



A Semantical View of Proof Systems

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Abstract. In this work, we explore proof theoretical connections between sequent, nested and labelled calculi. In particular, we show a semantical characterisation of intuitionistic, normal and non-normal modal logics for all these systems, via a case-by-case translation between labelled nested to labelled sequent systems.

1 Introduction

The quest of finding *analytic* proof systems for different logics has been the main research topic for proof theorists since Gentzen’s seminal work [5]. One of the best known formalisms for proposing analytic proof systems is Gentzen’s *sequent calculus*. While its simplicity makes it an ideal tool for proving meta-logical properties, sequent calculus is not expressive enough for constructing analytic calculi for many logics of interest. The case of modal logic is particularly problematic, since sequent systems for such logics are usually not modular, and they mostly lack relevant properties such as separate left and right introduction rules for the modalities. These problems are often connected to the fact that the modal rules in such calculi usually introduce more than one connective at a time, e.g. as in the rule k for modal logic K :

$$\frac{B_1, \dots, B_n \vdash A}{\Box B_1, \dots, \Box B_n \vdash \Box A} \text{ k}$$

One way of solving this problem is by considering extensions of the sequent framework that are expressive enough for capturing these modalities using separate left and right introduction rules. This is possible e.g. in *labelled sequents* [18] or in *nested sequents* [1]. In the labelled sequent framework, usually the semantical characterisation is explicitly added to sequents. In the nested framework in contrast, a single sequent is replaced with a tree of sequents, where successors of a sequent (nestings) are interpreted under a given modality. The nesting rules of these calculi govern the transfer of formulae between the different sequents, and they are *local*, in the sense that it is sufficient to transfer only one formula at a time. As an example, the labelled and nested versions for the *necessity right rule* (\Box_R) are

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$$\frac{\mathcal{R}, xRy, X \vdash Y, y:A}{\mathcal{R}, X \vdash Y, x:\Box A} \Box_R^l \quad \frac{\Gamma \vdash \Delta, [\cdot \vdash A]}{\Gamma \vdash \Delta, \Box A} \Box_R^n$$

where y is a fresh variable in the \Box_R^l rule. Reading bottom up, while the labelled system creates a new variable y related to x via a relation R and changes the label of A to y , in \Box_R^n a new nesting is created, and A is moved there. It seems clear that nestings and semantical structures are somehow related. Indeed, a direct translation between proofs in labelled and nested systems for the modal logic of provability (a.k.a. the Gödel-Löb provability logic) is presented in [6], while in [4] it is shown how to relate nestings with Kripke structures for intuitionistic logic (via indexed tableaux systems). In this work, we show this relationship for intuitionistic logic and some normal modal logics, using only sequent based systems.

Since nested systems have been also proposed for other modalities, such as the non-normal ones [2], an interesting question is whether this semantical interpretation can be generalised to other systems as well. In [15] a labelled approach was used for setting the grounds for proof theory of some non-normal modal systems based on *neighbourhood semantics*. In parallel, we have proposed [10] modular systems based on nestings for several non-normal modal logics. We will relate these two approaches for the logics **M** and **E**, hence clarifying the nesting-semantics relationship for such logics.

Finally, in [11], we showed that a class of nested systems can be transformed into sequent systems via a linearisation procedure, where sequent rules can be seen as nested *macro-rules*. By relating nested and sequent systems, we are able to extend the semantical interpretation also to the sequent case, hence closing the relationship between systems and shedding light on the semantical interpretation of several sequent based systems.

Organisation and Contributions. Section 2 presents the basic notation for sequent systems; Sect. 3 presents nested systems and summarizes the results for their sequentialisation; Sect. 4 presents the basic notation for labelled systems; Sects. 5, 6 and 7 show the results under the particular views of intuitionistic, normal and non-normal logics; Sect. 8 concludes the paper.

2 Sequent Systems

Contemporary proof theory started with Gentzen's work [5], and it has had a continuous development with the proposal of several proof systems for many logics.

Definition 1. A sequent is an expression of the form $\Gamma \vdash \Delta$ where Γ (the antecedent) and Δ (the succedent) are finite sets of formulae. A sequent calculus (SC) consists of a set of rule schemas, of the form

$$\frac{S_1 \quad \cdots \quad S_k}{S} r$$

$$\begin{array}{c}
 \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \rightarrow_L \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_R \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L \\
 \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R \quad \frac{\Gamma, A, \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R \quad \frac{}{\Gamma, \perp \vdash \Delta} \perp_L
 \end{array}$$

Fig. 1. Multi-conclusion intuitionistic calculus SC_{mLJ} .

where the sequent S is the conclusion inferred from the premise sequents S_1, \dots, S_k in the rule r . If the set of premises is empty, then r is an axiom. An instance of a rule is a rule application.

A derivation is a finite directed tree with nodes labelled by sequents and a single root, axioms at the top nodes, and where each node is connected with the (immediate) successor nodes (if any) according to the application of rules. The height of a derivation is the greatest number of successive applications of rules in it, where an axiom has height 0.

In this work we will consider only *fully structural* sequent systems, i.e. allowing freely applications of the schemas init and W below

$$\frac{}{\Gamma, P \vdash P, \Delta} \text{init} \quad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{W}$$

where P is atomic.

As an example, Fig. 1 presents SC_{mLJ} [12], a multiple conclusion sequent system for propositional intuitionistic logic. The rules are exactly the same as in classical logic, except for the implication right rule, that forces all formulae in the succedent of the conclusion sequent to be previously weakened. This guarantees that, on applying the (\rightarrow_R) rule on $A \rightarrow B$, the formula B should be proved assuming *only* the pre-existent antecedent context extended with the formula A , creating an interdependency between A and B .

3 Nested Systems

Nested systems [1, 16] are extensions of the sequent framework where a single sequent is replaced with a tree of sequents.

Definition 2. A nested sequent is defined inductively as follows:

- (i) if $\Gamma \vdash \Delta$ is a sequent, then it is a nested sequent;
- (ii) if $\Gamma \vdash \Delta$ is a sequent and G_1, \dots, G_k are nested sequents, then $\Gamma \vdash \Delta, [G_1], \dots, [G_k]$ is a nested sequent.

A nested system (NS) consists of a set of inference rules acting on nested sequents.

For readability, we will denote by Γ, Δ sequent contexts and by Λ sets of nestings. In this way, every nested sequent has the shape $\Gamma \vdash \Delta, \Lambda$ where elements of Λ

$$\begin{array}{c}
\frac{\mathcal{S}\{\Gamma \vdash \Delta, A, \Lambda\} \quad \mathcal{S}\{\Gamma, B \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, A \rightarrow B \vdash \Delta, \Lambda\}} \rightarrow_L^n \quad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [A \vdash B]\}}{\mathcal{S}\{\Gamma \vdash A \rightarrow B, \Delta, \Lambda\}} \rightarrow_R^n \\
\frac{\mathcal{S}\{\Gamma, A, B \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, A \wedge B \vdash \Delta, \Lambda\}} \wedge_L^n \quad \frac{\mathcal{S}\{\Gamma \vdash A, \Delta, \Lambda\} \quad \mathcal{S}\{\Gamma \vdash B, \Delta, \Lambda\}}{\mathcal{S}\{\Gamma \vdash A \wedge B, \Delta, \Lambda\}} \wedge_R^n \\
\frac{\mathcal{S}\{\Gamma, A \vdash \Delta, \Lambda\} \quad \mathcal{S}\{\Gamma, B \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, A \vee B \vdash \Delta, \Lambda\}} \vee_L^n \quad \frac{\mathcal{S}\{\Gamma \vdash A, B, \Delta, \Lambda\}}{\mathcal{S}\{\Gamma \vdash A \vee B, \Delta, \Lambda\}} \vee_R^n \\
\frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\Gamma', A \vdash \Delta', \Lambda']\}}{\mathcal{S}\{\Gamma, A \vdash \Delta, \Lambda, [\Gamma' \vdash \Delta', \Lambda']\}} \text{lift}^n \quad \frac{}{\mathcal{S}\{\Gamma, \perp \vdash \Delta, \Lambda\}} \perp_L^n
\end{array}$$

Fig. 2. Nested system NS_{mLJ} .

have the shape $[\Gamma' \vdash \Delta', \Lambda']$ and so on. We will denote by Υ an arbitrary nested sequent.

Application of rules and schemas in nested systems will be represented using *holed contexts*.

Definition 3. A nested-holed context is a nested sequent that contains a hole of the form $\{ \}$ in place of nestings. We represent such a context as $\mathcal{S}\{ \}$. Given a holed context and a nested sequent Υ , we write $\mathcal{S}\{\Upsilon\}$ to stand for the nested sequent where the hole $\{ \}$ has been replaced by $[\Upsilon]$, assuming that the hole is removed if Υ is empty and if \mathcal{S} is empty then $\mathcal{S}\{\Upsilon\} = \Upsilon$. The depth of $\mathcal{S}\{ \}$, denoted by $\text{dp}(\mathcal{S}\{ \})$, is the number of nodes on a branch of the nesting tree of $\mathcal{S}\{ \}$ of maximal length.

For example, $(\Gamma \vdash \Delta, \{ \})\{\Gamma' \vdash \Delta'\} = \Gamma \vdash \Delta, [\Gamma' \vdash \Delta']$ while $\{ \}\{\Gamma' \vdash \Delta'\} = \Gamma' \vdash \Delta'$.

The definition of application of nested rules and derivations in a NS are natural extensions of the one for SC, only replacing sequents by nested sequents. In this work we will assume that nested systems are *fully structural*, i.e., including the following nested versions for the initial axiom and weakening¹

$$\frac{}{\mathcal{S}\{\Gamma, P \vdash \Delta, P, \Lambda\}} \text{init}^n \quad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \Gamma' \vdash \Delta, \Delta', \Lambda, \Lambda'\}} W^n$$

Figure 2 presents the NS_{mLJ} [4], a nested system for mLJ.

3.1 Sequentialising Nested Systems

In [11] we identified general conditions under which a nested calculus can be transformed into a sequent calculus by restructuring the nested sequent derivation (proof) and shedding extraneous information to obtain a derivation of the same formula in the sequent calculus. These results were formulated generally

¹ All over this text, we will use n as a superscript, etc for indicating “nested”. Hence e.g., \rightarrow_R^n will be the designation of the implication right rule in the nesting framework.

so that they apply to calculi for intuitionistic, normal and non-normal modal logics. Here we will briefly explain the main ideas in that work.

First of all, we restrict our attention to *shallow directed* nested systems, in which rules are restricted so to falling in one of the following mutually exclusive schemas:

i. *sequent-like* rules

$$\frac{\mathcal{S}\{\Gamma_1 \vdash \Delta_1\} \quad \cdots \quad \mathcal{S}\{\Gamma_k \vdash \Delta_k\}}{\mathcal{S}\{\Gamma \vdash \Delta\}}$$

ii *nested-like* rules

ii.a *creation rules*

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, [\Gamma_1 \vdash \Delta_1]\}}{\mathcal{S}\{\Gamma \vdash \Delta\}}$$

ii.b *upgrade rules*

$$\frac{\mathcal{S}\{\Gamma' \vdash \Delta', [\Gamma'_1 \vdash \Delta'_1]\}}{\mathcal{S}\{\Gamma \vdash \Delta, [\Gamma_1 \vdash \Delta_1]\}}$$

The nesting in the premise of a creation rule is called the *auxiliary nesting*.

The following extends the definition of permutability to the nested setting.

Definition 4. Let \mathbf{NS} be shallow directed, r_1, r_2 be applications rules and \mathcal{Y} be a nested sequent. We say that r_2 permutes down r_1 ($r_2 \downarrow r_1$) if, for every derivation in which r_1 operates on \mathcal{Y} and r_2 operates on one or more of r_1 's premises (but not on auxiliary formulae/nesting of r_1), there exists another derivation of \mathcal{Y} in which r_2 operates on \mathcal{Y} and r_1 operates on zero or more of r_2 's premises (but not on auxiliary formulae/nesting of r_2). If $r_2 \downarrow r_1$ and $r_1 \downarrow r_2$ we will say that r_1, r_2 are permutable, denoted by $r_1 \updownarrow r_2$. Finally, \mathbf{NS} is said fully permutable if $r_1 \updownarrow r_2$ for any pair of rules.

Finally, the next result shows that fully permutable, shallow directed systems can be sequentialised.

Theorem 5. Let \mathbf{NS} be fully permutable, shallow and directed. There is a normalisation procedure of proofs in \mathbf{NS} transforming maximal blocks of applications of nested-like rules into sequent rules.

Next an example of such procedure.

Example 6. In \mathbf{NS}_{mLJ} , a nested block containing the creation rule \rightarrow_R^n and the upgrade rule lift^n has the shape

$$\frac{\frac{\mathcal{S}\{\Gamma' \vdash \Delta', [\Gamma, A \vdash B]\}}{\mathcal{S}\{\Gamma, \Gamma' \vdash \Delta', [A \vdash B]\}} \text{lift}^n}{\mathcal{S}\{\Gamma, \Gamma' \vdash \Delta', A \rightarrow B\}} \rightarrow_R^n$$

Observe that lift^n maps a left formula into itself and there are no context relations on right formulae. Hence the corresponding sequent rule is

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_R$$

which is the implication right rule for mLJ. That is, sequentialising the nested system NS_{mLJ} (Fig. 2) results in the sequent system mLJ (Fig. 1).

4 Labelled Proof Systems

While it is widely accepted that nested systems carry the Kripke structure on nestings for intuitionistic and normal modal logics, it is not clear what is the relationship between nestings and semantics for other systems. For example, in [9] we presented a linear nested system [8] for linear logic, but the interpretation of nestings for this case is still an open problem.

In this work we will relate labelled nested systems [6] with labelled systems [18]. While the results for intuitionistic and some normal modal logics are not new [4, 6], we give a complete different approach for these results, and present the first semantical interpretation for nestings in non-normal modal logics. In this section we shall recall some of the terminology for labelled systems.

Labelled Nested Systems. Let SV a countable infinite set of *state variables* (denoted by x, y, z, \dots), disjoint from the set of propositional variables. A *labelled formula* has the form $x : A$ where $x \in \text{SV}$ and A is a formula. If $\Gamma = \{A_1, \dots, A_k\}$ is a set of formulae, then $x : \Gamma$ denotes the set $\{x : A_1, \dots, x : A_k\}$ of labelled formulae. A (possibly empty) set of relation terms (*i.e.* terms of the form xRy , where $x, y \in \text{SV}$) is called a *relation set*. For a relation set \mathcal{R} , the *frame* $\text{Fr}(\mathcal{R})$ defined by \mathcal{R} is given by $(|\mathcal{R}|, \mathcal{R})$ where $|\mathcal{R}| = \{x \mid xRy \in \mathcal{R} \text{ or } yRx \in \mathcal{R} \text{ for some } y \in \text{SV}\}$. We say that a relation set \mathcal{R} is *treelike* if the frame defined by \mathcal{R} is a tree or \mathcal{R} is empty.

Definition 7. A labelled nested sequent LbNS is a labelled sequent $\mathcal{R}, X \vdash Y$ where

1. \mathcal{R} is treelike;
2. if $\mathcal{R} = \emptyset$ then X has the form $x : A_1, \dots, x : A_k$ and Y has the form $x : B_1, \dots, x : B_m$ for some $x \in \text{SV}$;
3. if $\mathcal{R} \neq \emptyset$ then every state variable y that occurs in either X or Y also occurs in \mathcal{R} .

A labelled nested sequent calculus is a labelled calculus whose initial sequents and inference rules are constructed from LbNS.

As in [6], labelled nested systems can be automatically generated from nested systems.

Definition 8. Given $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ sequents, we define $(\Gamma \vdash \Delta) \otimes (\Gamma' \vdash \Delta')$ to be $\Gamma, \Gamma' \vdash \Delta, \Delta'$. For a state variable x , define the mapping $\mathbb{T}\mathbb{L}_x$ from NS to LbLNS as follows

$$\begin{aligned} \mathbb{T}\mathbb{L}_x(\Gamma \vdash \Delta, [\Upsilon_1], \dots, [\Upsilon_n]) &= xRx_1, \dots, xRx_n, (x:\Gamma \vdash x:\Delta) \otimes \\ &\quad \mathbb{T}\mathbb{L}_{x_1}(\Upsilon_1) \otimes \dots \otimes \mathbb{T}\mathbb{L}_{x_n}(\Upsilon_n) \\ \mathbb{T}\mathbb{L}_x([\Gamma \vdash \Delta]) &= x:\Gamma \vdash x:\Delta \end{aligned}$$

with all state variables pairwise distinct.

For the sake of readability, when the state variable is not important, we will suppress the subscript, writing $\mathbb{T}\mathbb{L}$ instead of $\mathbb{T}\mathbb{L}_x$. We will shortly illustrate the procedure of constructing labelled nestings using the mapping $\mathbb{T}\mathbb{L}$. Consider the following application of the rule \rightarrow_R of Fig. 2:

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, A, [A \vdash B]\}}{\mathcal{S}\{\Gamma \vdash \Delta, A \rightarrow B, A\}} \rightarrow_R^n$$

Applying $\mathbb{T}\mathbb{L}$ to the conclusion we obtain $\mathcal{R}, X \vdash Y, x:A \rightarrow B$ where the variable x label formulae in two components of the NS, and X, Y are sets of labelled formulae. Applying $\mathbb{T}\mathbb{L}$ to the premise we obtain $\mathcal{R}, xRy, X, y:A \vdash Y, y:B$ where y is a fresh variable (*i.e.* different from x and not occurring in X, Y). We thus obtain an application of the LbLNS rule

$$\frac{\mathcal{R}, xRy, X, y:A \vdash Y, y:B}{\mathcal{R}, X \vdash Y, x:A \rightarrow B} \mathbb{T}\mathbb{L}(\rightarrow_R^n)$$

Some rules of the labelled nested system LbNS_{mLJ} are depicted in Fig. 3.

The following result follows readily by transforming derivations bottom-up [6].

Theorem 9. *The mapping $\mathbb{T}\mathbb{L}_x$ preserves open derivations, that is, there is a 1-1 correspondence between derivations in a nested sequent system NS and in its labelled translation LbNS.*

Labelled Sequent Systems. In the labelled sequent framework, a semantical characterisation of a logic is explicitly added to sequents via the labelling of formulae [3, 13–15, 18]. In the case of world based semantics, the forcing relation $x \Vdash A$ is represented as the labelled formula $x:A$ and sequents have the form $\mathcal{R}, X \vdash Y$, where \mathcal{R} is a relation set and X, Y are multisets of labelled formulae.

The rules of the labelled calculus G3I are obtained from the inductive definition of validity in a Kripke frame (Fig. 4a), together with the rules describing a partial order, presented in Fig. 4b. Note that the anti-symmetry rule does not need to be stated directly since, for any x , the formula $x = x$ is equivalent to true and hence can be erased from the left side of a sequent.

$$\begin{array}{c}
\frac{\mathcal{R}, xRy, X, y: A \vdash Y, y: B}{\mathcal{R}, X \vdash Y, x: A \rightarrow B} \text{TL}(\rightarrow_R^n) \quad \frac{\mathcal{R}, X, \vdash x: A, Y \quad X, x: B \vdash Y}{\mathcal{R}, X, x: A \rightarrow B \vdash Y} \text{TL}(\rightarrow_R^n) \\
\frac{\mathcal{R}, X, x: A, x: B \vdash Y}{\mathcal{R}, X, x: A \wedge B \vdash Y} \text{TL}(\wedge_L^n) \quad \frac{\mathcal{R}, X \vdash x: A, Y \quad X \vdash x: B, Y}{\mathcal{R}, X \vdash x: A \wedge B, Y} \text{TL}(\wedge_R^n) \\
\frac{\mathcal{R}, X, x: A, \vdash Y \quad X, x: B \vdash Y}{\mathcal{R}, X, x: A \vee B \vdash Y} \text{TL}(\vee_L^n) \quad \frac{\mathcal{R}, X \vdash x: A, x: B, Y}{\mathcal{R}, X \vdash x: A \vee x: B, Y} \text{TL}(\vee_R^n) \\
\frac{}{\mathcal{R}, X, x: A \vdash x: A, Y} \text{TL}(\text{init}^n) \quad \frac{}{\mathcal{R}, X, x: \perp \vdash Y} \text{TL}(\perp_L^n) \quad \frac{\mathcal{R}, xRy, X, y: A \vdash Y}{\mathcal{R}, xRy, X, x: A \vdash Y} \text{TL}(\text{lift}^n)
\end{array}$$

Fig. 3. Labelled nested system LbNS_{mLJ} .

$$\begin{array}{c}
\frac{\mathcal{R}, x \leq y, X, x: A \rightarrow B \vdash y: A, Y \quad \mathcal{R}, x \leq y, X, y: B \vdash Y}{\mathcal{R}, x \leq y, X, x: A \rightarrow B \vdash Y} \rightarrow_L^t \quad \frac{xRx, \mathcal{R}, X \vdash Y}{\mathcal{R}, X \vdash Y} \text{Ref} \\
\frac{\mathcal{R}, x \leq y, X, y: A \vdash Y, y: B}{\mathcal{R}, X \vdash Y, x: A \rightarrow B} \rightarrow_R^t \quad \frac{}{\mathcal{R}, X, x \leq y, x: P \vdash Y, y: P} \text{init}' \quad \frac{xRz, xRy, yRz, \mathcal{R}, X \vdash Y}{xRy, yRz, \mathcal{R}, X \vdash Y} \text{Trans} \\
\text{(a) } y \text{ is fresh in } \rightarrow_R \text{ and } P \text{ is atomic in } \text{init}' \quad \text{(b) Relation rules.}
\end{array}$$

Fig. 4. Some rules of the labelled system G3I

5 Intuitionistic Logic

In this section we will relate various proof systems for intuitionistic logic by applying the results presented in the last sections.

Theorem 10. *All rules in NS_{mLJ} are height-preserving invertible and NS_{mLJ} is fully permutable.*

Proof. The proofs of invertibility are by induction on the depth of the derivation, distinguishing cases according to the last applied rule. Permutability of rules is proven by a case-by-case analysis, using the invertibility results. \square

The results in the previous sections entail the following.

Theorem 11. *Systems NS_{mLJ} , mLJ and LbNS_{mLJ} are equivalent.*

Observe that the proof uses *syntactical arguments only*, differently from e.g. [4, 8].

For establishing a comparison between labels in G3I and LbNS_{mLJ} , first observe that applications of rule Trans in G3I can be restricted to the leaves (i.e. just before an instance of the initial axiom). Also, since weakening is admissible in G3I and the monotonicity property: $x \Vdash A$ and $x \leq y$ implies $y \Vdash A$ is derivable in G3I (Lemma 4.1 in [3]), the next result follows.

Lemma 12. *The following rules are derivable in G3I up to weakening.*

$$\frac{\mathcal{R}, X, x: A \rightarrow B \vdash x: A, Y \quad \mathcal{R}, X, x: B \vdash Y}{\mathcal{R}, X, x: A \rightarrow B \vdash Y} \rightarrow_{L'} \quad \frac{}{\mathcal{R}, X, x: P \vdash Y, x: P} \text{init}'$$

Moreover, the rule

$$\frac{\mathcal{R}, x \leq y, X, y: A \vdash Y}{\mathcal{R}, x \leq y, X, x: A \vdash Y} \text{lift}'$$

is admissible in G3I.

Proof. For the derivable rules, just note that

$$\frac{\frac{\mathcal{R}, X, x \leq x, x: A \rightarrow B \vdash Y, x: A}{\mathcal{R}, X, x \leq x, x: A \rightarrow B \vdash Y} \rightarrow_L^t}{\frac{\mathcal{R}, X, x: A \rightarrow B \vdash Y}{\mathcal{R}, X, x: A \rightarrow B \vdash Y} \text{Ref}} \text{Ref}$$

and

$$\frac{\frac{\mathcal{R}, x \leq x, X, x: P \vdash Y, x: P}{\mathcal{R}, X, x: P, \vdash Y, x: P} \text{Ref}}{\mathcal{R}, X, x: P, \vdash Y, x: P} \text{init}^t$$

□

Using an argument similar to the one in [6], it is easy to see that, in the presence of the primed rules shown above, the relational rules are admissible. Moreover, labels are preserved.

Theorem 13. *G3I is label-preserving equivalent to LbNS_{mLJ}.*

That is, nestings in NS_{mLJ} and LNS_{mLJ} correspond to worlds in the Kripke structure where the sequent is valid and this is the semantical interpretation of the nested system for intuitionistic logic [4].

Observe that, since mLJ derivations are equivalent to normal NS_{mLJ} derivations, the semantical analysis for LNS_{mLJ} also hold for mLJ, that is, an application of the \rightarrow_R rule over $\Gamma \vdash A \rightarrow B$ in mLJ corresponds to creating a new world w in the Kripke structure and setting the forcing relation to A , B and all the formulae in Γ .

In what follows, we will show how this approach on different proof systems can be smoothly extended to normal as well as non-normal modalities, using propositional classical logic as the base logic.

6 Normal Modal Logics

The next natural step on investigating the relationship between frame semantics and nested sequent systems is to consider modal systems.

The normal modal logic K is obtained from classical propositional logic by adding the unary modal connective \Box to the set of classical connectives, together with the necessitation rule and the K axiom (see Fig. 5 for the Hilbert-style axiom schemata) to the set of axioms for propositional classical logic.

$$K \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \quad \frac{A}{\Box A} \text{ nec} \quad D \quad \neg\Box\perp \quad T \quad \Box A \rightarrow A \quad 4 \quad \Box A \rightarrow \Box\Box A$$

Fig. 5. Modal axiom K, necessitation rule nec and extensions D, T, 4.

$$\frac{S\{\Gamma, A \vdash B, \Delta, \Lambda\}}{S\{\Gamma \vdash A \rightarrow B, \Delta, \Lambda\}} \rightarrow_R^n \quad \frac{S\{\Gamma \vdash \Delta, [\Gamma', A \vdash \Delta'], \Lambda\}}{S\{\Gamma, \Box A \vdash \Delta, [\Gamma' \vdash \Delta'], \Lambda\}} \Box_L^n \quad \frac{S\{\Gamma \vdash \Delta, \Lambda, [\vdash A]\}}{S\{\Gamma \vdash \Delta, \Box A, \Lambda\}} \Box_R^n$$

Fig. 6. Nested system NS_K . The rules $\rightarrow_L^n, \wedge_R^n, \wedge_L^n, \vee_R^n, \vee_L^n$ and \perp_L^n are the same as in Fig. 2.

The nested framework provides an elegant way of formulating modal systems, since no context restriction is imposed on rules. Figure 6 presents the modal rules for the nested sequent calculus NS_K for the modal logic K [1, 16].

Observe that there are two rules for handling the box operator (\Box_L and \Box_R), which allows the treatment of one formula at a time. Being able to separate the left/right behaviour of the modal connectives is the key to modularity for nested calculi [8, 17]. Indeed, K can be modularly extended by adding to NS_K the nested corresponding to other modal axioms. In this paper, we will consider the axioms D, T and 4 (Fig. 5). Figure 7 shows the modal nested rules for such extensions: for a logic $K\mathcal{A}$ with $\mathcal{A} \subseteq \{D, T, 4\}$ the calculus $\text{NS}_{K\mathcal{A}}$ extends NS_K with the corresponding nested modal rules.

Note that rule t^n is actually a sequent-like rule. On the other hand, \Box_R^n and d^n are creation rules while \Box_L^n and 4^n are upgrade rules. It is straightforward to verify that $\text{NS}_{K\mathcal{A}}$ is shallow directed and fully permutable. Moreover, a nested block containing the application of one of the creation rules and possible several applications of the upgrade rules has one of the following shapes

$$\frac{\frac{S\{\Gamma' \vdash \Delta', [\Box\Gamma_4, \Gamma_K \vdash A]\}}{S\{\Box\Gamma_4, \Box\Gamma_K, \Gamma' \vdash \Delta', [\vdash A]\}} \Box_L^n, 4^n}{S\{\Box\Gamma_4, \Box\Gamma_K, \Gamma' \vdash \Delta', \Box A\}} \Box_R^n \quad \frac{S\{\Gamma' \vdash \Delta', [\Box\Gamma_4, \Gamma_K, A \vdash]\}}{S\{\Box\Gamma_4, \Box\Gamma_K, \Gamma' \vdash \Delta', [A \vdash]\}} \Box_L^n, 4^n}{S\{\Box\Gamma_4, \Box\Gamma_K, \Box A, \Gamma' \vdash \Delta'\}} d^n$$

where \Box_L^n acted in the context Γ_K and 4^n in the context Γ_4 . Observe that 4^n maps a boxed left formula into itself, \Box_L^n maps left formulae into the boxed versions and there are no context relations on right formulae. Hence sequentialising the nested system $\text{NS}_{K\mathcal{A}}$ (Fig. 7) results in the sequent system $\text{SC}_{K\mathcal{A}}$ (shown as rule schemas in Fig. 8).

Finally, Definition 8 of Sect. 4 can be extended to the normal modal case in a trivial way, resulting in the labelled nested system $\text{LbNS}_{K\mathcal{A}}$ (Fig. 9).

Theorem 14. *Systems $\text{NS}_{K\mathcal{A}}$, $\text{SC}_{K\mathcal{A}}$ and $\text{LbNS}_{K\mathcal{A}}$ are equivalent.*

Figures 10a and b present the modal and relational rules of $\text{G3K}\mathcal{A}$ [14], a sound and complete labelled sequent system for $K\mathcal{A}$.

The next results follow the same lines as the ones in Sect. 5.

Lemma 15. *The rules $\text{TL}(d^n), \text{TL}(t^n), \text{TL}(4^n)$ are derivable in $\text{G3K}\mathcal{A}$.*

$$\frac{S\{\Gamma \vdash \Delta, [A \vdash], \Lambda\}}{S\{\Gamma, \Box A \vdash \Delta, \Lambda\}} \text{d}^n \quad \frac{S\{\Gamma, A \vdash \Delta, \Lambda\}}{S\{\Gamma, \Box A \vdash \Delta, \Lambda\}} \text{t}^n \quad \frac{S\{\Gamma \vdash \Delta, [\Gamma', \Box A \vdash \Delta'], \Lambda\}}{S\{\Gamma, \Box A \vdash \Delta, [\Gamma' \vdash \Delta'], \Lambda\}} \text{4}^n$$

$$\text{NS}_{\mathcal{K}\mathcal{A}} : \quad \{\Box_R^n, \Box_L^n\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\text{D}, \text{T}, \text{4}\}$$

Fig. 7. Nested sequent rules for extensions of \mathbf{K} .

$$\frac{\Gamma \vdash A}{\Box \Gamma \vdash \Box A} \text{k} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \text{t} \quad \frac{\Gamma, A \vdash}{\Box \Gamma, \Box A \vdash} \text{d} \quad \frac{\Box \Gamma_4, \Gamma_{\mathbf{K}} \vdash A}{\Box \Gamma_4, \Box \Gamma_{\mathbf{K}} \vdash \Box A} \text{k4} \quad \frac{\Box \Gamma_4, \Gamma_{\mathbf{K}}, A \vdash}{\Box \Gamma_4, \Box \Gamma_{\mathbf{K}}, \Box A \vdash} \text{d4}$$

$$\text{SC}_{\mathbf{K}} \quad \{\text{k}\} \quad \text{SC}_{\mathbf{KT}} \quad \{\text{k}, \text{t}\} \quad \text{SC}_{\mathbf{KD}} \quad \{\text{k}, \text{d}\} \quad \text{SC}_{\mathbf{K4}} \quad \{\text{k4}\} \quad \text{SC}_{\mathbf{KD4}} \quad \{\text{d4}\}$$

Fig. 8. Modal sequent rules for normal modal logics $\text{SC}_{\mathcal{K}\mathcal{A}}$, for $\mathcal{A} \subseteq \{\text{T}, \text{D}, \text{4}\}$.

Proof. The proof is straightforward. For example, for \mathbf{KD}

$$\frac{\mathcal{R}, X, xRy, y : A \vdash Y}{\mathcal{R}, X, xRy, x : \Box A \vdash Y} \Box_L^t \quad \frac{}{\mathcal{R}, X, x : \Box A \vdash Y} \text{Ser}$$

□

Theorem 16. $\mathbf{G3K}\mathcal{A}$ is label-preserving equivalent to $\text{LbNS}_{\mathcal{K}\mathcal{A}}$.

Proof. That every provable sequent in $\text{LbNS}_{\mathcal{K}\mathcal{A}}$ is provable in $\mathbf{G3K}\mathcal{A}$ is a direct consequence of Lemma 15. For the other direction, observe that rule relational rules can be restricted so to be applied just before a \Box_L^t rule. □

This means that labels in $\text{NS}_{\mathcal{K}\mathcal{A}}$ represent worlds in a Kripke-frame, and this extends the results in [6] for modal logic of provability to normal modal logics $\mathcal{K}\mathcal{A}$.

7 Non-normal Modal Systems

We now move our attention to non-normal modal logics, i.e., modal logics that are not extensions of \mathbf{K} . In this work, we will consider the *classical modal logic* \mathbf{E} and the *monotone modal logic* \mathbf{M} . Although our approach is general enough for considering nested, linear nested and sequent systems for several extensions of such logics (such as the *classical cube* or the *modal tesseract* – see [10]), there are no satisfactory labelled sequent calculi in the literature for such extensions.

For constructing nested calculi for these logics, the sequent rules should be decomposed into their different components. However, there are two complications compared to the case of normal modal logics: the need for (1) a mechanism for capturing the fact that exactly one boxed formula is introduced on the left hand side; and (2) a way of handling multiple premises of rules. The first problem is solved by introducing the indexed nesting $[\]^e$ to capture a state where a sequent rule has been partly processed; the second problem is solved by making

$$\frac{\mathcal{R}, xRy, X, y : A \vdash Y}{\mathcal{R}, xRy, X, x : \Box A \vdash Y} \text{TL}(\Box_L^n) \quad \frac{\mathcal{R}, xRy, X \vdash Y, y : A}{\mathcal{R}, X \vdash Y, x : \Box A} \text{TL}(\Box_R^n) \quad \frac{\mathcal{R}, X, x : A \vdash Y}{\mathcal{R}, X, x : \Box A \vdash Y} \text{TL}(t^n)$$

$$\frac{\mathcal{R}, xRy, X, y : A \vdash Y}{\mathcal{R}, X, x : \Box A \vdash Y} \text{TL}(d^n) \quad \frac{\mathcal{R}, xRy, X, y : \Box A \vdash Y}{\mathcal{R}, xRy, X, x : \Box A \vdash Y} \text{TL}(4^n)$$

Fig. 9. Modal rules for labelled indexed nested system $\text{LbNS}_{\mathcal{K}\mathcal{A}}$.

$$\frac{\mathcal{R}, xRy, y : A, \Gamma \vdash \Delta}{\mathcal{R}, xRy, x : \Box A, \Gamma \vdash \Delta} \Box'_L \quad \frac{\mathcal{R}, xRx, \Gamma \vdash \Delta}{\mathcal{R}, \Gamma \vdash \Delta} \text{Ref} \quad \frac{\mathcal{R}, xRz, \Gamma \vdash \Delta}{\mathcal{R}, xRy, yRz, \Gamma \vdash \Delta} \text{Trans}$$

$$\frac{\mathcal{R}, xRy, \Gamma \vdash \Delta, y : A}{\mathcal{R}, \Gamma \vdash \Delta, x : \Box A} \Box'_R \quad \frac{\mathcal{R}, xRy, \Gamma \vdash \Delta}{\mathcal{R}, \Gamma \vdash \Delta} \text{Ser}$$

(a) Modal rules. (b) Modal relational rules. y is fresh in Ser.

Fig. 10. Some rules of the labelled system $\text{G3K}\mathcal{A}$.

the nesting operator $[\cdot]^e$ *binary*, which permits the storage of more information about the premises. Figure 11 presents a unified nested system for logics NS_E and NS_M .

NS_E and NS_M are fully permutable but, since the nested-like rule \Box_L^{en} has two premises, it does not fall into the definitions of shallowness/directedness. However, since propositional rules cannot be applied inside the indexed nestings, the modal rules naturally occur in blocks. Hence the nested rules correspond to macro-rules equivalent to the sequent rules in Fig. 12 for SC_E and SC_M .

Finally, using the labelling method in Sect. 4, the rules in Fig. 11 correspond to the rules in Fig. 13, where xNy and $xN_e(y_1, y_2)$ are relation terms capturing the behaviour of the nestings $[\cdot]$ and $[\cdot]^e$ respectively.

The semantical interpretation of non-normal modalities E, M can be given via *neighbourhood semantics*, that smoothly extends the concept of Kripke frames in the sense that accessibility relations are substituted by a family of neighbourhoods.

Definition 17. A neighbourhood frame is a pair $\mathcal{F} = (W, N)$ consisting of a set W of worlds and a neighbourhood function $N : W \rightarrow \wp(\wp W)$. A neighbourhood model is a pair $\mathcal{M} = (\mathcal{F}, \mathcal{V})$, where \mathcal{V} is a valuation. We will drop the model symbol when it is clear from the context.

The truth description for the box modality in the neighbourhood framework is

$$w \Vdash \Box A \text{ iff } \exists X \in N(w). [(X \Vdash^\forall A) \wedge (A \triangleleft X)] \quad (1)$$

where $X \Vdash^\forall A$ is $\forall x \in X. x \Vdash A$ and $A \triangleleft X$ is $\forall y. [(y \Vdash A) \rightarrow y \in X]$. The rules for \Vdash^\forall and \triangleleft are obtained using the geometric rule approach [15] and are depicted in Fig. 14.

$$\frac{\frac{\Gamma \vdash \Delta, \Lambda, [\vdash B; B \vdash \cdot]^e}{\Gamma \vdash \Delta, \Lambda, \Box B} \Box_R^{en}}{\frac{\Gamma, \Box A, \vdash \Delta, \Lambda, [\Sigma, A \vdash \Pi]}{\Gamma, \Box A \vdash \Delta, \Lambda, [\Sigma \vdash \Pi; \Omega \vdash \Theta]^e} \Box_L^{en} \quad \frac{\Gamma \vdash \Delta, \Lambda, [\Sigma \vdash \Pi; \Omega, \perp \vdash \Theta]^e}{\Gamma \vdash \Delta, \Lambda, [\Sigma \vdash \Pi; \Omega \vdash \Theta]^e} M^n}{} \text{Fig. 11. Modal rules for systems NS}_E \text{ and NS}_M.$$

Fig. 11. Modal rules for systems NS_E and NS_M.

$$\frac{A \vdash B \quad B \vdash A}{\Box A \vdash \Box B} E \quad \frac{A \vdash B}{\Box A \vdash \Box B} M$$

Fig. 12. Modal sequent rules for non-normal modal logics SC_E and SC_M.

If the neighbourhood frame is monotonic (i.e. $\forall X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$ then $Y \in N(w)$), it is easy to see [15] that (1) is equivalent to

$$w \Vdash \Box A \text{ iff } \exists X \in N(w). X \Vdash^\forall A. \quad (2)$$

This yields the set of labelled rules presented in Fig. 15, where the rules are adapted from [15] by collapsing invertible proof steps. Intuitively, while the box left rules create a fresh neighbourhood to x , the box right rules create a fresh world in this new neighbourhood and move the formula to it.

Theorem 18. G3E (resp. G3M) is label-preserving equivalent to LbNS_E (resp. LbNS_M).

Proof. Let π be a normal proof of $\mathcal{N}, X \vdash Y$ in LbNS_E. An instance of the blocked derivation

$$\frac{\frac{\mathcal{N}, xNy_1, X, y_1 : A, y_2 : B \vdash Y, y_1 : B \quad \mathcal{N}, xNy_2, X, y_2 : B \vdash Y, y_1 : B, y_2 : A}{\mathcal{N}, xN_e(y_1, y_2), X, x : \Box A, y_2 : B \vdash Y, y_1 : B} \text{TL}(\Box_L^{en})}{\mathcal{N}, X, x : \Box A \vdash Y, x : \Box B} \text{TL}(\Box_R^{en})} \pi_1$$

is transformed into the labelled derivation

$$\frac{\frac{\frac{y_1 : A, X \vdash Y, y_1 : B}{y_1 \in a, a \Vdash^\forall A, X \vdash Y, y_1 : B} \Vdash^\forall \quad \frac{\frac{y_2 : B, X \vdash Y, y_2 : A}{y_2 : B, A \triangleleft a, X \vdash Y, y_2 \in a} \text{init} \quad \frac{}{y_2 \in a, X \vdash Y, y_2 \in a} \triangleleft}{\frac{y_2 : B, X \vdash Y, y_2 : A}{y_2 : B, A \triangleleft a, X \vdash Y, y_2 \in a} \Box_R^{et}}{\frac{a \in N(x), a \Vdash^\forall A, A \triangleleft a, X \vdash Y, x : \Box B}{X, x : \Box A \vdash Y, x : \Box B} \Box_L^{et}} \Box_R^{et}}{} \pi_2$$

Observe that, in π_1 , the label y_2 will no longer be active, hence the formula $y_2 : B$ can be weakened. The same with y_1 in π_2 . Hence π_1/π_2 corresponds to π'_1/π'_2 and the “only if” part holds. The “if” part uses similar proof theoretical arguments as in the intuitionistic or normal modal case, observing that applications of the forcing rules can be restricted so to be applied immediately after the modal rules. \square

$$\frac{N, xN_e(y_1, y_2), X, y_2 : B \vdash y_1 : B, Y}{N, X \vdash Y, x : \Box B} \text{TL}(\Box_R^{en}) \quad \frac{N, xN_e(y_1, y_2), X, y_2 : \perp \vdash Y}{N, xN_e(y_1, y_2), X \vdash Y} \text{TL}(M^n)$$

$$\frac{N, xNy_1, y_1 : A, X \vdash Y \quad N, xNy_2, X \vdash Y, y_2 : A}{N, xN_e(y_1, y_2), x : \Box A, X \vdash Y} \text{TL}(\Box_L^{en})$$

Fig. 13. Modal rules for LbNS_E and LbNS_M with y_1, y_2 fresh in \Box_R^e .

$$\frac{x \in a, x : A, a \Vdash^\vee A, X \vdash Y}{x \in a, a \Vdash^\vee A, X \vdash Y} \Vdash^\vee \quad \frac{A \triangleleft a, X \vdash Y, z : A \quad z \in a, A \triangleleft a, X \vdash Y}{A \triangleleft a, X \vdash Y} \triangleleft$$

$$\frac{}{x \in a, X \vdash Y, x \in a} \text{init}^f$$

Fig. 14. Forcing rules, with z arbitrary in \triangleleft_L .

$$\frac{a \in N(x), a \Vdash^\vee A, A \triangleleft a, X \vdash Y}{x : \Box A, X \vdash Y} \Box_L^{ef}$$

$$\frac{z \in a, a \in N(x), X \vdash Y, x : \Box B, z : B \quad y : B, a \in N(x), X \vdash Y, x : \Box B, y \in a}{a \in N(x), X \vdash Y, x : \Box B} \Box_R^{ef}$$

$$\frac{a \in N(x), y \in a, X \vdash Y, x : \Box B, y : B}{a \in N(x), X \vdash Y, x : \Box B} \Box_R^{mr} \quad \frac{a \in N(x), a \Vdash^\vee A, X \vdash Y}{x : \Box A, X \vdash Y} \Box_L^{mr}$$

Fig. 15. Labelled systems G3E and G3M. a fresh in \Box_L^e, \Box_L^m and y, z fresh in \Box_R^e, \Box_R^m .

Observe that the neighbourhood information is “hidden” in the nested approach. That is, creating of a nesting has a two-step interpretation: one of creating a neighbourhood and another of creating a world in it. These steps are separated in Negri’s labelled systems, but the nesting information becomes superfluous after the creation of the related world. This indicates that the nesting approach is more efficient proof theoretically speaking when compared to labelled systems. Also, it is curious to note that this “two-step” interpretation makes the nested system *external*, in the sense that nestings cannot be interpreted inside the syntax of the logic. In fact, it makes use of the $\langle \rangle$ modality [2].

8 Conclusion and Future Work

In this work we showed a semantical characterisation of intuitionistic, normal and non-normal modal systems, via a case-by-case translation between labelled nested to labelled sequent systems. In this way, we closed the cycle of syntax/semantic characterisation for a class of logical systems.

While some of the presented results are expected (or even not new as the semantical interpretation of nestings in intuitionistic logic), our approach is, as far as we know, the first done entirely using proof theoretical arguments. Indeed, the soundness and completeness results are left to the case of labelled systems, that carry within the syntax the semantic information explicitly. Using the results in [11], we were able to extend all the semantic discussion to the sequent case.

This work can be extended in a number of ways. First, it seems possible to propose nested and labelled systems for some paraconsistent systems and systems with negative modalities [7]. Our approach, both for sequentialising nested systems and for relating the so-called internal and external calculi could then be applied in such cases.

Another natural direction to follow is to complete this syntactical/semantical analysis for the classical cube [10]. This is specially interesting since $MNC = K$, that is, we should be able to smoothly collapse the neighbourhood approach into the relational one. We observe that nestings play an important role in these transformations, since it enables to modularly building proof systems.

Finally, it would be interesting to analyse to what extent the methodology of the present paper might be applied to shed light on the problem of finding intuitive semantics for substructural logics, like linear logic and its modal extensions [9].

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