APAL with Memory Is Better

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Abstract. We introduce Arbitrary Public Announcement Logic with Memory (APALM), obtained by adding to the models a 'memory' of the initial states, representing the information before any communication took place ("the prior"), and adding to the syntax operators that can access this memory. We show that APALM is recursively axiomatizable (in contrast to the original Arbitrary Public Announcement Logic, for which the corresponding question is still open). We present a complete recursive axiomatization, that uses a natural finitary rule, we study this logic's expressivity and the appropriate notion of bisimulation.

1 Introduction

Arbitrary Public Announcement Logic (APAL) and its relatives are natural extensions of Public Announcements Logic (PAL), involving the addition of operators $\Box\varphi$ and $\Diamond\varphi$, quantifying over public announcements $[\theta]\varphi$ of some given type. These logics are of great interest both philosophically and from the point of view of applications. Motivations range from supporting an analysis of Fitch's paradox [16] by modeling notions of 'knowability' (expressible as $\Diamond K\varphi$), to determining the existence of communication protocols that achieve certain goals (cf. the famous Russian Card problem, given at a mathematical Olympiad [17]), and more generally to epistemic planning [9], and to inductive learnability in empirical science [5]. Many such extensions have been investigated, starting with the original APAL [2], and continuing with its variants GAL (Group Announcement Logic) [1], Future Event Logic [20], FAPAL (Fully Arbitrary Public Announcement Logic) [24], APAL$^+$ (Positive Arbitrary Announcement Logic) [19], BAPAL (Boolean Arbitrary Public Announcement Logic) [18], etc.

One problem with the above formalisms, with the exception of BAPAL[1], is that they all use *infinitary* axiomatizations. It is therefore not guaranteed that the validities of these logics are recursively enumerable.[2] The seminal paper

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[1] BAPAL is a very weak version, allowing $\Box\varphi$ to quantify over only purely propositional announcements - no epistemic formulas.

[2] APAL$^+$ is known to be decidable, hence its validities must be r.e., but no recursive axiomatization is known. Also, note that APAL$^+$ is still very weak, in that it quantifies only over positive epistemic announcements, thus not allowing public announcements of ignorance, which are precisely the ones driving the solution process in puzzles such as the Muddy Children.

on APAL [2] proved completeness using an infinitary rule, and then went on to claim that in theorem-proving[3] this rule can be replaced by the following finitary rule: from $\chi \rightarrow [\theta][p]\varphi$, infer $\chi \rightarrow [\theta]\Box\varphi$, as long as the propositional variable p is "fresh". This is a natural $\Box$-introduction rule, similar to the introduction rule for universal quantifier in FOL, and it is based on the intuition that variables that do not occur in a formula are irrelevant for its truth value, and thus can be taken to stand for any "arbitrary" formula (via some appropriate change of valuation). However, the soundness of this rule was later disproved via a counterexample by Kuijer [11]. Thus, a long-standing open question concerns *finding a 'strong' version of APAL for which there exists a recursive axiomatization.*[4]

In this paper, we solve this open question. Our diagnosis of Kuijer's counterexample is that it makes an essential use of a known undesirable feature of PAL and APAL, namely their *lack of memory*: the updated models "forget" the initial states. As a consequence, the expressivity of the APAL $\Box$-modality reduces after any update. This is what invalidates the above rule. We fix this problem by adding to the models a memory of the *initial* epistemic situation W^0, representing the information *before any non-trivial communication took place* ("the prior"). Since communication - gaining more information - deletes possibilities, the set W of currently possible states is a (possibly proper) subset of the set W^0 of initial states. On the syntactic side, we add an operator φ^0 saying that "φ was initially the case" (before all communication). To mark the initial states, we also need a constant 0, stating that "no non-trivial communication has taken place yet". Therefore, 0 will be true only in the initial epistemic situation. It is convenient, though maybe not absolutely necessary, to add a universal modality $U\varphi$ that quantifies over all currently possible states.[5] In the resulting Arbitrary Public Announcement Logic with Memory (APALM), the arbitrary announcement operator $\Box$ quantifies over updates (not only of epistemic formulas but) of arbitrary formulas that do not contain the operator $\Box$ itself.[6] As a result, the range of $\Box$ is wider than in standard APAL, covering announcements that may

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[3] This means that from any proof of a theorem from the axioms that uses the infinitary rule we can obtain a finitary proof of the same theorem, by using the finitary rule instead.

[4] Here, by 'strong' version we mean one that allows quantification over a sufficiently wide range of announcements (sufficiently wide to avoid Liar-like circles) as intended by a similar restriction in the original APAL or in its group-restricted version GAL.

[5] From an epistemic point of view, it would be more natural to replace U by an operator Ck that describes current common knowledge and quantifies only over currently possible states that are accessible by epistemic chains from the actual state. We chose to stick with U for simplicity and leave the addition of Ck to APAL for future work.

[6] This restriction is necessary to produce a well-defined semantics that avoids Liar-like vicious circles. In standard APAL, the restriction is w.r.t. inductive construct $\Diamond\varphi$. Thus, formulas of the form $\langle\Diamond p\rangle\varphi$ are allowed in original APAL. APAL and APALM expressivities seem to be incomparable, and that would still be the case if we dropped the above restriction.

refer to the initial situation (by the use of the operators 0 and φ^0) or to all currently possible states (by the use of $U\varphi$).

We show that the original finitary rule proposed in [2] *is* sound for APALM and, moreover, it forms the basis of a *complete recursive axiomatization.*[7] Besides its technical advantages, APALM is valuable in its own respect. Maintaining a record of the initial situation in our models helps us to formalize updates that refer to the 'epistemic past' such as "what you said, I knew already" [15]. This may be useful in treating certain epistemic puzzles involving reference to past information states, e.g. "What you said did not surprise me" [12]. The more recent *Cheryl's Birthday* problem also contains an announcement of the form "At first I didn't know when Cheryl's birthday is, but now I know" (although in this particular puzzle the past-tense announcement is redundant and plays no role in the solution).[8] See [15] for more examples.

Note though that the 'memory' of APALM is of a very specific kind. Our models do not remember the whole history of communication, but only the initial epistemic situation (before any communication). Correspondingly, in the syntax we do *not* have a 'yesterday' operator $Y\varphi$, referring to the previous state just before the last announcement as in [14], but only the operator φ^0 referring to the initial state. We think of this 'economy' of memory as *a (positive) "feature, not a bug"* of our logic: a detailed record of all history is simply not necessary for solving the problem at hand. In fact, keeping all the history and adding a $Y\varphi$ operator would greatly complicate our task by invalidating some of the standard nice properties of PAL (e.g. the composition axiom, stating that any sequence of announcements is equivalent to a single announcement).[9]

So we opt for simplicity, enriching the models and language just enough to recover the full expressivity of $\Box$ after updates, and thus establish the soundness of the $\Box$-introduction rule. The minimalistic-memory character of our semantics is rather natural and it is similar to the one encountered in Bayesian models[10],

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[7] We use a slightly different version of this rule, which is easily seen to be equivalent to the original version in the presence of the usual PAL reduction axioms.

[8] Cheryl's Birthday problem was part of the 2015 Singapore and Asian Schools Math Olympiad, and became viral after it was posted on Facebook by Singapore TV presenter Kenneth Kong.

[9] As a consequence, having such a rich memory of history would destroy some of the appealing features of the APAL operator (e.g. its $S4$ character: $\Box\varphi \rightarrow \Box\Box\varphi$), and would force us to distinguish between "knowability via one communication step" $\Diamond K$ versus "knowability via a finite communication sequence" $\Diamond^* K$, leading to an unnecessarily complex logic.

[10] In such models, only the 'prior' and the 'posterior' information states are taken to be relevant, while all the intermediary steps are forgotten. As a consequence, all the evidence gathered in between the initial and the current state can be compressed into one set E, called "the evidence" (rather than keeping a growing tail-sequence of all past evidence sets). Similarly, in our logic, all the past communication is compressed in its end-result, namely in the set W of current possibilities, which plays the same role as the evidence set E in Bayesian models.

with their distinction between 'prior' and 'posterior' (aka current) probabilities.[11]

On the technical side, our completeness proof involves an essential detour into an alternative semantics for APALM ('pseudo-models'), in the style of Subset Space Logics (SSL) [10,13]. This reveals deep connections between apparently very different formalisms. Moreover, this alternative semantics is of independent interest, giving us a more general setting for modeling knowability and learnability (see, e.g., [5,7,8]). Various combinations of PAL or APAL with subset space semantics have been investigated in the literature [4–7,21,22,25,26]. Following SSL-style, our pseudo-models come with a given family of *admissible sets* of worlds, which in our context represent "publicly announceable" (or communicable) propositions.[12] We interpret □ in pseudo-models as the so-called 'effort' modality of SSL, which quantifies over updates with announceable propositions (regardless of whether they are syntactically definable or not). The finitary □-introduction rule is obviously *sound for the effort modality*, because of its more 'semantic' character. This observation, together with the fact that APALM models (unlike original APAL models) can be seen as a special case of pseudo-models, lie at the core of our soundness and completeness proof.

Due to the page limit, we skipped some of the proofs. Readers interested to see all the relevant proofs, as well as a other related results, may consult the extended online version of this paper at https://analuciavsblog.wordpress.com/page/.

2 Syntax, Semantics, and Axiomatization

Let $Prop$ be a countable set of propositional variables and $\mathcal{AG} = \{1, \ldots, n\}$ be a finite set of agents. The language $\mathcal{L}$ of APALM (Arbitrary Public Announcement Logic with Memory) is recursively defined by the grammar:

$$\varphi ::= \top \mid p \mid 0 \mid \varphi^0 \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \Diamond\varphi,$$

with $p \in Prop$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\Diamond}$ is a formula in the sublanguage $\mathcal{L}_{-\Diamond}$ obtained from $\mathcal{L}$ by removing the $\Diamond$ operator. Given a formula $\varphi \in \mathcal{L}$, we

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[11] Moreover, this limited form of memory is enough to 'simulate' the yesterday operator $Y\varphi$ by using context-dependent formulas. For instance, the dialogue in Cheryl's birthday puzzle (Albert: "I don't know when Cheryl's birthday is, but I know that Bernard doesn't know it either"; Bernard: "At first I didn't know when Cheryl's birthday is, but I know now"; Albert: "Now I also know"), can be simulated by the following sequence of announcements: first, the formula $0 \wedge \neg K_a c \wedge K_a \neg K_b c$ is announced (where 0 marks the fact that this is the first announcement), then $(\neg K_b c)^0 \wedge K_b c$ is announced, and finally $K_a c$ is announced. (Here, we encode e.g. 'Albert knows Cheryl's birthday' as $K_a c = \bigvee \{K_a(d \wedge m) : d \in D, m \in M\}$, where D is the set of possible days and M is the set of possible months.).

[12] In SSL, the set of admissible sets is sometimes, but not always, taken to be a topology. Here, it will be a Boolean algebra with epistemic operators.

denote by P_φ the set of all propositional variables occurring in φ. We employ the usual abbreviations for $\bot$ and the propositional connectives $\lor, \rightarrow, \leftrightarrow$. The dual modalities are defined as $\hat{K}_i\varphi := \neg K_i\neg\varphi$, $E\varphi := \neg U\neg\varphi$, $\Box\varphi := \neg\Diamond\neg\varphi$, and $[\theta]\varphi := \neg\langle\theta\rangle\neg\varphi$.[13]

We read $K_i\varphi$ as "φ is *known* by agent i"; $\langle\theta\rangle\varphi$ as "θ can be *truthfully announced*, and *after* this public announcement φ is true". U and E are the global universal and existential modalities quantifying over all *current* possibilities: $U\varphi$ says that "φ is true in *all current alternatives* of the actual state". $\Diamond\varphi$ and $\Box\varphi$ are the (existential and universal) arbitrary announcement operators, quantifying over updates with formulas in $\mathcal{L}_{-\Diamond}$. We can read $\Box\varphi$ as "φ is *stably true* (under public announcements)": i.e., φ stays true no matter what (true) announcements are made. The constant 0, meaning that "*no (non-trivial) announcements* took place yet", holds only at the *initial time*. Similarly, the formula φ^0 means that "*initially* (prior to all communication), φ was true".

Definition 1 (Model, Initial Model, and Relativized Model)

- *A* model *is a tuple* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, *where* $W \subseteq W^0$ *are non-empty sets of states,* $\sim_i \subseteq W^0 \times W^0$ *are equivalence relations labeled by 'agents'* $i \in \mathcal{AG}$, *and* $\|\cdot\| : Prop \rightarrow \mathcal{P}(W^0)$ *is a valuation function that maps every propositional atom* $p \in Prop$ *to a set of states* $\|p\| \subseteq W^0$. W^0 *is the* initial domain, *representing the initial informational situation before any communication took place; its elements are called* initial states. *In contrast, W is the* current domain, *representing the current information, and its elements are called* current states.
- *For every model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, *we define its* initial model $\mathcal{M}^0 = (W^0, W^0, \sim_1, \ldots, \sim_n, \|\cdot\|)$, *whose current and initial domains are both given by the initial domain of the original model* $\mathcal{M}$.
- *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *and a set* $A \subseteq W$, *we define the* relativized model $\mathcal{M}|A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$.

For states $w \in W$ and agents i, we will use the notation $w_i := \{s \in W : w \sim_i s\}$ to denote the restriction to W of w's equivalence class modulo $\sim_i$.

Definition 2 (Semantics). *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, *we define a* truth set $[\![\varphi]\!]_\mathcal{M}$ *for every formula* φ. *When the current model* $\mathcal{M}$ *is understood, we skip the subscript and simply write* $[\![\varphi]\!]$. *The definition of* $[\![\varphi]\!]$ *is by recursion on* φ:

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[13] The update operator $\langle\theta\rangle\varphi$ is often denoted by $\langle!\theta\rangle\varphi$ in Public Announcement Logic literature; we skip the exclamation sign, but we will use the notation $\langle!\rangle$ for this modality and $[!]$ for its dual when we do not want to specify the announcement formula θ (so that ! functions as a placeholder for the content of the announcement).

$$\llbracket \top \rrbracket = W \qquad\qquad\qquad \llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$$

$$\llbracket p \rrbracket = \|p\| \cap W \qquad\qquad \llbracket K_i \varphi \rrbracket = \{w \in W : w_i \subseteq \llbracket \varphi \rrbracket\}$$

$$\llbracket 0 \rrbracket = \begin{cases} W^0 & \text{if } W = W^0 \\ \emptyset & \text{otherwise} \end{cases} \qquad \llbracket U \varphi \rrbracket = \begin{cases} W & \text{if } \llbracket \varphi \rrbracket = W \\ \emptyset & \text{otherwise} \end{cases}$$

$$\llbracket \varphi^0 \rrbracket = \llbracket \varphi \rrbracket_{\mathcal{M}^0} \cap W \qquad\qquad \llbracket \langle \theta \rangle \varphi \rrbracket = \llbracket \varphi \rrbracket_{\mathcal{M}|\llbracket \theta \rrbracket}$$

$$\llbracket \neg \varphi \rrbracket = W - \llbracket \varphi \rrbracket \qquad\qquad \llbracket \Diamond \varphi \rrbracket = \bigcup \{ \llbracket \langle \theta \rangle \varphi \rrbracket \, : \, \theta \in \mathcal{L}_{-\Diamond} \}$$

Observation 1. *Note that we have*

$$w \in \llbracket \Box \varphi \rrbracket \quad \text{iff} \quad w \in \llbracket [\theta] \varphi \rrbracket \text{ for every } \theta \in \mathcal{L}_{-\Diamond}.$$

In some of our inductive proofs, we will need a complexity measure on formulas different from the standard one[14]:

Lemma 1. *There exists a well-founded strict partial order $<$ on $\mathcal{L}$, such that:*

1. if φ is a subformula of ψ,   *2. $\langle \theta \rangle \varphi < \Diamond \varphi$, for all $\theta \in \mathcal{L}_{-\Diamond}$.*
   then $\varphi < \psi$,

Proposition 1. *We have $\llbracket \varphi \rrbracket \subseteq W$, for all formulas $\varphi \in \mathcal{L}$.*

(The proof, in the online version, is by induction on the complexity $<$ from Lemma 1.)

Definition 3 (APALM Models and Validity). *An* APALM model *is a tuple $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \| \cdot \|)$ such that $W = \llbracket \theta \rrbracket_{\mathcal{M}^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$; i.e. $\mathcal{M}$ can be obtained by updating its initial model $\mathcal{M}^0$ with some formula in $\mathcal{L}_{-\Diamond}$. A formula φ is* valid on APALM models *if it is true in all current states $s \in W$ (i.e. $\llbracket \varphi \rrbracket_{\mathcal{M}} = W$) for every APALM model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \| \cdot \|)$.*

APALM models are our *intended* models, in which the current information range comes from updating the initial range with some public announcement.

We arrive now at the main result of our paper.

Theorem 1. *(*Soundness and Completeness*) APALM validities are recursively enumerable. Indeed, the following axiom system* **APALM** *(given in Table 1, where recall that P_φ is the set of propositional variables in φ) is sound and complete wrt APALM models:*

Intuitive Reading of the Axioms. Parts (I) and (II) should be obvious. The axiom R[$\top$] says that *updating with tautologies is redundant.* The reduction laws

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[14] The standard notion requires only that formulas are more complex than their subformulas, while we also need that $\Diamond \varphi$ is more complex than $\langle \theta \rangle \varphi$. To the best of our knowledge, such a complexity measure was first introduced in [3] for the original APAL language from [2]. Similar measures have later been introduced for topological versions of APAL in [22,23].

Table 1. The axiomatization **APALM**. (Here, $\varphi, \psi, \chi \in \mathcal{L}$, while $\theta, \rho \in \mathcal{L}_{-\diamond}$.)

	(I) Basic Axioms of system APALM:
(CPL)	all classical propositional tautologies and Modus Ponens
($S5_{K_i}$)	all $S5$ axioms and rules for knowledge operator K_i
($S5_U$)	all $S5$ axioms and rules for U operator
(U-K_i)	$U\varphi \to K_i\varphi$
	(II) Axioms and rules for dynamic modalities [!]:
($K_!$)	*Kripke's axiom for* [!]: $[\theta](\psi \to \varphi) \to ([\theta]\psi \to [\theta]\varphi)$
($\mathrm{Nec}_!$)	*Necessitation for* [!]: from φ, infer $[\theta]\varphi$.
(RE)	*Replacement of Equivalents* [!]: from $\theta \leftrightarrow \rho$, infer $[\theta]\varphi \leftrightarrow [\rho]\varphi$.
Reduction laws:	
($R[\top]$)	$[\top]\varphi \leftrightarrow \varphi$
(R_p)	$[\theta]p \leftrightarrow (\theta \to p)$
($R_\neg$)	$[\theta]\neg\psi \leftrightarrow (\theta \to \neg[\theta]\psi)$
(R_{K_i})	$[\theta]K_i\psi \leftrightarrow (\theta \to K_i[\theta]\psi)$
($R_{[!]}$)	$[\theta][\rho]\chi \leftrightarrow [\langle\theta\rangle\rho]\chi$
(R^0)	$[\theta]\varphi^0 \leftrightarrow (\theta \to \varphi^0)$
(R_U)	$[\theta]U\varphi \leftrightarrow (\theta \to U[\theta]\varphi)$
(R_0)	$[\theta]0 \leftrightarrow (\theta \to (U\theta \wedge 0))$
	(III) Axioms and rules for 0 and *initial* operator ^{0}:
(Ax_0)	0^0
(0-U)	$0 \to U0$
(0-eq)	$0 \to (\varphi \leftrightarrow \varphi^0)$
(Nec^0)	*Necessitation for* ^{0}: from φ, infer φ^0
Equivalences with ^{0}:	
(Eq^0_p)	$p^0 \leftrightarrow p$
($\mathrm{Eq}^0_\neg$)	$(\neg\varphi)^0 \leftrightarrow \neg\varphi^0$
($\mathrm{Eq}^0_\wedge$)	$(\varphi \wedge \psi)^0 \leftrightarrow (\varphi^0 \wedge \psi^0)$
Implications with ^{0}:	
(Imp^0_U)	$(U\varphi)^0 \to U\varphi^0$
(Imp^0_i)	$(K_i\varphi)^0 \to K_i\varphi^0$
($\mathrm{Imp}^0_\square$)	$(\square\varphi)^0 \to \varphi$
	(III) Elim-axiom and Intro-rule for $\square$:
([!]$\square$-elim)	$[\theta]\square\varphi \to [\theta \wedge \rho]\varphi$
([!]$\square$-intro)	from $\chi \to [\theta \wedge p]\varphi$, infer $\chi \to [\theta]\square\varphi$ (for $p \notin P_\chi \cup P_\theta \cup P_\varphi$).

that do not contain ^{0}, U or 0 are well-known PAL axioms. R_U is the natural reduction law for the universal modality. The axiom R^0 says that the truth value of φ^0 formulas stays the same in time (because the superscript ^{0} serves as a time stamp), so they can be treated similarly to atoms. Ax_0 says that 0 was initially the case, and R_0 says that at any later stage (after any update) 0 can only be true if it was already true before the update and the update was trivial (universally true). Together, these two say that the constant 0 characterizes states where no non-trivial communication has occurred. Axiom 0-U is a sychronicity constraint: *if no non-trivial communication has taken place yet, then this is the case in all*

the currently possible states. Axiom 0-*eq* says that *initially, φ is equivalent to its initial correspondent φ^0.* The Equivalences with ^{0} express that ^{0} distributes over negation and over conjunction. $\text{Imp}^0_\Box$ says that *if initially φ was stably true (under any further announcements), then φ is the case now.* Taken together, the elimination axiom [!]$\Box$-elim and introduction rule [!]$\Box$-intro say that *φ is a stable truth after an announcement θ iff φ stays true after any more informative announcement* (of the form $\theta \wedge \rho$).[15]

Proposition 2. *The following schemas and inference rules are derivable in* **APALM**, *where $\varphi, \psi, \chi \in \mathcal{L}$ and $\theta \in \mathcal{L}_{-\Diamond}$:*

1. *from* $\vdash \varphi \leftrightarrow \psi$, *infer* $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$
2. $\vdash \langle\theta\rangle 0 \leftrightarrow (0 \wedge U\theta)$
3. $\vdash \langle\theta\rangle\psi \leftrightarrow (\theta \wedge [\theta]\psi)$
4. $\vdash \Box\varphi \rightarrow [\theta]\varphi$
5. *from* $\vdash \chi \rightarrow [p]\varphi$, *infer* $\vdash \chi \rightarrow \Box\varphi$
   $(p \notin P_\chi \cup P_\varphi)$
6. $\vdash (\varphi \rightarrow \psi)^0 \leftrightarrow (\varphi^0 \rightarrow \psi^0)$
7. $\vdash \varphi^{00} \leftrightarrow \varphi^0$
8. $\vdash \Box\varphi^0 \leftrightarrow \varphi^0$, *and* $\vdash \Diamond\varphi^0 \leftrightarrow \varphi^0$

9. $\vdash (\Box\varphi)^0 \rightarrow \Box\varphi^0$
10. $\vdash (0 \wedge \Diamond\varphi^0) \rightarrow \varphi$
11. $\vdash \varphi \rightarrow (0 \wedge \Diamond\varphi)^0$
12. $\vdash \varphi \rightarrow \psi^0$ *if and only if*
    $\vdash (0 \wedge \Diamond\varphi) \rightarrow \psi$
13. $\vdash [\theta](\psi \wedge \varphi) \leftrightarrow ([\theta]\psi \wedge [\theta]\varphi)$
14. $\vdash [\theta][p]\psi \leftrightarrow [\theta \wedge p]\psi$
15. $\vdash [\theta]\bot \leftrightarrow \neg\theta$

Proposition 3. *All S4 axioms and inference rules for $\Box$ are derivable in* **APALM**.

3   An Analysis of Kuijer's Counterexample

To understand Kuijer's counterexample [11] to the soundness of the finitary $\Box$-elimination rule for the original APAL, recall that in the APAL $\Box$ quantifies only over updates with *epistemic* formulas. More precisely, the APAL semantics of $\Box$ is given by

$$w \in [\![\Box\varphi]\!] \quad \text{iff} \quad w \in [\![[\theta]\varphi]\!] \text{ for every } \theta \in \mathcal{L}_{epi},$$

where $\mathcal{L}_{epi}$ is the sublanguage generated from propositional atoms $p \in Prop$ using only the Boolean connectives $\neg\varphi$ and $\varphi \wedge \psi$ and the epistemic operators $K_i\varphi$.

Kuijer takes the formula $\gamma := p \wedge \hat{K}_b\neg p \wedge \hat{K}_a K_b p$, and shows that

$$[\hat{K}_b p]\Box\neg\gamma \rightarrow [q]\neg\gamma.$$

is valid on APAL models. (In fact, it is also valid on APALM models!) But then, by the [!]$\Box$-intro rule (or rather, by its weaker consequence in Proposition 2(5)), the formula

$$[\hat{K}_b p]\Box\neg\gamma \rightarrow \Box\neg\gamma$$

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[15] The "freshness" of the variable $p \in P$ in the rule [!]$\Box$-intro ensures that it represents any 'generic' announcement.

should also be valid. But this is contradicted by the model $\mathcal{M}$ in Fig. 1. The premise $[\hat{K}_b p]\square\neg\gamma$ is true at w in $\mathcal{M}$, since $\square\neg\gamma$ holds at w in $\mathcal{M}|[\![\hat{K}_b p]\!]$: indeed, the only way to falsify it would be by deleting the lower-right node from Fig. 2a while keeping (all other nodes, and in particular) the upper-right node. But in $\mathcal{M}|[\![\hat{K}_b p]\!]$ the upper and lower right nodes can't be separated by epistemic sentences: they are bisimilar.
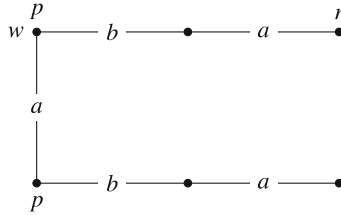
Fig. 1. An initial model $\mathcal{M}$. Worlds are nodes in the graph, valuation is given by labeling the node with the true atoms, and epistemic relations are given by labeled arrows.

(a) $\mathcal{M}|[\![\hat{K}_b p]\!]$ (b) $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$
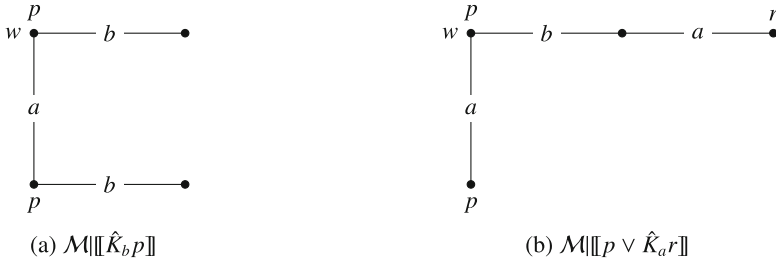
Fig. 2. Two updates of $\mathcal{M}$.

In contrast, the conclusion $\square\neg\gamma$ is *false* at w in $\mathcal{M}$, since in that original model the two mentioned nodes *could be separated*. Indeed, we could perform an alternative update with the formula $p \vee \hat{K}_a r$, yielding the epistemic model $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$ shown in Fig. 2b, in which γ *is true* at w (contrary to the assertion that $\square\neg\gamma$ was true in $\mathcal{M}$).

To see that the counterexample does not apply to APALM, notice that APALM models keep track of the initial states, so technically speaking the updated model $\mathcal{M}|[\![\hat{K}_b p]\!]$ consists now of the initial structure in Fig. 1 together with current set of worlds W in Fig. 2a. But in this model, $\square\neg\gamma$ is *no longer true* at w (and so the premise $[\hat{K}_b p]\square\neg\gamma$ was *not* true in $\mathcal{M}$ when considered as an APALM model!). Indeed, we *can* perform a new update of $\mathcal{M}|[\![\hat{K}_b p]\!]$ with the formula $(p \vee \hat{K}_a r)^0$, which yields an updated model whose current set of worlds is given in Fig. 3:
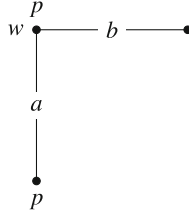
Fig. 3. The current worlds resulting from updating $\mathcal{M}|[\![\hat{K}_b p]\!]$ with $(p \vee \hat{K}_a r)^0$.

Note that, in this new model, γ *is* the case at w (-thus showing that $\Box \neg \gamma$ was *not* true at w in $\mathcal{M}|[\![\hat{K}_b p]\!]$). So the counterexample simply does not work for APALM.

Moreover, we can see that the un-soundness of [!]$\Box$-intro rule for APAL has to do with its lack of memory, which leads to information loss after updates: while initially (in $\mathcal{M}$) there were epistemic sentences (e.g. $p \vee \hat{K}_a r$) that could separate the two nodes mentioned above, there are no such sentences after the update.

APALM solves this by keeping track of the initial states, and referring back to them, as in $(p \vee \hat{K}_a r)^0$.

4   Soundness, via Pseudo-model Semantics

To start with, note that the even the soundness of our axiomatic system is not a trivial matter. As we saw from Kuijer's counterexample, the analogue of our finitary $\Box$-introduction rule was *not* sound for APAL. To prove its soundness on APALM models, we need a detour into an equivalent semantics, in the style of Subset Space Logics (SSL) [10,13]: *pseudo-models.*[16]

We first introduce an auxiliary notion: 'pre-models' are just SSL models, coming with a given family $\mathcal{A}$ of "admissible sets" of worlds (which can be thought of as the *communicable* propositions). We interpret $\Box$ in these structures as the so-called "effort modality" of SSL, which quantifies over updates with admissible propositions in $\mathcal{A}$. Our 'pseudo-models' are pre-models with additional closure conditions (saying that the family of admissible sets includes the valuations and is closed under complement, intersection and epistemic operators). These conditions imply that *every set definable by $\Diamond$-free formulas is admissible*, and this

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[16] A more direct soundness proof on APALM models is in principle possible, but would require at least as much work as our detour. Unlike in standard EL, PAL or DEL, the meaning of an APALM formula depends, not only on the valuation of the atoms occurring in it, but also on the family $\mathcal{A}$ of all sets definable by $\mathcal{L}_{-\Diamond}$-formulas. The move from models to pseudo-models makes explicit this dependence on the family $\mathcal{A}$, while also relaxing the demands on $\mathcal{A}$ (which is no longer required to be exactly the family of $\mathcal{L}_{-\Diamond}$-definable sets), and thus makes the soundness proof both simpler and more transparent. Since we will need pseudo-models for our completeness proof anyway, we see no added value in trying to give a more direct proof.

ensures the soundness of our $\Box$-elimination axiom on pseudo-models. As for the soundness of the long-problematic $\Box$-introduction rule on (both pre- and) pseudo-models, this is due to the fact that the effort modality has a more 'robust' range than the arbitrary announcement operator: it quantifies over admissible sets, regardless of whether these sets are syntactically definable or not. Soundness with respect to our intended (APALM) models then follows from the observation that they (in contrast to the original APAL models) are in fact equivalent to a *special case of pseudo-models*: the "standard" ones (in which the admissible sets in $\mathcal{A}$ are *exactly* the sets definable by $\Diamond$-free formulas).

Definition 4 (Pre-model). *A* pre-model *is a tuple* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \| \cdot \|)$, *where* W^0 *is the prior domain,* $\sim_i$ *are equivalence relations on* W^0, $\| \cdot \| : Prop \to \mathcal{P}(W^0)$ *is a valuation map, and* $\mathcal{A} \subseteq \mathcal{P}(W^0)$ *is a family of subsets of the initial domain, called* admissible sets *(representing the propositions that can be publicly announced).*

Given a set $A \subseteq W^0$ and a state $w \in A$, we use the notation $w_i^A := \{s \in A : w \sim_i s\}$ to denote the restriction to A of w's equivalence class modulo $\sim_i$.

Definition 5 (Pre-model Semantics). *Given a pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \| \cdot \|)$, *we recursively define a* truth set $[\![\varphi]\!]_A$ *for every formula* φ *and subset* $A \subseteq W^0$:

$$[\![\top]\!]_A = A \qquad\qquad [\![\varphi \wedge \psi]\!]_A = [\![\varphi]\!]_A \cap [\![\psi]\!]_A$$

$$[\![p]\!]_A = \|p\| \cap A \qquad\qquad [\![K_i\varphi]\!]_A = \{w \in A : w_i^A \subseteq [\![\varphi]\!]_A\}$$

$$[\![0]\!]_A = \begin{cases} A & if\ A = W^0 \\ \emptyset & otherwise \end{cases} \qquad\qquad [\![U\varphi]\!]_A = \begin{cases} A & if\ [\![\varphi]\!]_A = A \\ \emptyset & otherwise \end{cases}$$

$$[\![\varphi^0]\!]_A = [\![\varphi]\!]_{W^0} \cap A \qquad\qquad [\![\langle\theta\rangle\varphi]\!]_A = [\![\varphi]\!]_{[\![\theta]\!]_A}$$

$$[\![\neg\varphi]\!]_A = A - [\![\varphi]\!]_A \qquad\qquad [\![\Diamond\varphi]\!]_A = \bigcup\{[\![\varphi]\!]_B : B \in \mathcal{A}, B \subseteq A\}$$

Observation 2. *Note that, for all* $w \in A$, *we have*

$$w \in [\![\Box\varphi]\!]_A \quad iff \quad \forall B \in \mathcal{A}(w \in B \subseteq A \Rightarrow w \in [\![\varphi]\!]_B),$$

which fits with the semantics of the 'effort' modality in SSL [10,13]. *Moreover, it is easy to see that* $[\![\varphi]\!]_A \subseteq A$ *for all* $A \in \mathcal{A}$ *and* $\varphi \in \mathcal{L}$.

Definition 6 (Pseudo-models and Validity). *A* pseudo-model *is a pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \| \cdot \|)$, *satisfying the following closure conditions:*

1. $\|p\| \in \mathcal{A}$, for all $p \in Prop$,
2. $W^0 \in \mathcal{A}$,
3. if $A \in \mathcal{A}$ then $(W^0 - A) \in \mathcal{A}$,
4. if $A, B \in \mathcal{A}$ then $(A \cap B) \in \mathcal{A}$,
5. if $A \in \mathcal{A}$ then $K_i A \in \mathcal{A}$, where $K_i A := \{w \in W^0 : \forall s \in W^0 (w \sim_i s \Rightarrow s \in A)\}$.

A formula $\varphi \in \mathcal{L}$ *is* valid on pseudo-models *if it is true in all admissible sets* $A \in \mathcal{A}$ *of every pseudo-model* $\mathcal{M}$, *i.e,* $[\![\varphi]\!]_A = A$ *for all* $A \in \mathcal{A}$ *and all* $\mathcal{M}$.

Proposition 4. *Given a pseudo-model* $(W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, $A \in \mathcal{A}$ *and* $\theta \in \mathcal{L}_{-\diamond}$, *we have* $[\![\theta]\!]_A \in \mathcal{A}$.

Proof. The proof is by subformula induction on θ, where we denote by IH the induction hypothesis. The base cases and the inductive cases for the Booleans are immediate (using the conditions in Definition 6).

Case $\theta := \psi^0$. By the semantics, $[\![\psi^0]\!]_A = [\![\psi]\!]_{W^0} \cap A \in \mathcal{A}$, since $[\![\psi]\!]_{W^0} \in \mathcal{A}$ (by the fact that $W^0 \in \mathcal{A}$ and IH), $A \in \mathcal{A}$ (by assumption), and $\mathcal{A}$ is closed under intersection.

Case $\theta := K_i\psi$.

$$
\begin{aligned}
[\![K_i\psi]\!]_A &= \{w \in A : w_i^A \subseteq [\![\varphi]\!]_A\} \\
&= A \cap \{w \in W^0 : w_i^A \subseteq [\![\varphi]\!]_A\} \\
&= A \cap \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in [\![\varphi]\!]_A)\} \\
&= A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in A \Rightarrow s \in [\![\varphi]\!]_A))\} \\
&= A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in (X - A) \text{ or } s \in [\![\varphi]\!]_A))\} \\
&= A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((X - A) \cup [\![\varphi]\!]_A)\} \\
&= A \cap K_i((X - A) \cup [\![\varphi]\!]_A) \\
&\qquad \text{(by Definition 6(3–5), since } A \in \mathcal{A} \text{ and } [\![\varphi]\!]_A \in \mathcal{A}(\text{by IH}).)
\end{aligned}
$$

Case $\theta := U\psi$. By definition, $[\![U\psi]\!]_A \in \{\emptyset, A\} \subseteq \mathcal{A}$.

Case $\theta := \langle\varphi\rangle\psi$. Since $A \in \mathcal{A}$, we have $[\![\varphi]\!]_A \in \mathcal{A}$ (by IH), and hence $[\![\langle\varphi\rangle\psi]\!]_A = [\![\psi]\!]_{[\![\varphi]\!]_A} \in \mathcal{A}$ (by the semantics and IH again). $\qquad$

Proposition 5. *The system* **APALM** *is sound wrt pseudo-models.*

To prove Proposition 5, we need the following:

Lemma 2. *Let* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *and* $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ *be two pseudo-models and* $\varphi \in \mathcal{L}$ *such that* $\mathcal{M}$ *and* $\mathcal{M}'$ *differ only in the valuation of some* $p \notin P_\varphi$. *Then, for all* $A \in \mathcal{A}$, *we have* $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$.

Proof. The proof follows by subformula induction on φ. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ be two pseudo-models such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some $p \notin P_\varphi$ and let $A \in \mathcal{A}$. We want to show that $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$. The base cases $\varphi := q$, $\varphi := \top$, $\varphi := 0$, and the inductive cases for Booleans are immediate.

Case $\varphi := \psi^0$. Note that $P_{\psi^0} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$, in particular for $W^0 \in \mathcal{A}$. Thus $[\![\psi]\!]_{W^0}^{\mathcal{M}'} = [\![\psi]\!]_{W^0}^{\mathcal{M}}$. Then, $[\![\psi]\!]_{W^0}^{\mathcal{M}'} \cap A = [\![\psi]\!]_{W^0}^{\mathcal{M}} \cap A$ for all $A \in \mathcal{A}$. By the semantics of the initial operator on pseudo-models, we obtain $[\![\psi^0]\!]_A^{\mathcal{M}'} = [\![\psi^0]\!]_A^{\mathcal{M}}$.

Case $\varphi := K_i\psi$. Note that $P_{K_i\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$. Observe that $[\![K_i\psi]\!]_A^{\mathcal{M}} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}}\}$ and, similarly, $[\![K_i\psi]\!]_A^{\mathcal{M}'} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}'}\}$. Then, since $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$, we obtain $[\![K_i\psi]\!]_A^{\mathcal{M}} = [\![K_i\psi]\!]_A^{\mathcal{M}'}$.

Case $\varphi := U\psi$. Note that $P_{U\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. We have two case: (1) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} = A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = A = [\![U\psi]\!]_A^{\mathcal{M}}$. (2) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} \neq A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = [\![U\psi]\!]_A^{\mathcal{M}} = \emptyset$.

Case $\varphi := \langle\theta\rangle\psi$. Note that $P_{\langle\theta\rangle\psi} = P_\theta \cup P_\psi$. By IH, we have $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. By Proposition 4, we know that $[\![\theta]\!]_A^{\mathcal{M}} = [\![\theta]\!]_A^{\mathcal{M}'} \in \mathcal{A}$. Therefore, in particular, we have $[\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'} = [\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore, by the semantics of $\langle!\rangle$ on pseudo-models, we obtain $[\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}'} = [\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}}$.

Case $\varphi := \Diamond\psi$. Note that $P_{\Diamond\psi} = P_\psi$. Since *the same* family of sets $\mathcal{A}$ is carried by both models $\mathcal{M}$ and $\mathcal{M}'$ and since (by IH) $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for all $A \in \mathcal{A}$, we get:

$$[\![\Diamond\psi]\!]_A^{\mathcal{M}'} = \bigcup\{[\![\psi]\!]_B^{\mathcal{M}'} : B \in \mathcal{A}, B \subseteq A\} = \bigcup\{[\![\psi]\!]_B^{\mathcal{M}} : B \in \mathcal{A}, B \subseteq A\} = [\![\Diamond\psi]\!]_A^{\mathcal{M}}.$$

Proof of Proposition 5. The soundness of most of the axioms follows simply by spelling out the semantics. We present here only the soundness of the rule $[!]\Box$-intro:

Suppose $\models \chi \rightarrow [\theta \wedge p]\varphi$ and $\not\models \chi \rightarrow [\theta]\Box\varphi$, where $p \notin P_\chi \cup P_\theta \cup P_\varphi$. The latter means that there exists a pseudo model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin [\![\chi \rightarrow [\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Therefore $w \in [\![\chi \wedge \neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Thus we have (1) $w \in [\![\chi]\!]_A^{\mathcal{M}}$ and (2) $w \in [\![\neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Because of (2), $w \in [\![\langle\theta\rangle\Diamond\neg\varphi]\!]_A^{\mathcal{M}}$, and, by the semantics, $w \in [\![\Diamond\neg\varphi]\!]_{[\![\theta]\!]_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore, applying the semantics, we obtain (3) there exists $B \in \mathcal{A}$ s.t. $w \in B \subseteq [\![\theta]\!]_A^{\mathcal{M}} \subseteq A$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ such that $\|p\|' := B$ and $\|q\|' = \|q\|$ for any $q \neq p \in Prop$. In order to use Lemma 2 we must show that $\mathcal{M}'$ is a pseudo-model. For this we only need to verify that $\mathcal{M}'$ satisfies the closure conditions given in Definition 6. First note that $\|p\|' := B \in \mathcal{A}$ by the construction of $\mathcal{M}'$, so $\|p\|' \in \mathcal{A}$. For every $q \neq p$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$ we have $\|q\|' \in \mathcal{A}$. Since $\mathcal{A}$ is the same for both $\mathcal{M}$ and $\mathcal{M}'$, and $\mathcal{M}$ is a pseudo model, the rest of the closure conditions are already satisfied for $\mathcal{M}'$. Therefore $\mathcal{M}'$ is a pseudo model. Now continuing with our soundness proof, note that by Lemma 2 and since $p \notin P_\chi \cup P_\theta \cup P_\varphi$ we obtain $[\![\chi]\!]_A^{\mathcal{M}'} = [\![\chi]\!]_A^{\mathcal{M}}$, $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\neg\varphi]\!]_A^{\mathcal{M}'} = [\![\neg\varphi]\!]_A^{\mathcal{M}}$. Since $\|p\|' = B \subseteq [\![\theta]\!]_A^{\mathcal{M}'} \subseteq A$ we have $\|p\|' = [\![p]\!]_A^{\mathcal{M}'}$. Because of (3) we have that $w \in [\![\theta]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}'} = [\![\neg\varphi]\!]_{[\![p]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'} = [\![\langle p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$. Thus $w \in [\![p]\!]_A^{\mathcal{M}'}$, so $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}'} \cap [\![p]\!]_A^{\mathcal{M}'} = [\![p]\!]_A^{\mathcal{M}'}$ simply because $[\![p]\!]_A^{\mathcal{M}'} \subseteq [\![\theta]\!]_A^{\mathcal{M}'}$. Since $w \in [\![\neg\varphi]\!]_{[\![p]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'}$ we obtain $w \in [\![\neg\varphi]\!]_{[\![\theta \wedge p]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'}$. Putting everything together, $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]_{[\![\theta \wedge p]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'}$, we obtain that $w \in [\![\langle\theta \wedge p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$ and $w \in [\![\chi]\!]_A^{\mathcal{M}'}$. Therefore $\mathcal{M}', w \models \chi \wedge \langle\theta \wedge p\rangle\neg\varphi$, which contradicts the validity of $\chi \rightarrow [\theta \wedge p]\varphi$.

Definition 7 (Standard Pre-model). *A pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *is* standard *if and only if* $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$.

Proposition 6. *Every standard pre-model is a pseudo-model.*

Proof. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be a standard pre-model. This implies that $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. We need to show that $\mathcal{M}$ satisfies the closure conditions given in Definition 6. Conditions (1) and (2) are immediate.

For (3): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, we know that $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $\neg\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![\neg\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\neg\theta]\!]_{W^0} = W^0 - [\![\theta]\!]_{W^0}$, thus, we obtain $W^0 - A \in \mathcal{A}$.

For (4): let $A, B \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta_1]\!]_{W^0}$ and $B = [\![\theta_2]\!]_{W^0}$ for some $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$. Since $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$, we have $\theta_1 \wedge \theta_2 \in \mathcal{L}_{-\Diamond}$, thus, $[\![\theta_1 \wedge \theta_2]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\theta_1 \wedge \theta_2]\!]_{W^0} = [\![\theta_1]\!]_{W^0} \cap [\![\theta_2]\!]_{W^0} = A \cap B$, thus, we obtain $A \cap B \in \mathcal{A}$.

For (5): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $K_i\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![K_i\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![K_i\theta]\!]_{W^0} = \{w \in W^0 : \forall s \in W^0 (w \sim_i s \Rightarrow s \in [\![\theta]\!]_{W^0})\} = K_i [\![\theta]\!]_{W^0}$, thus, we obtain $K_i A \in \mathcal{A}$.

Equivalence Between the Standard Pseudo-models and APALM Models. For Proposition 7 only, we use the notation $[\![\varphi]\!]_A^{PS}$ to refer to pseudo-model semantics (as in Definition 5) and use $[\![\varphi]\!]_{\mathcal{M}}$ to refer to APALM semantics (as in Definition 1).

Proposition 7

1. *For every standard pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and every set $A \in \mathcal{A}$, we denote by $\mathcal{M}_A$ the model $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$. Then:*
   (a) For every $\varphi \in \mathcal{L}$, we have $[\![\varphi]\!]_{\mathcal{M}_A} = [\![\varphi]\!]_A^{PS}$ for all $A \in \mathcal{A}$.
   (b) $\mathcal{M}_A$ is an APALM model, for all $A \in \mathcal{A}$.
2. *For every APALM model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we denote by $\mathcal{M}'$ the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, where $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. Then*
   (a) $\mathcal{M}'$ is a standard pseudo-model.
   (b) For every $\varphi \in \mathcal{L}$, we have $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_W^{PS}$.

The proof of Proposition 7 needs the following lemma.

Lemma 3. *Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ a standard pseudo-model, $A \in \mathcal{A}$ and $\varphi \in \mathcal{L}$, then the following holds: $[\![\Diamond\varphi]\!]_A = \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$.*

Proof. For ($\subseteq$): Let $w \in [\![\Diamond\varphi]\!]_A$. Then, by the semantics of $\Diamond$ in pseudo-models. there exists some $B \in \mathcal{A}$ such that $w \in B \subseteq A$ and $w \in [\![\varphi]\!]_B$. Since $\mathcal{M}$ is standard, we know that $A = [\![\psi]\!]_{W^0}$ and $B = [\![\chi]\!]_{W^0}$ for some $\psi, \chi \in \mathcal{L}_{-\Diamond}$. Moreover, since $B = [\![\chi]\!]_{W^0} \subseteq A = [\![\psi]\!]_{W^0}$, we have $B = [\![\chi]\!]_{W^0} \cap [\![\psi]\!]_{W^0} = [\![\chi^0]\!]_{[\![\psi]\!]_{W^0}} = [\![\chi^0]\!]_A$, and so $w \in [\![\varphi]\!]_B = [\![\varphi]\!]_{[\![\chi^0]\!]_A} = [\![\langle\chi^0\rangle\varphi]\!]_A \subseteq \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$. For ($\supseteq$): Let $w \in \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$. Then we have $w \in [\![\langle\theta\rangle\varphi]\!]_A = [\![\varphi]\!]_{[\![\theta]\!]_A}$, for some $\theta \in \mathcal{L}_{-\Diamond}$, and since $[\![\theta]\!]_A \in \mathcal{A}$ (by Proposition 4) and $[\![\theta]\!]_A \subseteq A$ (by Observation 2), it follows that $w \in [\![\Diamond\varphi]\!]_A$ (by the semantics of $\Diamond$ in pseudo-models).

Proof of Proposition 7

1. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be a standard pseudo-model. $A \in \mathcal{A}$ implies $A = [\![\theta]\!]_{W^0}^{PS} \subseteq W$ for some θ, hence $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a model.

   (a) The proof is by induction on the complexity measure $<$ from Lemma 1. The base cases and the inductive cases for Booleans are straightforward.
   Case $\theta := \psi^0$. We have $[\![\psi]\!]_A^{PS} = [\![\psi]\!]_{W^0}^{PS} \cap A = [\![\psi]\!]_{\mathcal{M}_A^0} \cap A = [\![\psi^0]\!]_{\mathcal{M}_A}$ (by Definition 5, IH, and Definition 2).
   Case $\theta := K_i\psi$. We have $[\![K_i\psi]\!]_A^{PS} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{PS} = \{w \in A : w_i \subseteq [\![\psi]\!]_{\mathcal{M}_A}\} = [\![K_i\psi]\!]_{\mathcal{M}_A}$ (by Definition 5, IH, and Definition 2).
   Case $\theta := U\psi$. By Definitions 2 and 5, we have:

$$[\![U\psi]\!]_{\mathcal{M}_A} = \begin{cases} W & \text{if } [\![\psi]\!]_{\mathcal{M}_A} = A \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]_A^{PS} = \begin{cases} A & \text{if } [\![\psi]\!]_A^{PS} = A \\ \emptyset & \text{otherwise} \end{cases}$$

   By IH, $[\![\psi]\!]_A^{PS} = [\![\psi]\!]_{\mathcal{M}_A}$, therefore, $[\![U\psi]\!]_A^{PS} = [\![U\psi]\!]_{\mathcal{M}_A}$.
   Case $\varphi := \langle\psi\rangle\chi$. By Definition 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A}}$. Now consider the relativized model $\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A} = (W^0, [\![\psi]\!]_{\mathcal{M}_A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$. By Lemma 1(1) and IH, we have $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]_A^{PS}$. Moreover, by the definition of standard pseudo-models, we know that $A = [\![\theta]\!]_{W^0}^{PS}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]_A^{PS} = [\![\psi]\!]_{[\![\theta]\!]_{W^0}^{PS}}^{PS} = [\![\langle\theta\rangle\psi]\!]_{W^0}^{PS}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} \in \mathcal{A}$. We then have

$$[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]_{\mathcal{M}_A}}} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]_A^{PS}}} = [\![\chi]\!]_{[\![\psi]\!]_A^{PS}}^{PS} = [\![\langle\psi\rangle\chi]\!]_A^{PS},$$

   by the semantics and IH on ψ and on χ (since $[\![\psi]\!]_A^{PS} \in \mathcal{A}$).
   Case $\varphi := \Diamond\psi$. We have:

$$\begin{aligned}
[\![\Diamond\psi]\!]_{\mathcal{M}_A} &= \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}_A} : \chi \in \mathcal{L}_{-\diamond}\} & \text{(Defn.2)} \\
&= \bigcup\{[\![\langle\chi\rangle\psi]\!]_W^{PS} : \chi \in \mathcal{L}_{-\diamond}\} & \text{(Lemma 1.2, IH)} \\
&= [\![\Diamond\psi]\!]_A^{PS} & \text{(Lemma 3, since } \mathcal{M} \text{ is a standard pseudo model)}
\end{aligned}$$

   (b) By part (a), $[\![\varphi]\!]_{\mathcal{M}_A^0} = [\![\varphi]\!]_{\mathcal{M}_{W^0}} = [\![\varphi]\!]_{W^0}^{PS}$ for all φ. Since $\mathcal{M}$ is standard, we have $A = [\![\theta]\!]_{W^0}^{PS} = [\![\theta]\!]_{\mathcal{M}_A^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$, so $\mathcal{M}_A$ is an APALM model.

2. Let $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be an APALM model. Since $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\diamond}\} \subseteq \mathcal{P}(W^0)$, the model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a pre-model. Therefore, the semantics given in Definition 5 is defined on $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$.

   (a) By Proposition 6, it suffices to prove that the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is standard, i.e. that $\{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\diamond}\} = \{[\![\theta]\!]_{W^0}^{PS} : \theta \in \mathcal{L}_{-\diamond}\}$. For this, we need to show that for every APALM model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we have $[\![\theta]\!]_{\mathcal{M}} = [\![\theta]\!]_W^{PS}$ for all $\theta \in \mathcal{L}_{-\diamond}$.

We prove this by subformula induction on θ. The base cases and the inductive cases for Booleans are straightforward.

Case $\theta := \psi^0$. Then $[\![\psi^0]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0} \cap W = [\![\psi]\!]_{W^0}^{PS} \cap W = [\![\psi]\!]_W^{PS}$ (by Definition 2, IH, and Definition 5).

Case $\theta := K_i\psi$. We have $[\![K_i\psi]\!]_{\mathcal{M}} = \{w \in W : w_i \subseteq [\![\psi]\!]_{\mathcal{M}}\} = \{w \in W : w_i^W \subseteq [\![\psi]\!]_W^{PS}\} = [\![K_i\psi]\!]_W^{PS}$ (by Definition 2, IH, and Definition 5).

Case $\theta := U\psi$. By Definitions 2 and 5, we have:

$$[\![U\psi]\!]_{\mathcal{M}} = \begin{cases} W & \text{if } [\![\psi]\!]_{\mathcal{M}} = W \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]_W^{PS} = \begin{cases} W & \text{if } [\![\psi]\!]_W^{PS} = W \\ \emptyset & \text{otherwise} \end{cases}$$

By IH, $[\![\psi]\!]_W^{PS} = [\![\psi]\!]_{\mathcal{M}}$, therefore, $[\![U\psi]\!]_W^{PS} = [\![U\psi]\!]_{\mathcal{M}}$.

Case $\theta := \langle\psi\rangle\chi$. By Definition 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}}$. Now consider the relativized model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}} = (W^0, [\![\psi]\!]_{\mathcal{M}_A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$. By Lemma 1(1) and IH on ψ, we have $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_W^{PS}$. Moreover, by the definition of APALM models, we know that $W = [\![\theta]\!]_{\mathcal{M}^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0|[\![\theta]\!]_{\mathcal{M}^0}} = [\![\langle\theta\rangle\psi]\!]_{\mathcal{M}^0}$. Therefore, since $\langle\theta\rangle\psi \in \mathcal{L}_{-\diamond}$, the model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}$ is also an APALM model obtained by updating the initial model $\mathcal{M}^0$. We then have

$$\begin{aligned} [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} &= [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}} && \text{(Defn.2)} \\ &= [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_W^{PS}} && \text{(IH on } \psi\text{)} \\ &= [\![\chi]\!]_{[\![\psi]\!]_W^{PS}}^{PS} && \text{(IH on } \chi \text{ since } \mathcal{M}|[\![\psi]\!]_{\mathcal{M}} \text{ is an APALM model)} \\ &= [\![\langle\psi\rangle\chi]\!]_W^{PS} && \text{(Defn.5)} \end{aligned}$$

(b) The proof of this part follows by $<$-induction on φ (where $<$ is as in Lemma 1). All the inductive cases are similar to ones in the above proof, except for the case $\varphi := \diamond\psi$, shown below:

$$\begin{aligned} [\![\diamond\psi]\!]_{\mathcal{M}} &= \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}} : \chi \in \mathcal{L}_{-\diamond}\} && \text{(Defn.2)} \\ &= \bigcup\{[\![\langle\chi\rangle\psi]\!]_W^{PS} : \chi \in \mathcal{L}_{-\diamond}\} && \text{(Lemma 1.2, IH)} \\ &= [\![\diamond\psi]\!]_W^{PS} && \text{(Lemma 3, since } \mathcal{M}' \text{ is a standard pseudo model)} \end{aligned}$$

Corollary 1. *Validity on standard pseudo-models coincides with APALM validity.*

Proof. This is a straightforward consequence of Proposition 7.

Corollary 2. *The system* **APALM** *is sound wrt APALM models.*

This follows immediately from Proposition 5 and Corollary 1.

It is important to note that the equivalence between standard pseudo-models and APALM models (given by Proposition 7 above, and underlying our soundness result) is *not* trivial. It relies in particular on the equivalence between the effort modality and the arbitrary announcement operator □ on standard pseudo-models, which holds *only because our models and language retain the memory of the initial situation* (see Lemma 3). Hence, a similar equivalence fails for the original APAL.

5   Completeness

In this section we prove the completeness of **APALM**. First, we show completeness with respect to pseudo-models, via an innovative modification of the standard canonical model construction. This is based on a method previously used in [6], that makes an essential use of the finitary $\Box$-introduction rule, by requiring our canonical theories T to be (not only maximally consistent, but also) "witnessed". Roughly speaking, a theory T is witnessed if every $\Diamond\varphi$ occurring in every "existential context" in T is witnessed by some atomic formula p, meaning that $\langle p\rangle\varphi$ occurs in the same existential context in T. Our canonical pre-model will consist of all *initial, maximally consistent, witnessed theories* (where a theory is 'initial' if it contains the formula 0). A Truth Lemma is proved, as usual. Completeness for (both pseudo-models and) APALM models follows from the observation that our canonical pre-model *is standard*, hence it is (a standard pseudo-model, and thus) equivalent to a genuine APALM model.

We now proceed with the details. The appropriate notion of "existential context" is represented by *possibility forms*, in the following sense.

Definition 8 (Necessity forms and possibility forms). *For any finite string* $s \in (\{\bullet^0\} \cup \{\varphi\to \mid \varphi \in \mathcal{L}\} \cup \{K_i : i \in \mathcal{A}\} \cup \{U\} \cup \{\rho \mid \rho \in \mathcal{L}_{-\Diamond}\})^* = NF_{\mathcal{L}}$, *we define pseudo-modalities* $[s]$ *and* $\langle s\rangle$. *These pseudo-modalities are functions mapping any formula* $\varphi \in \mathcal{L}$ *to another formula* $[s]\varphi \in \mathcal{L}$ *(necessity form), respectively* $\langle s\rangle\varphi \in \mathcal{L}$ *(possibility form). The necessity forms are defined recursively as* $[\epsilon]\varphi = \varphi$, $[s, \bullet^0]\varphi = [s]\varphi^0$, $[s, \varphi\to]\varphi = [s](\varphi \to \varphi)$, $[s, K_i]\varphi = [s]K_i\varphi$, $[s, U]\varphi = [s]U\varphi$, $[s, \rho]\varphi = [s][\rho]\varphi$, *where* ϵ *is the empty string. For possibility forms, we set* $\langle s\rangle\varphi := \neg[s]\neg\varphi$.

Example: $[K_i, \bullet^0, \Diamond p\to, 0, U]$ *is a necessity form s.t.* $[K_i, \bullet^0, \Diamond p\to, 0, U]\varphi = K_i(\Diamond p \to [0]U\varphi)^0$.

Definition 9 (Theories: witnessed, initial, maximal). *Let* $\mathcal{L}^P$ *be the language of APALM based only on some countable set* P *of propositional variables. Similarly, let* NF^P *denote the corresponding set of strings defined based on* $\mathcal{L}^P$. *A* P-theory *is a consistent set of formulas in* $\mathcal{L}^P$ *(where "consistent" means consistent with respect to the axiomatization of* **APALM** *formulated for* $\mathcal{L}^P$). *A* maximal P-theory *is a P-theory* Γ *that is maximal with respect to* $\subseteq$ *among all P-theories; in other words,* Γ *cannot be extended to another P-theory. A* P-witnessed theory *is a P-theory* Γ *such that, for every* $s \in NF^P$ *and* $\varphi \in \mathcal{L}^P$, *if* $\langle s\rangle\Diamond\varphi$ *is consistent with* Γ *then there is* $p \in P$ *such that* $\langle s\rangle\langle p\rangle\varphi$ *is consistent with* Γ *(or equivalently: if* $\Gamma \vdash [s][p]\neg\varphi$ *for all* $p \in P$, *then* $\Gamma \vdash [s]\Box\neg\varphi$). *A* P-theory Γ *is called* initial *if* $0 \in \Gamma$. *A* maximal P-witnessed theory Γ *is a P-witnessed theory that is not a proper subset of any P-witnessed theory. A* maximal P-witnessed initial theory Γ *is a maximal P-witnessed theory such that* $0 \in \Gamma$.

The proofs of the following lemmas are in the online version.

Lemma 4. *For every necessity form $[s]$, there exist formulas $\theta \in \mathcal{L}_{-\Diamond}$ and $\psi \in \mathcal{L}$ such that for all $\varphi \in \mathcal{L}$, we have*

$$\vdash [s]\varphi \ \text{iff} \ \vdash \psi \rightarrow [\theta]\varphi.$$

Lemma 5. *The following rule is admissible in* **APALM***:*

$$\text{if} \ \vdash [s][p]\varphi \ \text{then} \ \vdash [s]\Box\varphi, \ \text{where} \ p \notin P_s \cup P_\varphi.$$

Lemma 6. *For every maximal P-witnessed theory Γ, and every formula $\varphi, \psi \in \mathcal{L}^P$:*

1. *$\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$*

2. *$\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$,*

3. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,*

4. *$\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$ implies $\psi \in \Gamma$.*

5. ***APALM**$_P \subseteq \Gamma$, where **APALM**$_P$ is **APALM** formulated for $\mathcal{L}^P$.*

Lemma 7. *For every $\Gamma \subseteq \mathcal{L}^P$, if Γ is a P-theory and $\Gamma \nvdash \neg\varphi$ for some $\varphi \in \mathcal{L}^P$, then $\Gamma \cup \{\varphi\}$ is a P-theory. Moreover, if Γ is P-witnessed, then $\Gamma \cup \{\varphi\}$ is also P-witnessed.*

Lemma 8. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is an increasing chain of P-theories such that $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a P-theory.*

Lemma 9. *For every maximal P-witnessed theory T, both $\{\theta \in \mathcal{L}^P : K_i\theta \in T\}$ and $\{\theta \in \mathcal{L}^P : U\theta \in T\}$ are P-witnessed theories.*

Lemma 10 (Lindenbaum's Lemma). *Every P-witnessed theory Γ can be extended to a maximal P-witnessed theory T_Γ.*

Lemma 11 (Extension Lemma). *Let P be a countable set of propositional variables and P' be a countable set of* fresh *propositional variables, i.e., $P \cap P' = \emptyset$. Let $\widetilde{P} = P \cup P'$. Then, every initial P-theory Γ can be extended to an initial $\widetilde{P}$-witnessed theory $\widetilde{\Gamma} \supseteq \Gamma$, and hence to a maximal $\widetilde{P}$-witnessed initial theory $T_\Gamma \supseteq \Gamma$.*

To define our canonical pseudo-model, we first put, for all maximal P-witnessed theories T, S:

$$T \sim_U S \ \text{iff} \ \forall\varphi \in \mathcal{L}^P \left(U\varphi \in T \ \text{implies} \ \varphi \in S\right).$$

Definition 10 (Canonical Pre-Model). *Given a maximal P-witnessed initial theory T_0, the canonical pre-model for T_0 is a tuple $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, \|\cdot\|^c)$ such that:*

– *$W^c = \{T : T$ is a maximal P-witnessed theory such that $T_0 \sim_U T\}$,*
– *$\mathcal{A}^c = \{\widehat{\theta} : \theta \in \mathcal{L}^P_{-\Diamond}\}$ where $\widehat{\varphi} = \{T \in W^c : \varphi \in T\}$ for any $\varphi \in \mathcal{L}^P$,*

– *for every $T, S \in W^c$ and $i \in \mathcal{AG}$ we define:*

$$T \sim_i^c S \quad iff \quad \forall \varphi \in \mathcal{L}^P \left(K_i \varphi \in T \ \text{implies} \ \varphi \in S \right).$$

– $\|p\|^c = \{T \in W^c \ : \ p \in T\} = \widehat{p}.$

As usual, it is easy to see (given the $S5$ axioms for K_i and for U) that $\sim_U$ and $\sim_i^c$ are equivalence relations.

The proofs of the first three results below are online.

Lemma 12 (Existence Lemma for K_i). *Let T be a maximal P-witnessed theory, $\alpha \in \mathcal{L}_{-\Diamond}^P$, and $\varphi \in \mathcal{L}^P$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. Then, there is a maximal P-witnessed theory S such that $T \sim_i S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Lemma 13 (Existence Lemma for U). *Let T be a maximal P-witnessed theory, $\alpha \in \mathcal{L}_{-\Diamond}^P$, and $\varphi \in \mathcal{L}^P$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. Then, there is a maximal P-witnessed theory S such that $T \sim_U S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Corollary 3. *For $\varphi \in \mathcal{L}$, we have $\widehat{U\varphi} = W^c$ if $\widehat{\varphi} = W^c$, and $\widehat{U\varphi} = \emptyset$ otherwise.*

Lemma 14. *Every element $T \in W^c$ is an initial theory (i.e. $0 \in T$).*

Proof. Let $T \in W^c$. By the construction of W^c, we have $T_0 \sim_U T$. Since $0 \to U0$ is an axiom and T_0 is maximal, $(0 \to U0) \in T_0$. Thus, since $0 \in T_0$, we obtain $U0 \in T_0$ (by Lemma 6(4)). Therefore, by the definition of $\sim_U$ and since $T_0 \sim_U T$, we have that $0 \in T$.

Corollary 4. *For all $\varphi \in \mathcal{L}^P$, we have $\widehat{\varphi} = \widehat{\varphi^0}$.*

Proof. Since $0 \in T$ for all $T \in W^c$, we obtain by axiom $(0\text{-}eq)$ that $\varphi \leftrightarrow \varphi^0 \in T$ for all $T \in W^c$. Therefore, $\widehat{\varphi} = \widehat{\varphi^0}$.

Lemma 15 (Truth Lemma). *Let $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, V^c)$ be the canonical pre-model for some T_0 and $\varphi \in \mathcal{L}^P$. Then, for all $\alpha \in \mathcal{L}_{-\Diamond}^P$, we have $[\![\varphi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\varphi}$.*

Proof. The proof is by $<$-induction on φ, using the following *induction hypothesis* (IH): for all $\psi < \varphi$, we have $[\![\psi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\psi}$ for all $\alpha \in \mathcal{L}_{-\Diamond}$. The cases for the Boolean connectives are straightforward. The cases for K_i and U are standard, using $\vdash \langle\alpha\rangle K_i \psi \leftrightarrow \alpha \wedge K_i[\alpha]\psi$ and Lemma 12 for K_i, and $\vdash \langle\alpha\rangle U\psi \leftrightarrow \alpha \wedge U[\alpha]\psi$ and Lemma 13 for U.

Base case $\varphi := \top$. Then $[\![\top]\!]_{\widehat{\alpha}} = \widehat{\alpha} = \widehat{\langle\alpha\rangle\top}$, by Definition 5 and the fact that $\vdash \alpha \leftrightarrow \langle\alpha\rangle\top$.

Base case $\varphi := p$. Then $[\![p]\!]_{\widehat{\alpha}} = \|p\|^c \cap \widehat{\alpha} = \widehat{p} \cap \widehat{\alpha} = \widehat{p \wedge \alpha} = \widehat{\langle\alpha\rangle p}$, by Definition 5, the defn. of $\| \cdot \|^c$, R_p, and Proposition 2(3).

Base case $\varphi := 0$. Then $[\![0]\!]_{\widehat{\alpha}} = W^c$ if $\widehat{\alpha} = W^c$, and $[\![0]\!]_{\widehat{\alpha}} = \emptyset$ otherwise. Also, $\widehat{\langle\alpha\rangle 0} = \widehat{0 \wedge U\alpha} = \widehat{0} \cap \widehat{U\alpha} = \widehat{0} \cap \widehat{U\alpha} = \widehat{U\alpha}$ (by Propositions 2(2) and Lemma 14). By Corollary 3, $\widehat{U\alpha} = W^c$ if $\widehat{\alpha} = W^c$, and $\widehat{U\alpha} = \emptyset$ otherwise. So $[\![0]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle 0}$.

Case $\varphi := \psi^0$. Follows easily from $\widehat{\top} = W^c$ and R[$\top$], Corollary 4, and R^0.

Case $\varphi := \langle\chi\rangle\psi$. Straightforward, using the fact that $\vdash \langle\alpha\rangle\langle\chi\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\chi\rangle\psi$ (by R$_{[!]}$).

Case $\varphi := \Diamond\psi$.

($\Rightarrow$) Suppose $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 5, that $\alpha \in T$ and there exists $B \in \mathcal{A}^c$ such that $T \in B \subseteq \widehat{\alpha}$ and $T \in [\![\psi]\!]_B$ (see Observation 2). By the construction of $\mathcal{A}^c$, we know that $B = \widehat{\theta}$ for some $\theta \in \mathcal{L}^P_{-\Diamond}$. Therefore, $T \in [\![\psi]\!]_B$ means that $T \in [\![\psi]\!]_{\widehat{\theta}}$. Moreover, since $\widehat{\theta} \subseteq \widehat{\alpha}$ and, thus, $\widehat{\theta} = \widehat{\alpha} \cap \widehat{\theta} = \widehat{\alpha \wedge \theta}$, we obtain $T \in [\![\psi]\!]_{\widehat{\alpha \wedge \theta}}$. By Lemma 1(1), we have $\psi < \Box\psi$. Therefore, by IH, we obtain $T \in \widehat{\langle\alpha \wedge \theta\rangle\psi}$. Then, by axiom ([!]$\Box$-elim) and the fact that T is maximal, we conclude that $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$, i.e., $\langle\alpha\rangle\Diamond\psi \in T$. Then, since T is a maximal P-witnessed theory, there is $p \in P$ such that $\langle\alpha\rangle\langle p\rangle\psi \in T$. By Lemma 1(2), we know that $\langle p\rangle\psi < \Diamond\psi$. Thus, by IH on $\langle p\rangle\psi$, we obtain that $T \in [\![\langle p\rangle\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 5 and Observation 2, that $T \in [\![\psi]\!]_{[\![p]\!]_{\widehat{\alpha}}} \subseteq [\![p]\!]_{\widehat{\alpha}}$. Since $p < \Diamond\psi$, by IH on p, we obtain that $[\![p]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$. By the construction of $\mathcal{A}^c$, we moreover have $\widehat{\langle\alpha\rangle p} \in \mathcal{A}^c$. Therefore, as $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle p}}$ and $\widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$, by Definition 5, we conclude that $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$.

Corollary 5. *The canonical pre-model $\mathcal{M}^c$ is standard (and hence a pseudo-model).*

Proof. $\mathcal{A}^c = \{\widehat{\theta} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{\widehat{\langle\top\rangle\theta} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{[\![\theta]\!]_{\widehat{\top}} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{[\![\theta]\!]_{W^c} : \theta \in \mathcal{L}^P_{-\Diamond}\}$.

Lemma 16. *For every $\varphi \in \mathcal{L}^P$, if φ is consistent then $\{0, \Diamond\varphi\}$ is an initial P_φ-theory.*

Proof. Let $\varphi \in \mathcal{L}^P$ s.t. $\varphi \nvdash \bot$. By the *Equivalences with* ^{0} in Table 1, we have $\vdash \bot^0 \leftrightarrow (p \wedge \neg p)^0 \leftrightarrow (p^0 \wedge \neg p^0) \leftrightarrow (p \wedge \neg p) \leftrightarrow \bot$. Therefore, $\vdash \psi \to \bot^0$ iff $\vdash \psi \to \bot$ for all $\psi \in \mathcal{L}^P$. Then, by Proposition 2(12), we obtain $\vdash \varphi \to \bot$ iff $\vdash (0 \wedge \Diamond\varphi) \to \bot$. Since $\varphi \nvdash \bot$, we have $0 \wedge \Diamond\varphi \nvdash \bot$, i.e., $\{0, \Diamond\varphi\}$ is a P_φ-theory. By definition, it is an initial one.

Corollary 6. **APALM** *is complete with respect to standard pseudo models.*

Proof. Let φ be a consistent formula. By Lemma 16, $\{0, \Diamond\varphi\}$ is an initial P_φ-theory. By Extension and Lindenbaum Lemmas, we can extend P_φ to some $P \supseteq P_\varphi$, and extend $\{0, \Diamond\varphi\}$ to some maximal P-witnessed theory T_0 such that $(0 \wedge \Diamond\varphi) \in T_0$. So T_0 is initial, and we can construct the canonical pseudo-model $\mathcal{M}^c$ for T_0. Since $\Diamond\varphi \in T_0$ and T_0 is P-witnessed, there exists $p \in P$ such that $\langle p\rangle\varphi \in T_0$. By Truth Lemma (applied to $\alpha := p$), we get $T_0 \in [\![\varphi]\!]_{\widehat{p}}$. Hence, φ is satisfied at T_0 in the set $\widehat{p} \in \mathcal{A}^c$.

Corollary 7. **APALM** *is complete with respect to APALM models.*

This follows immediately from Corollaries 1 and 6.

6   Expressivity

To compare APALM and its fragments with basic epistemic logic (and its extension with the universal modality), consider the *static fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle}$, obtained from $\mathcal{L}$ by removing both the $\Diamond$ operator and the dynamic modality $\langle\varphi\rangle\psi$; as well as the *present-only fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, obtained by removing the operators 0 and φ^0 from $\mathcal{L}_{-\Diamond,\langle!\rangle}$; and finally the *epistemic fragment* $\mathcal{L}_{epi}$, obtained by further removing the universal modality U from $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$. For every APALM model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, consider its *initial epistemic model* $\mathcal{M}^{initial} = (W^0, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and its *current epistemic model* $\mathcal{M}^{current} = (W, \sim_1 \cap W \times W, \ldots, \sim_n \cap W \times W, \|\cdot\| \cap W)$.

Proposition 8. *The fragment $\mathcal{L}_{-\Diamond}$ is co-expressive with the static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$. In fact, every formula $\varphi \in \mathcal{L}_{-\Diamond}$ is provably equivalent with some formula $\psi \in \mathcal{L}_{-\Diamond,\langle!\rangle}$ (by using **APALM** reduction laws to eliminate dynamic modalities, as in standard PAL).*

Proposition 9. *The static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$ (and hence, also $\mathcal{L}_{-\Diamond}$) is strictly more expressive than the present-only fragment $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, which in turn is more expressive than the epistemic fragment $\mathcal{L}_{epi}$. In fact, each of the operators 0 and φ^0 independently increase the logic's expressivity.*

Kuijer's counterexample shows that the standard epistemic bisimulation is not appropriate for APALM, so we now define a new such notion:

Definition 11 (APALM Bisimulation). *An APALM bisimulation between APALM models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a total bisimulation B (in the usual sense)[17] between the corresponding initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$, with the property that: if $s_1 B s_2$, then $s_1 \in W_1$ holds iff $s_2 \in W_2$ holds. Two current states $s_1 \in W_1$ and $s_2 \in W_2$ are APALM-bisimilar if there exists an APAL bisimulation B between the underlying APALM models such that $s_1 B s_2$.*

Since APALM models are always of the form $\mathcal{M} = (\mathcal{M}^0)|\theta$ for some $\theta \in \mathcal{L}_{-\Diamond}$, we have a characterization of APALM-bisimulation only in terms of the initial models:

Proposition 10. *Let $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be APALM models, and let $B \subseteq W_1^0 \times W_2^0$. The following are equivalent:*

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[17] A *total bisimulation* between epistemic models $(W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $(W', \sim_1', \ldots, \sim_n', \|\cdot\|')$ is an epistemic bisimulation relation (satisfying the usual valuation and back-and-forth conditions from Modal Logic) $B \subseteq W \times W'$, such that: for every $s \in W$ there exists some $s' \in W'$ with sBs'; and dually, for every $s' \in W'$ there exists some $s \in W$ with sBs'.

1. B is an APALM bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$;
2. B is a total bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM bisimulation between $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and $\mathcal{M}_1 = (\mathcal{M}_1^0)|\theta$, $\mathcal{M}_2 = (\mathcal{M}_2^0)|\theta$ for some common formula $\theta \in \mathcal{L}_{-\Diamond}$.

So, to check for APALM-bisimilarity, it is enough to check for total bisimilarity between the initial models and for both models being updates with the same formula.

Proposition 11. APALM formulas are invariant under APALM-bisimulation: if $s_1 B s_2$ for some APALM-bisimulation relation between APALM models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1, \ldots, \sim_n, \|\cdot\|)$, then: $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$. The (Hennessy-Milner) converse holds for finite models: if $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1, \ldots, \sim_n, \|\cdot\|)$ are APALM models with W_1^0, W_2^0 finite, then $s_1 \in W_1$ and $s_2 \in W_2$ satisfy the same APALM formulas iff they are APALM-bisimilar.

7 Conclusions and Future Work

This paper solves the open question of finding a strong variant of APAL that is recursively axiomatizable. Our system APALM is inspired by our analysis of Kuijer's counterexample [11], which lead us to add to APAL a 'memory' of the initial situation. The soundness and completeness proofs crucially rely on a Subset Space-like semantics and on the equivalence between the effort modality and the arbitrary announcement modality, thus revealing the strong link between these two formalisms.

It seems clear that our method works for other versions of APAL, and in ongoing work we are looking at a recursive axiomatization GALM for a memory-enhanced variant of GAL (Group Announcement Logic) [1]. As in GAL, the □ operator quantifies only over announcements that are known by some of the agents, so GALM seems better fit than APALM for treating puzzles involving epistemic dialogues.

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