Chapter 13
Copulae in High Dimensions: An Introduction

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Abstract This paper reviews the latest proceeding of research in high dimensional copulas. At the beginning the bivariate copulas are given as a fundamental followed with the multivariate copulas which are the concentration of the paper. In multivariate copula sections, the hierarchical Archimedean copula, the factor copula and vine copula are introduced. In the following section the estimation methods for multivariate copulas including parametric and nonparametric routines, are presented. Also the introduction of the goodness of fit tests in copula context is given. An empirical study of multivariate copulas in risk management is performed thereafter.

13.1 Introduction

Researches of dependence modeling were burgeoning during the last decade. The traditional approaches that concentrate on the elliptical distributions such as Gaussian models are giving way to copula-based models. Albeit these Gaussian models sometimes own the convenience in model construction and computation, yet an abundant amount of empirical evidences do not support the underlying assumptions. De facto, shortcomings in the elliptical and especially Gaussian family are mainly in lack of asymmetrical and tail dependence which have been deeply discussed in numer-

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ous papers. Furthermore and of great importance, margins of elliptical distributions belong to the same elliptical family.

The seminal result of Sklar (1959) provides a partial solution to these problems. It allows to separate the marginal distributions from the dependency structure between the random variables. Since the theory on modeling and estimation of univariate distributions is well established compared to the multivariate case, the initial problem reduces to modeling the dependency by copulas. In particular, this approach dramatically widens the class of candidate distributions and allows a simple construction of distributions with less parameters than imposed by elliptical models.

In the beginning of the copula study, researches were mainly focused on the bivariate dependence but as time passes problems raised by the financial, technological, biological industries dictated the rules of further developments, namely moves to higher dimensions. Nonetheless, it has been realized as clearly stated in Mai and Scherer (2013), that "the step from one-dimensional modeling is clearly large. But, unfortunately, the step from two to three (or even more) dimensions is not a bit smaller".

Numerous steps are accomplished in order to contribute to research on high-dimensional modeling approaches and these main branches have been established: pair copula construction, see Joe (1996), Bedford and Cooke (2001), Bedford and Cooke (2002) and Kurowicka and Cooke (2006), hierarchical Archimedean copula, see Savu and Trede (2010), Hofert (2011) and Okhrin et al. (2013a), and factor copula, see Krupskii and Joe (2013) and Oh and Patton (2015).

This chapter attempts at discussing such non-standard multivariate copula models and the subsequent sections are organized as follows. We introduce bivariate copulae and review modern multivariate copula families. Then, corresponding estimation methods and goodness of fit tests are presented. Last but not least, we study a risk management topic empirically.

13.2 Bivariate Copula

Modeling the dependence between only two random variables using copulae is the subject of this section. There are several equivalent definitions of the copula function. We define it as a bivariate distribution function and the simplest one is as follows:

Definition 13.1 The copula $C(u, v)$ is a bivariate distribution with margins being $U[0, 1]$.

Term copula was mentioned for the first time in the seminal result of Sklar (1959). The separation of the bivariate distribution function into the copula function and margins is formally stated in the subsequent theorem. One possible proof is presented in Nelsen (2006), for others we refer to Durante et al. (2012), Durante et al. (2013) and Durante and Sempi (2005)

Theorem 13.1 *Let F be a bivariate distribution function with margins F_1 and F_2, then there exists a copula C such that*

$$F(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\}, \quad x_1, x_2 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}. \qquad (13.1)$$

If F_1 and F_2 are continuous then C is unique. Otherwise C is uniquely determined on $F_1(\overline{\mathbb{R}}) \times F_2(\overline{\mathbb{R}})$.

Conversely, if C is a copula and F_1 and F_2 are univariate distribution functions, then function F in (13.1) is a bivariate distribution function with margins F_1 and F_2.

As indicated above, the theorem allows decomposing any continuous bivariate distribution into its marginal distributions and the dependency structure. Since by definition, the latter is the copula function with uniform margins, it follows that the copula density can be determined in the usual way

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}, \quad u_1, u_2 \in [0, 1]. \qquad (13.2)$$

Being armed with the Theorem 13.1 and (13.2), the density function $f(\cdot)$ of the bivariate distribution F can be rewritten in terms of copula

$$f(x_1, x_2) = c\{F_1(x_1), F_2(x_2)\} f_1(x_1) f_2(x_2), \quad x_1, x_2 \in \overline{\mathbb{R}}.$$

A very important property of copulae is given in Nelsen (2006) stating that copulae are invariant under strictly monotone transformations of margins. Seen from this angle, copulae capture only those features of the dependency which are invariant under increasing transformations.

13.2.1 Copula Families

Naturally, there is an infinite number of different copula functions satisfying the properties of Definition 13.1 and the number of them being deeply studied is expanding. In this section, we discuss three copula classes namely simple, elliptical and Archimedean copulae.

Simplest Copulae

To form basic intuition for copula functions, we first study some extreme special cases, like stochastically independent, perfect positive or negative dependent random variables. According to Theorem 13.1, the copula of two stochastically independent random variables X_1 and X_2 is given by the product (independence) copula defined as

$$\Pi(u_1, u_2) = u_1 u_2, \quad u_1, u_2 \in [0, 1].$$

The contour diagrams of the bivariate density function with product copula and either Gaussian or t_3-distributed margins are given in Fig. 13.1. Two additional extremes are the lower and upper Fréchet–Hoeffding bounds. They represent the perfect negative and positive dependence of two random variables respectively

$$W(u_1, u_2) = \max(0, u_1 + u_2 - 1) \quad \text{and} \quad M(u_1, u_2) = \min(u_1, u_2), \quad u_1, u_2 \in [0, 1].$$

If $C = W$ and $(X_1, X_2) \sim C(F_1, F_2)$ then X_2 is a decreasing function of X_1. Similarly, if $C = M$, then X_2 is an increasing function of X_1. In general, we can argue that an arbitrary copula which represents some dependency structure lies between these two bounds, i.e.

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2), \quad u_1, u_2 \in [0, 1].$$

The bounds serve as benchmarks for the evaluation of the dependency magnitude. There are numerous techniques for building new copulae by mixing at least two of the presented simplest copula. For example, copula families B11 and B12, see Joe (1997), arise as a combination of the upper Fréchet–Hoeffding bound and the product copula

$$C_{B11}(u_1, u_2, \theta) = \theta M(u_1, u_2) + (1 - \theta)\Pi(u_1, u_2) = \theta \min\{u_1, u_2\} + (1 - \theta)u_1 u_2,$$
$$C_{B12}(u_1, u_2, \theta) = M(u_1, u_2)^{\theta} \Pi(u_1, u_2)^{1-\theta} = (\min\{u_1, u_2\})^{\theta}(u_1 u_2)^{1-\theta}, \quad u_1, u_2, \theta \in [0, 1].$$

Family B11 builds on the fact that every convex combination of copulas is a copula as well. Family B12 is also known as Spearman or Cuadras–Augé copula, which is a weighted geometric mean of the upper Fréchet–Hoeffding bound and the product copula. Further generalization is done by using power mean over the upper Fréchet–Hoeffding bound and the product copula

$$C_p(u_1, u_2, \theta_1, \theta_2) = \{\theta_1 M^{\theta_2}(u_1, u_2) + (1 - \theta_1)\Pi^{\theta_2}(u_1, u_2)\}^{1/\theta_2}$$
$$= \{\theta_1 \min(u_1, u_2)^{\theta_2} + (1 - \theta_1)(u_1 u_2)^{\theta_2}\}^{1/\theta_2},$$

with $\theta_1 \in [0, 1]$, $\theta_2 \in \mathbb{R}$. Last but not least, a convex combination of the Fréchet–Hoeffding lower bound, upper bound and product copula forms the Fréchet copula

$$C_F(u_1, u_2, \theta_1, \theta_2) = \theta_1 W(u_1, u_2) + (1 - \theta_1 - \theta_2)\Pi(u_1, u_2) + \theta_2 M(u_1, u_2),$$

subject to $0 \leq \theta_1 + \theta_2 \leq 1$. Note that any bivariate copula can be approximated by the Fréchet family and a bound of the resulting approximation error can be estimated. Nelsen (2006) provides further methods for constructing multivariate copulas and discusses convex combination in more detail.
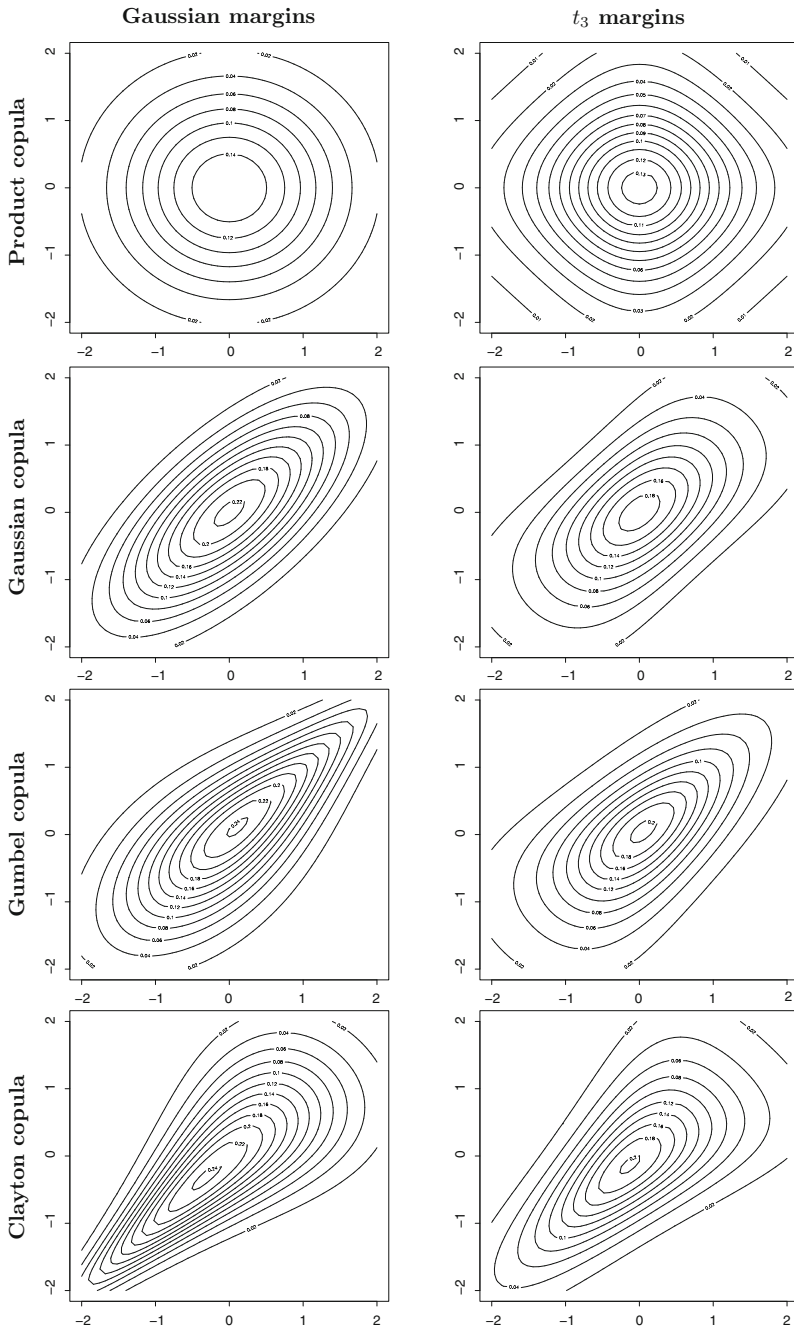
Fig. 13.1 Contour diagrams for product, Gaussian, Gumbel and Clayton copulae with Gaussian (*left column*) and t_3 distributed (*right column*) margins

Elliptical Family

Due to the popularity of the Gaussian and t-distribution in several applications, elliptical copulae play an important role as well. The construction of this type of copulae is directly based on Sklar's Theorem showing how new bivariate distributions can be constructed. The copula-based modeling approach substantially widens the family of elliptical distributions by keeping the same elliptical copula function and varying the marginal distributions or vice versa.

To determine the copula function of a given bivariate distribution, we employ the transformation

$$C(u_1, u_2) = F\{F_1^{-1}(u_1), F_2^{-1}(u_2)\}, \quad u_1, u_2 \in [0, 1], \tag{13.3}$$

where F_i^{-1}, $i = 1, 2$, are (generalized) inverses of the marginal distribution functions. Based on (13.3), arbitrary elliptical distributions can be derived. The problem, however, is that such copulae depend on the inverse distribution functions of the marginals which are rarely available in an explicit form.

For instance, from Formula 13.3 follows that the Gaussian copula and its density are given by

$$C_N(u_1, u_2, \delta) = \Phi_\delta(\Phi^{-1}(u_1), \Phi^{-1}(u_2)),$$
$$c_N(u_1, u_2, \delta) = (1 - \delta^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(1 - \delta^2)^{-1}(u_1^2 + u_2^2 - 2\delta u_1 u_2)\right\}$$
$$\times \exp\left\{\frac{1}{2}(u_1^2 + u_2^2)\right\}, \quad \text{for all } u_1, u_2 \in [0, 1], \ \delta \in [-1, 1],$$

where Φ is the distribution function of N(0, 1), Φ^{-1} is the functional inverse of Φ and Φ_δ denotes the bivariate standard normal distribution function with correlation coefficient δ. In the bivariate case, the t-copula and its density are given by

$$C_t(u_1, u_2, \nu, \delta) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{(1 - \delta^2)}}$$
$$\times \left\{1 + \frac{x_1^2 - 2\delta x_1 x_2 + x_2^2}{(1 - \delta^2)\nu}\right\}^{-\frac{\nu}{2} - 1} dx_1 dx_2,$$
$$c_t(u_1, u_2, \nu, \delta) = \frac{f_{\nu,\delta}\{t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\}}{f_\nu\{t^{-1}(u_1)\} f_\nu\{t^{-1}(u_2)\}}, \quad u_1, u_2, \delta \in [0, 1],$$

where δ denotes the correlation coefficient, ν is the number of degrees of freedom. $f_{\nu,\delta}$ and f_ν are joint and marginal t-distributions respectively, while t_ν^{-1} denotes the quantile function of the t_ν distribution. In-depth analysis of the t-copula is done in Rachev et al. (2008) and Luo and Shevchenko (2010). Long-tailed distributed margins lead to more mass and variability in the tail areas of the corresponding bivariate distribution. However, the contour-curves of the t-copula are symmetric, which reflects the ellipticity of the underlying copula. This property is theoretically

supported by Nelsen (2006), stating that a bivariate copula is elliptical and thus, has reflection symmetry, if and only if

$$C(u_1, u_2, \theta) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2, \theta), \quad u_1, u_2 \in [0, 1].$$

The next class of copulae and their generalizations provide an important flexible and rich family of alternatives to elliptical copulae.

Archimedean Family

In contrast to elliptical copulae, Archimedean copulae are not constructed via (13.3), but are related to Laplace transforms of bivariate distribution functions. The function $C : [0, 1]^2 \to [0, 1]$ defined as

$$C(u_1, u_2) = \phi\{\phi^{-1}(u_1) + \phi^{-1}(u_2)\}, \quad u_1, u_2 \in [0, 1],$$

is a 2-dimensional Archimedean copula, where $\phi \in \mathcal{L} = \{\phi : [0; \infty) \to [0, 1] \mid \phi(0) = 1, \ \phi(\infty) = 0; \ (-1)^j \phi^{(j)} \geq 0; \ j = 1, \ldots, \infty\}$ is referred to as the generator of the copula. The generator usually depends on some parameters, however, mostly generators with a single parameter θ are considered. Nelsen (2006) and Joe (2014) provide a thoroughly classified list of popular generators for Archimedean copulae and discuss their properties.

The useful applications in finance, see Patton (2012), appearing to be the Gumbel copula with the generator function $\phi(x, \theta) = \exp\{-x^{1/\theta}\}$, $1 \leq \theta < \infty$, $x \in [0, 1]$, leading to the copula function

$$C(u_1, u_2, \theta) = \exp\left\{-\left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta}\right\}, \quad u_1, u_2 \in [0, 1].$$

Genest and Rivest (1989) showed that a bivariate distribution based on the Gumbel copula with extreme valued marginal distributions is the only bivariate extreme value distribution belonging to the Archimedean family. Moreover, all distributions based on Archimedean copulae belong to its domain of attraction under common regularity conditions. In contrary to elliptical copulae, the Gumbel copula leads to asymmetric contour diagrams in Fig. 13.1. It exhibits a stronger linkage between positive values, however, more variability and more mass in the negative tail area. Opposite is observed for the Clayton copula with the generator $\phi(x, \theta) = (\theta x + 1)^{-\frac{1}{\theta}}$ with $-1 < \theta < \infty$, $\theta \neq 0$, $x \in [0, 1]$, and copula function

$$C(u_1, u_2, \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad u_1, u_2 \in [0, 1].$$

Also, the Frank generator $\phi(x, \theta) = \theta^{-1} \log\{1 - (1 - e^{-\theta})e^{-x}\}$ with $0 \leq \theta < \infty$, $x \in [0, 1]$, enjoys increased popularity and induces the copula function

$$C(u_1, u_2, \theta) = -\theta^{-1} \log\left\{\frac{1 - e^{-\theta} - (1 - e^{-\theta u_1})(1 - e^{-\theta u_2})}{1 - e^{-\theta}}\right\}, \quad u_1, u_2 \in [0, 1].$$

The respective Frank copula is the only elliptical Archimedean copula.

13.2.2 Bivariate Copula and Dependence Measures

Since copulae define the dependence structure between random variables, there is a relationship between copulae and different dependency measures. The classical measures for continuous random variables are Kendall's τ and Spearman's ρ. Similarly as copula functions, these measures are invariant under strictly increasing transformations. They are equal to 1 or -1 under perfect positive or negative dependence respectively. In contrast to τ and ρ, the Pearson correlation coefficient measures the linear dependence and, therefore, is not suitable for measuring non-linear relationships. Next, we discuss the relationship between τ, ρ and the underlying copula function.

Definition 13.2 Let F be a continuous bivariate cumulative distribution function with the copula C. Moreover, let $(X_1, X_2) \sim F$ and $(X_1', X_2') \sim F$ be independent random pairs. Then Kendall's τ is given by

$$
\begin{aligned}
\tau_2 &= P\{(X_1 - X_1')(X_2 - X_2') > 0\} - P\{(X_1 - X_1')(X_2 - X_2') < 0\} \\
&= 2\,P\{(X_1 - X_1')(X_2 - X_2') > 0\} - 1 = 4 \int_{[0,1]^2} C(u_1, u_2)\, dC(u_1, u_2) - 1.
\end{aligned}
$$

Kendall's τ represents the difference between the probability of two random concordant pairs and the probability of two random discordant pairs. For most copula functions with a single parameter θ there is a one-to-one relationship between θ and the Kendall's τ_2. For example, it holds that

$$
\tau_2(\text{Gaussian and } t) = \frac{2}{\pi} \arcsin \delta, \quad \tau_2(\text{Archimedean}) = 4 \int_0^1 \frac{\phi^{-1}(t)}{(\phi^{-1})'}\, dt + 1,
$$
$$
\tau_2(\Pi) = 0, \quad \tau_2(W) = 1, \quad \tau_2(M) = -1.
$$

For instance, this implies that an unknown copula parameter θ of the Gaussian, t and an arbitrary Archimedean copulae can be estimated using a type of method of moments procedure with a single moment condition. This requires, however, an estimator of τ_2, c.f. Kendall (1970). Naturally, it is computed by

$$
\tau_{2n} = \frac{4}{n(n-1)} P_n - 1,
$$

where n stands for the sample size and P_n denotes the number of concordant pairs, e.g. such pairs (X_1, X_2) and (X_1', X_2') that $(X_1 - X_1')(X_2 - X_2') > 0$. Next we provide the definition and similar results for the Spearman's ρ.

Definition 13.3 Let F be a continuous bivariate distribution function with the copula C and the univariate margins F_1 and F_2 respectively. Assume that $(X_1, X_2) \sim F$. Then the Spearman's ρ is given by

$$\rho_2 = 12 \int\limits_{\mathbb{R}^2} F_1(x_1) F_2(x_2) \, dF(x_1, x_2) = 12 \int\limits_{[0,1]^2} u_1 u_2 \, dC(u_1, u_2) - 3.$$

Similarly as for Kendall's τ, the relationship between Spearman's ρ and specific copulae is given through

$$\rho_2(\text{Gaussian and } t) = \frac{6}{\pi} \arcsin \frac{\delta}{2},$$
$$\rho_2(\Pi) = 0, \qquad \rho_2(W) = 1, \qquad \rho_2(M) = -1.$$

Unfortunately, there is no explicit representation of Spearman's ρ_2 for Archimedean in terms of generator functions as by Kendall's τ. The estimator of ρ is easily computed using

$$\rho_{2n} = \frac{12}{n(n+1)(n-1)} \sum_{i=1}^{n} R_i S_i - 3 \frac{n+1}{n-1},$$

where R_i and S_i denote the ranks of two samples. The exact regions determined by Kendall's τ and Spearman's ρ have been recently given by Schreyer et al. (2017).

13.3   Multivariate Copula: Primer and State-of-Art

As mentioned in the introduction, step from bivariate copulas to multivariate is large. Nevertheless, many works have been written properly different high-dimensional copulas. This section introduces simple multivariate models and most prominent families like hierarchical Archimedean copula (HAC), pair-copula constructions and factor copula.

A d-dimensional copula is also the distribution function on $[0, 1]^d$ having all marginal distributions uniform on $[0, 1]$. In Sklar's Theorem, the importance of copulas in the area of multivariate distributions is re-stated in an exquisite way.

Theorem 13.2 *Let F be a multivariate distribution function with margins $F_1, \ldots,$ F_d, then there exists the copula C such that*

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_k(x_d)\}, \quad x_1, \ldots, x_d \in \overline{\mathbb{R}}.$$

If F_i are continuous for $i = 1, \ldots, d$ then C is unique. Otherwise C is uniquely determined on $F_1(\overline{\mathbb{R}}) \times \cdots \times F_d(\overline{\mathbb{R}})$.

Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then function F defined above is a multivariate distribution function with margins $F_1, \ldots, F_d$.

As in the bivariate case, the representation in Sklar's Theorem can be used for constructing new multivariate distributions by changing either the copula function of

marginal distributions. For an arbitrary continuous multivariate distribution we can determine its copula from the transformation

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \quad u_1, \ldots, u_d \in [0, 1], \qquad (13.4)$$

where F_i^{-1} are inverse marginal distribution functions. Copula density and density of the multivariate distribution with respect to copula are

$$c(u_1, \ldots, u_d) = \frac{\partial^k C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}, \quad u_1, \ldots, u_d \in [0, 1],$$

$$f(x_1, \ldots, x_d) = c\{F_1(x_1), \ldots, F_d(x_d)\} \prod_{i=1}^{d} f_i(x_i), \quad x_1, \ldots, x_d \in \overline{\mathbb{R}}.$$

For the multivariate case as well as for the bivariate case copula functions are invariant under monotone transformations.

13.3.1 Extensions of Simple and Elliptical Bivariate Copulae

The independence copula and the upper and lower Fréchet–Hoeffding bounds can be straightforwardly generalized to the multivariate case. The independence copula is defined by the product $\Pi(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i$ and the bounds are given by

$$W(u_1, \ldots, u_d) = \max\Big(0, \sum_{i=1}^{d} u_i + 1 - d\Big),$$

$$M(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d), \quad u_1, \ldots, u_d \in [0, 1].$$

An arbitrary copula $C(u_1, \ldots, u_d)$ lies between the Fréchet–Hoeffdings bounds

$$W(u_1, \ldots, u_d) \leq C(u_1, \ldots, u_d) \leq M(u_1, \ldots, u_d),$$

where the Fréchet–Hoeffding lower bound is not a copula function for $d > 2$ though. The generalization of elliptical copulas to $d > 2$ is straightforward as well. For example, the Gaussian case yields

$$C_N(u_1, \ldots, u_d, \Sigma) = \Phi_\Sigma\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\},$$

$$c_N(u_1, \ldots, u_d, \Sigma) = |\Sigma|^{-1/2}$$
$$\exp\Big[-\frac{1}{2}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}^\top (\Sigma^{-1} - I)\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_k)\}\Big]$$

for all $u_1, \ldots, u_d \in [0, 1]$, where Φ_Σ is a d-dimensional Gaussian distribution with zero mean and correlation matrix Σ. Individual dispersion is imposed via the mar-

ginal distributions. Note that in the multivariate case the implementation of elliptical copulas can be involved due to technical difficulties with multivariate cdf's.

13.3.2 Hierarchical Archimedean Copula

A simple multivariate generalization of the Archimedean copulas is defined as

$$C(u_1, \ldots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \quad u_1, \ldots, u_d \in [0, 1], \quad (13.5)$$

where $\phi \in \mathcal{L}$. This definition provides a simple, but rather limited technique for the construction of multivariate copulas, since a possibly complicated multivariate dependence structure is determined by a single copula parameter. Furthermore, multivariate Archimedean copulas imply that the variables are exchangeable. This means, that the distribution of $(u_1, \ldots, u_d)$ is the same as of $(u_{j_1}, \ldots, u_{j_d})$ for all $j_\ell \neq j_v$. This is certainly not an acceptable assumption in practical applications.

A more flexible method is provided by hierarchical Archimedean copula (HAC) sometimes also called the nested Archimedean copula which replaces a uniform margin of a simple Archimedean copula by an additional Archimedean copula. The iterative substitution of margins by copulas widens the spectrum of attainable dependence structures. For example, the copula function for fully nested HAC is given by

$$\begin{aligned} C(u_1, \ldots, u_d) &= \phi_{d-1}\big\{\phi_{d-1}^{-1} \circ \phi_{d-2}\big(\ldots [\phi_2^{-1} \circ \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)\} \quad (13.6) \\ &\quad + \phi_2^{-1}(u_3)] + \cdots + \phi_{d-2}^{-1}(u_{d-1})\big) + \phi_{d-1}^{-1}(u_d)\big\} \\ &= \phi_{d-1}[\phi_{d-1}^{-1} \circ C(\{\phi_1, \ldots, \phi_{d-2}\})(u_1, \ldots, u_{d-1}) + \phi_{d-1}^{-1}(u_d)] \end{aligned}$$

for $\phi_{d-i}^{-1} \circ \phi_{d-j} \in \mathcal{L}^*$, $i < j$, where

$$\mathcal{L}^* = \{\omega : [0; \infty) \to [0, \infty) \,|\, \omega(0) = 0, \ \omega(\infty) = \infty; \ (-1)^{j-1}\omega^{(j)} \geq 0; \ j = 1, \ldots, \infty\},$$

As indicated above, contrarily to the usual Archimedean copula (13.5), HAC defines the dependency structure in a recursive way. At the lowest level of the so called HAC-tree, the dependency between the two variables is modeled by a copula function with the generator ϕ_1, i.e. $z_1 = C(u_1, u_2) = \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)\}$. At the second level, an another copula function is used to model the dependency between z_1 and u_3, etc. The generators ϕ_i can come from the same family and differ only through the parameter or, to introduce more flexibility, come from different generator families, c.f. Hofert (2011). As an alternative to the fully nested model, so-called partially nested copulas combine arbitrarily many copula functions at each copula level. For example the following 4-dimensional copula, where the first and the last two variables are joined by individual copulas with generators ϕ_{12} and ϕ_{34}. Further, the resulted copulas are combined by a copula with the generator ϕ.

$$C(u_1, u_2, u_3, u_4) = \phi\big(\phi^{-1}[\phi_{12}\{\phi_{12}^{-1}(u_1) + \phi_{12}^{-1}(u_2)\}] + \phi^{-1}[\phi_{34}\{\phi_{34}^{-1}(u_3) + \phi_{34}^{-1}(u_4)\}]\big).$$

The estimation of HAC is a challenging task, since both the copula structure and parameters of the generator functions have to be estimated. The variety of possible structures does not permit the enumeration of all possible structures and selecting that structure-parameter combination with the largest log-likelihood value.

Okhrin et al. (2013a) first propose methods for determining the optimal structure of HAC with (non-)parametrically estimated margins and provide asymptotic theory for the estimated parameters. The basic idea of the estimation procedure uses the fact that HAC are recursively defined and that dependencies decrease from the lowest to the highest hierarchical level for common parametric families. To sketch the procedure suppose margins are known: Parameters related to strongly dependent random variables are estimated first and the variables grouped at the bottom of the HAC-tree. The determined HAC-tree is spanned by at least two random variables and the tree itself determines a univariate random variable. After removing all random variables spanning the tree from the set of variables and adding the univariate random variable determined by the tree, the parameter of the subsequent level is determined by the selecting that pair of variables with the strongest dependency again. An additional level is added to the tree referring to the pair of variables with the strongest dependence and the set of variables is modified as explained above. The sketched steps are iteratively repeated until the HAC-tree is spanned by all random variables. This method is implemented in the `HAC` package for R, see Okhrin and Ristig (2014).

Segers and Uyttendaele (2014) introduce an algorithm for non-parametric structure determination by firstly decomposing the HAC's tree structure into four variants of trivariate structures. Then, the whole tree structure is subsequently determined based on testing the distance between trivariate copulas and Kendall's distribution function. Górecki et al. (2016) generalize the approach of Okhrin et al. (2013a) and propose an algorithm for simultaneous estimation of the structure and parameters based on the inversion of Kendall's τ_2, i.e. based on the link between Kendall's τ_2 and Archimedean generators.

Properties and simulation procedures are comprehensively studied in Joe (1997), Whelan (2004), Savu and Trede (2010), Hofert (2011), Okhrin et al. (2013b), Rezapour (2015) and Górecki et al. (2016). Note that HAC became a standard tool for pricing credit derivatives in academia such as collateralized debt obligations, see Hering et al. (2010), Hofert and Scherer (2011) and Choroś-Tomczyk et al. (2013).

Brechmann (2014) proposed hierarchical Kendall copula, which does not suffer from parameter restriction, but are slightly more complicated in estimation. Similar approach to avoid parameter restrictions and family limitations are proposed by using Lévy subordinated HAC, see Hering et al. (2010) and the corresponding application see Zhu et al. (2016).

13.3.3   Factor Copula

In classical factor analysis, a function links the observed and latent variables under the assumption that the latent variables explain the observed variables, e.g., see Johnson and Wichern (2013) and Härdle and Simar (2015). For example, a random variable $X_i, i = 1, \ldots, d$, is generated by an additive factor model, if

$$X_i = \sum_{j=1}^{m} \alpha_{ij} W_j + \varepsilon_i, \tag{13.7}$$

where $W_j, j = 1, \ldots, m$, are latent common factors and $\varepsilon_i, i = 1, \ldots, d$, are mutually independent idiosyncratic disturbances. The basic idea of factor models and their natural interpretation can be exported to the copula world in order to induce dependencies between independent idiosyncratic disturbances via common factors. Factor copula models, however, can be split into two complementary groups both having strengths and weaknesses. On the one hand, there are (implicit) factor copula models inducing dependencies among random variables via a functional which links latent factors and idiosyncratic disturbances. Such models are a straightforward extension of factor models from multivariate analysis. On the other hand, factor copulas and dependencies also arise from integrating the product of conditionally independent distributions –given a latent factor– with respect to this factor. This approach benefits from the fact, that the copula collapses to the product copula in case of known factors.

Oh and Patton (2015) concentrate on (implicit) factor copulas for $X = (X_1, \ldots, X_d)^{\top}$ arising from a functional relation between the factor(s) and mutual independent idiosyncratic errors. In this sense, the dependence component of the joint distribution of X is implied from the factors' distribution, the distribution of the idiosyncratic disturbances and the link function. In particular, X follows a multivariate distribution specified via a copula, i.e. $X \sim F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}$. For instance, the additive single factor copula model is represented as

$$X_i = W + \varepsilon_i, \ i = 1, \ldots, d, \tag{13.8}$$
$$W \sim F_W(\theta_W), \ \varepsilon_i \overset{i.i.d.}{\sim} F_\varepsilon(\theta_\varepsilon), \ W \perp \varepsilon_i, \quad \text{for all} \ \ i = 1, \ldots, d, ,$$

where W is the single common factor following the distribution of $F_W(\theta_W)$ and $\varepsilon_1, \ldots, \varepsilon_d$ are mutually independent shocks with distribution function $F_\varepsilon(\theta_\varepsilon)$. This model is extended to the non-linear factor copula based on the following representation,

$$Z_i = h(W, \varepsilon_i), \ i = 1, 2, \ldots, d, \tag{13.9}$$
$$W \sim F_W(\theta_W), \ \varepsilon_i \overset{i.i.d.}{\sim} F_\varepsilon(\theta_\varepsilon), \ W \perp \varepsilon_i, \quad \text{for all} \ \ i = 1, \ldots, d,$$

where h is a non necessarily linear link function. Thus, the dependence structure can be built in a more flexible way compared to the linear additive version. Model (13.8) implies a joint Gaussian random vector $X = (X_1, \ldots, X_d)^\top$, if the common factor and the idiosyncratic factor are both Gaussian. Therefore, a joint density function is available as well.

Nonetheless, a nice analytical expression of the joint density function for a factor copula with non-Gaussian margins and non-Gaussian factor is rarely available which makes parameter estimation demanding. Oh and Patton (2013) propose an estimation method for copula models without analytical form of the density function. This relies on a simulated method of moments approach building on the simplicity to draw random samples from a factor model. The proposed estimator for $(\theta_W^\top, \theta_\varepsilon^\top)^\top$ is found numerically by minimizing the distance between scale free empirical dependence measures between X_k and X_ℓ, such as $\tau_{2n}^{k\ell}$, $k = 1, \ldots, d$; $\ell = k+1, \ldots, d$, and those obtained from a drawn sample. Oh and Patton (2013) prove under weak regularity conditions that the simulated method of moment estimator is consistent and asymptotically normal. However, as argued by Genest et al. (1995), method of moment estimators of copula parameters can be highly inefficient.

Another form of factor copulae relies on the assumption that the observed variables $U_1, \ldots, U_d$ are conditionally independent given latent factors $V_1, \ldots, V_m$. Note that all random variables U_i, $i = 1, \ldots, d$, and V_j, $j = 1, \ldots, m$, are assumed to be uniformly distributed. Then, the conditional distribution of U_i given m factors $V_1, \ldots, V_m$ is given by $C_{U_i|V_1,\ldots,V_m}$. By using $C_{U_i|V_1,\ldots,V_m}$, the dependence structure of the observed variables $U_1, \ldots, U_d$ can be specified by the following copula function, such that

$$C(u_1, \ldots, u_d) = \int_{[0,1]^m} \prod_{i=1}^{d} C_{U_i|V_1,\ldots,V_m}(u_i|v_1, \ldots, v_m) dv_1 \cdots dv_m \quad \text{with} \quad u_i \in (0,1),$$

(13.10)

where the factors are out integrated. For the special case $m = 1$, the copula function (13.10) can be simplified to the form

$$C(u_1, \ldots, u_d) = \int_{[0,1]} \prod_{i=1}^{d} C_{U_i|V_1}(u_i|v_1) dv_1 \quad \text{with} \quad u_i \in (0,1).$$

(13.11)

Let C_{U_i,V_1} and c_{U_i,V_1} be the joint cdf and density of the pairs of random variables (U_i, V_1), $i = 1, \ldots, d$. Moreover, let the conditional distribution of U_i given V_1 be denoted by $C_{U_i|V_1}(u_i|v_1) = \partial C_{U_i,V_1}(u_i, v)/\partial v|_{v=v_1}$. Then, the copula density of $C(u_1, \ldots, u_d)$ can be represented by

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \cdots \partial u_d} = \int_{[0,1]} \prod_{i=1}^{d} c_{U_i,V_1}(u_i, v_1) dv_1 \quad \text{with} \quad u_i \in (0,1),$$

(13.12)

where $c_{U_i,V_1}(u_i, v_1) = \partial C(u_i|v_1)/\partial u_i$. Seen from this angle, the dependencies between d observed variables is determined by d bivariate copulas $C_{U_i,V_i}(u_i, v)$. Based on a parametric copula density $c(\cdot; \theta)$, Krupskii and Joe (2013) separate the parameter estimation into two steps. In the first step, the margins are estimated parametrically or non-parametrically. In the second step, the maximum likelihood (ML) method is employed to estimate the parameter θ.

Numerous literature about the factor copula's theory and applications can be referred to. Andersen et al. (2003), Hull and White (2004) and Laurent and Gregory (2005) have contributed works on generalization of one factor copula models. A comprehensive review of the factor copula theory is given in Joe (2014). Some applications by using factor copula models can be referred to Li (2000) for credit derivative pricing, Krupskii and Joe (2013) for fitting stock returns and Oh and Patton (2015) for measuring systemic risk.

13.3.4   Vine Copula

Vine copula or pair-copula constructions are originally proposed in Joe (1996) and developed in depth by Bedford and Cooke (2001), Bedford and Cooke (2002), Kurowicka and Cooke (2006) and Aas et al. (2009). The catchy name is due to similarities of the graphical representation of vine copulae and botanical vines. The fundamental idea of the vine copula is to construct a d-dimensional copula by decomposing the dependence structure into $d(d-1)/2$ bivariate copulas.

Let S be the index subset of $D = \{1, \ldots, d\}$ referring to the index set of conditioning variables and T be the index set of conditioned variables with $T \cup S = D$. Let $\sharp M$ denote the cardinality of set M. The cdf of variables with index in S is denoted by F_S, so that $F(x) = F_D(x)$. The conditional cdf of variables with index in T conditional on S is denoted $F_{T|S}$. A similar notation is used for the corresponding copulas. To derive a vine copula for a given $x = (x_1, \ldots, x_d)^\top$ in the spirit of Joe (2014), we start from a d-dimensional distribution function, i.e.

$$F(x) = \int_{(-\infty, x_S]} F_{T|S}(x_T|y_S) dF_S(y_S), \tag{13.13}$$

and replace the conditional distribution $F_{T|S}(x_T|x_S)$ by the corresponding $\sharp T$-dimensional copula $F_{T|S}(x_T|x_S) = C_{T;S}\{F_{j|S}(x_j|x_S) : j \in T\}$. The copula $C_{T;S}\{F_{j|S}(x_j|x_S) : j \in T\}$ is implied by Sklar's Theorem with margins $F_{j|S}(x_j|x_S)$, $j \in T$. It is not a conditional distribution although with conditional distribution as margins. This yields a copula-based representation of the joint d-dimensional distribution function from (13.13), which is given by

$$F(x) = \int_{(-\infty, x_S]} C_{T;S}\{F_{j|S}(x_j|y_S) : j \in T\} dF_S(y_S). \tag{13.14}$$

Note that the support of the integral in (13.13) and (13.14) is a cube $(-\infty, x_S] \in \mathbb{R}^{\sharp S}$. Converting all univariate margins to uniformly distributed random variables allows rewriting $F(x)$ as a d-dimensional copula

$$C(u) = \int_{[0,u_S]} C_{T;S}\{G_{j|S}(u_j|v_S) : j \in T\}dC_S(v_S), \qquad (13.15)$$

where $G_{j|S}(u_j|v_S)$ is a conditional distribution from copula $C_{S\cup\{j\}}$. If $T = \{i_1, i_2\}$, then

$$C_{S\cup\{i_1,i_2\}}(u_{S\cup\{i_1,i_2\}}) = \int_{[0,u_S]} C_{i_1,i_2;S}\{G_{i_1|S}(u_{i_1}|v_S), G_{i_2|S}(u_{i_2}|v_S)\}dC_S(v_S). \quad (13.16)$$

Since the essential idea of vine copula is based on building a joint dependence structure by $d(d-1)/2$ bivariate copulae, (13.16) is an important building block in the construction of vines referring to a $(\sharp S + 2)$-dimensional copula built from a bivariate copula $C_{i_1,i_2;S}$.

In case of continuous random variables, the d-dimensional distribution function from (13.13) admits a density function $f(x_1, \ldots, x_d)$, which can be decomposed and represented by bivariate copula densities in an analogue manner. Examples of density decompositions for the 6-dimensional case related to so called C-vine (canonical vine), D-vine (drawable vine) and R-vine (regular vine) copulas are given as follows.

The C-vine structure is illustrated in the left column of Fig. 13.2 and its density decomposition is

$$c\{F_1(x_1), \ldots, F_6(x_6)\} = c_{12}\{F_1(x_1), F_2(x_2)\} \cdot c_{13}\{F_1(x_1), F_3(x_3)\} \qquad (13.17)$$
$$\cdot c_{14}\{F_1(x_1), F_4(x_4)\} \cdot c_{15}\{F_1(x_1), F_5(x_5)\} \cdot c_{16}\{F_1(x_1), F_6(x_6)\}$$
$$\cdot c_{23;1}\{F(x_2|x_1), F(x_3|x_1)\} \cdot c_{24;1}\{F(x_2|x_1), F(x_4|x_1)\}$$
$$\cdot c_{25;1}\{F(x_2|x_1), F(x_5|x_1)\} \cdot c_{26;1}\{F(x_2|x_1), F(x_6|x_1)\}$$
$$\cdot c_{34;12}\{F(x_3|x_{12}), F(x_4|x_{12})\} \cdot c_{35;12}\{F(x_3|x_{12}), F(x_5|x_{12})\}$$
$$\cdot c_{36;12}\{F(x_3|x_{12}), F(x_6|x_{12})\} \cdot c_{45;123}\{F(x_4|x_{123}), F(x_5|x_{123})\}$$
$$\cdot c_{46;123}\{F(x_4|x_{123}), F(x_6|x_{123})\} \cdot c_{56;1234}\{F(x_5|x_{1234}), F(x_6|x_{1234})\}.$$

The density of the D-vine structure –given in the centred column of Fig. 13.2— is

$$c\{F_1(x_1), \ldots, F_6(x_6)\} = c_{12}\{F_1(x_1), F_2(x_2)\} \cdot c_{23}\{F_2(x_2), F_3(x_3)\} \qquad (13.18)$$
$$\cdot c_{34}\{F_3(x_3), F_4(x_4)\} \cdot c_{45}\{F_4(x_4), F_5(x_5)\} \cdot c_{56}\{F_5(x_5), F_6(x_6)\}$$
$$\cdot c_{13;2}\{F(x_1|x_3), F(x_2|x_3)\} \cdot c_{24;3}\{F(x_2|x_3), F(x_4|x_3)\}$$
$$\cdot c_{35;4}\{F(x_3|x_4), F(x_5|x_4)\} \cdot c_{46;5}\{F(x_4|x_5), F(x_6|x_5)\}$$
$$\cdot c_{14;23}\{F(x_2|x_{23}), F(x_4|x_{23})\} \cdot c_{25;34}\{F(x_2|x_{34}), F(x_5|x_{34})\}$$
$$\cdot c_{36;45}\{F(x_3|x_{45}), F(x_6|x_{45})\} \cdot c_{15;234}\{F(x_1|x_{234}), F(x_5|x_{234})\}$$
$$\cdot c_{26;345}\{F(x_2|x_{345}), F(x_6|x_{345})\} \cdot c_{16;2345}\{F(x_1|x_{2345}), F(x_6|x_{2345})\}.$$

Fig. 13.2 Vine tree structures of C-vine, D-vine and R-vine

The density of the R-vine structure illustrated in the right column of Fig. 13.2 is

$$
\begin{aligned}
c\{F_1(x_1), \ldots, F_6(x_6)\} = {} & c_{12}\{F_1(x_1), F_2(x_2)\} \cdot c_{23}\{F_2(x_2), F_3(x_3)\} \quad (13.19) \\
& \cdot c_{34}\{F_3(x_3), F_4(x_4)\} \cdot c_{25}\{F_2(x_2), F_5(x_5)\} \cdot c_{36}\{F_3(x_3), F_6(x_6)\} \\
& \cdot c_{13;2}\{F(x_1|x_2), F(x_3|x_2)\} \cdot c_{24;3}\{F(x_2|x_3), F(x_4|x_3)\} \\
& \cdot c_{26;3}\{F(x_2|x_3), F(x_6|x_3)\} \cdot c_{35;2}\{F(x_3|x_2), F(x_5|x_2)\} \\
& \cdot c_{15;23}\{F(x_1|x_{23}), F(x_5|x_{23})\} \cdot c_{56;23}\{F(x_5|x_{23}), F(x_6|x_{23})\} \\
& \cdot c_{46;23}\{F(x_4|x_{23}), F(x_6|x_{23})\} \cdot c_{16;235}\{F(x_1|x_{235}), F(x_6|x_{235})\} \\
& \cdot c_{45;236}\{F(x_4|x_{236}), F(x_5|x_{236})\} \cdot c_{14;2356}\{F(x_1|x_{2356}), F(x_4|x_{2356})\}.
\end{aligned}
$$

In particular, the C-vine and D-vine have an intuitive graphical representation which can be immediately related to the decomposition of the copula density function into the product of bivariate copula densities. For example, the product of bivariate copula densities from the first two lines of the right hand side of Eq. 13.17 refers

to a C-vine represented in the upper left graphic of Fig. 13.2. The formula and the corresponding graphic illustrate that the first variable X_1 is pairwise coupled with the second, third ... and sixth random variable. The subsequent two lines (3–4) of Eq. 13.17 are related to the second graphic of the left column of Fig. 13.2. Conditional on X_1, random variable X_2 is pairwise coupled with X_3, X_4, X_5 and X_6. Connecting the remaining graphics with formulas is left to the reader. While the "formula-graphic" matching follows a similar scheme in case of the D-vine, the R-vine belongs to a more general vine copula class and contains the C-vine and D-vine as special cases. A rigorous definition of an R-vine copula can be found in Joe (2014).

In fact, vines can be estimated by either full or stage-wise ML such as the inference function for margins (IFM) method discussed below in Sect. 13.4. Nonetheless, the inference approach derived in Haff (2013) namely the stepwise semi-parametric estimator deserves to be mentioned in more detail. Here, the marginal distributions are non-parametrically estimated by the empirical distribution function such as for factor copulae or HAC. In order to obtain a consistent and asymptotically Gaussian distributed estimator of a parametric vine copula, a so called simplifying assumption is required. The latter permits replacing "conditional" bivariate copula densities with unconditional densities. Then, it can be straightforwardly shown, that the log-likelihood can be maximized in a stage-wise manner. This is due to the decomposition of the density into the product of bivariate copula densities, so that the log-likelihood function is a sum of logarithmized copula densities. Coming back to the C-vine example from Fig. 13.2. At the first stage, all parameters of bivariate copulas represented in the upper left graphic of Fig. 13.2 are estimated, i.e. the parameters of the copulae for $(X_1, X_2), \ldots, (X_1, X_6)$. Keeping the corresponding parameters fixed at estimated values, the four parameters of copulae referring to the pairs from the second graphic of the left column of Fig. 13.2 are estimated. Holding these parameters fixed at estimated values again, all vine parameters of the remaining bivariate densities can be estimated iteratively. Literature on pair-copula construction is spreading steadily, and most recent information about it can be found on vine copula homepage http://www.statistics.ma.tum.de/en/research/vine-copula-models/.

13.4   Estimation Methods

The estimation of a copula-based multivariate distribution involves both the estimation of the copula parameters θ and the estimation of the margins F_j, $j = 1, \ldots, d$. The properties and goodness of the estimator of θ heavily depend on the estimators of F_j, $j = 1, \ldots, d$. We distinguish between a parametric and a non-parametric specification of the margins. If we are interested only in the dependency structure, the estimator of θ should be independent of any parametric models for the margins. However, Joe (1997) argues that complete distribution models and, therefore, parametric models for margins are actually more appropriate for applications.

In the bivariate case, a standard method of estimating the univariate parameter θ is based on Kendall's τ_2 statistic by Genest and Rivest (1993). The estimator of τ_2 complemented by the method of moments allows to estimate the parameters. However, as shown in Genest et al. (1995), the ML method leads to substantially more efficient estimators. For non-parametrically estimated margins, Genest et al. (1995) show the consistency and asymptotic normality of ML estimators and derive the moments of the asymptotic distribution. The ML procedure can be performed simultaneously for the parameters of the margins and of the copula function. Alternatively, a two-stage procedure can be applied, where the parameters of margins are estimated at the first stage and the copula parameters at the second stage, see Joe (1997) and Joe (2005). Chen and Fan (2006) and Chen et al. (2006) analyze the case of non-parametrically estimated margins. Fermanian and Scaillet (2003) and Chen and Huang (2007) consider a fully non-parametric estimation of the copula. Next we provide details on both approaches. Note that estimation procedures for HAC, conditional-independence-based factor copulas and vines are in fact generalizations of the subsequent approaches taking specific needs of the copula into account, e.g., parameter restrictions.

13.4.1 Parametric Margins

Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1^\top, \ldots, \boldsymbol{\alpha}_d^\top)^\top$ denote the vector of parameters of marginal distributions and $\boldsymbol{\theta}$ parameters of the copula. The classical full ML estimator $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\theta}^\top)^\top$ solves the system of equations

$$\frac{\partial \mathcal{L}(\boldsymbol{\eta}, \mathbf{X})}{\partial \boldsymbol{\eta}} = \mathbf{0},$$

$$\text{where} \quad \mathcal{L}(\boldsymbol{\eta}, \mathbf{X}) = \sum_{i=1}^{n} \log \left\{ c(F_1(x_{1i}, \boldsymbol{\alpha}_1), \ldots, F_d(x_{di}, \boldsymbol{\alpha}_d), \boldsymbol{\theta}) \prod_{j=1}^{d} f_j(x_{ji}, \boldsymbol{\alpha}_j) \right\}$$

$$= \sum_{i=1}^{n} \left\{ \log c(F_1(x_{1i}, \boldsymbol{\alpha}_1), \ldots, F_d(x_{di}, \boldsymbol{\alpha}_k), \boldsymbol{\theta}) + \sum_{j=1}^{d} \log f_j(x_{ji}, \boldsymbol{\alpha}_j) \right\}.$$

Following the standard theory on ML estimation, the estimator $\hat{\boldsymbol{\eta}}$ is efficient and asymptotically normal. However, it is often computationally demanding to solve the system simultaneously. Alternatively the multistage optimization proposed in Joe (1997), also known as inference functions for margins, can be applied: Firstly, the parameters of the margins are separately estimated under the assumption that the copula is the product copula. Secondly, the parameters of the copula are estimated replacing the parameters of margins by estimates from the first step and treating them as known quantities. The above optimization problem is then replaced by

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\alpha}_1^\top}, \ldots, \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\alpha}_d^\top}, \frac{\partial \mathcal{L}_{d+1}}{\partial \boldsymbol{\theta}^\top} \right)^\top = \mathbf{0}, \tag{13.20}$$

where $\mathcal{L}_j = \sum_{i=1}^n l_j(\mathbf{X}_i), \quad \text{for} \quad j = 1, \ldots, d+1,$

$l_j(\mathbf{X}_i) = \log f_j(x_{ji}, \boldsymbol{\alpha}_j), \quad \text{for} \quad j = 1, \ldots, d, i = 1, \ldots, n,$

and $l_{d+1}(\mathbf{X}_i) = \log c \big\{ F_1(x_{1i}, \boldsymbol{\alpha}_1), \ldots, F_d(x_{di}, \boldsymbol{\alpha}_d), \boldsymbol{\theta} \big\}, \quad \text{for} \quad i = 1, \ldots, n.$

The first d components in (13.20) correspond to the usual ML estimation of the parameters of the marginal distributions. The last component reflects the estimation of the copula parameters. Detailed discussion on this method can be found in Joe (1997). Note, that this procedure does not lead to efficient estimators, however, as argued by Joe (1997) the loss in the efficiency is modest and mainly depends on the strength of dependencies. This method is a special case of the generalized method of moments with an identity weighting matrix, see Cherubini et al. (2004). The advantage of the two-stage procedure lies in the dramatic reduction of the numerical complexity.

13.4.2 Non-parametric Margins

In this section, we consider a non-parametric estimation of the marginal distributions also referred to as canonical ML. The asymptotic properties of the multistage estimator for $\boldsymbol{\theta}$ do not depend explicitly on the type of the non-parametric estimator, but on its convergence properties. Here, we use the rectangular kernel (histogram) resulting in the estimator

$$\widehat{F}_j(x) = (n+1)^{-1} \sum_{i=1}^n \mathbf{1}(x_{ji} \leq x), \quad j = 1, \ldots, d.$$

The factor $n/(n+1)$ is used to restrict fitted values to the open unit interval. This is necessary as several copula densities are not bounded at zero and/or one. Let $\widehat{F}_1, \ldots, \widehat{F}_d$ denote the non-parametric estimators of $F_1, \ldots, F_d$. The canonical ML estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ solves the system $\partial \mathcal{L}/\partial \boldsymbol{\theta}^\top = \mathbf{0}$ by maximizing the pseudo log-likelihood with estimated margins $\widehat{F}_1, \ldots, \widehat{F}_d$, i.e.

$$\mathcal{L} = \sum_{i=1}^n l(\mathbf{X}_i) \quad \text{for} \quad j = 1, \ldots, p,$$

$$l(\mathbf{X}_i) = \log c \big\{ \widehat{F}_1(x_{1i}), \ldots, \widehat{F}_d(x_{di}), \boldsymbol{\theta} \big\}, \quad \text{for} \quad i = 1, \ldots, n.$$

As in the parametric case, the semi-parametric estimator $\hat{\boldsymbol{\theta}}$ is asymptotically normal under suitable regularity conditions. This method was first used in Oakes (1994)

and then investigated by Genest et al. (1995) and Shih and Louis (1995). Additional properties of the estimator, such as the covariance matrix, are stated in these papers.

13.5   Goodness-of-Fit Tests for Copulae

Having a dataset and an estimated copula at hand, it arises the natural question whether the selected copula describes the data properly. For this purpose, a series of different goodness-of-fit tests has been developed in the last decade. Under the H_0-hypothesis one assumes that the true copula belongs to some parametric family $H_0 : C \in C_0$.

The most natural test approach is to measure the deviation of the parametric copula from the empirical one given through

$$C_n(u_1, \ldots, u_d) = n^{-1} \sum_{i=1}^{n} \prod_{j=1}^{d} I\{\widehat{F}_j(x_{ij}) \leq u_j\}.$$

Gaensler and Stute (1987) and Radulovic and Wegkamp (2004) show that C_n is a consistent estimation of the true underlying copula. Several tests are based on the empirical copula process, which is defined as follows

$$\mathbb{C}_n(u_1, \ldots, u_d) = \sqrt{n}\{C_n(u_1, \ldots, u_d) - C_{\hat{\theta}}(u_1, \ldots, u_d)\}.$$

Fermanian (2005) and Genest and Rèmillard (2008) propose to compute different measures to quantify the deviation of the assumed parametric copula from the empirical copula, one of those is Cramér–von Mises distance

$$S_n^E = \int_{[0,1]^d} \mathbb{C}_n(u_1, \ldots, u_d)^2 dC_n(u_1, \ldots, u_d)$$

or the weighted Cramér–von Mises distance, with tuning parameters $m \geq 0$ and $\zeta_m \geq 0$ given as

$$R_n^E = \int_{[0,1]^d} \left\{ \frac{\mathbb{C}_n(u_1, \ldots, u_d)}{[C_{\hat{\theta}}(u_1, \ldots, u_d)\{1 - C_{\hat{\theta}}(u_1, \ldots, u_d)\} + \zeta_m]^m} \right\}^2 dC_n(u_1, \ldots, u_d).$$

The usual Kolmogorov–Smirnov distance as for classical univariate tests is also applicable here

$$T_n^E = \sup_{\{u_1, \ldots, u_d\} \in [0,1]^d} |\mathbb{C}_n(u_1, \ldots, u_d)|.$$

The other group of tests developed and investigated by Genest and Rivest (1993), Wang and Wells (2000), Genest et al. (2006) are based on the probability integral transform and in particular on so called Kendall's transform. Having

$$(X_1, \ldots, X_d) \sim F(x_1, \ldots, x_d) = C_\theta\{F_1(x_1), \ldots, F_d(x_d)\},$$

one concludes similar to $F_i(X_i) \sim U(0, 1)$ that the copula-based random variable is

$$C_\theta\{F_1(X_1), \ldots, F_d(X_d)\} \sim K_\theta(v)$$

where $K_\theta(v)$ is the univariate Kendall's distribution (not necessarily uniform), see Barbe et al. (1996), Jouini and Clemen (1996). Empirically, the distribution function K can be estimated as

$$K_n(v) = n^{-1} \sum_{i=1}^n I\left[C_n\{\widehat{F}_1(x_{i1}), \ldots, \widehat{F}_d(x_{id})\} \le v\right], \quad v \in [0, 1].$$

Further usual test statistics for the univariate distributions like Cramér–von Mises or Kolmogorov–Smirnov, see Genest et al. (2006), can be applied

$$S_n^{(K)} = \int_0^1 \mathbb{K}_n(v)^2 dK_{\hat{\theta}}(v), \qquad T_n^{(K)} = \sup_{v \in [0,1]} |\mathbb{K}_n(v)|,$$

where $\mathbb{K}_n = \sqrt{n}(K_n - K_{\hat{\theta}})$ is the Kendall's process. Here is, however, a little challenge in using this tests: as in testing for Kendall's distribution one tests in null hypothesis has $H_0'' : K \in \mathcal{K}_0 = \{K_\theta : \theta \in \Theta\}$, and as $H_0 \subset H_0''$, the non-rejection of H_0'' does not imply non rejection of H_0. For the bivariate Archimedean copulas H_0'' and H_0 are equivalent.

Another series of goodness-of-fit tests, is constructed via the other important integral transform, that dates back to Rosenblatt (1952). Based on the conditional distribution of U_i by

$$
\begin{aligned}
C_d(u_i|u_1, \ldots, u_{i-1}) &= \mathrm{P}\{U_i \le u_i | U_1 = u_1 \ldots U_{i-1} = u_{i-1}\} \\
&= \frac{\partial^{i-1} C(u_1, \ldots, u_i, 1, \ldots, 1)/\partial u_1 \ldots \partial u_{i-1}}{\partial^{i-1} C(u_1, \ldots, u_{i-1}, 1, \ldots, 1)/\partial u_1 \ldots \partial u_{i-1}},
\end{aligned}
$$

the Rosenblatt transform is defined as follows.

Definition 13.4 Rosenblatt's probability integral transform of a copula C is the mapping $\mathfrak{R} : (0, 1)^d \to (0, 1)^d$, $\mathfrak{R}(u_1, \ldots, u_d) = (e_1, \ldots, e_d)$ with $e_1 = u_1$ and $e_i = C_d(u_i|u_1, \ldots, u_{i-1})$, $\forall i = 2, \ldots, d$.

Under this definition, the null hypothesis $H_0 : C \in C_0$ can be rewritten as $H_{0R} :$ $(e_1, \ldots, e_d)^\top \sim \Pi$. The first test based on the Rosenblatt transform exploits information, that under H_0 transformed observations should be exactly uniform distributed

and independent, which is not the case, as those variables as not mutually independent and only approximately uniform. Nevertheless, two tests use Anderson–Darling test statistics, see Breymann et al. (2003), and are constructed as

$$T_n = -n - \sum_{i=1}^{n} \frac{2i - 1}{n} [\log G_{(i)} + \log\{1 - G_{(n+1-i)}\}]$$

where G_i might be constructed in two ways. In the first possibility

$$G_{i,Gamma} = \Gamma_d \left\{ \sum_{j=1}^{d} (-\log e_{ij}) \right\},$$

where $\Gamma_d(\cdot)$ is the Gamma distribution with shape d and scale 1. The second way takes

$$G_{i,\chi^2} = \chi_d^2 \left[\sum_{j=1}^{d} \{\Phi^{-1}(e_{ij})\}^2 \right],$$

where χ_d^2 refers to the Chi-squared distribution with d degrees of freedom and Φ is standard normal distribution. Another possibility compares the variables not via the Anderson–Darling test statistics, but by purely deviations between estimated density functions, as in Patton et al. (2004), where the test statistics is constructed by

$$C_n^{Ch} = \frac{n\sqrt{h}\hat{J}_n - c_n}{\sigma}$$

with c_n and σ are normalization factors and $\hat{J}_n = \int_0^1 \{\frac{1}{n} \sum_{i=1}^{n} K_h(w, G_{i,\chi^2}) - 1\}^2 dw$.

As discussed by Dobrić and Schmid (2007), the problem with those tests is that they have almost no power and even do not capture the type 1 error. Much better power have tests, that work directly on the copulas of the Rosenblatt transformed data, see Genest et al. (2009). The idea is to compute Cramer–von Mises statistics of the following form

$$S_n = n \int_{[0,1]^d} \{D_n(u) - \Pi(u)\}^2 du$$

$$S_n^{(C)} = n \int_{[0,1]^d} \{D_n(u) - \Pi(u)\}^2 dD_n(u)$$

where the empirical distribution function

$$D_n(u) = D_n(u_1, \ldots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} I(e_{ij} \leq u_j)$$

should be "close" to product copula Π under H_0.

Different from previous test are those based on the kernel density estimators, and just to mention one, let us consider test developed by Scaillet (2007), where the test statistics is given through

$$J_n = \int_{[0,1]^d} \{\hat{c}(u) - K_H * c(u; \hat{\theta})\} w(u) du,$$

with "$*$" being a convolution operator and $w(u)$ a weight function. The kernel function $K_H(y) = K(H^{-1}y)/\det(H)$ where K is the bivariate quadratic kernel with the bandwidth $H = 2.6073 n^{-1/6} \widehat{\Sigma}^{1/2}$ and $\widehat{\Sigma}$ being a sample covariance matrix. The copula density is estimated non-parametrically as

$$\hat{c}(u) = n^{-1} \sum_{i=1}^{n} K_H[u - \{\widehat{F}_1(X_{i1}), \ldots, \widehat{F}_d(X_{id})\}^\top],$$

where $\widehat{F}_j$ refers to an estimated marginal distribution, $j = 1, \ldots, d$. The most recent goodness of fit test for copulas have been proposed recently by Zhang et al. (2016), where one compares the two-step pseudo maximum likelihood:

$$\hat{\theta} = \operatorname*{argmax}_{\theta \in \Theta} \sum_{i=1}^{n} \mathcal{L}\{\widehat{F}_1(X_{i1}), \ldots, \widehat{F}_d(X_{id}); \theta\}.$$

with the delete-one-block pseudo maximum likelihood $\hat{\theta}_{-b}$, $1 \leq b \leq B$:

$$\hat{\theta}_{-b} = \operatorname*{argmax}_{\theta \in \Theta} \sum_{b' \neq b}^{B} \sum_{i=1}^{m} \mathcal{L}\{\widehat{F}_1(X_{i1}), \ldots, \widehat{F}_d(X_{id}); \theta\}, \quad b = 1, \ldots, B.$$

Further, "in-sample" and "out-of-sample" pseudo-likelihoods are compared with the following test statistic:

$$T_n(m) = \sum_{b=1}^{B} \sum_{i=1}^{m} \left[\mathcal{L}\{\widehat{F}_1(X_{i1}), \ldots, \widehat{F}_d(X_{id}); \hat{\theta}\} - \mathcal{L}\{\widehat{F}_1(X_{i1}), \ldots, \widehat{F}_d(X_{id}); \hat{\theta}_{-b}\} \right].$$

This leads to some challenges, like computation of $[\frac{n}{m}]$ dependence parameters, but Zhang et al. (2016) proposed an asymptotically equivalent test statistics based on variability and sensitivity matrices. As most of the above mentioned tests, have complicated asymptotic distributions, p-values of the tests can be performed via the parametric bootstrap sketched in the subsequent procedure:

Step 1   Generate bootstrap sample $\left\{\epsilon_i^{(k)}, i = 1, \ldots, n\right\}$ from copula $C(u; \hat{\theta})$ under H_0 with $\hat{\theta}$ and estimated marginal distribution $\widehat{F}$ obtained from original data;

Step 2   Based on $\left\{\epsilon_i^{(k)}, i = 1, \ldots, n\right\}$ from Step 1, estimate θ of the copula under H_0, and compute test statistics under consideration, say R_n^k;

Step 3   Repeat Steps (1–2) N-times and obtain N statistics $R_n^k, k = 1, \ldots, N$;

Step 4   Compute an empirical p-value as $p_e = N^{-1} \sum_{k=1}^N I\left(|R_n^k| \geq |R_n|\right)$ with R_n being the test statistic estimated from original data.

13.6 Empirical Study

Value-at-Risk (VaR) is an important measure in risk management. The traditional models for VaR estimation assume that the assets returns in a portfolio are jointly normally distributed. However, numerous empirical studies show that Gaussian based models are not sufficient to describe data characteristics, especially when extreme events happen such as financial crisis. The weak points of the Gaussian based models include the lack of asymmetry and tail dependence. Therefore copula methods come into the focus.

Twelve different copulas are used in this study to construct dependence structures. The employed families include the Gaussian copula, t-copula, Archimedean copulas (Clayton, Gumbel, Joe), HAC (Gumbel, Clayton, Frank), C- and D-vine structures and two factor copulas linked individually by a bivariate Gumbel and Clayton copula.

The data set utilized in this study includes five time series of stock close prices containing ADI (Analog Devices, Inc.), AVB (Avalonbay Communities Inc.), EQR (Equity Residential), LLY (Eli Lilly and Company) and TXN (Texas Instruments Inc.), from Yahoo finance. Here, ADI and TXN belong to high-tech industry, AVB and EQR to real estate industry and LLY to pharmacy industry. The time window spans from 20070113 to 20160116.

Let $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ denote the long position vector of a d-dimensional portfolio, $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$ stand for the vector of asset prices at time $t \in \{1, \ldots, T\}$ and $X_{i,t} = \log(S_{i,t}/S_{i,t-1})$ for the one period log-return of the i-th asset at time t. Then, $L_t = \sum_{i=1}^d w_i X_{i,t}$ denotes the portfolio return. The distribution function of the univariate random variable L_t is denoted by $F_{L_t}(x) = P(L_t \leq x)$ and the Value-at-Risk at level α for the portfolio is defined as the inverse of $F_{L_t}(x)$, namely $\text{VaR}_t(\alpha) = F_{L_t}^{-1}(\alpha)$.

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From the above formulations can be concluded that the idiosyncratic dependence of the log-return process $\{X_t\}_{t=1}^T$ is crucial for the appropriate estimation of the VaR. To remove temporal dependence from X_t, the single log-return processes are filtered through GARCH(1, 1) processes,
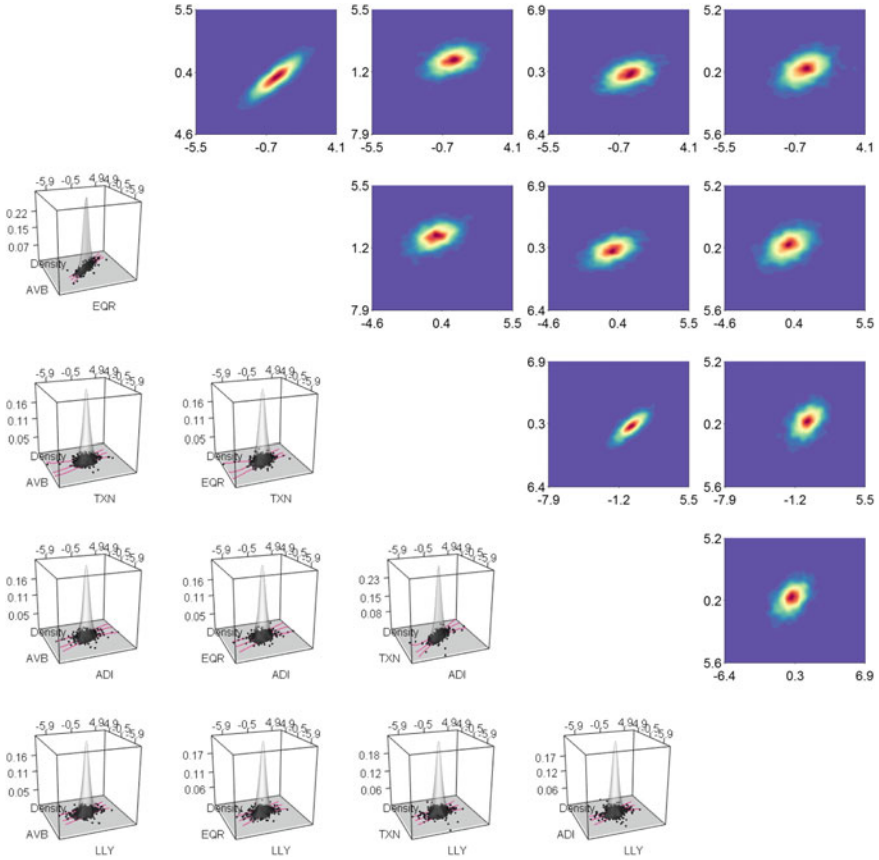
Fig. 13.3 The *lower* triangular plots give 2-dimensional kernel density estimations containing scatter plots of pairwise GARCH(1, 1)-filtered log-returns with quantile regressions under 0.05, 0.5, 0.95 quantiles. The *upper* triangular plots give pairwise contours of five variables

Table 13.1 Pairwise dependence measures including Pearson's correlation (*left*), Kendall's correlation (*center*) and Spearman's correlation (*right*)

	AVB	EQR	TXN	ADI	AVB	EQR	TXN	ADI	AVB	EQR	TXN	ADI
EQR	0.867				0.686				0.866			
TXN	0.359	0.375			0.260	0.264			0.376	0.381		
ADI	0.384	0.399	0.752		0.277	0.285	0.583		0.398	0.410	0.770	
LLY	0.358	0.370	0.358	0.362	0.268	0.260	0.272	0.270	0.390	0.376	0.393	0.391

Table 13.2 Exceeding ratios based on $\alpha \in \{0.05, \ 0.01, \ 0.005, \ 0.001\}$

Copula	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
Gaussian	**0.050**	0.018	0.009	0.004
t	0.048	0.014	0.011	0.005
Clayton	0.047	0.017	0.011	**0.002**
Gumbel	0.048	0.025	0.013	0.005
Joe	0.065	0.032	0.030	0.023
C-Vine	0.045	0.019	0.015	0.008
D-Vine	0.044	0.018	0.012	0.007
HAC-Clayton	0.044	**0.013**	**0.008**	0.003
HAC-Frank	0.055	0.033	0.026	0.016
HAC-Gumbel	0.070	0.036	0.028	0.017
Factor-Frank	0.046	0.026	0.017	0.015
Factor-Gumbel	0.086	0.042	0.032	0.024

$$X_{i,t} = \mu_{i,t} + \sigma_{i,t}\epsilon_{i,t}, \tag{13.21}$$

$$\sigma_{i,t}^2 = a_i + \alpha_i(X_{i,t-1} - \mu_{i,t-1})^2 + \beta_i\sigma_{i,t-1}^2. \tag{13.22}$$

The GARCH(1, 1)-filtered log-returns are illustrated in Fig. 13.3. Obviously, assets coming from the same sector have high correlation according to the GARCH residuals. For example, the AVB-EQR and TXN-ADI pairs have strong correlation coming from real estate industry and high technology industry respectively. The strong correlation is also observed in Table 13.1 presenting three dependence measures for pairs of AVB-EQR and TXN-ADI. LLY is from pharmacy industry and shows weak correlation with the other four companies according to the scatter-plots and the contours.

The performance of different copulas utilized for VaR estimation is evaluated via backtesting based on the exceeding ratio

$$\text{ER}^\alpha = (T - w)^{-1} \sum_{t=w}^{T} \mathbf{1}\{l_t < \widehat{\text{VaR}}_t(\alpha)\}, \tag{13.23}$$

where w is the sliding window size and l_t is the realization of L_t. For the twelve copulas, Table 13.2 presents the ERs which is optimal if it equals α. The Gaussian copula performs best for $\alpha = 0.05$, the HAC-Clayton copula has reached the most appropriate ER for $\alpha \in \{0.01, 0.005\}$ and the Clayton copula for $\alpha = 0.001$. The Factor-Gumbel copula provides the worst ER values for all values of α. Vines perform neither outstanding good nor bad. It deserves to be mentioned that copulas exhibiting upper-tail dependence show higher ER values, for instance, Joe copula, HAC-Gumbel copula and Factor-Gumbel copula. Even though some copulas are based on more parameters and thus, offer more flexibility, the increase of parameters does not essentially improve the ER (see Fig. 13.4).
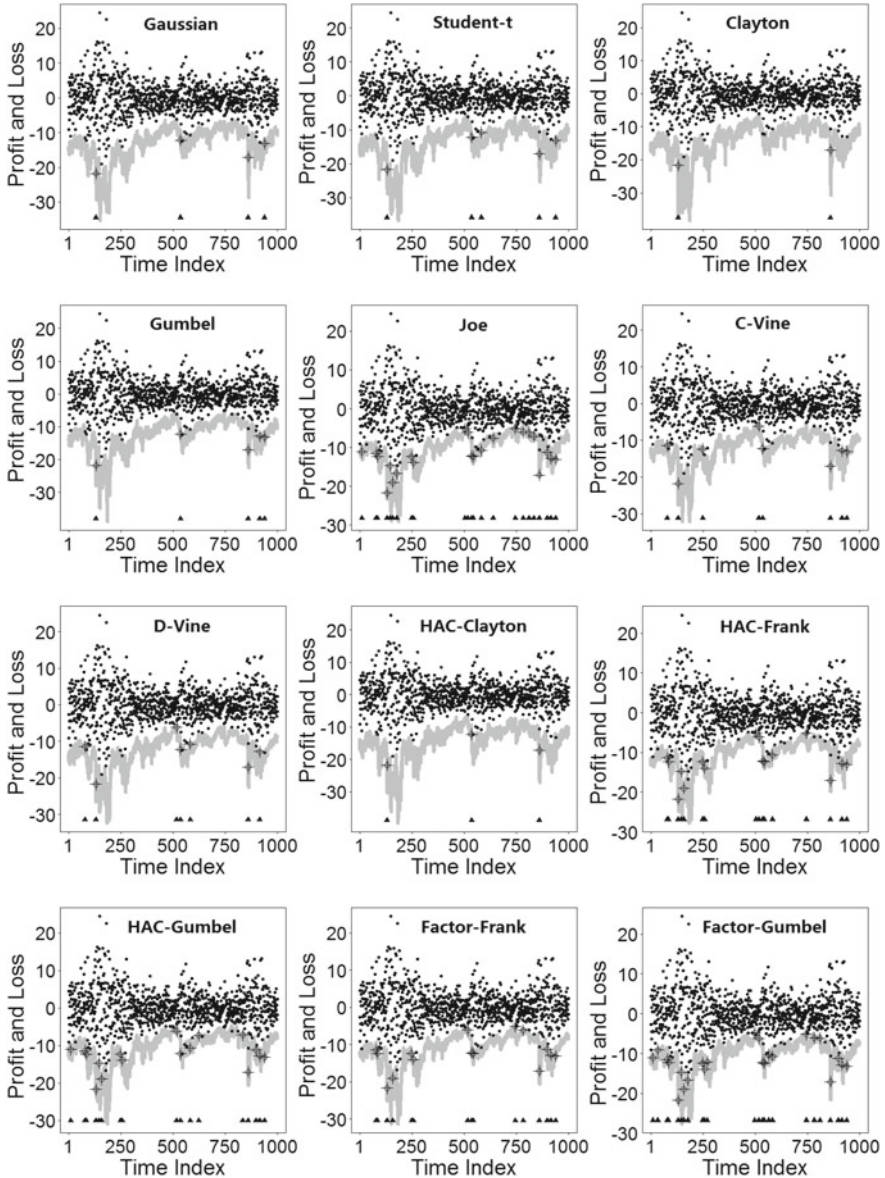
Fig. 13.4 VaRs for $\alpha = 0.001$ are constructed based on 1000 back-testing points with copulas of Gaussian, t, Clayton, Gumbel, Joe, C-Vine, D-Vine, HAC-Clayton, HAC-Frank, HAC-Gumbel, Factor-Frank, Factor-Gumbel, illustrated by row. XFGCHD_VaR_CVine, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_CVine, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_Clayton, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_DVine, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_Gaussian, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_Gumbel, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_Joe, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_StuT, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_hacClayton, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_hacFrank, https://github.com/QuantLet/XFG3/tree/master/XFGCHD_VaR_hacGumbel

13.7 Conclusion

This work discusses bivariate copula and focuses on three high dimensional copula models including the hierarchical Archimedean copula, the factor copula and the vine copula. The three models are developed in-depth with their advantages in modeling high dimensional data for diverse research fields. For the sake of comparison, an empirical study from risk management is presented. In this study, the estimation of Value-at-Risk is performed under 12 different copula models including the discussed state-of-art copulas as well as some classical benchmarks such as some of the elliptical and Archimedean family. Considered in toto, the hierarchical Archimedean copula with Clayton generator performs better than the alternatives in terms of the exceeding ratios measure.

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