



Algebraic Semantics for Nelson's Logic \mathcal{S}

Thiago Nascimento¹(✉), Umberto Rivieccio², João Marcos²,
and Matthew Spinks³

¹ Programa de Pós-graduação em Sistemas e Computação, UFRN, Natal, Brazil
thiagnascsilva@gmail.com

² Departamento de Informática e Matemática Aplicada, UFRN, Natal, Brazil
{urivieccio, jmarcos}@dimap.ufrn.br

³ Università degli Studi di Cagliari, Cagliari, Italy
mspinksau@yahoo.com.au

Abstract. Besides the better-known Nelson's Logic and Paraconsistent Nelson's Logic, in "Negation and separation of concepts in constructive systems" (1959), David Nelson introduced a logic called \mathcal{S} with the aim of analyzing the constructive content of provable negation statements in mathematics. Motivated by results from Kleene, in "On the Interpretation of Intuitionistic Number Theory" (1945), Nelson investigated a more symmetric recursive definition of truth, according to which a formula could be either primitively verified or refuted. The logic \mathcal{S} was defined by means of a calculus lacking the contraction rule and having infinitely many schematic rules, and no semantics was provided. This system received little attention from researchers; it even remained unnoticed that on its original presentation it was inconsistent. Fortunately, the inconsistency was caused by typos and by a rule whose hypothesis and conclusion were swapped. We investigate a corrected version of the logic \mathcal{S} , and focus on its propositional fragment, showing that it is algebraizable in the sense of Blok and Pigozzi (in fact, implicative) with respect to a certain class of involutive residuated lattices. We thus introduce the first (algebraic) semantics for \mathcal{S} as well as a finite Hilbert-style calculus equivalent to Nelson's presentation; we also compare \mathcal{S} with the other two above-mentioned logics of the Nelson family. Our approach is along the same lines of (and partly relies on) previous algebraic work on Nelson's logics due to M. Busaniche, R. Cignoli, S. Odintsov, M. Spinks and R. Veroff.

Keywords: Nelson's logics · Involutive residuated lattices
Algebraic semantics · Algebraic logic

1 Introduction

To study the notion of constructible falsity, David Nelson introduced a number of systems of non-classical logic that combine an intuitionistic approach to truth with a dual-intuitionistic treatment of falsity. Nelson's logics (\mathcal{S} , $\mathcal{N}3$, and $\mathcal{N}4$)

accept some notable theorems of classical logic, such as $\sim\sim\varphi \Leftrightarrow \varphi$, while rejecting others, such as $(\varphi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \psi)$ and $(\varphi \wedge \sim\varphi) \Rightarrow \psi$. Nelson introduced these logics with the aim of studying constructive proofs in Number Theory. To such an end, he gave a definition of truth [14, Definition 1] (analogous to Kleene’s [12, p. 112]) according to which either a formula or its negation should be realized by some natural number.

Nelson’s logic $\mathcal{N}3$ was introduced in [14] and $\mathcal{N}4$, a paraconsistent version of $\mathcal{N}3$, was introduced in [1]. $\mathcal{N}3$ is in fact an axiomatic extension of $\mathcal{N}4$ by the axiom¹ $\sim\varphi \rightarrow (\varphi \rightarrow \psi)$. The logic $\mathcal{N}3$ is by now well studied, both via a proof-theoretic approach and through algebraic methods; in particular, Odintsov [16] proved that $\mathcal{N}4$ (thus also $\mathcal{N}3$) is algebraizable à la Blok-Pigozzi [3].

In [15] Nelson also introduced the logic \mathcal{S} , aimed at the study of realizability. As suggested by Humberstone [11, Chap. 8.2, p. 1239–1240], the introduction of \mathcal{S} can perhaps also be viewed as an attempt to remedy what some logicians consider an undesirable feature of $\mathcal{N}3$ (and $\mathcal{N}4$), namely the fact that there are formulas φ, ψ in $\mathcal{N}3$ (and $\mathcal{N}4$) that are mutually interderivable but such that their negations $\sim\varphi, \sim\psi$ fail to be interderivable; and also a formula such as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ is a theorem of $\mathcal{N}3$ (and $\mathcal{N}4$) but $(\sim\varphi \rightarrow \sim\psi) \wedge (\sim\psi \rightarrow \sim\varphi)$ is not. It is useful to recall that these two phenomena are in general disassociated; the latter stems from the failure of the contraposition law for the so-called *weak implication* connective \rightarrow of $\mathcal{N}3$ (and $\mathcal{N}4$), while the former entails that $\mathcal{N}3$ and $\mathcal{N}4$ are non-congruential (or, as other authors say, non-self-extensional) logics: that is, the logical interderivability relation fails to be a congruence of the formula algebra. Now, while \mathcal{S} is also a non-congruential logic, its implication connective (here denoted \Rightarrow) does satisfy the contraposition law: in fact in \mathcal{S} one has that $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ is a theorem if and only if $(\sim\varphi \Rightarrow \sim\psi) \wedge (\sim\psi \Rightarrow \sim\varphi)$ is a theorem. In other words \mathcal{S} , although non-congruential if we look at its interderivability relation, enjoys at least \Leftrightarrow -congruentiality in Humberstone’s terminology (relative to the bi-implication \Leftrightarrow defined, in the usual way, as follows: $\varphi \Leftrightarrow \psi := (\sim\varphi \Rightarrow \sim\psi) \wedge (\sim\psi \Rightarrow \sim\varphi)$)².

Nelson’s original presentation of \mathcal{S} has infinitely many schematic rules and no algebraic semantics; [15] also leaves unclear whether $\mathcal{N}3$ is comparable with \mathcal{S} (and if so, which of the two is stronger). Unlike its relatives $\mathcal{N}3$ and $\mathcal{N}4$, the logic \mathcal{S} received little attention after [15] and basic questions about it were left open, for example: Is \mathcal{S} algebraizable? Can \mathcal{S} be finitely axiomatized? What are the exact relations between \mathcal{S} and $\mathcal{N}3$, and between \mathcal{S} and $\mathcal{N}4$? In the present paper we will use the modern techniques of algebraic logic to answer these questions.

Our study will follow the same lines of previous papers by Busaniche, Cignoli, Odintsov, Spinks and Veroff on (algebraic models of) $\mathcal{N}3$ and $\mathcal{N}4$ (see,

¹ The presence of two implications, the *strong* one (\Rightarrow) mentioned earlier and the *weak* one (\rightarrow), is a distinctive feature of Nelson’s logics; more on this below.

² Actually, we now know that $\mathcal{N}3$ (and $\mathcal{N}4$) are also \Leftrightarrow -congruential for a suitable choice of implication \Rightarrow (called *strong implication*) that can be defined using the weak one \rightarrow ; but Nelson may well not have been aware of this while writing [15].

e.g., [5, 6, 16, 19, 20]), which in turn rely on classic work by H. Rasiowa on the algebraization of non-classical logics. These investigations have shown that the algebraic approach to Nelson's logics may be particularly insightful, as it allows to view them as either conservative expansions of the negation-free fragment of intuitionistic logic by the addition of a new unary logical connective of *strong negation* (\sim) or as axiomatic extensions of well-known substructural/relevance logics. The first perspective allows us to establish a particularly useful link between algebraic models of $\mathcal{N}3/\mathcal{N}4$ and models of intuitionistic logic (via the so-called *twist-structure* construction—see especially [16]), while the second affords the possibility of exploiting general results and techniques that have been introduced in the study of residuated structures; this is the approach of [19, 20] as well as [13], and that we shall also take in the present paper (see especially Subsect. 3.2).

The paper is organized as follows. In Sect. 2 we present the propositional fragment of the logic \mathcal{S} and highlight some of its theorems, which will later be used to establish its algebraizability. In Sect. 3 we prove that \mathcal{S} is algebraizable and present its equivalent algebraic semantics. In Sect. 4 we provide another calculus for \mathcal{S} , one that has a finite number of schematic axioms and only one schematic rule (*modus ponens*). We point out that having only one rule makes it easy to prove the Deduction Metatheorem in the standard way using induction over derivations. In Sect. 5 we prove that $\mathcal{N}3$ is a proper axiomatic extension of \mathcal{S} , and that \mathcal{S} and $\mathcal{N}4$ are not extensions of each other. Proofs of some of the main new results are to be found in an Appendix to this paper.

2 Nelson's Logic \mathcal{S}

In this section we recall Nelson's original presentation of the propositional fragment of \mathcal{S} [15] and we highlight some theorems of \mathcal{S} that will be used further on to establish its algebraizability.

As is now usual, here we take a sentential logic \mathcal{L} to be a structure containing a substitution-invariant consequence relation $\vdash_{\mathcal{L}}$ defined over an algebra of formulas \mathbf{Fm} freely generated by a denumerable set of propositional variables $\{p, q, r, \dots\}$ over a given language Σ . We will henceforth refer to algebras using boldface strings (such as \mathbf{Fm} and \mathbf{A}), and use the corresponding italicized version of these same strings (such as Fm and A) to refer to their corresponding carriers. Fixing a given logic, we will use φ, ψ and γ , possibly decorated with subscripts, to refer to arbitrary formulas of it.

Definition 1. *Nelson's logic $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ is the sentential logic in the language $\langle \wedge, \vee, \Rightarrow, \sim, \perp \rangle$ of type $\langle 2, 2, 2, 1, 0 \rangle$ defined by the Hilbert-style calculus with the schematic axioms and rules listed below. As usual, $\varphi \Leftrightarrow \psi$ will be used to abbreviate $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.*

Axioms

$$(A1) \quad \varphi \Rightarrow \varphi$$

- (A2) $\perp \Rightarrow \varphi$
- (A3) $\sim\varphi \Rightarrow (\varphi \Rightarrow \perp)$
- (A4) $\sim\perp$
- (A5) $(\varphi \Rightarrow \psi) \Leftrightarrow (\sim\psi \Rightarrow \sim\varphi)$

Rules

$$\frac{\Gamma \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \gamma))}{\Gamma \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \gamma))} \text{ (P)} \quad \frac{\varphi \Rightarrow (\varphi \Rightarrow (\varphi \Rightarrow \gamma))}{\varphi \Rightarrow (\varphi \Rightarrow \gamma)} \text{ (C)} \quad \frac{\Gamma \Rightarrow \varphi \quad \varphi \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \text{ (E)}$$

$$\frac{\Gamma \Rightarrow \varphi \quad \psi \Rightarrow \gamma}{\Gamma \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \gamma)} \text{ (\Rightarrow 1)} \quad \frac{\gamma}{\varphi \Rightarrow \gamma} \text{ (\Rightarrow r)} \quad \frac{\varphi \Rightarrow \gamma}{(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\wedge 11)}$$

$$\frac{\psi \Rightarrow \gamma}{(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\wedge 12)} \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow (\varphi \wedge \psi)} \text{ (\wedge r)} \quad \frac{\varphi \Rightarrow \gamma \quad \psi \Rightarrow \gamma}{(\varphi \vee \psi) \Rightarrow \gamma} \text{ (\vee 11)}$$

$$\frac{\varphi \Rightarrow^2 \gamma \quad \psi \Rightarrow^2 \gamma}{((\varphi \vee \psi) \Rightarrow^2 \gamma)} \text{ (\vee 12)} \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow (\varphi \vee \psi)} \text{ (\vee r1)} \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow (\varphi \vee \psi)} \text{ (\vee r2)}$$

$$\frac{(\varphi \wedge \sim\psi) \Rightarrow \gamma}{\sim(\varphi \Rightarrow \psi) \Rightarrow \gamma} \text{ (\sim \Rightarrow 1)} \quad \frac{\Gamma \Rightarrow^2 (\varphi \wedge \sim\psi)}{\Gamma \Rightarrow^2 \sim(\varphi \Rightarrow \psi)} \text{ (\sim \Rightarrow r)} \quad \frac{(\sim\varphi \vee \sim\psi) \Rightarrow \gamma}{\sim(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\sim \wedge 1)}$$

$$\frac{\Gamma \Rightarrow (\sim\varphi \vee \sim\psi)}{\Gamma \Rightarrow \sim(\varphi \wedge \psi)} \text{ (\sim \wedge r)} \quad \frac{(\sim\varphi \wedge \sim\psi) \Rightarrow \gamma}{\sim(\varphi \vee \psi) \Rightarrow \gamma} \text{ (\sim \vee 1)} \quad \frac{\Gamma \Rightarrow (\sim\varphi \wedge \sim\psi)}{\Gamma \Rightarrow \sim(\varphi \vee \psi)} \text{ (\sim \vee r)}$$

$$\frac{\varphi \Rightarrow \gamma}{\sim\sim\varphi \Rightarrow \gamma} \text{ (\sim\sim 1)} \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \sim\sim\varphi} \text{ (\sim\sim r)}$$

In the above rules, following Nelson’s notation, $\Gamma = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a finite set of formulas and the following abbreviations are employed:

- $\Gamma \Rightarrow \varphi := \varphi_1 \Rightarrow (\varphi_2 \Rightarrow (\dots \Rightarrow (\varphi_n \Rightarrow \varphi) \dots))$
- $\varphi \Rightarrow^2 \psi := \varphi \Rightarrow (\varphi \Rightarrow \psi)$
- $\Gamma \Rightarrow^2 \varphi := \varphi_1 \Rightarrow^2 (\varphi_2 \Rightarrow^2 (\dots \Rightarrow^2 (\varphi_n \Rightarrow^2 \varphi) \dots))$

Moreover, when $\Gamma = \emptyset$, we take $\Gamma \Rightarrow \varphi := \varphi$.

Notice that we have fixed obvious infelicities in the rules $(\wedge 12)$, $(\wedge r)$ and $(\sim \Rightarrow r)$ as they appear in [15, pp. 214–215]. For example, the original rule $(\wedge 12)$ in Nelson’s paper was:

$$\frac{(\varphi \wedge \psi) \Rightarrow \gamma}{\psi \Rightarrow \gamma} \text{ (\wedge 12)}$$

This clearly makes the logic inconsistent. Indeed, taking $\varphi = \gamma$, we have:

$$\frac{(\gamma \wedge \psi) \Rightarrow \gamma}{\psi \Rightarrow \gamma} \text{ (\wedge 12)}$$

Now, since $(\gamma \wedge \psi) \Rightarrow \gamma$ is a theorem (see Proposition 1, below), $\psi \Rightarrow \gamma$ is a theorem too. Choosing ψ as an axiom, we would conclude thus that γ is a theorem for any formula γ .

We note in passing that the rule (C), called *weak condensation* by Nelson, replaces (and is indeed a weaker form of) the usual contraction rule:

$$\frac{\varphi \Rightarrow (\varphi \Rightarrow \psi)}{\varphi \Rightarrow \psi}$$

Rule (C) is also known in the literature as *3-2 contraction* [17, p. 389] and corresponds, on algebraic models, to the property of *three-potency* (see Subsect. 3.2).

Also, do note that we obtain *modus ponens*, (MP), by taking $\Gamma = \emptyset$ in rule (E):

$$\frac{\varphi \quad \varphi \Rightarrow \gamma}{\gamma}$$

It is worth noticing that, despite appearances, Nelson's system \mathcal{S} is a Hilbert-style calculus, rather than a sequent system. Its underlying notion of derivation, $\vdash_{\mathcal{S}}$, is the usual one. Henceforth, for any logic \mathcal{L} and any set of formulas $\Gamma \cup \Pi$, we shall write $\Gamma \vdash_{\mathcal{L}} \Pi$ to say that $\Gamma \vdash_{\mathcal{L}} \pi$ for every $\pi \in \Pi$. By $\Gamma \dashv\vdash_{\mathcal{L}} \Pi$ we will abbreviate the double assertion $\Gamma \vdash_{\mathcal{L}} \Pi$ and $\Pi \vdash_{\mathcal{L}} \Gamma$.

One of the crucial steps in proving that a logic is algebraizable (in the sense of Blok and Pigozzi [3, Definition 2.2]) is to prove that it satisfies certain congruence properties. In the present context, this entails checking that $\varphi \Leftrightarrow \psi \vdash_{\mathcal{S}} \sim\varphi \Leftrightarrow \sim\psi$ and $\{\varphi_1 \Leftrightarrow \psi_1, \varphi_2 \Leftrightarrow \psi_2\} \vdash_{\mathcal{S}} (\varphi_1 \bullet \varphi_2) \Leftrightarrow (\psi_1 \bullet \psi_2)$ for each connective $\bullet \in \{\wedge, \vee, \Rightarrow\}$. The following auxiliary results will be used to prove that much, in the next section.

Proposition 1. *The following formulas are theorems of \mathcal{S} :*

1. $(\varphi \wedge \psi) \Rightarrow \varphi$
2. $(\varphi \wedge \psi) \Rightarrow \psi$
3. $\varphi \Rightarrow (\varphi \vee \psi)$
4. $\psi \Rightarrow (\varphi \vee \psi)$
5. $(\varphi \Rightarrow (\psi \Rightarrow \gamma)) \Leftrightarrow (\psi \Rightarrow (\varphi \Rightarrow \gamma))$

Proof. All justifying derivations are straightforward. We detail the first item, as an example:

$$\frac{\overline{\varphi \Rightarrow \varphi} \text{ (A1)}}{(\varphi \wedge \psi) \Rightarrow \varphi} \text{ (\wedge 11)}$$

Proposition 2. $\{\varphi \Leftrightarrow \psi\} \dashv\vdash_{\mathcal{S}} \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$.

Proof. Such a logical equivalence is easily justified by Proposition 1(1–2) and by considering the rule $(\wedge \mathbf{r})$ with $\Gamma = \emptyset$.

3 \mathcal{S} Is Algebraizable

In this section we prove that \mathcal{S} is algebraizable in the sense of Blok and Pigozzi (it is, in fact, implicative [8, Definition 2.3]), and we give two alternative presentations for its equivalent algebraic semantics (to be called ‘ \mathcal{S} -algebras’). The first one is obtained via the algorithm of [3, Theorem 2.17], while the second one is closer to the usual axiomatizations of classes of residuated lattices, which are the algebraic counterparts of many logics in the substructural family. As a particular advantage, the second presentation of \mathcal{S} -algebras will allow us to see at a glance that they form an equational class, and will also make it easier to compare them with other known classes of algebras for substructural logics.

Definition 2. *An implicative logic is a sentential logic \mathcal{L} whose underlying algebra of formulas in a language Σ has a term $\alpha(p, q)$ in two variables that satisfies the following conditions:*

- [IL1] $\vdash_{\mathcal{L}} \alpha(p, p)$
- [IL2] $\alpha(p, q), \alpha(q, r) \vdash_{\mathcal{L}} \alpha(p, r)$
- [IL3] $p, \alpha(p, q) \vdash_{\mathcal{L}} q$
- [IL4] $q \vdash_{\mathcal{L}} \alpha(p, q)$
- [IL5] *for each n -ary $\bullet \in \Sigma$,*
 $\bigcup_{i=1}^n \{ \alpha(p_i, q_i), \alpha(q_i, p_i) \} \vdash_{\mathcal{L}} \alpha(\bullet(p_1, \dots, p_n), \bullet(q_1, \dots, q_n))$

We call any such α an \mathcal{L} -implication.

Given an algebra of formulas \mathbf{Fm} of the language Σ , the associated set $Fm \times Fm$ of equations will henceforth be denoted by \mathbf{Eq} ; we will write $\varphi \approx \psi$ rather than $(\varphi, \psi) \in \mathbf{Eq}$. Let \mathbf{A} be an algebra with the same similarity type as \mathbf{Fm} . A homomorphism $\mathcal{V}: \mathbf{Fm} \rightarrow \mathbf{A}$ is called a *valuation in \mathbf{A}* . We say that a valuation \mathcal{V} in \mathbf{A} *satisfies $\varphi \approx \psi$ in \mathbf{A}* when $\mathcal{V}(\varphi) = \mathcal{V}(\psi)$; we say that an algebra \mathbf{A} *satisfies $\varphi \approx \psi$* when all valuations in \mathbf{A} satisfy it.

Definition 3. *A logic \mathcal{L} is algebraizable if and only if there are equations $\mathbf{E}(\varphi) \subseteq \mathbf{Eq}$ and a transform $\mathbf{Eq} \xrightarrow{\rho} 2^{Fm}$, denoted by $\Delta(\varphi, \psi) := \rho(\varphi \approx \psi)$, such that \mathcal{L} respects the following conditions:*

- [Alg] $\varphi \dashv\vdash_{\mathcal{L}} \Delta(\mathbf{E}(\varphi))$
- [Ref] $\vdash_{\mathcal{L}} \Delta(\varphi, \varphi)$
- [Sym] $\Delta(\varphi, \psi) \vdash_{\mathcal{L}} \Delta(\psi, \varphi)$
- [Trans] $\Delta(\varphi, \psi) \cup \Delta(\psi, \gamma) \vdash_{\mathcal{L}} \Delta(\varphi, \gamma)$
- [Cong] *for each n -ary $\bullet \in \Sigma$,*
 $\bigcup_{i=1}^n \Delta(\varphi_i, \psi_i) \vdash_{\mathcal{L}} \Delta(\bullet(\varphi_1, \dots, \varphi_n), \bullet(\psi_1, \dots, \psi_n))$

We call any such $\mathbf{E}(\varphi)$ the set of defining equations and any such $\Delta(\varphi, \psi)$ the set of equivalence formulas of \mathcal{L} .

Clarifying the notation in [Alg], recall that the set $\mathbf{E}(\varphi)$ contains pairs of formulas and we write $\varphi \approx \psi$ simply as syntactic sugar for a pair (φ, ψ) belonging to this set. Now, $\Delta(\varphi, \psi)$ transforms an equation into a set of formulas. Accordingly, we take $\Delta(\mathbf{E}(\varphi))$ as $\bigcup \{ \Delta(\varphi_1, \varphi_2) \mid (\varphi_1, \varphi_2) \in \mathbf{E}(\varphi) \}$. Similarly, we shall let $\mathbf{E}(\Delta(\varphi, \psi))$ stand for $\bigcup \{ \mathbf{E}(\chi) \mid \chi \in \Delta(\varphi, \psi) \}$.

Definition 4. Let \mathcal{L} be an implicative logic in the language Σ , having an \mathcal{L} -implication α . An \mathcal{L} -algebra \mathbf{A} is a Σ -algebra such that $1 \in A$ and:

[LALG1] For all $\Gamma \cup \{\varphi\} \subseteq Fm$ and every valuation \mathcal{V} in \mathbf{A} ,

if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\mathcal{V}(\Gamma) \subseteq \{1\}$, then $\mathcal{V}(\varphi) = 1$.

[LALG2] For all $a, b \in A$, if $\alpha(a, b) = 1$ and $\alpha(b, a) = 1$, then $a = b$.

The class of \mathcal{L} -algebras is denoted by $Alg^* \mathcal{L}$.

Every implicative logic \mathcal{L} is algebraizable with respect to the class $Alg^* \mathcal{L}$ [8, Proposition 3.15], and such algebraizability is witnessed by the defining equations $E(\varphi) := \{\varphi \approx \alpha(\varphi, \varphi)\}$ and the equivalence formulas $\Delta(\varphi, \psi) := \{\alpha(\varphi, \psi), \alpha(\psi, \varphi)\}$. These are in fact the sets of defining equations and of equivalence formulas that we will use in the remainder of the present paper.

We can now prove (the details are to be found in the Appendix) that:

Theorem 1. The calculus $\vdash_{\mathcal{S}}$ is implicative and thus algebraizable. The \mathcal{S} -implication is given by \Rightarrow , that is, $\alpha(p, q) := p \Rightarrow q$.

In the case of \mathcal{S} we have thus that $E(\varphi) = \{\varphi \approx \varphi \Rightarrow \varphi\}$ and $\Delta(\varphi, \psi) = \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$.

3.1 \mathcal{S} -algebras

By Blok-Pigozzi's algorithm ([3, Theorem 2.17], see also [8, Proposition 3.44]), we know that the equivalent algebraic semantics of \mathcal{S} is the class of algebras given by Definition 5 below. We denote by Ax the set of axioms and denote by Inf R the set the inference rules of \mathcal{S} , given in Definition 1.

Definition 5. An \mathcal{S} -algebra is a structure $\mathbf{A} = \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 2, 1, 0, 0 \rangle$ that satisfies the following equations and quasiequations:

1. $E(\Delta(\varphi, \varphi))$
2. $E(\Delta(\varphi, \psi))$ implies $\varphi \approx \psi$
3. $E(\varphi)$, for each $\varphi \in Ax$
4. $\bigcup_{i=1}^n E(\gamma_i)$ implies $\varphi \approx 1$ for each $\gamma_1, \dots, \gamma_n \vdash_{\mathcal{S}} \varphi \in \text{Inf R}$

Regarding the notation in the above definition, $E(\Delta(\varphi, \varphi))$ stands for the equation $\varphi \Rightarrow \varphi \approx (\varphi \Rightarrow \varphi) \Rightarrow (\varphi \Rightarrow \varphi)$. Item 2 is the quasiequation: $(\varphi \Rightarrow \psi) \approx (\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$ and $(\psi \Rightarrow \varphi) \approx (\psi \Rightarrow \varphi) \Rightarrow (\psi \Rightarrow \varphi)$ implies $\varphi \approx \psi$; $E(\varphi)$ is the equation $\varphi \approx \varphi \Rightarrow \varphi$ for each axiom φ of \mathcal{S} . In fact, these conditions are telling us that for each axiom φ of \mathcal{S} we have the equation $\varphi \approx 1$, and for each rule $\varphi \vdash_{\mathcal{S}} \psi$ of \mathcal{S} , in the corresponding algebras we have the quasiequation: if $\varphi \approx 1$, then $\psi \approx 1$.

We shall denote by $E(A_n)$ the equation given in Definition 5(3) for the axiom A_n (for $1 \leq n \leq 5$), and by $Q(R)$ the quasiequation given in Definition 5(4) for the rule R of \mathcal{S} . From this point on, in this subsection, in order to make the

propositions and their proofs shorter, we shall also use the following abbreviations:

$$x * y := \sim(x \Rightarrow \sim y)$$

$$x^2 := x * x$$

$$x^n := x * (x^{n-1}), \text{ for } n > 2$$

The following result, whose proof may be found in the Appendix, will help us in checking that the class of \mathcal{S} -algebras forms a variety.

Proposition 3. *Let \mathbf{A} be an \mathcal{S} -algebra and let $a, b, c \in A$. Then:*

1. $a \Rightarrow a = 1 = \sim 0$.
2. The relation \leq defined by setting $a \leq b$ iff $a \Rightarrow b = 1$, is a partial order with maximum 1 and minimum 0.
3. $a \Rightarrow b = \sim b \Rightarrow \sim a$.
4. $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$.
5. $\sim \sim a = a$ and $a \Rightarrow 0 = \sim a$.
6. $\langle A, *, 1 \rangle$ is a commutative monoid.
7. $(a * b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$.
8. The pair $(*, \Rightarrow)$ is residuated with respect to \leq , i.e., $a * b \leq c$ iff $b \leq a \Rightarrow c$.
9. $a^2 \leq a^3$.
10. $\langle A, \wedge, \vee \rangle$ is a lattice with order \leq .
11. $(a \vee b)^2 \leq a^2 \vee b^2$.

In the next section we introduce an equivalent presentation of \mathcal{S} -algebras which takes precisely the properties of Proposition 3 above as postulates.

3.2 Alternative Presentation of \mathcal{S} -algebras

We start here by recalling the following standard definition [9, p. 185]:

Definition 6. *A commutative integral bounded residuated lattice (CIBRL) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that:*

1. $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice with ordering \leq , minimum element 0 and maximum element 1.
2. $\langle A, *, 1 \rangle$ is a commutative monoid.
3. The pair $(*, \Rightarrow)$ is residuated with respect to \leq , i.e., $a * b \leq c$ iff $b \leq a \Rightarrow c$.

In the context of the above definition, the *integrality* condition corresponds to having 1 not only as a maximum but also as the multiplicative unit of the operation $*$, that is, $x * 1 = x$. For a CIBRL this condition immediately follows from Definition 6(1–2).

Setting $\sim x := x \Rightarrow 0$, we say that a residuated lattice is *involutive* [10, p. 186] when $\sim \sim a = a$ (in such a case, it follows that $a \Rightarrow b = \sim b \Rightarrow \sim a$). We say that a residuated lattice is *3-potent* when it satisfies the equation $x^2 \leq x^3$. While we have earlier defined $*$ from \Rightarrow , and now $*$ is a primitive operation, now we can show that every CIBRL satisfies $x * y = \sim(x \Rightarrow \sim y)$ (see [10, Lemma 5.1]).

Definition 7. *An \mathcal{S}' -algebra is an involutive 3-potent CIBRL.*

The proof of the following result may be found in the Appendix:

- Lemma 1.** 1. Any CIBRL satisfies the equation $(x \vee y) * z \approx (x * z) \vee (y * z)$.
 2. Any CIBRL satisfies $x^2 \vee y^2 \approx (x^2 \vee y^2)^2$.
 3. Any 3-potent CIBRL satisfies $(x \vee y^2)^2 \approx (x \vee y)^2$.
 4. Any 3-potent CIBRL satisfies $(x \vee y)^2 \approx x^2 \vee y^2$.

Since involutive residuated lattices form an equational class [9, Theorem 2.7], it is obvious that \mathcal{S}' -algebras are also an equational class. From Proposition 3, we immediately conclude the following:

Proposition 4. Let $\mathbf{A} = \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ be an \mathcal{S} -algebra. Defining $x * y := \sim(x \Rightarrow \sim y)$, we have that $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ is an \mathcal{S}' -algebra.

Conversely, we are going to see that every \mathcal{S}' -algebra gives rise to an \mathcal{S} -algebra by checking that all (quasi) equations introduced in Definition 5 are satisfied (the proof may be found in the Appendix):

Proposition 5. Let $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ be an \mathcal{S}' -algebra. Defining $\sim x := x \Rightarrow 0$, we have that $\mathbf{A} = \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ is an \mathcal{S} -algebra.

Thus, the classes of \mathcal{S} -algebras and of \mathcal{S}' -algebras are term-equivalent.

The presentation given in Definition 7 has several advantages in what concerns the study of the semantics of \mathcal{S} . For example, it is now straightforward to check that the three-element MV-algebra [7] is a model of Nelson's logic \mathcal{S} . This in turn allows one to prove that the formulas which Nelson claims not to be derivable in \mathcal{S} [15, p. 213] are indeed not valid (see [13]).

4 A Finite Hilbert-Style Calculus for \mathcal{S}

In this section we introduce a finite Hilbert-style calculus (which is an extension of the calculus $IPC^* \setminus c$, called *intuitionistic logic without contraction*, of [4]) that is algebraizable with respect to the class of \mathcal{S}' -algebras.

We are thus going to have two logics that are both algebraizable with respect to the same variety with the same defining equations and equivalence formulas; from this we will obtain an equivalence between our calculus and Nelson's.

The logic $\mathcal{S}' = \langle \mathbf{Fm}, \vdash_{\mathcal{S}'} \rangle$ is the sentential logic in the language $\langle \wedge, \vee, \Rightarrow, *, \sim, \perp, \top \rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$ defined by the Hilbert-style calculus with the following schematic axioms and with *modus ponens* as the only rule:

- (A1') $(\varphi \Rightarrow \psi) \Rightarrow ((\gamma \Rightarrow \varphi) \Rightarrow (\gamma \Rightarrow \psi))$
 (A2') $(\varphi \Rightarrow (\psi \Rightarrow \gamma)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \gamma))$
 (A3') $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
 (A4') $(\varphi \Rightarrow \gamma) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow ((\varphi \vee \psi) \Rightarrow \gamma))$
 (A5') $\varphi \Rightarrow (\varphi \vee \psi)$
 (A6') $\psi \Rightarrow (\varphi \vee \psi)$
 (A7') $(\varphi \wedge \psi) \Rightarrow \varphi$

- (A8') $(\varphi \wedge \psi) \Rightarrow \psi$
 (A9') $\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi))$
 (A10') $((\gamma \Rightarrow \varphi) \wedge (\gamma \Rightarrow \psi)) \Rightarrow (\gamma \Rightarrow \varphi(\wedge\psi))$
 (A11') $\varphi \Rightarrow (\psi \Rightarrow (\varphi * \psi))$
 (A12') $(\varphi \Rightarrow (\psi \Rightarrow \gamma)) \Rightarrow ((\varphi * \psi) \Rightarrow \gamma)$
 (A13') $\sim\varphi \Rightarrow (\varphi \Rightarrow \psi)$
 (A14') $(\varphi \Rightarrow \psi) \Leftrightarrow (\sim\psi \Rightarrow \sim\varphi)$
 (A15') $\varphi \Leftrightarrow \sim\sim\varphi$
 (A16') $\perp \Rightarrow \varphi$
 (A17') $\varphi \Rightarrow \top$
 (A18') $\varphi^2 \Rightarrow \varphi^3$

As before, $\varphi \Leftrightarrow \psi$ abbreviates $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$, while the connective $*$ is here taken as primitive.

Axioms (A1')–(A13'), (A14'(\Rightarrow)), (A15'(\Rightarrow)), (A16') and (A17') of our calculus are the same as those of $IPC^*\setminus c$ as presented in [4, Table 3.2], where it is proven that $IPC^*\setminus c$ is algebraizable. We added the converse implication in axioms (A14') and (A15') to characterize involution and we added the axiom (A18') to characterize 3-potency. As algebraizability is preserved by axiomatic extensions (cf. [8, Proposition 3.31]) we have the following results:

Theorem 2. *The calculus \mathcal{S}' is algebraizable (with the same defining equations and equivalence formulas as \mathcal{S}) with respect to the class of \mathcal{S}' -algebras.*

Proof. We know from [4, Theorem 5.1] that $IPC^*\setminus c$ is algebraizable with respect to the class of commutative integral bounded residuated lattices with the same defining equations and equivalence formulas already considered above. The axioms that were now added imply that the algebraic semantics of our extension is involutive and 3-potent, i.e., it is an \mathcal{S}' -algebra.

Corollary 1. *\mathcal{S} and \mathcal{S}' define the same logic.*

Proof. Let $K_{\mathcal{S}}$ be the class of \mathcal{S} -algebras. Thanks to Propositions 4 and 5 we know that $K_{\mathcal{S}}$ is also the class of \mathcal{S}' -algebras. The result follows now from [8, Proposition 3.47], that gives us an algorithm to find a Hilbert-style calculus for an algebraizable logic from its quasivariety, defining equations and equivalence formulas. As \mathcal{S} -algebras and \mathcal{S}' -algebras are the same class of algebras and their defining equations and equivalence formulas are the same, the Hilbert-style calculus given by the algorithm must do the same job as the one we had before.

Working with Nelson's original presentation of \mathcal{S} , it can be hard to directly prove some version of the Deduction Metatheorem. Indeed, if we prove it, as usual, by way of an induction over the structure of the derivations, we need to apply the inductive hypothesis over each rule of the system. The advantage of \mathcal{S}' , in employing such a strategy, is that it has only one inference rule. This allows us to establish:

Theorem 3 (Deduction Metatheorem). *If $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi^2 \Rightarrow \psi$.*

Proof. Thanks to [9, Corollary 2.15] we have a version of the Deduction Metatheorem for substructural logics which says that $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi^n \Rightarrow \psi$ for some n . In view of (A'18) it is easy to see that in \mathcal{S} we can always choose $n = 2$.

5 Comparing \mathcal{S} with $\mathcal{N}3$ and $\mathcal{N}4$

As mentioned before, Nelson introduced two other better-known logics, $\mathcal{N}3$ and $\mathcal{N}4$, which are also algebraizable with respect to classes of residuated structures (namely, the so-called $\mathcal{N}3$ -lattices and $\mathcal{N}4$ -lattices). A question that immediately arises concerns the precise relation between \mathcal{S} and these other logics, or (equivalently) between \mathcal{S} -algebras and $\mathcal{N}3$ - and $\mathcal{N}4$ -lattices. In what follows it is worth taking into account that not all \mathcal{S} -algebras are distributive (see [13, Example 5.1]).

5.1 $\mathcal{N}4$

Definition 8. $\mathcal{N}4 = \langle \mathbf{Fm}, \vdash_{\mathcal{N}4} \rangle$ is the sentential logic in the language $\langle \wedge, \vee, \rightarrow, \sim \rangle$ of type $\langle 2, 2, 2, 1 \rangle$ defined by the Hilbert-style calculus with the following schematic axioms and modus ponens as the only schematic rule. Below, $\varphi \leftrightarrow \psi$ will be used to abbreviate $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

- (N1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (N2) $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$
- (N3) $(\varphi \wedge \psi) \rightarrow \varphi$
- (N4) $(\varphi \wedge \psi) \rightarrow \psi$
- (N5) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \gamma) \rightarrow (\varphi \rightarrow (\psi \wedge \gamma)))$
- (N6) $\varphi \rightarrow (\varphi \vee \psi)$
- (N7) $\psi \rightarrow (\varphi \vee \psi)$
- (N8) $(\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\varphi \vee \psi) \rightarrow \gamma))$
- (N9) $\sim \sim \varphi \leftrightarrow \varphi$
- (N10) $\sim(\varphi \vee \psi) \leftrightarrow (\sim \varphi \wedge \sim \psi)$
- (N11) $\sim(\varphi \wedge \psi) \leftrightarrow (\sim \varphi \vee \sim \psi)$
- (N12) $\sim(\varphi \rightarrow \psi) \leftrightarrow (\varphi \wedge \sim \psi)$

The implication \rightarrow in $\mathcal{N}4$ is usually called *weak implication*, in contrast to the *strong implication* \Rightarrow that is defined in $\mathcal{N}4$ as follows:

$$\varphi \Rightarrow \psi := (\varphi \rightarrow \psi) \wedge (\sim \psi \rightarrow \sim \varphi).$$

As the notation suggests, it is the strong implication that we shall compare with the implication of \mathcal{S} . This appears indeed to be the most meaningful choice, for otherwise, since the weak implications of both $\mathcal{N}4$ and $\mathcal{N}3$ fail to satisfy contraposition (which holds in \mathcal{S}), we would have to say that \mathcal{S} is incomparable with both logics.

The logic $\mathcal{N}4$ is algebraizable (though not implicative) with equivalence formulas $\{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$ and defining equation $\varphi \approx \varphi \rightarrow \varphi$ [18, Theorem 2.6].

We notice in passing that the implication in this defining equation could as well be taken to be the strong one, so $\varphi \approx \varphi \Rightarrow \varphi$ would work too; in contrast, $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi\}$ would not be a set of equivalence formulas, due precisely to the failure of contraposition. The equivalent algebraic semantics of $\mathcal{N}4$ is the class of $\mathcal{N}4$ -lattices defined below [16, Definition 8.4.1]:

Definition 9. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, \sim \rangle$ is an $\mathcal{N}4$ -lattice if it satisfies the following properties:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra.
2. The relation \preceq defined, for all $a, b \in A$, by $a \preceq b$ iff $(a \rightarrow b) \rightarrow (a \rightarrow b) = (a \rightarrow b)$ is a pre-order on \mathbf{A} .
3. The relation \equiv defined, for all $a, b \in A$ as $a \equiv b$ iff $a \preceq b$ and $b \preceq a$ is a congruence relation with respect to $\wedge, \vee, \rightarrow$ and the quotient algebra $\mathbf{A}_{\boxtimes} := \langle A, \vee, \wedge, \rightarrow \rangle / \equiv$ is an implicative lattice.
4. For any $a, b \in A$, $\sim(a \rightarrow b) \equiv a \wedge \sim b$.
5. For any $a, b \in A$, $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$, where \leq is the lattice order for \mathbf{A} .

A very simple example of an $\mathcal{N}4$ -lattice is the four-element algebra \mathbf{A}_4 whose lattice reduct is the four-element diamond De Morgan algebra. This algebra has carrier $A_4 = \{0, 1, b, n\}$, the maximum element of the lattice order being 1, the minimum 0, and b and n being incomparable. The negation (Fig. 1) is given by $\sim b := b$, $\sim n := n$, $\sim 1 := 0$ and $\sim 0 := 1$. The weak implication is given, for all $a \in A_4$, by $1 \rightarrow a = b \rightarrow a := a$ and $0 \rightarrow a = n \rightarrow a := 1$. One can check that \mathbf{A}_4 satisfies all properties of Definition 9 (in particular, the quotient \mathbf{A}_4 / \equiv is the two-element Boolean algebra).

Proposition 6. $\mathcal{N}4$ and \mathcal{S} are incomparable, that is, neither of them extends the other.

Proof. We show that not every \mathcal{S} -algebra is an $\mathcal{N}4$ -lattice, and that no $\mathcal{N}4$ -lattice is an \mathcal{S} -algebra. The first claim follows from the fact that $\mathcal{N}4$ -lattices have a distributive lattice reduct, whereas \mathcal{S} -algebras need not be distributive. As to the second, it is sufficient to observe that the equation $x \Rightarrow x \approx y \Rightarrow y$ is satisfied in all \mathcal{S} -algebras but does not hold in the four-element $\mathcal{N}4$ -lattice \mathbf{A}_4 . There we have $1 \Rightarrow 1 \neq b \Rightarrow b$ because $1 \Rightarrow 1 = (1 \rightarrow 1) \wedge (\sim 1 \rightarrow \sim 1) = 1 \wedge 1 = 1$ but $b \Rightarrow b = (b \rightarrow b) \wedge (\sim b \rightarrow \sim b) = b \wedge b = b$. Since both $\mathcal{N}4$ and \mathcal{S} are algebraizable logics, this immediately entails that neither $\mathcal{N}4 \leq \mathcal{S}$ nor $\mathcal{S} \leq \mathcal{N}4$. In logical terms, one can check that the distributivity axiom is valid in $\mathcal{N}4$ but not in \mathcal{S} , whereas the formula $(\varphi \Rightarrow \varphi) \Rightarrow (\psi \Rightarrow \psi)$ is valid in \mathcal{S} but not in $\mathcal{N}4$.



Fig. 1. A_4

5.2 $\mathcal{N}3$

Definition 10. *Nelson's logic $\mathcal{N}3 = \langle \mathbf{Fm}, \vdash_{\mathcal{N}3} \rangle$ is the axiomatic extension of $\mathcal{N}4$ obtained by adding the following axiom:*

$$(N13) \quad \sim\varphi \rightarrow (\varphi \rightarrow \psi).$$

Proposition 7. *$\mathcal{N}3$ is a proper extension of \mathcal{S} .*

Proof. It is known from [19] that every $\mathcal{N}3$ -lattice (the algebraic counterpart of $\mathcal{N}3$) satisfies all properties of our Definition 7, and therefore every $\mathcal{N}3$ -lattice is an \mathcal{S} -algebra. On the other hand, the logic $\mathcal{N}3$ was defined as an axiomatic extension of $\mathcal{N}4$, therefore it is distributive too, whereas \mathcal{S} -algebras need not be distributive (see [13, Example 5.1]).

6 Future Work

We have studied \mathcal{S} in two directions, through a proof-theoretic approach and through algebraic methods. Concerning the proof-theoretic approach, we have introduced a finite Hilbert-style calculus for \mathcal{S} . An interesting question that still remains is about other types of calculi. In this sense we would find it attractive to be able to present a sequent calculus for \mathcal{S} enjoying a cut-elimination theorem, so that it could be used to determine, among other things, whether \mathcal{S} is decidable and enjoys the Craig interpolation theorem.

As observed in Theorem 3, if we let $\varphi \rightarrow \psi := \varphi \Rightarrow (\varphi \Rightarrow \psi)$, then the weak implication \rightarrow enjoys a version of the Deduction Metatheorem; this suggests that the connective \rightarrow has a special logical role within \mathcal{S} , whereas \Rightarrow is the key operation on the corresponding algebras. It is well known that the logic $\mathcal{N}3$ as well as its algebraic counterpart can be equivalently axiomatized by taking either the weak or the strong implication as primitive, defining \rightarrow from \Rightarrow as shown above. The analogous result for $\mathcal{N}4$ has been harder to prove (see [20]), and the corresponding definition is $\varphi \rightarrow \psi := (\varphi \wedge (((\varphi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi) \Rightarrow ((\varphi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi))) \Rightarrow ((\varphi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi)$.

We can ask a similar question about the logic \mathcal{S} and its algebraic counterpart: namely, given that Nelson's axiomatization as well as ours have \Rightarrow as primitive, is it also possible to axiomatize \mathcal{S} -algebras by taking the weak implication \rightarrow as primitive? This question is related to certain algebraic properties that \rightarrow enjoys on \mathcal{S} -algebras. In fact, we have shown in [13, Theorem 4.5] that, analogously to $\mathcal{N}3$ -lattices, \mathcal{S} -algebras are a variety of *weak Brouwerian semilattices with filter-preserving operations* [2, Definition 2.1], which means that they possess an intuitionistic-like internal structure, where a *weak relative pseudo-complementation* operation (an intuitionistic-like implication) is given precisely by the weak implication. This suggests that one may in fact hope to be able to view (and axiomatize) \mathcal{S} as a conservative expansion of some intuitionistic-like positive logic by a strong (involutive) negation, as has been the case of $\mathcal{N}3$ and $\mathcal{N}4$.

As hinted above, a more detailed study of \mathcal{S} -algebras can be found in the companion paper [13]. Some questions regarding the variety of \mathcal{S} -algebras, its extensions, congruences and more relations between \mathcal{S} -algebras and other well-known algebras are investigated there. Another question that is still open is which logic (class of algebras) is the infimum of \mathcal{S} (-algebras) and $\mathcal{N}4$ (-lattices)—it is easy to see that the least logic extending \mathcal{S} and $\mathcal{N}4$ is precisely $\mathcal{N}3$.

Appendix: Proofs of the Some of the Main Results

Theorem 1. The calculus $\vdash_{\mathcal{S}}$ is implicative, and thus algebraizable.

Proof. In the case of \mathcal{S} , the term $\alpha(\varphi, \psi)$ may be chosen to be $\varphi \Rightarrow \psi$. We will make below free use of Proposition 2.

IL1 follows immediately from axiom (A1), while **IL2** follows from rule (E). **IL3** follows from (MP) and **IL4** follows from $(\Rightarrow r)$. We are left with proving that \Rightarrow respects **IL5** for each connective $\bullet \in \{\wedge, \vee, \Rightarrow, \sim\}$.

(\sim) $\{(\varphi \Leftrightarrow \psi), (\psi \Leftrightarrow \varphi)\} \vdash_{\mathcal{S}} \sim\varphi \Leftrightarrow \sim\psi$ holds by axiom (A5) and the (derived) rule (MP).

(\wedge) We must prove that $\{(\varphi_1 \Leftrightarrow \psi_1), (\varphi_2 \Leftrightarrow \psi_2)\} \vdash_{\mathcal{S}} (\varphi_1 \wedge \varphi_2) \Leftrightarrow (\psi_1 \wedge \psi_2)$.

From Proposition 1(1–2) we have $\vdash_{\mathcal{S}} (\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_1$ and $\vdash_{\mathcal{S}} (\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_2$. Then:

$$\frac{\frac{(\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_1 \quad \varphi_1 \Rightarrow \psi_1}{(\varphi_1 \wedge \varphi_2) \Rightarrow \psi_1} \text{ (E)} \quad \frac{(\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_2 \quad \varphi_2 \Rightarrow \psi_2}{(\varphi_1 \wedge \varphi_2) \Rightarrow \psi_2} \text{ (E)}}{(\varphi_1 \wedge \varphi_2) \Rightarrow (\psi_1 \wedge \psi_2)} \text{ (\wedge r)}$$

The remainder of the proof is analogous.

(\vee) We must prove that $\{(\varphi_1 \Leftrightarrow \psi_1), (\varphi_2 \Leftrightarrow \psi_2)\} \vdash_{\mathcal{S}} (\varphi_1 \vee \varphi_2) \Leftrightarrow (\psi_1 \vee \psi_2)$.

From Proposition 1(3–4), $\psi_1 \Rightarrow (\psi_1 \vee \psi_2)$ and $\psi_2 \Rightarrow (\psi_1 \vee \psi_2)$ are derivable. Then:

$$\frac{\frac{\varphi_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow (\psi_1 \vee \psi_2)}{\varphi_1 \Rightarrow (\psi_1 \vee \psi_2)} \text{ (E)} \quad \frac{\varphi_2 \Rightarrow \psi_2 \quad \psi_2 \Rightarrow (\psi_1 \vee \psi_2)}{\varphi_2 \Rightarrow (\psi_1 \vee \psi_2)} \text{ (E)}}{(\varphi_1 \vee \varphi_2) \Rightarrow (\psi_1 \vee \psi_2)} \text{ (\vee l1)}$$

The remainder of the proof is analogous.

(\Rightarrow) We must prove that $\{(\theta \Leftrightarrow \varphi), (\psi \Leftrightarrow \gamma)\} \vdash_{\mathcal{S}} (\theta \Rightarrow \psi) \Leftrightarrow (\varphi \Rightarrow \gamma)$. This time, we have:

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \gamma}{\varphi \Rightarrow ((\theta \Rightarrow \psi) \Rightarrow \gamma)} \text{ (\Rightarrow 1)}$$

Taking ψ as $\theta \Rightarrow \psi$ in Proposition 1(5), we have:

$$\frac{\varphi \Rightarrow ((\theta \Rightarrow \psi) \Rightarrow \gamma) \quad \overline{\overline{\varphi \Rightarrow ((\theta \Rightarrow \psi) \Rightarrow \gamma)} \Rightarrow ((\theta \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \gamma))}}{(\theta \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \gamma)} \text{ (MP)}$$

The remainder of the proof is analogous.

Proposition 3. Let \mathbf{A} be an \mathcal{S} -algebra and let $a, b, c \in A$. Then:

1. $a \Rightarrow a = 1 = \sim 0$.
2. The relation \leq defined by setting $a \leq b$ iff $a \Rightarrow b = 1$, is a partial order with maximum 1 and minimum 0.
3. $a \Rightarrow b = \sim b \Rightarrow \sim a$.
4. $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$.
5. $\sim \sim a = a$ and $a \Rightarrow 0 = \sim a$.
6. $\langle A, *, 1 \rangle$ is a commutative monoid.
7. $(a * b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$.
8. The pair $(*, \Rightarrow)$ is residuated with respect to \leq , i.e., $a * b \leq c$ iff $b \leq a \Rightarrow c$.
9. $a^2 \leq a^3$.
10. $\langle A, \wedge, \vee \rangle$ is a lattice with order \leq .
11. $(a \vee b)^2 \leq a^2 \vee b^2$.

Proof. 1. This follows from the fact that \mathcal{S} is an implicative logic, see [8, Lemma 2.6]. In particular, $\sim 0 = 0 \Rightarrow 0 = 1$.

2. By E(A2) we have that 0 is the minimum element with respect to the order \leq . The rest easily follows from the fact that \mathcal{S} is implicative.
3. This follows from E(A5) and item 2 above.
4. By Q(P) and item 2 above, we have that $d \leq a \Rightarrow (b \Rightarrow c)$ implies $d \leq b \Rightarrow (a \Rightarrow c)$ for all $d \in A$. Then, taking $d = a \Rightarrow (b \Rightarrow c)$, we have $a \Rightarrow (b \Rightarrow c) \leq b \Rightarrow (a \Rightarrow c)$, which easily implies the desired result.
5. The identity $\sim \sim a = a$ follows from item 2 above together with Q($\sim \sim 1$) and Q($\sim \sim \mathbf{r}$). By item 3 above, $a \Rightarrow 0 = \sim 0 \Rightarrow \sim a = 1 \Rightarrow \sim a = \sim a$. The last identity holds good because, on the one hand, by Q($\Rightarrow 1$) we have that $1 \leq 1$ and $\sim a \leq \sim a$ implies $1 \Rightarrow \sim a \leq \sim a$. On the other hand, by item 1 we have $\sim a \Rightarrow \sim a \leq 1$ and so we can apply Q($\Rightarrow \mathbf{r}$) to obtain $1 \Rightarrow (\sim a \Rightarrow \sim a) = 1$. By item 4, we have $1 \Rightarrow (\sim a \Rightarrow \sim a) = \sim a \Rightarrow (1 \Rightarrow \sim a)$, hence we conclude that $\sim a \Rightarrow (1 \Rightarrow \sim a) = 1$ and so, by item 2, $\sim a \leq 1 \Rightarrow \sim a$.
6. As to commutativity, using items 3 and 5 above, we have $a * b = \sim(a \Rightarrow \sim b) = \sim(\sim \sim b \Rightarrow \sim a) = \sim(b \Rightarrow \sim a) = b * a$. As to associativity, using 3, 5, Q($\sim \sim \mathbf{r}$) and Q($\sim \sim 1$), we have $(a * b) * c = \sim(\sim(a \Rightarrow \sim b) \Rightarrow \sim c) = \sim(\sim \sim c \Rightarrow \sim \sim(a \Rightarrow \sim b)) = \sim(c \Rightarrow (a \Rightarrow \sim b)) = \sim(a \Rightarrow (c \Rightarrow \sim b)) = \sim(a \Rightarrow (b \Rightarrow \sim c)) = \sim(a \Rightarrow \sim \sim(b \Rightarrow \sim c)) = a * (b * c)$. As to 1 being the neutral element, using items 1 and 5 above, we have $a * 1 = a * \sim 0 = \sim(a \Rightarrow \sim 0) = \sim(a \Rightarrow 0) = \sim \sim a = a$.
7. Using items 2, 3, 5 and 6 above, we have $(a * b) \Rightarrow c = \sim(a \Rightarrow \sim b) \Rightarrow c = \sim c \Rightarrow \sim \sim(a \Rightarrow \sim b) = \sim c \Rightarrow (a \Rightarrow \sim b) = a \Rightarrow (\sim c \Rightarrow \sim b) = a \Rightarrow (\sim \sim b \Rightarrow \sim c) = a \Rightarrow (b \Rightarrow c)$.
8. By item 2 above, we have $a * b \leq c$ iff $(a * b) \Rightarrow c = 1$ iff, by item 7, $a \Rightarrow (b \Rightarrow c) = 1$ iff, by item 6, $b \Rightarrow (a \Rightarrow c) = 1$ iff, by 2 again, $b \leq a \Rightarrow c$.
9. By Q(C) we have that $a^3 \leq c$ implies $a^2 \leq c$ for all $c \in A$. Then, taking $c = a^3$, we have $a^2 \leq a^3$.
10. We check that $a \wedge b$ is the infimum of the set $\{a, b\}$ with respect to \leq . First of all, we have $a \wedge b \leq a$ and $a \wedge b \leq b$ by Q($\wedge 11$), Q($\wedge 12$) and item 2 above. Then, assuming $c \leq a$ and $c \leq b$, we have $c \leq a \wedge b$ by Q($\wedge \mathbf{r}$). An

analogous reasoning, using $Q(\vee r1)$, $Q(\vee r2)$ and $Q(\vee l1)$ shows that $a \vee b$ is the supremum of $\{a, b\}$.

11. By item 10 we have that $a^2 \leq a^2 \vee b^2$ and $b^2 \leq a^2 \vee b^2$. Hence, by item 8, we have $a \leq a \Rightarrow (a^2 \vee b^2)$ and $b \leq b \Rightarrow (a^2 \vee b^2)$. By item 2 we have then $a \Rightarrow (a \Rightarrow (a^2 \vee b^2)) = b \Rightarrow (b \Rightarrow (a^2 \vee b^2)) = 1$, hence we can use $Q(\vee l2)$ to obtain $(a \vee b) \Rightarrow ((a \vee b) \Rightarrow (a^2 \vee b^2)) = 1$. Then items 2 and 8 give us $(a \vee b)^2 \leq a^2 \vee b^2$, as was to be proved.

- Lemma 1.**
1. Any CIBRL satisfies the equation $(x \vee y) * z \approx (x * z) \vee (y * z)$.
 2. Any 3-potent CIBRL satisfies $x^2 \vee y^2 \approx (x^2 \vee y^2)^2$.
 3. Any 3-potent CIBRL satisfies $(x \vee y^2)^2 \approx (x \vee y)^2$.
 4. Any 3-potent CIBRL satisfies $(x \vee y)^2 \approx x^2 \vee y^2$.

Proof. 1. See [9, Lemma 2.6].

2. Let a, b be arbitrary elements of a given 3-potent CIBRL. From $a^2 \leq (a^2 \vee b^2)$ and $b^2 \leq (a^2 \vee b^2)$, using monotonicity of $*$, we have $a^4 \leq (a^2 \vee b^2)^2$ and $b^4 \leq (a^2 \vee b^2)^2$. Using 3-potency, the latter inequalities simplify to $a^2 \leq (a^2 \vee b^2)^2$ and $b^2 \leq (a^2 \vee b^2)^2$. Thus, $a^2 \vee b^2 \leq (a^2 \vee b^2)^2$.
3. We have $(a \vee b^2) \leq (a \vee b)$ from monotonicity of $*$ and supremum of \vee , therefore $(a \vee b^2)^2 \leq (a \vee b)^2$. For the converse, we have that $(a * b) \leq a$, whence $(a * b) \leq (a \vee b^2)$. Also $a^2 \leq (a \vee b^2)$ and $b^2 \leq (a \vee b^2)$. By supremum of \vee , $(a^2 \vee (a * b) \vee b^2) \leq (a \vee b^2)$. But $(a^2 \vee (a * b) \vee b^2) = (a \vee b)^2$ by Lemma 1(1), so $(a \vee b)^2 \leq (a \vee b^2)$. Using the monotonicity of $*$, $(a \vee b)^4 \leq (a \vee b^2)^2$ and from 3-potency we have $(a \vee b)^2 \leq (a \vee b^2)^2$.
4. From Lemma 1(2) we have $(a^2 \vee b^2) = (a^2 \vee b^2)^2$, and from Lemma 1(3) we have $(a^2 \vee b^2)^2 = (a^2 \vee b)^2 = (b \vee a^2)^2 = (b \vee a)^2$.

Proposition 5. Let $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ be an \mathcal{S}' -algebra. Defining $\sim x := x \Rightarrow 0$, we have that $\mathbf{A} = \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ is an \mathcal{S} -algebra.

Proof. Let \mathbf{A} be an \mathcal{S}' -algebra. We first consider the equations corresponding to the axioms of \mathcal{S} . As $a \leq b$ iff $a \Rightarrow b = 1$, we will write the former rather than the latter.

Equations

The equation E(A1) easily follows from integrality. We have E(A2) from the fact that 0 is the minimum element of \mathbf{A} . From the definition of \sim in \mathcal{S}' and from E(A1) we see that E(A3) holds. We know that $1 := \sim 0$, therefore we have E(A4). As \mathbf{A} is involutive, it follows that E(A5) holds. We are still to prove the equation E($\Delta(\varphi, \varphi)$). For that, see that we need to prove the identity $(\varphi \Rightarrow \varphi) \wedge (\varphi \Rightarrow \varphi) = 1$, and we already know that $\varphi \Rightarrow \varphi = 1$, therefore also $(\varphi \Rightarrow \varphi) \wedge (\varphi \Rightarrow \varphi) = 1$.

Quasiequations

Q(P) follows from the commutativity of $*$ and from the identity $(a * b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$. Q(C) follows from 3-potency: since $a^2 \leq a^3$, we have that $a^3 \Rightarrow b = 1$ implies $a^2 \Rightarrow b = 1$.

Q(E) follows from the fact that \mathbf{A} comprises a partial order \leq that is determined by the implication \Rightarrow . To prove Q($\Rightarrow 1$), suppose $a \leq b$ and $c \leq d$. From

$c \leq d$, as $b \Rightarrow c \leq b \Rightarrow c$, using residuation we have that $b * (b \Rightarrow c) \leq c \leq d$, therefore $b * (b \Rightarrow c) \leq d$ and therefore $b \Rightarrow c \leq b \Rightarrow d$. Note that as $a \leq b$, using residuation we have that $a * (b \Rightarrow d) \leq b * (b \Rightarrow d) \leq d$, therefore $b \Rightarrow d \leq a \Rightarrow d$ and then $b \Rightarrow c \leq a \Rightarrow d$. Now, since $b \Rightarrow c \leq a \Rightarrow d$ iff $a * (b \Rightarrow c) \leq d$ iff $a \leq (b \Rightarrow c) \Rightarrow d$, we obtain thus the desired result.

For $\mathbf{Q}(\Rightarrow \mathbf{r})$ we need to prove that if $d = 1$, then $b \Rightarrow d = 1$. This follows immediately from integrality.

Quasiequations $\mathbf{Q}(\wedge 11)$, $\mathbf{Q}(\wedge 12)$, $\mathbf{Q}(\wedge \mathbf{r})$, $\mathbf{Q}(\vee 11)$, $\mathbf{Q}(\vee \mathbf{r}1)$ and $\mathbf{Q}(\vee \mathbf{r}2)$ follow straightforwardly from the fact that \mathbf{A} is partially ordered and the order is determined by the implication.

In order to prove $\mathbf{Q}(\vee 12)$, notice that $(b \vee c)^2 \leq b^2 \vee c^2$ by Lemma 1(4). Suppose $b^2 \leq d$ and $c^2 \leq d$, then since \mathbf{A} is a lattice, we have $b^2 \vee c^2 \leq d$ and as $(b \vee c)^2 \leq b^2 \vee c^2$ we conclude that $(b \vee c)^2 \leq d$ and thus $(b \vee c)^2 \Rightarrow d = 1$.

As to $\mathbf{Q}(\sim \Rightarrow 1)$, by integrality we have $b * c \leq b$ and $b * c \leq c$. Thus $b * c \leq b \wedge c$. Now, if $b \wedge c \leq d$, then $b * c \leq d$.

In order to prove $\mathbf{Q}(\sim \Rightarrow \mathbf{r})$, suppose $d^2 \leq b \wedge c$. Using monotonicity of $*$, we have $d^2 * d^2 \leq (b \wedge c) * (b \wedge c)$, i.e., $d^4 \leq (b \wedge c)^2$. Using 3-potency, we have $d^4 = d^2$, therefore $d^2 \leq (b \wedge c)^2$. Since $(b \wedge c)^2 \leq b * c$, we have $d^2 \leq (b \wedge c)^2 \leq (b * c)$, i.e., $d^2 \leq (b * c)$.

$\mathbf{Q}(\sim \wedge 1)$, $\mathbf{Q}(\sim \wedge \mathbf{r})$, $\mathbf{Q}(\sim \vee 1)$ and $\mathbf{Q}(\sim \vee \mathbf{r})$ follow from the De Morgan's Laws (cf. [9, Lemma 3.17]).

Finally, we have $\mathbf{Q}(\sim \sim 1)$ and $\mathbf{Q}(\sim \sim \mathbf{r})$ from \mathbf{A} being involutive.

It remains to be proven that the quasiequation $\mathbf{E}(\Delta(\varphi, \psi))$ implies $\varphi \approx \psi$, that is, if $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)) = 1$, then $\varphi = \psi$. As 1 is the maximum of the algebra, we have that $(\varphi \Rightarrow \psi) = 1$ and $(\psi \Rightarrow \varphi) = 1$, therefore $\varphi \leq \psi$ and $\psi \leq \varphi$. As \leq is an order relation, it follows that $\varphi = \psi$.

References

1. Almkudad, A., Nelson, D.: Constructible falsity and inexact predicates. *J. Symb. Log.* **49**(1), 231–233 (1984)
2. Blok, W.J., Köhler, P., Pigozzi, D.: On the structure of varieties with equationally definable principal congruences II. *Algebra Universalis* **18**, 334–379 (1984)
3. Blok, W.J., Pigozzi, D.: *Algebraizable Logics*. Memoirs of the American Mathematical Society, vol. 396. A.M.S., Providence (1989)
4. Bou, F., García-Cerdaña, À., Verdú, V.: On two fragments with negation and without implication of the logic of residuated lattices. *Arch. Math. Log.* **45**(5), 615–647 (2006)
5. Busaniche, M., Cignoli, R.: Constructive logic with strong negation as a substructural logic. *J. Log. Comput.* **20**, 761–793 (2010)
6. Cignoli, R.: The class of Kleene algebras satisfying an interpolation property and Nelson algebras. *Algebra Universalis* **23**(3), 262–292 (1986)
7. Cignoli, R., D'Ottaviano, I.M.L., Mundici, D.: *Algebraic Foundations of Many-Valued Reasoning*. Trends in Logic—Studia Logica Library, vol. 7. Kluwer Academic Publishers, Dordrecht (2000)
8. Font, J.M.: *Abstract Algebraic Logic. An Introductory Textbook*. College Publications, London (2016)

9. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Studies in Logic and the Foundations of Mathematics, vol. 151. Elsevier, Amsterdam (2007)
10. Galatos, N., Raftery, J.G.: Adding involution to residuated structures. *Studia Logica* **77**(2), 181–207 (2004)
11. Humberstone, L.: The Connectives. MIT Press, Cambridge (2011)
12. Kleene, S.C.: On the interpretation of intuitionistic number theory. *J. Symb. Log.* **10**, 109–124 (1945)
13. Nascimento, T., Marcos, J., Riviuccio, U., Spinks, M.: Nelson’s logic \mathcal{S} . <https://arxiv.org/abs/1803.10851>
14. Nelson, D.: Constructible falsity. *J. Symb. Log.* **14**, 16–26 (1949)
15. Nelson, D.: Negation and separation of concepts in constructive systems. *Studies in Logic and the Foundations of Mathematics*, vol. 39, pp. 205–225 (1959)
16. Odintsov, S.P.: Constructive Negations and Paraconsistency. Springer, Dordrecht (2008). <https://doi.org/10.1007/978-1-4020-6867-6>
17. Restall, G.: How to be *really* contraction free. *Studia Logica* **52**(3), 381–391 (1993)
18. Riviuccio, U.: Paraconsistent modal logics. *Electron. Notes Theor. Comput. Sci.* **278**, 173–186 (2011)
19. Spinks, M., Veroff, R.: Constructive logic with strong negation is a substructural logic. II. *Studia Logica* **89**(3), 401–425 (2008)
20. Spinks, M., Veroff, R.: Paraconsistent constructive logic with strong negation as a contraction-free relevant logic. In: Czelakowski, J. (ed.) *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science. Outstanding Contributions to Logic*, vol. 16, pp. 323–379. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-74772-9_13