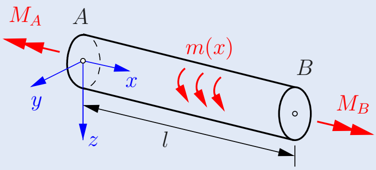
Chapter 4

Torsion

4

Torsion

If an external load causes an internal moment M_x along the longitudinal axis, the bar is loaded by *torsion* (twisting). In the following we refer to the moment M_x as *torque* or *torsional moment* M_T.



Prerequisites, assumptions:

- Warping of the cross sections is not constrained (*pure torsion*),
- The shape of the cross sections does not change during rotation.

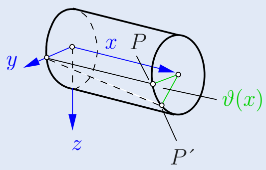
Equilibrium conditions

$$\frac{\mathrm{d}M_T}{\mathrm{d}x} = -m\,, \qquad m(x) = \text{external moment per unit length.}$$

Differential equation for the angle of twist

$$GI_T\frac{\mathrm{d}\vartheta}{\mathrm{d}x} = M_T\,,$$

ϑ = angle of twist,
GI_T = torsional rigidity,
G = shear modulus,
I_T = torsional constant.



Twist of end sections

$$\Delta\vartheta = \vartheta(l) - \vartheta(0) = \int_0^l \vartheta'(x)\mathrm{d}x = \int_0^l \frac{M_T}{GI_T}\mathrm{d}x\,.$$

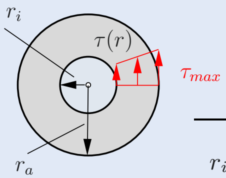
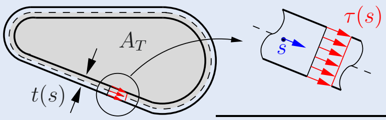
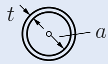
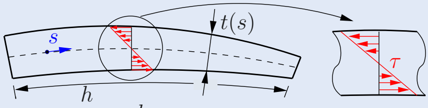
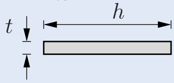
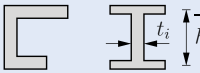
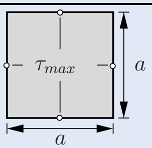
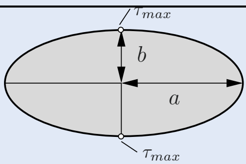
Special case: $GI_T = \text{const}$, $M_T = \text{const}$

$$\Delta\vartheta = \frac{M_T l}{GI_T}\,.$$

Maximum shear stress

$$\tau_{\max} = \frac{M_T}{W_T}\,, \qquad W_T = \text{sectional moment of torsion.}$$

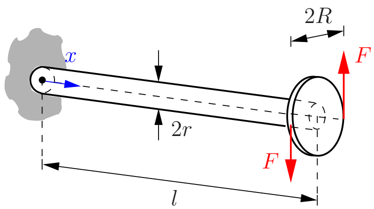
The location of the maximum shear stress is provided in the following table.

Cross section	I_T
	$I_T = I_p = \frac{\pi}{2}(r_a^4 - r_i^4)$
$r_i = 0$ (full circle)	$I_T = \frac{\pi}{2} r_a^4$
thin-walled, closed profile 	$I_T = \frac{4A_T^2}{\oint \frac{\mathrm{d}s}{t(s)}}$
 $a = \text{const}$ $t = \text{const}$	$I_T = 2\pi a^3 t$
thin-walled, open profile 	$I_T = \frac{1}{3}\int_0^h t^3(s)\mathrm{d}s$
 $t = \text{const}$	$I_T = \frac{1}{3} h t^3$
 $t_i = \text{const}$	$I_T = \frac{1}{3}\sum h_i t_i^3$
square 	$I_T = 0,141 a^4$
ellipse 	$I_T = \pi \frac{a^3 b^3}{a^2 + b^2}$

W_T	Remarks
$W_T = \frac{I_T}{r_a} = \frac{\pi}{2} \frac{r_a^4 - r_i^4}{r_a}$	The shear stresses are distributed linearly across the cross section: $$\tau(r) = \frac{M_T}{I_T} r \ .$$ Cross sections remain plane during deformation.
$W_T = \frac{\pi}{2} r_a^3$	
$W_T = 2A_T \, t_{\min}$	τ is constant across the wall-thickness t. The shear flow $$T = \tau t = \frac{M_T}{2A_T}$$ is constant. $\tau_{\max}$ occurs at the smallest wall-thickness $t_{\min}$. A_T is the area encircled by the central line of the profile.
$W_T = 2\pi a^2 t$	
$W_T = \frac{I_T}{t_{\max}}$	τ is linearly distributed across the wall-thickness. $\tau_{\max}$ occurs at the largest wall-thickness $t_{\max}$.
$W_T = \frac{1}{3} h t^2$	
$W_T = \frac{I_T}{t_{\max}}$	
$W_T = 0.208 \ a^3$	$\tau_{\max}$ occurs at in the middle of the lateral lengths.
$W_T = \frac{\pi}{2} \, a b^2$	$\tau_{\max}$ occurs at the ends of the smaller semi-axis.

P4.1

Problem 4.1 A shaft with circular cross section is clamped at one end and loaded by a pair of forces.



Determine F such that the admissible shear stress τ_{admis} is not exceeded. Compute for this case the twist of the end section.

Given: $R = 200$ mm, $r = 20$ mm, $l = 5$ m, $\tau_{\text{zul}} = 150$ MPa,
$G = 0.8 \cdot 10^5$ MPa.

Solution The torque (torsional moment)

$$M_T = 2RF$$

is constant along the bar. The maximum shear stress in the cross section is given with

$$W_T = \frac{\pi}{2} r^3$$

by

$$\tau_{\max} = \frac{M_T}{W_T} = \frac{4RF}{\pi r^3} .$$

In order not to exceed the admissible shear stress,

$$\tau_{\max} \leq \tau_{\text{admis}} \qquad \rightsquigarrow \qquad F \leq \frac{\pi r^3}{4R} \tau_{\text{admis}} .$$

must hold and we obtain

$$\underline{\underline{F_{\max}}} = \frac{\pi r^3}{4R} \tau_{\text{allow}} = \frac{\pi \cdot 8000 \cdot 150}{4 \cdot 200} = \underline{\underline{4712 \text{ N}}} .$$

For this load the twist (in radians) can be computed using

$$I_T = \frac{\pi}{2} r^4 \qquad \text{and} \qquad M_T = 2RF_{\max} .$$

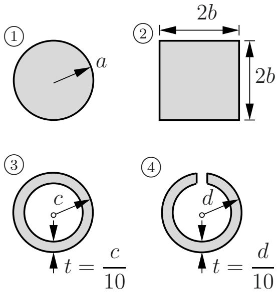
Inserting yields

$$\underline{\underline{\Delta\vartheta}} = \frac{M_T l}{G I_T} = \frac{\tau_{\text{zul}} l}{G r} = \frac{150 \cdot 5000}{0.8 \cdot 10^5 \cdot 20} = \underline{\underline{0.47}} .$$

This value is equivalent to an angle of 27°.

Problem 4.2 A shaft has to carry the torque $M_T = 12 \cdot 10^3$ Nm. Select a cross section from the depicted group. **P4.2**

Dimension the cross sections such that the admissible shear stress $\tau_{\mathrm{admis}} = 50$ MPa is not exceeded. Which cross section is the most efficient in terms of material usage?



Solution The admissble shear stress is reached for

$$\tau_{\max} = \frac{M_T}{W_T} = \tau_{\mathrm{admis}} \ .$$

With the section moment for torsion

$$W_{T_1} = \frac{\pi}{2} a^3 \,, \qquad W_{T_2} = 0.208 \cdot 8\, b^3 = 1.664\, b^3 \,,$$

$$W_{T_3} = 2\pi c^2 t = \frac{\pi}{5} c^3 \,, \qquad W_{T_4} = \frac{2\pi}{3} d\, t^2 = \frac{\pi}{150} d^3$$

we determine with the given numerical values

$$\underline{\underline{a}} = \sqrt[3]{\frac{2M_T}{\pi \tau_{\mathrm{zul}}}} = \underline{\underline{53.5 \text{ mm}}} \,, \qquad \underline{\underline{b}} = \sqrt[3]{\frac{M_T}{1.664\, \tau_{\mathrm{zul}}}} = \underline{\underline{52.4 \text{ mm}}} \,,$$

$$\underline{\underline{c}} = \sqrt[3]{\frac{5\, M_T}{\pi\, \tau_{zul}}} = \underline{\underline{72,6 \text{ mm}}} \,, \qquad \underline{\underline{d}} = \sqrt[3]{\frac{150\, M_T}{\pi\, \tau_{zul}}} = \underline{\underline{225.5 \text{ mm}}} \,.$$

The cross section areas are

$$A_1 = \pi a^2 = 89.8 \text{ cm}^2 \,, \qquad A_2 = 4b^2 = 110.0 \text{ cm}^2 \,,$$

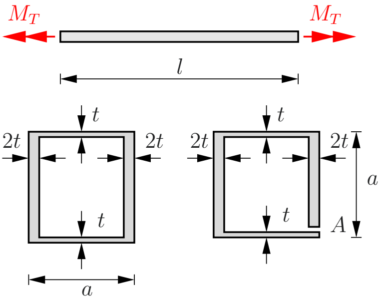
$$\underline{\underline{A_3}} = \frac{\pi}{5} c^2 = \underline{\underline{33.1 \text{ cm}^2}} \,, \qquad A_4 = \frac{\pi}{5} d^2 = 319.4 \text{ cm}^2 \,.$$

Therefore, the third cross section (i. e. the thin-walled *closed* profile) is the most material efficient profile.

P4.3

Problem 4.3 Determine the maximum admissible torque (torsional moment) and the corresponding admissible twist for the closed profile and the profile that is slit at A.

Given: $a = 10$ cm, $t = 2$ mm,
$\tau_{\text{admis}} = 20$ MPa,
$l = 5$ m,
$G = 0.8 \cdot 10^5$ MPa.



Solution The admissible torque and the admissible twist are computed for both profiles via

$$M_{T_{\text{admis}}} = \tau_{\text{admis}} W_T \, , \qquad \Delta\vartheta_{\text{admis}} = \frac{M_{T_{\text{admis}}} l}{G I_T} = \frac{\tau_{\text{admis}} W_T l}{G I_T} \, .$$

In the case of the *closed* profile with $t \ll a$ it holds

$$A_T = a^2 \, , \qquad \oint \frac{\mathrm{d}s}{t(s)} = 2\left(\frac{a}{2t} + \frac{a}{t}\right) = 3\frac{a}{t} \, ,$$

$$I_T = \frac{4A_T^2}{\oint \frac{\mathrm{d}s}{t(s)}} = \frac{4}{3} t a^3 \, , \qquad W_T = 2A_T t_{\min} = 2a^2 t$$

and we obtain

$$\underline{\underline{M_{T_{\text{admis}}}}} = \tau_{\text{admis}} 2a^2 t = \underline{\underline{800 \text{ Nm}}} \, ,$$

$$\underline{\underline{\Delta\vartheta_{\text{allow}}}} = \frac{3\tau_{\text{admis}} l}{2Ga} = \underline{\underline{0.01875}} \quad (\widehat{=} 1,07^\circ) \, .$$

If the profile is *open* (slit at position A), we compute with

$$I_T = \frac{1}{3}\sum_i t_i^3 h_i = 6t^3 a \, , \qquad W_T = \frac{I_T}{t_{\max}} = 3t^2 a$$

the torque and twist

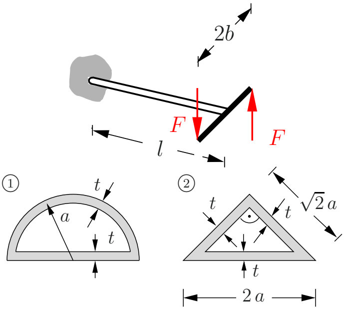
$$\underline{\underline{M_{T_{\text{admis}}}}} = \tau_{\text{admis}} 3t^2 a = \underline{\underline{24 \text{ Nm}}} \, ,$$

$$\underline{\underline{\Delta\vartheta_{\text{admis}}}} = \frac{\tau_{\text{admis}} l}{2Gt} = \underline{\underline{0.3125}} \quad (\widehat{=} 17.9^\circ).$$

Note: The closed profile is much stiffer with respect to torsion than the open profile.

Problem 4.4 A shaft is loaded by a pair of forces. The shaft is assmbled from two different thin-walled cross sections ($t \ll a$) of the same material (shear modulus G). **P4.4**

Determine in both cases the admissible forces and the corresponding twist such that the shear stress τ_{admis} is not exceeded.



Solution The torque $M_T = 2bF$ is constant along the length of the shaft. Stress and twist are determined from

$$\tau = \frac{M_T}{W_T} = \frac{2bF}{W_T} \quad , \qquad \Delta\vartheta = \frac{M_T l}{G I_T} = \frac{2bFl}{G I_T} \, .$$

The admissible shear stress will not be exeeded for

$$\tau \le \tau_{\text{admis}} \quad \rightsquigarrow \quad F \le \frac{W_T \tau_{\text{admis}}}{2b} \quad \rightsquigarrow \quad F_{\text{admis}} = \frac{W_T \tau_{\text{admis}}}{2b} \, ,$$

$$\Delta\vartheta_{\text{admis}} = \frac{2bl F_{\text{admis}}}{G I_T} = \frac{\tau_{\text{admis}} W_T l}{G I_T} \, .$$

With the values for the two different cross sections

$$① \; A_T = \frac{\pi}{2} a^2 \, , \quad \oint \frac{\mathrm{d}s}{t} = \frac{a}{t}(2+\pi) \, , \qquad W_T = \pi a^2 t \, , \quad I_T = \frac{\pi^2}{2+\pi} a^3 t \, ,$$

$$② \; A_T = a^2 \, , \quad \oint \frac{\mathrm{d}s}{t} = \frac{a}{t}(2 + 2\sqrt{2}) \, , \quad W_T = 2a^2 t \, , \quad I_T = \frac{2}{1+\sqrt{2}} a^3 t$$

we obtain

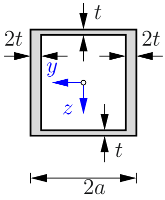
$$\underline{\underline{F_{\text{admis}_1} = \frac{\pi}{2} \frac{a^2 t}{b} \tau_{\text{admis}}}} \, , \qquad \underline{\underline{F_{\text{admis}_2} = \frac{a^2 t}{b} \tau_{\text{admis}}}} \quad ,$$

$$\underline{\underline{\Delta\vartheta_{\text{admis}_1} = \frac{2+\pi}{\pi} \frac{l \tau_{\text{admis}}}{aG}}} \, , \qquad \underline{\underline{\Delta\vartheta_{\text{admis}_2} = (1+\sqrt{2}) \frac{l \tau_{\text{admis}}}{aG}}} \quad .$$

Note: The admissible force is larger for the first profile, while the admissible twist is larger for the second profile.

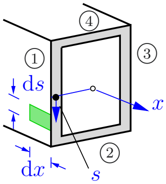
P4.5

Problem 4.5 The thin-walled box girder is loaded by a torque M_T.



Determine the warping of the cross section.

Solution The warping $u(s)$ (displacement in longitudinal direction) is computed from the shear strain



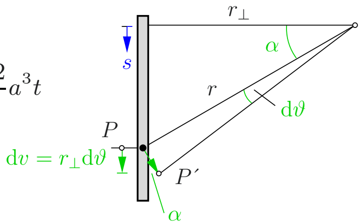
$$\gamma = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}$$

of the wall segments. With

$$\gamma = \frac{\tau}{G} = \frac{M_T}{G 2 A_T t(s)} ,$$

$$\frac{\partial v}{\partial x} = r_\perp \frac{\mathrm{d}\vartheta}{\mathrm{d}x} = r_\perp(s) \frac{M_T}{G I_T} ,$$

$$A_T = 4a^2 , \qquad I_T = \frac{4 \cdot 16 a^4}{\frac{4a}{t} + \frac{4a}{2t}} = \frac{32}{3} a^3 t$$



we obtain

$$\frac{\partial u}{\partial s} = \frac{M_T}{8Ga^2 t} \left[\frac{t}{t(s)} - \frac{3 r_\perp(s)}{4a} \right] .$$

Integration in region ① provides ($t(s) = 2t$, $r = a$) with $u(s=0) = 0$ (then u vanishes on average)

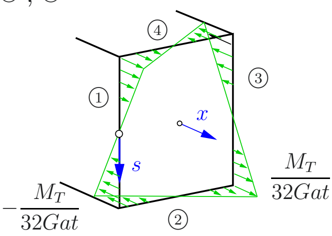
$$\underline{\underline{u_1(s)}} = \frac{M_T}{8Ga^2 t} \left[\frac{1}{2} - \frac{3}{4} \right] s = \underline{\underline{-\frac{M_T}{32Ga^2 t}}} \, s .$$

Analogously, we obtain in regions ② , ③ , ④

$$\underline{\underline{u_2(s) = \frac{M_T}{32Ga^2 t}[s - 2a]}} ,$$

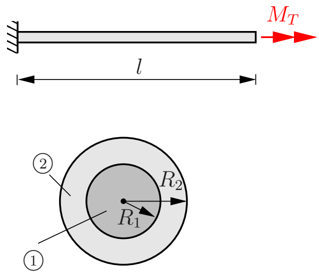
$$\underline{\underline{u_3(s) = -\frac{M_T}{32Ga^2 t}[s - 4a]}} ,$$

$$\underline{\underline{u_4(s) = \frac{M_T}{32Ga^2 t}[s - 6a]}} .$$



Problem 4.6 A tube ② is mounted by heat shrinking on a shaft ① with circular cross section of different material. **P4.6**

Determine the maximum shear stresses in ① and ② as well as the twist under the application of a torque M_T.



Solution First we consider shaft ① and pipe ② independently. For the angle of twist and the stress it yields

$$\vartheta_1 = \frac{M_{T_1} l}{G_1 I_{p_1}} , \qquad \tau_{\max_1} = \frac{M_{T_1}}{W_{T_1}} ,$$

$$\vartheta_2 = \frac{M_{T_2} l}{G_2 I_{p_2}} , \qquad \tau_{\max_2} = \frac{M_{T_2}}{W_{T_2}}$$

with

$$I_{p_1} = \frac{\pi}{2} R_1^4 , \quad I_{p_2} = \frac{\pi}{2} \left(R_2^4 - R_1^4\right) , \quad W_{T_1} = \frac{I_{p_1}}{R_1} , \quad W_{T_2} = \frac{I_{p_2}}{R_2} .$$

Together with equilibrium

$$M_T = M_{T_1} + M_{T_2}$$

and geometric compatibilty

$$\vartheta_1 = \vartheta_2 = \vartheta$$

we obtain

$$M_{T_1} = M_T \frac{G_1 I_{p_1}}{G_1 I_{p_1} + G_2 I_{p_2}} , \qquad M_{T_2} = M_T \frac{G_2 I_{p_2}}{G_1 I_{p_1} + G_2 I_{p_2}}$$

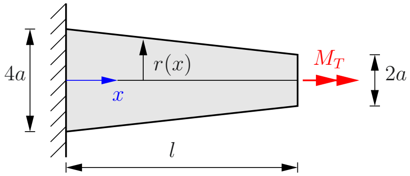
and

$$\underline{\underline{\tau_{\max_1} = \frac{M_T G_1 r_1}{G_1 I_{p_1} + G_2 I_{p_2}}}} , \qquad \underline{\underline{\tau_{\max_2} = \frac{M_T G_2 r_2}{G_1 I_{p_1} + G_2 I_{p_2}}}} ,$$

$$\underline{\underline{\vartheta = \frac{M_T l}{G_1 I_{p_1} + G_2 I_{p_2}}}} .$$

P4.7 **Problem 4.7** A conical shaft with varying radius is loaded by a torque M_T.

Determine the twist and the peripheral stress as a function of x.



Solution The differential equation for the twist angle is given with

$$r(x) = a\left(2 - \frac{x}{l}\right) , \qquad I_p(x) = \frac{\pi}{2} r^4 = \frac{\pi}{2} a^4 \left(2 - \frac{x}{l}\right)^4$$

by

$$\vartheta' = \frac{M_T}{GI_p} = \frac{2M_T}{\pi G a^4} \frac{1}{\left(2 - \frac{x}{l}\right)^4} .$$

Integration with respect to x yields

$$\vartheta(x) = \frac{2M_T l}{3\pi G a^4} \frac{1}{\left(2 - \frac{x}{l}\right)^3} + C .$$

The integration constants are determined from the boundary conditions

$$\vartheta(0) = 0 \quad \rightsquigarrow \quad C = -\frac{2M_T l}{3\pi G a^4} \frac{1}{8} .$$

Thus the twist results in

$$\underline{\underline{\vartheta(x) = \frac{M_T l}{12\pi G a^4} \left\{ \frac{1}{\left(1 - \frac{x}{2l}\right)^3} - 1 \right\}}} .$$

The peripheral shear stress is computed with

$$W_T(x) = \frac{I_p}{r} = \frac{\pi}{2} a^3 \left(1 - \frac{x}{l}\right)^3$$

as

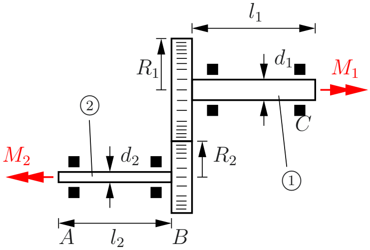
$$\underline{\underline{\tau_P(x) = \frac{M_T}{W_T} = \frac{2M_T}{\pi a^3 \left(2 - \frac{x}{l}\right)^3}}} .$$

Twist and stress have a maximum at $x = l$:

$$\vartheta(l) = \frac{7M_T l}{12\pi G a^4} , \qquad \tau_P(l) = \frac{2M_T}{\pi a^3} .$$

Problem 4.8 The depicted gear-system consists of two shafts (lengths l_1, l_2) of identical material, that are connected by two gear wheels (radii R_1, R_2). The shaft ① is loaded by an external torque M_1.

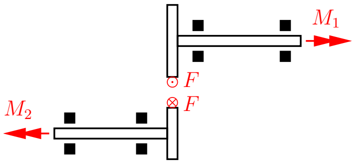
P4.8



a) Determine M_2 such that equilibrium is fulfilled.
b) Choose the diameters d_1 and d_2 such that the admissible shear stress τ_{admis} is not exceeded?
c) Compute the angle of twist at position C, if shaft ② is fixed at position A.

Solution **to a)** Equilibrium of moments

$$M_1 = R_1 F\ , \qquad M_2 = -R_2 F$$



yields

$$\underline{\underline{M_2 = -\frac{R_2}{R_1} M_1}}\ .$$

to b) The critical value of the shear stress is reached in each shaft for:

$$\tau_{\max_1} = \frac{|M_1|}{W_1} = \frac{16 M_1}{\pi d_1^3} = \tau_{\text{admis}} \quad \leadsto \quad \underline{\underline{d_1 = \sqrt[3]{\frac{16 M_1}{\pi \tau_{\text{admis}}}}}}\ ,$$

$$\tau_{\max_2} = \frac{|M_2|}{W_2} = \frac{R_2}{R_1} \frac{16 M_1}{\pi d_2^3} = \tau_{\text{admis}} \quad \leadsto \quad \underline{\underline{d_2 = \sqrt[3]{\frac{R_2}{R_1}}\, d_1}}\ .$$

to c) For the twist angle in ① and ② we obtain

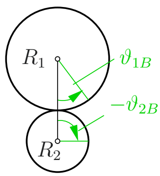
$$\Delta\vartheta_1 = \frac{l_1 M_1}{G I_{T_1}} = \frac{32 M_1 l_1}{\pi G d_1^4}\ , \qquad \Delta\vartheta_2 = \vartheta_{2B} = \frac{32 M_2 l_2}{\pi G d_2^4}\ .$$

With the continuity of the rotations

$$\vartheta_{1B} R_1 = -\vartheta_{2B} R_2$$

and

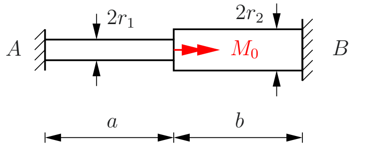
$$\vartheta_C = \vartheta_{1B} + \Delta\vartheta_1$$



we compute

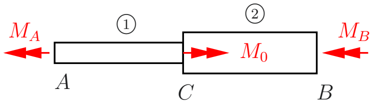
$$\underline{\underline{\vartheta_C = \frac{32 M_1}{G \pi d_1^4} \left\{ l_1 + \left(\frac{R_2}{R_1}\right)^{\frac{2}{3}} l_2 \right\}}}\ .$$

P4.9 **Problem 4.9** A homogeneous, graded shaft with circular cross section is clamped at both ends and loaded by the torque M_0.



Compute the torques at the support positions A and B as well as the twist at the point where M_0 is applied.

Solution The system is statically indeterminate because the support torques M_A and M_B cannot be computed solely from the equilibrium conditions.



$$M_A + M_B = M_0$$

By cutting the shaft at C constant torques are obtained in the regions ① and ② . This results in the following twists

$$\vartheta_1 = \frac{M_A a}{G I_{p_1}} \,, \qquad \vartheta_2 = \frac{M_B b}{G I_{p_2}} \,.$$

Geometric compatibility requires that the two angles of twist are identical:

$$\vartheta_C = \vartheta_1 = \vartheta_2 \,.$$

Together with

$$I_{p_1} = \frac{\pi}{2} r_1^4 \,, \qquad I_{p_2} = \frac{\pi}{2} r_2^4$$

we obtain

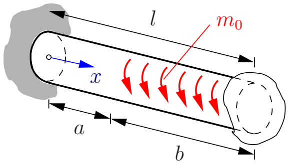
$$\underline{\underline{M_A = M_0 \frac{1}{1 + \frac{r_2^4 a}{r_1^4 b}}}} \,, \qquad \underline{\underline{M_B = M_0 \frac{1}{1 + \frac{r_1^4 b}{r_2^4 a}}}} \,,$$

$$\underline{\underline{\vartheta_C = \frac{2 M_0 a b}{\pi G \left(b r_1^4 + a r_2^4 \right)}}} \,.$$

Problem 4.10 A shaft is clamped at both ends and loaded along part b of its length l by a constant distributed torque m_0.

Determine the function of twist angle and torque.

P4.10



Solution The external torque $m(x)$ has a jump at position $x = a$. We use the Macauley bracket to incorparate the discontinuous function. With

$$m(x) = m_0 < x - a >^0$$

the differential equation for the twist angle follows

$$GI_T \vartheta'' = -m(x) = -m_0 < x - a >^0 \quad .$$

Integrating twice yields

$$GI_T \vartheta' = M_T = -m_0 < x - a >^1 + C_1$$

$$GI_T \vartheta = -\frac{1}{2} m_0 < x - a >^2 + C_1 x + C_2 \; .$$

The constants folllow from the boundary conditions

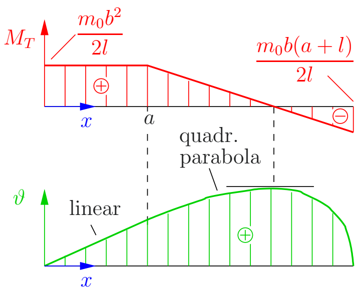
$$\vartheta(0) = 0 \quad \rightsquigarrow \quad C_2 = 0 \; ,$$

$$\vartheta(l) = 0 \quad \rightsquigarrow \quad C_1 = \frac{1}{2} \frac{m_0 b^2}{l} \; .$$

Finally we obtain

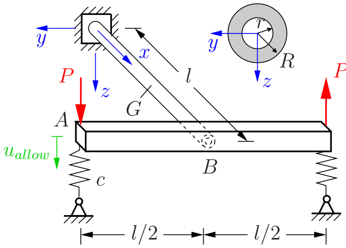
$$\underline{\underline{M_T(x) = m_0 b \left\{ \frac{b}{2l} - \frac{< x - a >^1}{b} \right\}}} \; ,$$

$$\underline{\underline{\vartheta(x) = \frac{1}{2} \frac{m_0 b^2}{GI_T} \left\{ \frac{x}{l} - \frac{< x - a >^2}{b^2} \right\}}} \; .$$



P4.11 **Problem 4.11** The depicted shaft with ring-shaped cross section is clamped at one end. At the other end a rigid beam is attached. The beam is supported by two springs and loaded by the forces P. Determine

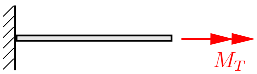
a) the maximum force $P_{\max}$ for a prescribed admissible displacement u_{admis} (in z-direction) at point A,
b) position and value of the maximum shear stress in the cross section of the truss for $P = P_{\max}$.



Given : $u_{\text{admis}} = 2$ cm , $\quad$ l = 2 m
$r = 5$ cm , $\quad$ R = 10 cm
$c = 10^6$ N/m
$G = 8 \cdot 10^{10}$ N/m^2

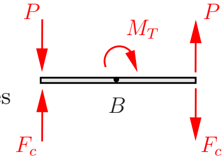
Solution **to a)** The system is statically indeterminate. We free the system at point B leading to the twist of the shaft

$$\Delta\varphi = \frac{M_T\, l}{GI_p} \quad \rightsquigarrow \quad M_T = \frac{GI_p}{l}\,\Delta\varphi$$



with (small twist angles)

$$\Delta\varphi = \frac{u_{\text{admis}}}{l/2} = 0.2\ .$$



Equilibrium of moments for the beam provides

$$\overset{\curvearrowleft}{B}:\ M_T = lP_{\max} - lF_c\,,\ \text{where}\ F_c = c\,u_{\text{admis}}\ .$$

Eliminating $\Delta\varphi$, M_T and F_c yields

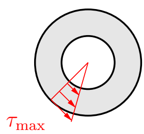
$$\underline{\underline{P_{\max} = \Big(2\,\frac{GI_p}{l^3} + c\Big)u_{\text{admis}}}}\ .$$

With $I_p = \pi(R^4 - r^4)/2 = 1.47 \cdot 10^{-4}$ m^4 and the given numerical values we obtain

$$\underline{\underline{P_{\max}}} = \left(\frac{2 \cdot 8 \cdot 10^{10} \cdot 1.47}{10^4 \cdot 8} + 10^6\right) 2 \cdot 10^{-2} = \underline{\underline{78.7\,\text{kN}}}$$

to b) The shear stress assumes its maximum value at the outer perimeter of the cross section. The absolute value is computed by

$$M_T = P_{\max}\, l - c\, u_{\text{admis}}\, l$$
$$= (78.7 - 10^3 \cdot 0.02)\, 2 = 117.4\,\text{kNm}$$



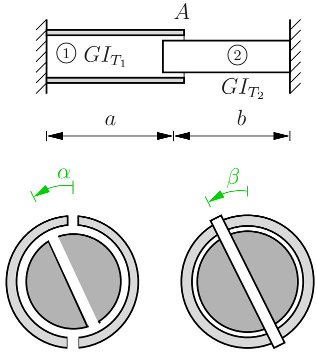
and

$$\underline{\underline{\tau_{\max}}} = \frac{M_T R}{I_p} = \frac{117.4 \cdot 0.1}{1.47 \cdot 10^{-4}} = \underline{\underline{79.8\ \text{MN/m}^2.}}$$

Problem 4.12 The hollow shaft ① and the solid shaft ② are joint by a bolt at A.

P4.12

Determine the torque M_T and the twist angle β of the bolt after assembly for the case that the ends of the shafts have an angular difference of α in the stress-free state.



Solution In the assembled state both shafts are loaded by the torque M_T. We cut the system at position A and determine the angle of twist of ① and ② separately:

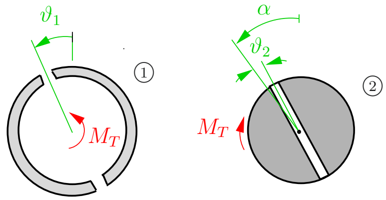
$$\vartheta_1 = \frac{M_T a}{GI_{T_1}}\,, \qquad \vartheta_2 = \frac{M_T b}{GI_{T_2}}\,.$$

From the geometric compatibility in the assembled state

$$\alpha - \vartheta_2 = \vartheta_1$$

and

$$\beta = \vartheta_1$$

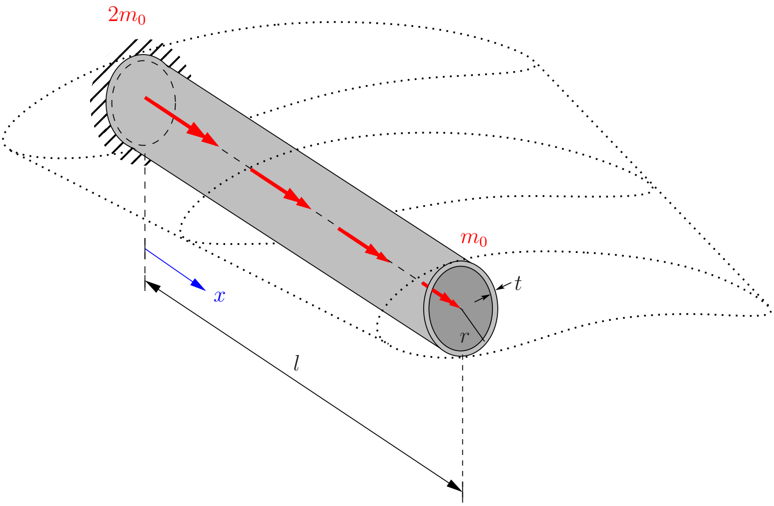


we obtain for M_T and β

$$\underline{\underline{M_T = GI_{T_1}\frac{\alpha}{a}\frac{1}{1+\frac{b}{a}\frac{I_{T_1}}{I_{T_2}}}}}\,,$$

$$\underline{\underline{\beta = \vartheta_1 = \frac{\alpha}{1+\frac{b}{a}\frac{I_{T_1}}{I_{T_2}}}}}\,.$$

P4.13 **Problem 4.13** The thin-walled spar with ring-shaped cross section (length l, shear modulus G, radius r, thickness $t \ll r$) is located in the interior of an airplane wing. It is loaded by a distributed torque $m_T(x)$ with $m_T(0) = 2m_0$ and $m_T(l) = m_0$. The spar is clamped at the fuselage.



Determine
a) the torque $M_T(x)$ in the spar,
b) the distribution of the shear stress $\tau(x)$ and the maximum shear stress $\tau_{\max}$ due to torsion,
c) the angle ϑ_l, by which the end of the wing at $x = l$ rotates with regard to the fuselage.

Solution to a) The distributed torque is given by

$$m_T(x) = \left(2 - \frac{x}{l}\right) m_0 \,.$$

The torque follows by integration

$$M_T(x) = -\int m_T(x)\,\mathrm{d}x + C_1 = \left(\frac{x^2}{2l} - 2x\right) m_0 + C_1$$

which leads with the boundary condition

$$M_T(l) = 0$$

$$\leadsto \qquad \left(\frac{l}{2} - 2l\right) m_0 + C_1 = 0 \qquad \leadsto \qquad C_1 = \frac{3}{2} m_0 l$$

to

$$\underline{\underline{M_T(x) = \left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right) m_0 l}}\ .$$

to b) For the thin-walled spar cross section the shear stresses are computed using the second moment of area for torsion $I_T = 2\pi r^3 t$:

$$\underline{\underline{\tau(x)}} = \frac{M_T}{I_T} r = \underline{\underline{\frac{m_0 l}{2\pi r^2 t} \left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right)}}\ .$$

The maximum shear stress occurs at position $x = 0$ and its value is given by

$$\underline{\underline{\tau_{\max} = \frac{3}{4} \frac{m_0 l}{\pi r^2 t}}}\ .$$

to c) With the second moment of area for torsion I_T and the shear modulus G we obtain for the twist

$$\vartheta'(x) = \frac{M_T(x)}{G I_T} = \frac{m_0 l}{2 G \pi r^3 t} \left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right)$$

as well as for the edge rotation

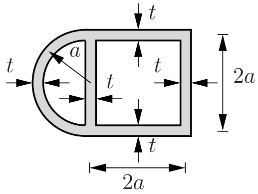
$$\vartheta(x) = \frac{m_o l}{2 G \pi r^3 t} \left(\frac{x^3}{6l^2} - \frac{x^2}{l} + \frac{3}{2} x\right) + C_2\ .$$

The integration constant is determined from the boundary condition $\vartheta(0) = 0$ to be $C_2 = 0$. Thus the edge rotation ϑ_l at the end of the wing yields ($x = l$):

$$\vartheta_l = \vartheta(l) = \frac{m_o l^2}{2 G \pi r^3 t} \left(\frac{1}{6} - 1 + \frac{3}{2}\right) \qquad \leadsto \qquad \underline{\underline{\vartheta_l = \frac{m_0 l^2}{3 G \pi r^3 t}}}\ .$$

P4.14

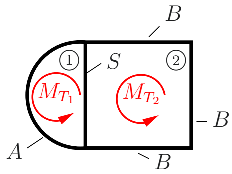
Problem 4.14 A shaft with the depicted thin-walled profile is loaded by a torque M_T.



a) Determine the shear stress in different sections of the profile.

b) Compute the maximum admissible torque, such that the admissible shear stress $\tau_{\rm admis}$ is not exceeded.

Solution The profile consists of two parts. For each part the following holds:



$$T = \tau(s) \cdot t(s) = \frac{M_{T_i}}{2A_{T_i}} \,,$$

$$\vartheta_i' = \frac{M_{T_i}}{GI_{T_i}} = \frac{1}{2GA_{T_i}} \oint\limits_i \frac{T}{t} \mathrm{d}s \,.$$

With the given values

$$A_{T_1} = \frac{\pi}{2} a^2 \,, \qquad A_{T_2} = 4a^2$$

we obtain by considerating that the shear flux in section S is composed of the contributions from the torques M_{T_1} and M_{T_2}:

$$\vartheta_1' = \frac{1}{\pi a^2 G} \left\{ \frac{M_{T_1}}{\pi a^2} \frac{\pi a}{t} + \left[\frac{M_{T_1}}{\pi a^2} - \frac{M_{T_2}}{8a^2} \right] \frac{2a}{t} \right\} \,,$$

$$\vartheta_2' = \frac{1}{8a^2 G} \left\{ \frac{M_{T_2}}{8a^2} \frac{6a}{t} + \left[\frac{M_{T_2}}{8a^2} - \frac{M_{T_1}}{\pi a^2} \right] \frac{2a}{t} \right\} \,.$$

Inserting this result into the geometric compatibility

$$\vartheta' = \vartheta_1' = \vartheta_2'$$

yields

$$\frac{M_{T_1}}{M_{T_2}} = \frac{2+\pi}{10+\frac{16}{\pi}}$$

with

$$M_T = M_{T_1} + M_{T_2}$$

the torques

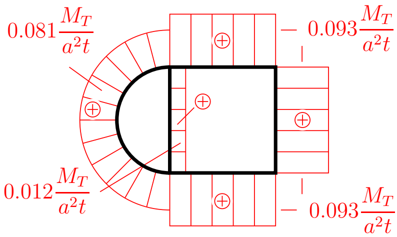
$$M_{T_1} = \frac{2+\pi}{12+\pi+\frac{16}{\pi}} M_T = 0.254\, M_T \ , \quad M_{T_2} = 0.746\, M_T \ .$$

Now the stresses in the sections A, B and S follow

$$\underline{\underline{\tau_A}} = \frac{M_{T_1}}{2A_{T_1} t} = \underline{\underline{0.081 \frac{M_T}{a^2 t}}} \ ,$$

$$\underline{\underline{\tau_B}} = \frac{M_{T_2}}{2A_{T_2} t} = \underline{\underline{0.093 \frac{M_T}{a^2 t}}} \ ,$$

$$\underline{\underline{\tau_S}} = \tau_B - \tau_A = \underline{\underline{0.012 \frac{M_T}{a^2 t}}} \ .$$



Equalizing the maximum shear stress with the admissible shear stress

$$\tau_{\max} = \tau_B = 0.093\, \frac{M_T}{a^2 t} = \tau_{\text{admis}} \ ,$$

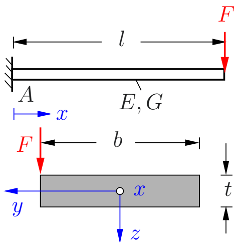
provides the maximum admissible torque

$$\underline{\underline{M_{T_{\text{admis}}} = 10-75\, \frac{\tau_{\text{admis}} a^2 t}{M_T}}} \ .$$

Note: Inserting M_{T_1} and M_{T_2} in ϑ' determines the second moment of area for torsion $I_T = 13.7 a^3 t$. Neglecting the section S, we obtain $I_T = 13.6\, a^3 t$. Thus section S only contributes a small amount to the torsional rigidity.

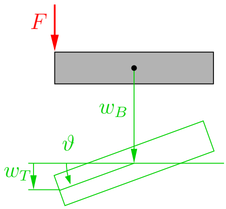
P4.15 **Problem 4.15** The fixed leaf spring ($t \ll b$) is eccentrically loaded by a force F.

Compute the deflection at the point loading. Determine the maximum normal and shear stress.



Solution The leaf spring is subjected to a bending and a torsion load. Due to bending the deflection is given by the table on page 62.

$$w_B = \frac{Fl^3}{3EI} \quad \text{with} \quad I = \frac{bt^3}{12}\,.$$



The constant torque

$$M_T = Fb/2$$

causes a rotation at the end of the spring

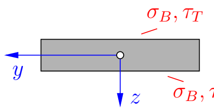
$$\vartheta = \frac{M_T l}{G I_T} \quad \text{with} \quad I_T = \frac{1}{3} bt^3$$

and the corresponding displacement $w_T = \frac{b}{2}\vartheta$. The total deflection is thus obtained by

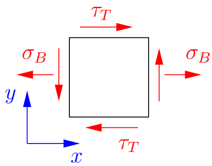
$$\underline{\underline{w}} = w_B + w_T = \underline{\underline{\frac{4Fl^3}{Ebt^3}\left(1 + \frac{3Eb^2}{16Gl^2}\right)}}\,.$$

Bending and torsion cause stress in the extreme fibre of the fixed cross section

$$\sigma_B = \frac{M}{W} = \frac{6lF}{bt^2}\,, \qquad \tau_T = \frac{M_T}{W_T} = \frac{3bF}{2bt^2}\,.$$



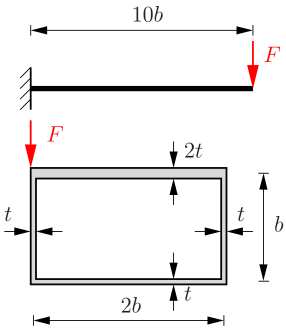
An area element at the top surface ($z = -t/2$) is loaded as sketched. Thus the maximum normal and shear stress follow



$$\underline{\underline{\sigma_1}} = \frac{\sigma_B}{2} + \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau_T^2} = \underline{\underline{\frac{3Fl}{bt^2}\left(1 + \sqrt{1 + \frac{b^2}{4l^2}}\right)}}\,,$$

$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau_T^2} = \underline{\underline{\frac{3Fl}{bt^2}\sqrt{1 + \frac{b^2}{4l^2}}}}\,.$$

Problem 4.16 An element of a bridge is constructed as a thin-walled ($t \ll b$) box girder. During construction the box girder is eccentrically loaded. P4.16



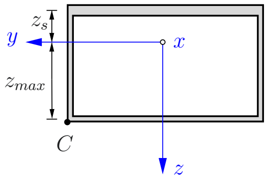
Determine the location and value of the maximum normal and shear stress.

Solution Section properties of the profile are

$$z_s = \frac{2b^2 t + 2 \cdot \frac{b}{2}(b \cdot t)}{8bt} = \frac{3}{8} b \,, \qquad S_y(z_{max}) = b\,t\,\frac{5}{8}\,b = \frac{5}{8}\,b^2\,t$$

$$I_y = 2\left(\frac{tb^3}{12} + \frac{tb^3}{64}\right) + 4bt\left(\frac{3}{8}b\right)^2 + 2bt\left(\frac{5}{8}b\right)^2$$

$$= \frac{37}{24} tb^3 \,,$$



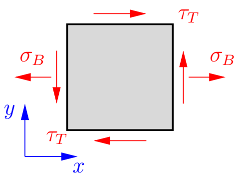
$$W = \frac{I_y}{z_{\max}} = \frac{37}{15} tb^2 \,,$$

$$W_T = 2A_T t_{\min} = 4b^2 t \,.$$

Using bending moment, torque, shear force in the clamped support

$$M_B = -10\,b\,F \,, \qquad M_T = b\,F \,, \qquad V_z = F$$

yields for the lower section



$$\sigma_B = \frac{M_B}{W} = -\frac{150}{37}\,\frac{F}{bt} \,,$$

$$\tau_T = \frac{M_T}{W_T} = \frac{1}{4}\,\frac{F}{bt} \,, \qquad \tau_Q = \frac{V_z\,S_y}{I_y\,t} = \frac{15}{37}\,\frac{F}{bt} \,.$$

The largest absolute value for the normal stress and the shear stress are obtained by $\tau = \tau_T + \tau_Q$ at location C

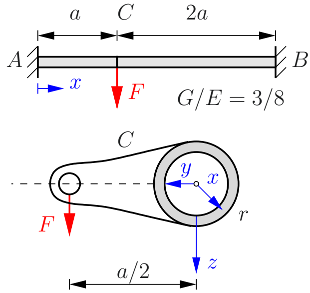
$$\underline{\underline{\sigma_2}} = \frac{\sigma_B}{2} - \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau^2} = \underline{\underline{-4.16\,\frac{F}{bt}}} \,,$$

$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau^2} = \underline{\underline{2.13\,\frac{F}{bt}}} \,.$$

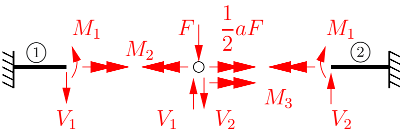
P4.17

Problem 4.17 The depicted cantilever with thin-walled circular cross section is clamped at both ends and loaded eccentrically at point C.

Determine the deflection at the point where the load is applied and compute the normal stress and the shear stresses due to torsion.



Solution The cantilever is cut at point C. Equilibrium yields



$$M_2 = M_3 + \frac{1}{2}aF\,, \quad V_1 = V_2 + F\,.$$

The deflection, the angle of bending, and the angle of twist are given at point C by (see table on page 62):

$$w_{C_1} = \frac{V_1 a^3}{3EI} - \frac{M_1 a^2}{2EI}\,, \quad w_{C_2} = -\frac{8V_2 a^3}{3EI} - \frac{4M_1 a^2}{2EI}\,,$$

$$w'_{C_1} = \frac{V_1 a^2}{2EI} - \frac{M_1 a}{EI}\,, \quad w'_{C_2} = +\frac{4V_2 a^2}{2EI} + \frac{2M_1 a}{EI}\,,$$

$$\vartheta_{C_1} = \frac{M_2 a}{GI_T}\,, \quad \vartheta_{C_2} = -\frac{2M_3 a}{GI_T}\,.$$

Compatibility demands

$$w_{C_1} = w_{C_2}\,, \quad w'_{C_1} = w'_{C_2}\,, \quad \vartheta_{C_1} = \vartheta_{C_2}$$

which renders

$$V_1 = \frac{20}{27}F\,, \quad {}_2 = -\frac{7}{27}F\,, \quad M_1 = \frac{8}{27}aF\,,$$

$$M_2 = \frac{1}{3}aF\,, \quad M_3 = -\frac{1}{6}aF\,.$$

The second moments of area and the elasticity constants

$$I_T = 2I = 2\pi r^3 t \qquad \text{und} \qquad \frac{G}{E} = \frac{3}{8}$$

yield the deflection at the point of loading

$$\underline{\underline{w_F}} = w_{C_1} + \frac{a}{2}\vartheta_{C_1} = \underline{\underline{\frac{26Fa^3}{81EI}}} .$$

To compute the stresses, we need the bending moments at A and B:

$$M_A = M_1 - V_1 a = -\tfrac{4}{9}aF ,$$

$$M_B = M_1 + V_2 2a = -\tfrac{2}{9}aF .$$

The maximum normal stresses due to bending in A, B and C are given with the section modulus $W = I\,/\,r$

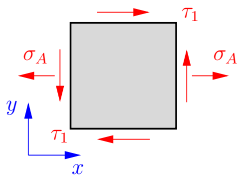
$$\sigma_A = \frac{|M_A|}{W} = \frac{4arF}{9\ I} , \quad \sigma_B = \frac{2arF}{9\ I} ,$$

$$\sigma_C = \frac{|M_1|}{W} = \frac{8arF}{27\ I} .$$

The shear stresse in secion ① or ② are calculated with $W_T = 2W = \frac{2I}{r}$:

$$\tau_1 = \frac{M_2}{W_T} = \frac{arF}{6\ I} , \quad \tau_2 = \frac{M_3}{W_T} = \frac{arF}{12\ I} .$$

The largest stresses occur at the point A. An area element at the top surface (analogously on the bottom surface) is loaded as sketched. For the principal stress and the maximum shear stress we obtain

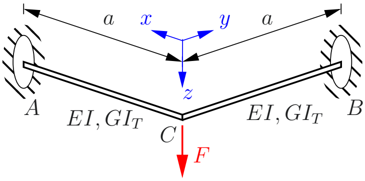


$$\underline{\underline{\sigma_1}} = \frac{\sigma_A}{2} + \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2} = \underline{\underline{\frac{arF}{2\ I}}} ,$$

$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2} = \underline{\underline{\frac{5arF}{18\ I}}} .$$

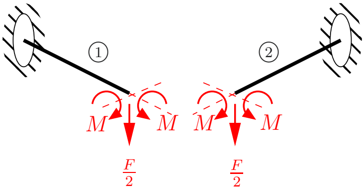
P4.18

Problem 4.18 The depicted cantilever is fixed at both ends and bent by 90°. The cantilever is loaded at point C by the force F.



Compute the deflection at the point C.

Solution To solve the problem we use superposition. We cut the system at point C and apply symmetry arguments for the depicted loading with respect to bending and torsion. At this stage the moment M is unknown. From the table on page 62 we deduce



$$w'_C = \frac{Fa^2}{4EI} - \frac{Ma}{EI}\,, \qquad w_C = \frac{Fa^3}{6EI} - \frac{Ma^2}{2EI}\,.$$

The angle of twist due to torsion at C is given by

$$\vartheta_C = \frac{Ma}{GI_T}\,.$$

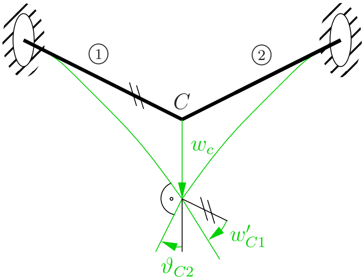
The geometric compatibility

$$w'_{C1} = \vartheta_{C2}$$

yields

$$M = \frac{Fa}{4}\,\frac{GI_T}{EI + GI_T}$$

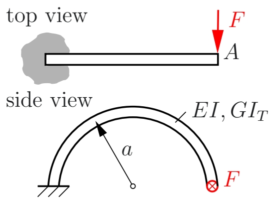
and the final result



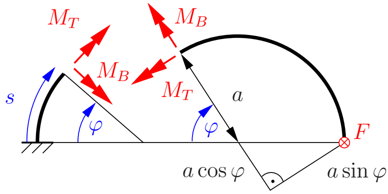
$$\underline{\underline{w_C = \frac{Fa^3}{24EI}\,\frac{4EI + GI_T}{EI + GI_T}}}\,.$$

Problem 4.19 The depicted semi-circular support is loaded at point A by a force F. **P4.19**

Determine the deflection at the point A.



Solution Equilibrium of moments provides the bending moment M_B and the torque M_T



$$M_B(\varphi) = -aF\sin\varphi\ ,$$

$$M_T(\varphi) = a(1+\cos\varphi)F\ .$$

The angle of twist is given by

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}s} = \frac{M_T}{GI_T} \qquad \text{mit} \qquad \mathrm{d}s = a\mathrm{d}\varphi\ .$$

The twist $\mathrm{d}\vartheta$ at position φ causes the deflection at A

$$\mathrm{d}w_{TA} = a\sin\varphi\,\mathrm{d}\vartheta\ .$$

Combining the previous results and integration yields the deflection due to torsion

$$w_{TA} = \int \mathrm{d}w_{TA} = \frac{Fa^3}{GI_T}\int\limits_0^\pi \sin\varphi(1+\cos\varphi)\mathrm{d}\varphi = \frac{2Fa^3}{GI_T}\ .$$

The deflection due to bending is follows from

$$EI\frac{\mathrm{d}^2 w_B}{\mathrm{d}s^2} = -M_B \quad \rightsquigarrow \quad \frac{\mathrm{d}^2 w_B}{\mathrm{d}\varphi^2} = \frac{Fa^3}{EI}\sin\varphi\ ,$$

$$\frac{\mathrm{d}w_B}{\mathrm{d}\varphi} = \frac{Fa^3}{EI}(-\cos\varphi + C_1)\ , \qquad w_B(\varphi) = \frac{Fa^3}{EI}(-\sin\varphi + C_1\varphi + C_2)$$

and the boundary conditions

$$w_B'(0) = 0 \quad \rightsquigarrow \quad C_1 = 1\ , \qquad w_B(0) = 0 \quad \rightsquigarrow \quad C_2 = 0\,.$$

Using these constants yields

$$w_B(\varphi) = \frac{Fa^3}{EI}(\varphi - \sin\varphi)\ .$$

Finally the total deflection at A is given at position $\varphi = \pi$

$$\underline{\underline{w_A}} = w_{TA} + w_B(\pi) = \underline{\underline{\frac{Fa^3}{EI}\left(\pi + 2\frac{EI}{GI_T}\right)}}\ .$$

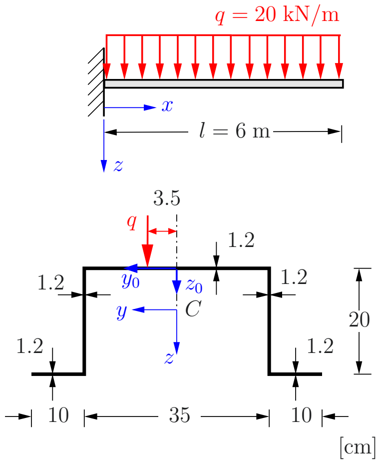
P4.20

Problem 4.20 A cantilever beam with the depicted profile is subjected to an eccentric line load q. Determine at the clamped support

a) the largest shear stress due to the shear force and its position,

b) the shear stress due to torsion.

c) the distribution of the shear stresses due to shear force and torsion across the profile. Determine position and value of the largest shear stress.



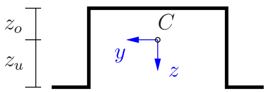
Solution We start by computing the stress resultants at the clamped support:

$$
\begin{aligned}
V_z &= q\,l &&= 20 \cdot 6 &&= 120 \text{ kN} \,, \\
M_y &= -\frac{q\,l^2}{2} &&= -20 \cdot \frac{6^2}{2} &&= -360 \text{ kNm} \,, \\
M_T &= q\,l \cdot 3.5\,\text{cm} &&= 20 \cdot 6 \cdot 0.035 &&= 4.2 \text{ kNm} \,.
\end{aligned}
$$

With the geometric data of the profile we calculate the position of the centroid C and the second moment of area I_y:

$$
z_o = \frac{\sum z_i A_i}{\sum A_i} = \frac{2 \cdot (20 \cdot 1.2) \cdot 10 + 2 \cdot (10 \cdot 1.2) \cdot 20}{35 \cdot 1.2 + 2 \cdot 20 \cdot 1.2 + 2 \cdot 10 \cdot 1.2} = 8.42 \text{ cm} \,,
$$

$$
z_u = 20 - z_o = 11.58 \text{ cm} \,,
$$



$$
\begin{aligned}
I_y &= \sum \frac{b_i h_i^3}{12} + \sum A_i \bar{z}_i^2 \\
&= (35 \cdot 1.2) \cdot 8.42^2 + 2 \cdot \frac{20^3 \cdot 1.2}{12} \\
&\quad + 2 \cdot (20 \cdot 1,2) \cdot 1.58^2 + 2 \cdot (10 \cdot 1.2) \cdot 11.58^2 \\
&= 7915.8 \text{ cm}^4 \,.
\end{aligned}
$$

to a) The shear stress due to the shear force is obtained by

$$\tau = \frac{V_z\, S_y}{I_y\, h} = \frac{120}{7915.8 \cdot 1.2}\, S_y = 0.01263\, S_y \,.$$

The static moment S_y reaches its maximum at $z = 0$:

$$S_{y\,\max} = S(z=0) = 8.4 \cdot 1.2 \cdot \frac{35}{2} + \frac{1}{2}\, 8.4^2 \cdot 1.2 = 218.7 \text{ cm}^3 \,.$$

From this result the maximum shear stress due to shear force follows

$$\tau_{V\,\max} = 0.01263 \cdot 218.7$$

$$\leadsto \quad \underline{\underline{\tau_{V\,\max} = 2.76 \text{ kN/cm}^2 = 27.6 \text{ N/mm}^2}} \,.$$

to b) The shear stress due to torsion is calculated using the second moment of area for torsion respectively the torsion modulus of the profile:

$$I_T = \frac{1}{3} \sum h_i t_i^3 = \frac{1}{3}(35 + 2 \cdot 20 + 2 \cdot 10) \cdot 1.2^3 = 54.7 \text{ , cm}^4$$

$$W_T = \frac{1}{3} \frac{\sum h_i t_i^3}{t_{\max}} = \frac{54.7}{1.2} = 45.6 \text{ cm}^3 \,.$$

With the already calculated torque M_T we obtain

$$\tau_T = \frac{M_T}{W_T} = \frac{4.2 \cdot 10^2}{45.6}$$

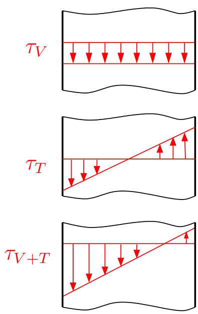
$$\leadsto \quad \underline{\underline{\tau_T = 9.21 \text{ kN/cm}^2 = 92.1 \text{ N/mm}^2}} \,.$$

to c) The largest shear stress occurs at the position $z = 0$. It is distributed linearly across the wall thickness with the following extreme values:

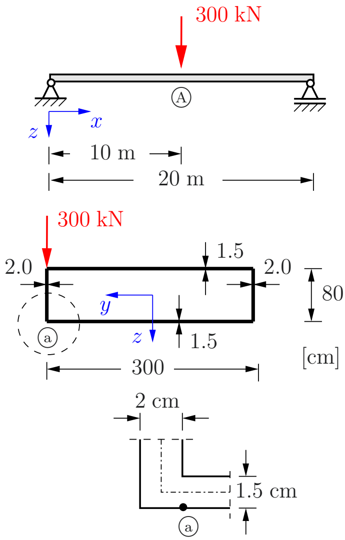
$$\tau_{\text{inside}} = 27.6 - 92.1 = -64.5 \text{ N/mm}^2,$$

$$\tau_{\text{outside}} = 27.6 + 92.1 = 119.7 \text{ N/mm}^2$$

$$\leadsto \quad \underline{\underline{\tau_{\max} = 119.9 \text{ N/mm}^2}} \,.$$



P4.21 **Problem 4.21** A thin-walled box girder is loaded by a force of 300 kN. Determine for the cross section at position Ⓐ



a) the stress distribution (normal and shear stresses) due to shear force and torsion,

b) the position of the maximum principal stress and

c) the value and direction of the principal stress at the vertex ⓐ of the profile.

Remark: Assume for the torsional load case a fork bearing at the left end.

Solution The second moment of area is given by

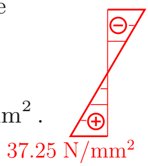
$$I_y = \sum_i \frac{b_i h_i^3}{12} + \sum_i A_i \bar{z}_i^2 = 2 \cdot \frac{2 \cdot 80^3}{12} + 2 \cdot (1.5 \cdot 300) \cdot 40^2 = 1.611 \cdot 10^6 \text{ cm}^4 .$$

The stress resultants at position Ⓐ (or directly left of it) are

$$V_z = \frac{300}{2} = 150\,\text{kN}\,, \quad M_y = \frac{300 \cdot 20}{4} = 1500\text{ kNm}\,,$$

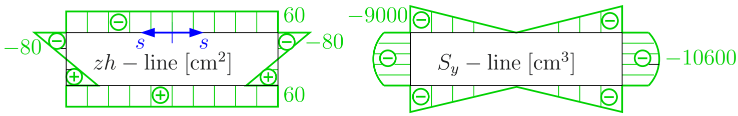
$$M_T = 300 \cdot 1.5 = 450\,\text{kNm}\,.$$

to a) The normal stress is linear across the height of the cross section and reaches in point ⓐ the value



$$\sigma_x = \frac{M_y}{I_y} z_a = \frac{1500 \cdot 1000 \cdot 1000}{1.611 \cdot 10^6 \cdot 10^4} \cdot 40 \cdot 10 = 37.25 \text{ N/mm}^2 .$$

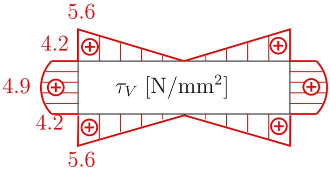
The shear stresses due to V_z are determined by the zh-line and S_y-line.



By using the S_y-line we obtain

$$\tau_V = \frac{V_z S_y}{I_y h} = \frac{150}{1.611 \cdot 10^6}\frac{S_y}{h} = 9.3 \cdot 10^{-5}\frac{S_y}{h}\ \mathrm{kN/cm^2}\,.$$

At position ⓐ they assume the value



$$\tau_{Va} = \frac{150 \cdot 9000}{1.611 \cdot 10^6 \cdot 1.5}$$

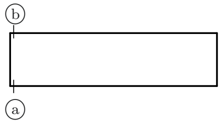
$$= 0.56\ \mathrm{kN/cm^2} = 5.6\ \mathrm{N/mm^2}\,.$$

The shear stresses due to torsion are given by

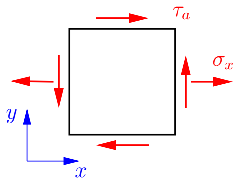
$$\tau_T = \frac{M_T}{2A_T h}\,, \qquad A_T = 300 \cdot 80 = 24000\ \mathrm{cm^2}$$

$$\rightsquigarrow \underline{\underline{\tau_{Ta}}} = \frac{450 \cdot 10^3 \cdot 10^3}{2 \cdot 24000 \cdot 1.5 \cdot 10^3} = \underline{\underline{6.25\ \mathrm{N/mm^2}}}\,.$$

to b) The maximum shear stresses occur at points ⓐ and ⓑ, the maximum normal stresses at point ⓐ. Thus the principal stresses assume the largest value at ⓐ.



to c) In point ⓐ the shear and normal stresses are:



$$\tau_a = \tau_{Va} + \tau_{Ta} = 5.6 + 6.25 = 11.85\ \mathrm{N/mm^2}\,,$$

$$\sigma_x = 37.25\ \mathrm{N/mm^2}\,.$$

The principal stresses are given by

$$\underline{\underline{\sigma_1}} = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_a^2} = \underline{\underline{40.7\ \mathrm{N/mm^2}}},$$

$$\underline{\underline{\sigma_2}} = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_a^2} = \underline{\underline{-3.45\ \mathrm{N/mm^2}}}.$$

For the direction of the principal stress σ_1 we compute

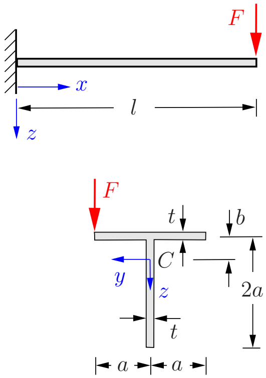
$$\tan 2\alpha_0 = \frac{2\tau}{\sigma_x} = 0.636 \quad \rightsquigarrow \quad \underline{\underline{\alpha_0 = 16.23^\circ}}\,.$$

P4.22

Problem 4.22 A cantilever beam with thin-walled T-profile ($t \ll a$) is eccentrically loaded by a force F. The clamped support is designed such that warping is allowed.

Determine the maximum stresses due to bending, shear force and torsion. At which position do they occur?

Given: $t = a/10$, $l = 20\,a$



Solution We start by determining the following geometric properties of the profile:

$$b = \frac{a}{2}\,,$$

$$I = b^2 2at + \left[\frac{t(2a)^3}{12} + b^2 2at\right] = \frac{1}{6}\,a^4\,, \quad W = \frac{I}{3a/2} = \frac{1}{9}\,a^3\,,$$

$$S_C = b\,2at + \frac{b}{2}\,\frac{at}{2} = \frac{9}{80}\,a^3\,,$$

$$I_T = \frac{1}{3}\,2(2a)t^3 = \frac{4}{3000}\,a^4\,, \qquad W_T = \frac{I_T}{t} = \frac{4}{300}\,a^3\,.$$

The bending moment reaches its maximum at the clamped support ($x = 0$), while shear force and torque are constant along the beam:

$$M_{\max} = -lF = -20aF\,, \qquad V = F\,, \qquad M_T = aF\,.$$

We compute the maximum bending stress (compression, at the lower surface, at $x = 0$), the maximum shear stress due to shear force (at the centroid C), and the shear stress due to torsion (at the outer boundary of the flanges):

$$\underline{\underline{\sigma_{\max}}} = \frac{|M_{\max}|}{W} = \frac{20aF}{\frac{1}{9}a^3} = \underline{\underline{180\,\frac{F}{a^2}}}\,,$$

$$\underline{\underline{\tau_V^C}} = \frac{C\,S_C}{I\,t} = \frac{F\,\frac{9}{80}a^3}{\frac{1}{6}a^4\,\frac{1}{10}a} = \underline{\underline{\frac{27}{4}\,\frac{F}{a^2}}}\,,$$

$$\underline{\underline{\tau_{M_T}}} = \frac{M_T}{W_T} = \frac{aF}{\frac{4}{300}a^3} = \underline{\underline{75\,\frac{F}{a^2}}}\,.$$

Note: The shear stress due to shear force is small compared to the shear stress due to torsion.