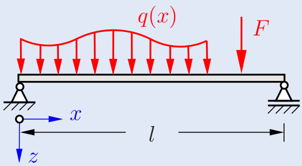
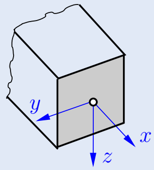
Chapter 3

Bending of Beams

3

Beam = straight structural element, length l large compared to dimensions of the cross section, perpendicular loads.





3.1

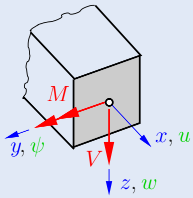
3.1 Ordinary bending

nomenclature and assumptions:

- x = axis of cross section centroids; y, z = principal axis of the second moment of area (moment of inertia).
- kinematic assumption: *plane cross sections remain plane*

$$w = w(x)\,, \qquad u = z\,\psi(x)\,,$$

w = displacement in z-direction,
u = displacement in x-direction,
ψ = rotation angle of cross section.

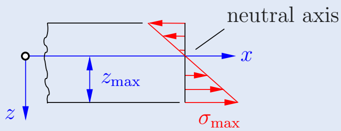


- stress resultants:

V = V_z = shear force,
M = M_y = bending moment.

Normal stress

$$\sigma(z) = \frac{M}{I}\,z$$



I = moment of inertia with respect to y-axis,
z = distance to *neutral axis* (= axis of centroids).

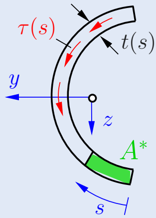
The largest absolute value of the stress occurs in the extreme fibre:

$$\sigma_{\max} = \frac{M}{W}\,, \qquad W = \frac{I}{|z_{\max}|} \quad = \text{ section modulus.}$$

Shear stress

a) thin-walled, open profile

$$\tau(s) = \frac{V\,S(s)}{I\,t(s)}\,,$$

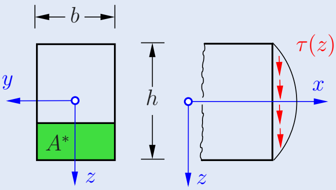


$S(s)$ = static moment of A^* with regard to y-axis,

$t(s)$ = thickness of profile at position s.

b) compact cross section

$$\tau(z) = \frac{V\,S(z)}{I\,b(z)}\,.$$



special case: rectangle

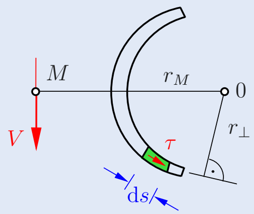
$$\tau = \frac{3}{2}\frac{Q}{A}\left(1 - \frac{4z^2}{h^2}\right)\,.$$

Note: $\tau_{\mathrm{max}} = \tau(z=0) = \frac{3}{2}\frac{Q}{bh}$ is 50% larger than $\tau_{\mathrm{mean}} = \frac{Q}{bh}$.

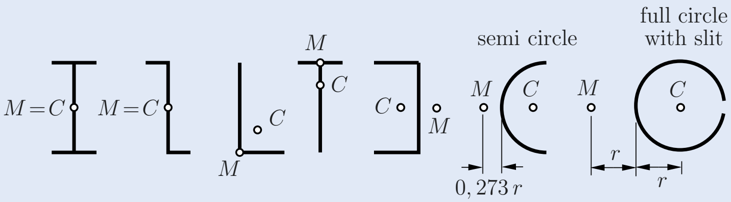
Shear center M of singly symmetrical cross sections.

moment of V with regard to 0 = moment of distributed shear stresses with regard to 0:

$$r_M Q = \int \tau(s)\, r_\perp(s)\, t(s)\, \mathrm{d}s$$



Position of centriod C und shear center M for selected profiles:



Basic equations

equilibrium conditions	$\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q\,, \quad \dfrac{\mathrm{d}M}{\mathrm{d}x} = V\,,$
Hooke's law, kinematics	$M = EI\psi'$
	$V = GA_S(\psi + w')\,,$

EI = bending stiffness,
GA_S = shear stiffness,
A_S = κA = shear area (κ = shear correction factor).

Rigid with respect to shear (Bernoulli beam): If we additionally assume, that cross sections perpendicular to the undeformed beam axis remain perpendicular to the deflection curve during the deformation, it follows from Hooke's law for the shear force ($GA_S \to \infty$)

$$\psi = -w'\,.$$

Differential equation of the deflection curve for the Bernoulli beam: Inserting into Hooke's law for M yields

$$EIw'' = -M\,.$$

This leads with the equilibrium conditions to

$$(EIw'')'' = q\,,$$

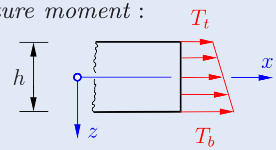
or for $EI = \text{const}$

$$EIw^{IV} = q\,.$$

Temperature induced moment

A linearly, across the height h, varying temperature field (= temperature gradient) can be treated by a *temperature moment* :

$$M_T = EI\alpha_T\,\frac{T_b - T_t}{h}\,,$$

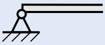
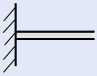
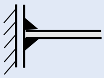


α_T = coefficient of thermal expansion.

In this case, the differential equation for the deflection curve yields

$$EIw'' = -(M + M_T)\,.$$

Table of boundary conditions

support	w	w'	M	V
	0	$\neq 0$	0	$\neq 0$
	0	0	$\neq 0$	$\neq 0$
 free end	$\neq 0$	$\neq 0$	0	0
	$\neq 0$	0	$\neq 0$	0

Solution methods

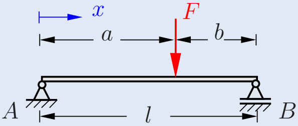
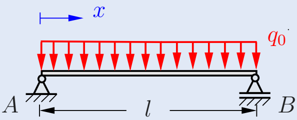
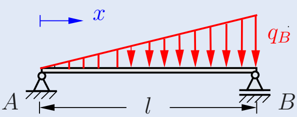
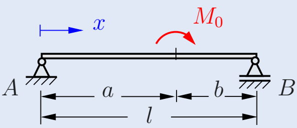
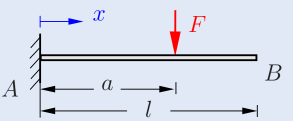
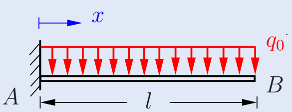
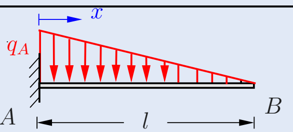
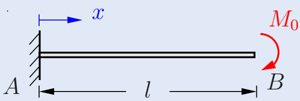
1. For continuous functions of $q(x)$ or $M(x)$, four or two times integration of the corresponding differential equation yields the deflection curve $w(x)$. The four or two integration constants are obtained by the boundary conditions (see table of boundary conditions).

2. For several regions (discontinuities in the loads, deformation, concentrated forces or concentrated moments), the integration has to be performed piecewise. The integration constants are determined from boundary and matching (continuity) conditions. The computation can by simplified by using the Macauley bracket (see Engineering Mechanics 1):

$$< x - a >^n = \begin{cases} 0 & \text{für } x < a\,, \\ (x-a)^n & \text{für } x > a\,. \end{cases}$$

3. Statically indeterminate problems can be solved by using *superposition* of known deflections and rotations. For this purpose, deflection and rotations of the most frequent load cases and support situations can be found in the table on page 62/63.

4. Statically indeterminate problems can also be solved by using the *principle of virtual forces (energy method)* (see chapter 5).

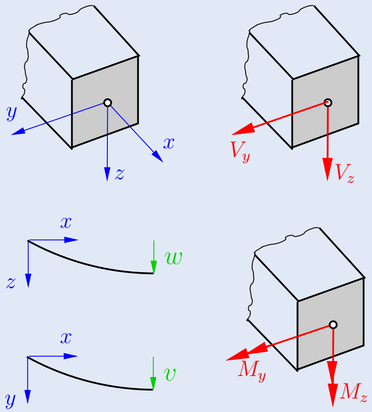
no.	load case	EIw'_A	EIw'_B
1		$\frac{Fl^2}{6}(\beta-\beta^3)$	$-\frac{Fl^2}{6}(\alpha-\alpha^3)$
2		$\frac{q_0 l^3}{24}$	$-\frac{q_0 l^3}{24}$
3		$\frac{7}{360}q_B l^3$	$-\frac{1}{45}q_B l^3$
4		$\frac{M_0 l}{6}(3\beta^2-1)$	$\frac{M_0 l}{6}(3\alpha^2-1)$
5		0	$\frac{Fa^2}{2}$
6		0	$\frac{q_0 l^3}{6}$
7		0	$\frac{q_A l^3}{24}$
8		0	$M_0 l$

explanations:
$$\xi=\frac{x}{l}\,,\quad \alpha=\frac{a}{l}\,,\quad \beta=\frac{b}{l}\,,\quad (\;)'\,\widehat{=}\,\frac{\mathrm{d}}{\mathrm{d}x}(\;)=\frac{1}{l}\,\frac{\mathrm{d}}{\mathrm{d}\xi}(\;)\,,$$

$EIw(x)$	$EIw_{\max}$
$\frac{Fl^3}{6}[\beta\xi(1-\beta^2-\xi^2)+<\xi-\alpha>^3]$	$\frac{Fl^3}{48}$ for $\alpha=\beta=1/2$
$\frac{q_0l^4}{24}(\xi-2\xi^3+\xi^4)$	$\frac{5}{384}q_0l^4$
$\frac{q_Bl^4}{360}(7\xi-10\xi^3+3\xi^5)$	see problem 3.13
$\frac{M_0l^2}{6}[\xi(3\beta^2-1)+\xi^3-3<\xi-\alpha>^2]$	$\frac{M_0l^2}{27}\sqrt{3}$ for $a=0$
$\frac{Fl^3}{6}[3\xi^2\alpha-\xi^3+<\xi-\alpha>^3]$	$\frac{Fl^3}{3}$ for $a=l$
$\frac{q_0l^4}{24}(6\xi^2-4\xi^3+\xi^4)$	$\frac{q_0l^4}{8}$
$\frac{q_Al^4}{120}(10\xi^2-10\xi^3+5\xi^4-\xi^5)$	$\frac{q_Al^4}{30}$
$M_0\frac{x^2}{2}$	$M_0\frac{l^2}{2}$

$<\xi-\alpha>^n \;\widehat{=}\;$ Macauley bracket

3.2 Biaxial bending



x = axis of centroids,
y, z = arbitrary orthogonal axis.

shear forces V_y , V_z

and

bending moments M_y , M_z (positive when positive right-hand screw at positive intersection).

Differential equation of the deflection for *shear rigid* beams:

$$Ew'' = \frac{1}{\Delta}(-M_y I_z + M_z I_{yz})$$

$$Ev'' = \frac{1}{\Delta}(M_z I_y - M_y I_{yz})$$

$$\Delta = I_y I_z - I_{yz}^2 \; ,$$

$I_y,\ I_z,\ I_{yz}$ = second order area moments.

Normal stress

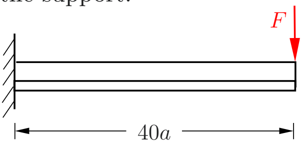
$$\sigma = \frac{1}{\Delta}[(M_y I_z - M_z I_{yz})z - (M_z I_y - M_y I_{yz})y] \; .$$

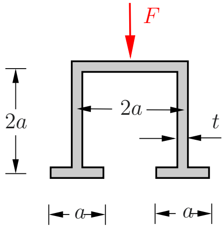
Special case: If y, z are *principal axis* ($I_{yz} = 0$), then

$$EI_y w'' = -M_y \; , \quad EI_z v'' = M_z \; , \quad \sigma = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y \; .$$

Problem 3.1 A cantilever beam with the depicted cross section (constant wall thickness t, $t \ll a$ is subjected to a concentrated force F at one end. **P3.1**

Determine the maximum stress in the cross section at the support.

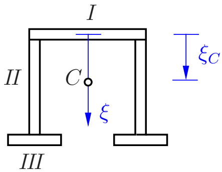




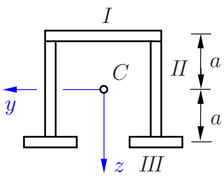
Solution The distance of the centroid ξ_C from the top surface is obtained from the sub-areas by using $t \ll a$

$$\xi_C = \frac{\Sigma \xi_i A_i}{\Sigma A_i} = \frac{2\overbrace{(2at \cdot a)}^{II} + 2\overbrace{(at \cdot 2a)}^{III}}{\underbrace{2at}_{I} + 2 \cdot \underbrace{2at}_{II} + 2 \cdot \underbrace{at}_{III}} = \frac{8a^2 t}{8at}$$

$$= a\,.$$



The second moment of area with regard to the y-axis is computed by using the parallel-axis theorem.



$$I_y = \overbrace{a^2 \cdot 2at}^{I} + 2\overbrace{\left\{\frac{t(2a)^3}{12}\right\}}^{II} + 2\overbrace{\{a^2 \cdot at\}}^{III} = \frac{16}{3} t a^3 ,$$

Thus we obtain for the section modulus

$$W = \frac{I_y}{z_{\max}} = \frac{\frac{16}{3} t a^3}{a} = \frac{16}{3} t a^2 \,.$$

The stress in the cross section at the support is calculated using the bending moment at this position

$$M = -40\, aF$$

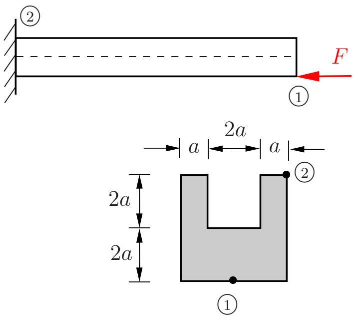
to be

$$\underline{\underline{\sigma_{\max}}} = \frac{|M|}{W} = \frac{40aF}{\frac{16}{3} t a^2} = \underline{\underline{\frac{30}{4} \frac{F}{at}}}$$

(the upper fibre is in tension, the lower under compression).

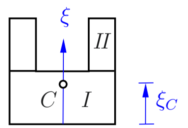
P3.2 **Problem 3.2** A cantilever beam with the sketched cross section is loaded by the force F at point ① .

Determine the normal stresses at point ② at the support.



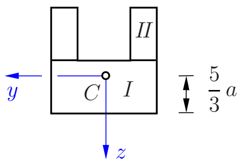
Solution As the neutral axis is passing trough the centroids of the cross sections, we first determine the position of the centroid:

$$\xi_C = \frac{\Sigma A_i \xi_i}{\Sigma A_i} = \frac{\overbrace{8a^2 \cdot a}^{I} + 2\overbrace{\left\{2a^2 \cdot 3a\right\}}^{II}}{8a^2 + 4a^2} = \frac{5}{3}a\,.$$



The second moment of area with respect to the y-axis is computed by summing up the contributions of the sub-areas:

$$I_y = \left[\frac{4a(2a)^3}{12} + \left(\frac{2}{3}\,a\right)^2 8a^2\right] + \\ +2\left[\frac{a(2a)^3}{12} + \left(\frac{4}{3}a\right)^2 2a^2\right] = \frac{44}{3}a^4\,.$$



The following stress resultants are present in the cross section at the support

$$N = -F \qquad \text{and} \qquad M_y = -\frac{5}{3}aF\,.$$

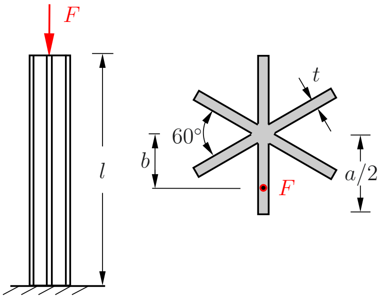
The associated stresses are (σ_N due to normal force, σ_M due to bending moment)

$$\sigma_N = \frac{N}{A} = -\frac{F}{12a^2} \qquad \text{and} \qquad \sigma_M = \frac{M_y}{I_y}z = -\frac{5}{3}\,\frac{aFz}{\frac{44}{3}a^4} = -\frac{5}{44}\,\frac{Fz}{a^3}\,.$$

At point ② superposition with $z_2 = -\frac{7}{3}a$ yields

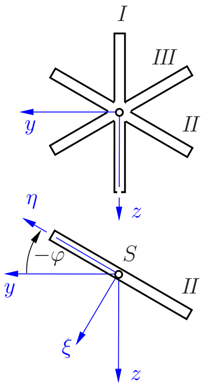
$$\underline{\underline{\sigma}} = \sigma_N + \sigma_M(z_2) = -\frac{F}{12a^2} + \frac{5}{44}\,\frac{F}{a^3}\,\frac{7}{3}a = \underline{\underline{\frac{2}{11}\,\frac{F}{a^2}}}\,.$$

Problem 3.3 The column with a star-shaped cross section ($t \ll a$) is loaded by a force F, applied off center. **P3.3**



Determine
a) the maximum absolute value of the stress,
b) the maximal value of b such that nowhere in the cross section tensile stresses occur.

Solution **to a)** Due to the load and the symmetry of the cross section it is convenient to introduce the following y,z-coordinate system. This yields



$$I_{yI} = \frac{ta^3}{12}\,.$$

The second moments of area for the sub-areas *II* and *III* with respect to the y-axis are determined by the transformation equations

$$I_\eta = \frac{at^3}{12}\,, \quad I_\zeta = \frac{ta^3}{12}\,, \quad I_{\eta\zeta} = 0\,, \quad \varphi = -30^\circ\,.$$

Using $t \ll a$ we obtain

$$I_{yII} = I_{yIII} = \frac{I_\eta + I_\zeta}{2} + \frac{I_\eta - I_\zeta}{2}\cos 2\varphi + I_{\eta\zeta}\sin 2\varphi = \frac{ta^3}{24} - \frac{ta^3}{24}\,\frac{1}{2} = \frac{ta^3}{48}\,.$$

This leads to

$$I_y = I_{yI} + 2I_{yII} = \frac{ta^3}{12} + 2\frac{ta^3}{48} = \frac{ta^3}{8}\,.$$

Together with the stress resultants $N = -F$ and $M_y = -bF$ it follows

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y} z = -\frac{F}{3at} - \frac{8bF}{ta^3} z\,.$$

The largest stress (compression) occurs at $z = a/2$:

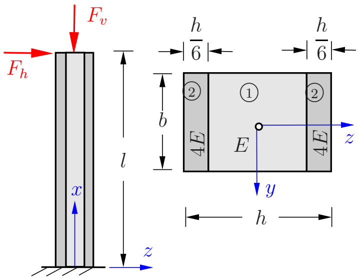
$$\underline{\underline{\sigma_{\max} = -\frac{F}{at}\left(\frac{1}{3} + 4\frac{b}{a}\right)}}\,.$$

to b) Tensile stress occurs first at $z = -a/2$:

$$\sigma(-\frac{a}{2}) = 0 \quad \rightsquigarrow \quad -\frac{F}{3at} + 4\frac{Fb}{ta^2} = 0 \quad \rightsquigarrow \quad \underline{\underline{b = \frac{a}{12}}}\,.$$

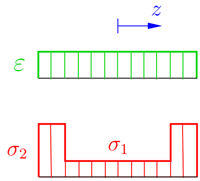
P3.4 **Problem 3.4** A column is clamped at the bottom and is carrying a vertical load F_v at the center of the top cross section and a horizontal load F_h in the middle of edge b. The column is made of 3 layers with different Young's moduli.

Determine the normal stress distribution in the cross section at the clamping.



Solution We consider the different load cases independently.

to a) With the vertical load F_v, we obtain from



equilibrium $\sigma_1 A_1 + \sigma_2 A_2 = -F_v\,,$

Hooke's law $\sigma_i = E_i \varepsilon_i$

and geometry $\varepsilon_1 = \varepsilon_2 = \varepsilon$

the strain

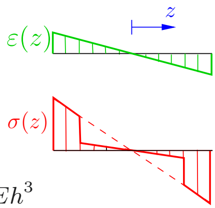
$$E_1\varepsilon_1 A_1 + E_2\varepsilon_2 A_2 = E\varepsilon\frac{2}{3}bh + 4E\varepsilon\frac{1}{3}bh = -F_v \quad \rightsquigarrow \quad \varepsilon = -\frac{F_v}{2Ebh}$$

and the associated stresses

$$\underline{\underline{\sigma_1 = -\frac{F_v}{2bh}}}\,, \qquad \underline{\underline{\sigma_2 = -2\frac{F_v}{bh}}}\,.$$

to b) F_h causes a moment $M_S = -F_h l$ at the support. Then geometry (assume: cross sections remain plane)

$$u = \psi \cdot z \quad \rightsquigarrow \quad \varepsilon = \psi' \cdot z\,,$$



Hooke's law $\sigma(z) = E(z)\varepsilon(z)$

and

$$M = \int \sigma z \mathrm{d}A = 2b\psi'[E_1\int_0^{h/3} z^2\mathrm{d}z + E_2\int_{h/3}^{h/2} z^2\mathrm{d}z]$$

$$= 2b\psi' E[\frac{1}{3}(\frac{h}{3})^3 + \frac{4}{3}((\frac{h}{2})^3 - (\frac{h}{3})^3)] = \frac{7}{27}b\psi' E h^3$$

lead to (using $M = M_S$)

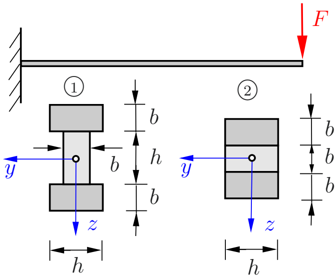
$$\psi' = -\frac{27}{7}\frac{F_h l}{Ebh^3}\,.$$

Finally, the stresses follow as

$$\sigma_1 = E_1\psi' z = E\frac{27}{7}\frac{M}{Ebh^3}z \quad \rightsquigarrow \quad \underline{\underline{\sigma_1(\frac{h}{3}) = -\frac{9F_h l}{7bh^2}}}\,,$$

$$\sigma_2 = E_2\psi' z = 4E\frac{27}{7}\frac{M}{Ebh^3}z \quad \rightsquigarrow \quad \underline{\underline{\sigma_2(\frac{h}{2}) = -\frac{54F_h l}{7bh^2}}}\,.$$

Problem 3.5 A wooden cantilever can be assembled from 3 beams (dimensions of the cross section $b = a$ and $h = 2a$) in different ways. **P3.5**



What is the maximal force F for the two variants ① and ② , if the maximal allowed shear stress in the bonding layer is given by τ_{allow}?

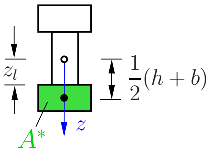
Solution With $V = F$ the shear stress in the bonding layer becomes in general $(z = z_l)$

$$\tau(z_l) = \frac{F\,S(z_l)}{I\;b(z_l)}\,.$$

This yields with $\tau(z_l) = \tau_{\text{allow}}$ the maximal load $F_{\max}$

$$F_{\max} = \frac{\tau_{\text{allow}} I\;b(z_l)}{S(z_l)}\,.$$

For variant ① we obtain

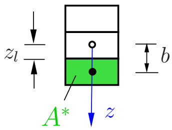


$$I = \frac{bh^3}{12} + 2\Big[\frac{hb^3}{12} + \Big(\frac{h}{2}+\frac{b}{2}\Big)^2 bh\Big] = 10\,a^4\,,$$
$$b(z_l) = b = a\,,$$
$$S(z_l) = \int_{A^*} z\,\mathrm{d}A = \frac{1}{2}(h+b)bh = 3\,a^3\,.$$

which leads to the force

$$\underline{\underline{F_{1\max}}} = \tau_{\text{allow}}\frac{10a^4\cdot a}{3a^3} = \underline{\underline{\frac{10}{3}\,\tau_{\text{allow}}\;a^2}}\,.$$

Analogously we obtain for variant ②



$$I = \frac{h(3b)^3}{12} = \frac{9}{2}a^4\,,\qquad b(z_l) = h = 2a\,,$$
$$S(z_l) = \int_{A^*} z\,\mathrm{d}A = b\cdot bh = 2a^3$$

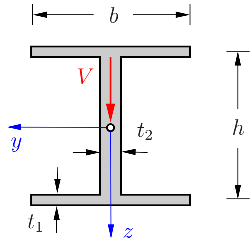
and the force

$$\underline{\underline{F_{2\max}}} = \tau_{\text{allow}}\frac{9a^4\cdot 2a}{2\cdot 2a^3} = \underline{\underline{\frac{9}{2}\,\tau_{\text{allow}}\;a^2}}\,.$$

Note: The shear stresses in the cross section at $z = z_l$ and in the corresponding perpendicular bonding interface are equal (*associated shear stresses!*).

P3.6

Problem 3.6 Determine the shear stress due to an applied shear resultant force V in the depicted thin- walled **I**-profile.



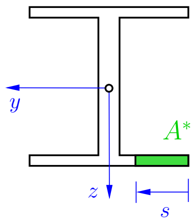
Solution The shear stresses are computed from

$$\tau = \frac{V\ S(s)}{I\ t(s)}$$

Thus we need to determine the second moment of area I with regard to the y-axis. With $t_1 \ll b$ and $t_2 \ll h$ we obtain

$$I = I_1 + I_2 = 2\,t_1 b\left(\frac{h}{2}\right)^2 + t_2\frac{h^3}{12}$$

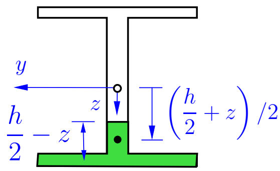
$$= \frac{h^2}{12}(t_2 h + 6t_1 b) = \frac{h^2}{12}(A_1 + 6A_2)\,.$$



The static moment of sub-area A^* for a position s in the lower sub-area is given by

$$S(s) = \frac{h}{2}\,t_1 s$$

and for a position z in the second sub-area it follows



$$S(z) = 2\left(\frac{h}{2}\,t_1\,\frac{b}{2}\right) + \frac{\frac{h}{2}+z}{2}\left(\frac{h}{2}-z\right)t_2$$

$$= A_1\frac{h}{2} + \frac{t_2}{8}(h^2 - 4z^2)\,.$$

These relations yield the shear stress in the upper sub-area

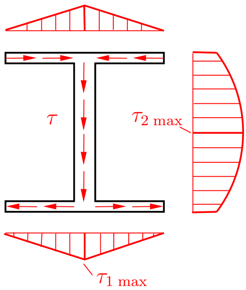
$$\tau_1(s) = \frac{V\,\frac{h}{2}\,t_1 s}{\frac{h^2}{12}(A_2+6A_1)t_1} = \frac{V}{A_2}\,\frac{\frac{A_2}{A_1}}{1+\frac{A_2}{6A_1}}\,\frac{s}{h}$$

and in the second sub-area

$$\tau_2(z) = \frac{V\left[A_1\frac{h}{2}+\frac{t_2}{8}(h^2-4z^2)\right]}{\frac{h^2}{12}(A_2+6A_1)\;t_2} = \frac{V}{A_2}\,\frac{1+\frac{A_2}{4A_1}\left[1-\left(\frac{2z}{h}\right)^2\right]}{1+\frac{A_2}{6A_1}}\,.$$

The maximum shear stress occurs at the center of the profile,

$$\tau_{2\,\max} = \tau_2(z=0) = \frac{V}{A_2}\,\frac{1+\frac{A_2}{4A_1}}{1+\frac{A_2}{6A_1}}\,,$$



it depends on the area ratio A_2/A_1. The maximum shear stress in the first sub-area is given by

$$\tau_{1\,\max} = \tau_G(s=b/2) = \frac{V}{A_2}\,\frac{\frac{A_2}{A_1}}{1+\frac{A_2}{6A_1}}\,\frac{b}{2h}\,.$$

For example $A_1 = A_2$ and $b = h$ yields $\tau_{2\,\max} = \dfrac{15}{14}\dfrac{V}{A_2}$ at the center and $\tau_{1\,\max} = \dfrac{6}{14}\dfrac{V}{A_2}$. For this situation the smallest value in the vertical sub-area

$$\tau_{2\,\min} = \tau_2(z=h/2) = \frac{V}{A_2}\,\frac{1}{1+\frac{A_2}{6A_1}} = \frac{12}{14}\,\frac{V}{A_2}\,,$$

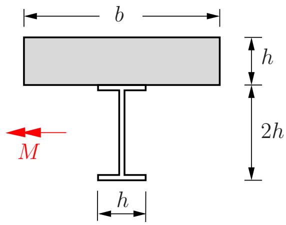
is only 20% smaller than $\tau_{2\,\max}$. As a rough estimate we can use the average shear stress $\tau_{\text{ave}} = V/A_2$ in the central sub-area.

P3.7

Problem 3.7 A composite beam consists of an upper concrete slab and a steel I beam. The structure is loaded by a bending moment M.

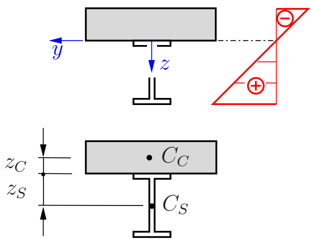
a) Determine the width b of the concrete slab, such that compressive stresses occur only in the concrete part, while the tension is present in the steel part.

b) For this case compute the stresses in the extreme fibres of the two materials.



Given : $M = 1000$ kNm
$E_C = 3.5 \cdot 10^4$ N/mm^2
$E_S = 2.1 \cdot 10^5$ N/mm^2
$h = 40$ cm
$A_S = h^2 / 6$
$I_S = h^4 / 18$

Solution to a) For the case that compression occurs only in the concrete and tension only in the steel sub-area the strain in the bonding layer has do be zero (=neutral fibre). With the chosen coordinate system we have



$$\varepsilon = az\,,$$

where a is not yet determined. The stresses in steel and concrete are

$$\sigma_S = E_S\,\varepsilon = a\,E_S\,z\,, \qquad \sigma_C = E_C\,\varepsilon = a\,E_C\,z\,.$$

As the beam is loaded only by a bending moment, the normal force N has to vanish:

$$N = \int_{A_S} \sigma_S\,\mathrm{d}A + \int_{A_C} \sigma_C\,\mathrm{d}A = 0 \quad \rightsquigarrow \quad E_S \int_{A_S} z\,\mathrm{d}A + E_C \int_{A_C} z\,\mathrm{d}A = 0\,.$$

With

$$\int_{A_S} z\,\mathrm{d}A = z_S A_S = h\,\frac{h^2}{6} = \frac{h^3}{6}\,, \qquad \int_{A_C} z\,\mathrm{d}A = z_C A_C = -\,\frac{h}{2}hb = -\,\frac{h^2 b}{2}$$

and $E_S/E_C = 6$ the required width b is obtained:

$$6\,\frac{h^3}{6} - \frac{h^2 b}{2} = 0 \qquad \rightsquigarrow \qquad \underline{\underline{b = 2h = 80\,\mathrm{cm}}}\,.$$

to b) The unknown factor a follows from the prescribed bending moment.

From the definitions

$$M = \int\limits_{A_S} z\,\sigma_S\,\mathrm{d}A + \int\limits_{A_C} z\,\sigma_C\,\mathrm{d}A = a\,E_S \int\limits_{A_S} z^2 \mathrm{d}A + a\,E_C \int\limits_{A_C} z^2 \mathrm{d}A\,.$$

and the evaluation of the integrals

$$\int\limits_{A_S} z^2 \mathrm{d}A = I_S + h^2 A_S = \frac{h^4}{18} + \frac{h^4}{6} = \frac{2}{9}\,h^4$$

$$\int\limits_{A_C} z^2 \mathrm{d}A = \frac{bh^3}{3} = \frac{2}{3}\,h^4$$

it follows

$$M = \frac{a h^4 E_C}{9}\left[2\,\frac{E_S}{E_C} + 6\right] = 2 a h^4 E_C \qquad \rightsquigarrow \qquad a = \frac{M}{2h^4 E_C}\,.$$

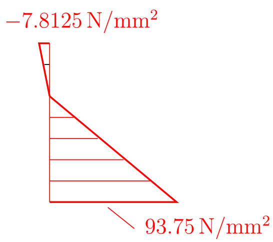
With this result the stresses in the steel and concrete are

$$\sigma_S = \frac{E_S M}{2 E_C h^4}\,z = 3\,\frac{M}{h^4}\,z\,, \qquad\qquad \sigma_C = \frac{M}{2h^4}\,z\,.$$

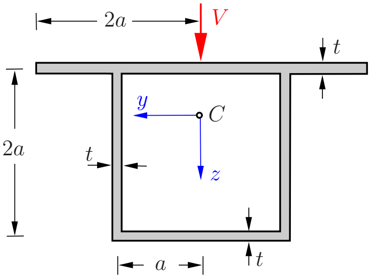
For the top extreme fibre in concrete ($z^t = -h$) and the bottom extrem fibre in steel ($z^b = 2h$) we obtain

$$\underline{\underline{\sigma_C^t}} = -\,\frac{M}{2h^3} = \underline{\underline{-7.8125\,\mathrm{N/mm^2}}}\,,$$

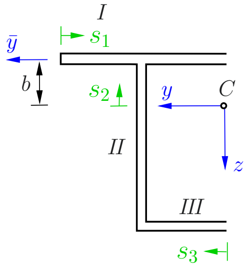
$$\underline{\underline{\sigma_S^b}} = 6\,\frac{M}{h^3} = \underline{\underline{93.75\,\mathrm{N/mm^2}}}\,,$$



P3.8 **Problem 3.8** Determine the shear stresses due to a shear force V for the depicted thin-walled beam cross section ($t \ll a$).



Solution At first we compute the cross section area, the location of the centroid and the second moment of area:

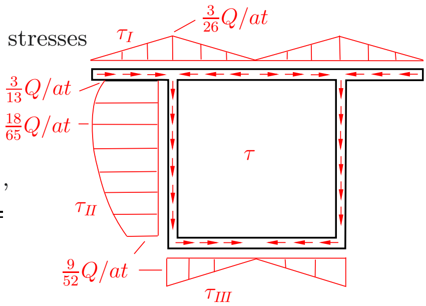


$$A = 4at + 2 \cdot 2at + 2at = 10\,at\,,$$
$$bA = 2a \cdot 2at + 2a \cdot 2at \quad \rightsquigarrow \quad b = \frac{4}{5}\,a\,,$$
$$I_{\bar{y}} = (2a)^2 2at + 2\,\frac{t(2a)^3}{3} = \frac{40}{3}ta^3\,,$$
$$I = I_y = I_{\bar{y}} - b^2 A = \frac{104}{15}ta^3\,.$$

Due to symmetry of the cross section the shear stress is symmetric to the z-axis. Thus only half of the cross section has to be considered. With the coordinantes s_1 to s_3 we obtain for the static moments in the sub-areas *I* to *III*

$$S_I = b\,s_1 t = \frac{4}{5}\,at\,s_1\,,$$
$$S_{II} = b\,2at + \left(s_2 + \frac{b - s_2}{2}\right)(b - s_2)\,t = \frac{48}{25}a^2 t - \frac{1}{2}t\,s_2^2\,,$$
$$S_{III} = (2a - b)t\,s_3 = \frac{6}{5}\,at\,s_3\,.$$

These relations result in the shear stresses

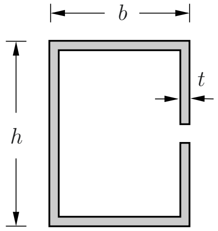


$$\underline{\underline{\tau_I}} = \frac{Q\,S_I}{I\,t} = \underline{\underline{\frac{3}{26}\,\frac{Q}{at}\,\frac{s_1}{a}}}\,,$$

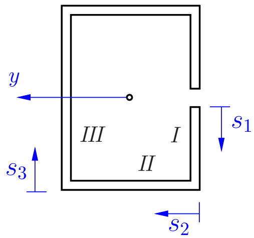
$$\underline{\underline{\tau_{II}}} = \frac{Q\,S_{II}}{I\,t} = \underline{\underline{\frac{Q}{at}\left(\frac{18}{65} - \frac{15}{208}\,\frac{s_2^2}{a^2}\right)}},$$

$$\underline{\underline{\tau_{III}}} = \frac{Q\,S_{III}}{I\,t} = \underline{\underline{\frac{9}{52}\,\frac{Q}{at}\,\frac{s_3}{a}}}\,.$$

Problem 3.9 Locate the shear center for the depicted thin-walled ($t \ll b, h$) box profile with a slit. P3.9



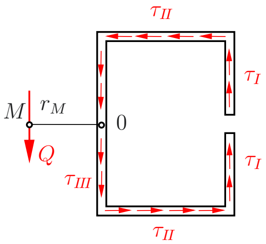
Solution We start by computing the static moments with respect to the y-axis of the three sub-areas:



$$S_I = t\frac{s_1^2}{2}\,, \quad S_{II} = t\frac{h^2}{8} + \frac{h}{2}ts_2\,,$$
$$S_{III} = t\frac{h^2}{8} + \frac{h}{2}bt + s_3 t\left(\frac{h}{2} - \frac{s_3}{2}\right)\,.$$

Thus the shear stresses become

$$\tau_I = \frac{Q}{I}\frac{s_1^2}{2}\,,$$
$$\tau_{II} = \frac{Q}{I}\left(\frac{h^2}{8} + \frac{h}{2}s_2\right)\,,$$
$$\tau_{III} = \frac{Q}{I}\left(\frac{h^2}{8} + \frac{h}{2}b + \frac{s_3}{2}(h - s_3)\right)\,.$$



The equivalency of moments with respect to 0 provides

$$Q\, r_M = 2\int_0^{h/2} \tau_I bt\, \mathrm{d}s_1 + 2\int_0^b \tau_{II}\frac{h}{2}t\, \mathrm{d}s_2 = \frac{Qt}{I}\left(b\frac{h^3}{24} + \frac{1}{8}bh^3 + \frac{1}{4}h^2 b^2\right)$$
$$= \frac{Qtbh^2}{I}\left(\frac{1}{6}h + \frac{1}{4}b\right)\,.$$

With the second moment of area for the thin-walled profile

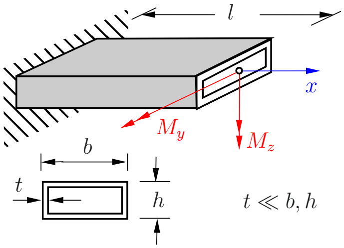
$$I = 2\left[\frac{th^3}{12} + bt\left(\frac{h}{2}\right)^2\right] = th^2\left(\frac{h}{6} + \frac{b}{2}\right)$$

we obtain the distance r_M of the shear center M to the reference point 0

$$\underline{\underline{r_M}} = \frac{tbh^2}{th^2}\,\frac{\frac{1}{6}h + \frac{1}{4}b}{\frac{1}{6}h + \frac{1}{2}b} = \underline{\underline{b\,\frac{2h + 3b}{2h + 6b}}}\,.$$

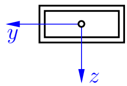
P3.10 **Problem 3.10** The cantilever with thin-walled box cross section is loaded by two bending moments $M_y = Fl$ and $M_z = 2Fl$.

Determine the distribution of the normal stresses in the cross section for $b = 2h$.



Solution Because of symmetry y and z are principal axes. The stress distribution is computed from

$$\sigma = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y \, .$$



With

$$I_y = 2 \cdot \frac{th^3}{12} + 2 \cdot \left(\frac{h}{2}\right)^2 tb = \frac{1}{6} th^2 (h + 3b) \, ,$$

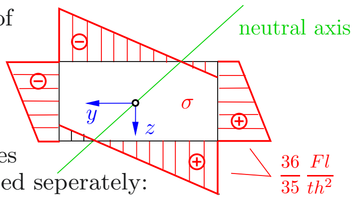
$$I_z = 2 \cdot \frac{tb^3}{12} + 2 \cdot \left(\frac{b}{2}\right)^2 ht = \frac{1}{6} tb^2 (b + 3h)$$

and the given bending moments we find

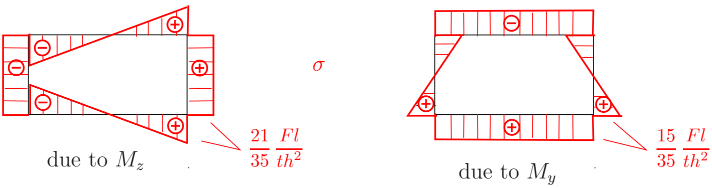
$$\underline{\underline{\sigma}} = \frac{Fl}{\frac{1}{6} th^2 \cdot 7h} z - \frac{2Fl}{\frac{1}{6} t \, 4h^2 \cdot 5h} y = \underline{\underline{\frac{6Fl}{th^3} \left(\frac{z}{7} - \frac{y}{10}\right)}} \, .$$

The equation of the neutral axis (line of zero stress) is computed from $\sigma = 0$

$$z = \frac{7}{10} \, y \, .$$

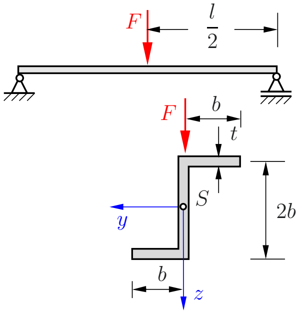


To clarify the representation the stresses due to the two loading cases are depicted seperately:



Problem 3.11 A beam, simply supported at both ends, with a thin-walled profile ($t \ll b$) is loaded by a force F in the middle. **P3.11**

Determine the stress distribution under the load as well as the location and value of the maximum stress.



Solution For the unsymmetrical profile the principal axes are not known. We have to use the equations for biaxial bending. Thus we obtain for the stresses with $M_z = 0$

$$\sigma = \frac{M_y}{\Delta}(I_z z + I_{yz} y)\,.$$

The moment due to the load is given by

$$M_y = M_{\max} = \frac{Fl}{4}\,.$$

Together with the geometric quantities of the cross section

$$I_y = \frac{t(2b)^3}{12} + 2\cdot b^2(bt) = \frac{8}{3}\,tb^3\,, \quad I_z = 2\Big[\frac{tb^3}{12} + \Big(\frac{b}{2}\Big)^2 bt\Big] = \frac{2}{3}\,t\,b^3\,,$$

$$I_{yz} = -2\cdot b\cdot\frac{b}{2}\cdot bt = -tb^3\,,$$

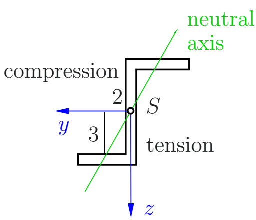
$$\Delta = I_y I_z - I_{yz}^2 = \frac{16}{9}t^2b^6 - t^2b^6 = \frac{7}{9}\,t^2b^6$$

we obtain the stress

$$\underline{\underline{\sigma}} = \frac{Fl}{4\cdot\frac{7}{9}t^2b^6}\left(\frac{2}{3}t\,b^3 z - t\,b^3 y\right) = \underline{\underline{\frac{3}{28}\,\frac{Fl}{t\,b^3}\,(2z-3y)}}\,.$$

The neutral axis follows from the condition

$$\sigma = 0 \quad\leadsto\quad z = \frac{3}{2}\,y\,.$$

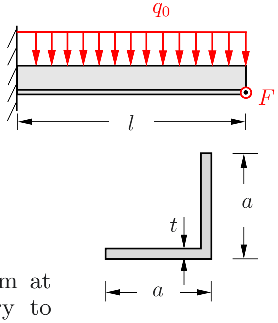


The maximal stresses occur at points with the largest distance to the neutral axis ($y = 0$, $z = \pm b$):

$$\underline{\underline{\sigma_{\max} = \pm\frac{3}{14}\,\frac{Fl}{t\,b^2}}}\,.$$

P3.12

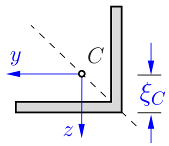
Problem 3.12 A cantilever beam with thin-walled profile ($t \ll a$) is subjected to a constant line load q_0 and a concentrated force F.



Determine the distribution of the normal stress in the cross section at the support.

Given: $F = 2q_0 l$.

Solution We place a y, z-coordinate system at the not yet known centroid. By symmetry to the 45°-axis the distance ξ_C to both sub-areas is identical. As the static moment vanishes with regard to the symmetry axis, we have



$$\xi_C\, at = \left(\frac{a}{2} - \xi_C\right) a\, t \quad \rightsquigarrow \quad \xi_C = \frac{a}{4}\,.$$

With regard to the symmetry axis we find

$$I_y = I_z = \frac{ta^3}{12} + \left(\frac{a}{4}\right)^2 a\, t + \left(\frac{a}{4}\right)^2 a\, t = \frac{5}{24} ta^3\,,$$

$$I_{yz} = -\frac{a}{4}\,\frac{a}{4} a\, t - \left(-\frac{a}{4}\right)\left(-\frac{a}{4}\right) a\, t = -\frac{1}{8} ta^3\,.$$

This yields

$$\Delta = I_y I_z - I_{yz}^2 = \left(\frac{5}{24}\right)^2 t^2 a^6 - \frac{1}{64} t^2 a^6 = \frac{1}{36} t^2 a^6\,.$$

The internal moments at the support are given by

$$M_y = -\frac{q_0 l^2}{2} \qquad \text{and} \qquad M_z = Fl = +2q_0 l^2\,.$$

Finally we obtain for the stress

$$\underline{\underline{\sigma}} = \frac{1}{\Delta}\left\{\left[M_y I_z - M_z I_{yz}\right] z - \left[M_z I_y - M_y I_{yz}\right] y\right\}$$

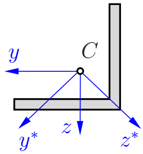
$$= \frac{36}{t^2 a^6}\left\{\left[-\frac{q_0 l^2}{2}\,\frac{5}{24} ta^3 - 2q_0 l^2\left(-\frac{ta^3}{8}\right)\right] z - \left[2q_0 l^2 \frac{5}{24} ta^3 + \frac{q_0 l^2}{2}\left(-\frac{ta^3}{8}\right)\right] y\right\}$$

$$= \underline{\underline{\frac{3}{4}\,\frac{q_0 l^2}{ta^3}\,(7z - 17y)}}\,.$$

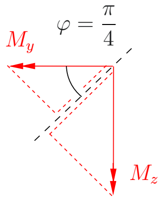
Alternatively we can describe the stress distribution with respect to the principal axes y^*, z^*, which we know from symmetry considerations. The principal values of the second moments of area follow with $I_y = I_z$ and $\varphi = 45°$

$$I_y^* = \frac{I_y + I_z}{2} + I_{yz} = \frac{5}{24}ta^3 - \frac{1}{8}ta^3 = \frac{1}{12}ta^3 \,,$$

$$I_z^* = \frac{I_y + I_z}{2} - I_{yz} = \frac{5}{24}ta^3 + \frac{1}{8}ta^3 = \frac{1}{3}ta^3 \,.$$



Decomposition of the loading in the principal directions yields



$$M_y^* = -\frac{q_0 l^2}{2}\cos\varphi + Fl\sin\varphi$$

$$= q_0 l^2 \left(2 - \frac{1}{2}\right)\frac{1}{2}\sqrt{2}\,,$$

$$M_z^* = \frac{q_0 l^2}{2}\sin\varphi + Fl\cos\varphi$$

$$= q_0 l^2 \left(\frac{1}{2} + 2\right)\frac{1}{2}\sqrt{2}\,,$$

which leads to the stresses in the principal directions

$$\sigma = \frac{M_y^*}{I_y^*}z^* - \frac{M_z^*}{I_z^*}y^* = \frac{3\sqrt{2}}{4}\frac{q_0 l^2}{ta^3}(12z^* - 5y^*)\,.$$

To check the result we transform with

$$z^* = -y\sin\varphi + z\cos\varphi = (z - y)\frac{1}{2}\sqrt{2}\,,$$

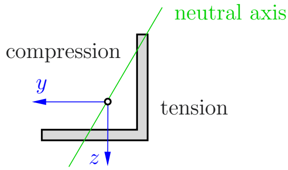
$$y^* = y\cos\varphi + z\sin\varphi = (z + y)\frac{1}{2}\sqrt{2}$$

back and find by re-substitution

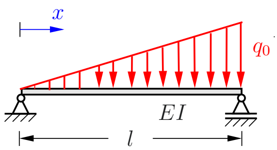
$$\sigma = \frac{3}{4}\,\frac{q_0 l^2}{ta^3}[12(z - y) - 5(z + y)] = \frac{3}{4}\,\frac{q_0 l^2}{ta^3}(7z - 17y)\,.$$

The neutral axis satisfies the equation

$$z = \frac{17}{7}y\,.$$



P3.13 **Problem 3.13** The beam is simply supported at both ends. Determine
a) location and value of maximal moment,
b) location and value of maximal deflection,
c) the slope of the deflection curve at both supports.



Solution Bending moment and deflection curve can be computed independently, because the beam is statically determinate.

to a) The given loading provides

$$q = q_0 \frac{x}{l}$$

by twice integration

$$V = -q_0 \frac{x^2}{2l} + C_1 ,$$
$$M = -q_0 \frac{x^3}{6l} + C_1 x + C_2 .$$

With the *static* boundary conditions

$$M(0) = 0 \quad \rightsquigarrow \quad C_2 = 0 , \qquad M(l) = 0 \quad \rightsquigarrow \quad C_1 = \frac{q_0 l}{6}$$

we obtain

$$V = \frac{q_0 l}{6}\Big[1 - 3\big(\frac{x}{l}\big)^2\Big] , \qquad M = \frac{q_0 l^2}{6}\Big[\frac{x}{l} - \big(\frac{x}{l}\big)^3\Big] .$$

Location and value of the maximal moment are determined by the condition $M' = 0$:

$$M' = V = 0 \quad \rightsquigarrow \quad 1 - 3\big(\frac{x^*}{l}\big)^2 = 0 \quad \rightsquigarrow \quad \underline{\underline{x^* = \frac{1}{3}\sqrt{3}\, l = 0,577\, l}} ,$$

$$\underline{\underline{M_{\max}}} = M(x^*) = \frac{1}{18}\sqrt{3}\, q_0 l^2 \big(1 - \frac{1}{3}\big) = \underline{\underline{\frac{1}{27}\sqrt{3}\, q_0 l^2}} .$$

to b) With the known function of the moment

$$M = \frac{q_0 l^2}{6}\left[\frac{x}{l} - \left(\frac{x}{l}\right)^3\right]$$

we derive from $EI\, w'' = -M$ by twice integration

$$EI\, w' = -\frac{q_0 l^2}{6}\left(\frac{x^2}{2l} - \frac{1}{4}\frac{x^4}{l^3}\right) + C_3 ,$$
$$EI\, w = -\frac{q_0 l^2}{6}\left(\frac{x^3}{6l} - \frac{1}{20}\frac{x^5}{l^3}\right) + C_3 x + C_4 .$$

The new integration constants are determined from the *geometric* boundary conditions

$$w(0) = 0 \quad \leadsto \quad C_4 = 0\,,$$

$$w(l) = 0 \quad \leadsto \quad C_3 = \frac{q_0 l^3}{6}\left(\frac{1}{6} - \frac{1}{20}\right) = \frac{7}{360} q_0 l^3\,.$$

Finally we obtain (cf. table on page 62, load case no. 3)

$$EI\,w = \frac{q_0 l^4}{360}\left[7\,\frac{x}{l} - 10\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^5\right].$$

The maximal deflection is computed by using the condition $w' = 0$:

$$EI\,w' = 0 \quad \leadsto \quad 7 - 30\left(\frac{x^{**}}{l}\right)^2 + 15\left(\frac{x^{**}}{l}\right)^4 = 0$$

$$\leadsto \quad \left(\frac{x^{**}}{l}\right)^4 - 2\left(\frac{x^{**}}{l}\right)^2 + \frac{7}{15} = 0\,,$$

$$\leadsto \quad \underline{\underline{x^{**}}} = \underline{\underline{\sqrt{1 \overset{(+)}{-} \sqrt{\frac{8}{15}}}\,l}} = 0,519\,l\,.$$

(The (+)-sign provides an x-value outside of the range of validity.) Thus we have

$$\underline{\underline{w_{\max}}} = w(x^{**}) = \frac{q_0 l^4}{360 EI}\sqrt{1 - \sqrt{\frac{8}{15}}}\left[7 - 10\left(1 - \sqrt{\frac{8}{15}}\right) + 3\left(1 - \sqrt{\frac{8}{15}}\right)^2\right]$$

$$= \underline{\underline{0,0065\,\frac{q_0 l^4}{EI}}}\,.$$

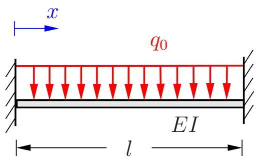
to c) The slope of the deflection curve follows as

$$\underline{\underline{w'(0)}} = \frac{C_3}{EI} = \underline{\underline{\frac{7}{360}\,\frac{q_0 l^3}{EI}}}\,,$$

$$\underline{\underline{w'(l)}} = -\frac{q_0 l^2}{6EI}\left(\frac{l}{2} - \frac{l}{4}\right) + \frac{7}{360}\,\frac{q_0 l^3}{EI} = \underline{\underline{-\frac{8}{360}\,\frac{q_0 l^3}{EI}}}\,.$$

Note: Maximal moment and maximal deflection occur at different locations: $x^* \neq x^{**}$.

P3.14 **Problem 3.14** Determine the function of the bending moment for the depicted beam.



Solution The beam is statically *indeterminate*. Thus the function of the moment needs to be computed with help of the deflection curve. From the differential equation we derive by integration

$$EI\,w^{IV} = q = q_0\,,$$

$$-EI\,w''' = Q = -q_0 x + C_1\,,$$

$$-EI\,w'' = M = -q_0\frac{x^2}{2} + C_1 x + C_2\,,$$

$$EI\,w' = q_0\frac{x^3}{6} - C_1\frac{x^2}{2} - C_2 x + C_3\,,$$

$$EI\,w = q_0\frac{x^4}{24} - C_1\frac{x^3}{6} - C_2\frac{x^2}{2} + C_3 x + C_4\,.$$

The 4 integration constants follow from the 4 geometric boundary conditions:

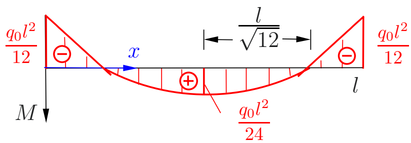
$$w'(0) = 0 \;\rightsquigarrow\; C_3 = 0\,,$$

$$w(0) = 0 \;\rightsquigarrow\; C_4 = 0\,,$$

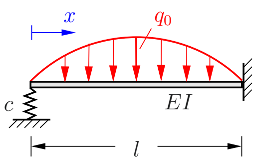
$$\left.\begin{aligned} w'(l) = 0 \;&\rightsquigarrow\; \frac{q_0 l^3}{6} - C_1\frac{l^2}{2} - C_2 l = 0 \\ w(l) = 0 \;&\rightsquigarrow\; \frac{q_0 l^4}{24} - C_1\frac{l^3}{6} - C_2\frac{l^2}{2} = 0 \end{aligned}\right\} \rightsquigarrow \quad \begin{aligned} C_1 &= \frac{q_0 l}{2} \\ C_2 &= -\frac{q_0 l^2}{12}\,. \end{aligned}$$

This yields

$$M = -\frac{q_0 l^2}{12}\left[1 - 6\,\frac{x}{l} + 6\left(\frac{x}{l}\right)^2\right].$$



Problem 3.15 Determine the deflection of the depicted beam. The left end of the beam is elastically supported by a spring, the right end is clamped, and the load has the shape of a quadratic parabola. **P3.15**



Solution We start by computing the quadratic equation for the line load. From the general equation $q = A + Bx + Cx^2$ and

$$\left.\begin{aligned} q(0) &= 0 \quad \rightsquigarrow \quad A = 0\,, \\ q(l) &= 0 \quad \rightsquigarrow \quad Bl + Cl^2 = 0\,, \\ q(\tfrac{l}{2}) &= q_0 \quad \rightsquigarrow \quad B\frac{l}{2} + C\frac{l^2}{4} = q_0\,, \end{aligned}\right\} \rightsquigarrow \; C = -\frac{B}{l}\,, \quad B = 4\frac{q_0}{l}$$

it follows $\quad q(x) = 4q_0\left[\frac{x}{l} - (\frac{x}{l})^2\right].$

Four times integration of $EI\,w^{IV} = q$ yields

$$\begin{aligned} -EI\,w''' &= V = -4q_0\left(\frac{x^2}{2l} - \frac{x^3}{3l^2}\right) + C_1\,, \\ -EI\,w'' &= M = -4q_0\left(\frac{x^3}{6l} - \frac{x^4}{12l^2}\right) + C_1 x + C_2\,, \\ EI\,w' &= 4q_0\left(\frac{x^4}{24l} - \frac{x^5}{60l^2}\right) - C_1\frac{x^2}{2} - C_2 x + C_3\,, \\ EI\,w &= 4q_0\left(\frac{x^5}{120l} - \frac{x^6}{360l^2}\right) - C_1\frac{x^3}{6} - C_2\frac{x^2}{2} + C_3 x + C_4\,. \end{aligned}$$

The boundary conditions provide

$$\begin{aligned} M(0) &= 0 && \rightsquigarrow C_2 = 0\,, \\ V(0) &= c\cdot w(0) && \rightsquigarrow C_1 = c\,\frac{C_4}{EI}\,, \\ w'(l) &= 0 && \rightsquigarrow \frac{q_0 l^3}{10} - C_1\frac{l^2}{2} + C_3 = 0\,, \\ w(l) &= 0 && \rightsquigarrow \frac{q_0 l^4}{45} - C_1\frac{l^3}{6} + C_3 l + C_4 = 0\,. \end{aligned}$$

The 3 equations for C_1, C_3, and C_4 yield with the abbreviation $\Delta = 1 + cl^3/3EI$

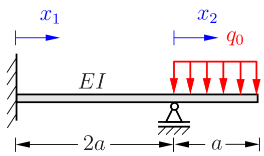
$$C_1 = \frac{7}{90}\frac{c}{\Delta}\frac{q_0 l^4}{EI}\,, \quad C_3 = -\frac{q_0 l^3}{10\Delta}\left(1 - \frac{1}{18}\frac{cl^3}{EI}\right), \quad C_4 = \frac{7}{90}\frac{q_0 l^4}{\Delta}$$

which leads to the final result

$$\underline{\underline{w = \frac{q_0 l^4}{10EI}\left[\frac{1}{3}(\frac{x}{l})^5 - \frac{1}{9}(\frac{x}{l})^6 - \frac{7}{54}\frac{cl^3}{\Delta EI}(\frac{x}{l})^3 - \left(1 - \frac{1}{18}\frac{cl^3}{EI}\right)\frac{1}{\Delta}(\frac{x}{l}) + \frac{7}{9\Delta}\right]}}.$$

P3.16 **Problem 3.16** A cantilever beam is subjected to a constant distributed load q_0.

Determine the deflection at the free end.



Solution We solve the problem in two different ways.
1st solution: Due to the discontinuity of $q(x)$ we have to consider two domains:

$$
\begin{aligned}
0 \le x_1 < 2a \qquad q_1 &= 0\,,\\
V_1 &= C_1\,,\\
M_1 &= C_1 x_1 + C_2\,,\\
EI\,w_1' &= -C_1\frac{x_1^2}{2} - C_2 x_1 + C_3\,,\\
EI\,w_1 &= -C_1\frac{x_1^3}{6} - C_2\frac{x_1^2}{2} + C_3 x_1 + C_4\,,\\
0 < x_2 \le a \qquad q_2 &= q_0\,,\\
V_2 &= -q_0 x_2 + C_5\,,\\
M_2 &= -q_0\frac{x_2^2}{2} + C_5 x_2 + C_6\,,\\
EI\,w_2' &= q_0\frac{x_2^3}{6} - C_5\frac{x_2^2}{2} - C_6 x_2 + C_7\,,\\
EI\,w_2 &= q_0\frac{x_2^4}{24} - C_5\frac{x_2^3}{6} - C_6\frac{x_2^2}{2} + C_7 x_2 + C_8\,.
\end{aligned}
$$

The 8 integration constants C_i follow from:

4 boundary conditons
$$
\begin{cases}
w_1'(0) = 0 \rightsquigarrow \quad C_3 = 0\,, \quad w_1(0) = 0 \rightsquigarrow \quad C_4 = 0\,,\\
Q_2(a) = 0 \rightsquigarrow \quad C_5 = q_0 a\,,\ M_2(a) = 0 \rightsquigarrow \quad C_6 = -\dfrac{q_0 a^2}{2}
\end{cases}
$$

and 4 continuity conditions
$$
\begin{cases}
M_1(2a) = M_2(0) \quad \rightsquigarrow C_1 2a + C_2 = C_6\,,\\
w_1'(2a) = w_2'(0) \quad \rightsquigarrow -C_1\dfrac{(2a)^2}{2} - C_2 2a + C_3 = C_7\,,\\
w_1(2a) = w_2(0) = 0 \rightsquigarrow -C_1\dfrac{(2a)^3}{6} - C_2\dfrac{(2a)^2}{2}\\
\qquad\qquad + C_3 2a + C_4 = C_8 = 0
\end{cases}
$$

$$
\rightsquigarrow \quad C_1 = -\frac{3}{8}q_0 a\,, \quad C_2 = \frac{1}{4}q_0 a^2\,, \quad C_7 = \frac{1}{4}q_0 a^3\,, \quad C_8 = 0\,.
$$

(For the shear force no continuity condition is available because it experiences a jump related to the unknown reaction force B). The deflection at the free end yields

$$\underline{\underline{w_2(a)}} = \frac{q_0}{EI}\left\{\frac{a^4}{24} - \frac{a^4}{6} + \frac{a^4}{4} + \frac{a^4}{4}\right\} = \underline{\underline{\frac{3}{8}\frac{q_0 a^4}{EI}}}\,.$$

2nd solution: Using the Macauley bracket we can describe both domains by a *single* equation. We introduce x from the left end and have to consider the jump in the shear resultant at B (assumed to be positive in upward direction):

$$\begin{aligned}
q &= q_0 < x - 2a >^0 ,\\
V &= -q_0 < x - 2a >^1 + B < x - 2a >^0 + C_1 ,\\
M &= -\frac{1}{2} q_0 < x - 2a >^2 + B < x - 2a >^1 + C_1 x + C_2 ,\\
EI\,w' &= \frac{1}{6} q_0 < x - 2a >^3 - \frac{1}{2} B < x - 2a >^2 - \frac{1}{2} C_1 x^2 - C_2 x + C_3 ,\\
EI\,w &= \frac{1}{24} q_0 < x - 2a >^4 - \frac{1}{6} B < x - 2a >^3 - \frac{1}{6} C_1 x^3 - \frac{1}{2} C_2 x^2 + C_3 x + C_4 .
\end{aligned}$$

The 5 unknowns C_i and B follow from

$$\text{4 boundary conditions and}\left\{\begin{aligned}
&w'(0) = 0 \ \rightsquigarrow C_3 = 0 ,\\
&w(0) = 0 \ \rightsquigarrow C_4 = 0 ,\\
&Q(3a) = 0 \ \rightsquigarrow -q_0 a + B + C_1 = 0 ,\\
&M(3a) = 0 \rightsquigarrow -q_0 \frac{a^2}{2} + Ba + C_1 3a + C_2 = 0
\end{aligned}\right.$$

$$\text{1 reaction condition}\ \left\{ w(2a) = 0 \rightsquigarrow -C_1 \frac{(2a)^3}{6} - C_2 \frac{(2a)^2}{2} + C_3 2a + C_4 = 0 .\right.$$

Solving yields:

$$C_1 = -\frac{3}{8} q_0 a , \quad C_2 = \frac{1}{4} q_0 a^2 , \quad C_3 = 0 , \quad C_4 = 0 , \quad B = \frac{11}{8} q_0 a .$$

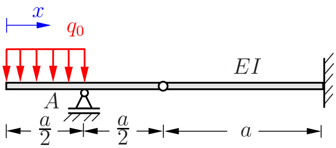
Thus the deflection at the free end is given by

$$\underline{\underline{w(3a)}} = \frac{q_0}{EI}\left[\frac{a^4}{24} - \frac{11}{8} a \frac{a^3}{6} + \frac{3}{8} a \frac{(3a)^3}{6} - \frac{1}{4} a^2 \frac{(3a)^2}{2}\right] = \underline{\underline{\frac{3}{8}\frac{q_0 a^4}{EI}}}\,.$$

Note: The computation of displacements at designated locations is less complex with methods discussed in chapter 5.

P3.17 **Problem 3.17** The depicted beam is loaded on its cantilever part by a constant line load.

Compute the deflection at the hinge and determine the slope difference at the hinge.



Solution With the help of the Macauley bracket the entire domain can be descibed by a *single* equation. During integration the jump in the slope $\Delta\varphi$ at the hinge has to be considered separately.

$$q = q_0 - q_0 < x - \frac{a}{2} >^0 ,$$

$$V = -q_0 x + q_0 < x - \frac{a}{2} >^1 + A < x - \frac{a}{2} >^0 + C_1 ,$$

$$M = -q_0 \frac{x^2}{2} + \frac{q_0}{2} < x - \frac{a}{2} >^2 + A < x - \frac{a}{2} >^1 + C_1 x + C_2 ,$$

$$EI\, w' = q_0 \frac{x^3}{6} - \frac{q_0}{6} < x - \frac{a}{2} >^3 - \frac{A}{2} < x - \frac{a}{2} >^2 - C_1 \frac{x^2}{2} - C_2 x + EI\Delta\varphi < x - a >^0 + C_3 ,$$

$$EI\, w = q_0 \frac{x^4}{24} - \frac{q_0}{24} < x - \frac{a}{2} >^4 - \frac{A}{6} < x - \frac{a}{2} >^3 - C_1 \frac{x^3}{6} - C_2 \frac{x^2}{2} + EI\Delta\varphi < x - a >^1 + C_3 x + C_4 .$$

The 4 integration constants C_i, the unknown reaction force A, and the slope difference $\Delta\varphi$ at the hinge are determined from the following 6 conditions

$$V(0) = 0 \ \rightsquigarrow \ C_1 = 0 , \qquad M(0) = 0 \ \rightsquigarrow \ C_2 = 0 ,$$

$$M(a) = 0 \ \rightsquigarrow \ A = \frac{3}{4} q_0 a , \qquad w(\frac{a}{2}) = 0 \ \rightsquigarrow \ \frac{1}{384} q_0 a^4 + C_3 \frac{a}{2} + C_4 = 0 ,$$

$$w'(2a) = 0 \rightsquigarrow \ \frac{4}{3} q_0 a^3 - \frac{27}{48} q_0 a^3 - \frac{27}{32} q_0 a^3 + EI\Delta\varphi + C_3 = 0 ,$$

$$w(2a) = 0 \ \rightsquigarrow \ \frac{2}{3} q_0 a^4 - \frac{81}{384} q_0 a^4 - \frac{81}{192} q_0 a^4 + EI\Delta\varphi\, a + C_3 2a + C_4 = 0.$$

This yields the solution

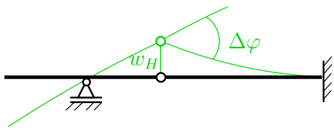
$$C_3 = -\frac{5}{24} q_0 a^3 , \quad C_4 = \frac{39}{384} q_0 a^4 , \quad EI\Delta\varphi = \frac{9}{32} q_0 a^3 .$$

Thus we obtain for the deflection at the hinge

$$\underline{\underline{w_H = w(a) = -\frac{1}{12} \frac{q_0 a^4}{EI}}}$$

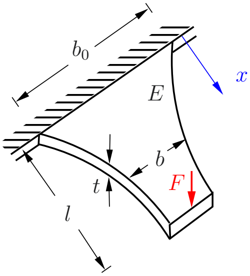
and for the slope difference

$$\underline{\underline{\Delta\varphi = \frac{9}{32} \frac{q_0 a^3}{EI}}} .$$



Problem 3.18 A leaf spring with constant thickness t and variable width $b = b_0 l/(l+x)$ is fixed at one side and loaded at one edge by F. **P3.18**

Determine the deflection at the position of the load.



Solution The system is statically determinate. Hence the function of the moment follows from equilibrium considerations:

$$V = F = \text{const}\,, \qquad M = Fx + C\,.$$

The condition $M(l) = 0$ yields $C = -Fl$ and thus

$$M = -F(l-x)\,.$$

Use of the differential equation $EI\,w'' = -M$ yields with

$$I(x) = b(x)\frac{t^3}{12} = \frac{b_0 t^3}{12}\,\frac{l}{l+x}$$

and the abbreviation $I_0 = b_0 t^3/12$:

$$w'' = \frac{F(l-x)(l+x)}{EI_0 l} = \frac{F}{EI_0 l}(l^2 - x^2)\,.$$

By integration we obtain

$$w' = \frac{F}{EI_0 l}\left(l^2 x - \frac{x^3}{3} + C_1\right),$$
$$w = \frac{F}{EI_0 l}\left(l^2\frac{x^2}{2} - \frac{x^4}{12} + C_1 x + C_2\right).$$

The boundary conditions

$$w'(0) = 0 \quad \rightsquigarrow \quad C_1 = 0\,, \qquad w(0) = 0 \quad \rightsquigarrow \quad C_2 = 0$$

render the solution

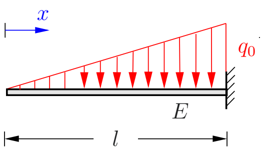
$$\underline{\underline{w(l) = w_{\max} = \frac{5}{12}\,\frac{Fl^3}{EI_0}}}\,.$$

Note: For a beam with *constant* width b_0 the same load results in a smaller deflection

$$w(l) = \frac{Fl^3}{3EI_0} = \frac{4}{12}\,\frac{Fl^3}{EI_0}\,.$$

P3.19 **Problem 3.19** A cantilever beam with rectangular cross section (width b, height $h(x)$) is subjected to a linear varying load such that the extreme fibre experiences a stress σ_0.

Determine the deflection of the left end.



Solution First we have to compute the unknown cross section height. Using

$$\sigma_{\max} = \frac{|M|}{W} = \sigma_0$$

together with

$$M = -\frac{q_0 x^3}{6l}, \quad I = \frac{b\,h^3(x)}{12}, \quad W(x) = \frac{I}{h/2} = \frac{b\,h^2(x)}{6}$$

yields $h(x)$

$$h(x) = \sqrt{\frac{q_0}{\sigma_0 b l}}\, x^{3/2}.$$

This leads to

$$I(x) = \frac{q_0}{12\sigma_0 l}\sqrt{\frac{q_0}{b\sigma_0 l}}\, x^{9/2}.$$

Integration of $EI\,w'' = -M$ provides together with the boundary conditions $w'(l) = w(l) = 0$:

$$w'' = -\frac{M}{EI} = \frac{q_0 x^3 12\sigma_0 l}{6lEq_0}\sqrt{\frac{b\sigma_0 l}{q_0}}\, x^{-9/2} = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\, x^{-3/2},$$

$$w' = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\left(-2x^{-1/2} + 2l^{-1/2}\right),$$

$$w = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\left(-4x^{1/2} + 2l^{-1/2}x + 2l^{1/2}\right).$$

Evaluation at $x = 0$ yields the deflection at the left end

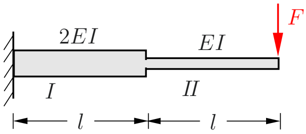
$$\underline{\underline{w(0) = 4\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l^2}{q_0}}}}.$$

As a test we check the physical dimensions ($F\widehat{=}$force, $L\widehat{=}$length):

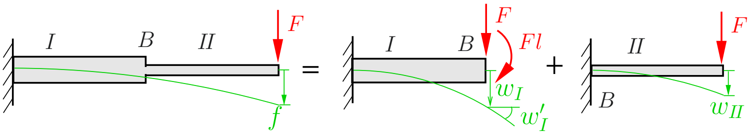
$$[w] = \frac{FL^{-2}}{FL^{-2}}\sqrt{\frac{LFL^{-2}L^2}{FL^{-1}}} = L.$$

Problem 3.20 The depicted beam is assembled from two parts with different bending stiffness. **P3.20**

Determine the deflection at the free end.



Solution We use superposition together with the tabulated results on page 62. First we assume that beam II is fixed at point B and compute the defection w_{II}. To this we have to add the deflection w_I of the left beam I due to F and $M = Fl$. Finally we have to consider the slope w_I', that appears at the left beam. This slop has to be multiplied by the length l and added as an additional deflection at the right end:



$$f = w_{II} + w_I + w_I' l = w_{II} + (w_{I_F} + w_{I_M}) + (w_{I_F}' + w_{I_M}')l\,.$$

According to load case no. 5

$$w_{II} = \frac{Fl^3}{3EI}\,, \qquad w_{I_F} = \frac{Fl^3}{3(2EI)}\,, \qquad w_{I_F}' = \frac{Fl^2}{2(2EI)}$$

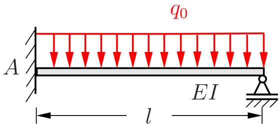
and load case no. 8

$$w_{I_M} = \frac{(Fl)l^2}{2(2EI)}\,, \quad w_{I_M}' = \frac{(Fl)l}{(2EI)}\,.$$

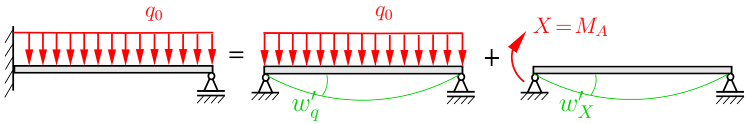
superposition yields the deflection at the end

$$\underline{\underline{f}} = \frac{Fl^3}{3EI}\left\{1 + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{2}\right\} = \underline{\underline{\frac{3}{2}\,\frac{Fl^3}{EI}}}\,.$$

P3.21 **Problem 3.21** Determine the deflection curve for the depicted beam.



Solution The beam is statically indeterminate. We free the support moment at the left end and introduce the unknown moment X:



From the table on page 62 we obtain for the slope:

load case no. 2 $\qquad w'_q = \dfrac{q_0 l^3}{24EI}\,,$

load case no. 4 (with $\beta = 1$) $\qquad w'_X = \dfrac{Xl}{3EI}\,.$

The total slope at the left support has to vanish. Thus compatibility provides

$$w'_q + w'_X = 0 \quad \rightsquigarrow \quad X = M_A = -\frac{1}{8} q_0 l^2\,.$$

Superposition of the deflection curves in table on page 62 yields the deflection curve of the system

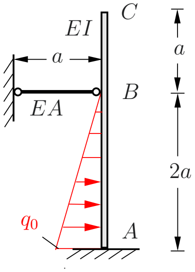
$$\underline{\underline{EI\,w}} = EI(w_q + w_X)$$

$$= \frac{q_0 l^4}{24}(\xi - 2\xi^3 + \xi^4) - \frac{1}{8} q_0 l^2 \frac{l^2}{6}(2\xi + \xi^3 - 3\xi^2)$$

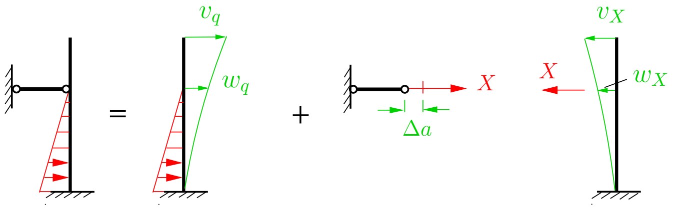
$$\underline{\underline{= \frac{q_0 l^4}{48}(3\xi^2 - 5\xi^3 + 2\xi^4)}}\,.$$

Problem 3.22 A pole is clamped at A and supported at B by an elastic rope. The pole is subjected to a horizontal linearily varying load. **P3.22**

Compute the horizontal displacement v at point C for $\dfrac{EI}{a^2EA} = \dfrac{1}{3}$.



Solution We disconnect rope and pole:



Compatibility at the connection of the rope requires

$$w_q - w_X = \Delta a \,, \qquad \text{where} \qquad \Delta a = \frac{Xa}{EA} \quad \text{(see chapter 2)}\,.$$

With the table on page 62 we obtain:

$$\text{load case no. 7} \qquad w_q = \frac{q_0(2a)^4}{30EI} = \frac{8}{15}\,\frac{q_0a^4}{EI}\,,$$

$$\text{load case no. 5} \qquad w_X = \frac{X(2a)^3}{3EI} = \frac{8}{3}\,\frac{Xa^3}{EI}\,.$$

Using these values in the compatibility condition provides

$$\frac{8}{15}\,\frac{q_0a^4}{EI} - \frac{8}{3}\,\frac{Xa^3}{EI} = \frac{Xa}{EA} \qquad \leadsto \qquad X = \frac{\frac{1}{5}q_0a}{1 + \frac{3}{8}\,\frac{EI}{a^2EA}} = \frac{8}{45}\,q_0a\,.$$

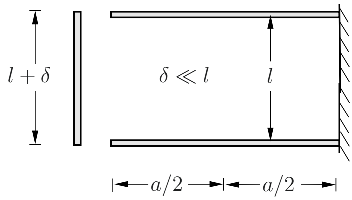
The displacement v results from superposition (for the linear varying load we have to consider the displacement w_q and the slope w_q': $v_q = w_q + w_q'a$):

$$\underline{\underline{EI\,v}} = EI(v_q + v_X) = \frac{q_0(2a)^4}{30} + \frac{q_0(2a)^3}{24}\,a - \underbrace{\frac{X(3a)^3}{6}\left[3\cdot\frac{2}{3} - 1 + \left(\frac{1}{3}\right)^3\right]}_{\text{load case no. 5 with } \alpha = 2/3}$$

$$= \frac{13}{15}q_0a^4 - \frac{14}{3}Xa^3 = \underline{\underline{\frac{q_0a^4}{27}}}\,.$$

P3.23

Problem 3.23 Two parallel beams (bending stiffness EI, length a) have a distance of l and are clamped at the left support. An elastic bar (axial rigidity EA) of length $l+\delta$ is force fitted at $a/2$ between the two beams.



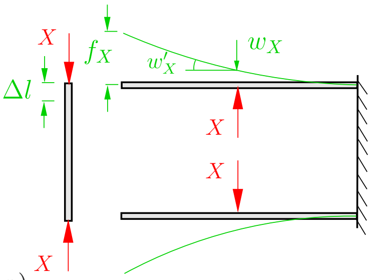
a) Determine the force in the bar?

b) Compute the change e by which the distance l at the beam ends is changed.

Solution **to a)** From geometry (compatibility)

$$l + 2w_X = (l+\delta) - \Delta l$$

$$\rightsquigarrow \quad 2w_X + \Delta l = \delta$$



we obtain (see table on page 62, load case no. 5)

$$w_X = \frac{X\left(\frac{a}{2}\right)^3}{3EI} \quad \text{und} \quad \Delta l = \frac{Xl}{EA}$$

and the force in the bar (compression)

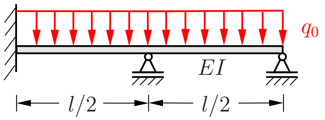
$$\underline{\underline{S}} = X = \frac{\delta}{\frac{l}{EA} + \frac{a^3}{12EI}} = \underline{\underline{\delta\,\frac{EA}{l}\,\frac{1}{1+\frac{a^3EA}{12\,l\,EI}}}}\,.$$

to b) The opening e is computed with help of the table on page 62 from load case no. 5

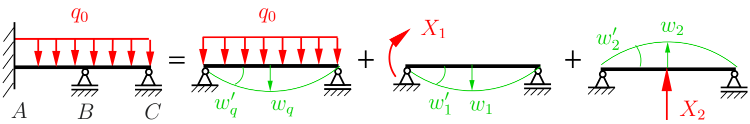
$$\underline{\underline{e}} = 2\,f_X = 2\,\frac{Xa^3}{6\,EI}\left[3\cdot 1\cdot\frac{1}{2} - 1 + \left(\frac{1}{2}\right)^3\right] = \underline{\underline{\frac{5}{24}\,\frac{a^3EA}{l\,EI}\,\frac{\delta}{1+\frac{a^3EA}{12\,l\,EI}}}}\,.$$

Note: In the limit case $EI \to \infty$ one obtains $S = \delta\,\dfrac{EA}{l}$ and $e = 0$.

Problem 3.24 Compute the reaction forces for the depicted beam. **P3.24**



Solution The system is *twice* statically indeterminate. We treat the support moment $M_A = X_1$ and the reaction force $B = X_2$ as static redundant quantities and use superposition:



Considering the (arbitrary chosen) directions yields for the compatibility

$$w'_q + w'_1 - w'_2 = 0\,,$$

$$w_q + w_1 - w_2 = 0\,.$$

From the table on page 62 (no. 2, 4 and 1) we obtain

$$\frac{q_0 l^3}{24} + \frac{X_1 l}{3} - \frac{X_2 l^2}{16} = 0\,,$$

$$\frac{5}{384} q_0 l^4 + \frac{1}{16} X_1 l^2 - \frac{X_2 l^3}{48} = 0\,,$$

which yields

$$X_1 = -\frac{1}{56}\, q_0 l^2\,, \qquad X_2 = \frac{4}{7}\, q_0 l\,.$$

The support reactions are determined by superposition of the 3 load cases

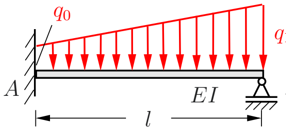
$$\underline{\underline{A}} = \frac{q_0}{2} - \frac{X_1}{l} - \frac{X_2}{2} = \underline{\underline{\frac{13}{56}\, q_0 l}}\,,$$

$$\underline{\underline{B}} = X_2 = \underline{\underline{\frac{4}{7}\, q_0 l}}\,,$$

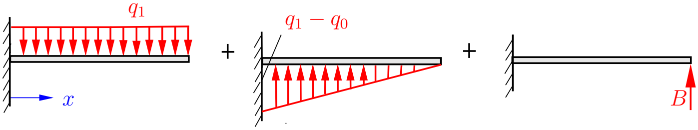
$$\underline{\underline{C}} = \frac{q_0 l}{2} + \frac{X_1}{l} - \frac{X_2}{2} = \underline{\underline{\frac{11}{56}\, q_0 l}}\,,$$

$$\underline{\underline{M_A}} = X_1 = \underline{\underline{-\frac{1}{56}\, q_0 l^2}}\,.$$

P3.25 **Problem 3.25** Determine the deflection curve for the depicted beam subjected to a trapezoidal load.



Solution The beam is statically indeterminate. We choose B as the static redundant quantity and use superposition of 3 load cases (the trapezoidal load is replaced by an equivalent constant and linearly varying load)



The table on page 62 (load case no. 6, 7 and 5) provides

$$EI\,w(x) = \frac{q_1 l^4}{24}(6\xi^2 - 4\xi^3 + \xi^4) - \frac{(q_1 - q_0)l^4}{120}(10\xi^2 - 10\xi^3 + 5\xi^4 - \xi^5) - \frac{Bl^3}{6}(3\xi^2 - \xi^3)\,.$$

The support condition at B yields the reaction force B

$$w(l) = 0 \quad \rightsquigarrow \quad B = \frac{3}{8}q_1 l - \frac{(q_1 - q_0)l}{10}\,.$$

By recasting the above equations

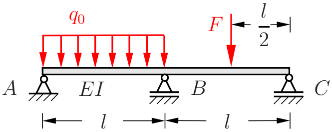
$$\frac{q_1 l^4}{24} = \frac{(q_1 - q_0)l^4}{24} + \frac{q_0 l^4}{24}$$

we determine the deflection curve

$$\underline{\underline{EI\;w(x) = \frac{q_0 l^4}{24}\left\{\xi^4 - \frac{5}{2}\xi^3 + \frac{3}{2}\xi^2\right\} + \frac{(q_1 - q_0)l^4}{120}\left\{\xi^5 - \frac{9}{2}\xi^3 + \frac{7}{2}\xi^2\right\}}}\,.$$

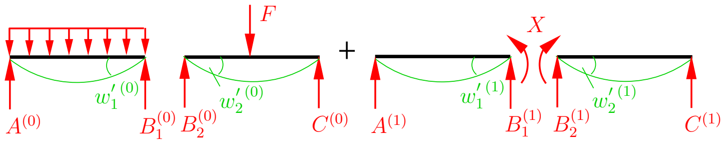
Problem 3.26 For the beam with two domains determine the support reactions and the deflection at the center of each domain.

Given: $F = 2q_0 l$.



P3.26

Solution We divide the beam into 2 separate (hinged at both ends) beams and introduce the moment at the central support as statically redundant quantity:



Equilibrium yields

$$A^{(0)} = B_1^{(0)} = \frac{1}{2} q_0 l\,, \qquad B_2^{(0)} = C^{(0)} = \frac{F}{2}\,,$$
$$A^{(1)} = C^{(1)} = -B_1^{(1)} = -B_2^{(1)} = \frac{X}{l}\,.$$

The table on page 62 provides

$$w_1'^{(0)} = -\frac{q_0 l^3}{24EI}\,, \quad w_2'^{(0)} = \frac{Fl^2}{16EI}\,, \quad w_1'^{(1)} = -w_2'^{(1)} = -\frac{Xl}{3EI}\,.$$

Compatibility can be formulated as

$$w_1'^{(0)} + w_1'^{(1)} = w_2'^{(0)} + w_2'^{(1)}$$

which yields together with the tabulated results

$$X = -\frac{1}{16} q_0 l^2 - \frac{3}{32} Fl = -\frac{1}{4} q_0 l^2 = M_B\,.$$

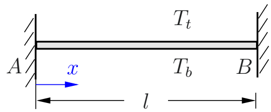
The support reactions are computed by superposition

$$\underline{\underline{A}} = A^{(0)} + A^{(1)} = \frac{1}{2} q_0 l - \frac{1}{4} q_0 l = \underline{\underline{\frac{1}{4} q_0 l}}\,,$$
$$\underline{\underline{B}} = B_1^{(0)} + B_1^{(1)} + B_2^{(0)} + B_2^{(1)} = \underline{\underline{2 q_0 l}}\,,$$
$$\underline{\underline{C}} = C^{(0)} + C^{(1)} = \frac{F}{2} - \frac{1}{4} q_0 l = \underline{\underline{\frac{3}{4} q_0 l}}\,.$$

For the deflections at the center of the domains we compute

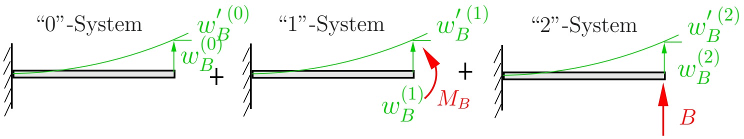
$$\underline{\underline{f_1}} = f_1^{(0)} + f_1^{(1)} = \frac{5}{384}\frac{q_0 l^4}{EI} + \frac{Xl^2}{6EI}\left(\frac{1}{2} - \frac{1}{8}\right) = \underline{\underline{-\frac{q_0 l^4}{384\,EI}}}\,,$$
$$\underline{\underline{f_2}} = f_2^{(0)} + f_2^{(1)} = \frac{Fl^3}{48\,EI} + \frac{Xl^2}{6\,EI}\left(\frac{1}{2} - \frac{1}{8}\right) = \underline{\underline{\frac{5\,q_0 l^4}{192\,EI}}}\,.$$

P3.27 **Problem 3.27** A beam (rectangular cross section, width b, height h) that is clamped at both ends is subjected along its length l to a constant temperature difference $T_t - T_b$.



Determine the defection of the beam and the maximum stresses.

Solution The beam is twice statically indeterimante. We choose as statically redundant quantities the reaction moment $X_1 = M_B$ and the reaction force $X_2 = B$. We use superpostion of the three (statically determinate) systems:



The deflection in the "0"-System is computed by the temperature moment

$$M_{\Delta T} = EI\alpha_T(T_b - T_t)/h$$

using the differential equation $w''^{(0)} = -M_{\Delta T}/EI$ and considering the boundary conditions $w^{(0)}(0) = 0$, $w'^{(0)}(0) = 0$:

$$w'^{(0)}(x) = -\frac{M_{\Delta T}}{EI}\,x\,, \qquad w^{(0)}(x) = -\frac{M_{\Delta T}}{EI}\,\frac{x^2}{2}\,.$$

Due to the clamping at B compatibility requires

$$w_B = w_B^{(0)} + w_B^{(1)} + w_B^{(2)} = 0\,, \qquad w_B' = w_B'^{(0)} + w_B'^{(1)} + w_B'^{(2)} = 0\,.$$

From the table on page 62 we obtain

$$-\frac{M_{\Delta T}}{EI}\,l - \frac{M_B l}{EI} - \frac{Bl^2}{2EI} = 0\,, \qquad -\frac{M_{\Delta T}}{EI}\,\frac{l^2}{2} - \frac{M_B l^2}{2EI} - \frac{Bl^3}{3EI} = 0\,,$$

with the solution

$$B = 0\,, \qquad M_B = -M_{\Delta T}\,.$$

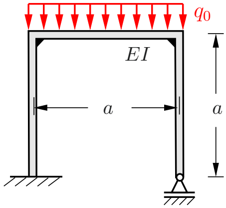
As $M_B = M$ is constant along the entire length of the beam the deflection becomes

$$w'' = -\frac{M + M_{\Delta T}}{EI} = 0 \qquad \text{i. e.} \qquad \underline{\underline{w \equiv 0}}\,.$$

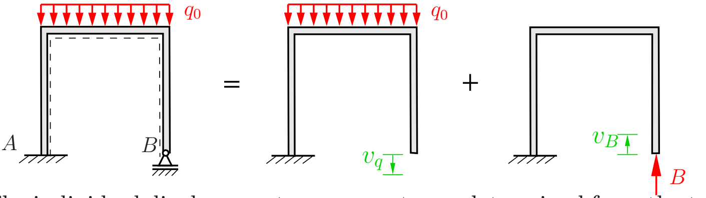
The maximum stress is computed with the section modulus $W = bh^2/6$

$$\underline{\underline{|\sigma_{\max}|}} = \frac{|M|}{W} = \underline{\underline{6\,\frac{M_{\Delta T}}{bh^2}}}\,.$$

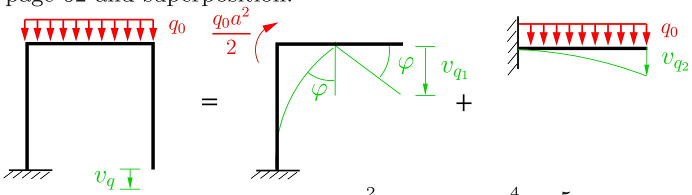
Problem 3.28 Determine the support reactions for the depicted frame. **P3.28**



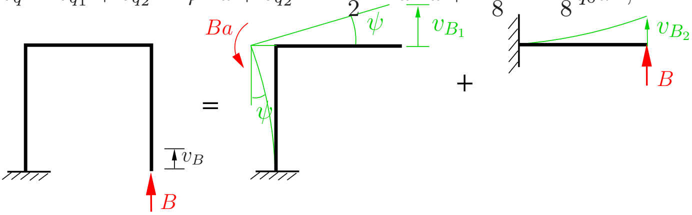
Solution We free the right support and use B as static redundant quantity



The individual displacement components are determined from the table on page 62 and superposition:



$$v_q = v_{q_1} + v_{q_2} = \varphi \cdot a + v_{q_2} = \frac{q_0 a^2}{2} \cdot a \cdot a + \frac{q_0 a^4}{8} = \frac{5}{8} q_0 a^4 \,,$$



$$v_B = v_{B_1} + v_{B_2} = \psi \cdot a + v_{B_2} = Ba \cdot a \cdot a + B\frac{a^3}{3} = \frac{4}{3} B a^3 \,.$$

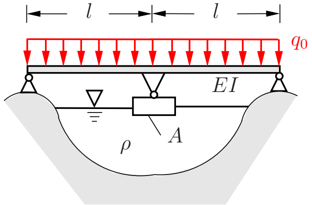
The compatibility at B provides the reaction force B:

$$v_q = v_B \quad \rightsquigarrow \quad \underline{\underline{B = \frac{15}{32} q_0 a}} \,.$$

The other support reactions follow from equilibrium

$$\underline{\underline{A = \frac{17}{32} q_0 a}} \quad \text{and} \quad \underline{\underline{M_A = -\frac{1}{32} q_0 a^2}} \,.$$

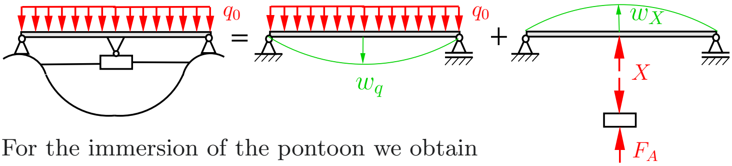
P3.29 **Problem 3.29** An auxiliary bridge, that is resting on the river banks, is supported in the middle by an additional pontoon (block with cross section A at the water line). The bridge is subjected to a constant load q_0.



Given: water density ρ, $EI/Al^3\rho g = 1/24$.

Determine the immersion depth f of the pontoon due to q_0.

Solution The system is statically *indeterminately* supported. We use the pontoon force as statically redundant force and apply superposition:



For the immersion of the pontoon we obtain

$$f = w_q - w_X \,.$$

Archimedes' principle yields the buoyant force F_A that is equal to the weight of displaced fluid (see also chapter 7), i. e. we have

$$X = F_A = \rho g f A \quad \rightsquigarrow \quad f = \frac{X}{\rho g A} \,.$$

The table on page 62 provides

$$\text{no. 2}: \quad w_q = \frac{5}{384}\frac{q_0(2l)^4}{EI} \,, \qquad \text{no. 1}: \quad w_X = \frac{X(2l)^3}{48EI} \,.$$

Using the above results

$$\frac{X}{\rho g A} = \frac{5}{384}\frac{q_0 16 l^4}{EI} - \frac{X 8 l^3}{48EI} \quad \rightsquigarrow \quad X = \frac{\dfrac{5}{24}\dfrac{q_0 l^4}{EI}}{\dfrac{1}{6}\dfrac{l^3}{EI} + \dfrac{1}{\rho g A}} = q_0 l \,.$$

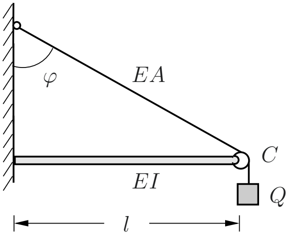
the immersion depth is given by

$$\underline{\underline{f}} = \frac{X}{\rho g A} = \frac{q_0 l}{\rho g A}\frac{EI}{EI}\frac{l^3}{l^3} = \underline{\underline{\frac{1}{24}\frac{q_0 l^4}{EI}}} \,.$$

Problem 3.30 An elastic rope (length s) is fixed to the wall and in C frictionless redirected by a pulley. The pulley is attached to a beam (axial rigidity $\to\infty$),

Determine the displacement of the load Q.

P3.30



Solution The displacement of Q is computed by the length change

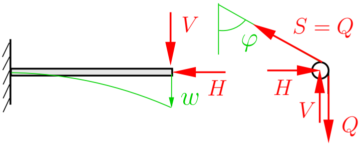
$$\Delta s = \frac{Q\,s}{EA}$$

of the rope and a contributions δ of the deflection of the pulley. The deflection is calculated by the vertical load on the beam

$$V = Q - S\cos\varphi = Q(1-\cos\varphi)$$

to be

$$w = \frac{Vl^3}{3EI} = \frac{Q(1-\cos\varphi)l^3}{3EI}\,.$$

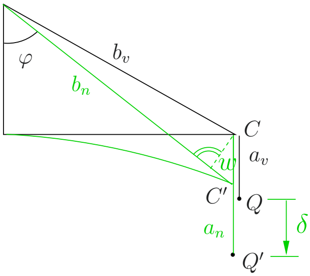


The deflection δ of the load Q follows from

$$\begin{aligned}\delta &= w + a_n - a_v\\ &= w + (s-b_n) - (s-b_v)\\ &= w + b_v - b_n\end{aligned}$$

with

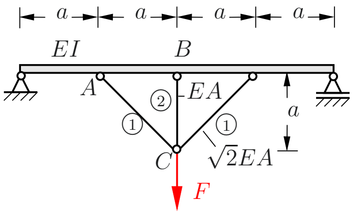
$$b_n - b_v = w\cos\varphi \qquad (\text{for } w \ll b_v)\,.$$



This leads to the deflection of Q

$$\underline{\underline{v_Q}} = \delta + \Delta s = w(1-\cos\varphi) + \frac{Qs}{EA} = \underline{\underline{Q\left[\frac{s}{EA} + \frac{l^3(1-\cos\varphi)^2}{3EI}\right]}}\,.$$

P3.31 **Problem 3.31** The depicted structure consists of a beam and bars with stiffness ratio $\alpha = EI/a^2EA$. The structure is loaded by the force F.

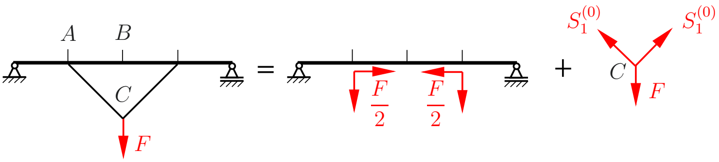


a) Determine the forces in the bars for $\alpha = 1/8$

b) For which value of α vanishes the force S_2?

c) For which α follows $M_B = 0$?

Solution The system is statically indeterminate in the interior. We free the middle bar (basic system):



Equilibrium in C yields $S_1^{(0)} = \sqrt{2}F/2$. The beam is loaded by the components $F/2$. With the table on page 62 (load case no. 1) the displacement at A is given by

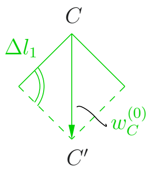
$$EI\, w_A^{(0)} = \frac{F}{2}\frac{(4a)^3}{6}\left[\frac{3}{4}\cdot\frac{1}{4}\left(1-\frac{9}{16}-\frac{1}{16}\right)+\frac{1}{4}\cdot\frac{1}{4}\left(1-\frac{1}{16}-\frac{1}{16}\right)\right] = \frac{2}{3}Fa^3\ ,$$

and at location B

$$EI\, w_B^{(0)} = 2\cdot\frac{F}{2}\frac{(4a)^3}{6}\frac{1}{4}\cdot\frac{1}{2}\left(1-\frac{1}{16}-\frac{1}{4}\right) = \frac{11}{12}Fa^3\,.$$

Due to the truss elongation Δl_1 point C experiences the displacement

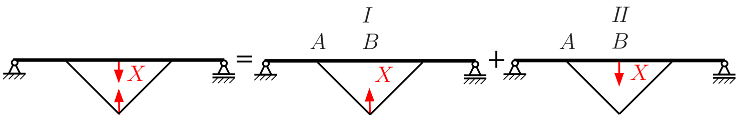
$$w_C^{(0)} = \Delta l_1\sqrt{2} = \frac{S_1 l_1}{\sqrt{2}EA}\sqrt{2} = \frac{\frac{1}{2}\sqrt{2}\,Fa\sqrt{2}}{\sqrt{2}\,EA}\sqrt{2} = \frac{Fa}{EA}\,.$$



Hence the total displacement of C is given by

$$v_C^{(0)} = w_B^{(0)} + w_C^{(0)} = \frac{2}{3}\frac{Fa^3}{EI} + \frac{Fa}{EA}\,.$$

Now we load the system by the unknown normal force $S_2 = X$ and consider the two load cases independently:



In sub-system I the deformation is analogous to the basic system, if F is replaced by $-X$, i. e.

$$v_C^{(I)} = -\frac{2}{3}\frac{Xa^3}{EI} - \frac{Xa}{EA}\,, \qquad w_B^{(I)} = -\frac{11}{12}\frac{Xa^3}{EI}\,.$$

The displacement in sub-system II is again determined from the table on page 62

$$w_B^{(II)} = \frac{X(4a)^3}{48EI} = \frac{4}{3}\frac{Xa^3}{EI}\,,$$

$$v_C^{(II)} = w_A^{(II)} = \frac{X(4a)^3}{6EI}\left\{\frac{1}{2}\frac{1}{4}\left(1-\frac{1}{4}-\frac{1}{16}\right)\right\} = \frac{11}{12}\frac{Xa^3}{EI}\,.$$

Compatibility requires that the difference in the total displacement at points C und B are equal to the elongation of bar 2:

$$v_C^{(0)} + v_C^{(I)} + v_C^{(II)} - \left[w_B^{(0)} + w_B^{(I)} + w_B^{(II)}\right] = \frac{Xa}{EA}$$

or

$$\frac{2Fa^3}{3EI} + \frac{Fa}{EA} - \frac{2Xa^3}{3EI} - \frac{Xa}{EA} + \frac{11Xa^3}{12EI} - \left(\frac{11Fa^3}{12EI} - \frac{11Xa^3}{12EI} + \frac{4Xa^3}{3EI}\right) = \frac{Xa}{EA}$$

$$\rightsquigarrow \qquad \underline{\underline{X = \frac{\alpha - \frac{1}{4}}{2\alpha + \frac{1}{6}}F}}\,.$$

With this result the answers to the questions are:

to a) $X = \underline{\underline{S_2}} = \dfrac{\frac{1}{8}-\frac{1}{4}}{\frac{1}{4}+\frac{1}{6}}F = \underline{\underline{-\dfrac{3}{10}F}}, \quad \underline{\underline{S_1}} = \dfrac{1}{2}\sqrt{2}\,(F - X) = \underline{\underline{\dfrac{13}{20}\sqrt{2}\,F}},$

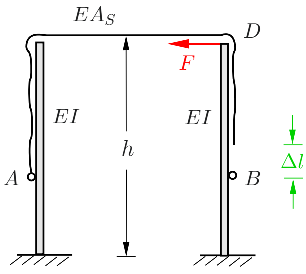
to b) $S_2 = X = 0 \quad \rightsquigarrow \quad \underline{\underline{\alpha = \dfrac{1}{4}}},$

to c) $M_B = \dfrac{F}{2}2a - \left(\dfrac{F}{2} - \dfrac{X}{2}\right)a = 0 \quad \rightsquigarrow \quad X = -F\,,$

$$\rightsquigarrow \quad \frac{\alpha - \frac{1}{4}}{2\alpha + \frac{1}{6}}F = -F \quad \rightsquigarrow \quad \underline{\underline{\alpha = \frac{1}{36}}}\,.$$

P3.32

Problem 3.32 The two depicted posts have to be connected by a rope. The rope has to be fixed at points A and B. The rope is too short by Δl.



a) Determine the horizontal force F at the right post that is required to fix the rope stress-free.

b) The force F is removed after assembly. Determine the force in the rope and the moments at both supports.

Solution **to a)** The force F has to bend the post by Δl to the left. From the table on page 62 (load case no. 5) we obtain

$$\Delta l = \frac{Fh^3}{3EI} \quad \rightsquigarrow \quad \underline{\underline{F = \frac{3EI}{h^3}\Delta l}}\,.$$

to b) The length Δl follows from the extension Δl_S of the rope due to a yet unknown force S in the rope and the deflection f_S of both posts due to the same unknown force S. Compatibility states

$$\Delta l = \Delta l_S + f_S + f_S$$

which yields

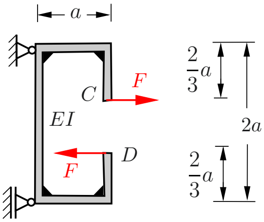
$$\Delta l = \frac{Sl}{EA_S} + \frac{Sh^3}{3EI} + \frac{Sh^3}{3EI} \qquad \rightsquigarrow \qquad \underline{\underline{S = \frac{\Delta l}{l} EA_S \frac{1}{1 + \frac{2}{3}\frac{h^3 EA_S}{lEI}}}}\,.$$

Finally the moments at the support follow from equilibrium

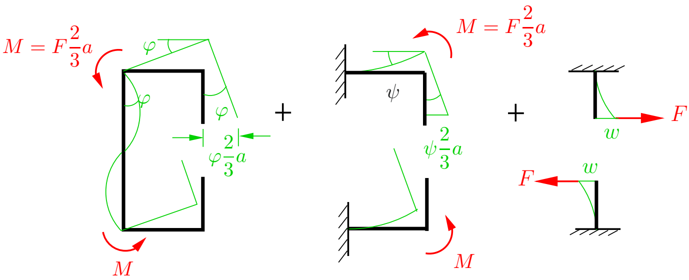
$$\underline{\underline{M}} = hS = \underline{\underline{\frac{\Delta l}{l} EA_S h \frac{1}{1 + \frac{2}{3}\frac{h^3 EA_S}{lEI}}}}\,.$$

Problem 3.33 A plane frame is loaded in C and D by two forces. **P3.33**

Determine the reciprocative horizontal displacement Δu of C und D.



Solution To apply the table on page 62 we have to separate the deformation of the individual beams and use superposition.



$$C \text{ is moved by} \quad \varphi \cdot \frac{2}{3}a + \psi \cdot \frac{2}{3}a + w \quad \text{to the right,}$$

$$D \text{ is moved by} \quad \varphi \cdot \frac{2}{3}a + \psi \cdot \frac{2}{3}a + w \quad \text{to the left.}$$

Thus, the reciprocative displacement follows

$$\Delta u = 2\left[\varphi \cdot \frac{2}{3}a + \psi \cdot \frac{2}{3}a + w\right].$$

With the table on page 62 it follows:

$$\text{load case no. 2} \quad EI\,\varphi = \left(\frac{2}{3}Fa\right)\frac{2a}{3} - \left(\frac{2}{3}Fa\right)\frac{2a}{6} = \frac{2}{9}Fa^2,$$

$$\text{load case no. 8} \quad EI\,\psi = \left(\frac{2}{3}Fa\right)a = \frac{2}{3}Fa^2,$$

$$\text{load case no. 5} \quad EI\,w = \frac{F\left(\frac{2}{3}a\right)^3}{3} = \frac{8}{81}Fa^3,$$

which yields

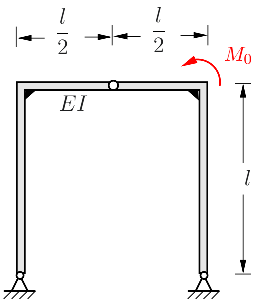
$$\underline{\underline{\Delta u}} = 2\left(\frac{4}{27} + \frac{4}{9} + \frac{8}{81}\right)\frac{Fa^3}{EI} = \underline{\underline{\frac{112}{81}\frac{Fa^3}{EI}}}.$$

Note: Due to the antisymmetry of the system the vertical displacements of C and D are the same.

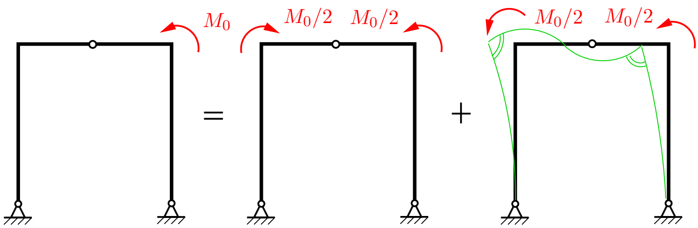
P3.34

Problem 3.34 The depicted frame is loaded by a moment M_0.

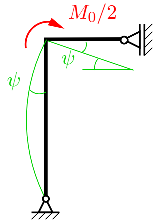
Determine the reciprocative rotation $\Delta\varphi_H$ at the hinge.



Solution It is reasonable to split the loading into a symmetric and antisymmetric contribution:



The *antisymmetric* loading causes *no* reciprocative rotation at the hinge. For the *symmetric* loadign it suffices to consider half of the frame structure. The rotation ψ results solely from the bending of the vertical post (only a normal force occurs in the horizontal beam). Thus from the table on page 62 (load case no. 4 with $\beta = 1$ and $\alpha = 0$) we obtain

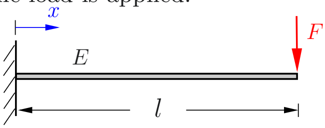


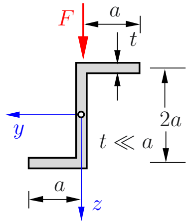
$$\psi = \frac{\frac{M_0}{2} l}{3EI} = \frac{M_0 l}{6EI} .$$

Hence the reciprocative rotation follows

$$\underline{\underline{\Delta\varphi_H}} = 2\psi = \underline{\underline{\frac{M_0 l}{3EI}}} .$$

Problem 3.35 Determine for the depicted beam with a thin-walled profile the displacement at the point where the load is applied. P3.35





Solution Due to the unsymmetrical profile oblique bending occurs. The displacements are computed using the two related differential equations. The bending moments are given by

$$M_y = -F(l-x)\,, \qquad M_z = 0\,,$$

and the second moments of area for the thin-walled profile follow from

$$I_y = \frac{t(2a)^3}{12} + 2(at)a^2 = \frac{8}{3}ta^3, \qquad I_z = \frac{2}{3}ta^3,$$
$$I_{yz} = -2(ta)a\frac{a}{2} = -ta^3, \qquad \Delta = I_y I_z - I_{yz}^2 = \frac{7}{9}t^2a^6.$$

Thus the two differential equations can be integrated for the z-direction

$$Ew'' = -\frac{M_y I_z}{\Delta} = \frac{6}{7}\frac{F}{ta^3}(l-x)\,,$$
$$Ew' = -\frac{3}{7}\frac{F}{ta^3}(l-x)^2 + C_1\,,$$
$$Ew = \frac{1}{7}\frac{F}{ta^3}(l-x)^3 + C_1 x + C_2$$

and the y-direction

$$Ev'' = -\frac{M_y I_{yz}}{\Delta} = -\frac{9}{7}\frac{F}{ta^3}(l-x)\,,$$
$$Ev' = \frac{9}{14}\frac{F}{ta^3}(l-x)^2 + C_3\,,$$
$$Ev = -\frac{3}{14}\frac{F}{ta^3}(l-x)^3 + C_3 x + C_4\,.$$

The boundary conditions at the support yield

$$v'(0) = 0 \;\rightsquigarrow\; C_3 = -\frac{9}{14}\frac{Fl^2}{ta^3}\,, \quad w'(0) = 0 \;\rightsquigarrow\; C_1 = \frac{3}{7}\frac{Fl^2}{ta^3}\,,$$
$$v(0) = 0 \;\rightsquigarrow\; C_4 = \frac{3}{14}\frac{Fl^3}{ta^3}\,, \quad w(0) = 0 \;\rightsquigarrow\; C_2 = -\frac{1}{7}\frac{Fl^3}{ta^3}\,.$$

Thus the displacements at the point, where the load is applied $x = l$, are

$$\underline{\underline{w(l) = \frac{2}{7}\frac{Fl^3}{Eta^3}}}\,, \qquad \underline{\underline{v(l) = -\frac{3}{7}\frac{Fl^3}{Eta^3}}}\,.$$

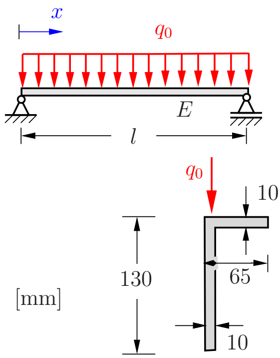
Note: Although the load is acting in *vertical* direction a displacement in *horizontal* direction occurs. The profile preferably deforms in the direction which is related to the smaller second moment of area.

P3.36 **Problem 3.36** The simply supported beam is loaded by a constant distributed load.

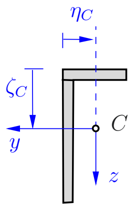
Determine the displacement of the centroid of the cross section in the middle of the beam (only deformation due to bending).

Given: $l = 2$ m ,

$$E = 2.1 \cdot 10^5 \text{ MPa} ,$$
$$q_0 = 10^4 \text{ N/m} .$$



Solution We compute the geometric quantities of the cross section:



$$A = 65 \cdot 10 + 120 \cdot 10 = 1850 \text{ mm} ,$$

$$\zeta_C = \frac{(65 \cdot 10) \cdot 5 + (120 \cdot 10) \cdot 70}{1850} = 47.16 \text{ mm} ,$$

$$\eta_C = \frac{(65 \cdot 10) \cdot 32.5 + (120 \cdot 10) \cdot 5}{1850} = 14.66 \text{ mm} ,$$

$$I_y = \frac{65 \cdot 10^3}{12} + (42.16)^2 (65 \cdot 10) + \frac{10 \cdot 120^3}{12} + (22.84)^2 (10 \cdot 120)$$
$$= 322.7 \text{ cm}^4 ,$$

$$I_z = \frac{10 \cdot 65^3}{12} + (17.84)^2 (65 \cdot 10) + \frac{120 \cdot 10^3}{12} + (9.66)^2 (10 \cdot 120)$$
$$= 55.8 \text{ cm}^4 ,$$

$$I_{yz} = -(-17.84)(-42.16)(65 \cdot 10) - (22.84)(9.66)(10 \cdot 120)$$
$$= -75.4 \text{ cm}^4 ,$$

$$\Delta = I_y I_z - I_{yz}^2 = 12321.5 \text{ cm}^8 .$$

The loading causes only a moment along the y-axis:

$$M_y(x) = \frac{q_0 l}{2} x - q_0 \frac{x^2}{2} .$$

The basic equations simplify to

$$Ew'' = -\frac{M_y I_z}{\Delta}\,, \qquad Ev'' = -\frac{M_y I_{yz}}{\Delta}\,.$$

Integrating twice yields

$$Ew' = -\frac{I_z}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^2}{2} - \frac{x^3}{3} + C_1\right)\,,$$

$$Ew = -\frac{I_z}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^3}{6} - \frac{x^4}{12} + C_1 x + C_2\right)\,,$$

$$Ev' = -\frac{I_{yz}}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^2}{2} - \frac{x^3}{3} + C_3\right)\,,$$

$$Ev = -\frac{I_{yz}}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^3}{6} - \frac{x^4}{12} + C_3 x + C_4\right)\,.$$

The boundary conditions

$$w(0) = 0 \;\rightsquigarrow\; C_2 = 0\,, \qquad v(0) = 0 \;\rightsquigarrow\; C_4 = 0\,,$$

$$w(l) = 0 \;\rightsquigarrow\; C_1 = -\frac{l^3}{12}\,, \quad v(l) = 0 \;\rightsquigarrow\; C_3 = -\frac{l^3}{12}$$

together with the abbreviation $\xi = \dfrac{x}{l}$ yield

$$Ew = \frac{q_0 l^4}{24}\left\{\xi^4 - 2\xi^3 + \xi\right\}\frac{I_z}{\Delta}\,,$$

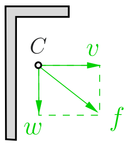
$$Ev = \frac{q_0 l^4}{24}\left\{\xi^4 - 2\xi^3 + \xi\right\}\frac{I_{yz}}{\Delta}\,.$$

In the middle of the beam ($\xi = 1/2$) the curly brackets attain the value 5/16 which leads with the given numerical values (converted to cm) to

$$\underline{\underline{w}} = 10^2 \cdot 200^4\,\frac{5}{384}\,\frac{55.8}{12321.5}\cdot\frac{1}{2.1\cdot 10^7} = \underline{\underline{0.45\text{ cm}}}\,,$$

$$\underline{\underline{v}} = 10^2 \cdot 200^4\,\frac{5}{384}\,\frac{-75.4}{12321.5}\cdot\frac{1}{2.1\cdot 10^7} = \underline{\underline{-0.61\text{ cm}}}\,,$$

$$\underline{\underline{f}} = \sqrt{w^2 + v^2} = \underline{\underline{0.76\text{ cm}}}\,.$$



P3.37

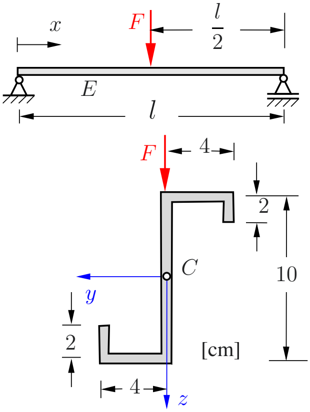
Problem 3.37 In the middle of a beam the force F is applied. The thin-walled profile is produced from an aluminium sheet of 2 mm thickness.

Compute the deformation at the point where the force is applied.

Given: $l = 2$ m ,

$E = 7 \cdot 10^4$ MPa ,

$F = 1200$ N .



Solution The displacement can be determined with regards to the y, z-axes, or with regard to the principal axes. We want to consider both possibilities.

1st solution: The position of the centroid is known. With regard to the y, z-axes we find

$$I_y = \frac{0.2 \cdot 10^3}{12} + \left(\frac{0.2 \cdot 10^3}{12} - \frac{0.2 \cdot 6^3}{12}\right) + 2 \cdot 5^2 \cdot 0.2 \cdot 4 = 69.73 \text{ cm}^4 ,$$

$$I_z = \frac{0.2 \cdot 8^3}{12} + 2 \cdot 4^2 \cdot 0.2 \cdot 2 = 21.33 \text{ cm}^4 ,$$

$$I_{yz} = -2\{5 \cdot 2 \cdot 0.2 \cdot 4 + 4 \cdot 4 \cdot 0.2 \cdot 2\} = -28.8 \text{ cm}^4 ,$$

$$\Delta = I_y I_z - I_{yz}^2 = 657.9 \text{ cm}^8 .$$

With the bending moments $M_y = \frac{F}{2}x\,, \quad M_z = 0$ für $0 \leq x \leq l/2$ (symmetry) the differential equations are given by

$$Ew'' = -\frac{F\,I_z}{2\,\Delta}\,x\,, \qquad Ev'' = -\frac{F\,I_{yz}}{2\,\Delta}\,x\,.$$

After integration and incorporation of the boundary conditions we obtain in the middle of the beam (see also table on page 62):

$$\underline{\underline{w}} = \frac{Fl^3}{48E}\,\frac{I_z}{\Delta} = \frac{1200 \cdot 200^3}{48 \cdot 7 \cdot 10^6} \cdot \frac{21.33}{657.9} = \underline{\underline{0.93 \text{ cm}}} ,$$

$$\underline{\underline{v}} = \frac{Fl^3}{48E}\,\frac{I_{yz}}{\Delta} = \frac{1200 \cdot 200^3}{48 \cdot 7 \cdot 10^6} \cdot \frac{(-28.8)}{657.9} = \underline{\underline{-1.25 \text{ cm}}} ,$$

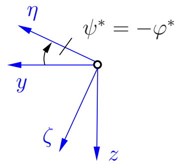
$$\underline{\underline{f}} = \sqrt{w^2 + v^2} = \underline{\underline{1.56 \text{ cm}}} .$$

2nd solution: We refer to the principal axes. The principal directions and values of the second moment of area are given by

$$\tan 2\varphi^* = \frac{2I_{yz}}{I_y - I_z} = -1.19 \quad \rightsquigarrow \quad \varphi^* = -24.98^\circ$$

$$I_{1,2} = \frac{91.06}{2} \pm \sqrt{24.2^2 + 28.8^2}$$

$$\rightsquigarrow \quad I_1 = I_\eta = 83.15 \text{ cm}^4\,, \quad I_2 = I_\zeta = 7.91 \text{ cm}^4\,.$$



Decomposition of the load into principal directions yields

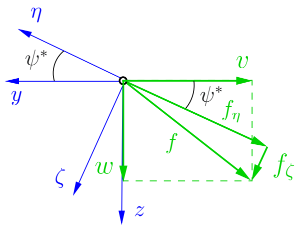
$$F_\zeta = F\cos\psi^* = 0.906\ F\,, \qquad F_\eta = -F\sin\psi^* = 0.422\ F\,,$$

and the displacements follow from the table on page 62 (load case no. 1)

$$f_\eta = \frac{F_\eta l^3}{48EI_\zeta} = -\frac{1200\cdot 0.422\cdot 200^3}{48\cdot 7\cdot 10^6\cdot 7.91} = -1.52 \text{ cm}\,,$$

$$f_\zeta = \frac{F_\zeta l^3}{48EI_\eta} = \frac{1200\cdot 0.906\cdot 200^3}{48\cdot 7\cdot 10^6\cdot 83.15} = 0.31 \text{ cm}\,,$$

$$\underline{\underline{f}} = \sqrt{f_\eta^2 + f_\zeta^2} = \underline{\underline{1.55 \text{ cm}}}\,.$$



For comparison with the 1st solution we transfer the displacements into the y, z-coordinate system:

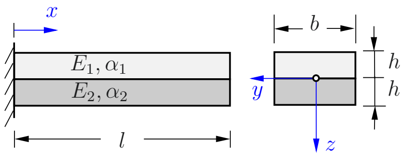
$$\underline{\underline{|v|}} = |f_\eta|\cos\psi^* - f_\zeta\sin\psi^* = \underline{\underline{1.25 \text{ cm}}}\,,$$

$$\underline{\underline{w}} = |f_\eta|\sin\psi^* + f_\zeta\cos\psi^* = \underline{\underline{0.93 \text{ cm}}}\,.$$

Note: We used in the computations numerical values up to the second digit. Thus the numerical value for the total displacement f differs in the second digit.

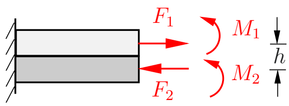
P3.38

Problem 3.38 A beam composed of two different materials (a bi-metal beam to measure temperature) is heated uniformly by a temperature difference ΔT.



Determine the deformation at the free end.

Solution We assume a linear stress distribution in each material and replace the stresses by a resultant force F_i and a resulting moment M_i. If we suppose $\alpha_2 > \alpha_1$ the lower part wants expand more. As this is prevented by the upper part, the lower part is under compression, while tension prevails in the upper part. F_1 and F_2 cause a moment in the composite beam which is in equilibrium with M_1 and M_2 (no external loads). Thus the following equations hold:



$$\text{statics} \qquad N = 0 \quad \rightsquigarrow \quad F_1 = F_2 = F \,,$$

$$M = 0 \quad \rightsquigarrow \quad Fh = M_1 + M_2 \,,$$

$$\text{Hooke's law} \quad w_1'' = -\frac{M_1}{E_1}\frac{12}{bh^3} \,, \quad w_2'' = -\frac{M_2}{E_2}\frac{12}{bh^3} \,.$$

Kinematic compatibility demands

$$w_1'' = w_2'' = w'' \,.$$

Additionally the strains have to match at the interface. They consist of three contributions: temperature $\alpha_i \Delta T$, normal force F/EA and bending M/EW. Considering tension and compression we formulate

$$\alpha_1 \Delta T + \frac{F}{bhE_1} + \frac{M_1 6}{E_1 bh^2} = \alpha_2 \Delta T - \frac{F}{bhE_2} - \frac{M_2 6}{E_2 bh^2} \,.$$

Eliminating the moments M_i and rearrangement to get w'' yields

$$w'' = -\frac{12E_1E_2(\alpha_2 - \alpha_1)\Delta T}{h(E_1^2 + 14E_1E_2 + E_2^2)} = -C \,.$$

Integration, by incorporating the boundary conditions at the left end, provides the displacement at the free end

$$\underline{\underline{w = -C\,\frac{l^2}{2}}} \,.$$