughout the paper. For any relation $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, let

$$T^{(0)}[V'] := \{u \mid \forall v (v \in V' \Rightarrow uTv)\} \quad T^{(1)}[U'] := \{v \mid \forall u (u \in U' \Rightarrow uTv)\}.$$

It can be easily verified that $U' \subseteq T^{(0)}[V']$ iff $V' \subseteq T^{(1)}[U']$, that $V_1 \subseteq V_2 \subseteq V$ (resp. $U_1 \subseteq U_2 \subseteq U$) implies that $T^{(0)}[V_2] \subseteq T^{(0)}[V_1]$ (resp. $T^{(1)}[U_2] \subseteq T^{(1)}[U_1]$), and $S \subseteq T \subseteq U \times V$ implies that $S^{(0)}[V'] \subseteq T^{(0)}[V']$ and $S^{(1)}[U'] \subseteq T^{(1)}[U']$ for all $V' \subseteq V$ and $U' \subseteq U$.

Formal contexts, or *polarities*, are structures $\mathbb{P} = (A, X, I)$ such that A and X are sets, and $I \subseteq A \times X$ is a binary relation. Intuitively, formal contexts can be understood as abstract representations of databases [8], so that A represents a collection of *objects*, X

¹ In [22], sequent calculi for non-distributive versions of the logics associated with varieties of ‘rough algebras’ are introduced, which are sound and complete but without cut elimination.

as a collection of *features*, and for any object a and feature x , the tuple (a, x) belongs to I exactly when object a has feature x . In what follows, we use a, b (resp. x, y) for elements of A (resp. X), and B (resp. Y) for subsets of A (resp. of X).

As is well known, for every formal context $\mathbb{P} = (A, X, I)$, the pair of maps

$$(\cdot)^\uparrow : \mathcal{P}(A) \rightarrow \mathcal{P}(X) \quad \text{and} \quad (\cdot)^\downarrow : \mathcal{P}(X) \rightarrow \mathcal{P}(A),$$

respectively defined by the assignments $B^\uparrow := I^{(1)}[B]$ and $Y^\downarrow := I^{(0)}[Y]$, form a Galois connection and hence induce the closure operators $(\cdot)^{\uparrow\downarrow}$ and $(\cdot)^{\downarrow\uparrow}$ on $\mathcal{P}(A)$ and on $\mathcal{P}(X)$ respectively.² Moreover, the fixed points of these closure operators form complete sub- \cap -semilattices of $\mathcal{P}(A)$ and $\mathcal{P}(X)$ respectively, and hence are complete lattices which are dually isomorphic to each other via the restrictions of the maps $(\cdot)^\uparrow$ and $(\cdot)^\downarrow$. This motivates the following.

Definition 1. For every formal context $\mathbb{P} = (A, X, I)$, a formal concept of \mathbb{P} is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^\uparrow = Y$ and $Y^\downarrow = B$. The set B is the extension of c , which we will sometimes denote $\llbracket c \rrbracket$, and Y is the intension of c , sometimes denoted $\langle\langle c \rangle\rangle$. Let $L(\mathbb{P})$ denote the set of the formal concepts of \mathbb{P} . Then the concept lattice of \mathbb{P} is the complete lattice

$$\mathbb{P}^+ := (L(\mathbb{P}), \wedge, \vee),$$

where for every $\mathcal{X} \subseteq L(\mathbb{P})$,

$$\wedge \mathcal{X} := (\cap_{c \in \mathcal{X}} \llbracket c \rrbracket, (\cap_{c \in \mathcal{X}} \llbracket c \rrbracket)^\uparrow) \quad \text{and} \quad \vee \mathcal{X} := ((\cap_{c \in \mathcal{X}} \langle\langle c \rangle\rangle)^\downarrow, \cap_{c \in \mathcal{X}} \langle\langle c \rangle\rangle).$$

Then clearly, $\top^{\mathbb{P}^+} := \wedge \emptyset = (A, A^\uparrow)$ and $\perp^{\mathbb{P}^+} := \vee \emptyset = (X^\downarrow, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad \llbracket c \rrbracket \subseteq \llbracket d \rrbracket \quad \text{iff} \quad \langle\langle d \rangle\rangle \subseteq \langle\langle c \rangle\rangle.$$

Theorem 1 (Birkhoff's theorem, main theorem of FCA). Any complete lattice \mathbb{L} is isomorphic to the concept lattice \mathbb{P}^+ of some formal context \mathbb{P} .

Definition 2. An enriched formal context is a tuple $\mathbb{F} = (\mathbb{P}, R_\square, R_\diamond)$ such that $\mathbb{P} = (A, X, I)$ is a formal context, and $R_\square \subseteq A \times X$ and $R_\diamond \subseteq X \times A$ are I -compatible relations, that is, $R_\square^{(0)}[x]$ (resp. $R_\diamond^{(0)}[a]$) and $R_\square^{(1)}[a]$ (resp. $R_\diamond^{(1)}[x]$) are Galois-stable for all $x \in X$ and $a \in A$. The complex algebra of \mathbb{F} is

$$\mathbb{F}^+ = (\mathbb{P}^+, [R_\square], \langle R_\diamond \rangle),$$

where \mathbb{P}^+ is the concept lattice of \mathbb{P} , and $[R_\square]$ and $\langle R_\diamond \rangle$ are unary operations on \mathbb{P}^+ defined as follows: for every $c \in \mathbb{P}^+$,

$$[R_\square]c := (R_\square^{(0)}[\llbracket c \rrbracket], (R_\square^{(0)}[\llbracket c \rrbracket])^\uparrow) \quad \text{and} \quad \langle R_\diamond \rangle c := ((R_\diamond^{(0)}[\langle\langle c \rangle\rangle])^\downarrow, R_\diamond^{(0)}[\langle\langle c \rangle\rangle]).$$

Since R_\square and R_\diamond are I -compatible, $[R_\square], \langle R_\diamond \rangle, [R_\square^{-1}], \langle R_\diamond^{-1} \rangle : \mathbb{P}^+ \rightarrow \mathbb{P}^+$ are well-defined.

² When $B = \{a\}$ (resp. $Y = \{x\}$) we write $a^{\uparrow\downarrow}$ for $\{a\}^{\uparrow\downarrow}$ (resp. $x^{\downarrow\uparrow}$ for $\{x\}^{\downarrow\uparrow}$).

Lemma 1 (cf. [18, Lemma 3]). *For any enriched formal context $\mathbb{F} = (\mathbb{P}, R_{\square}, R_{\diamond})$, the algebra $\mathbb{F}^+ = (\mathbb{P}^+, [R_{\square}], \langle R_{\diamond} \rangle)$ is a complete lattice expanded with normal modal operators such that $[R_{\square}]$ is completely meet-preserving and $\langle R_{\diamond} \rangle$ is completely join-preserving.*

Definition 3. *For any formal context $\mathbb{P} = (A, X, I)$ and any I -compatible relations $R, T \subseteq A \times X$, the composition $R; T \subseteq A \times X$ is defined as follows: for any $a \in A$ and $x \in X$,*

$$(R; T)^{(1)}[a] = R^{(1)}[I^{(0)}[T^{(1)}[a]]] \text{ or equivalently } (R; T)^{(0)}[x] = R^{(0)}[I^{(1)}[T^{(0)}[x]]].$$

3 Motivation: Kent's Rough Concept Analysis

Below, we report on the basic definitions and constructions in Rough Concept Analysis [15], cast in the notational conventions of Sect. 2.

Rough formal contexts (abbreviated as *Rfc*) are tuples $\mathbb{G} = (\mathbb{P}, E)$ such that $\mathbb{P} = (A, X, I)$ is a polarity (cf. Sect. 2), and $E \subseteq A \times A$ is an equivalence relation (the *indiscernibility* relation between objects). For every $a \in A$ we let $(a)_E := \{b \in A \mid aEb\}$. The relation E induces two relations $R, S \subseteq A \times I$ approximating I , defined as follows: for every $a \in A$ and $x \in X$,

$$aRx \text{ iff } bIx \text{ for some } b \in (a)_E; \quad aSx \text{ iff } bIx \text{ for all } b \in (a)_E. \quad (1)$$

By definition, R, S are E -definable (i.e. $R^{(0)}[x] = \cup_{aRx}(a)_E$ and $S^{(0)}[x] = \cup_{aSx}(a)_E$ for any $x \in X$), and E being reflexive immediately implies that

Lemma 2. *For any Rfc $\mathbb{G} = (\mathbb{P}, E)$, if R and S are defined as in (1), then*

$$S \subseteq I \quad \text{and} \quad I \subseteq R. \quad (2)$$

Intuitively, we can think of R as the *lax* version of I determined by E , and S as its *strict* version determined by E . Following the methodology introduced in [4] and applied in [2, 3] to introduce a polarity-based semantics for the modal logics of formal concepts, under the assumption that R and S are I -compatible (cf. Definition 2), the relations R and S can be used to define normal modal operators $[R], \langle R \rangle, [S], \langle S \rangle$ on \mathbb{P}^+ defined as follows: for any $c \in \mathbb{P}^+$,

$$[[[R]c]] := R^{(0)}[[[c]]] = \{a \in A \mid \forall x(x \in [[c]] \Rightarrow aRx)\} \quad (3)$$

$$[[[S]c]] := S^{(0)}[[[c]]] = \{a \in A \mid \forall x(x \in [[c]] \Rightarrow aSx)\}. \quad (4)$$

That is, the members of $[R]c$ are exactly those objects that satisfy (possibly by proxy of some object equivalent to them) all features in the description of c , while the members of $[S]c$ are exactly those objects that not only satisfy all features in the description of c , but that ‘force’ all their equivalents to also satisfy them. The assumption that $S \subseteq I$ implies that $[[[S]c]] = S^{(0)}[[[c]]] \subseteq I^{(0)}[[[c]]] = [[c]]$, hence $[S]c$ is a sub-concept of c . The assumption that $I \subseteq R$ implies that $[[c]] = I^{(0)}[[[c]]] \subseteq R^{(0)}[[[c]]] = [[[R]c]]$, hence $[R]c$ is a super-concept of c . Moreover, for any $c \in \mathbb{P}^+$,

$$[[\langle R \rangle c]] := R^{(1)}[[[c]]] = \{x \in X \mid \forall a(a \in [[c]] \Rightarrow aRx)\} \quad (5)$$

$$(\langle S \rangle c) := S^{(1)}[\llbracket c \rrbracket] = \{x \in X \mid \forall a(a \in \llbracket c \rrbracket \Rightarrow aSx)\}. \quad (6)$$

That is, $\langle R \rangle c$ is the concept described by those features shared not only by each member of c but also by their equivalents, while $\langle S \rangle c$ is the concept described by the common features of those members of c which ‘force’ each of their equivalents to share them. The assumption that $I \subseteq R$ implies that $(\llbracket c \rrbracket) = I^{(1)}[\llbracket c \rrbracket] \subseteq R^{(1)}[\llbracket c \rrbracket] = (\langle R \rangle c)$, and hence $\langle R \rangle c$ is a sub-concept of c . The assumption that $S \subseteq I$ implies that $(\langle S \rangle c) = S^{(1)}[\llbracket c \rrbracket] \subseteq I^{(1)}[\llbracket c \rrbracket] = (\llbracket c \rrbracket)$, and hence $\langle S \rangle c$ is a super-concept of c . Summing up the discussion above, we have verified that the conditions $I \subseteq R$ and $S \subseteq I$ imply that the following sequents of the modal logic of formal concepts are valid on Kent’s basic structures:

$$\Box_s \phi \vdash \phi \quad \phi \vdash \Box_\ell \phi \quad \phi \vdash \Diamond_s \phi \quad \Diamond_\ell \phi \vdash \phi, \quad (7)$$

where \Box_s is interpreted as $[S]$, \Box_ℓ as $[R]$, \Diamond_s as $\langle S \rangle$ and \Diamond_ℓ as $\langle R \rangle$. Translated algebraically, these conditions say that \Box_s and \Diamond_ℓ are *deflationary*, as *interior* operators are, \Diamond_s and \Box_ℓ are *inflationary*, as *closure* operators are. Hence, it is natural to ask under which conditions they (i.e. their semantic interpretations) are indeed closure/interior operators. The next definition and lemma provide answers to this question.

Definition 4. An Rfc $\mathbb{G} = (\mathbb{P}, E)$ is amenable if E , R and S (defined as in (1)) are *I-compatible*.³

Lemma 3. For any amenable Rfc $\mathbb{G} = (\mathbb{P}, E)$, if and R and S are defined as in (1), then

$$R; R \subseteq R \quad \text{and} \quad S \subseteq S; S. \quad (8)$$

Proof. Let $x \in X$. To show that $R^{(0)}[I^{(1)}[R^{(0)}[x]]] \subseteq R^{(0)}[x]$, let $a \in R^{(0)}[I^{(1)}[R^{(0)}[x]]]$. By adjunction, this is equivalent to $I^{(1)}[R^{(0)}[x]] \subseteq R^{(1)}[a]$, which implies that $I^{(0)}[R^{(1)}[a]] \subseteq I^{(0)}[I^{(1)}[R^{(0)}[x]]] = R^{(0)}[x]$, the last equality holding since R is *I-compatible* by assumption. Moreover, $I \subseteq R$ (cf. Lemma 2) implies that $I^{(1)}[a] \subseteq R^{(1)}[a]$, which implies that $I^{(0)}[R^{(1)}[a]] \subseteq I^{(0)}[I^{(1)}[a]] \subseteq (a)_E$, the last inclusion holding since E is *I-compatible* by assumption. Hence, $I^{(0)}[R^{(1)}[a]] \subseteq R^{(0)}[x] \cap (a)_E$. Suppose for contradiction that $a \notin R^{(0)}[x]$. By the E -definability of R , this is equivalent to $R^{(0)}[x] \cap (a)_E = \emptyset$. Hence $I^{(0)}[R^{(1)}[a]] = \emptyset$, from which it follows that $R^{(1)}[a] = I^{(1)}[I^{(0)}[R^{(1)}[a]]] = I^{(1)}[\emptyset] = X$. Hence, $x \in R^{(1)}[a]$, i.e. $a \in R^{(0)}[x]$, against the assumption that $a \notin R^{(0)}[x]$.

Let $x \in X$. To show that $S^{(0)}[x] \subseteq S^{(0)}[I^{(1)}[S^{(0)}[x]]]$, assume that $a \in S^{(0)}[x]$. Since S is E -definable by construction, this is equivalent to $(a)_E \subseteq S^{(0)}[x]$. To show that $a \in S^{(0)}[I^{(1)}[S^{(0)}[x]]]$, we need to show that bIy for any $b \in (a)_E$ and any $y \in I^{(1)}[S^{(0)}[x]]$. Let $y \in I^{(1)}[S^{(0)}[x]]$. Hence, by definition, $b'Iy$ for every $b' \in S^{(0)}[x]$. Since $(a)_E \subseteq S^{(0)}[x]$, this implies that bIy for any $b \in (a)_E$, as required.

By the general theory developed in [4] and applied to enriched formal contexts in [18, Proposition 5], properties (8) guarantee that the following sequents of the modal logic of formal concepts are also valid on amenable Rfc’s:

$$\Box_s \phi \vdash \Box_s \Box_s \phi \quad \Box_\ell \Box_\ell \phi \vdash \Box_\ell \phi \quad \Diamond_s \Diamond_s \phi \vdash \Diamond_s \phi \quad \Diamond_\ell \phi \vdash \Diamond_\ell \Diamond_\ell \phi. \quad (9)$$

³ The assumption that E is *I-compatible* does not follow from R and S being *I-compatible*. Let $\mathbb{G} = (\mathbb{P}, Id_A)$ for any polarity \mathbb{P} such that not all singleton sets of objects are Galois-stable. Hence $E = Id_A$ is not *I-compatible*. However, if $E = Id_A$, then $R = S = I$ are *I-compatible*.

Finally, again by [18, Proposition 5], the fact that by construction \Box_s and \Diamond_s (resp. \Box_ℓ and \Diamond_ℓ) are interpreted by operations defined in terms of the same relation guarantees the validity of the following sequents on amenable Rfc's:

$$\phi \vdash \Box_s \Diamond_s \phi \quad \Diamond_s \Box_s \phi \vdash \phi \quad \phi \vdash \Box_\ell \Diamond_\ell \phi \quad \Diamond_\ell \Box_\ell \phi \vdash \phi. \quad (10)$$

Axioms (7), (9) and (10) constitute the starting point and motivation for the proof-theoretic investigation of the logics associated to varieties of algebraic structures which can be understood as abstractions of amenable Rfc's. We define these varieties in the next section.

4 Kent Algebras

In the present section, we introduce *basic Kent algebras* (and the variety of *abstract Kent algebras* (aKa) to which they naturally belong), as algebraic generalizations of amenable Rfc's, and then introduce some subvarieties of aKas in the style of [20, 21].

Definition 5. A basic Kent algebra is a structure $\mathbb{A} = (\mathbb{L}, \Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell)$ such that \mathbb{L} is a complete lattice, and $\Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell$ are unary operations on \mathbb{L} such that for all $a, b \in \mathbb{L}$,

$$\Diamond_s a \leq b \text{ iff } a \leq \Box_s b \quad \text{and} \quad \Diamond_\ell a \leq b \text{ iff } a \leq \Box_\ell b, \quad (11)$$

and for any $a \in \mathbb{L}$,

$$\Box_s a \leq a \quad a \leq \Diamond_s a \quad a \leq \Box_\ell a \quad \Diamond_\ell a \leq a \quad (12)$$

$$\Box_s a \leq \Box_s \Box_s a \quad \Diamond_s \Diamond_s a \leq \Diamond_s a \quad \Box_\ell \Box_\ell a \leq \Box_\ell a \quad \Diamond_\ell a \leq \Diamond_\ell \Diamond_\ell a \quad (13)$$

We let \mathbf{KA}^+ denote the class of basic Kent algebras.

From (11) it follows that, in basic Kent algebras, \Box_s and \Box_ℓ are completely meet-preserving, \Diamond_s and \Diamond_ℓ are completely join-preserving. For any amenable Rfc $\mathbb{G} = (\mathbb{P}, E)$, if R and S are defined as in (1), then

$$\mathbb{G}^+ := (\mathbb{P}^+, [S], \langle S \rangle, [R], \langle R \rangle)$$

where \mathbb{P}^+ is the concept lattice of the formal context \mathbb{P} and $[S], \langle S \rangle, [R], \langle R \rangle$ are defined as in (3)–(6). The following proposition is an immediate consequence of [18, Proposition 5], using Lemmas 2 and 3, and the fact that $[R]$ and $\langle R \rangle$ (resp. $[S]$ and $\langle S \rangle$) are defined using the same relation.

Proposition 1. If $\mathbb{G} = (\mathbb{P}, E)$ is an amenable Rfc, then \mathbb{G}^+ is a basic Kent algebra.

The natural variety containing basic Kent algebras is defined as follows.

Definition 6. An abstract Kent algebra (aKa) is a structure $\mathbb{A} = (\mathbb{L}, \Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell)$ such that \mathbb{L} is a lattice, and $\Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell$ are unary operations on \mathbb{L} validating (11), (12) and (13). We let \mathbf{KA} denote the class of abstract Kent algebras.

From (11) it follows that, in aKas, \Box_s and \Box_ℓ are finitely meet-preserving, \Diamond_s and \Diamond_ℓ are finitely join-preserving.

Lemma 4. For any aKa $\mathbb{A} = (\mathbb{L}, \square_s, \diamond_s, \square_\ell, \diamond_\ell)$ and every $a \in \mathbb{L}$,

$$\square_s a \vee \diamond_\ell a \leq \square_\ell a \wedge \diamond_s a. \quad (14)$$

$$a \leq \square_s \diamond_s a \quad \diamond_s \square_s a \leq a \quad a \leq \square_\ell \diamond_\ell a \quad \diamond_\ell \square_\ell a \leq a \quad (15)$$

$$\square_s a \leq \square_s \diamond_s a \quad \diamond_s \square_s a \leq \diamond_s a \quad \diamond_\ell a \leq \square_\ell \diamond_\ell a \quad \diamond_\ell \square_\ell a \leq \square_\ell a. \quad (16)$$

$$\diamond_s \square_s a \leq \square_s a \quad \diamond_s a \leq \square_s \diamond_s a \quad \diamond_\ell \square_\ell a \leq \square_\ell a \quad \diamond_\ell a \leq \square_\ell \diamond_\ell a. \quad (17)$$

Proof. The inequalities in (15) are straightforward consequences of (11). The inequalities in (14) and (16) follow from (12) and (15), using the transitivity of the order. The inequalities in (17) follow from those in (13) using (11).

Conditions (17) define the ‘Kent algebra’ counterparts of topological quasi Boolean algebras 5 (tqBa5) [21]. In the next definition, we introduce ‘Kent algebra’ counterparts of some other varieties considered in [21] (omitting those the axiomatization of which involves negation and those that cannot be captured by multi-type analytic rules in the present setting), and also varieties characterized by interaction axioms between lax and strict connectives which follow the pattern of the 5-axioms in rough algebras.

Definition 7. An aKa \mathbb{A} as above is an aKa5’ if for any $a \in \mathbb{L}$,

$$\diamond_\ell a \leq \square_s \diamond_\ell a \quad \diamond_s \square_\ell a \leq \square_\ell a \quad \square_s a \leq \diamond_\ell \square_s a \quad \square_\ell \diamond_s a \leq \diamond_s a; \quad (18)$$

is a K-IA $_s$ if for any $a, b \in \mathbb{L}$,

$$\square_s a \leq \square_s b \text{ and } \diamond_s a \leq \diamond_s b \text{ imply } a \leq b, \quad (19)$$

and is a K-IA $_\ell$ if for any $a, b \in \mathbb{L}$,

$$\square_\ell a \leq \square_\ell b \text{ and } \diamond_\ell a \leq \diamond_\ell b \text{ imply } a \leq b. \quad (20)$$

Notice that the axioms above do not need to be analytic inductive in order for the resulting logic to be (multi-type) properly displayable: interestingly, the third and fourth inequality in (18) are not analytic inductive (cf. [7, Definition 55]), but are equivalent to analytic inductive inequalities in the multi-type language of the heterogeneous algebras discussed in the next section. This is an illustration of the technical advantage of moving to the multi-type setting (see also [10, Introduction, Sect. 4], where it is discussed how the multi-type approach was key in overcoming the difficulties encountered by the authors of [17] in introducing an analytic calculus for IA3).

5 Multi-type Presentation of Kent Algebras

Similarly to what holds for rough algebras (cf. [10, Sect. 3]), since the modal operations of any aKa $\mathbb{A} = (\mathbb{L}, \square_s, \diamond_s, \square_\ell, \diamond_\ell)$ are either interior operators or closure operators, each of them factorizes into a pair of adjoint normal modal operators which are retractions or co-retractions, as illustrated in the following table:

$\Box_s = \circ_I \cdot \blacksquare_I$	$\blacksquare_I \cdot \circ_I = id_{S_I}$	$\Diamond_s = \circ_C \cdot \blacklozenge_C$	$\blacklozenge_C \cdot \circ_C = id_{S_C}$
$\circ_I : S_I \hookrightarrow \mathbb{L}$	$\blacksquare_I : \mathbb{L} \twoheadrightarrow S_I$	$\blacklozenge_C : \mathbb{L} \twoheadrightarrow S_C$	$\circ_C : S_C \hookrightarrow \mathbb{L}$
$\Box_\ell = \Box_C \cdot \bullet_C$	$\bullet_C \cdot \Box_C = id_{L_C}$	$\Diamond_\ell = \Diamond_I \cdot \bullet_I$	$\bullet_I \cdot \Diamond_I = id_{L_I}$
$\bullet_C : \mathbb{L} \twoheadrightarrow L_C$	$\Box_C : L_C \hookrightarrow \mathbb{L}$	$\Diamond_I : L_I \hookrightarrow \mathbb{L}$	$\bullet_I : \mathbb{L} \twoheadrightarrow L_I$

where $S_I := \Box_s[\mathbb{L}]$, $S_C := \Diamond_s[\mathbb{L}]$, $L_C := \Box_\ell[\mathbb{L}]$, and $L_I := \Diamond_s[\mathbb{L}]$, and such that for all $\alpha \in S_I$, $\delta \in S_C$, $a \in \mathbb{L}$, $\pi \in L_I$, $\sigma \in L_C$,

$$\circ_I \alpha \leq a \text{ iff } \alpha \leq \blacksquare_I a \quad \blacklozenge_C a \leq \delta \text{ iff } a \leq \circ_C \delta \quad \bullet_C a \leq \pi \text{ iff } a \leq \Box_C \pi \quad \Diamond_I \sigma \leq a \text{ iff } \sigma \leq \bullet_I a. \quad (21)$$

Again similarly to what observed in [10], the lattice structure of \mathbb{L} can be exported to each of the sets S_I, S_C, L_C and L_I via the corresponding pair of modal operators as follows.

Definition 8. For any aKa \mathbb{A} , the strict interior kernel $\mathbf{S}_I = (S_I, \cup_I, \cap_I, t_I, f_I)$ and the strict closure kernel $\mathbf{S}_C = (S_C, \cup_C, \cap_C, t_C, f_C)$ are such that, for all $\alpha, \beta \in S_I$, and all $\delta, \gamma \in S_C$,

$$\begin{aligned} \alpha \cup_I \beta &:= \blacksquare_I (\circ_I \alpha \vee \circ_I \beta) & \delta \cup_C \gamma &:= \blacklozenge_C (\circ_C \delta \vee \circ_C \gamma) \\ \alpha \cap_I \beta &:= \blacksquare_I (\circ_I \alpha \wedge \circ_I \beta) & \delta \cap_C \gamma &:= \blacklozenge_C (\circ_C \delta \wedge \circ_C \gamma) \\ t_I &:= \blacksquare_I \top, f_I := \blacksquare_I \perp & t_C &:= \blacklozenge_C \top, f_C := \blacklozenge_C \perp \end{aligned}$$

The lax interior kernel $\mathbf{L}_I = (L_I, \sqcup_I, \sqcap_I, 1_I, 0_I)$ and the lax closure kernel $\mathbf{L}_C = (L_C, \sqcup_C, \sqcap_C, 1_C, 0_C)$ are such that, for all $\pi, \xi \in L_I$, and all $\sigma, \tau \in L_C$,

$$\begin{aligned} \pi \sqcup_I \xi &:= \bullet_I (\Diamond_I \pi \vee \Diamond_I \xi) & \sigma \sqcup_C \tau &:= \bullet_C (\Box_C \sigma \vee \Box_C \tau) \\ \pi \sqcap_I \xi &:= \bullet_I (\Diamond_I \pi \wedge \Diamond_I \xi) & \sigma \sqcap_C \tau &:= \bullet_C (\Box_C \sigma \wedge \Box_C \tau) \\ 1_I &:= \bullet_I \top, 0_I := \bullet_I \perp & 1_C &:= \bullet_C \top, 0_C := \bullet_C \perp \end{aligned}$$

Similarly to what observed in [10], it is easy to verify that the algebras defined above are lattices, and the operations indicated with a circle (either black or white) are lattice homomorphisms (i.e. are both normal box-type and normal diamond-type operators). The construction above justifies the following definition of class of heterogeneous algebras equivalent to aKas:

Definition 9. A heterogeneous aKa (*haKa*) is a tuple

$$\mathbb{H} = (\mathbb{L}, \mathbf{S}_I, \mathbf{S}_C, \mathbf{L}_I, \mathbf{L}_C, \circ_I, \blacksquare_I, \circ_C, \blacklozenge_C, \bullet_I, \Diamond_I, \bullet_C, \Box_C)$$

such that:

H1 $\mathbb{L}, \mathbf{S}_I, \mathbf{S}_C, \mathbf{L}_I, \mathbf{L}_C$ are bounded lattices;

H2 $\circ_I : \mathbf{S}_I \hookrightarrow \mathbb{L}$, $\circ_C : \mathbf{S}_C \hookrightarrow \mathbb{L}$, $\bullet_I : \mathbb{L} \twoheadrightarrow \mathbf{L}_I$, $\bullet_C : \mathbb{L} \twoheadrightarrow \mathbf{L}_C$ are lattice homomorphisms;

H3 $\circ_I \dashv \blacksquare_I$ $\blacklozenge_C \dashv \circ_C$ $\bullet_C \dashv \Box_C$ $\Diamond_I \dashv \bullet_I$;

H4 $\blacksquare_I \circ_I = id_{S_I}$ $\blacklozenge_C \circ_C = id_{S_C}$ $\bullet_C \Box_C = id_{L_C}$ $\bullet_I \Diamond_I = id_{L_I}$ ⁴

The haKas corresponding to the varieties of Definition 7 are defined as follows:

⁴ Condition H3 implies that $\blacksquare_I : \mathbb{L} \twoheadrightarrow \mathbf{S}_I$ and $\Box_I : \mathbf{L}_I \hookrightarrow \mathbb{L}$ are \wedge -hemimorphisms and $\blacklozenge_C : \mathbb{L} \twoheadrightarrow \mathbf{S}_C$ and $\Diamond_C : \mathbf{L}_C \hookrightarrow \mathbb{L}$ are \vee -hemimorphisms; condition H4 implies that the black connectives are surjective and the white ones are injective.

Algebra	Acronym	Conditions
heterogeneous aKa5'	haKa5'	$\diamond_I \pi \leq \circ_I \blacksquare_I \diamond_I \pi \quad \circ_C \blacklozenge_C \square_C \sigma \leq \square_C \sigma$ $\circ_I \alpha \leq \diamond_I \bullet_I \circ_I \alpha \quad \square_C \bullet_C \circ_C \delta \leq \circ_C \delta$
heterogeneous K-IA3 _s	hK-IA3 _s	$\blacksquare_I a \leq \blacksquare_I b$ and $\blacklozenge_C a \leq \blacklozenge_C b$ imply $a \leq b$
heterogeneous K-IA3 _ℓ	hK-IA3 _ℓ	$\square_C \bullet_C a \leq \square_C \bullet_C b$ and $\diamond_I \bullet_I a \leq \diamond_I \bullet_I b$ imply $a \leq b$

Notice that the inequalities defining haKa5' are all analytic inductive. A heterogeneous algebra \mathbb{H} is perfect if every lattice in the signature of \mathbb{H} is perfect (cf. [4, Definition 1.8]), and every homomorphism (resp. hemimorphism) in the signature of \mathbb{H} is a complete homomorphism (resp. hemimorphism).

Similarly to what discussed in [10, Sect. 3], one can readily show that the classes of haKas defined above correspond to the varieties defined in Sect. 4. That is, for any aKa \mathbb{A} one can define its corresponding haKa \mathbb{A}^+ using the factorizations described at the beginning of the present section and Definition 8, and conversely, given a haKa \mathbb{H} , one can define its corresponding aKa \mathbb{H}_+ by endowing its first domain \mathbb{L} with modal operations defined by taking the appropriate compositions of pairs of heterogeneous maps of \mathbb{H} . Then, for every $\mathbb{K} \in \{\text{aKa}, \text{aKa5}', \text{K-IA3}_s, \text{K-IA3}_\ell\}$, letting \mathbb{HK} denote its corresponding class of heterogeneous algebras, the following holds:

Proposition 2. 1. If $\mathbb{A} \in \mathbb{K}$, then $\mathbb{A}^+ \in \mathbb{HK}$.

2. If $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}_+ \in \mathbb{K}$.

3. $\mathbb{A} \cong (\mathbb{A}^+)_+$ and $\mathbb{H} \cong (\mathbb{H}_+)^+$.

4. The isomorphisms of the previous item restrict to perfect members of \mathbb{K} and \mathbb{HK} .

5. If $\mathbb{A} \in \mathbb{K}$, then $\mathbb{A}^\delta \cong ((\mathbb{A}^+)^\delta)_+$ and if $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}^\delta \cong ((\mathbb{H}_+)^\delta)^+$.

6 Multi-type Calculi for the Logics of Kent Algebras

In the present section, we introduce the multi-type calculi associated with each class of algebras $\mathbb{K} \in \{\text{aKa}, \text{aKa5}', \text{K-IA3}_\ell\}$. The language of these logics matches the language of haKas, and is built up from structural and operational (i.e. logical) connectives. Each structural connective is denoted by decorating its corresponding logical connective with \wedge (resp. \vee or \sim). In what follows, we will adopt the convention that unary connectives bind more strongly than binary ones.

general lattice \mathbb{L}

$$\begin{aligned} A &::= p \mid \top \mid \perp \mid \circ_I \alpha \mid \circ_C \delta \mid \diamond_I \pi \mid \square_C \sigma \mid A \wedge A \mid A \vee A \\ X &::= A \mid \dot{\downarrow} \mid \dot{\uparrow} \mid \check{\circ}_I \Gamma \mid \check{\circ}_C \Delta \mid \hat{\diamond}_I \Pi \mid \check{\diamond}_I \Pi \mid \hat{\diamond}_C \Sigma \mid \check{\diamond}_C \Sigma \mid X \wedge X \mid X \vee X \end{aligned}$$

strict-interior kernel \mathbb{S}_I

$$\alpha ::= \blacklozenge_I A \mid \blacksquare_I A$$

$$\Gamma ::= \alpha \mid \blacklozenge_I X \mid \blacksquare_I X \mid \check{\uparrow}_I \mid \hat{\uparrow}_I \mid \Gamma \hat{\wedge}_I \Gamma \mid \Gamma \check{\vee}_I \Gamma$$

lax-interior kernel \mathbb{L}_I

$$\pi ::= \bullet_I A$$

$$\Pi ::= \pi \mid \bullet_I X \mid \check{\circ}_I \mid \hat{\circ}_I \mid \Pi \hat{\wedge}_I \Pi \mid \Pi \check{\vee}_I \Pi$$

strict-closure kernel \mathbb{S}_C

$$\delta ::= \blacklozenge_C A \mid \blacksquare_C A$$

$$\Delta ::= \delta \mid \blacklozenge_C X \mid \blacksquare_C X \mid \check{\uparrow}_C \mid \hat{\uparrow}_C \mid \Delta \hat{\wedge}_C \Delta \mid \Delta \check{\vee}_C \Delta$$

lax-closure kernel \mathbb{L}_C

$$\sigma ::= \bullet_C A$$

$$\Sigma ::= \sigma \mid \bullet_C X \mid \check{\circ}_C \mid \hat{\circ}_C \mid \Sigma \hat{\wedge}_C \Sigma \mid \Sigma \check{\vee}_C \Sigma$$

– Interpretation of structural connectives as their logical counterparts⁵

1. structural and operational pure L-type connectives:

structural operations	$\hat{\top}$	$\check{\perp}$	$\hat{\wedge}$	$\check{\vee}$
logical operations	\top	\perp	\wedge	\vee

2. structural and operational pure S_I -type and S_C -type connectives:

structural operations	\hat{t}_I	\check{f}_I	$\hat{\cap}_I$	$\check{\cup}_I$	\hat{t}_C	\check{f}_C	$\hat{\cap}_C$	$\check{\cup}_C$
logical operations	t_I	f_I	\cap_I	\cup_I	t_C	f_C	\cap_C	\cup_C

3. structural and operational pure L_I -type and L_C -type connectives:

structural operations	$\hat{1}_I$	$\check{0}_I$	$\hat{\sqcap}_I$	$\check{\sqcup}_I$	$\hat{1}_C$	$\check{0}_C$	$\hat{\sqcap}_C$	$\check{\sqcup}_C$
logical operations	1_I	0_I	\sqcap_I	\sqcup_I	1_C	0_C	\sqcap_C	\sqcup_C

4. structural and operational multi-type strict connectives:

types	$L \rightarrow S_I$	$L \rightarrow S_C$	$S_I \rightarrow L$	$S_C \rightarrow L$		
structural operations	$\hat{\blacklozenge}_I$	$\check{\blacklozenge}_I$	$\hat{\blacklozenge}_C$	$\check{\blacklozenge}_C$	$\tilde{\circ}_I$	$\tilde{\circ}_C$
logical operations	\blacklozenge_I	\blacksquare_I	\blacklozenge_C	\blacksquare_C	\circ_I	\circ_C

5. structural and operational multi-type lax connectives:

types	$L_I \rightarrow L$	$L_C \rightarrow L$	$L \rightarrow L_I$	$L \rightarrow L_C$
structural operations	$\hat{\blacklozenge}_I$	$\check{\blacklozenge}_I$	$\hat{\blacklozenge}_C$	$\check{\blacklozenge}_C$
logical operations	\blacklozenge_I	\square_I	\blacklozenge_C	\square_C

In what follows, we will use x, y, z as structural variables of arbitrary types, a, b, c as term variables of arbitrary types.

The calculus D.AKA consists of the following axiom and rules.

– Identity and Cut:

$$Id_L \frac{}{p \vdash p} \quad \frac{x \vdash a \quad a \vdash y}{x \vdash y} \text{Cut}$$

– Multi-type display rules (we omit the display rules capturing the adjunctions $\blacklozenge_I \dashv \bullet_I \dashv \square_I$ and $\blacklozenge_I \dashv \bullet_I \dashv \square_I$):

$$ad_{LS_I} \frac{\tilde{\circ}_I \Gamma \vdash X}{\Gamma \vdash \check{\blacklozenge}_I X} \quad \frac{X \vdash \tilde{\circ}_I \Gamma}{\hat{\blacklozenge}_I X \vdash \Gamma} ad_{LS_I} \quad ad_{LS_C} \frac{X \vdash \tilde{\circ}_C \Delta}{\hat{\blacklozenge}_C X \vdash \Delta} \quad \frac{\tilde{\circ}_C X \vdash \Delta}{X \vdash \check{\blacklozenge}_C \Delta} ad_{LS_C}$$

– Multi-type structural rules for strict-kernel operators:

⁵ The connectives which appear in a grey cell in the synoptic tables will only be included in the present language at the structural level.

$$\begin{array}{c}
\begin{array}{c} \bar{\sigma}_I \hat{\iota}_I \frac{\bar{\sigma}_I \hat{\iota}_I \vdash X}{\hat{\iota}_I \vdash X} \quad \frac{X \vdash \bar{\sigma}_I \check{f}_I}{X \vdash \perp} \quad \bar{\sigma}_I \check{f}_I \end{array} \\
\hat{\diamond}_I \bar{\sigma}_I \frac{\hat{\diamond}_I \bar{\sigma}_I \Gamma \vdash \Gamma'}{\Gamma \vdash \Gamma'} \quad \frac{\Gamma' \vdash \check{\mathfrak{N}}_I \bar{\sigma}_I \Gamma}{\Gamma' \vdash \Gamma} \quad \check{\mathfrak{N}}_I \bar{\sigma}_I \\
\bar{\sigma}_I \hat{\diamond}_I \frac{\bar{\sigma}_I \hat{\diamond}_I X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash \bar{\sigma}_I \check{\mathfrak{N}}_I X}{Y \vdash X} \quad \bar{\sigma}_I \check{\mathfrak{N}}_I \\
\bar{\sigma}_C \hat{\iota}_C \frac{\bar{\sigma}_C \hat{\iota}_C \vdash X}{\hat{\iota}_C \vdash X} \quad \frac{X \vdash \bar{\sigma}_C \check{f}_C}{X \vdash \perp} \quad \bar{\sigma}_C \check{f}_C \\
\hat{\diamond}_C \bar{\sigma}_C \frac{\hat{\diamond}_C \bar{\sigma}_C \Delta \vdash \Delta'}{\Delta \vdash \Delta'} \quad \frac{\Delta' \vdash \check{\mathfrak{N}}_C \bar{\sigma}_C \Delta}{\Delta' \vdash \Delta} \quad \check{\mathfrak{N}}_C \bar{\sigma}_C \\
\bar{\sigma}_C \hat{\diamond}_C \frac{\bar{\sigma}_C \hat{\diamond}_C X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash \bar{\sigma}_C \check{\mathfrak{N}}_C X}{Y \vdash X} \quad \bar{\sigma}_C \check{\mathfrak{N}}_C
\end{array}$$

– Multi-type structural rules for lax-kernel operators:

$$\begin{array}{c}
\begin{array}{c} \bar{\sigma}_I \hat{\iota}_I \frac{\bar{\sigma}_I \hat{\iota}_I \vdash \Pi}{\hat{\iota}_I \vdash \Pi} \quad \frac{\Pi \vdash \bar{\sigma}_I \check{\iota}}{\Pi \vdash \check{\theta}_I} \quad \bar{\sigma}_I \check{\theta}_I \\
\hat{\sigma}_I \bar{\sigma}_I \frac{\Pi \vdash \Pi'}{\hat{\sigma}_I \bar{\sigma}_I \Pi \vdash \Pi'} \quad \frac{\Pi' \vdash \Pi}{\Pi' \vdash \check{\iota}_I \bar{\sigma}_I \Pi} \quad \check{\iota}_I \bar{\sigma}_I \\
\bar{\sigma}_I \hat{\sigma}_I \frac{\bar{\sigma}_I \hat{\sigma}_I \Pi \vdash \Pi'}{\Pi \vdash \Pi'} \quad \frac{\Pi' \vdash \bar{\sigma}_I \check{\iota}_I \Pi}{\Pi' \vdash \Pi} \quad \bar{\sigma}_I \check{\iota}_I \\
\hat{\iota}_C \bar{\sigma}_C \frac{\bar{\sigma}_C \hat{\iota}_C \vdash \Sigma}{\hat{\iota}_C \vdash \Sigma} \quad \frac{\Sigma \vdash \bar{\sigma}_C \check{\iota}}{\Sigma \vdash \check{\theta}_C} \quad \bar{\sigma}_C \check{\theta}_C \\
\hat{\sigma}_C \bar{\sigma}_C \frac{\Sigma \vdash \Sigma'}{\hat{\sigma}_C \bar{\sigma}_C \Sigma \vdash \Sigma'} \quad \frac{\Sigma' \vdash \Sigma}{\Sigma' \vdash \check{\iota}_C \bar{\sigma}_C \Sigma} \quad \check{\iota}_C \bar{\sigma}_C \\
\bar{\sigma}_C \hat{\sigma}_C \frac{\bar{\sigma}_C \hat{\sigma}_C \Sigma \vdash \Sigma'}{\Sigma \vdash \Sigma'} \quad \frac{\Sigma' \vdash \bar{\sigma}_C \check{\iota}_C \Sigma}{\Sigma' \vdash \Sigma} \quad \bar{\sigma}_C \check{\iota}_C
\end{array}
\end{array}$$

– Multi-type structural rules for the correspondence between kernels:

$$\bar{\sigma} \hat{\diamond} \frac{\bar{\sigma}_I \hat{\diamond}_I X \vdash Y}{\bar{\sigma}_C \hat{\diamond}_C X \vdash Y} \quad \frac{Y \vdash \check{\iota}_I \bar{\sigma}_I X}{Y \vdash \check{\iota}_C \bar{\sigma}_C X} \quad \check{\iota} \bar{\sigma}$$

– Logical rules for multi-type connectives related to strict kernels:

$$\begin{array}{c}
\begin{array}{c} \hat{\diamond}_I \frac{\hat{\diamond}_I A \vdash \Gamma}{\hat{\diamond}_I A \vdash \Gamma} \quad \frac{X \vdash A}{\hat{\diamond}_I X \vdash \hat{\diamond}_I A} \quad \hat{\diamond}_I \\
\bar{\sigma}_I \frac{\bar{\sigma}_I \alpha \vdash X}{\bar{\sigma}_I \alpha \vdash X} \quad \frac{X \vdash \bar{\sigma}_I \alpha}{X \vdash \bar{\sigma}_I \alpha} \quad \bar{\sigma}_I \\
\bar{\sigma}_C \frac{A \vdash X}{\bar{\sigma}_C A \vdash \bar{\sigma}_C X} \quad \frac{\Delta \vdash \check{\mathfrak{N}}_C A}{\Delta \vdash \bar{\sigma}_C A} \quad \bar{\sigma}_C \\
\bar{\sigma}_C \frac{\bar{\sigma}_C \delta \vdash X}{\bar{\sigma}_C \delta \vdash X} \quad \frac{X \vdash \bar{\sigma}_C \delta}{X \vdash \bar{\sigma}_C \delta} \quad \bar{\sigma}_C
\end{array}
\end{array}$$

– Logical rules for lattice connectives:

$$\begin{array}{c}
\begin{array}{c} \hat{\diamond}_I \frac{\hat{\diamond}_I \pi \vdash X}{\hat{\diamond}_I \pi \vdash X} \quad \frac{\Pi \vdash \pi}{\hat{\diamond}_I \Pi \vdash \hat{\diamond}_I \pi} \quad \hat{\diamond}_I \\
\bar{\sigma}_I \frac{\bar{\sigma}_I A \vdash \Pi}{\bar{\sigma}_I A \vdash \Pi} \quad \frac{\Pi \vdash \bar{\sigma}_I A}{\Pi \vdash \bar{\sigma}_I A} \quad \bar{\sigma}_I \\
\hat{\sigma}_C \frac{\hat{\sigma}_C \sigma \vdash \Sigma}{\hat{\sigma}_C \sigma \vdash \check{\iota}_C \Sigma} \quad \frac{X \vdash \check{\iota}_C \sigma}{X \vdash \hat{\sigma}_C \sigma} \quad \hat{\sigma}_C \\
\bar{\sigma}_C \frac{\bar{\sigma}_C A \vdash \Sigma}{\bar{\sigma}_C A \vdash \Sigma} \quad \frac{\Sigma \vdash \bar{\sigma}_C A}{\Sigma \vdash \bar{\sigma}_C A} \quad \bar{\sigma}_C
\end{array}
\end{array}$$

– Logical rules for lattice connectives:

$$\begin{array}{c}
\begin{array}{c} \top \frac{\hat{\iota}_I \vdash X}{\top \vdash X} \quad \frac{}{\hat{\iota}_I \vdash \top} \quad \top \\
\wedge_i \frac{A_{i \in \{1,2\}} \vdash X}{A_1 \wedge A_2 \vdash X} \quad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \quad \wedge \\
\perp \frac{}{\perp \vdash \perp} \quad \frac{X \vdash \check{\iota}}{X \vdash \perp} \quad \perp \\
\vee \frac{A \vdash X \quad B \vdash X}{A \vee B \vdash X} \quad \frac{X \vdash A_{i \in \{1,2\}}}{X \vdash A_1 \vee A_2} \quad \vee_i
\end{array}
\end{array}$$

The proper display calculi for the subvarieties of aKa discussed in Sect. 4 are obtained by adding the following rules:

Logic	Calculus	Rules
H.aKa5'	D.aKa5'	$\begin{array}{c} \tilde{\sigma}_I \hat{\diamond}_I \hat{\diamond}_I \frac{\hat{\diamond}_I \Pi \vdash X}{\tilde{\sigma}_I \hat{\diamond}_I \hat{\diamond}_I \Pi \vdash X} \quad \frac{X \vdash \check{\sigma}_C \Sigma}{X \vdash \check{\sigma}_C \check{\sigma}_C \check{\sigma}_C \Sigma} \quad \check{\sigma}_C \check{\sigma}_C \check{\sigma}_C \\ \hat{\diamond}_I \tilde{\sigma}_I \tilde{\sigma}_I \frac{\hat{\diamond}_I \tilde{\sigma}_I \tilde{\sigma}_I \Gamma \vdash X}{\tilde{\sigma}_I \Gamma \vdash X} \quad \frac{X \vdash \check{\sigma}_C \check{\sigma}_C \tilde{\sigma}_C \Delta}{X \vdash \check{\sigma}_C \Delta} \quad \check{\sigma}_C \tilde{\sigma}_C \check{\sigma}_C \end{array}$
K-IA3 _ℓ	D.K-IA3 _ℓ	$\frac{X \vdash \check{\sigma}_I \tilde{\sigma}_I Y \quad \hat{\diamond}_C \tilde{\sigma}_C X \vdash Y}{X \vdash Y} \quad k\text{-ia}3_\ell$

These calculi enjoy the properties of soundness, completeness, conservativity, cut elimination and subformula property the verification of which is standard and follows from the general theory of proper display calculi (cf. [6, 11–14, 16, 23]). These verifications are discussed in the appendix.

A Properties

Throughout this section, we let $\mathbf{K} \in \{aKa, aKa5', K\text{-IA}3_\ell\}$, and \mathbf{HK} the class of heterogeneous algebras corresponding to \mathbf{K} . Further, we let $\mathbf{D.K}$ denote the multi-type calculus for the logic $\mathbf{H.K}$ canonically associated with \mathbf{K} .

A.1 Soundness for Perfect HK Algebras

The verification of the soundness of the rules of $\mathbf{D.K}$ w.r.t. the semantics of *perfect* elements of \mathbf{HK} (see Definition 9) is analogous to that of many other multi-type calculi (cf. [6, 11–14, 16, 23]). Here we only discuss the soundness of the rule $k\text{-ia}3_\ell$. By definition, the following quasi-inequality is valid on every $\mathbf{K}\text{-IA}3_\ell$:

$$\square_\ell a \leq \square_\ell b \text{ and } \diamond_\ell a \leq \diamond_\ell b \text{ imply } a \leq b.$$

This quasi-inequality equivalently translates into the multi-type language as follows:

$$\square_C \bullet_C a \leq \square_C \bullet_C b \text{ and } \diamond_I \bullet_I a \leq \diamond_I \bullet_I b \text{ imply } a \leq b.$$

By adjunction, the quasi-inequality above can be equivalently rewritten as follows:

$$\diamond_C \bullet_C \square_C \bullet_C a \leq b \text{ and } a \leq \square_I \bullet_I \diamond_I \bullet_I b \text{ imply } a \leq b,$$

which, thanks to a well known property of adjoint maps, simplifies as:

$$\diamond_C \bullet_C a \leq b \text{ and } a \leq \square_I \bullet_I b \text{ imply } a \leq b.$$

Hence, the quasi-inequality above is equivalent to the following inequality:

$$a \wedge \square_I \bullet_I b \leq \diamond_C \bullet_C a \vee b.$$

The inequality above is analytic inductive (cf. [7, Definition 55]), and hence running ALBA on this inequality produces:

$$\begin{aligned}
& \forall a \forall b [a \wedge \square_I \bullet_I b \leq \diamond_C \bullet_C a \vee b] \\
& \text{iff } \forall p \forall q \forall a \forall b [(p \leq a \wedge \square_I \bullet_I b \ \& \ \diamond_C \bullet_C a \vee b \leq q) \Rightarrow p \leq q] \\
& \text{iff } \forall p \forall q \forall a \forall b [(p \leq a \ \& \ p \leq \square_I \bullet_I b \ \& \ b \leq q \ \& \ \diamond_C \bullet_C a \leq q) \Rightarrow p \leq q] \\
& \text{iff } \forall p \forall q [(p \leq \square_I \bullet_I q \ \& \ \diamond_C \bullet_C p \leq q) \Rightarrow p \leq q].
\end{aligned}$$

The last quasi-inequality above is the semantic translation of the rule $k\text{-ia}3_\ell$:

$$\frac{X \vdash \check{\square}_I \check{\bullet}_I Y \quad \hat{\diamond}_C \check{\bullet}_C X \vdash Y}{X \vdash Y} \text{ } k\text{-ia}3_\ell$$

which we then proved to be sound on every perfect heterogeneous K-IA3 $_\ell$, by the soundness of the ALBA steps. Likewise, the defining condition of K-IA3 $_\ell$ translates into the inequality

$$a \wedge \circ_C \blacklozenge_C b \leq \circ_I \blacksquare_I a \vee b,$$

which, however, is not analytic inductive, and hence it cannot be transformed into an analytic rule via ALBA.

A.2 Completeness

Let $A^\tau \vdash B^\tau$ be the translation of any sequent $A \vdash B$ in the language of H.K into the language of D.K induced by the correspondence between K and HK described in Sect. 5.

Proposition 3. *For every H.K-derivable sequent $A \vdash B$, the sequent $A^\tau \vdash B^\tau$ is derivable in D.K.*

Below we provide the multi-type translations of the single-type sequents corresponding to inequalities (11). All of them are derivable in D.AKA by logical introduction rules, display rules, and the rules $\check{\square} \check{\bullet}$ and $\hat{\circ} \hat{\blacklozenge}$.

$$\begin{aligned}
\diamond_s A \vdash B \text{ iff } A \vdash \square_s B & \rightsquigarrow \circ_C \blacklozenge_C A \vdash B \text{ iff } A \vdash \circ_I \blacksquare_I B \\
\diamond_\ell A \vdash B \text{ iff } A \vdash \square_\ell B & \rightsquigarrow \diamond_I \bullet_I A \vdash B \text{ iff } A \vdash \square_C \bullet_C B
\end{aligned}$$

Below we provide the multi-type translations of the single-type sequents corresponding to inequalities (12) and (13), respectively. All of them are derivable in D.AKA by logical introduction rules and display rules.

$$\begin{array}{llll}
\square_s A \vdash A & \rightsquigarrow & \circ_I \blacksquare_I A \vdash A & \quad \square_s A \vdash \square_s \square_s A & \rightsquigarrow & \circ_I \blacksquare_I A \vdash \circ_I \blacksquare_I \circ_I \blacksquare_I A \\
A \vdash \diamond_s A & \rightsquigarrow & A \vdash \circ_C \blacklozenge_C A & \quad \diamond_s \diamond_s A \vdash \diamond_s A & \rightsquigarrow & \circ_C \blacklozenge_C \circ_C \blacklozenge_C A \vdash \circ_C \blacklozenge_C A \\
A \vdash \square_\ell A & \rightsquigarrow & A \vdash \square_C \bullet_C A & \quad \square_\ell \square_\ell A \vdash \square_\ell A & \rightsquigarrow & \square_C \bullet_C \square_C \bullet_C A \vdash \square_C \bullet_C A \\
\diamond_\ell A \vdash A & \rightsquigarrow & \diamond_I \bullet_I A \vdash A & \quad \diamond_\ell A \vdash \diamond_\ell \diamond_\ell A & \rightsquigarrow & \diamond_I \bullet_I A \vdash \diamond_I \bullet_I \diamond_I \bullet_I A
\end{array}$$

Below we provide the multi-type translation of the single-type sequents corresponding to inequalities (18). All of them are derivable in D.AKA5'.

$$\begin{array}{ll}
\diamond_\ell A \vdash \square_s \diamond_\ell A & \rightsquigarrow \diamond_I \bullet_I A \vdash \circ_I \blacksquare_I \diamond_I \bullet_I A \\
\diamond_s \square_\ell A \vdash \square_\ell A & \rightsquigarrow \circ_C \blacklozenge_C \square_C \bullet_C A \vdash \square_C \bullet_C A \\
\square_s A \vdash \diamond_\ell \square_s A & \rightsquigarrow \circ_I \blacksquare_I A \vdash \diamond_I \bullet_I \circ_I \blacksquare_I A \\
\square_\ell \diamond_s A \vdash \diamond_s A & \rightsquigarrow \square_C \bullet_C \circ_C \blacklozenge_C A \vdash \circ_C \blacklozenge_C A
\end{array}$$

Below we provide the multi-type translations of the single-type rules corresponding to quasi-inequality (20), respectively.

$$\begin{aligned} \diamond_\ell A \vdash \diamond_\ell B \text{ and } \square_\ell A \vdash \square_\ell B &\text{ imply } A \vdash B \rightsquigarrow \\ \diamond_I \bullet_I A \vdash \diamond_I \bullet_I B \text{ and } \square_C \bullet_C A \vdash \square_C \bullet_C B &\text{ imply } A \vdash B \end{aligned}$$

Below, we derive (20). Firstly, $A \wedge \square_C \bullet_C B \vdash \diamond_I \bullet_I A \vee B$ is derivable via $k\text{-ia}3_\ell$ by means of the following derivation \mathcal{D} :

$$\frac{\frac{\frac{\frac{B \vdash B}{B \vdash \diamond_I \bullet_I A \check{V} B}}{B \vdash \diamond_I \bullet_I A \vee B}}{\check{\bullet}_C B \vdash \check{\bullet}_C (\diamond_I \bullet_I A \vee B)}}{\bullet_C B \vdash \check{\bullet}_C (\diamond_I \bullet_I A \vee B)}}{\square_C \bullet_C B \vdash \check{\bullet}_C \check{\bullet}_C (\diamond_I \bullet_I A \vee B)} \quad \frac{\frac{\frac{A \vdash A}{A \wedge \square_C \bullet_C B \vdash A}}{A \wedge \square_C \bullet_C B \vdash A}}{\check{\bullet}_I (A \wedge \square_C \bullet_C B) \vdash \check{\bullet}_I A}}{\check{\bullet}_I (A \wedge \square_C \bullet_C B) \vdash \check{\bullet}_I A}}{\frac{\frac{A \wedge \square_C \bullet_C B \vdash \check{\bullet}_C \check{\bullet}_C (\diamond_I \bullet_I A \vee B)}{A \wedge \square_C \bullet_C B \vdash \check{\bullet}_C \check{\bullet}_C (\diamond_I \bullet_I A \vee B)}}{\hat{\diamond}_I \check{\bullet}_I (A \wedge \square_C \bullet_C B) \vdash \diamond_I \bullet_I A}}{\hat{\diamond}_I \check{\bullet}_I (A \wedge \square_C \bullet_C B) \vdash \diamond_I \bullet_I A \check{V} B}}{\hat{\diamond}_I \check{\bullet}_I (A \wedge \square_C \bullet_C B) \vdash \diamond_I \bullet_I A \vee B} \quad k\text{-ia}3_\ell$$

Assuming $\diamond_I \bullet_I A \vdash \diamond_I \bullet_I B$ and $\square_C \bullet_C A \vdash \square_C \bullet_C B$, we derive $A \vdash B$ via cut as follows:

$$\frac{\frac{A \vdash A}{C_L} \frac{\frac{\frac{\square_C \bullet_C A \vdash \square_C \bullet_C B}{Id_L + \bullet_C + adj_{JLSc} + Cut_L}}{A \vdash \square_C \bullet_C B} \wedge}{A \wedge \square_C \bullet_C B \vdash \square_C \bullet_C B} \quad \frac{\mathcal{D}}{A \wedge \square_C \bullet_C B \vdash \diamond_I \bullet_I A \vee B} \quad \frac{\frac{\frac{\frac{\diamond_I \bullet_I A \vdash \diamond_I \bullet_I B}{Id_L + \bullet_I + adj_{JLSI} + Cut_L}}{\diamond_I \bullet_I A \vdash B} \vee}{\diamond_I \bullet_I A \vee B \vdash B} \quad \frac{B \vdash B}{C_L}}{\diamond_I \bullet_I A \vee B \vdash B} \quad \frac{C_L}{Cut_L}}{A \vdash B} \quad Cut_L$$

A.3 Conservativity

To argue that D.K is conservative w.r.t. H.K, we follow the standard proof strategy discussed in [7, 9]. We need to show that, for all formulas A and B in the language of H.K, if $A^\tau \vdash B^\tau$ is a D.K-derivable sequent, then $A \vdash B$ is derivable in H.K. This claim can be proved using the following facts: (a) The rules of D.K are sound w.r.t. perfect members of HK (cf. Sect. A.1); (b) H.K is complete w.r.t. the class of perfect algebras in K; (c) A perfect element of K is equivalently presented as a perfect member of HK so that the semantic consequence relations arising from each type of structures preserve and reflect the translation. Let A, B be as above. If $A^\tau \vdash B^\tau$ is D.K-derivable, then by (a), $\models_{HK} A^\tau \vdash B^\tau$. By (c), this implies that $\models_K A \vdash B$, where \models_K denotes the semantic consequence relation arising from the perfect members of class K. By (b), this implies that $A \vdash B$ is derivable in H.K, as required.

A.4 Cut Elimination and Subformula Property

Cut elimination and subformula property for each D.K are obtained by verifying the assumptions of [5, Theorem 4.1]. All of them except C'_8 are readily satisfied by inspecting the rules. Condition C'_8 requires to check that reduction steps can be performed for

every application of cut in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we only show C'_8 for some heterogeneous connectives.

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \\
 \Gamma \vdash \check{\Box}_I A \\
 \hline
 \Gamma \vdash \blacksquare_I A
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \\
 A \vdash Y \\
 \hline
 \blacksquare_I A \vdash \blacksquare_I Y
 \end{array} \\
 \hline
 \Gamma \vdash \check{\Box}_I Y
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_1 \\
 \Gamma \vdash \blacksquare_I A \\
 \hline
 \check{\delta}_I \Gamma \vdash A
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \\
 A \vdash Y \\
 \hline
 \check{\delta}_I \Gamma \vdash Y
 \end{array} \\
 \hline
 \Gamma \vdash \check{\Box}_I Y
 \end{array}$$

$$\begin{array}{c}
 \vdots \pi_1 \\
 X \vdash \check{\delta}_I \alpha \\
 \hline
 X \vdash \circ_I \alpha
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \\
 \check{\delta}_I \alpha \vdash Y \\
 \hline
 \circ_I \alpha \vdash Y
 \end{array} \\
 \hline
 X \vdash Y
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_1 \\
 X \vdash \check{\delta}_I \alpha \\
 \hline
 \hat{\diamond}_I X \vdash \alpha
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \\
 \check{\delta}_I \alpha \vdash Y \\
 \hline
 \alpha \vdash \check{\Box}_I Y
 \end{array} \\
 \hline
 \hat{\diamond}_I X \vdash \check{\Box}_I Y \\
 \hline
 X \vdash Y
 \end{array}$$

The remaining cases are analogous.

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