7

Chapter 7

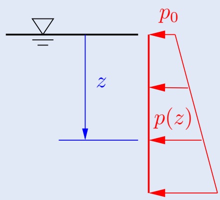
Hydrostatics

Prerequisite: The density ρ (unit: $\mathrm{kg/m^3}$) of the fluid is constant.

Pressure: The pressure p (unit: $\mathrm{Pa} \equiv \mathrm{N/m^2}$) is a force per area, that is identical for all cross sections and always acts normal to the cross section (hydrostatic stress state).

Pressure in a fluid under the action of gravity and a surface pressure p_0 is given by:

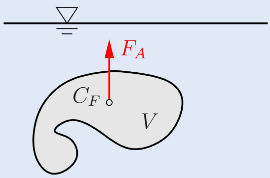
$$p\,(z) = p_0 + \varrho\, g\, z\,.$$



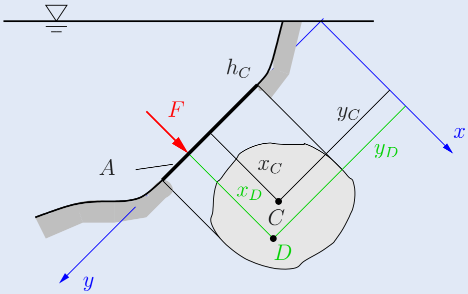
The **buoyancy** force acting on a body (volume V) immersed in a fluid is equal to the weight of the displaced fluid volume.
Buoyancy force:

$$F_A = \rho\, g\, V\,.$$

The line of action related to the buoyancy force passes through the center of gravity C_F of the displaced fluid volume.



Fluid pressure on plane surfaces



Resulting force

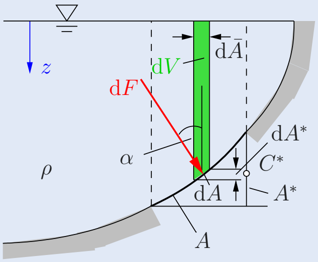
$$F = p\,(y_C)\,A = \rho\, g\, h_C\, A\,.$$

Center of pressure D

$$y_D = \frac{I_x}{S_x}\,,$$

$$x_D = -\frac{I_{xy}}{S_x}\,.$$

Fluid pressure on curved surfaces



$$\mathrm{d}F_V = p\,\mathrm{d}A\cos\alpha = \rho\, g\,\mathrm{d}V$$

$$\mathrm{d}F_H = p\,\mathrm{d}A\sin\alpha = p\,\mathrm{d}A^*$$

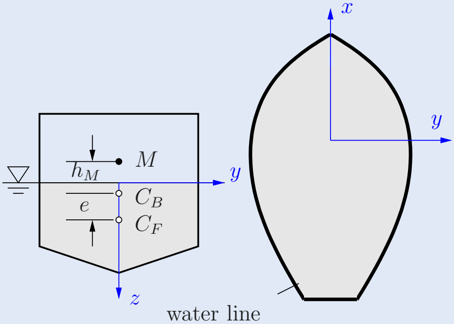
Integration yields

$$F_V = \rho\, g\, V\,,$$

$$F_H = p_{C^*}\, A^*\,.$$

The resulting horizontal component of the fluid pressure F_H is equal to the product of the vertically projected area A^* and the pressure p_{C^*} in the centroid of the projected area.

Stability of a floating body: The equilibrium state is stable if the meta center M is above the centroid C_B of the body:



$$h_M = \begin{cases} > 0 & : \text{stable} \\ < 0 & : \text{unstable} \end{cases}$$

with the position of the meta center

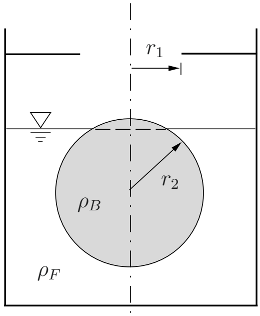
$$h_M = \frac{I_x}{V} - e\,.$$

Here the following data are used

- I_x : second moment of area defined by the water line,
- V : volume of the displaced fluid,
- e : distance of the centroid of the body centroid C_B from the centroid of the displaced fluid C_F.

P7.1

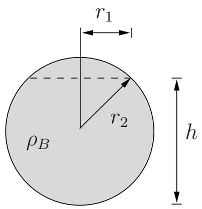
Problem 7.1 A container is closed during filling by a ball valve.



Determine the density ρ_B of the ball, such that no air remains in the container when the ball closes the valve.

Given.: ρ_F, r_1, r_2.

Solution The ball has to submerge to a depth that just closes the opening when the container is full. The buoyancy force is than $\rho_F\, g\, V_1$, where V_1 is the volume of the displaced fluid (spherical segment). The buoyancy force has to be equal to the weight of the ball



$$\rho_F\, g\, V_1 = \rho_B\, g\, V.$$

With the volume of a sphere

$$V = \frac{4}{3}\,\pi\, r_2^3$$

and the spherical section

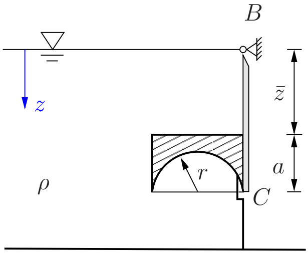
$$V_1 = \pi\, h^2\,(r_2 - \frac{h}{3})\,, \qquad h = r_2 + \sqrt{r_2^2 - r_1^2}$$

we compute for the density of the ball

$$\underline{\underline{\rho_B}} = \rho_F\,\frac{V_1}{V} = \rho_F\,\frac{\pi\, h^2\,(r_2 - \frac{h}{3})}{\frac{4}{3}\,\pi\, r_2^3} = \underline{\underline{\rho_F\,\frac{3}{4}\left(\frac{h}{r_2}\right)^2\left(1 - \frac{h}{3\, r_2}\right)}}\,.$$

P7.2

Problem 7.2 The design of the depicted valve of a water basin ensures that the valve opens if the water level reaches the hinge at point B. The flap valve is assumed to be massless.

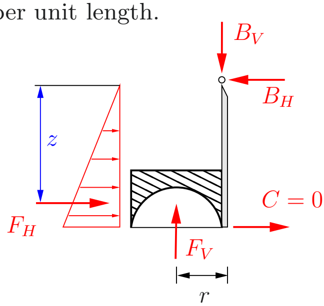


Determine $\bar{z}$ for the valve to function in the described way.

Given: ρ, a, r.

Solution The thickness of the flap valve is irrelevant for the following considerations, as all forces are assumed per unit length.

We compute the resulting horizontal force from the linear pressure distribution:



$$F_H = \frac{1}{2}\,\rho\,g\,(\bar{z}+a)^2$$

with

$$z = \frac{2}{3}\,(\bar{z}+a)\;.$$

The vertical buoyancy force can be computed from the weight of the displaced water by using the area of the dashed region:

$$F_V = \rho\,g\left(2\,a\,r - \frac{\pi}{2}\,r^2\right)\;.$$

The flap valve just opens if the reaction force in C vanishes. Equilibrium of moments with regard to B provides:

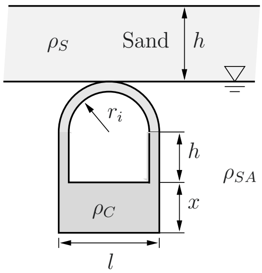
$$\overset{\curvearrowleft}{B}\;:\;-rF_V + zF_H = 0$$

$$\leadsto\quad -\rho\,g\left(2\,a\,r - \frac{\pi}{2}\,r^2\right)r + \frac{1}{2}\,\rho\,g\,(\bar{z}+a)^2\,\frac{2}{3}\,(a+\bar{z}) = 0\;.$$

The solution of this equation with respect to $\bar{z}$ yields the water level

$$\underline{\underline{\bar{z} = \sqrt[3]{3\,(2\,a\,r - \frac{\pi}{2}\,r^2\,)\,r} - a\;.}}$$

P7.3 **Problem 7.3** The depicted cross section of a tunnel is immersed in water saturated "liquid" sand (density ρ_{SA}). Above resides a layer of dry sand (density ρ_S).



Determine the thickness x of the concrete base (density ρ_C), such that a safety factor $\eta = 2$ against lifting is reached. It is assumed that the weight of the dry sand is acting on the cross section of the tunnel.

Given: $\rho_B = 2.5 \cdot 10^3$ kg/m^3, $\rho_S = 2.0 \cdot 10^3$ kg/m^3,
$\rho_{SA} = 1.0 \cdot 10^3$ kg/m^3, $l = 10$ m, $r_i = 4$ m, $h = 7$ m.

Solution The weight (per unit length) of the tunnel cross section and sand load is given by

$$G = \rho_C\, g \left[x\, l + (\frac{l}{2} - r_i)\, 2\, h + \frac{\pi}{2}(\frac{l^2}{4} - r_i^2) \right] + \rho_S\, g\, l\, h\,.$$

With the buoyancy force (per unit length)

$$B = \rho_{SA}\, g \left[(h + x)\, l + \frac{\pi}{2}\frac{l^2}{4} \right]$$

we can determine the height of the concrete base, such that a safety factor against lifting

$$\eta = 2 = \frac{G}{B}$$

is achieved. Solving for x yields:

$$(2\rho_{SA}\, l - \rho_B l)x = \rho_S\, lh + \rho_B \left[(\frac{l}{2} - r_i)2h + \frac{\pi}{2}(\frac{l^2}{4} - r_i^2) \right]$$
$$-2\,\rho_{SA}\big(hl + \frac{\pi}{2}\frac{l^2}{4}\big).$$

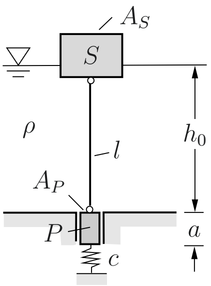
With the given data we get

$$(20 - 25)\, x = 2 \cdot 70 + 2.5\, \left[14 + \frac{\pi}{2}(25 - 16) \right] - 2\, \left(70 + \frac{\pi}{2}\, 25 \right)$$

$$\rightsquigarrow \quad -5\, x = 210.34 - 218.54$$

$$\rightsquigarrow \quad \underline{\underline{x\ =\ 1.64\,\mathrm{m}}}\,.$$

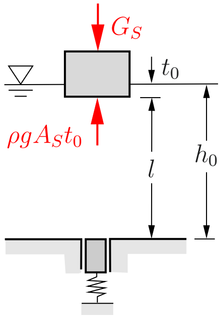
Problem 7.4 A cylindrical plug P (cross section A_P, length a) is elastically supported and closes straight with the bottom of a basin for the water line h_0. In this situation the force vanishes in the rope (length l) to which a floater S is attached (cross section $A_S > A_P$). P7.4



a) Determine the weight G_S of the floater.

b) Which maximal water height h_1 can be reached before leaking occurs?

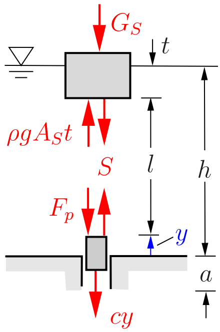
Solution to a) The weight G_S of the floater is computed from equilibrium and geometry in the reference situation:



$$\left.\begin{aligned} \rho g A_S\, t_0 &= G_S \\ h_0 &= l + t_0 \end{aligned}\right\} \quad \rightsquigarrow \quad \underline{\underline{G_S = (h_0 - l)\rho g A_S}}\,.$$

to b) For a water line h the plug is elevated by a distance y due to the force in the rope S. The equilibrium conditions for the floater, for the plug, and the geometric conditions are

$$\rho g A_S\, t = G_S + S\,, \qquad S - F_p = cy\,,$$
$$h = l + t + y\,.$$



In the equilibrium expression, F_p is the difference in the pressure force in the displaced and the reference situation (the forces due to lateral pressure are in equilibrium):

$$F_p = \rho g(h - y)A_p - \rho g h_0 A_p = \rho g(h - y - h_0)A_p\,.$$

Eliminating G_S, S, F_p, and t yields

$$h - h_0 = y\left[1 + \frac{c}{\rho g(A_S - A_P)}\right].$$

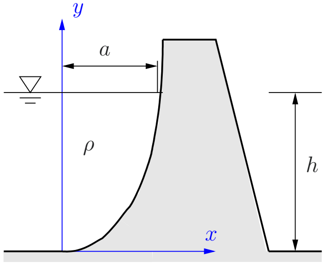
The maximal heighth $h = h_1$ is reached, if $y = a$ is attained:

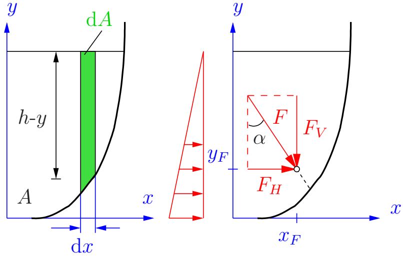
$$\underline{\underline{h_1 = h_0 + a\left[1 + \frac{c}{\rho g(A_S - A_P)}\right]}}\,.$$

P7.5 **Problem 7.5** A dam of length l has a surface of parabolic shape with a horizontal tangent at the bottom of the water basin.

Determine the force resulting from the pressure, the position of the point of action, and the line of action for a water height h.

Given: h, l, $a = h/4$, ρ.





Solution The vertical component of the force is $F_V = \rho\, g\, V$ with the volume $V = l\, A$. The area is determined by the function $y(x) = 16\, x^2/h$ of the parabola

$$A = \int_0^a (h - y)\, \mathrm{d}x$$

$$= \int_0^a (h - \frac{16}{h}\, x^2)\, \mathrm{d}x = \left[\, h\, x - \frac{16}{3\, h}\, x^3 \right]_0^a = \frac{h^2}{6}\,.$$

Thus the vertical component of the pressure force becomes:

$$F_V = \frac{1}{6}\, \rho\, g\, h^2\, l.$$

The vertical force acts at the centroid C of the area

$$\underline{\underline{x_F}} = \frac{1}{A} \int_0^a x\, (h - \frac{16}{h}\, x^2)\, \mathrm{d}x = \left[\, h\, \frac{x^2}{2} - \frac{16}{h}\, \frac{x^4}{4} \right]_0^a = \underline{\underline{\frac{3}{32}\, h}}\,.$$

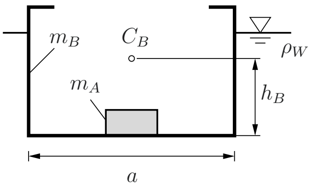
The horizontal component of the fluid pressure is computed by the projected area $A^* = h\, l$ and the pressure $p_{S^*} = \frac{1}{2}\, \rho\, g\, h$ in the centroid of the projected area:

$$F_H = \frac{1}{2}\, \rho\, g\, h^2\, l \qquad \text{with} \quad y_F = \frac{1}{3}\, h\,.$$

By the theorem of Pythagoras, we obtain the resulting force, its line of action passes through the point $(x_F\,, y_F)$ and forms an angle α to the y-axis:

$$\underline{\underline{F}} = \sqrt{F_H^2 + F_V^2} = \underline{\underline{\frac{1}{6}\, \sqrt{10}\, \rho\, g\, h^2\, l}}, \quad \underline{\underline{\alpha}} = \arctan \frac{F_H}{F_V} = \arctan 3 = \underline{\underline{71.5^o}}.$$

Problem 7.6 A prismatic body with the mass m_B, width a, and length l is floating in the water. Its centroid C_B is in the height h_B. **P7.6**



Determine the additional point mass m_A, such that the body floats in a stable manner.

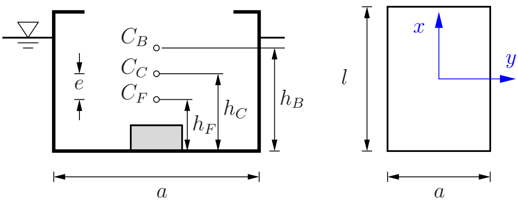
Given: ρ_W, m_B, h_{SB}, l, a.

Solution Stable floating of the body is defined by the position of the meta center $h_M = I_x \,/\, V - e > 0$. For $h_M = 0$ the limit of the stable state is reached.

The volume V of the displaced fluid is obtained by equilibrium (buoyancy = weight of the body and added mass):

$$\rho_W \, g \, V = (m_B + m_A) \, g \quad \leadsto \quad V = \frac{1}{\rho_W} (m_B + m_A)$$

The second moment of area is



$$I_x = \frac{l \, a^3}{12} \, .$$

For $e = h_C - h_F$ we need the center of gravity h_C of the floating construction and h_F of the displaced fluid. They are determined by

$$h_C \, (m_B + m_A) = h_B \, m_B \quad \leadsto \quad h_C = h_B \, \frac{m_B}{m_B + m_A} \, ,$$

$$V = a \, l \, (2 \, h_F) \quad \leadsto \quad h_F = \frac{m_B + m_A}{2 \, a \, l \, \rho_W} \, .$$

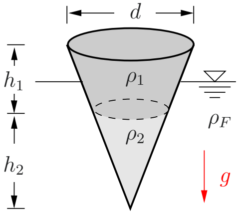
The limit for stable floating is reached if $h_M = 0$:

$$1 - 12 \, h_C \, \frac{m_B}{l \, a^3 \, \rho_W} + \frac{12 \, (m_B + m_A)^2}{2 \, l^2 \, a^4 \, {\rho_W}^2} = 0 \, .$$

Solving for the required additional mass m_A yields

$$\underline{\underline{m_A = \frac{l \, a^2 \, \rho_W}{\sqrt{6}} \sqrt{12 \, h_{CB} \, \frac{m_B}{l \, a^3 \, \rho_W} - 1} - m_B \, .}}$$

P7.7 **Problem 7.7** A cone-shaped floating device is made of two materials with densities ρ_1 and ρ_2.



Determine the diameter d of the cone, such that it floats stable in a fluid of density ρ_F.

Given:

$$\rho_1 = \frac{2}{3}\,\rho_F\,, \quad \rho_2 = \frac{1}{3}\,\rho_F\,,$$

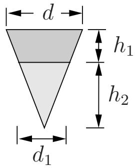
$$h_1 = 2\,h\,, \qquad h_2 = 4\,h\,.$$

Solution The cone has a stable floating position, if the following conditions are met:

$$(1): \quad G = A,$$

$$(2): \quad h_M = \frac{I_x}{V} - e > 0.$$

(1) Floating condition:



$$\frac{d}{h_1 + h_2} = \frac{d_1}{h_2} \quad \rightsquigarrow \quad d_1 = d\,\frac{h_2}{h_1 + h_2} = \frac{2}{3}\,d.$$

The force due to weight is

$$G = V_1\,\rho_1\,g + V_2\,\rho_2\,g$$

$$= \frac{1}{12}\pi\,h_1\,(d^2 + dd_1 + d_1^2)\,\rho_1\,g + \frac{1}{12}\,\pi\,h_2\,d_1^2\,\rho_2\,g$$

$$= \frac{23}{81}\,\pi\,h\,d^2\,\rho_F\,g = 0.892\,h\,d^2\,\rho_F g\,.$$

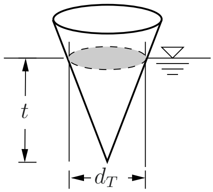
The immersion depth t and the diameter $d_T = d\,t/(h_1 + h_2)$ of the cone at the water line of the fluid follows the buoyancy force

$$A = \frac{1}{12}\,\pi\,t\,d_T^2\,\rho_F\,g$$

$$= \frac{1}{432}\,\pi\,\frac{d^2}{h^2}\,\rho_F\,g\,t^3\,.$$

For $G = A$ we obtain

$$t^3 = \frac{368}{3}\,h^3 \quad \rightsquigarrow \quad t = 4.969\,h\,.$$



(2) Stability condition:

The volume of the displaced fluid is given by

$$V = \frac{1}{432}\,\pi\,\frac{d^2}{h^2}\,t^3 = \frac{23}{81}\,\pi\,h\,d^2 = 0.892\,h\,d^2\,,$$

and the second moment of area I_x is

$$I_x = \frac{d_T^4\,\pi}{64} = \frac{(0.828\,d)^4\,\pi}{64} = 0.023\,d^4\,.$$

The distance of the centroid of the body from the centroid of the displaced fluid is provided by

$$e = x_S - \frac{3}{4}\,t$$

with

$$x_S = \frac{\frac{3}{4}\,(h_1+h_2)\,\rho_1\,\frac{1}{16}\,\pi\,d^2\,(h_1+h_2) + \frac{3}{4}\,h_2\,(\rho_2-\rho_1)\,\frac{1}{16}\,\pi\,d_1^2\,h_2}{\rho_1\,\frac{1}{16}\,\pi\,d^2\,(h_1+h_2) + (\rho_2-\rho_1)\,\frac{1}{16}\,\pi\,d_1^2\,h_2}$$

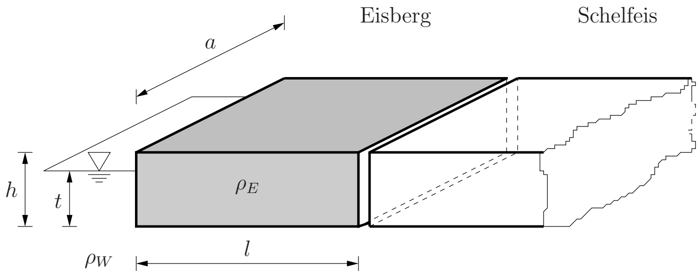
$$= \frac{18\,h - \frac{16}{9}\,h}{4 - \frac{16}{27}} = 4.761\,h$$

$$\rightsquigarrow \quad e = 4.761\,h - \frac{3}{4}\cdot 4.969\,h = 1.034\,h\,.$$

For the diameter of the cone we finally obtain

$$h_M = \frac{0.023\,d^4}{0.892\,h\,d^2} - 1.034 > 0 \quad \rightsquigarrow \quad \underline{\underline{d > 6.333\,h}}\,.$$

P7.8 **Problem 7.8** A block-shaped iceberg of dimensions $a \times h \times l$ calves of a floating ice shelf. It is assumed that $a \gg h$. The density of the water is ρ_W, the density of the ice $\rho_I = \frac{9}{10}\rho_W$.



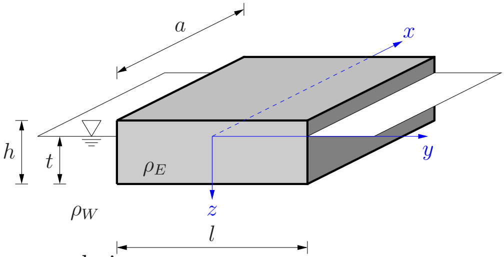
For which length l does the iceberg float in a stable way?

Solution We start by determining the immersion depth t of the iceberg. Equilibrium between iceberg and buoyancy force renders for the given density ratio the immersion depth

$$\rho_I ghla = \rho_W gtla \quad \rightsquigarrow \quad t = \frac{9}{10}h\ .$$

To analyze the floating stability we consider the position h_M of the meta center:

$$h_M = \frac{I_x}{V} - e\ ,$$
$$I_x = \frac{al^3}{12}\ ,$$
$$V = alt = \frac{9}{10}alh\ ,$$
$$e = \frac{h}{2} - \frac{t}{2} = \frac{h}{20}\ .$$



By combining all relations we derive

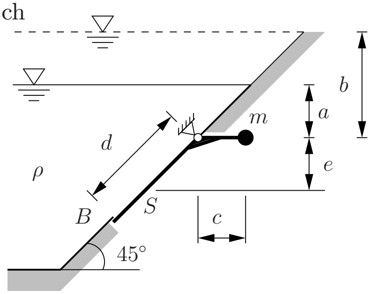
$$h_M = \frac{5}{54}\frac{l^2}{h} - \frac{h}{20}\ .$$

We consider the limit of floating stability ($h_M = 0$). This determines the length l_0 :

$$l_0^2 = \frac{27}{50}h^2 \quad \rightsquigarrow \quad l_0 = \sqrt{\frac{27}{50}}h \approx 0.735h\ .$$

In a stable floating state, we must have $h_M > 0$. Thus, the iceberg floats stable for $\underline{\underline{l > l_0}}$. For $l < l_0$ the iceberg tips over.

Problem 7.9 A circular shaped hatch closes the outflow of a tank. **P7.9**

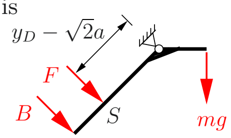


a) Determine the mass m, such that the hatch opens if m is attached in the distance c from the hinge point.

b) Determine the distance by which the mass m has to be shifted, for the hatch to open when the water level reaches the height b.

Given: a, b, c, d, e, m, ρ.

Solution **zu a)** The force acting on the hatch is



$$F = \rho\, g\, A\, h_S = \rho\, g\, \frac{\pi\, d^2}{4}\, (a + e)\, .$$

The point of action of F is determined by

$$y_D = y_S + \frac{I_\xi}{y_S\, A} = \sqrt{2}\, (a + e) + \frac{d^2}{16\sqrt{2}\, (a + e)}\, .$$

The hatch opens, if $B = 0$. Equilibrium of moments provides

$$F\, (\, y_D - \sqrt{2}\, a\,) - m\, g\, c = 0\, .$$

From this we compute the required mass

$$m = \rho\, \frac{\pi\, d^2}{4\, c}\, (a + e)\, \left[\sqrt{2}\, e + \frac{d^2}{16\, \sqrt{2}\, (a + e)} \right] .$$

to b) For the water level b the force acting on the hatch is

$$F = \rho\, g\, A\, h_s = \rho\, g\, \frac{\pi\, d^2}{4}\, (b + e)\, .$$

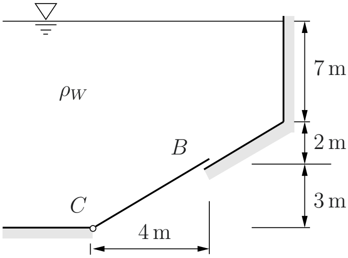
With the point of action

$$y_D = \sqrt{2}\, (b + e) + \frac{d^2}{16\, \sqrt{2}\, (b + e)}$$

of F the equilibrium condition $F\, (\, y_D - \sqrt{2}\, b\,) - m\, g\, c = 0$ yields the distance c:

$$c = \rho\, \frac{\pi\, d^2}{4}\, (b + e)\, \left[\sqrt{2}\, e + \frac{d^2}{16\, \sqrt{2}\, (b + e)} \right] \frac{1}{m}\, .$$

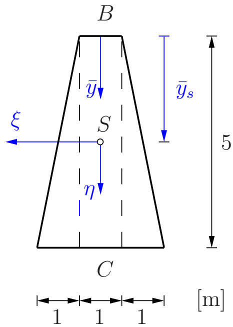
P7.10 **Problem 7.10** A trapezoidal hatch closes the outflow of the depicted basin.



Determine the resulting force on the hatch together with the support reactions in point B.

Given: $\rho_W = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$, $g = 9.81\,\dfrac{\text{m}}{\text{s}^2}$.

Solution The area $A = 10\,\text{m}^2$, the centroid of the hatch



$$\bar{y}_s = \left(5{\cdot}2,5+5{\cdot}\frac{2}{3}{\cdot}5\right)\frac{1}{10} = \frac{35}{12}\,\text{m}$$

and the pressure

$$p\,(\bar{y}_s) = \rho\,g\left[9+\frac{3}{5}\cdot\frac{35}{12}\right] = \frac{43}{4}\,\rho\,g$$

are used to compute the resulting force

$$\underline{\underline{F}} = \rho\,g\,A\,p\,(\bar{y}_s) = 10^3{\cdot}9,81{\cdot}10{\cdot}\frac{43}{4} = \underline{\underline{1.05\,\text{MN}}}\,.$$

The position of the line of action follows from

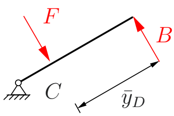
$$I_\xi = \frac{5^3\cdot 1}{12} + 5{\cdot}1\left(\frac{35}{12}-2,5\right)^2 + 2\,\frac{5^3\cdot 1}{36} + 5{\cdot}1\left(\frac{35}{12}-\frac{10}{3}\right)^2 = 19.1\,\text{m}^4\,,$$

$y_s = \bar{y}_s + 15\,\text{m}$ and $y_D = \bar{y}_D + 15\,\text{m}$ to be

$$y_D = \frac{I_x}{S_x} = \frac{y_s^2\,A + I_\xi}{y_s\,A} \quad\rightsquigarrow\quad \bar{y}_D = \bar{y}_s + \frac{I_\xi}{y_s\,A} = \frac{35}{12} + \frac{19,1}{(\frac{35}{12}+15)\,10}$$

$$= 3.02\,\text{m}\,.$$

The support reaction is determined by equilibrium of moments with regard to the hinge point C of the hatch



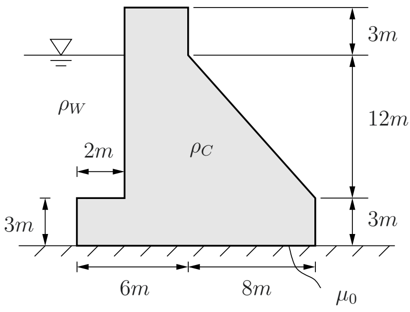
$$\overset{\curvearrowleft}{C}: \quad B\cdot 5 - F\,(\,5-3,02\,) = 0$$

$$\rightsquigarrow \quad \underline{\underline{B}} = 1,05\,\frac{5-3,02}{5} = \underline{\underline{0.415\,\text{MN}\,.}}$$

Problem 7.11 A concrete dam (density ρ_C) closes a basin that is filled up to the height $h = 15$ m. **P7.11**

Determine

a) the safety factor against sliding at the bottom (adhesion coefficient μ_0),
b) the safety against tilting,
c) the stress distribution at the bottom, if it assumed to be a linear distribution.



Given: $\rho_C = 2.5 \cdot 10^3$ kg/m^3, $\rho_W = 10^3$ kg/m^3, $\mu_0 = 0.5$, $g = 10$ m/s^2

Solution **to a)** To determine the safety factor against sliding we compute the horizontal forces due to the water pressure and compare them to the adhesion forces acting at the bottom. The horizontal force due to water pressure is computed from

$$F_H = \frac{1}{2}\,\rho_W\, g\, h\, A = \frac{1}{2}\, 10^3 \cdot 10 \cdot 15 \cdot 15 \cdot 1 = 1125 \text{ kN/m}\,.$$

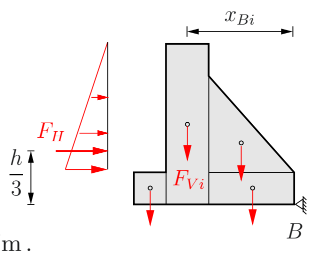
The resulting force due to the weight of the concrete and the water pressure is

$$F_V = 2.5 \cdot 10^3\,(3 \cdot 2 + 4 \cdot 18 + 3 \cdot 8 + \frac{1}{2} \cdot 12 \cdot 8) + 10^3\,(2 \cdot 12) = 3990\,\text{kN/m}\,.$$

Using Coulomb's friction law we determine the safety factor η_S against the onset of sliding

$$\underline{\underline{\eta_S}} = \frac{\mu_0\, F_V}{F_H} = \frac{0.5 \cdot 3990}{1125} = \underline{\underline{1.77}}\,.$$

to b) The dam can tilt around point B. The safety against tilting is determined by comparing the moment of forces. The moment of the water pressure is given by



$$M_{BW} = F_H\,\frac{h}{3} = 1125 \cdot \frac{15}{3} = 5625\,\text{kNm}\,.$$

The moment related to the weights is

$$
\begin{aligned}
M_{BG} &= \sum_i F_{Vi}\, x_{Bi} \\
&= 2.5 \cdot 10^3\, (3 \cdot 2 \cdot 13 + 4 \cdot 18 \cdot 10 + 3 \cdot 8 \cdot 4 \\
&\quad + \frac{1}{2} \cdot 12 \cdot 8 \cdot \frac{2}{3} \cdot 8) + 10^3\, (2 \cdot 12 \cdot 13) = 31870\,\text{kNm}\,.
\end{aligned}
$$

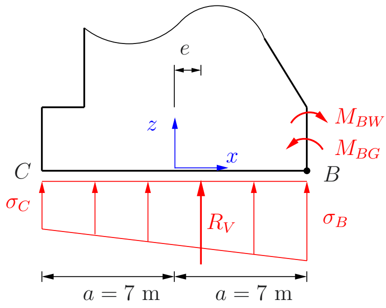
This results in a safety factor against tilting

$$
\underline{\underline{\eta_T}} = \frac{M_{BG}}{M_{BW}} = \frac{31780}{5625} = \underline{\underline{5.67}}\,.
$$

to c) To compute the stress distribution in the bottom gap of the dam we determine the excentricity of the resulting force $R_V = \sum_i F_{Vi}$. The vertical component of the force acting in the gap yields, according to the sketch below,

$$
R_V\,(a - e) = M_{BG} - M_{BW}
$$

$$
\leadsto \quad e = a - \frac{M_{BG} - M_{BW}}{R_V} = 7 - \frac{31870 - 5625}{3990} = 0.422\,\text{m}\,.
$$



With the introduced coordinate-system we compute the normal stresses in the bottom gap (like in a beam cross section)

$$
\sigma = \frac{N}{A} + \frac{M_y}{I_y}\, x\,.
$$

Here we have to insert the following data: $A = 14\,\text{m}^2$, $I_y = 1 \cdot 14^3 / 12 = 288.67\,\text{m}^4$, $N = -R_V = -3990$ kN, $M_y = N \cdot e = -1685$ kNm. As a result we obtain for the stress distribution

$$
\underline{\underline{\sigma}} = \frac{-3990}{14} + \frac{-1685}{228.67}\, x = \underline{\underline{-285 - 7.37\, x\ \text{kN/m}^2}}\,.
$$

For the selected points C and B evaluation yields

$$
\sigma_C = -0.23\,\text{MPa} \quad \text{and} \quad \sigma_B = -0.34\,\text{MPa}\,.
$$

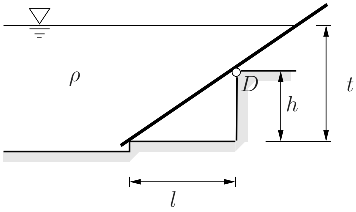
Problem 7.12 A rectangular plate of width b closes the outlet of a basin. It is hinged at point D.

P7.12

a) Determine the water height t, for which the plate starts to rotate around point D.

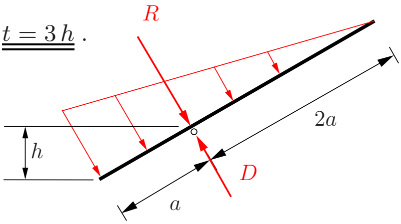
b) Compute the bending moment at point D for this situation.

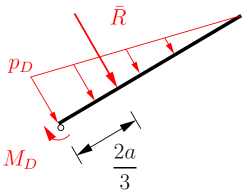
Given: b, l, h, ρ.



to a) The plate starts to rotate, if the resulting force R of the water pressure is above point D. In the limit case the resulting force of the water pressure passes through point D. From this we can determine the water height

$$\underline{\underline{t = 3\,h}}\,.$$





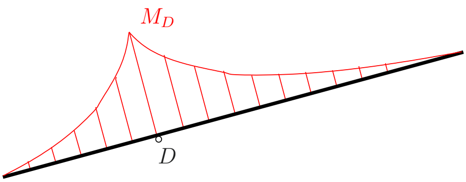
to b) To compute the bending moment in the plate we start with the moment at point D. With the resultant $\bar{R}$ of the upper plate and the pressure at point D,

$$\bar{R} = \frac{1}{2}\,p_D\,2\,a\,b\,, \quad p_D = \rho\,g\,2\,h\,,$$

we obtain

$$\underline{\underline{M_D}} = -\bar{R}\,\frac{2}{3}\,a = -\frac{2}{3}\,p_D\,b\,a^2 = \underline{\underline{-\frac{4}{3}\,\rho\,g\,(l^2 + h^2)h\,b}}\,.$$

The distribution of the bending moment is cubic for a linearly varying load. The maximum occurs at the hinge point D.



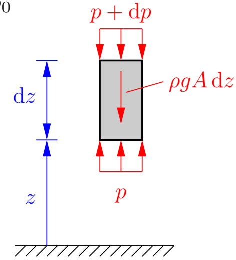
P7.13

Problem 7.13 The pressure p in gases depends on the density ρ. The relation between the two state variables is provided by the universal gas equation $p = \rho RT$ (universal gas constant R, temperature T). E. g. for air at sea level and at $T = 0^\circ$ it holds: $p_0 = 101325\,\text{Pa}$ and $\rho_0 = 1.293\,\text{kg/m}^3$.

Determine the dependency of air pressure on height for the case of a constant temperature (barometric height relation).

Solution First, we apply the universal gas law at sea level. This yields

$$p_0 = \rho_0 RT \qquad \text{or} \qquad RT = \frac{p_0}{\rho_0}\,.$$



Equilibrium of an infinitesimal air column with cross section A and height $\mathrm{d}z$

$$\uparrow: \quad pA - \rho g A\,\mathrm{d}z - (p + \mathrm{d}p)\,A = 0$$

leads to

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g\,.$$

Using the universal gas equation yields

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{pg}{RT}\,.$$

By separation of variables and integration we obtain:

$$\frac{\mathrm{d}p}{p} = -\frac{g}{RT}\mathrm{d}z \;\leadsto\; \int_{p_0}^{p} \frac{\mathrm{d}\bar{p}}{\bar{p}} = -\int_0^z \frac{g}{RT}\mathrm{d}\bar{z} \;\leadsto\; \ln\frac{p}{p_0} = -\frac{g}{RT}z\,.$$

This renders the air pressure as a function of the height

$$\underline{\underline{p = p_0\,\mathrm{e}^{-\frac{gz}{RT}}}}\,.$$

The air pressure decreases exponentially with the height. From the relation $RT = p_0/\rho_0$ and the gravity constant $g = 9.80665\,\text{m/s}^2$ we deduce

$$p = 101325\,\text{Pa}\,\mathrm{e}^{-\frac{z}{7991\,\text{m}}}\,.$$

Note: In a height of $5{,}5\,\text{km}$ the pressure has dropped to one half of its original value.