Three applications of Euler's formula

Chapter 13

A graph is *planar* if it can be drawn in the plane $\mathbb{R}^2$ without crossing edges (or, equivalently, on the 2-dimensional sphere S^2). We talk of a *plane* graph if such a drawing is already given and fixed. Any such drawing decomposes the plane or sphere into a finite number of connected regions, including the outer (unbounded) region, which are referred to as *faces*. Euler's formula exhibits a beautiful relation between the number of vertices, edges and faces that is valid for any plane graph. Euler mentioned this result for the first time in a letter to his friend Goldbach in 1750, but he did not have a complete proof at the time. Among the many proofs of Euler's formula, we present a pretty and "self-dual" one that gets by without induction. It can be traced back to von Staudt's book "Geometrie der Lage" from 1847.

> **Euler's formula.** *If G is a connected plane graph with n vertices, e edges and f faces, then*
> $$n - e + f = 2.$$

■ **Proof.** Let $T \subseteq E$ be the edge set of a spanning tree for G, that is, of a minimal subgraph that connects all the vertices of G. This graph does not contain a cycle because of the minimality assumption.

We now need the *dual graph* G^* of G: To construct it, put a vertex into the interior of each face of G, and connect two such vertices of G^* by edges that correspond to common boundary edges between the corresponding faces. If there are several common boundary edges, then we draw several connecting edges in the dual graph. (Thus G^* may have multiple edges even if the original graph G is simple.)

Consider the collection $T^* \subseteq E^*$ of edges in the dual graph that corresponds to edges in $E \backslash T$. The edges in T^* connect all the faces, since T does not have a cycle; but also T^* does not contain a cycle, since otherwise it would separate some vertices of G inside the cycle from vertices outside (and this cannot be, since T is a spanning subgraph, and the edges of T and of T^* do not intersect). Thus T^* is a spanning tree for G^*.

For every tree the number of vertices is one larger than the number of edges. To see this, choose one vertex as the root, and direct all edges "away from the root": this yields a bijection between the non-root vertices and the edges, by matching each edge with the vertex it points at. Applied to the tree T this yields $n = e_T + 1$, while for the tree T^* it yields $f = e_{T^*} + 1$. Adding both equations we get $n + f = (e_T + 1) + (e_{T^*} + 1) = e + 2$. □
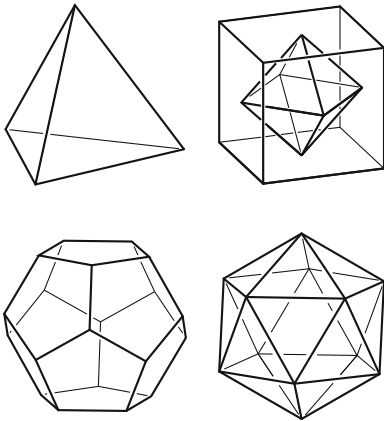

Leonhard Euler
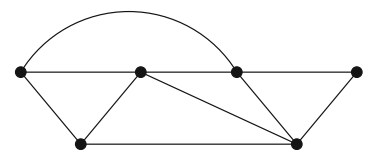
A plane graph G: $n = 6$, $e = 10$, $f = 6$
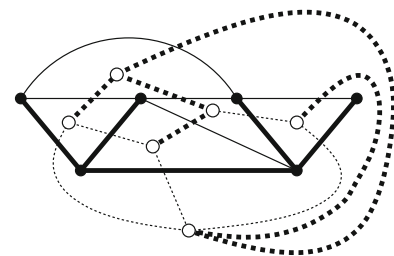
Dual spanning trees in G and in G^*

The five platonic solids

Euler's formula thus produces a strong *numerical* conclusion from a *geometric-topological* situation: the numbers of vertices, edges, and faces of a finite graph G satisfy $n - e + f = 2$ whenever the graph is *or can be* drawn in the plane or on a sphere.

Many well-known and classical consequences can be derived from Euler's formula. Among them are the classification of the regular convex polyhedra (the platonic solids), the fact that K_5 and $K_{3,3}$ are not planar (see below), and the five-color theorem that every planar map can be colored with at most five colors such that no two adjacent countries have the same color. But for this we have a much better proof, which does not even need Euler's formula — see Chapter 39.

This chapter collects three other beautiful proofs that have Euler's formula at their core. The first two — a proof of the Sylvester–Gallai theorem, and a theorem on two-colored point configurations — use Euler's formula in clever combination with other arithmetic relationships between basic graph parameters. Let us first look at these parameters.

The *degree* of a vertex is the number of edges that end in the vertex, where loops count double. Let n_i denote the number of vertices of degree i in G. Counting the vertices according to their degrees, we obtain

Here the degree is written next to each vertex. Counting the vertices of given degree yields $n_2 = 3$, $n_3 = 0$, $n_4 = 1$, $n_5 = 2$.

$$n = n_0 + n_1 + n_2 + n_3 + \cdots \tag{1}$$

On the other hand, every edge has two ends, so it contributes 2 to the sum of all degrees, and we obtain

$$2e = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots \tag{2}$$

You may interpret this identity as counting in two ways the ends of the edges, that is, the edge-vertex incidences. The *average degree* $\overline{d}$ of the vertices is therefore

$$\overline{d} = \frac{2e}{n}.$$

Next we count the faces of a plane graph according to their number of sides: a *k-face* is a face that is bounded by k edges (where an edge that on both sides borders the same region has to be counted twice!). Let f_k be the number of k-faces. Counting all faces we find
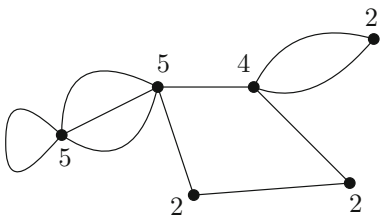
$$f = f_1 + f_2 + f_3 + f_4 + \cdots \tag{3}$$

Counting the edges according to the faces of which they are sides, we get

The number of sides is written into each region. Counting the faces with a given number of sides yields $f_1 = 1$, $f_2 = 3$, $f_4 = 1$, $f_9 = 1$, and $f_i = 0$ otherwise.
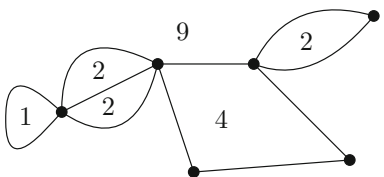
$$2e = f_1 + 2f_2 + 3f_3 + 4f_4 + \cdots \tag{4}$$

As before, we can interpret this as double-counting of edge-face incidences. Note that the average number of sides of faces is given by

$$\overline{f} = \frac{2e}{f}.$$

Let us deduce from this — together with Euler's formula — quickly that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar. For a hypothetical plane drawing of K_5 we calculate $n = 5$, $e = \binom{5}{2} = 10$, thus $f = e + 2 - n = 7$ and $\overline{f} = \frac{2e}{f} = \frac{20}{7} < 3$. But if the average number of sides is smaller than 3, then the embedding would have a face with at most two sides, which cannot be.

Similarly for $K_{3,3}$ we get $n = 6$, $e = 9$, and $f = e + 2 - n = 5$, and thus $\overline{f} = \frac{2e}{f} = \frac{18}{5} < 4$, which cannot be since $K_{3,3}$ is simple and bipartite, so all its cycles have length at least 4.

It is no coincidence, of course, that the equations (3) and (4) for the f_i's look so similar to the equations (1) and (2) for the n_i's. They are transformed into each other by the dual graph construction $G \to G^*$ explained above.

From the double counting identities, we get the following important "local" consequences of Euler's formula.

K_5 drawn with one crossing

$K_{3,3}$ drawn with one crossing

Proposition. *Let G be any simple plane graph with $n > 2$ vertices. Then*

(A) *G has at most $3n - 6$ edges.*

(B) *G has a vertex of degree at most 5.*

(C) *If the edges of G are two-colored, then there is a vertex of G with at most two color-changes in the cyclic order of the edges around the vertex.*

■ **Proof.** For each of the three statements, we may assume that G is connected.

(A) Every face has at least 3 sides (since G is simple), so (3) and (4) yield

$$f \quad = \quad f_3 + \quad f_4 + \quad f_5 + \cdots$$

and

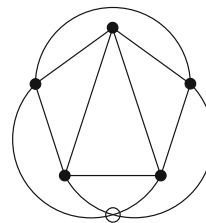$$2e \quad = \quad 3f_3 + 4f_4 + 5f_5 + \cdots$$

and thus $2e - 3f \geq 0$. Euler's formula now gives
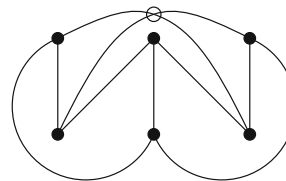
$$3n - 6 = 3e - 3f \; \geq \; e.$$

(B) By part (A), the average degree $\overline{d}$ satisfies

$$\overline{d} \;=\; \frac{2e}{n} \;\leq\; \frac{6n - 12}{n} \;<\; 6.$$

So there must be a vertex of degree at most 5.

Arrows point to the corners with color changes.
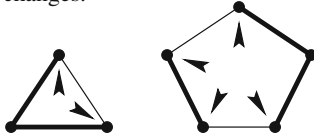
(C) Let c be the number of corners where color changes occur. Suppose the statement is false, then we have $c \geq 4n$ corners with color changes, since at every vertex there is an even number of changes. Now every face with $2k$ or $2k+1$ sides has at most $2k$ such corners, so we conclude that

$$
\begin{aligned}
4n \;\leq\; c \;&\leq\; 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots \\
&\leq\; 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7 + \cdots \\
&=\; 2(3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + \cdots) \\
&\quad\; -4(f_3 + f_4 + f_5 + f_6 + f_7 + \cdots) \\
&=\; 4e - 4f
\end{aligned}
$$

using again (3) and (4). So we have $e \geq n + f$, again contradicting Euler's formula. □

1. The Sylvester–Gallai theorem, revisited

It was first noted by Norman Steenrod, it seems, that part (B) of the proposition yields a strikingly simple proof of the Sylvester–Gallai theorem (see Chapter 11).

The Sylvester–Gallai theorem. *Given any set of $n \geq 3$ points in the plane, not all on one line, there is always a line that contains exactly two of the points.*

■ **Proof.** (Sylvester–Gallai via Euler)
If we embed the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ near the unit sphere S^2 as indicated in our figure, then every point in $\mathbb{R}^2$ corresponds to a pair of antipodal points on S^2, and the lines in $\mathbb{R}^2$ correspond to great circles on S^2. Thus the Sylvester–Gallai theorem amounts to the following:

Given any set of $n \geq 3$ pairs of antipodal points on the sphere, not all on one great circle, there is always a great circle that contains exactly two of the antipodal pairs.

Now we dualize, replacing each pair of antipodal points by the corresponding great circle on the sphere. That is, instead of points $\pm v \in S^2$ we consider the orthogonal circles given by $C_v := \{x \in S^2 : \langle x, v \rangle = 0\}$. (This C_v is the equator if we consider v as the north pole of the sphere.)

Then the Sylvester–Gallai problem asks us to prove:

Given any collection of $n \geq 3$ great circles on S^2, not all of them passing through one point, there is always a point that is on exactly two of the great circles.

But the arrangement of great circles yields a simple plane graph on S^2, whose vertices are the intersection points of two of the great circles, which divide the great circles into edges. All the vertex degrees are even, and they are at least 4 — by construction. Thus part (B) of the proposition yields the existence of a vertex of degree 4. That's it! □
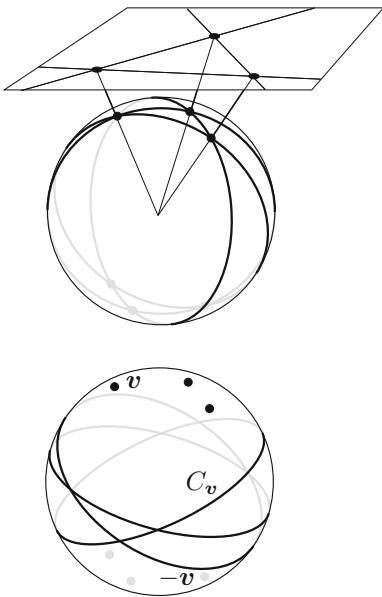
2. Monochromatic lines

The following proof of a "colorful" relative of the Sylvester–Gallai theorem is due to Don Chakerian.

Theorem. *Given any finite configuration of "black" and "white" points in the plane, not all on one line, there is always a "monochromatic" line: a line that contains at least two points of one color and none of the other.*

■ **Proof.** As for the Sylvester–Gallai problem, we transfer the problem to the unit sphere and dualize it there. So we must prove:

Given any finite collection of "black" and "white" great circles on the unit sphere, not all passing through one point, there is always an intersection point that lies either only on white great circles, or only on black great circles.

Now the (positive) answer is clear from part (C) of the proposition, since in every vertex where great circles of different colors intersect, we always have at least 4 corners with sign changes.   □
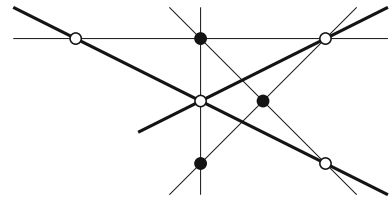
3. Pick's theorem

Pick's theorem from 1899 is a beautiful and surprising result in itself, but it is also a "classical" consequence of Euler's formula. For the following, call a convex polygon $P \subseteq \mathbb{R}^2$ *elementary* if its vertices are integral (that is, they lie in the *lattice* $\mathbb{Z}^2$), but if it does not contain any further lattice points.

Lemma. *Every elementary triangle $\Delta = \mathrm{conv}\{p_0, p_1, p_2\} \subseteq \mathbb{R}^2$ has area $A(\Delta) = \frac{1}{2}$.*

■ **Proof.** Both the parallelogram P with corners $p_0, p_1, p_2, p_1 + p_2 - p_0$ and the lattice $\mathbb{Z}^2$ are symmetric with respect to the map

$$\sigma : \ x \ \longmapsto \ p_1 + p_2 - x,$$

which is the reflection with respect to the center of the segment from p_1 to p_2. Thus the parallelogram $P = \Delta \cup \sigma(\Delta)$ is elementary as well, and its integral translates tile the plane. Hence $\{p_1 - p_0, p_2 - p_0\}$ is a basis of the lattice $\mathbb{Z}^2$, it has determinant ± 1, P is a parallelogram of area 1, and Δ has area $\frac{1}{2}$. (For an explanation of these terms see the box on the next page.)   □

Theorem. *The area of any (not necessarily convex) polygon $Q \subseteq \mathbb{R}^2$ with integral vertices is given by*

$$A(Q) \ = \ n_{int} + \frac{1}{2} n_{bd} - 1,$$

where n_{int} and n_{bd} are the numbers of integral points in the interior respectively on the boundary of Q.

$n_{int} = 11$, $n_{bd} = 8$, so $A = 14$
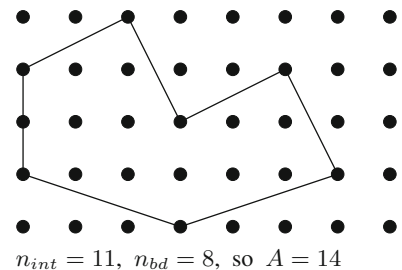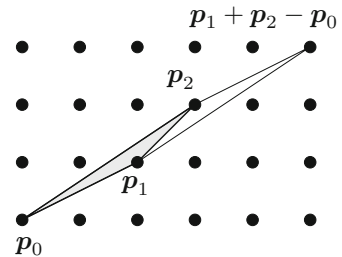
Lattice bases

A *basis* of $\mathbb{Z}^2$ is a pair of linearly independent vectors e_1, e_2 such that

$$\mathbb{Z}^2 = \{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_1, \lambda_2 \in \mathbb{Z}\}.$$

Let $e_1 = \binom{a}{b}$ and $e_2 = \binom{c}{d}$, then the area of the parallelogram spanned by e_1 and e_2 is given by $A(e_1, e_2) = |\det(e_1, e_2)| = |\det \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)|$. If $f_1 = \binom{r}{s}$ and $f_2 = \binom{t}{u}$ is another basis, then there exists an invertible $\mathbb{Z}$-matrix Q with $\left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)Q$. Since $QQ^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the determinants are integers, it follows that $|\det Q| = 1$, and hence $|\det(f_1, f_2)| = |\det(e_1, e_2)|$. Therefore all basis parallelograms have the same area 1, since $A\left(\binom{1}{0}, \binom{0}{1}\right) = 1$.
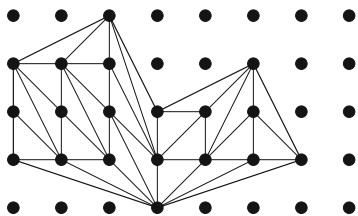
■ **Proof.** Every such polygon can be triangulated using all the n_{int} lattice points in the interior, and all the n_{bd} lattice points on the boundary of Q. (This is not quite obvious, in particular if Q is not required to be convex, but the argument given in Chapter 40 on the art gallery problem proves this.)

Now we interpret the triangulation as a plane graph, which subdivides the plane into one unbounded face plus $f - 1$ triangles of area $\frac{1}{2}$, so

$$A(Q) = \frac{1}{2}(f - 1).$$

Every triangle has three sides, where each of the e_{int} interior edges bounds two triangles, while the e_{bd} boundary edges appear in one single triangle each. So $3(f-1) = 2e_{int} + e_{bd}$ and thus $f = 2(e-f) - e_{bd} + 3$. Also, there is the same number of boundary edges and vertices, $e_{bd} = n_{bd}$. These two facts together with Euler's formula yield

$$\begin{aligned} f &= 2(e - f) - e_{bd} + 3 \\ &= 2(n - 2) - n_{bd} + 3 = 2n_{int} + n_{bd} - 1, \end{aligned}$$

and thus

$$A(Q) = \tfrac{1}{2}(f - 1) = n_{int} + \tfrac{1}{2}n_{bd} - 1. \qquad \square$$

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