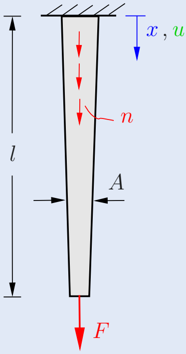
Chapter 2

Tension and Compression in Bars

2

Tensile or compressive loading in bars

Assumptions:



- Length l of the bar is large compared to characteristic dimensions of the cross section $A(x)$.
- Axis of the bar (line connecting centroids of the cross sections) is a straight line.
- Common line of action (external loads F and $n(x)$ are aligned with the axis of the bar).
- Cross section $A(x)$ can only vary slightly.

Stress: Assuming a constant stress σ across the section A the following relation with the normal force N holds:

$$\sigma(x) = \frac{N(x)}{A(x)} .$$

Basic equations of a deformable bar:

equilibrium condition	$\dfrac{\mathrm{d}N}{\mathrm{d}x} = -n ,$
Hooke's law	$\varepsilon = \dfrac{\sigma}{E} + \alpha_T \, \Delta T ,$
Kinematic relation	$\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

E = Young's modulus,

α_T = coefficient of thermal expansion,

ΔT = temperature difference with respect to a reference state,

$u(x)$ = displacement of a point x within the bar.

The basic equations lead to a *single* differential equation for the displacements ($\{\cdot\}' := \mathrm{d}\{\cdot\}/\mathrm{d}x$):

$$(EAu')' = -n + (EA\alpha_T \Delta T)' .$$

Elongation of a bar: $$\Delta l = u(l) - u(0) = \int_0^l \varepsilon \, \mathrm{d}x \, .$$

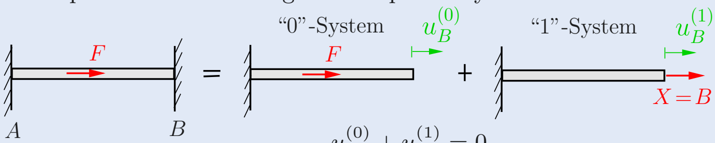
special cases:

$$\Delta l = \int_0^l \frac{N}{EA} \mathrm{d}x \qquad (\Delta T = 0) \, ,$$

$$\Delta l = \frac{Fl}{EA} \qquad (N = F = \text{const}, \; EA = \text{const}, \; \Delta T = 0),$$

$$\Delta l = \alpha_T \Delta T \, l \qquad (N = 0, \; EA = \text{const}, \; \alpha_T \Delta T = \text{const}).$$

Superposition: The solution of a statically indeterminate problem can be achieved by *superposition* of solutions of associated statically determinate problems considering the compatibility conditions.

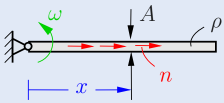


$$u_B^{(0)} + u_B^{(1)} = 0 \, .$$

Rotating bar: A bar rotating with the angluar velocity ω experiences an axial loading per unit length of

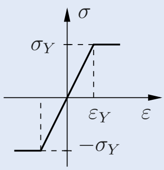
$$n = \rho A \, x \omega^2 \, .$$

Here ρ is the density and x represents the distance of the cross section A from the center of rotation.



Elastic-plastic bar: For an elastic-ideal-plastic material behavior, Hooke's law is valid only until a certain *yield limit* σ_Y :

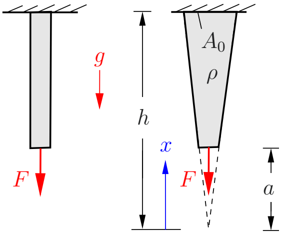
$$\sigma = \begin{cases} E\,\varepsilon \, , & |\varepsilon| \leq \varepsilon_Y \, , \\ \sigma_Y \, \text{sign}(\varepsilon) \, , & |\varepsilon| \geq \varepsilon_Y \, . \end{cases}$$



System of bars: The displacements are obtained by "disconnecting" and "reconnecting" of the bars from the nodes using a *displacement diagram*.

Note: In areas with rapidly changing cross sections (notches, holes) the above theory for bars is not applicable.

P2.1 **Problem 2.1** Determine the stresses distribution $\sigma(x)$ in the homogeneous bar due to its weight. The bar has constant thickness and a linear varying width. Furthermore, identify the location and value of the smallest stress.



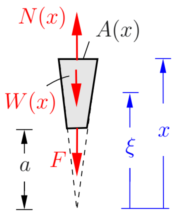
It is reasonable to introduce the x-coordinate at the intersection of the extended edges of the trapeziod. The x dependent cross section area follows then as

$$A(x) = A_0 x/l\,.$$

With the weight

$$W(x) = \rho g V(x) = \rho g \int_a^x A(\xi)\mathrm{d}\xi = \rho g A_0 \frac{x^2 - a^2}{2l}$$

of the lower part equilibrium provides



$$N(x) = F + W(x) = F + \rho g A_0 \frac{x^2 - a^2}{2h}\,.$$

This leads to the stress

$$\underline{\underline{\sigma(x)}} = \frac{N(x)}{A(x)} = \underline{\underline{\frac{Fh + \rho g \frac{A_0}{2}\left(x^2 - a^2\right)}{A_0 x}}}\,.$$

The location x^* of the minimum is determined by condition $\sigma' = 0$:

$$\sigma' = -\frac{Fh}{A_0}\frac{1}{x^2} + \frac{\rho g}{2}\left(1 + \frac{a^2}{x^2}\right) = 0 \quad \rightsquigarrow \quad \underline{\underline{x^* = \sqrt{\frac{2Fh}{\rho g A_0} - a^2}}}\,.$$

The value of the minimum stress is

$$\underline{\underline{\sigma_{\text{min}}}} = \sigma(x^*) = \rho g \sqrt{\frac{2Fh}{\rho g A_0} - a^2} = \underline{\underline{\rho g x^*}}\,.$$

Note:

- For $\rho g = 0$ ("weight-less bar") no minimum exists. The largest stress occurs at $x = a$.
- The minimum will be located within the bar, only if $a < x^* < h$ or $\rho g A_0 a^2/(2h) < F < \rho g A_0 (h^2 + a^2)/(2h)$ holds.

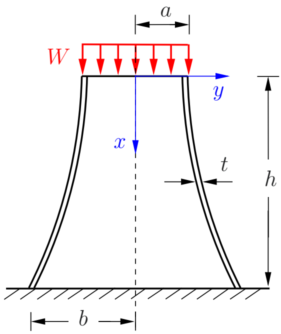
Problem 2.2 The contour of a lighthouse with circular thin-walled cross section follows a hyperbolic equation

P2.2

$$y^2 - \frac{b^2 - a^2}{h^2}\, x^2 = a^2\,.$$

Determine the stress distribution as a consequence of weight W of the lighthouse head (the weight of the structure can be neglected).

Given: $b = 2a$, $t \ll a$.

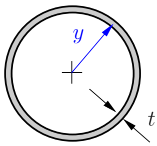


Solution As the weight W is the only acting external load, the normal force N is constant (compression):

$$N = -W\,.$$

The cross section area A is changing. It can be approximated by (thin-walled structure with $t \ll y$)

$$\begin{aligned} A(x) = 2\pi y t &= 2\pi t\sqrt{a^2 + \frac{b^2-a^2}{h^2}\,x^2} \\ &= 2\pi t\sqrt{a^2 + 3\frac{a^2}{h^2}\,x^2} \\ &= 2\pi a t\sqrt{1 + 3\,\frac{x^2}{h^2}}\,. \end{aligned}$$



The stress follows now as

$$\underline{\underline{\sigma(x) = \frac{N}{A} = -\frac{W}{2\pi a t\sqrt{1 + 3\,\frac{x^2}{h^2}}}}}\,.$$

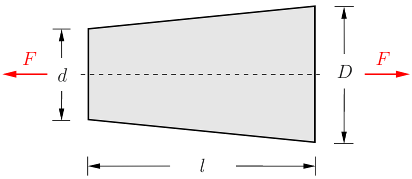
Especially at the top and bottom position we get

$$\sigma(x=0) = -\frac{W}{2\pi a t} \qquad \text{bzw.} \qquad \sigma(x=h) = -\frac{W}{4\pi a t}\,.$$

Note: The stress at the top is twice as large as the stress at the bottom, which is a inefficient use of material. This situation changes if the weight of the thin-walled structure is included in the analysis.

P2.3

Problem 2.3 Determine the elongation Δl of the conical shaft (Young's modulus E) under the application of a tensile force F.



Solution The normal force $N = F$ is constant, while the cross section area A varies. With $\sigma = N/A$ the elongation is computed by

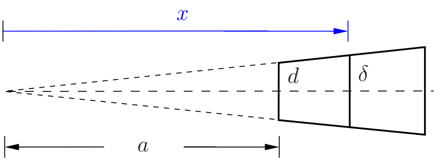
$$\Delta l = \int_0^l \varepsilon\,\mathrm{d}x = \frac{1}{E}\int_0^l \sigma\,\mathrm{d}x = \frac{1}{E}\int_0^l \frac{N\mathrm{d}x}{A} = \frac{F}{E}\int_0^l \frac{\mathrm{d}x}{A(x)}\,.$$

To describe the change of the cross section area $A(x)$ we start the x-axis at the peak of the frustum. Using the intercept theorem and the auxiliary variable a we obtain for the diameter

$$\delta(x) = d\,\frac{x}{a}$$

and for the area

$$A(x) = \frac{\pi}{4}\,\delta^2(x) = \frac{\pi}{4}\,d^2\,\frac{x^2}{a^2}\,.$$



Introducing this in the relation for the elongation, then integration provides (integration limits!):

$$\Delta l = \frac{F}{E}\int_a^{a+l} \frac{\mathrm{d}x}{\frac{\pi}{4}\,d^2\,\frac{x^2}{a^2}} = \frac{4Fa^2}{\pi E\,d^2}\left(-\frac{1}{x}\right)\Bigg|_a^{a+l}\,.$$

With

$$\frac{a+l}{D} = \frac{a}{d} \quad \leadsto \quad a = \frac{d}{D}\,\frac{l}{1-\frac{d}{D}}$$

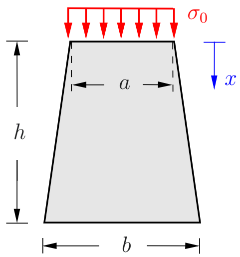
the elongation is

$$\underline{\underline{\Delta l = \frac{4Fl}{\pi EDd}}}\,.$$

Test: For $D = d$ (constant cross section) we obtain $\Delta l = \dfrac{4Fl}{\pi E d^2} = \dfrac{Fl}{EA}$
.

Problem 2.4 A homogeneous frustum of a pyramid (Young's modulus E) with a square cross section is loaded on its top surface by a stress σ_0. **P2.4**

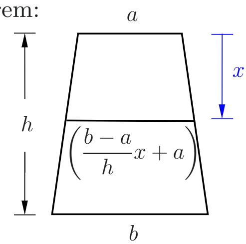
Determine the displacement field $u(x)$ of a cross section at position x.



Solution The normal force $N = -\sigma_0 a^2$ is constant. From the kinematic relation $\varepsilon = \mathrm{d}u/\mathrm{d}x$ and Hooke's law $\varepsilon = \sigma/E = N/EA$ we obtain a differential equation for the displacement u

$$EA(x)\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\sigma_0 a^2\,.$$

The area $A(x)$ follows from the intercept theorem:



$$A(x) = [a + (b-a)\frac{x}{h}]^2\,.$$

Thus we have

$$E\big(a + \frac{b-a}{h}\,x\big)^2\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\sigma_0 a^2\,.$$

Separation of variables yields

$$\mathrm{d}u = -\frac{\sigma_0 a^2}{E}\,\frac{\mathrm{d}x}{\left(\frac{b-a}{h}\,x + a\right)^2} \quad\leadsto\quad \int\limits_{u(0)}^{u(x)} \mathrm{d}u = -\frac{\sigma_0 a^2}{E}\int\limits_0^x \frac{\mathrm{d}\xi}{\left(\frac{b-a}{h}\,\xi + a\right)^2}.$$

Using the substitution $z = a + (b-a)\,\xi/h$, $\quad \mathrm{d}z = (b-a)\,\mathrm{d}\xi/h$ leads to

$$u(x) - u(0) = -\,\frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(-\frac{1}{z}\right)\Bigg|_a^{\frac{b-a}{h}x+a} = -\frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(\frac{1}{a} - \frac{1}{\frac{b-a}{h}\,x + a}\right).$$

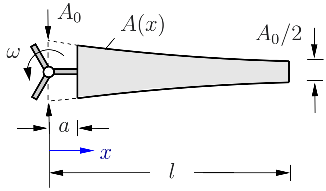
The displacement $u(0)$ of the top surface follows from the boundary condition that the displacement has to vanish on the bottom edge $x = h$:

$$u(h) = 0 \quad\leadsto\quad u(0) = \frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\sigma_0 a h}{Eb}\,.$$

From this relation the displacement follows

$$\underline{\underline{u(x) = \frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(-\,\frac{1}{b} + \frac{1}{\frac{b-a}{h}x + a}\right)}}.$$

P2.5 **Problem 2.5** The cross section of a solid helicopter blade (density ρ, Young's modulus E) is described by the equation $A(x) = A_0 e^{-\alpha x/l}$.



Determine the stress distribution $\sigma(x)$, if the blade is rotating with a constant angular velocity ω. Compute the elongation Δl under the assumption $a = 0$.

Solution First, the sketched geometry $A(l) = A_0/2$ yields

$$A_0 e^{-\alpha} = A_0/2 \quad \leadsto \quad e^{\alpha} = 2 \quad \leadsto \quad \alpha = \ln 2 = 0.693\,.$$

The rotation causes a distributed load per unit length

$$n = \rho\omega^2 x A(x) = \rho\omega^2 A_0 x e^{-\alpha x/l}\,.$$

The equilibrium condition $N' = -n$ provides the normal force by integration

$$N = -\int n\,dx = -\frac{\rho\,\omega^2 A_0 l^2}{\alpha^2}\left[-\frac{\alpha x}{l}e^{-\alpha x/l} - e^{-\alpha x/l} + C\right].$$

The integration constant C is determined by the boundary condition:

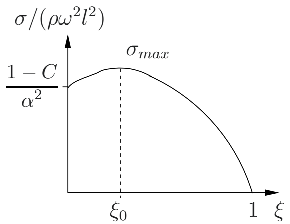
$$N(l) = 0 \quad \leadsto \quad C = (1+\alpha)e^{-\alpha} = 0.847\,.$$

Introducing the dimensionless coordinate $\xi = x/l$ yields

$$N(\xi) = \frac{\rho\,\omega^2 A_0 l^2}{\alpha^2}[(1+\alpha\xi)e^{-\alpha\xi} - C]\,,$$

and for the stress distribution

$$\underline{\underline{\sigma(\xi) = \frac{N}{A} = \frac{\rho\,\omega^2 l^2}{\alpha^2}[1 + \alpha\xi - Ce^{\alpha\xi}]}}\,.$$



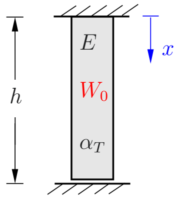
The elongation is calculated from

$$\underline{\underline{\Delta l}} = \int_0^l \varepsilon\,dx = \frac{l}{E}\int_0^1 \sigma\,d\xi = \frac{\rho\,\omega^2 l^3}{\alpha^2 E}\left[\xi + \frac{\alpha\xi^2}{2} - \frac{C}{\alpha}e^{\alpha\xi}\right]_0^1$$

$$\underline{\underline{= \frac{\rho\,\omega^2 l^3}{E\alpha^2}\left[1 + \frac{\alpha}{2} - \frac{C}{\alpha}e^{\alpha} + \frac{C}{\alpha}\right] = 0.258\,\frac{\rho\omega^2 l^3}{E}}}\,.$$

Note: Due to the varying cross section the maximum stress occurs at the position $\xi_0 = -(\ln C)/\alpha = 0.24$ and attains the maximum value $\sigma_{\max} = -(\rho\omega^2 l^2 \ln C)/\alpha^2 = 0.347\,\rho\omega^2 l^2$.

Problem 2.6 A massive bar (weight W_0, cross section area A, thermal expansion coefficient α_T) is fixed at $x = 0$ and just touches the ground in a stress-free manner. **P2.6**

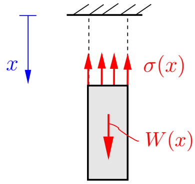


Determine the stress distribution $\sigma(x)$ in the bar after a uniform heating by ΔT.

Which ΔT causes compression everywhere in the bar?

Solution We investigate the "two load cases", weight und heating. The weight causes a a normal force

$$N(x) = W(x) = W_0 \frac{h - x}{h} = W_0 \left(1 - \frac{x}{h}\right)$$



which is related to the stress distribution

$$\sigma_1(x) = \frac{N(x)}{A} = \frac{W_0}{A} \left(1 - \frac{x}{h}\right)$$

The *heating* produces an additional strain, which is blocked by the support on the bottom. The relation

$$\varepsilon = \frac{\sigma_2(x)}{E} + \alpha_T \Delta T = 0$$

yields

$$\sigma_2(x) = -E\alpha_T \Delta T \,.$$

Thus the total stress is computed by

$$\underline{\underline{\sigma(x)}} = \sigma_1 + \sigma_2 = \underline{\underline{\frac{W_0}{A} \left(1 - \frac{x}{h}\right) - E\alpha_T \Delta T}} \,.$$

Due to the blocked temperature strain, there exists a compressive stress at the end of the bar $(x = h)$ at all times. As the stress distribution is linear, the stress will be compressive everywhere, if compression is present at the top edge. Thus the relation

$$\sigma(x = 0) < 0 \qquad \text{bzw.} \qquad \frac{W_0}{A} - E\alpha_T \Delta T < 0$$

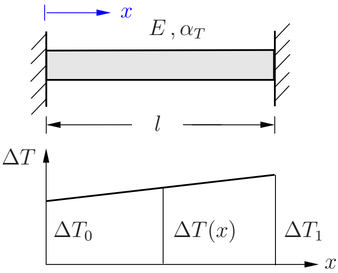
provides the necessary temperature difference

$$\underline{\underline{\Delta T > \frac{W_0}{EA\alpha_T}}} \,.$$

P2.7

Problem 2.7 An initially stress-free fixed bar (cross section area A) experiences a temperature increase varying linearly in x.

Determine the stress and strain distribution.



Solution The bar is supported in a statically indeterminate way. Thus we use equilibrium, kinematics and Hooke's law for the solution of the problem. With $n = 0$ and $\sigma = N/A$ these equations read

$$\sigma' = 0\,, \qquad \varepsilon = u'\,, \qquad \varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T(x)$$

with

$$\Delta T(x) = \Delta T_0 + (\Delta T_1 - \Delta T_0)\frac{x}{l}\,.$$

Combining the above relations renders the differential equation for the displacements

$$u'' = \alpha_T \Delta T' = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,.$$

Integrating twice yields

$$u' = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,x + C_1\,,$$
$$u = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,\frac{x^2}{2} + C_1 x + C_2\,.$$

The two integration constants follow from the boundary conditions:

$$u(0) = 0 \;\leadsto\; C_2 = 0\,, \qquad u(l) = 0 \;\leadsto\; C_1 = -\frac{\alpha_T}{2}(\Delta T_1 - \Delta T_0)\,.$$

We obtain the displacement field

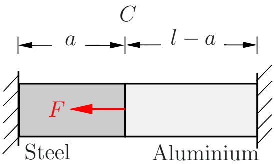
$$\underline{\underline{u(x) = \frac{\alpha_T l}{2}(\Delta T_1 - \Delta T_0)\left(\frac{x^2}{l^2} - \frac{x}{l}\right)}}$$

together with the (constant) stress

$$\underline{\underline{\sigma}} = E(u' - \alpha_T \Delta T) = \underline{\underline{-\frac{\alpha_T}{2}(\Delta T_1 + \Delta T_0)E}}\,.$$

Note: With constant heating $\Delta T_1 = \Delta T_0$ the displacement $u(x)$ vanishes. In this situation the stress is $\sigma = -\alpha_T \Delta T_0 E$.

Problem 2.8 A bar with a constant cross section A is fixed at both ends. The bar is made of two different materials, that are joint together at point C. **P2.8**



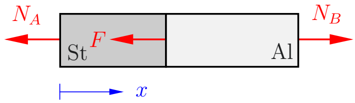
a) What are the reaction forces, if an external force F is applied at point C ?

b) Determine the normal force that is caused by a pure heating by ΔT?

Given: $E_{St}/E_{Al} = 3, \;\; \alpha_{St}/\alpha_{Al} = 1/2$.

Solution We treat the system as two joint bars with constant normal forces.

to a)



equilibrium: $-N_A + N_B = F$,

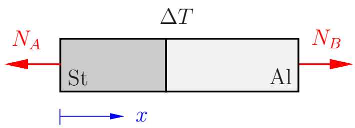
kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \dfrac{N_A a}{E_{St} A}$, $\quad \Delta l_{Al} = \dfrac{N_B (l-a)}{E_{Al} A}$.

The 4 equations for the 4 unknowns (N_A, N_B, Δl_{St}, Δl_{Al}) yield with the given numerical values

$$\underline{\underline{N_A = -F\,\frac{3(l-a)}{3l-2a}}}\,, \qquad \underline{\underline{N_B = F\,\frac{a}{3l-2a}}}\,.$$

to b)



equilibrium: $N_A = N_B = N$,

kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \dfrac{N\,a}{E_{St} A} + \alpha_{St} \Delta T\, a$,

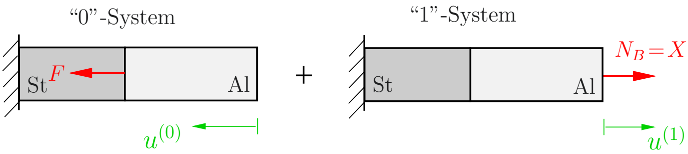
$$\Delta l_{Al} = \frac{N(l-a)}{E_{Al} A} + \alpha_{Al} \Delta T\,(l-a)\,.$$

Solving the system of equations for the normal force N yields with the given numerical values

$$\underline{\underline{N = -\frac{2l-a}{3l-2a}\, E_{St}\, \alpha_{St}\, A\, \Delta T}}\,.$$

P2.9 **Problem 2.9** Solve Problem 2.8 by superposition.

Solution **to a)** We choose the reaction force N_B as statically redundant quantity.



Hooke's law provides

$$u^{(0)} = \frac{Fa}{E_{St}A} , \qquad u^{(1)} = \frac{X(l-a)}{E_{Al}A} + \frac{Xa}{E_{St}A} .$$

As the right edge is fixed compatibility requires

$$u^{(0)} = u^{(1)} .$$

This condition yields

$$\underline{\underline{N_B}} = X = \frac{Fa}{a + (l-a)\dfrac{E_{St}A}{E_{Al}A}} = \underline{\underline{F\,\frac{a}{3l-2a}}} .$$

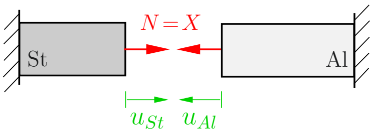
From equilibrium we have

$$\underline{\underline{N_A = N_B - F = -F\,\frac{3(l-a)}{3l-2a}}} .$$

to b) In the free body diagram we choose the normal force N as statically redundant quantity X. From Hooke's law

$$u_{St} = \frac{Xa}{E_{St}A} + \alpha_{St}\Delta T a ,$$

$$u_{Al} = \frac{X(l-a)}{E_{Al}A} + \alpha_{Al}\Delta T(l-a)$$



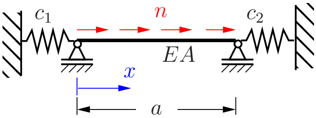
and the compatibility

$$u_{St} + u_{Al} = 0$$

we obtain

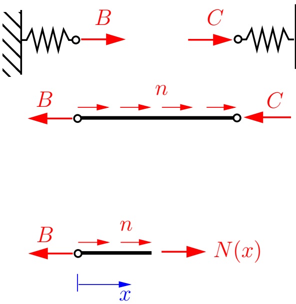
$$\underline{\underline{N}} = X = -\frac{\alpha_{St}a + \alpha_{Al}(l-a)}{\dfrac{a}{E_{St}A} + \dfrac{(l-a)}{E_{Al}A}} = \underline{\underline{-\frac{2l-a}{3l-2a}\,E_{St}\,\alpha_{St}\,A\,\Delta T}} .$$

Problem 2.10 An elastically supported bar ($c_1 = 2c_2 = EA/2a$) is loaded by a constant axial load n. **P2.10**



Compute the distribution of the normal force $N(x)$ in the bar.

Solution Using the free body diagram with the forces B and C at the ends of the bar, the equilibrium conditions can be formulated



$$B + C = na\,, \qquad N(x) = B - nx\,.$$

The elongation/shortening of the springs is given by

$$\Delta u_1 = \frac{B}{c_1}\,, \qquad \Delta u_2 = \frac{C}{c_2}\,.$$

The elongation of the bar is computed from

$$\Delta u_{\mathrm{St}} = \int_0^a \varepsilon\,\mathrm{d}x = \int_0^a \frac{N}{EA}\,\mathrm{d}x\,.$$

With $N = B - nx$ we obtain

$$\Delta u_{\mathrm{St}} = \frac{Ba}{EA} - \frac{na^2}{2\,EA}\,.$$

Finally, the kinematic relation

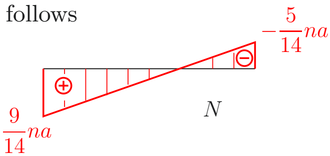
$$\Delta u_1 + \Delta u_{\mathrm{St}} = \Delta u_2 \qquad \rightsquigarrow \qquad \frac{B}{c_1} + \frac{Ba}{EA} - \frac{na^2}{2\,EA} = \frac{C}{c_2}$$

with $C = -B + na$ and the given value for c_1 and c_2 yields

$$B\left(\frac{2a}{EA} + \frac{4a}{EA} + \frac{a}{EA}\right) = na\left(\frac{a}{2\,EA} + \frac{4a}{EA}\right) \quad \rightsquigarrow \quad B = \frac{9}{14}\,na$$

and the distribution of the normal force follows

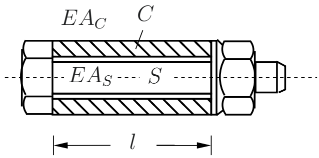
$$\underline{\underline{N(x) = \frac{9}{14}\,na - nx\,.}}$$



P2.11

Problem 2.11 Determine the compression Δl_C of a casing C of length l, if the nut of screw S (lead h) is turned by one revolution.

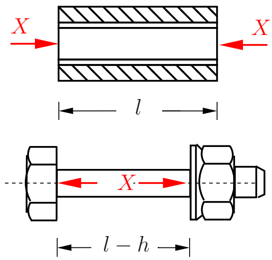
Given: $\dfrac{EA_C}{EA_S} = \dfrac{4}{3}$.



Solution After the revolution of the nut we cut the system of screw and casing and introduce the statically indeterminate force F between the two parts.

The casing experiences a compression

$$\Delta l_C = \frac{Xl}{EA_C} \,.$$



For the screw we obtain an elongation

$$\Delta l_S = \frac{Xl}{EA_S} \,.$$

The length changes have to be adjusted in such a way that casing and screw have the same length. Therefore compatibility can be written as

$$h = \Delta l_C + \Delta l_S \,.$$

Inserting the length changes yields the force

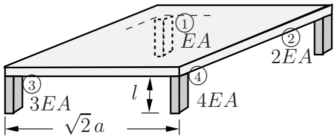
$$X = \frac{h}{l}\,\frac{1}{\dfrac{1}{EA_C}+\dfrac{1}{EA_S}}$$

and the compression of the casing

$$\underline{\underline{\Delta l_C}} = \frac{Xl}{EA_C} = h\,\frac{1}{1+\dfrac{EA_C}{EA_S}} = h\,\frac{1}{1+\frac{4}{3}} = \underline{\underline{\frac{3}{7}\,h}} \,.$$

Note: As the axial rigidity of the casing is larger than the one of the screw, the compression is only 3/7 of the lead. If equal axial rigidities are present $EA_C = EA_S$, the length change of both parts will be equal, i. e. $\Delta l_C = \Delta l_S = h/2$.

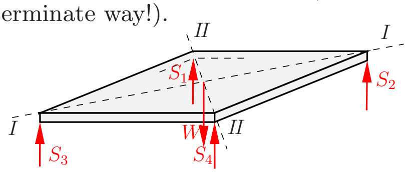
Problem 2.12 A *rigid* quadratic plate (weight W, edge length $\sqrt{2}\,a$) is supported on 4 elastic posts. The posts are of equal length l, but possess different axial rigidities. **P2.12**



Determine the weight distribution on the 4 posts?

Determine the displacement f in the middle of the plate.

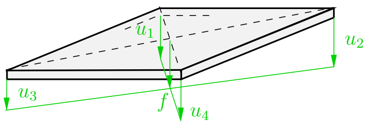
Solution The system is statically indeterminate of degree one (a table on 3 posts rests in a statically determinate way!).
Equilibrium yields



$$\uparrow: \quad S_1 + S_2 + S_3 + S_4 = W\,,$$
$$\overset{\curvearrowright}{I}: \quad aS_4 = aS_1\,,$$
$$\overset{\curvearrowright}{II}: \quad aS_2 = aS_3\,.$$

The displacement f in the middle is obtained from the average value of the displacements u_i (= length change of the posts) at opposite corners (rigid plate). Accordingly the compatibility reads:

$$f = \frac{1}{2}(u_1 + u_4) = \frac{1}{2}(u_2 + u_3)\,.$$



With Hooke's law

$$u_i = \frac{S_i l}{EA_i}$$

and $S_1 = S_4$, $S_2 = S_3$ we obtain as intermediate result

$$\frac{S_1 l}{EA} + \frac{S_1 l}{4EA} = \frac{S_2 l}{2EA} + \frac{S_2 l}{3EA} \quad \rightsquigarrow \quad \frac{5}{4}S_1 = \frac{5}{6}S_2\,.$$

Inserting this into the first equilibrium condition yields

$$S_1 + \frac{3}{2}S_1 + \frac{3}{2}S_1 + S_1 = G \quad \rightsquigarrow \quad \underline{\underline{S_1 = S_4 = \frac{1}{5}G}}\,, \qquad \underline{\underline{S_2 = S_3 = \frac{3}{10}G}}\,.$$

form which the displacement follows:

$$\underline{\underline{f = \frac{1}{2}\left(\frac{S_1 l}{EA} + \frac{S_1 l}{4EA}\right) = \frac{1}{8}\frac{Gl}{EA}}}\,.$$

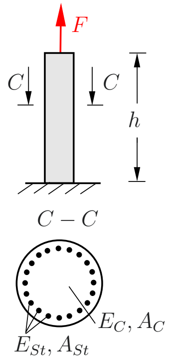
P2.13

Problem 2.13 A column of steel reinforced concrete is loaded by a tensile force F.

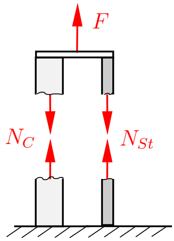
What are the stresses in the concrete and the steel as well as the height change Δh of the column, if we assume

a) a perfect bonding between steel and concrete?
b) the concrete is cracked and does not carry any load?

Given: $E_{St}/E_C = 6, \;\; A_{St}/A_C = 1/9$.



Solution **to a)** We consider the composite as a system of two "bars" of different materials, which experience under load F the same length change Δl. With this the basic equations of the system are:



equilibrium: $N_{St} + N_C = F$,

kinematics: $\Delta h_{St} = \Delta h_C = \Delta h$,

Hooke's law: $\Delta h_{St} = \dfrac{N_{St} h}{EA_{St}}, \quad \Delta h_C = \dfrac{N_C h}{EA_C}$.

Solution of the system of equation yields – with the stiffness ratio $EA_C/EA_{St} = 3/2$ – the normal forces

$$N_{St} = F \frac{1}{1 + \dfrac{EA_C}{EA_{St}}} = \frac{2}{5} F , \qquad N_C = F \frac{\dfrac{EA_C}{EA_{St}}}{1 + \dfrac{EA_C}{EA_{St}}} = \frac{3}{5} F$$

and the height change

$$\underline{\underline{\Delta h = \frac{Fh}{EA_{St} + EA_C}}} \qquad \text{i. e.} \qquad \underline{\underline{\Delta h}} = \frac{Fh}{EA_{St}} \frac{1}{1 + \dfrac{EA_C}{EA_{St}}} = \underline{\underline{\frac{2}{5} \frac{Fl}{EA_{St}}}} .$$

The stresses result from $A = A_C + A_{St}$ and $A_{St} = A/10$ and $A_C = 9A/10$

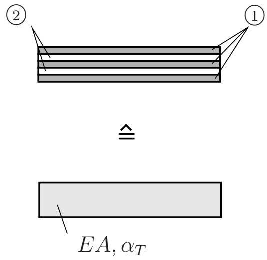
$$\underline{\underline{\sigma_{St}}} = \frac{N_{St}}{A_{St}} = \underline{\underline{4 \frac{F}{A}}} , \qquad \underline{\underline{\sigma_C}} = \frac{N_C}{A_C} = \underline{\underline{\frac{2}{3} \frac{F}{A}}} .$$

to b) If only the steel carries load, we will obtain with $N_{St} = F$

$$\underline{\underline{\sigma_{St}}} = \frac{F}{A_{St}} = 10 \underline{\underline{\frac{F}{A}}} , \qquad \underline{\underline{\Delta h = \frac{Fh}{EA_{St}}}} .$$

Problem 2.14 A laminated bar made of bonded layers of two different materials (respective axial rigidities EA_1, EA_2 and coefficients of thermal expansion α_{T1}, α_{T2}) is to be replaced by a bar made of a homogeneous material. **P2.14**

Determine EA and α_T such that the homogeneous bar experiences the same elongation as the laminated bar under application of a force and a temperature change ?



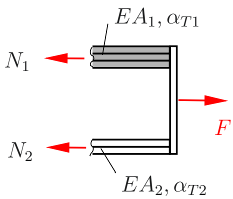
Solution For the laminated bar, subjected to a force F and a temperature increase ΔT, the basic equation yield

equilibrium: $N_1 + N_2 = F$,

kinematics: $\Delta l_1 = \Delta l_2 = \Delta l_{\text{lam}}$,

Hooke's law: $\Delta l_1 = \dfrac{N_1 l}{EA_1} + \alpha_{T1}\Delta T\, l$,

$$\Delta l_2 = \frac{N_2 l}{EA_2} + \alpha_{T2}\Delta T\, l\,.$$



This yields

$$\Delta l_{\text{lam}} = \frac{Fl}{EA_1 + EA_2} + \frac{EA_1\alpha_{T1} + EA_2\alpha_{T2}}{EA_1 + EA_2}\Delta T\, l\,.$$

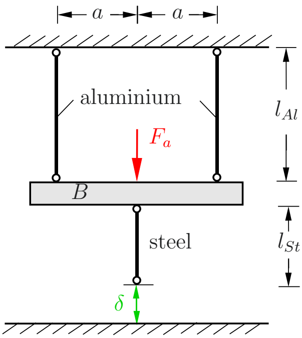
For a homogeneous bar under identical loading conditions, we have

$$\Delta l_{\text{hom}} = \frac{Fl}{EA} + \alpha_T \Delta T\, l\,.$$

The length changes Δl_{lam} and Δl_{hom} agree for arbitrary F and ΔT only, if

$$\underline{\underline{EA = EA_1 + EA_2}}\,, \qquad \underline{\underline{\alpha_T = \frac{EA_1\alpha_{T1} + EA_2\alpha_{T2}}{EA_1 + EA_2}}}\,.$$

P2.15 **Problem 2.15** In the depicted support construction for the *rigid* body B the lower support bar is too short by the length δ. In order to assemble the structure a force F_a is applied, such that the end of the bar just touches the ground. After assembly the force F_a is removed. The diameters of all bars d_i are identical.



a) Compute the required assembly force F_a.

b) Determine the displacement v_B of the body and the forces in the bars after assembly.

Given: $l_{Al} = 1\,\text{m}$, $d_{Al} = 2\,\text{mm}$, $E_{Al} = 0.7 \cdot 10^5\,\text{MPa}$, $l_{St} = 1.5\,\text{m}$, $d_{St} = 2\,\text{mm}$, $E_{St} = 2.1 \cdot 10^5\,\text{MPa}$, $\delta = 5\,\text{mm}$.

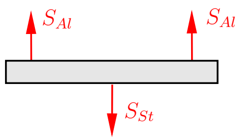
Solution to a) Each aluminium bar carries half of the assembly force (equilibrium) and elongates by the amount delta δ. This yields

$$S_{Al} = \frac{F_a}{2}\,, \qquad \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \frac{F_a l_{Al}}{2EA_{Al}} = \delta\,,$$

$$\leadsto \quad \underline{\underline{F_a}} = 2\frac{\delta}{l_{Al}} EA_{Al} = 2 \cdot \frac{5}{1000} \cdot 0,7 \cdot 10^5 \cdot \pi \cdot 1^2 = \underline{\underline{2200\,\text{N}}}\,.$$

to b) After removal of the force F_a new forces S_{Al} and S_{St} are present. This leads to the equilibrium condition

$$S_{St} = 2S_{Al}\,,$$



Hooke's law

$$\Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}}\,, \qquad \Delta l_{St} = \frac{S_{St} l_{St}}{EA_{St}}$$

and the compatibility condition

$$\Delta l_{Al} + \Delta l_{St} = \delta\,.$$

Solving the 4 equations yields

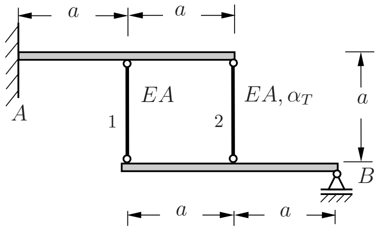
$$\underline{\underline{S_{Al}}} = \frac{\delta}{l_{Al}} \frac{EA_{Al}}{1 + 2\,\frac{l_{St}}{l_{Al}}\,\frac{EA_{Al}}{EA_{St}}} = \frac{5}{1000}\,\frac{0,7 \cdot 10^5 \cdot \pi \cdot 1^2}{1 + 2 \cdot \frac{3}{2} \cdot \frac{1}{3}} = \underline{\underline{550\,\text{N}}}\,,$$

$$\underline{\underline{S_{St}}} = 2S_{Al} = \underline{\underline{1100\,\text{N}}}\,, \qquad \underline{\underline{v_K}} = \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \underline{\underline{2.5\,\text{mm}}}\,.$$

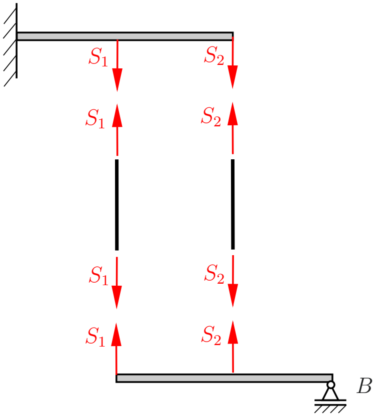
Problem 2.16 Two *rigid* beams are connected by two elastic bars. The first beams is fixed at point A, while the second is simply supported at point B. Bar 2 is heated by a temperature ΔT.

P2.16

Compute the forces in the two bars.



Solution We cut the system and use the following free body diagram to formulate the equilibrium conditions



$$\overset{\curvearrowright}{B}: \quad 2aS_1 + aS_2 = 0\,,$$

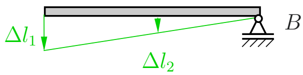
Hooke's law

$$\Delta l_1 = \frac{S_1 a}{EA}\,,$$

$$\Delta l_2 = \frac{S_2 a}{EA} + \alpha_T \Delta T \cdot a$$

and the compatibility condition

$$\Delta l_1 = 2\Delta l_2\,.$$



Solving for the unknown forces in the bars yields

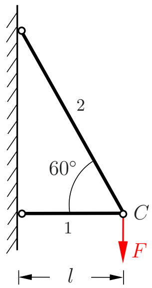
$$\underline{\underline{S_1 = \frac{2}{5}\, EA\, \alpha_T \Delta T}}\,, \qquad \underline{\underline{S_2 = -\frac{4}{5}\, EA\, \alpha_T \Delta T}}\,.$$

Note: In the heated bar compressive forces are generated due to the constrained deformations.

P2.17

Problem 2.17 In the depicted two bar system both bars have the same axial rigidity EA.

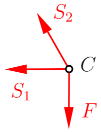
Determine the displacement of point C where the load is applied.



Solution From equilibrium we have

$$\uparrow: \quad S_2 \sin 60^\circ = F \quad \leadsto \quad S_2 = \frac{2}{3}\sqrt{3}\, F\,,$$

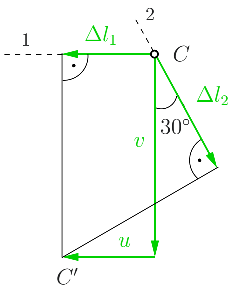
$$\rightarrow: \quad -S_1 - S_2 \cos 60^\circ = 0 \leadsto \quad S_1 = -\frac{1}{3}\sqrt{3}\, F\,.$$



Thus the elongation and shrinking of the bars follow as

$$\Delta l_2 = \frac{S_2 l_2}{EA} = \frac{\frac{2}{3}\sqrt{3}\frac{l}{\cos 60^\circ}F}{EA} = \frac{4\sqrt{3}}{3}\frac{Fl}{EA}\,, \quad \Delta l_1 = \frac{S_1 l_1}{EA} = -\frac{\sqrt{3}}{3}\frac{Fl}{EA}\,.$$

To determine the displacements of point C we construct the displacement diagram. In this diagram the length changes are introduced. As the length changes are small $\Delta l_i \ll l$ they are not drawn to the scale. In this example Δl_1 is a shrinkage (to the left) and Δl_2 an elongation. Considering that the bars can only rotate around the hinge points we introduce the right angles and read off the displacement diagram:



$$\underline{\underline{u}} = |\Delta l_1| = \underline{\underline{\frac{\sqrt{3}}{3}\frac{Fl}{EA}}}\,,$$

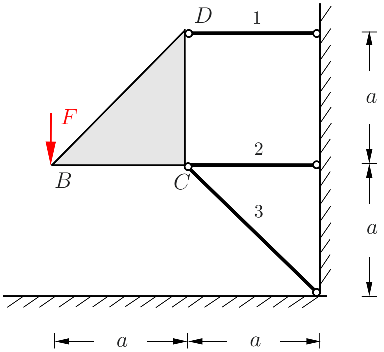
$$\underline{\underline{v}} = \frac{\Delta l_2}{\cos 30^\circ} + \frac{u}{\tan 60^\circ} = \frac{4\sqrt{3}}{3}\frac{Fl}{EA}\frac{1}{\frac{1}{2}\sqrt{3}} + \frac{\sqrt{3}}{3}\frac{Fl}{EA}\frac{1}{\sqrt{3}} = \underline{\underline{3\,\frac{Fl}{EA}}}\,.$$

Problem 2.18 A rigid weightless triangle is supported by 3 bars with the axial rigidity EA. The triangle is loaded in point B by the force F.

P2.18

a) Determine the forces S_i in the 3 bars and their elongations Δl_i.

b) Compute the displacement of point C.

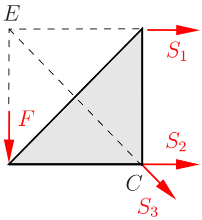


Solution **to a)** The system is statically determinately supported. The forces in the bars follow immediately from the equilibrium conditions:

$$\stackrel{\curvearrowleft}{C}:\ aS_1 = aF \quad \leadsto \quad \underline{\underline{S_1 = F}}\,,$$

$$\stackrel{\curvearrowleft}{E}:\ aS_2 = 0 \quad \leadsto \quad \underline{\underline{S_2 = 0}}\,,$$

$$\uparrow:\ S_3 \sin 45^\circ + F = 0 \quad \leadsto \quad \underline{\underline{S_3 = -\sqrt{2}\,F}}\,.$$

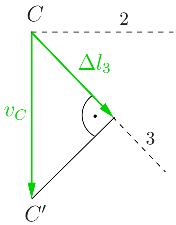


Related to these forces are the following elongations

$$\underline{\underline{\Delta l_1}} = \frac{S_1 l_1}{EA} = \underline{\underline{\frac{Fa}{EA}}}\,, \qquad \underline{\underline{\Delta l_2 = 0}}\,,$$

$$\underline{\underline{\Delta l_3}} = \frac{S_3 l_3}{EA} = -\frac{\sqrt{2}\,F \cdot \sqrt{2}\,a}{EA} = \underline{\underline{-2\frac{Fa}{EA}}}\,.$$

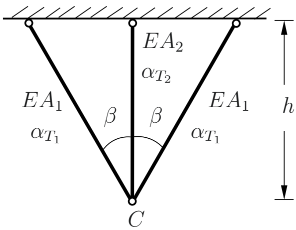
to b) The displacement of point C is sketched in the displacement diagram. As bar 2 experiences no force and thus no length change, the horizontal displacement vanishes. From the displacement diagram we obtain for the vertical displacement v_C:



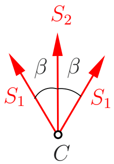
$$\underline{\underline{v_C}} = \sqrt{2}\,|\Delta l_3| = \underline{\underline{2\sqrt{2}\,\frac{Fa}{EA}}}\,.$$

P2.19 **Problem 2.19** In the depicted truss the members have the axial rigidities EA_1, EA_2 and the coefficients of thermal expansion α_{T1}, α_{T2}.

Determine the axial forces in the trusses, if the system is heated by ΔT?



Solution As the system is statically indeterminate, we have to use all basic equations. We start with the equilibrium



$$2S_1 \cos\beta + S_2 = 0$$

and continue with Hooke's law

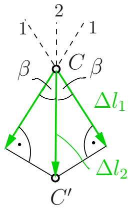
$$\Delta l_1 = \frac{S_1 l_1}{EA_1} + l_1 \alpha_{T1} \Delta T \ ,$$

$$\Delta l_2 = \frac{S_2 l_2}{EA_2} + l_2 \alpha_{T2} \Delta T \ ,$$

where

$$l_1 = \frac{h}{\cos\beta} \ , \qquad l_2 = h \, .$$

The compatibility of the displacements is according to the the displacement diagram:



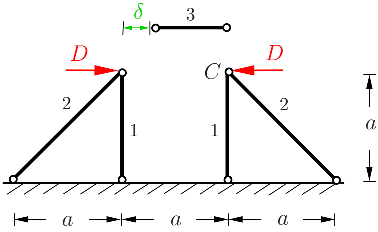
$$\Delta l_1 = \Delta l_2 \cos\beta \, .$$

Solving the 4 equations for the two truss forces and the two elongations yields

$$\underline{\underline{S_1 = EA_1 \frac{\alpha_{T2}\cos^2\beta - \alpha_{T1}}{1 + 2\cos^3\beta \dfrac{EA_1}{EA_2}} \Delta T}} \ , \qquad \underline{\underline{S_2 = -2\cos\beta \; S_1}} \, .$$

Note: For $\cos\beta = \sqrt{\alpha_{T1}/\alpha_{T2}}$ we obtain $S_1 = S_2 = 0$: the trusses can than expand without causing forces! (special case $\alpha_{T1} = \alpha_{T2} \rightsquigarrow \beta = 0$)

Problem 2.20 Truss member 3 was produced too short to be assembled between two identical trusses. **P2.20**



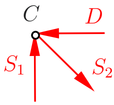
a) Determine the required assembly force D?
b) Calculate the normal force S_3 after the assembly ($D = 0$)?

Given: $EA_1 = EA_3 = EA$, $EA_2 = \sqrt{2}\,EA$.

Solution to a) The force D has to move point C by $\delta/2$ in horizontal direction during assembly. From equilibrium

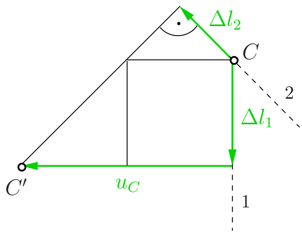
$$\rightarrow: \quad S_2 \cos 45^\circ = D\,,$$

$$\uparrow: \quad S_1 = S_2 \cos 45^\circ\,,$$



kinematics (S_1 was positively introduced as compressive force!!) with the prescribed displacement

$$u_C = \Delta l_1 + \Delta l_2 \sqrt{2}\,, \qquad u_C = \frac{\delta}{2}\,,$$



and Hooke's law

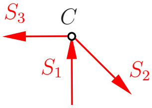
$$\Delta l_1 = \frac{S_1 a}{EA}\,, \qquad \Delta l_2 = \frac{S_2 a \sqrt{2}}{\sqrt{2}\,EA}$$

we obtain

$$\underline{\underline{D = \frac{1}{6}\,\frac{\delta}{a}\,EA\,.}}$$

to b) Equlibrium, kinematics and Hooke's law are as in a), but D has to be replaced by S_3. With the known compatibility condition

$$2u_C + \Delta l_3 = \delta \qquad \text{and} \qquad \Delta l_3 = \frac{S_3 a}{EA}$$



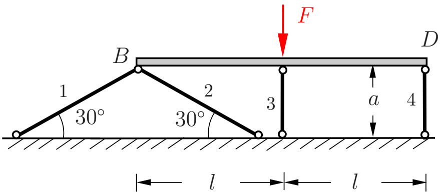
it follows

$$\underline{\underline{S_3 = \frac{1}{7}\,\frac{\delta}{a}\,EA\,.}}$$

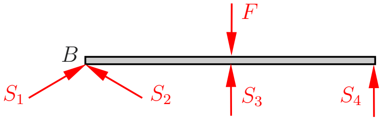
P2.21

Problem 2.21 A centric loaded *rigid* beam is supported by 4 elastic bars of equal axial rigidity EA.

Determine the forces in the bars?



Solution **a)** First, we solve the statically indeterminate system by applying all basic equations simultaneously. Using equilibrium



$$\rightarrow: \quad S_1 = S_2 \,,$$

$$\uparrow: \quad (S_1 + S_2)\sin 30^\circ + S_3 + S_4 = F \,,$$

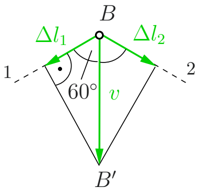
$$\overset{\curvearrowleft}{B}: \quad lS_3 + 2lS_4 = lF \,,$$

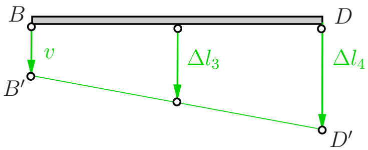
Hooke's laws

$$\Delta l_1 = \Delta l_2 = \frac{S_1 2a}{EA} \,,$$

$$\Delta l_3 = \frac{S_3 a}{EA} \,, \qquad \Delta l_4 = \frac{S_4 a}{EA}$$

and the geometry of the deformation



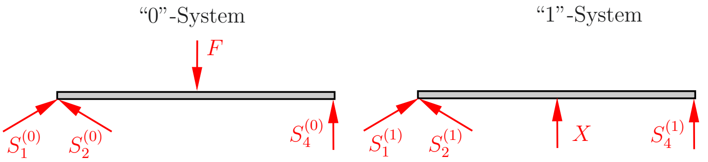


$$v = \frac{\Delta l_1}{\cos 60^\circ} \qquad\qquad \Delta l_3 = \frac{1}{2}(v + \Delta l_4)$$

we obtain as solution

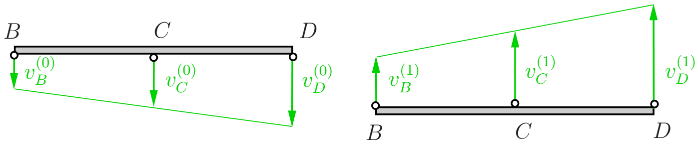
$$\underline{\underline{S_1 = S_2 = S_4 = \frac{2}{9}F \,,}} \qquad \underline{\underline{S_3 = \frac{5}{9}F \,.}}$$

b) Now, we solve the problem by superposition. The system is divided into two statically determinate basic systems:



Equilibrium yields

$$S_1^{(0)} = S_2^{(0)} = S_4^{(0)} = \frac{F}{2}\,, \qquad S_1^{(1)} = S_2^{(1)} = S_4^{(1)} = \frac{X}{2}\,.$$



From geometry and Hooke’s laws it follows

$$v_B^{(0)} = \frac{\Delta l_1^{(0)}}{\cos 60^\circ} = \frac{F\,2a}{EA}\,, \qquad v_B^{(1)} = \frac{X\,2a}{EA}\,,$$

$$v_D^{(0)} = \Delta l_4^{(0)} = \frac{Fa}{2EA}\,, \qquad v_D^{(1)} = \frac{Xa}{2EA}\,,$$

$$v_C^{(0)} = \frac{1}{2}\left(v_B^{(0)} + v_D^{(0)}\right) = \frac{5}{4}\frac{Fa}{EA}\,, \qquad v_C^{(1)} = \frac{5}{4}\frac{Xa}{EA}\,,$$

$$\Delta l_3^{(1)} = \frac{Xa}{EA}\,.$$

The kinematic compatibility requires the total displacement of point C to coincide with the shortening of truss 3:

$$v_C^{(0)} - v_C^{(1)} = \Delta l_3^{(1)}\,.$$

Inserting the displacements yields

$$\underline{\underline{X = S_3 = \frac{5}{9}\,F}}$$

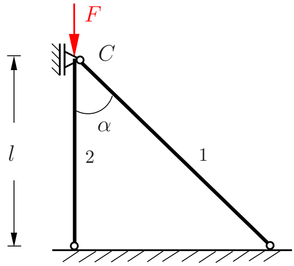
and

$$\underline{\underline{S_1}} = S_1^{(0)} - S_1^{(1)} = \underline{\underline{\frac{2}{9}\,F}}\,, \qquad \underline{\underline{S_4}} = S_4^{(0)} - S_4^{(1)} = \underline{\underline{\frac{2}{9}\,F}}\,.$$

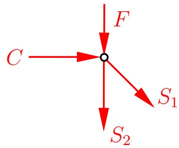
P2.22 **Problem 2.22** The depicted truss system (axial rigidity EA) is loaded by the external force F and additionally pinned at point C.

a) Determine the reaction force at point C.

b) Calculate the vertical displacement of point C.



Solution **to a)** Using equilibrium

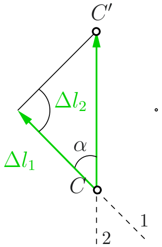


$$\downarrow: \quad F + S_2 + S_1 \cos\alpha = 0\,,$$

$$\rightarrow: \quad C + S_1 \sin\alpha = 0\,,$$

Hooke's laws

$$\Delta l_1 = \frac{S_1 l_1}{EA}\,, \qquad \Delta l_2 = \frac{S_2 l_2}{EA}\,,$$



and kinematics

$$\Delta l_1 = \Delta l_2 \cos\alpha$$

yields

$$\underline{\underline{C = \frac{\sin\alpha\cos^2\alpha}{1+\cos^3\alpha}\,F}}\,, \quad \underline{\underline{S_1 = -\frac{\cos^2\alpha}{1+\cos^3\alpha}\,F}}\,, \quad \underline{\underline{S_2 = -\frac{1}{1+\cos^3\alpha}\,F}}\,.$$

to b) Knowing S_2 the vertical displacement of point C follows as

$$\underline{\underline{v_C}} = \Delta l_2 = \frac{S_2 l}{EA} = \underline{\underline{-\frac{1}{1+\cos^3\alpha}\,\frac{Fl}{EA}}}\,.$$

In contrast to the displacement diagram, in which tensile forces (elongations) are assumed, compressive force occur in the system. Due to shortening point C moves in downwards direction.

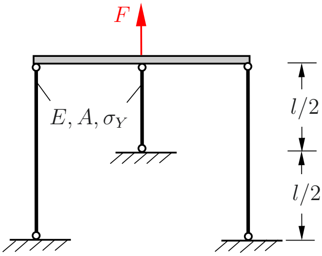
Test: $\alpha = \pi/2$ yields $S_1 = 0$ and $S_2 = -F$.

$\alpha = 0$ yields $S_1 = S_2 = -F/2$.

Problem 2.23 A *rigid* beam is supported by three bars of elastic-ideal-plastic material.

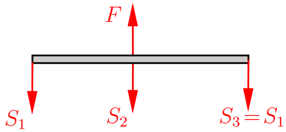
a) At what force F^{el}_{max} and at which location in the bars is the yield stress σ_Y reached at first?
b) At what force F^{pl}_{max} occurs plastic yielding in all bars of the system?

P2.23



Solution to a) The system is statically indeterminate. Using symmetry equilibrium provides

$$2S_1 + S_2 = F$$



Kinematics is expressed by

$$\Delta l_1 = \Delta l_2 \,.$$

Until plastic yielding Hooke's law can be used

$$\Delta l_1 = \frac{S_1 l}{EA}\,, \qquad \Delta l_2 = \frac{S_2 l}{2EA}\,.$$

The solution provides forces and stresses in the bars

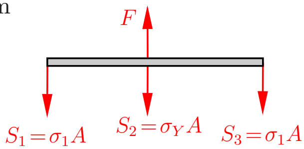
$$S_1 = \frac{F}{4}\,, \qquad S_2 = \frac{F}{2} \qquad \leadsto \qquad \sigma_1 = \frac{F}{4A}\,, \qquad \sigma_2 = \frac{F}{2A}\,.$$

As the stress in bar 2 is the highest, the yield limit is reached first there during load increase:

$$\sigma_2 = \sigma_Y \qquad \leadsto \qquad \underline{\underline{F^{el}_{max} = 2\,\sigma_Y A}}\,.$$

to b) For a load increase above F^{el}_{max} bar 1 and bar 3 still respond elastically, while bar two undergoes plastic deformation: $\sigma_2 = \sigma_Y$. Thus with $S_i = \sigma_i A$ it follows from equilibrium

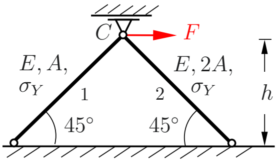
$$2\sigma_1 A + \sigma_Y A = F$$
$$\leadsto \quad \sigma_1 = \frac{F}{2A} - \frac{\sigma_Y}{2}\,.$$



All bars are undego plastic deformation if

$$\sigma_1 = \sigma_Y \quad \leadsto \quad \frac{F}{2A} - \frac{\sigma_Y}{2} = \sigma_Y \qquad \leadsto \qquad \underline{\underline{F^{pl}_{max} = 3\,\sigma_Y A}}\,.$$

P2.24 **Problem 2.24** In the depicted symmetric system all bars are made of the same elastic-ideal-plastic material, but have different cross sections.



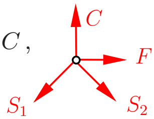
a) At what force F^{el}_{max} and at which location in the bars is the yield stress σ_Y reached at first? Determine the reaction force at C for this situation.

b) Determine the force F^{pl}_{max} when both bars deform plastically?

c) Calculate the displacement u^{el}_{max} of point C for case a)?

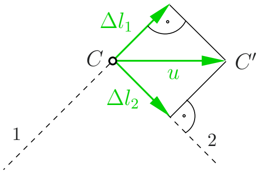
Solution **to a)** Until reaching the force F^{el}_{max} the system responds elastically. Therefore the equilibrium conditions are given by

$$\rightarrow: \ \frac{\sqrt{2}}{2} S_1 - \frac{\sqrt{2}}{2} S_2 = F, \qquad \uparrow: \ \frac{\sqrt{2}}{2} S_1 + \frac{\sqrt{2}}{2} S_2 = C,$$



together with Hooke's law

$$\Delta l_1 = \frac{S_1 \sqrt{2}\, h}{EA}, \qquad \Delta l_2 = \frac{S_2 \sqrt{2}\, h}{2EA}$$



and the kinematics (bar 2 will shorten)

$$\Delta l_1 = -\Delta l_2 .$$

From the above relation we obtain

$$S_1 = \frac{\sqrt{2}}{3} F, \quad S_2 = -\frac{2\sqrt{2}}{3} F, \qquad C = -\frac{F}{3}, \quad \Delta l_1 = -\Delta l_2 = \frac{2Fh}{3EA}$$

$$\rightsquigarrow \qquad \sigma_1 = \frac{S_1}{A} = \frac{\sqrt{2}}{3}\frac{F}{A}, \qquad \sigma_2 = \frac{S_2}{2A} = -\frac{\sqrt{2}}{3}\frac{F}{A}.$$

The absolute value of the stresses is identical in both bars. Yielding will occur if

$$\sigma_1 = |\sigma_2| = \sigma_Y \quad \rightsquigarrow \quad \underline{\underline{F^{el}_{max} = \frac{3}{2}\sqrt{2}\,\sigma_Y A}}, \quad \rightsquigarrow \quad \underline{\underline{C^{el}_{max} = -\frac{\sqrt{2}}{2}\sigma_Y A.}}$$

to b) As at F^{el}_{max} plastic yielding occurs in both bars, we have

$$\underline{\underline{F^{el}_{max} = F^{pl}_{max}}}.$$

to c) Until the yield limit is reached the displacement of C is given by

$$u = \sqrt{2}\,\Delta l_1 = \frac{2\sqrt{2}}{3}\frac{Fh}{EA}, \qquad \rightsquigarrow \qquad \underline{\underline{u^{el}_{max}}} = u(F^{el}_{max}) = \underline{\underline{2\,\frac{\sigma_Y}{E}\,h}}.$$