Beliefs Based on Evidence and Argumentation

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Abstract. In this paper, we study doxastic attitudes that emerge on the basis of argumentational reasoning. In order for an agent's beliefs to be called 'rational', they ought to be well-grounded in strong arguments that are constructed by combining her available evidence in a specific way. A study of how these rational and grounded beliefs emerge requires a new logical setting. The language of the logical system in this paper serves this purpose: it is expressive enough to reason about concepts such as *factive combined evidence*, *correctly grounded belief*, and *infallible knowledge*, which are the building blocks on which our notions of *argument* and *grounded belief* can be defined. Building further on previous work, we use a topological semantics to represent the structure of an agent's collection of evidence, and we use input from abstract argumentation theory to single out the relevant sets of evidence to construct the agent's beliefs. Our paper provides a sound and complete axiom system for the presented logical language, which can describe the given models in full detail, and we show how this setting can be used to explore more intricate epistemic notions.

Keywords: Evidence-based beliefs · Topological models
Abstract argumentation theory · Doxastic logic

1 Introduction

Propositional attitudes such as knowledge and belief are extensively studied in a number of areas, ranging from artificial intelligence (in particular in multi-agent systems) and computer science (in the study of distributed systems) to philosophy (in epistemology). In these studies we encounter the need to find a mechanism for distinguishing between different attitudes, ranging from weak and stronger forms of belief to infallible knowledge. In order to answer this question, philosophers have proposed a number of additional ingredients (including justifications, evidence, arguments) as well as principles (e.g. safety or stability) on the basis of which one can conduct this comparison and ultimately decide when an item of belief is strong enough to qualify as a piece of knowledge. The topic of this paper relates directly to this discussion on distinguishing different

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propositional attitudes. Our starting point is based on making explicit the *reasons* (e.g., evidence, arguments, justifications) on which beliefs are grounded. In this way, our work contributes to the different representations of beliefs in the literature, which includes not only purely qualitative structures (e.g., the *KD45* approach in doxastic logic [1]; the plausibility models of [2,3]), but also quantitative frameworks (e.g., ranking-based plausibility representations [4]; conditional probabilistic spaces [5,6]). Yet, while the mentioned representations are useful for discussing the properties of belief, they cannot capture the support the agent has for such beliefs, i.e. her available *evidence* and the way she combines it to build *arguments* for and against a given proposition, and how such arguments may yield *justifications* to adopt certain beliefs.[1]

In order to explicitly represent the reasons on which beliefs are based, this paper brings together two different formal frameworks. On the one hand we use input from abstract argumentation theory [9] and relate our work to the use of abstract argumentation within modal logic. As such our work relates to [10–12]. On the other hand we contribute to the semantic approaches for representing evidence-based reasoning; this stands in contrast to the syntactic approach of e.g. justification logic [13]. The semantic approach traces back to [14,15], representing evidence as a set of possible worlds (in terms of so-called *evidence models*) and defining beliefs in terms of the maximally consistent ways in which evidence can be combined. The work in [16] follows the latter direction, adding a topological structure to describe how combined evidence is generated, then using topological notions to single out relevant sets of *combined* pieces of evidence. Note that the two mentioned semantic proposals, using the agent's available evidence to define her beliefs, consider all pieces of combined evidence equally important. This simplifies some definitions, as contradictory evidence is kept separated, yet it also means that in the presence of contradicting evidence, the agent will not make a choice. As such the existing work on evidence-based beliefs doesn't capture one important aspect: the *argumentative* stage in which the agent weighs her (possibly contradicting) evidence in order to make sense of it. Indeed, in [9]'s words (p. 323),

> *[...] a statement is believable if it can be argued successfully against attacking arguments. In other words, whether or not a rational agent believes in a statement depends on whether or not [an] argument supporting this statement can be successfully defended against the counterarguments.*

The work in this paper explicitly incorporates argumentational reasoning. We build on the investigation that was first initiated in [17], by equipping our models with an extra argumentative layer over the topological setting of [16]. This allows us to single out (even in the presence of conflict) a meaningful family of combined pieces of evidence (i.e., arguments) on which *grounded beliefs* can be defined. The combination of a topological semantics with abstract argumentation theory gives

[1] An exception are the so called *truth maintenance systems* [7,8], which keep track of natural-deduction-style *syntactic* justifications.

raise to a wide spectrum of epistemic notions, including not only known concepts such as *evidence*, *argument*, *justified belief* and *infallible knowledge*, but also new ones, such as *(correctly) grounded belief* and *full support belief.*

On the syntactic side, our formal language differs from the logic in [17] as it has the expressive power to reason explicitly about concepts such as *factive combined evidence*, *correctly grounded belief*, and *infallible knowledge*, which are the building blocks for our notions of *argument* and *grounded belief.* Even more: the main technical result of this paper, a sound and complete axiom system for this new language with respect to the given structures, is useful in two important ways. First, it characterises the basic properties of the language's primitive concepts (e.g., grounded beliefs are mutually consistent) as well as the essential relationship between them (e.g., justified beliefs are grounded beliefs), which allow us to find further connections. Second, the axiomatisation can be used as a tool for exploring more intricate epistemic notions and their relationship with grounded belief and justified belief.

The paper starts with Sect. 2 recalling the frameworks on which this proposal is based; then Sect. 3 introduces a topological argumentation model and a formal language to describe it, together with a sound and complete axiom system. Sect. 4 uses the axiomatisation to explore further epistemic notions, and Sect. 5 summarises the proposal, outlining some directions for further work.

2 Preliminaries

Evidence-Based Belief. Let At be a countable set of atomic propositions.

Definition 1 (Evidence model [14]). *A (uniform)* evidence model *is a tuple* $M = (W, \mathcal{E}_0, V)$ *where* ***(i)*** $W \neq \varnothing$ *is a set of possible worlds;* ***(ii)*** $\mathcal{E}_0 \subseteq 2^W - \{\varnothing\}$ *is a family of non-empty subsets of* W *(with* $W \in \mathcal{E}_0$*) called the collection of* pieces of basic evidence*;* ***(iii)*** $V : \mathsf{At} \to 2^W$ *is a valuation function.*

Intuitively, $\mathcal{E}_0$ contains the pieces of evidence the agent has acquired. The requirements over $\mathcal{E}_0$ state that a contradiction cannot be taken as evidence ($\varnothing \notin \mathcal{E}_0$) and that, if knowledge is defined as truth in all possible worlds, the agent knows what is the full range of possibilities ($W \in \mathcal{E}_0$).

Note that some pieces of basic evidence can contradict each other: there may be $P, Q \in \mathcal{E}_0$ with $P \cap Q = \varnothing$. However, this does not mean that the agent accepts contradictions. Collecting evidence is not the end of the story: the agent should be able to *combine* her basic evidence in a meaningful way, and thus the agent's beliefs should ideally not be taken directly from her basic pieces of evidence. The strategy in [14] is that beliefs arise from the maximal consistent ways these basic pieces of evidence can be combined.

Definition 2 (Body of evidence). *Let* $M = (W, \mathcal{E}_0, V)$ *be an evidence model.*

- *A family* $\mathcal{U} \subseteq 2^W$ *has the* finite intersection property *iff the intersection of every finite subset of* $\mathcal{U}$ *is non-empty.*

- *A* body of evidence *is a subfamily* $\mathcal{F} \subseteq \mathcal{E}_0$ *satisfying the finite intersection property.*
- *A body of evidence is* maximal *iff it cannot be properly extended.*

By combining her available evidence in this way, the agent gets a set

$$\mathcal{MC} := \{\bigcap \mathcal{F} \subseteq W \mid \mathcal{F} \text{ is a maximal body of evidence}\}$$

with all its elements in conflict with each other. It is precisely this set which will define the agent's beliefs, yet there are different ways in which this can be done. The choice in [14] is, in some sense, conservative: the agent will believe only what is *supported* by *all* the elements of $\mathcal{MC}$.

Definition 3 (Evidence-based belief [14]). *Let* $M = (W, \mathcal{E}_0, V)$ *be an evidence model. The agent* believes *a proposition* $P \subseteq W$ *(notation:* $B^e\, P$*) if and only if the combination of* every *maximal body of evidence* supports P*, i.e.,*

$$B^e\, P \quad \textit{iff}_{\,def} \quad E \subseteq P \textit{ for all } E \in \mathcal{MC}$$

Justified Belief. Even though the agent's beliefs are given by the combination of maximally consistent pieces of evidence, contradictions may occur.

Example 1 ([18]). Consider the evidence model $(\mathbb{N}, \mathcal{E}_0 = \{[n, +\infty) \mid n \in \mathbb{N}\}, \varnothing)$. Note how $\mathcal{E}_0$ itself is a body of evidence and, moreover, is the unique maximal one. But $\bigcap \mathcal{E}_0 = \varnothing$, and thus the agent believes $\varnothing$.

The reason for this is that maximal bodies of evidence $\mathcal{F}$ are only *finitely* consistent. However, in order to determine whether they support a given proposition, the agent uses arbitrary intersections ($\bigcap \mathcal{F}$ in the definition of $\mathcal{MC}$). In order to reconcile this discrepancy, [16] uses a different strategy; it uses the *topology* generated by $\mathcal{E}_0$.[2]

Definition 4 (Topological evidence model [16]). *A* topological evidence model $M = (W, \mathcal{E}_0, \tau_{\mathcal{E}_0}, V)$ *extends an evidence model* $(W, \mathcal{E}_0, V)$ *(Definition 1) with* $\tau_{\mathcal{E}_0}$*, the topology over* W *generated by* $\mathcal{E}_0$*.*[3]

Arguments. Open sets in τ are unions of *finite* intersections of elements of $\mathcal{E}_0$, and can be seen as the agent's logical manipulation of her basic evidence. Following [16], non-empty open sets in τ are called *arguments* ([19, Subsect. 5.2.2] justifies the use of this term). Note how not every maximal body of evidence defines an argument, as even if all finite intersections of its elements are non-empty, arbitrary intersections might not (Example 1). Thus, the definition of beliefs changes in [16]: instead of asking for all maximal bodies of evidence to support P, it is required that all *arguments* (i.e., finite bodies of evidence) can be strengthened (i.e., combined with further evidence) to yield an argument supporting P.

[2] A *topology* over a non-empty domain X is a family $\tau \subseteq 2^X$ containing both X and $\varnothing$, and is closed under both *finite* intersections and *arbitrary* unions. The elements of a topology are called *open sets*. The *topology generated by a given* $\mathcal{Y} \subseteq 2^X$ is the smallest topology $\tau_{\mathcal{Y}}$ over X such that $\mathcal{Y} \subseteq \tau_{\mathcal{Y}}$.

[3] When no confusion arises, $\tau_{\mathcal{E}_0}$ will be denoted simply by τ.

Definition 5 (Justified belief [16]). *Let $M = (W, \mathcal{E}_0, \tau, V)$ be a topological evidence model. The agent has a* justified belief *of a proposition $P \subseteq W$ (notation: $\mathsf{B}^j P$) if and only if every argument T can be* strengthened *to an argument T' that supports P, that is,*

$$\mathsf{B}^j P \quad \textit{iff}_{\,def} \quad \text{for all } T \in \tau \setminus \{\varnothing\} \text{ there is } T' \in \tau \setminus \{\varnothing\} \text{ s.t. } T' \subseteq T \text{ and } T' \subseteq P$$

Given a topology τ over a set X, an open $T \in \tau$ is *dense* if and only if it has a non-empty intersection with all the other non-empty opens. Then, as stated in [16, Proposition 2].

Proposition 1. *Let $M = (W, \mathcal{E}_0, \tau, V)$ be a topological evidence model. Then, $\mathsf{B}^j P$ holds in M if and only if there is a dense open $T \in \tau$ such that $T \subseteq P$.*

Hence from Definition 5 and Proposition 1 it follows that the agent justifiably believes P if and only if P is supported by an argument that is consistent with any other argument. In this setting, every argument is equally important when deciding what to believe. In order for an agent to weigh her arguments differently we bring in argumentational reasoning in the next section.

3 Belief, Evidence, Argumentation, and Their Logic

The proposal of [17] extends the topological evidence model of [16] with a further semantic component coming from the abstract argumentation framework of [9] to elaborate on relations between arguments.

Definition 6 (Topological argumentation model [17]). *A* topological argumentation *(TA)* model *$M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ extends a topological evidence model $(W, \mathcal{E}_0, \tau, V)$ (Definition 4) with a relation $\leftarrowtail \subseteq (\tau \times \tau)$, the* attack relation *on τ (where $T_1 \leftarrowtail T_2$ reads as "T_2* attacks *T_1"), required to satisfy the following:*

1. *for every $T_1, T_2 \in \tau$: $T_1 \cap T_2 = \varnothing$ if and only if $T_1 \leftarrowtail T_2$ or $T_2 \leftarrowtail T_1$;*
2. *for every $T, T_1, T_1' \in \tau$: if $T_1 \leftarrowtail T$ and $T_1' \subseteq T_1$, then $T_1' \leftarrowtail T$;*
3. *for every $T \in \tau \setminus \{\varnothing\}$: $\varnothing \leftarrowtail T$ and $T \not\leftarrowtail \varnothing$.*

The first condition states (right to left) that attack implies conflict (i.e., empty intersection), but also (left to right) that, while conflict implies attack, the attack does not need to be mutual. The second asks that, if T attacks T_1, then it should also attack any stronger T_1'. The last establishes that, while the empty set is attacked by all non-empty opens, it does not attack any of them.[4]

In a *TA* model, the topology τ represents the arguments the agent has in her mind, and the attack relation $\leftarrowtail$ can be understood as inducing a form of preference over conflicting combined evidence. Together, τ and $\leftarrowtail$ form the basis of the agent's *argumentation framework*. Yet how can the agent use this framework to form her beliefs? Abstract argumentation theory [9] provides useful tools; here are the required notions.

[4] In fact, as the first condition implies, it only attacks itself.

Definition 7 (Characteristic (defense) function). *Let* $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ *be a TA model, with* $A_\tau = (\tau, \leftarrowtail)$ *its argumentation framework. A subset* $\mathcal{T} \subseteq \tau$ *is said to* defend $T \in \tau$ *iff any open* T' *attacking* T *(i.e., for all* $T' \in \tau$ *with* $T \leftarrowtail T'$*) is attacked by some open in* $\mathcal{T}$ *(i.e., there is* $T'' \in \mathcal{T}$ *with* $T' \leftarrowtail T''$*). Then, the* characteristic function of A_τ*, denoted by* d_τ*, also called the* defense function*, receives a set of opens* $\mathcal{T} \subseteq \tau$ *and returns the set of opens it defends:*

$$d_\tau(\mathcal{T}) := \{T \in \tau \mid T \textit{ is defended by } \mathcal{T}\}$$

The characteristic function d_τ is monotonic [9, Lemma 19], so it has a least fixed point $\mathsf{LFP}_\tau \subseteq \tau$ (i.e., LFP_τ is the smallest subset of τ satisfying $\mathsf{LFP}_\tau = d_\tau(\mathsf{LFP}_\tau)$ [20,21]). Since LFP_τ can defend all ($\subseteq$) and only ($\supseteq$) its members against any attack, and it is also conflict-free (i.e., there are no $T, T' \in \mathsf{LFP}_\tau$ such that $T \leftarrowtail T'$), it provides a reasonable definition for the relevant family of open sets in τ over which beliefs will be defined. This set LFP_τ, called *the grounded extension* in abstract argumentation, is never empty in this setting, as W is never attacked (it is in conflict only with the empty set, which does not attack anybody) and therefore it is always in LFP_τ.

Definition 8 (Grounded belief [17]). *Let* $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ *be a TA model. The agent* believes a proposition $P \subseteq W$ *(notation:* $\mathfrak{B}^g\, P$*) iff there is an open set in* LFP_τ *supporting* P*, that is*

$$\mathfrak{B}^g\, P \quad \textit{iff}_{\,def} \quad \textit{there exists } F \in \mathsf{LFP}_\tau \textit{ such that } F \subseteq P.$$

Grounded belief is a fully introspective and mutually consistent notion, closed under conjunction elimination, but not under conjunction introduction [17].

To illustrate grounded belief's failure of conjunction introduction while also showing how a *TA* model can be used to model 'real life' scenarios, we recall Example 3.1 of [17].

Example 2. The zoo in Tom's town bought a new animal and will show it soon to the public. Tom is curious about what species the animal is, so he asks his colleagues. However, he gets different answers. Some tell him that the animal is a penguin ($\{1\}$), some tell him that the animal is a pterosaur ($\{2\}$) and some tell him that the animal is a bat ($\{3\}$). Moreover, two other colleagues, who he really trusts, tell him that the animal can fly ($\{2,3\}$) and the animal is not a mammal ($\{1,2\}$). After receiving all these pieces of information, Tom is very puzzled. Although "the animal can fly" and "the animal is not a mammal" imply that the animal is a neither a penguin nor a bat, it is still hard to imagine that there can be a pterosaur living in the world. Intuitively, in such a situation, Tom comes to believe that the animal can fly and the animal is not a mammal. However, it seems that his evidence is not strong enough to support the claim that the animal is a pterosaur.

Let $\mathsf{At} = \{p, t, b\}$ be a set of atomic propositions (p: "the animal is a penguin"; t: "the animal is a pterosaur"; b: "the animal is a bat"). The following *TA* model M describes Tom's evidence, arguments and doxastic situation.

$$M = (W = \{1,2,3\}, \mathcal{E}_0 = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}, \tau = 2^W, \leftarrowtail, V) \tag{1}$$

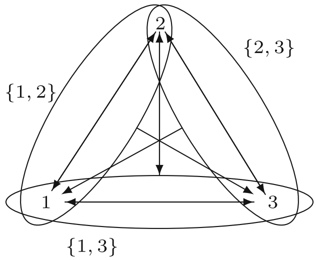

Fig. 1. Grounded beliefs are not closed under conjunction.

with $V = \{(p, \{1\}), (t, \{2\}), (b, \{3\})\}$ and $\leftarrowtail$ given by the union of ***(i)***singletons attacking one another, ***(ii)*** $\{\varnothing \leftarrowtail T \mid T \in \tau\}$ and ***(iii)*** $\{\{3\} \leftarrowtail \{1,2\}, \{1\} \leftarrowtail \{2,3\}, \{2\} \leftarrowtail \{1,3\}, \{1,3\} \leftarrowtail \{2\}\}$, as shown in Fig. 1.[5] According to the definition, $\mathsf{LFP}_\tau = \{\{1,2\}, \{2,3\}, \{1,2,3\}\}$ (a set that is not closed under intersection); together with the definition of grounded belief, this confirms the intuition that Tom can come to believe that the animal can fly ($\{2,3\} \in \mathsf{LFP}_\tau$) and that the animal is not a mammal ($\{1,2\} \in \mathsf{LFP}_\tau$), but he does not come to believe that the animal is a pterosaur (no subset of $\{2\}$ is in LFP_τ).

3.1 The Logic of Belief, Evidence and Argumentation

In order to reveal the relationship between evidence, arguments, and grounded beliefs, this paper introduces a richer language to describe *TA* models compared to the logic studied in [17]. It relies on the notions of *infallible knowledge*, *factive combined evidence* and *correctly grounded belief*, from which notions as *argument* and *grounded belief* can be defined. This language can be used not only for providing a more detailed description of the models; it can be also used to explore more intricate epistemic notions and their interrelationship, as in Sect. 4.

Definition 9 (Language $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$). *The language $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ is generated by:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\,\varphi \mid \mathrm{K}\,\varphi \mid \mathfrak{T}\,\varphi$$

with atoms $p \in \mathsf{At}$ (define $\Diamond\,\varphi := \neg\,\Box\,\neg\varphi$, $\widehat{\mathrm{K}}\,\varphi := \neg\,\mathrm{K}\,\neg\varphi$, and $\widehat{\mathfrak{T}}\,\varphi := \neg\,\mathfrak{T}\,\neg\varphi$). For its semantics, given a TA model M, atoms and Boolean operators are interpreted as usual. For cases involving operators, we first define $\mathcal{F}^c$, the set containing the combination of finite bodies of evidence:

$$\mathcal{F}^c := \{\textstyle\bigcap \mathcal{F} \subseteq W \mid \mathcal{F} \textit{ is a } \text{finite } \textit{body of evidence}\}.$$

$$\begin{array}{lll}
M, w \models \Box\,\varphi & \textit{iff}_{\,def} & \textit{there exists } E \in \mathcal{F}^c \textit{ such that } w \in E \textit{ and } E \subseteq [\![\varphi]\!] \\
M, w \models \mathrm{K}\,\varphi & \textit{iff}_{\,def} & W \subseteq [\![\varphi]\!] \\
M, w \models \mathfrak{T}\,\varphi & \textit{iff}_{\,def} & \textit{there exists } F \in \mathsf{LFP}_\tau \textit{ such that } w \in F \textit{ and } F \subseteq [\![\varphi]\!]
\end{array}$$

[5] Attack edges involving the empty set are not drawn.

The modality K can be understood as describing the agent's *infallible knowledge*. The operator $\Box$ indicates the existence of combined evidence ($E \in \mathcal{F}^c$) that is *factive* ($w \in E$) [16], so $\Box\,\varphi$ denotes *"the agent has factive combined evidence for* φ*"*. Note the following equivalence.

Proposition 2. *Given a TA model* M *and any world* w *in it,*

$$M, w \models \Box\,\varphi \quad \textit{iff} \quad \textit{there is an argument } T \in \tau\backslash\{\varnothing\} \textit{ such that } w \in T \subseteq [\![\varphi]\!].$$

Hence, $\Box\,\varphi$ also expresses that *"the agent has a correct argument for* φ *"*. Finally, to understand the interpretation of the operator $\mathfrak{T}$, compare it to the operator for grounded belief $\mathfrak{B}^g$ in Definition 8. The difference is that, while the truth condition of $\mathfrak{T}\,\varphi$ requires that there is a *correct* argument F in LFP_τ ($w \in F$), the truth condition of $\mathfrak{B}^g\,\varphi$ does not require correctness. Thus, $\mathfrak{T}\,\varphi$ is read as *"the agent has correctly grounded belief of* φ", from which the operator for grounded belief can be defined as

$$\mathfrak{B}^g\,\varphi := \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi.$$

Choosing $\mathfrak{T}$ instead of $\mathfrak{B}^g$ as a basic operator in the language is only a matter of technical convenience. There is no difference between the two choices, as

Proposition 3. $\models \mathfrak{T}\,\varphi \leftrightarrow (\mathfrak{B}^g\,\varphi \wedge \Box\varphi)$.

This equivalence also shows that correctly grounded belief $\mathfrak{T}\,\varphi$ is different from *grounded true belief*, which can be expressed in the language as $\mathfrak{B}^g\,\varphi \wedge \varphi$. While the latter only requires the belief to be true, a correctly grounded belief requires a *correct argument* in LFP supporting the belief.

Axiom System. It has been proved [16, Theorem 4] that the validities of $\mathcal{L}_{\Box,\mathrm{K}}$ with respect to topological evidence models (Definition 4) are characterised by ***(i)*** propositional tautologies and Modus Ponens, ***(ii)*** the S4 axioms and rules for $\Box$; ***(iii)*** the S5 axioms and rules for K, ***(iv)*** $\mathrm{K}\,\varphi \rightarrow \Box\,\varphi$. The challenge here is to find a proper axiom system characterising the validities of $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ (which extends $\mathcal{L}_{\Box,\mathrm{K}}$ with $\mathfrak{T}$) over *TA* models (which extend topological evidence models with an attack relation on τ); Table 1 shows our proposal.

Axioms and rules in the upper block of Table 1 are self-explanatory, with exception of the last two, describing the interaction between $\mathfrak{T}$ and K. For the first, recall that $\widehat{\mathrm{K}}\,\mathfrak{T}$ is the operator for grounded belief; then, $\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \rightarrow \neg\,\widehat{\mathrm{K}}\,\mathfrak{T}\,\neg\varphi$ states that grounded beliefs are mutually consistent. The second, $(\mathfrak{T}\,\varphi \wedge \mathrm{K}\,\psi) \rightarrow \mathfrak{T}(\varphi \wedge \mathrm{K}\,\psi)$, is the 'pullout' axiom[6] for $\mathfrak{T}$, and states that a correctly grounded belief of φ and infallible knowledge of ψ give the agent a correctly grounded belief of the conjunction of φ and her infallible knowledge of ψ. The axiom can be used to derive easier-to-read validities describing the interaction between $\mathfrak{T}$ and K. An example is the following one, a variation of the famous K axiom, indicating that infallible knowledge of an implication and a correctly grounded belief of the antecedent gives the agent a correctly grounded belief of the consequent.

6 The 'pullout' axiom is from [14], where it is used with the operator for evidence $\Box$.

Table 1. Axiom system $\mathsf{L}_{\Box,\mathrm{K},\mathfrak{T}}$, for $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ w.r.t. topological argumentation models.

• Propositional Tautologies and Modus Ponens	
• The S5 axioms and rules for K	• The S4 axioms and rules for $\Box$
• $\mathfrak{T}\top$	• $\mathfrak{T}\varphi \rightarrow \varphi$
• $\mathfrak{T}\varphi \rightarrow \mathfrak{T}\mathfrak{T}\varphi$	• From $\varphi \rightarrow \psi$ infer $\mathfrak{T}\varphi \rightarrow \mathfrak{T}\psi$
• $\widehat{\mathrm{K}}\mathfrak{T}\varphi \rightarrow \neg\widehat{\mathrm{K}}\mathfrak{T}\neg\varphi$	• $(\mathfrak{T}\varphi \wedge \mathrm{K}\psi) \rightarrow \mathfrak{T}(\varphi \wedge \mathrm{K}\psi)$
• $\mathfrak{T}\varphi \rightarrow \Box\varphi$	
• $\mathfrak{T}\varphi \rightarrow \mathrm{K}(\Box\varphi \rightarrow \mathfrak{T}\varphi)$	• $\mathrm{K}\Diamond\Box\varphi \rightarrow \widehat{\mathrm{K}}\mathfrak{T}\varphi$
• $\big(\widehat{\mathrm{K}}\mathfrak{T}\varphi \wedge \neg\widehat{\mathrm{K}}\mathfrak{T}\psi \wedge \mathrm{K}((\varphi\wedge\psi) \rightarrow \Box(\varphi\wedge\psi))\big) \rightarrow \widehat{\mathrm{K}}\Box(\varphi\wedge\neg\psi)$	

Proposition 4. $\vdash \mathrm{K}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\varphi \rightarrow \mathfrak{T}\psi)$.

Proof

(1) $\vdash (\mathfrak{T}\varphi \wedge \mathrm{K}(\varphi \rightarrow \psi)) \rightarrow \mathfrak{T}(\varphi \wedge \mathrm{K}(\varphi \rightarrow \psi))$	Instance of the 'pullout' axiom
(2) $\vdash (\varphi \wedge \mathrm{K}(\varphi \rightarrow \psi)) \rightarrow \psi$	Axioms T for K, $\mathfrak{T}$; Modus Ponens
(3) $\vdash \mathfrak{T}(\varphi \wedge \mathrm{K}(\varphi \rightarrow \psi)) \rightarrow \mathfrak{T}\psi$	(2) and rule for $\mathfrak{T}$
(4) $\vdash \mathrm{K}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\varphi \rightarrow \mathfrak{T}\psi)$	(1), (3) and Modus Ponens

The four axioms in the lower block of Table 1 describe the relationship between different modalities. Axiom $\mathfrak{T}\varphi \rightarrow \Box\varphi$, tells us that a correctly grounded belief of φ implies that the agent has a correct argument for φ. The axiom, $\mathfrak{T}\varphi \rightarrow \mathrm{K}(\Box\varphi \rightarrow \mathfrak{T}\varphi)$, states that if the agent has a correctly grounded belief of φ then she infallibly knows that a correct argument for φ implies a correctly grounded belief of φ; it describes a form of strong (K) introspection of grounded arguments ($\mathfrak{T}$) in the presence of 'normal' arguments ($\Box$).

To understand axiom $\mathrm{K}\Diamond\Box\varphi \rightarrow \widehat{\mathrm{K}}\mathfrak{T}\varphi$, recall first that $\widehat{\mathrm{K}}\mathfrak{T}$ characterises grounded belief. Then, note that justified belief (Definition 5) is characterised by $\mathrm{K}\Diamond\Box$ [16, Proposition 2], that is, $\mathsf{B}^j\varphi := \mathrm{K}\Diamond\Box\varphi$. Hence, the axiom indicates that justified belief implies grounded belief. Finally, the fourth axiom, $\big(\widehat{\mathrm{K}}\mathfrak{T}\varphi \wedge \neg\widehat{\mathrm{K}}\mathfrak{T}\psi \wedge \mathrm{K}((\varphi\wedge\psi) \rightarrow \Box(\varphi\wedge\psi))\big) \rightarrow \widehat{\mathrm{K}}\Box(\varphi\wedge\neg\psi)$, states that if the agent has a grounded belief of φ but not of ψ, and she at the same time has argument $[\![\varphi\wedge\psi]\!]$,[7] then the agent must have an argument for $\varphi\wedge\neg\psi$.

The soundness of the axiom system is proved by verifying that the axioms are valid and the rules are validity-preserving. Most of the cases are relatively simple; here we focus on the last three axioms, to give the reader a better grasp of the modalities' semantic interpretation.

Proposition 5. $\models \mathfrak{T}\varphi \rightarrow \mathrm{K}(\Box\varphi \rightarrow \mathfrak{T}\varphi)$.

[7] Note that "the agent has argument $[\![\varphi]\!]$" is different from "the agent has *an* argument *for* $[\![\varphi]\!]$". The former is expressed by $\mathrm{K}(\varphi \rightarrow \Box\varphi)$, semantically stating that there is an argument T such that $T = [\![\varphi]\!]$; the latter corresponds to $\widehat{\mathrm{K}}\Box\varphi$, semantically stating that there is an argument T such that $T \subseteq [\![\varphi]\!]$.

Proof. Let $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ be a *TA* model; take $w \in W$, and suppose $M, w \models \mathfrak{T}\,\varphi$. Then, there exists $F \in \mathsf{LFP}_\tau$ such that $w \in F$ and $F \subseteq [\![\varphi]\!]$, that is, there is $F \in \mathsf{LFP}_\tau$ such that $F \subseteq [\![\varphi]\!]$. Now, take any $u \in W$: if there is $T_u \in \mathcal{F}^c$ such that both $u \in T_u$ and $T_u \subseteq [\![\varphi]\!]$, then $F \cup T_u$ is not only a factive (at u, as $u \in (F \cup T_u)$) argument supporting φ (clearly, $(F \cup T_u) \subseteq [\![\varphi]\!]$); it is also in LFP_τ, as $F \in \mathsf{LFP}_\tau$ and [17, Proposition 3.1] indicates that, for all $T, T' \in \tau$, if $T \subseteq T'$ and $T \in \mathsf{LFP}_\tau$ then $T' \in \mathsf{LFP}_\tau$. Therefore, $M, u \models \mathfrak{T}\,\varphi$.

Proposition 6. $\models \mathrm{K} \Diamond \Box \varphi \to \widehat{\mathrm{K}}\, \mathfrak{T}\, \varphi$.

Proof. By Proposition 1, justified belief of φ, $\mathrm{K} \Diamond \Box \varphi$, implies the existence of a dense open T supporting φ. But dense opens intersect with all non-empty opens, so they are not attacked at all; hence, T must be in LFP_τ.

Proposition 7. $\models \big(\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \wedge \neg \widehat{\mathrm{K}}\,\mathfrak{T}\,\psi \wedge \mathrm{K}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))\big) \to \widehat{\mathrm{K}}\,\Box(\varphi \wedge \neg\psi)$.

Proof. Let $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ be a *TA* model; take $w \in W$. Suppose

$$M, w \models \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \wedge \neg \widehat{\mathrm{K}}\,\mathfrak{T}\,\psi \wedge \mathrm{K}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))$$

From the first conjunct, there is $F \in \mathsf{LFP}_\tau$ such that $F \subseteq [\![\varphi]\!]$. But, from the second, no $F' \in \mathsf{LFP}_\tau$ is such that $F' \subseteq [\![\psi]\!]$; in particular, $[\![\varphi \wedge \psi]\!] \notin \mathsf{LFP}$. However, by the third conjunct, $[\![\varphi \wedge \psi]\!] \in \tau$. So there must be an argument keeping $[\![\varphi \wedge \psi]\!]$ out of LFP, that is, there is a non-empty $T \in \tau$ such that $[\![\varphi \wedge \psi]\!] \leftarrowtail T$ and T intersects with all arguments in LFP_τ, with the former implying that $T \subseteq W \backslash [\![\varphi \wedge \psi]\!]$ and the latter implying that $T \cap F \neq \varnothing$. Moreover, $F \subseteq [\![\varphi]\!]$, so $T \cap F \subseteq (W \backslash [\![\varphi \wedge \psi]\!]) \cap [\![\varphi]\!] = [\![\varphi \wedge \neg\psi]\!]$; together with $T \cap F \neq \varnothing$, $T \cap F \in \tau$, and Proposition 2, it implies that $M, w \models \widehat{\mathrm{K}}\,\Box(\varphi \wedge \neg\psi)$.

As for the completeness of the system, the reader can find (for space reasons, an abridged version of) the proof in the Appendix A.

Theorem 1. *The axiom system of Table 1 is sound and strongly complete for the language $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ w.r.t. topological argumentation models.*

4 Further Epistemic Notions

We have seen in Sect. 3 that the language $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ can express several epistemic notions, such as arguments ($\widehat{\mathrm{K}}\,\Box$), grounded belief ($\widehat{\mathrm{K}}\,\mathfrak{T}$) and justified belief ($\mathrm{K} \Diamond \Box \varphi$). This section applies the logic of belief, evidence and argumentation to explore further epistemic notions and the way they relate to each other.

Recall the result on the definability of justified belief and grounded belief: $\mathsf{B}^j \varphi := \mathrm{K} \Diamond \Box \varphi$ and $\mathfrak{B}^g \varphi := \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$. Thus, while justified belief is defined by infallible knowledge and factive combined evidence, grounded belief is defined by infallible knowledge and correctly grounded belief.

From these definitions, one may wonder about the relationship of the given concepts with the ones given by

$$\mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi \qquad \text{and} \qquad \widehat{\mathrm{K}}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi$$

The first substitutes $\widehat{\mathfrak{T}}\,\mathfrak{T}$ for $\Diamond\,\Box$ in $\mathrm{K}\,\Diamond\,\Box\,\varphi$; it can be intuitively read as *"the agent knows that it is consistent with her correctly grounded beliefs that she has a correctly grounded belief of φ"*. The second substitutes $\Diamond\,\Box\,\varphi$ for φ in $\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$; it can be read as *"the agent has a grounded belief of the possibility of having a correct argument for φ"*.

Now, do these two formulas describe epistemic notions different from justified belief and grounded belief? The axiom system shows us that the answer is no: they are just two alternative ways of characterising grounded belief:

Proposition 8. $\vdash \mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi \leftrightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$ *and* $\vdash \widehat{\mathrm{K}}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi \leftrightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$.

Proof. For space reasons, here we only prove the first.

(1) $\vdash \widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi \rightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$
(2) $\vdash \mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi \rightarrow \mathrm{K}\,\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$
(3) $\vdash \mathrm{K}\,\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \rightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$
(4) $\vdash \mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi \rightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi$

(1) $\vdash \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \rightarrow \widehat{\mathrm{K}}\,\mathfrak{T}\,\mathfrak{T}\,\varphi$
(2) $\vdash \widehat{\mathrm{K}}\,\mathfrak{T}\,\mathfrak{T}\,\varphi \rightarrow \neg\,\widehat{\mathrm{K}}\,\mathfrak{T}\,\neg\,\mathfrak{T}\,\varphi$
(3) $\vdash \neg\,\widehat{\mathrm{K}}\,\mathfrak{T}\,\neg\,\mathfrak{T}\,\varphi \rightarrow \mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi$
(4) $\vdash \widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \rightarrow \mathrm{K}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi$

We close this section with a new epistemic notion, obtained on the basis of a semantic argument:

Definition 10 (Full-support belief). *Let $M = (W, \mathcal{E}_0, \tau, \hookleftarrow, V)$ be a topological argumentation model. The agent has* full-support belief *of a proposition $P \subseteq W$ (notation: $\mathcal{B}^f\,P$) if and only if every argument in LFP_τ can be strengthened to an argument in LFP_τ which supports P.*

Compare the definition of full-support belief with the definition of justified belief (Definition 5): the only difference is that all the arguments involved in defining full-support belief need to be members of LFP_τ. On one hand, the similarities between the definitions of these two concepts suggest that they may share the same properties, and indeed this is the case: within *TA* models, full-support belief $\mathcal{B}^f$ is a *KD45* operator. Here we only prove that it is closed under conjunction introduction.

Proposition 9. *Given a TA model, for any $P, Q \subseteq W$ we have that.*

$$(\mathcal{B}^f\,P \wedge \mathcal{B}^f\,Q) \rightarrow \mathcal{B}^f\,(P \wedge Q)$$

Proof. Given a topological argumentation model, assume that, for all $F \in \mathsf{LFP}_\tau$, not only there is $F' \subseteq F$ such that $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P$, but also there is $F'' \subseteq F$ such that $F'' \in \mathsf{LFP}_\tau$ and $F'' \subseteq Q$.

Take an arbitrary $T \in \mathsf{LFP}_\tau$. By the assumption, there is $T' \subseteq T$ such that $T' \in \mathsf{LFP}_\tau$ and $T' \subseteq P$. By the assumption again, from $T' \in \mathsf{LFP}_\tau$ it follows that there is $T'' \subseteq T'$ such that $T'' \in \mathsf{LFP}_\tau$ and $T'' \subseteq Q$. But $T'' \subseteq T'$ and $T' \subseteq P$ imply $T'' \subseteq P$. Hence, $T'' \subseteq P \wedge Q$, Thus, for all $F \in \mathsf{LFP}_\tau$, we can find an argument $F' \subseteq F$ such that $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P \wedge Q$.

On the other hand, the similarity between the semantic definition of full-support belief (Definition 10) and the semantic definition of justified belief (Definition 5) seems to suggest that because the latter can be expressed as $\mathrm{K} \Diamond \Box \varphi$, also the former can be written as $\mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} \varphi$. However, this is not the case. By Proposition 8, $\mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} \varphi$ is a syntactical definition of grounded belief. Full-support belief $\mathcal{B}^f$ and grounded belief $\mathfrak{B}^g$ are different, as the latter is not closed under conjunction [17] while the former is (Proposition 9). So full-support belief cannot be syntactically defined by $\mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} \varphi$. Why is there such a discrepancy? It is due to the lack of closure under finite intersection in LFP_τ, as the following proposition shows.

Proposition 10. *Given a TA model, $\mathcal{B}^f P \leftrightarrow \mathfrak{B}^g P$ holds for any $P \subseteq W$ if and only if LFP_τ is closed under finite intersections.*

Proof. From left to right: if grounded belief and full-support belief are equivalent in the given model, then grounded beliefs should be closed under conjunction: for any $P, Q \subseteq W$, if $\mathfrak{B}^g P \wedge \mathfrak{B}^g Q$ holds then $\mathfrak{B}^g(P \wedge Q)$ also holds. Now, if LFP_τ is not closed under finite intersection, it is easy to find P and Q such that the above fact fails. Thus, LFP_τ has to be closed under finite intersection.

From right to left: by Proposition 8, we only need to prove that $\mathcal{B}^f P \leftrightarrow \mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} \varphi$ holds when LFP_τ is closed under finite intersection. For the first direction, assume $\mathcal{B}^f P$; then, for all $F \in \mathsf{LFP}_\tau$ there is $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$. Now take an arbitrary $w \in W$ and an arbitrary $F \in \mathsf{LFP}_\tau$ with $w \in F$; then there is $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$. But $\varnothing \notin \mathsf{LFP}_\tau$ so $F' \neq \varnothing$; there is $v \in F'$ with $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P$. Hence, for all $w \in W$ and all $F \in \mathsf{LFP}_\tau$ such that $w \in F$, we can find a $v \in F'$ such that $\mathfrak{T} P$ holds on v; then, $\mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} P$ holds in the model. Note how we did not use LFP_τ's closure under finite intersections.

For the second direction, assume $\mathrm{K} \widehat{\mathfrak{T}} \mathfrak{T} P$; then, for all $w \in W$ and all $F \in \mathsf{LFP}_\tau$ with $w \in F$, there is $v \in F$ such that there is an argument $F' \in LFP_\tau$ such that $v \in F'$ and $F' \subseteq P$. Note that F' is not required to be a subset of F; still, LFP_τ is closed under finite intersections, so $F \cap F'$ is also in LFP_τ, which gives us an argument in LFP_τ that is a subset of F ($F \cap F' \in \mathsf{LFP}_\tau$) and supports P ($F \cap F' \subseteq P$). So, for all $F \in \mathsf{LFP}_\tau$, we can find an $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$, which implies that $\mathcal{B}^f P$ holds in the model.

A more detailed study of the relationship between grounded, justified and full-support belief will be provided in this paper's full-version, where we can show that B^j implies $\mathcal{B}^f$, which in turn implies $\mathfrak{B}^g$, but not the other way around.

5 Conclusion and Future Work

Continuing the series of works [14,16,17] on models providing an explicit representation of the 'reasons' supporting an agent's beliefs, this paper focuses explicitly on the concepts of evidence and argumentation. On the first, it relies on the topological extension [16] of the so-called evidence models [14,15], representing a piece of evidence as a set of possible worlds. With respect to 'argumentation', we use tools from abstract argumentation theory [9] to single out arguments on which a notion of *grounded belief* is defined. This combination of topological semantics with abstract argumentation theory gives raise to a wide spectrum of epistemic notions, including not only known concepts as *evidence*, *argument*, *justified belief* and *infallible knowledge*, but also new ones, such as *(correctly) grounded belief* and *full support belief*.

The main technical contribution of this paper is the logic of belief, evidence and argumentation, via which the semantic analysis on grounded belief and its relationship with justified belief of [17] is fully characterized. The logic is useful not only for characterising the relationship between the mentioned epistemic notions; it also helps to find new epistemic concepts, deepening our understanding of the notion of grounded belief that is central to this work.

The presented setting opens several interesting alternatives for further research. An immediate one follows from the fact that *full-support belief* has been semantically characterized but not syntactically defined in $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$. Further research on its syntactic definability is necessary and may as well require an extension of the language. Moreover, other interesting notions of belief may arise by using further tools from abstract argumentation theory. Indeed, grounded belief relies on the grounded extension of the argumentation framework, but other extensions might be considered, as *preferred* extension, *stable* extension and so on. They would give raise to further types of belief that can be compared with the ones studied here.

Equally interesting is a move to a multi-agent scenario, with different agents considering possibly different attack relations. This would give rise to a more 'real' argumentation setting, with argumentation taking place not only within an agent's mind, but also between different agents. In turn, this emphasises the importance of a further dynamic layer, exploring the different epistemic actions that might affect the agent's epistemic state. In line with other work on evidence-dynamics in [14], the emergence of new evidence is interesting (as is the dismissal of existing ones); our setting also allows for changes in an agent's attack relation (arising, e.g., from her interaction with others). By providing the formal tools to study such scenarios, one will be able to truly understand how interaction in multi-agent argumentation affects the epistemic state of the involved agents.

A Completeness for $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$

The proof shows that any $\mathsf{L}_{\Box,\mathrm{K},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$-formulas is satisfiable. Satisfiability will be proved in an *Alexandroff qTA models* (see below), which is $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$-equivalent to its corresponding *TA* model.[8] Here are the details.

Definition 11 (qTA model). *A* quasi-topological argumentation model *(qTA) is a tuple* $\mathcal{M} = (W, \mathcal{E}_0, \leqslant, \leftarrowtail, V)$ *in which* $(W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ *is a TA model (with*

[8] A similar strategy is used in [16]: show that any consistent set of formulas is satisfiable in a quasi-model, then turn it into a modally-equivalent topological evidence model.

τ generated by $\mathcal{E}_0$, as before) and $\leqslant \subseteq (W \times W)$ a preorder such that, for every $E \in \mathcal{E}_0$, if $u \in E$ and $u \leqslant v$, then $v \in E$.

Formulas in $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ are interpreted in qTA models just as in *TA* models. The only difference is $\Box$, which becomes a normal universal modality for $\leqslant$. More precisely, $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if $w \leqslant v$ then $\mathcal{M}, w \models \Box\varphi$. Now, two topological definitions, a refined qTA model, and the connection.

Definition 12 (Specification preorder). *Let (X,τ) be a topological space. Its specification preorder $\sqsubseteq_\tau \subseteq (X \times X)$ is defined, for any $x, y \in X$, as $x \sqsubseteq_\tau y$ iff for all $T \in \tau$, $x \in T$ implies $y \in T$.*

Definition 13 (Alexandroff space). *A topological space (X,τ) is* Alexandroff *iff τ is closed under* arbitrary *intersections (i.e., $\bigcap T \in \tau$ for any $T \subseteq \tau$).*

Definition 14 (Alexandroff qTA model). *A qTA-model $\mathcal{M} = (W, \mathcal{E}_0, \leqslant, \hookleftarrow, V)$ is called* Alexandroff *iff **(i)** $(W, \tau_{\mathcal{E}_0})$ is* Alexandroff, *and **(ii)** $\leqslant = \sqsubseteq_\tau$.*

Proposition 11. *Given an Alexandroff qTA model $\mathcal{M} = (W, \mathcal{E}_0, \leqslant, \hookleftarrow, V)$, take $M = (W, \mathcal{E}_0, \tau, \hookleftarrow, V)$. Then, $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_M$ for every $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$.*

Proof. Exactly as that of [19, Proposition 5.6.14] for topological evidence models and $\mathcal{L}_{\Box,\mathrm{K}}$, as $\mathfrak{T}$ has the same truth condition in qTA and *TA* models.

For notation, define $\Gamma^{\bigcirc} = \{\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}} \mid \bigcirc\varphi \in \Gamma\}$ for $\Gamma \subseteq \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ and $\bigcirc \in \{\Box, \mathrm{K}, \mathfrak{T}\}$. For the proof, let Φ_0 be a $\mathsf{L}_{\Box,\mathrm{K},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$-formulas. A slightly modified version of Lindenbaum Lemma shows that it can be extended to a maximal consistent one. Let MCS be the family of all maximally $\mathsf{L}_{\Box,\mathrm{K},\mathfrak{T}}$-consistent sets of $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$-formulas; let Φ be an element of MCS extending Φ_0.

Definition 15 (Canonical qTA model). *The canonical qTA model for Φ, $\mathcal{M}^\Phi = (W^\Phi, \mathcal{E}_0^\Phi, \leqslant^\Phi, \hookleftarrow^\Phi, V^\Phi)$, is defined as follows.*

- *$W^\Phi := \{\Gamma \in \mathrm{MCS} \mid \Gamma^{\mathrm{K}} = \Phi^{\mathrm{K}}\}$ and $V^\Phi(p) := \{\Gamma \in W^\Phi \mid p \in \Gamma\}$.*
- *For $\Gamma, \Delta \in W^\Phi$, $\Gamma \leqslant^\Phi \Delta$ iff$_{def}$ for any $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$, $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$.*
- *For any $\Gamma \in W^\Phi$, define the set $\leqslant^\Phi[\Gamma] := \{\Omega \in W^\Phi \mid \Gamma \leqslant^\Phi \Omega\}$. Then, let $\mathcal{E}_0^\Phi := \{\bigcup_{\Gamma \in U} \leqslant^\Phi[\Gamma] \mid U \subseteq W^\Phi\} \setminus \{\varnothing\}$.*

While $\leqslant^\Phi$ and V^Φ are standard (recall: $\Box$ is a normal universal modality for $\leqslant$), each $E \in \mathcal{E}_0^\Phi$ is a non-empty union of the $\leqslant^\Phi$-upwards closure of the elements of some subset of W^Φ. The last component, the attack relation $\hookleftarrow^\Phi$, is the novel one in this model, and it requires more care. First, define $\{\!|\varphi|\!\}_M := \{\Gamma \in W^\Phi \mid \varphi \in \Gamma\}$. Then, by taking τ^Φ to be the topology generated by $\mathcal{E}_0^\Phi$ define, for any $T, T' \in \tau^\Phi$,

- $T \hookleftarrow^\Phi T'$ *iff$_{def}$* $\begin{cases} T = \varnothing & \text{if } T' = \varnothing \\ T \cap T' = \varnothing \text{ and there is no } \varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}} \text{ s.t.} & \text{otherwise} \\ \qquad \text{both } \{\!|\mathfrak{T}\varphi|\!\} \subseteq T \text{ and } \widehat{\mathrm{K}}\mathfrak{T}\varphi \in \Phi & \end{cases}$

In the rest, and when no confusion arises, the superscript Φ will be omitted.

Note how $\mathcal{M}^{\Phi}$ is indeed a qTA model (Definition 11). First, it is clear that $\varnothing \notin \mathcal{E}_0$ and $W \in \mathcal{E}_0$. Moreover, $\leqslant$ is indeed a preorder (see its axioms) satisfying the extra condition. Finally, it can be proved that $\leftarrowtail$ satisfies the three conditions.

Lemma 1. *Let $\mathcal{M}^{\Phi} = (W, \mathcal{E}_0, \leqslant, \leftarrowtail, V)$ be the model of Definition 15. Then,*

1. *for every $T_1, T_2 \in \tau$: $T_1 \cap T_2 = \varnothing$ if and only if $T_1 \leftarrowtail T_2$ or $T_2 \leftarrowtail T_1$;*
2. *for every $T, T_1, T_1' \in \tau$: if $T_1 \leftarrowtail T$ and $T_1' \subseteq T_1$, then $T_1' \leftarrowtail T$;*
3. *for every $T \in \tau \setminus \{\varnothing\}$: $\varnothing \leftarrowtail T$ and $T \not\leftarrowtail \varnothing$.*

Thus, $\mathcal{M}^{\Phi}$ is a qTA model. The next proposition (standard proof) provides existence lemmas for the standard modality $\Box$ and the global modality $\widehat{\mathrm{K}}$.

Proposition 12. *For any $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ and any $\Gamma \in W$:*

- *$\Diamond \varphi \in \Gamma$ iff there is $\Delta \in W$ s.t. $\Gamma \leqslant \Delta$ and $\varphi \in \Delta$.*
- *$\widehat{\mathrm{K}} \varphi \in \Gamma$ iff there is $\Delta \in W$ s.t. $\varphi \in \Delta$.*

Now, tools to prove a similar result for the operator $\mathfrak{T}$, whose truth clause relies on LFP, given by $\leftarrowtail$. First, some useful properties of the model.

Fact 1. ***(1)*** *$\tau = \mathcal{E}_0 \cup \{\varnothing\}$.* ***(2)*** *If $\widehat{\mathrm{K}} \Box \varphi \in \Phi$, then $\{\!|\Box \varphi|\!\} \in \tau$.* ***(3)*** *If $\widehat{\mathrm{K}} \mathfrak{T} \varphi \in \Phi$, then $\{\!|\mathfrak{T} \varphi|\!\} \in \tau$.* ***(4)*** *For any $T \in \tau$ and any $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$: if $T \subseteq \{\!|\varphi|\!\}$, then $T \subseteq \{\!|\Box \varphi|\!\}$.*

Here are the first steps towards locating LFP.

Definition 16 (Semi-acceptable and Acceptable). *Define $\mathcal{C}_1$ as*

$$\mathcal{C}_1 = \{T \in \tau \mid \text{there exists } \varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}} \text{ such that } \{\!|\mathfrak{T} \varphi|\!\} \subseteq T \text{ and } \widehat{\mathrm{K}} \mathfrak{T} \varphi \in \Phi\}$$

- *An open $T \in \tau$ is* semi-acceptable *if and only if, for any $\psi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ with $T \subseteq \{\!|\Box \psi|\!\}$, there is $\xi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ such that $\{\!|\mathfrak{T} \xi|\!\} \subseteq \{\!|\Box \psi|\!\}$ and $\widehat{\mathrm{K}} \mathfrak{T} \xi \in \Phi$.*
- *An open $T \in \tau$ is* acceptable *if and only if T is semi-acceptable and there is no $T' \in \tau$ such that $T \cap T' = \varnothing$ and $T' \cap T'' \neq \varnothing$ for all $T'' \in \mathcal{C}_1$.*

Define $\mathcal{C}_2$ as $\mathcal{C}_2 = \{T \in \tau \setminus \mathcal{C}_1 \mid T \text{ is acceptable}\}$.

Note that no element of $\mathcal{C}_1$ is attacked by elements of τ. Moreover,

Fact 2. ***(i)*** *For any $T \in \tau$, if $T \in \mathcal{C}_1$, then T is acceptable.* ***(ii)*** *If $T \in \tau$ is semi-acceptable, then $T \cap T' \neq \varnothing$ for all $T' \in \mathcal{C}_1$.*

Lemma 2. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then, $\mathsf{LFP} = \mathcal{C}$.*

Proof. ($\supseteq$) The proof of this direction can be fulfilled by checking two cases ***(i)*** $T \in \mathcal{C}_1$ and ***(ii)*** $T \in \mathcal{C}_2$, which is relatively simple, so we turn to the details of the other direction's proof.

($\subseteq$) Take now $T \in \tau$ such that $T \notin \mathcal{C}$; it will be shown that $T \notin \mathsf{LFP}$. The case with $T = \varnothing$ is immediate, as $\varnothing \leftarrowtail \varnothing$. Thus, suppose $T \neq \varnothing$.

From $T \notin \mathcal{C}$ it follows that $T \notin \mathcal{C}_1$, so there is no $\phi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ such that $\{\!|\mathfrak{T} \phi|\!\} \subseteq T$ and $\widehat{\mathrm{K}} \mathfrak{T} \phi \in \Phi$; hence, from $\leftarrowtail$'s definition, every $T' \in \tau$ with

$T \cap T' = \varnothing$ is such that $T \leftarrowtail T'$. It can be proved by using axiom $\mathrm{K}\Diamond\Box\varphi \to \widehat{\mathrm{K}}\mathfrak{T}\varphi$ that there is at least one $T' \in \tau$ with $T \cap T' = \varnothing$. Thus, the rest of the proof is divided into two cases: either there is $T' \in \tau$ with $T \cap T' = \varnothing$ and $T' \in \mathcal{C}$ (at least one T' contradicting T is in $\mathcal{C}$), or else for any $T' \in \tau$ with $T \cap T' = \varnothing$, $T' \notin \mathcal{C}$ (no T' contradicting T is in $\mathcal{C}$). In the first case, take any $T' \in \tau$ such that $T \cap T' = \varnothing$ and $T' \in \mathcal{C}$. Then, as it has been argued, $T \leftarrowtail T'$; moreover, as it has been proved, $\mathcal{C} \subseteq \mathsf{LFP}$. Thus, $T \notin \mathsf{LFP}$, as LFP has to be conflict-free.

In the second case, it follows that $C \in \mathcal{C}$ implies $T \cap C \neq \varnothing$. Now, consider the following two sub-cases: either T is semi-acceptable, or it is not. Next, we prove that in both two cases, $T \notin d(\mathcal{C})$. The case where T is semi-acceptable is relatively easy, so we focus on the case where T is not semi-acceptable.

If T is not semi-acceptable, there is $\varphi_T \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ such that $T \subseteq \{\!|\Box\varphi_T|\!\}$ and there is no $\psi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ such that both $\{\!|\mathfrak{T}\psi|\!\} \subseteq \{\!|\Box\varphi_T|\!\}$ and $\widehat{\mathrm{K}}\mathfrak{T}\psi \in \Phi$. In particular, φ_T itself cannot be such ψ, so either $\{\!|\mathfrak{T}\varphi_T|\!\} \nsubseteq \{\!|\Box\varphi_T|\!\}$ or else $\widehat{\mathrm{K}}\mathfrak{T}\varphi_T \notin \Phi$. But axiom $\mathfrak{T}\varphi \to \Box\varphi$ implies $\{\!|\mathfrak{T}\varphi_T|\!\} \subseteq \{\!|\Box\varphi_T|\!\}$, so $\widehat{\mathrm{K}}\mathfrak{T}\varphi_T \notin \Phi$. Now, take any $C \in \mathcal{C}_1$; let $\varphi_C \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ be one of the formulas satisfying both $\{\!|\mathfrak{T}\varphi_C|\!\} \subseteq C$ and $\widehat{\mathrm{K}}\mathfrak{T}\varphi_C \in \Phi$ (by $\mathcal{C}$'s definition, there is at least one). From theorem $\mathfrak{T}\varphi \to \Box\mathfrak{T}\varphi$, it follows that $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to (\Box\varphi_T \wedge \Box\mathfrak{T}\varphi_C)$ is a theorem too, and thus so are $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to (\Box\Box\varphi_T \wedge \Box\mathfrak{T}\varphi_C)$ (by axiom $\Box\varphi \to \Box\Box\varphi$) and $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)$ (axiom K for $\Box$). Hence, by Proposition 12, $\mathrm{K}\big((\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)\big) \in \Phi$.

So far we have $\widehat{\mathrm{K}}\mathfrak{T}\varphi_T \notin \Phi$ and, for every $C \in \mathcal{C}_1$, not only $\widehat{\mathrm{K}}\mathfrak{T}\varphi_C \in \Phi$ but also $\mathrm{K}\big((\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)\big) \in \Phi$. The first and theorem $\mathfrak{T}\varphi \leftrightarrow \mathfrak{T}\Box\varphi$ imply $\widehat{\mathrm{K}}\mathfrak{T}\Box\varphi_T \notin \Phi$; the second and axiom $\mathfrak{T}\varphi \to \mathfrak{T}\mathfrak{T}\varphi$ imply $\widehat{\mathrm{K}}\mathfrak{T}\mathfrak{T}\varphi_C \in \Phi$. These two, the third, and axiom $\big(\widehat{\mathrm{K}}\mathfrak{T}\varphi \wedge \neg\widehat{\mathrm{K}}\mathfrak{T}\psi \wedge \mathrm{K}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))\big) \to \widehat{\mathrm{K}}\Box(\varphi \wedge \neg\psi)$ imply $\widehat{\mathrm{K}}\Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \in \Phi$. For the final part, take the union of $\{\!|\Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T)|\!\}$ for all $C \in \mathcal{C}_1$, i.e.

$$S = \bigcup_{C \in \mathcal{C}_1} \{\!|\Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T)|\!\}$$

The following two facts about S (whose proof we omit here) are key to what we want to prove ($T \notin d(\mathcal{C})$): ***(i)*** $S \cap T = \varnothing$. ***(ii)*** For any $C' \in \mathcal{C}$, $C' \cap S \neq \varnothing$. Since $S \cap T = \varnothing$ and T is not semi-acceptable (so there is no $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ s.t. both $\{\!|\mathfrak{T}\varphi|\!\} \subseteq T$ and $\widehat{\mathrm{K}}\mathfrak{T}\varphi \in \Phi$), we have found an open S in τ with $T \leftarrowtail S$, according to the definition of $\leftarrowtail$. But $S \cap C \neq \varnothing$ for all $C \in \mathcal{C}$, so $S \not\leftarrowtail C$ for all $C \in \mathcal{C}$: no open in $\mathcal{C}$ attacks S. Hence, $T \notin d(\mathcal{C})$.

Therefore, regardless of whether T is semi-acceptable or not, we have $T \notin d(\mathcal{C})$. Since d is monotonic and $\mathcal{C} \subseteq \mathsf{LFP}$ (as it has been shown), it follows that $d(\mathcal{C}) \subseteq d(\mathsf{LFP}) = \mathsf{LFP}$, which implies $T \notin \mathsf{LFP}$.

Thus, in both cases $T \notin \mathcal{C}$ implies $T \notin \mathsf{LFP}$. This completes the proof.

Proposition 13 (Truth lemma). *For any $\varphi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ and any $\Gamma \in W$,*

$$\Gamma \in \{\!|\varphi|\!\}_{\mathcal{M}^\Phi} \textit{ if and only if } \Gamma \in [\![\varphi]\!]_{\mathcal{M}^\Phi}$$

Proof. The proof proceeds by induction, with the cases for atomic propositions and Boolean connectives being routine, and those for and $\Box$ and K relying on Proposition 12. Here we focus on the case for $\mathfrak{T}$.

From left to right, suppose $\Gamma \in \{\!|\, \mathfrak{T}\,\varphi |\!\}$. Then, $\mathfrak{T}\,\varphi \in \Gamma$ so, by Proposition 12, $\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \in \Phi$ which, by item 3 of Fact 1, implies $\{\!|\, \mathfrak{T}\,\varphi |\!\} \in \tau$. Now, let $T = \{\!|\, \mathfrak{T}\,\varphi |\!\}$. Then, ***(i)*** from $\{\!|\, \mathfrak{T}\,\varphi |\!\} \subseteq T$ and $\widehat{\mathrm{K}}\,\mathfrak{T}\,\varphi \in \Phi$, it follows that $T \in \mathcal{C}_1$ which, by Lemma 2, implies $T \in \mathsf{LFP}$; ***(ii)*** $\Gamma \in T$, as $\Gamma \in \{\!|\, \mathfrak{T}\,\varphi |\!\}$; ***(iii)*** from axiom $\mathfrak{T}\,\varphi \rightarrow \varphi$ it follows that $T \subseteq \{\!|\varphi|\!\}$ which, by inductive hypothesis $\{\!|\varphi|\!\} = [\![\varphi]\!]$, implies $T \subseteq [\![\varphi]\!]$. Hence, by $\mathfrak{T}$'s truth condition, $\Gamma \in [\![\mathfrak{T}\,\varphi]\!]$.

From right to left, suppose $\Gamma \in [\![\mathfrak{T}\,\varphi]\!]$. Then, by $\mathfrak{T}$'s truth condition, there is $T \in \mathsf{LFP}$ with $\Gamma \in T$ and $T \subseteq [\![\varphi]\!]$. But, from LFP's definition, $\Gamma \in T$ implies $\leqslant[\Gamma] \subseteq T$; hence, by $\Box$'s truth condition, $T \subseteq [\![\Box\,\varphi]\!]$. The inductive hypothesis implies $[\![\Box\,\varphi]\!] = \{\!|\, \Box\,\varphi |\!\}$, so then we have $\Gamma \in T$ and $T \subseteq \{\!|\, \Box\,\varphi |\!\}$.

By Lemma 2, $\mathsf{LFP} = \mathcal{C}_1 \cup \mathcal{C}_2$; thus, $T \in \mathcal{C}_1 \cup \mathcal{C}_2$. Suppose $T \in \mathcal{C}_1$; then there is $\psi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ with $\{\!|\, \mathfrak{T}\,\psi |\!\} \subseteq T$ and $\widehat{\mathrm{K}}\,\mathfrak{T}\,\psi \in \Phi$. Thus, $\{\!|\, \mathfrak{T}\,\psi |\!\} \subseteq T \subseteq \{\!|\, \Box\,\varphi |\!\}$, so $\mathrm{K}(\mathfrak{T}\,\psi \rightarrow \Box\,\varphi) \in \Phi$. Now, take any $\Delta \in \{\!|\, \mathfrak{T}\,\psi |\!\}$; then, $\mathrm{K}(\mathfrak{T}\,\psi \rightarrow \Box\,\varphi) \in \Delta$. This, together with theorem $\mathrm{K}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\psi)$ (Proposition 4), implies $\mathfrak{T}\,\mathfrak{T}\,\psi \rightarrow \mathfrak{T}\,\Box\,\varphi \in \Delta$. Moreover: $\Delta \in \{\!|\, \mathfrak{T}\,\psi |\!\}$ implies $\Delta \in \{\!|\, \mathfrak{T}\,\mathfrak{T}\,\psi |\!\}$, so $\Delta \in \{\!|\, \mathfrak{T}\,\Box\,\varphi |\!\}$, that is, $\mathfrak{T}\,\Box\,\varphi \in \Delta$. The latter, together with theorem $\mathfrak{T}\,\varphi \leftrightarrow \mathfrak{T}\,\Box\,\varphi$ and axiom $\mathfrak{T}\,\varphi \rightarrow \mathrm{K}(\Box\,\varphi \rightarrow \mathfrak{T}\,\varphi)$, imply $\mathrm{K}(\Box\,\varphi \rightarrow \mathfrak{T}\,\varphi) \in \Delta$, and thus $\mathrm{K}(\Box\,\varphi \rightarrow \mathfrak{T}\,\varphi) \in \Phi$. Hence, $\{\!|\, \Box\,\varphi |\!\} \subseteq \{\!|\, \mathfrak{T}\,\varphi |\!\}$ and thus, since $\Gamma \in T$ and $T \subseteq \{\!|\, \Box\,\varphi |\!\}$, we have $\Gamma \in \{\!|\, \mathfrak{T}\,\varphi |\!\}$. Otherwise, $T \in \mathcal{C}_2$, and hence for any $\psi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ with $T \subseteq \{\!|\, \Box\,\psi |\!\}$ there is $\xi \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ with $\{\!|\, \mathfrak{T}\,\xi |\!\} \subseteq \{\!|\, \Box\,\psi |\!\}$ and $\widehat{\mathrm{K}}\,\mathfrak{T}\,\xi \in \Phi$. Thus, since φ is such that $T \subseteq \{\!|\, \Box\,\varphi |\!\}$, there is $\eta \in \mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$ such that $\{\!|\, \mathfrak{T}\,\eta |\!\} \subseteq \{\!|\, \Box\,\varphi |\!\}$ and $\widehat{\mathrm{K}}\,\mathfrak{T}\,\eta \in \Phi$. From here we can repeat the argument used in the case of $T \in \mathcal{C}_1$ in order to get $\Gamma \in \{\!|\, \mathfrak{T}\,\varphi |\!\}$ again. Thus, in both cases, $\Gamma \in \{\!|\, \mathfrak{T}\,\varphi |\!\}$, which completes the proof.

Lemma 3. $\mathcal{M}^{\Phi}$ *is Alexandroff.*

Proof. Whether $\mathcal{M}^{\Phi}$ is Alexandroff has nothing to do with $\leftarrowtail$; thus, we can apply Proposition 5.6.15 in [19], which states that if $\tau = \{\bigcup_{\Gamma \in U} \leqslant[\Gamma] \mid U \subseteq W\}$ then $\mathcal{M}^{\Phi}$ is Alexandroff. But item 1 of Fact 1 and the definition of $\mathcal{E}_0$ imply the required condition; then, $\mathcal{M}^{\Phi}$ is Alexandroff.

Since $\mathcal{M}^{\Phi}$ is Alexandroff, Proposition 11 tells us it has a modally equivalent topological argumentation model. Hence, the $\mathsf{L}_{\Box,\mathrm{K},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{K},\mathfrak{T}}$-formulas Φ_0 is satisfiable in a topological argumentation model.

References

1. Hintikka, J.: Knowledge and Belief. Cornell University Press, Ithaca (1962)
2. Board, O.: Dynamic interactive epistemology. Games Econ. Behav. **49**(1), 49–80 (2004)
3. Baltag, A., Smets, S.: A qualitative theory of dynamic interactive belief revision. Texts Log. Games **3**, 9–58 (2008)

4. Spohn, W.: Ordinal conditional functions: a dynamic theory of epistemic states. In: Harper, W.L., Skyrms, B. (eds.) Causation in Decision, Belief Change, and Statistics, pp. 105–134. Kluwer, Dordrecht (1988)
5. van Fraassen, B.C.: Fine-grained opinion, probability, and the logic of full belief. J. Philos. Log. **24**(4), 349–377 (1995)
6. Baltag, A., Smets, S.: Probabilistic dynamic belief revision. In: Johan van Benthem, S.J., Veltman, F. (eds.) A Meeting of the Minds: Proceedings of the Workshop on Logic, Rationality and Interaction, vol. 8 (2007)
7. Doyle, J.: A truth maintenance system. Artif. Intell. **12**(3), 231–272 (1979)
8. de Kleer, J.: An assumption-based TMS. Artif. Intell. **28**(2), 127–162 (1986)
9. Dung, P.M.: On the acceptability of arguments and its fundamental role in non-monotonic reasoning, logic programming and n-person games. Artif. Intell. **77**, 321–357 (1995)
10. Grossi, D.: Abstract argument games via modal logic. Synthese **190**, 5–29 (2013)
11. Caminada, M.W.A., Gabbay, D.M.: A logical account of formal argumentation. Stud. Log. **93**(2–3), 109 (2009)
12. Grossi, D., van der Hoek, W.: Justified beliefs by justified arguments. In: Proceedings of the Fourteenth International Conference on Principles of Knowledge Representation and Reasoning, pp. 131–140. AAAI Press (2014)
13. Artemov, S.N.: The logic of justification. Rev. Symb. Log. **1**(4), 477–513 (2008)
14. van Benthem, J., Pacuit, E.: Dynamic logics of evidence-based beliefs. Stud. Log. **99**(1–3), 61–92 (2011)
15. van Benthem, J., Duque, D.F., Pacuit, E.: Evidence and plausibility in neighborhood structures. Ann. Pure Appl. Log. **165**(1), 106–133 (2014)
16. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: Justified belief and the topology of evidence. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 83–103. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_6
17. Shi, C., Smets, S., Velázquez-Quesada, F.R.: Argument-based belief in topological structures. In: Lang, J. (ed.) Proceedings Sixteenth Conference on Theoretical Aspects of Rationality and Knowledge, TARK 2017, University of Liverpool, Liverpool, UK, 24–26 June 2017, Volume 251 of Electronic Proceedings in Theoretical Computer Science, pp. 489–503. Open Publishing Association (2017)
18. van Benthem, J., Fernández-Duque, D., Pacuit, E.: Evidence logic: a new look at neighborhood structures. In: Bolander, T., Braüner, T., Ghilardi, S., Moss, L. (eds.) Proceedings of Advances in Modal Logic, vol. 9, pp. 97–118. King's College Press (2012)
19. Özgün, A.: Evidence in epistemic logic: a topological perspective. Ph.D. thesis. Institute for Logic, Language and Computation, Amsterdam, The Netherlands, ILLC Dissertation series DS-2017-07, October 2017
20. Knaster, B.: Un théorème sur les fonctions d'ensembles. Annales de la Société Polonaise de Mathématiques **6**, 133–134 (1928)
21. Tarski, A.: A lattice-theoretical theorem and its applications. Pac. J. Math. **5**(2), 285–309 (1955)