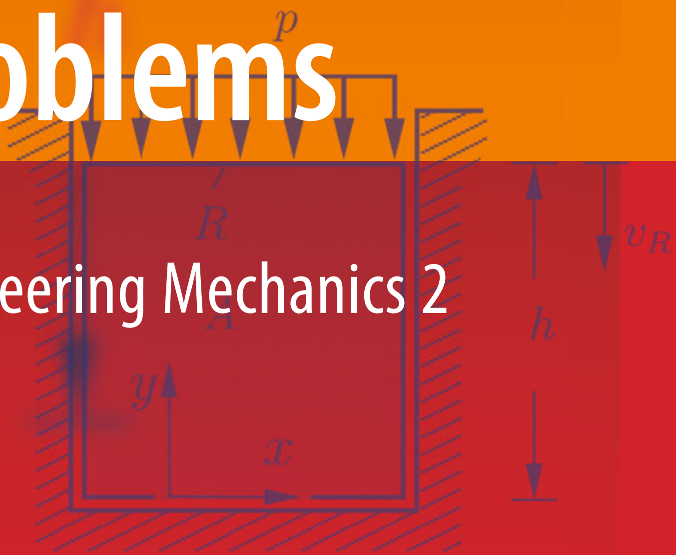
Dietmar Gross · Wolfgang Ehlers
Peter Wriggers · Jörg Schröder
Ralf Müller

Mechanics of Materials – Formulas and Problems



Engineering Mechanics 2

Springer

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ISBN 978-3-662-53879-1 ISBN 978-3-662-53880-7 (eBook)
DOI 10.1007/978-3-662-53880-7

Library of Congress Control Number: 2016956827

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer-Verlag GmbH Germany
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface

This collection of problems results from the demand of students for supplementary problems and support in the preparation for examinations. With the present collection 'Engineering Mechanics 2 - Formulas and Problems, Mechanics of Materials' we provide more additional exercise material.

The subject 'Mechanics of Materials' is commonly taught in the second course of Engineering Mechanics classes at universities. The problems analyzed within these courses use equilibrium conditions and kinematic relations in conjunction with constitutive relations. As we want concentrate more on basic concepts and solution procedures the focus lies on linear elastic material behavior and the small strain regime. However, this covers a wide range of elasto-static problems with relevancy in engineering applications. Special attention is given to structural elements like bars, beams and shafts as well as plane stress and strain situations.

Following the warning in the first collection, we would like to make the reader aware that pure reading and trying to comprehend the presented solutions will not provide a deeper understanding of mechanics. Neither does it improve the problem solving skills. Using this collection wisely, one has to try to solve the problems independently. The proposed solution should only be considered when experiencing major problems in solving an exercise.

Obviously this collection cannot substitute a full-scale textbook. If not familiar with the formulae, explanations, or technical terms the reader has to consider his or her course material or additional textbooks on mechanics of materials. An incomplete list is provided on page IX.

Darmstadt, Hannover, Stuttgart, Essen and
Kaiserslautern, Summer 2016

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Literature

Textbooks

Gross, D., Hauger, W., Schröder, J., Wall, W., Bonet, J., Engineering Mechanics 2: Mechanics of Materials, 1st edition, Springer 2011

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Notation

The following symbols are used in the solutions to the problems:

$\uparrow$: short notation for *sum of all forces in the direction of the arrow equals zero.*

$\overset{\curvearrowleft}{A}$: short notation for *sum of all moments with reference to point A equals zero.*

$\leadsto$ short notation for *it follows.*

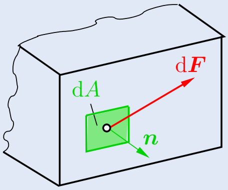
Chapter 1

1

Stress, Strain, Hooke's Law

1.1

1.1 Stress, Equilibrium conditions



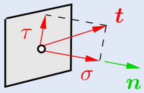
Stress is related to forces distributed over the area of a cross section. The *stress vector* $\boldsymbol{t}$ is defined as

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}A} \,,$$

where $\mathrm{d}\boldsymbol{F}$ is the force acting on the area element $\mathrm{d}A$ (unit: $1\ \mathrm{Pa} = 1\ \mathrm{N/m^2}$).

Note: The stress vector and its components depend on the orientation of the area element (with its normal $\boldsymbol{n}$).

Components of the stress vector:

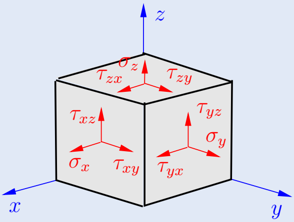


σ – normal stress (perpendicular to the plane)

τ – shear stress (in plane)

Sign convention: Positive stresses at a positive (negative) face point in positive (negative) coordinate directions.

Spatial stress state: is uniquely defined by the components of the stress vectors in three mutually perpendicular sections. The stress components are the components of the stress tensor



$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

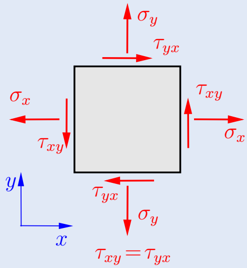
Equilibrium of moments yields the following relations

$$\tau_{xy} = \tau_{yx} \,, \qquad \tau_{xz} = \tau_{zx} \,, \qquad \tau_{yz} = \tau_{zy} \,.$$

Hence the stress tensor is a *symmetric tensor of second order*: $\tau_{ij} = \tau_{ji}$.

Plane stress state: is uniquely defined by the stress components of two mutually perpendicular sections. The stress components in the third direction (here z-direction) vanish ($\sigma_z = \tau_{yz} = \tau_{xz} = 0$)

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} .$$

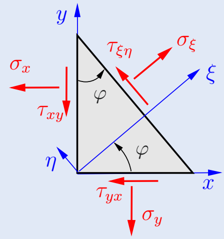


Coordinate transformation

$$\sigma_\xi = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi ,$$

$$\sigma_\eta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi ,$$

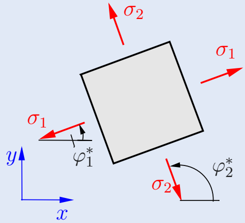
$$\tau_{\xi\eta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi .$$



Principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

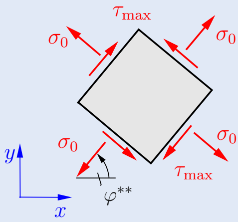
$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



Note:
- The shear stresses vanish in these directions!
- The principal directions are perpendicular to each other: $\varphi_2^* = \varphi_1^* \pm \pi/2$.

Maximum shear stresses

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\,, \qquad \varphi^{**} = \varphi^* \pm \frac{\pi}{4} .$$



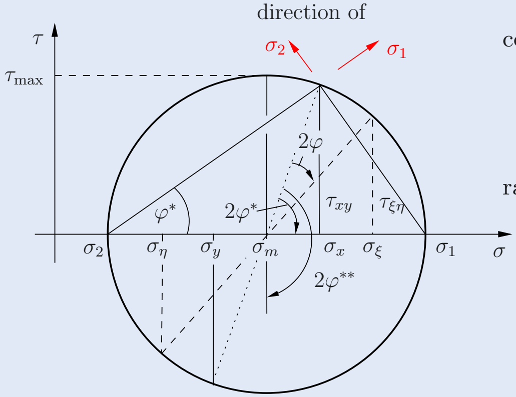
In these sections the normal stresses reach the value $\sigma_0 = (\sigma_x + \sigma_y)/2$.

Invariants

$$I_\sigma = \sigma_x + \sigma_y = \sigma_\xi + \sigma_\eta = \sigma_1 + \sigma_2 ,$$

$$II_\sigma = \sigma_x\sigma_y - \tau_{xy}^2 = \sigma_\xi\sigma_\eta - \tau_{\xi\eta}^2 = \sigma_1\sigma_2 .$$

Mohr's circle



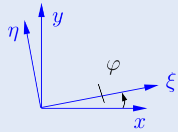
center:

$$\sigma_m = \tfrac{1}{2}(\sigma_x + \sigma_y)\,,$$

$$\tau = 0$$

radius:

$$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



- The construction of Mohr's circle is always possible, provided three independent quantities are known (e. g. σ_x, σ_y, τ_{xy} or σ_x, σ_y, φ^*).
- The shear stress τ_{xy} is plotted over σ_x ($\tau_{\xi\eta}$ over σ_ξ).
- The angle of transformation φ is doubled in the circle (2φ) and oriented in opposite direction.

Equilibrium conditions

in space (3D)

$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0\,,\\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0\,,\\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0\,,
\end{aligned}\right\}\ \mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}\,.$$

in plane (2D)

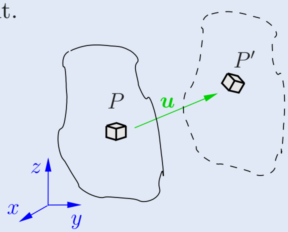
$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0\,,\\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0\,,
\end{aligned}\right\}\ \mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}\,.$$

where

$$\mathrm{div}\boldsymbol{\sigma} = \sum_i \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z}\right)\boldsymbol{e}_i\,.$$

1.2 Strain

The *strains* describe changes in the edge lengths (stretching) and in the angles (shearing) of a cubic volume element.



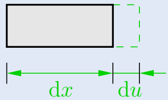
Displacement vector

$$\boldsymbol{u} = u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z$$

u, v, w = displacement components

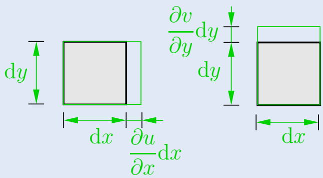
Uniaxial strain state

strain $\boxed{\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}}$

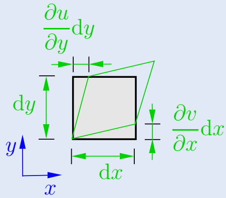


Biaxial strain state

normal strains



shear strains



$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}\,, \qquad \varepsilon_y = \frac{\partial v}{\partial y}\,, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,.}$$

Triaxial strain state $\varepsilon_x = \dfrac{\partial u}{\partial x}, \ \varepsilon_y = \dfrac{\partial v}{\partial y}, \ \varepsilon_z = \dfrac{\partial w}{\partial z},$

strain tensor: $$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{pmatrix}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,,$$
$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\,,$$
$$\gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\,.$$

Remark:

- The strains are, like the stresses, components of a *symmetric tensor of second order*. Thus all properties (coordinate transformation, principal values etc.) of the stress tensor can be used analogously. $\sigma_x \to \varepsilon_x$, $\tau_{xy} \to \gamma_{xy}/2, \ldots$
- In a *plane strain state* the following holds: $\varepsilon_z = 0, \ \gamma_{xz} = 0, \ \gamma_{yz} = 0.$

1.3
1.3 Hooke's law

Hooke's law describes the experimentally observed linear relation between stresses and strains. The validity of Hooke's law is restricted by the *proportionality limit* (uniaxial σ_p). In elastic-plastic materials this limit frequently conincides with the *yield limit* (uniaxial σ_Y).

Uniaxial stress state (bar, beam)

$$\boxed{\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T\ .}$$

E	–	Young's modulus,
α_T	–	coefficient of thermal expansion,
ΔT	–	temperature change.

Plane stress state

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha_T \Delta T\ ,\\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha_T \Delta T\ ,\\ \gamma_{xy} &= \frac{1}{G}\tau_{xy}\quad ,\end{aligned}$$

$$\text{shear modulus:}\quad G = \frac{E}{2(1+\nu)}\ ,\qquad \text{Poisson's ratio}:\quad \nu\ .$$

Triaxial stress state

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha_T \Delta T\ , & \gamma_{xy} &= \frac{1}{G}\tau_{xy}\ ,\\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha_T \Delta T\ , & \gamma_{yz} &= \frac{1}{G}\tau_{yz}\ ,\\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha_T \Delta T\ , & \gamma_{zx} &= \frac{1}{G}\tau_{zx}\ .\end{aligned}$$

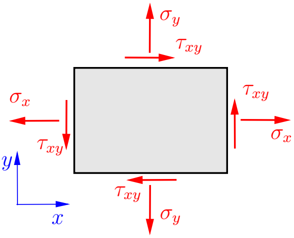
Selected material data

material	E [MPa]	ν	α_T $[1/^\circ C]$
steel	$2,1 \cdot 10^5$	$0,3$	$12 \cdot 10^{-6}$
aluminium	$0,7 \cdot 10^5$	$0,3$	$23 \cdot 10^{-6}$
copper	$1,2 \cdot 10^5$	$0,3$	$16 \cdot 10^{-6}$
concrete	$0,3 \cdot 10^5$	$0,15 \dots 0,3$	$10 \cdot 10^{-6}$
wood	$0,1 \cdot 10^5$		$3 \dots 9 \cdot 10^{-6}$

Remark: $1\text{MPa} = 10^3\text{kPa} = 10^6\text{Pa}\,,\ 1\text{Pa} = 1\text{N/m}^2$

Problem 1.1 In a thin metal sheet the stresses σ_x, σ_y, τ_{xy} are given. Determine value and direction of the principal stresses. **P1.1**

Given: $\sigma_x = 20$ MPa, $\sigma_y = 30$ MPa,
$\tau_{xy} = 10$ MPa.



Solution We start with the analytical method. The principal stresses are computed by

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 25 \pm \sqrt{25 + 100} = 25 \pm 11.18$$

leading to

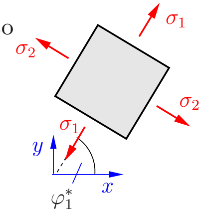
$$\underline{\underline{\sigma_1 = 36.18 \text{ MPa}}}\,, \qquad \underline{\underline{\sigma_2 = 13.82 \text{ MPa}}}\,.$$

For the principal directions, we obtain according to

$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -2$$

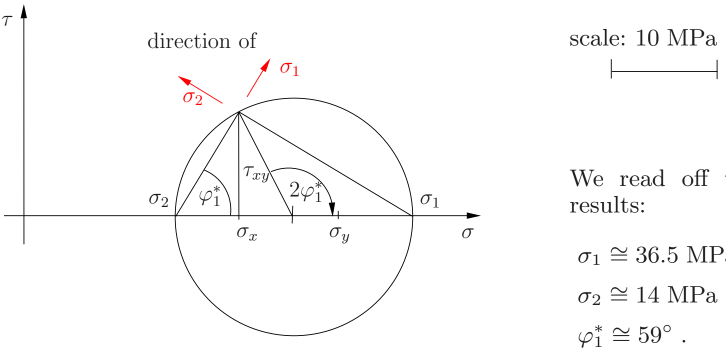
the results

$$\underline{\underline{\varphi_1^* = 58.28^\circ}}\,, \qquad \underline{\underline{\varphi_2^* = 148.28^\circ}}.$$



To illustrate the results an element loaded by the principal stresses is sketched.

We can also solve the problem graphically by using Mohr's circle:



We read off the results:

$$\sigma_1 \cong 36.5 \text{ MPa}\,,$$

$$\sigma_2 \cong 14 \text{ MPa}\,,$$

$$\varphi_1^* \cong 59^\circ\,.$$

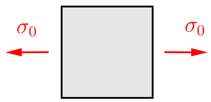
P1.2 **Problem 1.2** Determine the stress components, the principal stresses, and the principal directions, as well as the maximum shear stress in any cross section for the given special cases of plane stress states :

a) $\sigma_x = \sigma_0,\ \ \sigma_y = 0,\ \ \tau_{xy} = 0$ (uniaxial tension),

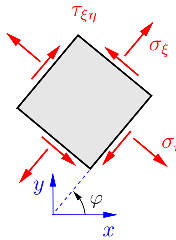
b) $\sigma_x = \sigma_y = \sigma_0,\ \ \tau_{xy} = 0$ (biaxial, equal tension),

c) $\sigma_x = \sigma_y = 0,\ \ \tau_{xy} = \tau_0$ (pure shear).

Solution **to a)** The stress components are obtained for any cross section which has the angle φ to the x- and y-direction by inserting σ_x, σ_y and τ_{xy} into the transformation relations

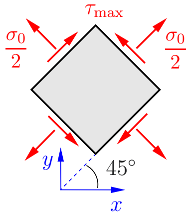


$$\begin{aligned}\underline{\underline{\sigma_\xi}} &= \tfrac{1}{2}\,(\sigma_0+0) + \tfrac{1}{2}\,(\sigma_0-0)\cos 2\varphi + 0\cdot\sin 2\varphi\\ &= \underline{\underline{\tfrac{1}{2}\sigma_0(1+\cos 2\varphi)}}\ ,\\ \underline{\underline{\sigma_\eta}} &= \tfrac{1}{2}\,(\sigma_0+0) - \tfrac{1}{2}\,(\sigma_0-0)\cos 2\varphi - 0\cdot\sin 2\varphi\\ &= \underline{\underline{\tfrac{1}{2}\sigma_0(1-\cos 2\varphi)}}\ ,\\ \underline{\underline{\tau_{\xi\eta}}} &= -\tfrac{1}{2}\,(\sigma_0-0)\sin 2\varphi + 0\cdot\cos 2\varphi\\ &= \underline{\underline{\tfrac{1}{2}\sigma_0\sin 2\varphi}}\ .\end{aligned}$$



Due to $\tau_{xy} = 0$ the stresses σ_x, σ_y are principal stresses, and the x- as well as y-direction are principal directions:

$$\underline{\underline{\sigma_1}} = \sigma_x = \underline{\underline{\sigma_0}},\ \ \underline{\underline{\sigma_2}} = \sigma_y = \underline{\underline{0}},\qquad \varphi_1^* = 0,\ \ \varphi_2^* = \pm\frac{\pi}{2}.$$



The maximum shear stress and the corresponding direction is determined by the following relations

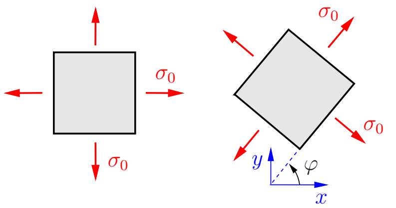
$$\underline{\underline{\tau_{\max}}} = \frac{1}{2}|\sigma_1 - \sigma_2| = \underline{\underline{\frac{1}{2}\sigma_0}}\ ,\qquad \varphi^{**} = \pm\frac{\pi}{4}\ .$$

Remark: A plate made from a material that supports only limited *shear-stresses* will fail along lines under an angle of $\pm 45°$ to the x-axis.

to b) Inserting the given values into the coordinate transformation yields

$$\underline{\underline{\sigma_\xi = \sigma_0}}\ ,\qquad \underline{\underline{\sigma_\eta = \sigma_0}}\ ,\qquad \underline{\underline{\tau_{\xi\eta} = 0}}\ .$$

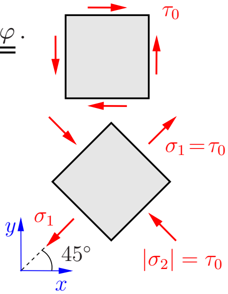
Therefore the normal stress σ_0 is acting in any section, and the shear stress vanishes. There is no distinguished principal direction, any section is a principal direction:



$$\underline{\underline{\sigma_1 = \sigma_2 = \sigma_0}} \,.$$

to c) In this case the coordinate transformation yields

$$\underline{\underline{\sigma_\xi = \tau_0 \sin 2\varphi}} \,, \quad \underline{\underline{\sigma_\eta = -\tau_0 \sin 2\varphi}} \,, \quad \underline{\underline{\tau_{\xi\eta} = \tau_0 \cos 2\varphi}} \,.$$



The principal stresses and directions are

$$\underline{\underline{\sigma_1 = +\tau_0}} \,, \quad \underline{\underline{\sigma_2 = -\tau_0}} \,, \quad \varphi_1^* = \frac{\pi}{4} \,, \quad \varphi_2^* = -\frac{\pi}{4} \,.$$

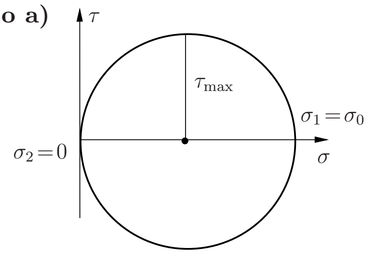
For the maximum shear stress and the corresponding directions we obtain

$$\underline{\underline{\tau_{\max} = \tau_0}} \,, \qquad \varphi_1^{**} = 0 \,, \qquad \varphi_2^{**} = \pi/2 \,.$$

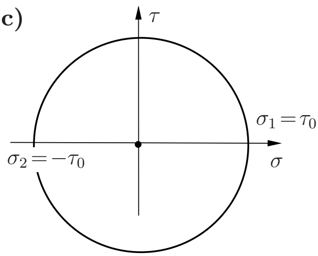
Remark: A plate made from a material that supports limited *normal stresses* will fail along lines under an angle of $\pm 45°$ to the x-axis.

The results of all three stress states can be illustrated by the corresponding Mohr's circles:

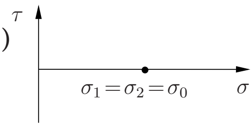
to a)



to c)



to b)

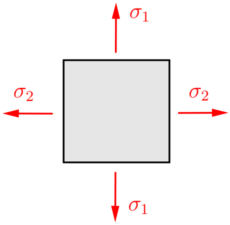


Note: In case **b)** Mohr's circle degenerates to a single point along the σ-axis!

P1.3

Problem 1.3 In a plane section the following principal stresses are present

$$\sigma_1 = 96 \text{ MPa} \quad \text{and} \quad \sigma_2 = -52 \text{ MPa} .$$

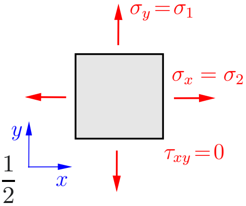


a) Determine the stresses in sections which are inclined by $\varphi^a = 60°$ with regard to the principal axes?

b) In which section φ^b does the normal stress vanish? What are the values of the shear and normal stresses in a direction perpendicular to the direction φ^b?

c) In which directions do the maximal shear stresses appear, and what are the corresponding normal stresses?

Solution **to a)** According to the sketch we use a coordinate system x, y that coincides with the principal axes. The stresses in the cross sections inclined by $\varphi^a = 60°$ follow from the coordinate transformation



$$\sigma_\xi^a = \frac{\sigma_2 + \sigma_1}{2} + \frac{\sigma_2 - \sigma_1}{2} \cos 2\varphi^a = 22 + 74 \cdot \frac{1}{2}$$

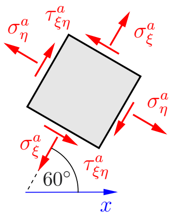
$$= \underline{\underline{59 \text{ MPa}}} ,$$

$$\sigma_\eta^a = \frac{\sigma_2 + \sigma_1}{2} - \frac{\sigma_2 - \sigma_1}{2} \cos 2\varphi^a = 22 - 74 \cdot \frac{1}{2}$$

$$= \underline{\underline{-15 \text{ MPa}}} ,$$

$$\tau_{\xi\eta}^a = -\frac{\sigma_2 - \sigma_1}{2} \sin 2\varphi^a = 74 \cdot \frac{1}{2}\sqrt{3}$$

$$= \underline{\underline{64.1 \text{ MPa}}} .$$



to b) For the normal stress σ_ξ to vanish the following must hold

$$\sigma_\xi^b = \frac{\sigma_2 + \sigma_1}{2} + \frac{\sigma_2 - \sigma_1}{2} \cos 2\varphi^b = 0$$

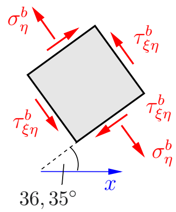
$$\rightsquigarrow \quad \cos 2\varphi^b = \frac{22}{74} = 0.297 \quad \rightsquigarrow \quad 2\varphi^b = 72.7° \quad \rightsquigarrow \quad \underline{\underline{\varphi^b = 36.35°}} .$$

For σ_η^b and $\tau_{\xi\eta}^b$ we obtain

$$\underline{\underline{\sigma_\eta^b}} = \frac{\sigma_2+\sigma_1}{2} - \frac{\sigma_2-\sigma_1}{2}\cos 2\varphi^b = \underline{\underline{44\text{ MPa}}}\,,$$

$$\underline{\underline{\tau_{\xi\eta}^b}} = -\frac{\sigma_2-\sigma_1}{2}\sin 2\varphi^b = 74\cdot 0.955$$

$$= \underline{\underline{70.7\text{ MPa}}}\,.$$



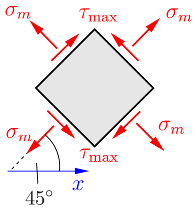
to c) The maximum shear stress occurs in directions of $\pm 45°$ with regard to the principal axes. This results in

$$\underline{\underline{\tau_{\max}}} = \frac{\sigma_1-\sigma_2}{2} = \underline{\underline{74\text{ MPa}}}\,.$$

The corresponding normal stresses are

$$\underline{\underline{\sigma_m}} = \frac{\sigma_1+\sigma_2}{2} = \underline{\underline{22\text{ MPa}}}$$

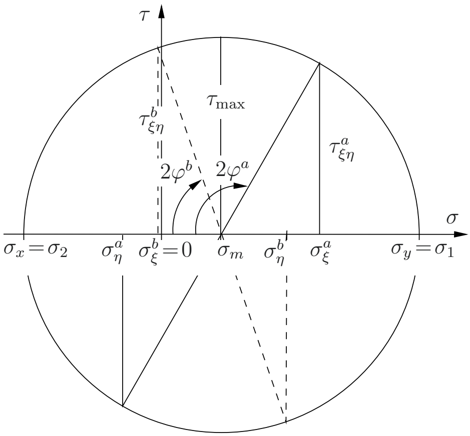
for the given data.



All informations can be illustrated by use of Mohr's circle for the given stress state

scale: 50 MPa

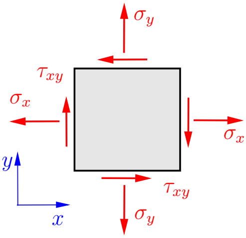
$$\sigma_\xi^a \cong 59\text{ MPa}\,,$$
$$\sigma_\eta^a \cong -15\text{ MPa}\,,$$
$$\tau_{\xi\eta}^a \cong 64\text{ MPa}\,,$$
$$\varphi^b \cong 37°\,,$$
$$\sigma_\eta^b \cong 44\text{ MPa}\,,$$
$$\tau_{\xi\eta}^b \cong 71\text{ MPa}\,,$$
$$\tau_{\max} \cong 74\text{ MPa}\,,$$
$$\sigma_m \cong 22\text{ MPa}\,.$$



P1.4

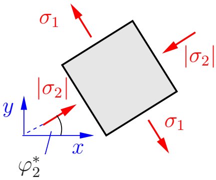
Problem 1.4 The following stresses are acting in a panel.
$\sigma_x = 20$ MPa, $\sigma_y = 60$ MPa and $\tau_{xy} = -40$ MPa.

Determine analytically and graphically the principal stresses, the maximum shear stress, and the corresponding directions. Sketch the related sections.



Solution The principal stresses and their directions are calculated analytically by

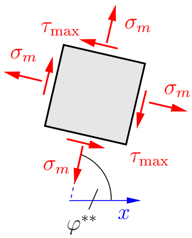
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= 40 \pm \sqrt{(20)^2 + (40)^2} \ ,$$



$$\leadsto \quad \underline{\underline{\sigma_1 = 84.72 \text{ MPa}}} \ , \quad \underline{\underline{\sigma_2 = -4.72 \text{ MPa}}} \ ,$$

$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 2 \quad \leadsto \quad \underline{\underline{\varphi_1^* = 121.7^\circ}} \ , \quad \underline{\underline{\varphi_2^* = 31.7^\circ}} \ .$$

To determine which principal stress is associated with which direction, the transformation relations or Mohr's circle has to be used.
For the maximum stress the following result is obtained



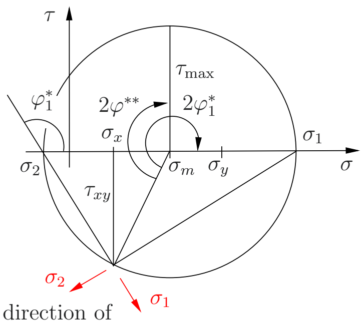
$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \underline{\underline{44.72 \text{ MPa}}} \ ,$$

$$\underline{\underline{\varphi^{**}}} = \varphi^* \pm 45^\circ = \underline{\underline{31.7^\circ \pm 45^\circ}} \ .$$

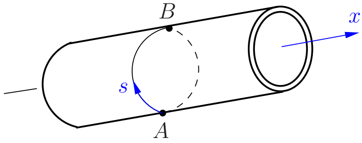
The graphic solution by Mohr's circle is sketched below:

scale: 20 MPa

$$\sigma_1 \cong 85 \text{ MPa} \ ,$$
$$\sigma_2 \cong -5 \text{ MPa} \ ,$$
$$\tau_{\max} \cong 45 \text{ MPa} \ ,$$
$$\varphi_1^* \cong 122^\circ \ ,$$
$$\varphi^{**} \cong 77^\circ \ .$$



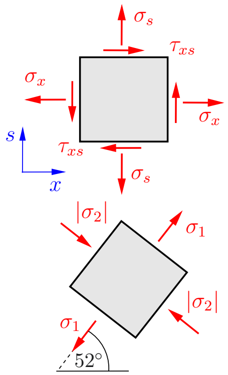
Problem 1.5 A thin-walled tube is loaded by a bending moment, an internal pressure, and a torsional moment. In points A and B the following stresses occur due to the loading: **P1.5**



$$\sigma_x^{A,B} = \pm 50 \text{ MPa}\,, \quad \sigma_s^{A,B} = 100 \text{ MPa}\,, \quad \tau_{xs}^{A,B} = 100 \text{ MPa}\,.$$

Determine value and direction of the principal stresses in the points A and B.

Solution For point A the principal stresses are computed by



$$\begin{aligned}\sigma_{1,2} &= \tfrac{1}{2}(\sigma_x + \sigma_s) \pm \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_s)]^2 + \tau_{xs}^2}\\ &= 75 \pm \sqrt{(-25)^2 + 100^2}\\ &= 75 \pm 103.08\end{aligned}$$

yielding

$$\underline{\underline{\sigma_1 = 178.08 \text{ MPa}}}\,, \qquad \underline{\underline{\sigma_2 = -28.08 \text{ MPa}}}\,.$$

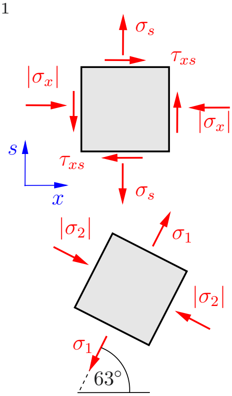
For the principal directions we obtain

$$\tan 2\varphi^* = \frac{2\tau_{xs}}{\sigma_x - \sigma_s} = \frac{2\cdot 100}{50-100} = -4 \quad \rightsquigarrow \quad \underline{\underline{\varphi_1^* = 52.02^\circ}}\,, \quad \underline{\underline{\varphi_2^* = -37.98^\circ}}\,.$$

From the coordinate transformation it is obvious that direction φ_1^* is associated with the principal stress σ_1:

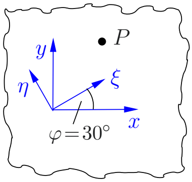
$$\begin{aligned}\sigma_\xi &= \tfrac{1}{2}(\sigma_x + \sigma_s) + \tfrac{1}{2}(\sigma_x - \sigma_s)\cos 2\varphi_1^* + \tau_{xs}\sin 2\varphi_1^*\\ &= 75 - 25 \cdot (-0.242) + 100 \cdot 0.970\\ &= 178.08 \text{ MPa} = \sigma_1\,.\end{aligned}$$

In an analogous way the principal stresses and their directions in point B are obtained:



$$\begin{aligned}\sigma_{1,2} &= 25 \pm \sqrt{(-75)^2 + 100^2}\\ &= 25 \pm 125\\ \rightsquigarrow \quad & \underline{\underline{\sigma_1 = 150 \text{ MPa}}}\,, \qquad \underline{\underline{\sigma_2 = -100 \text{ MPa}}}\,.\\ & \tan 2\varphi^* = \frac{2\cdot 100}{-50-100} = -1.33\\ \rightsquigarrow \quad & \underline{\underline{\varphi_1^* = 63.4^\circ}}\,, \qquad \underline{\underline{\varphi_2^* = -26.6^\circ}}\,.\end{aligned}$$

P1.6 **Problem 1.6** In a thin aluminium sheet ($E = 0.7 \cdot 10^5$ MPa , $\nu = 0.3$) the following strains $\varepsilon_x = 0.001$, $\varepsilon_y = 0.0005$, $\gamma_{xy} = 0$ are experimentally measured in point P.



What are the principal stresses, the maximum shear stress, and the stresses in a sections, that are inclined by $\varphi = 30°$ with regard to the principal directions?

Solution In the aluminium sheet a state of plane stress is present. From Hooke's law

$$E\varepsilon_x = \sigma_x - \nu\sigma_y\,, \qquad E\varepsilon_y = \sigma_y - \nu\sigma_x\,, \qquad G\gamma_{xy} = \tau_{xy}$$

the following stresses can be computed

$$\underline{\underline{\sigma_x}} = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{0.7 \cdot 10^5}{1 - 0.09}(0.001 + 0.00015) = \underline{\underline{88.5 \text{ MPa}}}\,,$$

$$\underline{\underline{\sigma_y}} = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = \frac{0.7 \cdot 10^5}{1 - 0.09}(0.0005 + 0.0003) = \underline{\underline{61.5 \text{ MPa}}}\,,$$

$$\underline{\underline{\tau_{xy} = 0}}\,.$$

As the shear stress τ_{xy} is equal to zero, σ_x, σ_y are principal stresses, and axes x, y are principal axes:

$$\sigma_x = \sigma_1 \qquad \sigma_y = \sigma_2\,.$$

Therefore, the maximum shear stress is

$$\underline{\underline{\tau_{\max}}} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(\sigma_x - \sigma_y) = \underline{\underline{13.5 \text{ MPa}}}\,.$$

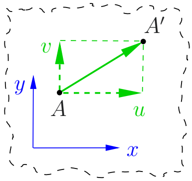
For the sections inclined by $\varphi = 30°$, the stresses follow with $\tau_{xy} = 0$ from the transformation relations

$$\underline{\underline{\sigma_\xi}} = \frac{\sigma_x + \sigma_x}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi = 75 + 13.5\cos 60° = \underline{\underline{81.75 \text{ MPa}}}\,,$$

$$\underline{\underline{\sigma_\eta}} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi = 75 - 13.5\cos 60° = \underline{\underline{68.25 \text{ MPa}}}\,,$$

$$\underline{\underline{\tau_{\xi\eta}}} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\varphi = -13.5\sin 60° = \underline{\underline{-11.69 \text{ MPa}}}\,.$$

Problem 1.7 In a thin sheet the following plane displacement field was obtained by measurements: **P1.7**



$$u(x,y) = 3.5 \cdot 10^{-3} x + 2 \cdot 10^{-3} y\,,$$
$$v(x,y) = 1 \cdot 10^{-3} x - 0.5 \cdot 10^{-3} y\,.$$

a) Determine the state of strain.

b) What are principal strains, and under which angle to the x-axis do they appear?

c) What is the maximum shear strain $\gamma_{\max}$?

Solution **to a)** The strains are computed by differentiation of the displacement components:

$$\underline{\underline{\varepsilon_x}} = \frac{\partial u}{\partial x} = \underline{\underline{3.5 \cdot 10^{-3}}}\,, \qquad \underline{\underline{\varepsilon_y}} = \frac{\partial v}{\partial y} = \underline{\underline{-0.5 \cdot 10^{-3}}}\,,$$
$$\underline{\underline{\gamma_{xy}}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2 \cdot 10^{-3} + 1 \cdot 10^{-3} = \underline{\underline{3 \cdot 10^{-3}}}\,.$$

The strains are constant in the entire sheet (=homogeneous strain state).

to b) The principal strains and their corresponding directions are calculated from the relations for the principal stresses by using the replacements ($\sigma_x \to \varepsilon_x$, $\tau_{xy} \to \gamma_{xy}/2$ etc.). This yields the principal strains

$$\begin{aligned}\varepsilon_{1,2} &= \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 1.5 \cdot 10^{-3} \pm \sqrt{(2 \cdot 10^{-3})^2 + (1.5 \cdot 10^{-3})^2} = 1.5 \cdot 10^{-3} \pm 2.5 \cdot 10^{-3}\end{aligned}$$

$$\leadsto \quad \underline{\underline{\varepsilon_1 = 4 \cdot 10^{-3}}}\,, \qquad \underline{\underline{\varepsilon_2 = -1 \cdot 10^{-3}}}\,,$$

and the principal directions

$$\tan 2\varphi^* = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{3}{4} \quad \leadsto \quad \underline{\underline{\varphi_1^* = 18.4^\circ}}\,, \quad \underline{\underline{\varphi_2^* = 108.4^\circ}}\,.$$

to c) The maximum shear strain is

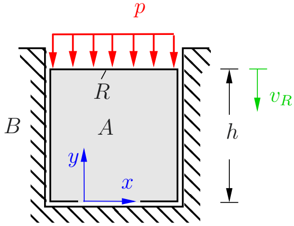
$$\underline{\underline{\gamma_{\max}}} = \varepsilon_1 - \varepsilon_2 = 4 \cdot 10^{-3} + 1 \cdot 10^{-3} = \underline{\underline{5 \cdot 10^{-3}}}\,.$$

It occurs at angles, which are inclined by $\pm 45^\circ$ with regard to the principal directions.

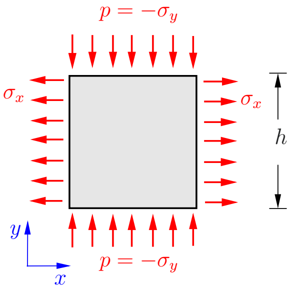
P1.8

Problem 1.8 An elastic panel A fits into the rigid socket B of height h (Young's modulus E, Poisson's ratio $\nu > 0$).

Determine the stress σ_x and the value of the displacement v_R at the top edge R for a constant pressure p. It is assumed that the elastic panel can move frictionless in the socket mounting.



Solution In the panel a uniform plane stress state is present, where the stress component σ_y is known: $\sigma_y = -p$. Thus Hooke's law yields



$$E\varepsilon_x = \sigma_x - \nu\sigma_y = \sigma_x + \nu p\,,$$

$$E\varepsilon_y = \sigma_y - \nu\sigma_x = -p - \nu\sigma_x\,.$$

As the panel cannot expand in x-direction, it holds

$$\varepsilon_x = 0\,.$$

Inserting this into Hooke's law provides the stress σ_x and the normal strain in y-direction:

$$\underline{\underline{\sigma_x = -\nu p}}\,, \qquad \varepsilon_y = -p\,\frac{1-\nu^2}{E}\,.$$

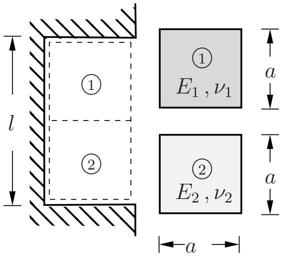
Knowing the strain ε_y we compute the displacement v by integration:

$$\frac{\partial v}{\partial y} = \varepsilon_y \quad \rightsquigarrow \quad v(y) = \int \varepsilon_y \mathrm{d}y = -p\,\frac{1-\nu^2}{E}\,y + C\,.$$

The lower edge of the panel does not experience a displacement, i. e. $v(0) = 0$, and $C = 0$. For the value of the displacement at the top edge we obtain

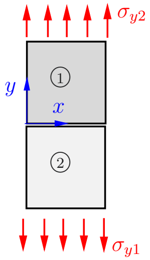
$$\underline{\underline{v_R = |v(h)| = \frac{1-\nu^2}{E}\,ph}}\,.$$

Problem 1.9 Two quadratic panels made of different materials have both the edge length a in the unloaded state. As sketched, they are inserted into a rigid socket, which has an opening l, that is smaller than $2a$. **P1.9**



What are the stresses and the changes of the edge lengths, if it is assumed that the panel can slide frictionless in the rigid socket?

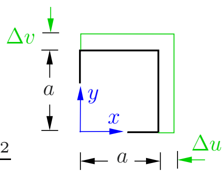
Solution After force fitting into the rigid socket the panels experience a uniform plane stress state. Equilibrium in vertical direction yields $\sigma_{y1} = \sigma_{y2} = \sigma_y$. Considering the condition $\sigma_{x1} = \sigma_{x2} = 0$ in Hooke's law for both panels provides



$$① \qquad E_1\varepsilon_{y1} = \sigma_y \,, \qquad E_1\varepsilon_{x_1} = -\nu_1\sigma_y \,,$$

$$② \qquad E_2\varepsilon_{y2} = \sigma_y \,, \qquad E_2\varepsilon_{x_2} = -\nu_2\sigma_y \,.$$

With the strain-displacement relation (constant strains)



$$\varepsilon_{x1} = \frac{\Delta u_1}{a} \,, \; \varepsilon_{y1} = \frac{\Delta v_1}{a} \,, \; \varepsilon_{x2} = \frac{\Delta u_2}{a} \,, \; \varepsilon_{y2} = \frac{\Delta v_2}{a}$$

and the kinematic compatibility

$$(a + \Delta v_1) + (a + \Delta v_2) = l$$

we obtain for the normal stress in y-direction

$$\underline{\underline{\sigma_y = -\frac{2a - l}{a}\,\frac{E_1 E_2}{E_1 + E_2}}} \,.$$

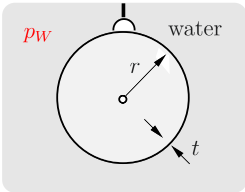
This stress leads to the following length changes

$$\underline{\underline{\Delta v_1 = -(2a - l)\frac{E_2}{E_1 + E_2}}} \,, \qquad \underline{\underline{\Delta v_2 = -(2a - l)\frac{E_1}{E_1 + E_2}}} \,,$$

$$\underline{\underline{\Delta u_1 = -\nu_1 \Delta v_1}} \,, \qquad \underline{\underline{\Delta u_2 = -\nu_2 \Delta v_2}} \,.$$

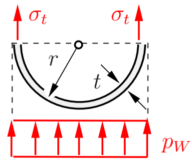
P1.10

Problem 1.10 A thin-walled diving sphere (radius $r = 500$ mm, wall thickness $t = 12.5$ mm) is submerged 1000 m under the water surface (pressure $p_W = 10$ MPa).



Compute the stresses in the wall of the sphere.

Solution We cut the sphere with a section perpendicular to the surface of the sphere, resulting in two hemispheres. The equilibrium conditions



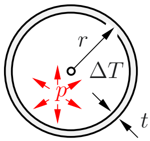
$$\uparrow: \quad \sigma_t 2\pi r t + p_W r^2 \pi = 0$$

provide for any section (spherical symmetry) the stresses

$$\underline{\underline{\sigma_t}} = -p_W \frac{r}{2t} = -10 \frac{500}{2 \cdot 12,5} = \underline{\underline{-200 \text{ MPa}}} .$$

P1.11

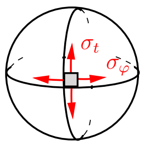
Problem 1.11 A spherical steel tank is heated by a hot gas ($\Delta T = 300\,°\text{C}$) and additionally subjected to an internal pressure ($p = 1.5$ MPa).



Compute the change of the radius.

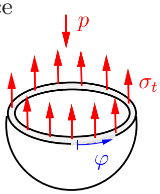
Given: $r = 2\,\text{m}$, $t = 10\,\text{mm}$, $E = 2.1 \cdot 10^5$ MPa,
$\nu = 0.3$, $\alpha_T = 12 \cdot 10^{-6}\,°\text{C}^{-1}$.

Solution For any cross section perpendicular to the surface of the sphere equilibrium provides



$$\sigma_t = \sigma_\varphi = p \frac{r}{2t} .$$

The strain is computed by the change of circumference



$$\varepsilon_t = \varepsilon_\varphi = \frac{2\pi(r + \Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r} .$$

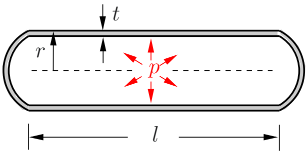
Using Hooke's law

$$E\varepsilon_t = \sigma_t - \nu\sigma_\varphi + E\alpha_T \Delta T$$

yields

$$\underline{\underline{\Delta r}} = r \left[\frac{p\,r(1-\nu)}{2Et} + \alpha_T \Delta T \right] = 2000 \left[\frac{1.5 \cdot 10^{-3}}{3} + 3.6 \cdot 10^{-3} \right] = \underline{\underline{8.25 \text{ mm}}} .$$

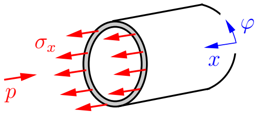
Problem 1.12 A thin-walled cylindrical pressure vessel made of steel is subjected to an internal pressure p. **P1.12**



What is the maximum value of the pressure such that the normal stresses in the central part do not to exceed the limit stress $\sigma_{\lim}$?

Compute for this case the change of radius r and length l.

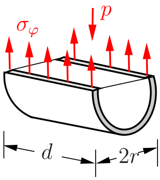
Given: $l = 5$ m, $r = 1$ m, $t = 1$ cm, $E = 2.1 \cdot 10^5$ MPa,
$\nu = 0.3$, $\sigma_{\lim} = 100$ MPa.



Solution The stresses are determined by equilibrium conditions at suitable sections:

$$\rightarrow: \quad pr^2\pi - \sigma_x 2r\pi t = 0 \quad \rightsquigarrow \quad \underline{\underline{\sigma_x = p\,\frac{r}{2t}}}\ ,$$

$$\uparrow: \quad \sigma_\varphi 2d\, t - p2rd = 0 \quad \rightsquigarrow \quad \underline{\underline{\sigma_\varphi = p\,\frac{r}{t}}}\ .$$



These stresses are principal stresses, as the shear stress vanish in these sections. The largest normal stress exceeds the limit stress for

$$\sigma_\varphi \leq \sigma_{\lim} \quad \rightsquigarrow \quad p \leq \frac{t}{r}\sigma_{\lim} = 1 \text{ MPa} \quad \rightsquigarrow \quad \underline{\underline{p_{\max} = 1 \text{ MPa}}}\ .$$

The related hoop strain ε_φ results from the circumferential change:

$$\varepsilon_\varphi = \frac{2\pi(r + \Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r}\ .$$

Hooke's law $E\varepsilon_\varphi = \sigma_\varphi - \nu\sigma_t$ provides

$$\underline{\underline{\Delta r}} = r\,\frac{p_{\max} r}{Et}\left(1 - \frac{\nu}{2}\right) = \underline{\underline{0.41 \text{ mm}}}\ .$$

In an analogous way the strain $\varepsilon_t = \Delta l / l$ and Hooke's law $E\varepsilon_t = \sigma_t - \nu\sigma_\varphi$ provide the length change

$$\underline{\underline{\Delta l}} = l\,\frac{p_{\max} r}{Et}\left(\frac{1}{2} - \nu\right) = \underline{\underline{0.47 \text{ mm}}}\ .$$

Note: The caps at the ends of the pressure vessel are excluded- i. e. the solution for the stresses is only valid in a sufficient distance from the caps.

P1.13 **Problem 1.13** The rails of a train track are installed at a temperature of 15°C such that no internal forces are present.

Determine the stress at a temperature of −25°C, if it is assumed that the rails cannot experience any length change?
Given: $E = 2.1 \cdot 10^5$ MPa, $\alpha_T = 12 \cdot 10^{-6}\,{}^\circ\mathrm{C}^{-1}$.

Solution In the rail exists a uniaxial stress state and Hooke's law provides

$$E\,\varepsilon = \sigma + E\,\alpha_T\,\Delta T\,.$$

As displacements are suppressed, ε has to be zero. Using $\Delta T = -40°$C yields for the stresses

$$\underline{\underline{\sigma}} = -E\,\alpha_T\,\Delta T = 2.1 \cdot 10^5 \cdot 12 \cdot 10^{-6} \cdot 40 = \underline{\underline{100.8\ \mathrm{MPa}}}\,.$$

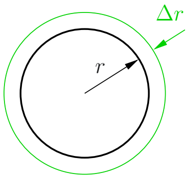
Note: In rails the stresses due to temperature changes can become considerably large.

P1.14 **Problem 1.14** A thin copper ring of radius r is heated due to the temperature difference ΔT.

What are the changes in radius and circumference if it is assumed that the ring can deform freely?
Given: $r = 100$ mm, $\alpha_T = 16 \cdot 10^{-6}\,{}^\circ\mathrm{C}^{-1}$, $\Delta T = 50°$C.

Solution A uniform, stress-free uniaxial strain state exists in the ring after heating. The strain is determined by the change in circumferencial direction (change in length) Δl:

$$\varepsilon = \frac{\Delta l}{l} = \frac{2\pi(r+\Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r}\,.$$



Using the Hooke's law for uniaxial states

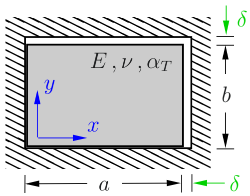
$$\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T$$

and the stress-free condition $\sigma = 0$, leads to

$$\underline{\underline{\Delta r}} = r\,\alpha_T \Delta T = 100 \cdot 16 \cdot 10^{-6} \cdot 50 = \underline{\underline{0.08\ \mathrm{mm}}}\,,$$

$$\underline{\underline{\Delta l}} = \frac{l}{r}\Delta r = 2\pi \Delta r = \underline{\underline{0.50\ \mathrm{mm}}}\,.$$

Problem 1.15 A rectangular plate ($a > b$) is inserted into a rigid oversized opening, such that spacings of size δ are present. Subsequently the plate is heated. It is assumed, that the plate can move frictionless along its edges. **P1.15**



a) Which temperature increase ΔT_a is required to just close the spacing on the right?
b) For which temperature increase ΔT_b is the upper spacing just closing? What is the value of σ_x in this situation?
c) What are the stresses in the plate for $\Delta T > \Delta T_b$?

Solution **to a)** For $\Delta T < \Delta T_a$ the plate expands in a stress-free way. With $\sigma_x = \sigma_y = 0$ Hooke's law reduces to

$$\varepsilon_x = \varepsilon_y = \alpha_T \Delta T\,.$$

The spacing on the right is closing, if the condition $\varepsilon_x = \delta/a$ is met. Introducing this result yields the temperature increase:

$$\underline{\underline{\Delta T_a = \frac{\delta}{\alpha_T a}}}\,.$$

to b) At a temperature increase $\Delta T_a \leq \Delta T \leq \Delta T_b$ the plate can only expand freely in y-direction, while the strain in x-direction remains constant. With $\sigma_y = 0$ and $\varepsilon_x = \delta/a$ it follows

$$\frac{\delta}{a} = \frac{\sigma_x}{E} + \alpha_T \Delta T\,, \qquad \varepsilon_y = -\nu\,\frac{\sigma_x}{E} + \alpha_T \Delta T\,.$$

The upper spacing is closing, if the the condition $\varepsilon_y = \delta/b$ is satisfied. All above relations provide

$$\underline{\underline{\Delta T_b = \frac{\delta}{\alpha_T a}\,\frac{a + \nu b}{(1+\nu)b}}}\,, \qquad \underline{\underline{\sigma_x = -\frac{E}{1+\nu}\,\frac{\delta(a-b)}{ab}}}\,.$$

to c) For $\Delta T > \Delta T_b$ the strains in both directions remain constant: $\varepsilon_x = \delta/a$, $\varepsilon_y = \delta/b$. Then

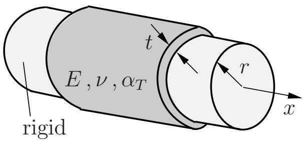
$$E\varepsilon_x = \sigma_x - \nu\sigma_y + E\alpha_T\Delta T\,, \quad E\varepsilon_y = \sigma_y - \nu\sigma_x + E\alpha_T\Delta T$$

provide the stresses

$$\underline{\underline{\sigma_x = E\left[\frac{\delta(\nu a + b)}{(1-\nu^2)ab} - \frac{\alpha_T \Delta T}{1-\nu}\right]}}, \quad \underline{\underline{\sigma_y = E\left[\frac{\delta(\nu b + a)}{(1-\nu^2)ab} - \frac{\alpha_T \Delta T}{1-\nu}\right]}}.$$

P1.16

Problem 1.16 A thin-walled bushing has to be heated by the temperature difference ΔT^*, to be mounted on a shaft.

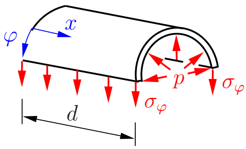


What are the stresses in the bushing, and what is the pressure p between bushing and shaft after cooling? It is assumed that the shaft is *rigid* and that the displacements of the bushing in x-directions are blocked by friction.

Solution Before cooling the bushing is stress-free. The stresses after cooling are obtained by equilibrium, Hooke's law, and kinematics. The equilibrium condition provides

$$p \cdot 2rd = \sigma_\varphi 2t\,d \quad \leadsto \quad \sigma_\varphi = p\,\frac{r}{t}\,.$$

Hooke's law with $\Delta T = -\Delta T^*$ (cooling!) states



$$E\varepsilon_\varphi = \sigma_\varphi - \nu\sigma_x - E\alpha_T\Delta T^*\,,$$

$$E\varepsilon_x = \sigma_x - \nu\sigma_\varphi - E\alpha_T\Delta T^*\,.$$

During cooling the strains in the bushing (shrinking) are blocked by the shaft and friction. Thus the kinematic relations are given by

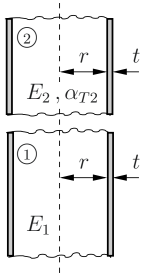
$$\varepsilon_\varphi = 0\,, \qquad \varepsilon_x = 0\,.$$

Combining the above relations and solving for stresses and pressure yields

$$\underline{\underline{\sigma_x = \sigma_\varphi = \frac{E}{1-\nu}\,\alpha_T\Delta T^*}}\,, \qquad \underline{\underline{p = \frac{t}{r}\,\frac{E}{1-\nu}\,\alpha_T\Delta T^*}}\,.$$

Note:
- In the bushing a plane stress state is present with equal normal stresses: $\sigma_x = \sigma_\varphi$.
- If the bushing can deform freely in x-direction (no friction, $\varepsilon_x \neq 0$), then $\sigma_x = 0$ and $\sigma_\varphi = E\alpha_T\Delta T^*$ follow.

Problem 1.17 On the thin-walled elastic shaft ① a pipe ② will be mounted by heat shrinking. Before heat shrinking both parts have identical geometrical dimensions, but are made of different materials. **P1.17**



Which temperature difference is required to the mount pipe ② on the shaft ①?

What is the pressure p between the shaft and the pipe after cooling, if it is assumed that no stresses are present in axial direction?

Solution For the pipe ② to be mounted on the shaft ① its radius has to increase by thermal expansion by t. Thus in the heated state, the hoop strain has to assume the value

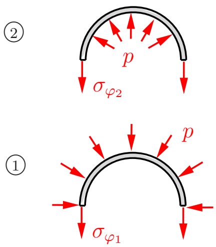
$$\varepsilon_{\varphi 2} = \frac{2\pi(r+t) - 2\pi r}{2\pi r} = \frac{t}{r}$$

Now Hooke's law yields with $\sigma_{\varphi 2} = 0$ (the pipe is stress free in the heated state!)

$$\varepsilon_{\varphi 2} = \alpha_{T2}\Delta T \quad \leadsto \quad \underline{\underline{\Delta T = \frac{1}{\alpha_{T2}}\,\frac{t}{r}}}\,.$$

The pressure after cooling is obtained from the equilibrium equations

$$\sigma_{\varphi 1} = -\frac{r}{t}\,p\,, \qquad \sigma_{\varphi 2} = +\frac{r}{t}\,p\,,$$



Hooke's laws,

$$E_1\varepsilon_{\varphi 1} = \sigma_{\varphi 1}\,, \qquad E_2\varepsilon_{\varphi 2} = \sigma_{\varphi 2}\,,$$

the strains

$$\varepsilon_{\varphi 1} = \frac{\Delta r_1}{r}\,, \qquad \varepsilon_{\varphi 2} = \frac{\Delta r_2}{r}$$

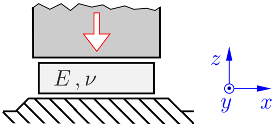
and the kinematic compatibility

$$\Delta r_2 = \Delta r_1 + t\,.$$

Combining the above equations yields

$$\underline{\underline{p = \frac{E_1 E_2}{E_1 + E_2}\left(\frac{t}{r}\right)^2}}\,.$$

P1.18 **Problem 1.18** A block is subjected to a pressure p_0 in z-direction by a rigid press.



Determine the strains and stresses, if

a) the deformations in x- and y-direction are restrained,
b) only the deformation in y-direction is restrained,
c) the deformations in $x-$ and $y-$direction are *not* restrained?

Solution In the above cases a homogeneous, triaxial stress and strain state is present in the plate. With $\sigma_z = -p_0$ Hooke's law yields (there are no shear stresses present!):

$$E\varepsilon_x = \sigma_x - \nu\sigma_y + \nu p_0, \quad E\varepsilon_y = \sigma_y + \nu p_0 - \nu\sigma_x, \quad E\varepsilon_z = -p_0 - \nu\sigma_x - \nu\sigma_y.$$

For case **a)** we have $\varepsilon_x^a = \varepsilon_y^a = 0$, and from

$$0 = \sigma_x^a - \nu\sigma_y^a + \nu p_0, \quad 0 = \sigma_y^a + \nu p_0 - \nu\sigma_x^a, \quad E\varepsilon_z^a = -p_0 - \nu\sigma_x - \nu\sigma_y$$

it follows

$$\underline{\underline{\varepsilon_z^a = -\frac{1-\nu-2\nu^2}{1-\nu}\,\frac{p_0}{E}}}\,, \qquad \underline{\underline{\sigma_x^a = \sigma_y^a = -\frac{\nu}{1-\nu}\,p_o}}\,.$$

In case **b)** $\varepsilon_y^b = 0$ and $\sigma_x^b = 0$ holds (free deformation, i. e. no stresses in x-direction). With Hooke's law

$$E\varepsilon_x^b = -\nu\sigma_y^b + \nu p_0\,, \quad 0 = \sigma_y^b + \nu p_0\,, \quad E\varepsilon_z = -p_0 - \nu\sigma_y^b$$

we obtain

$$\underline{\underline{\varepsilon_x^b = \nu(1+\nu)\,\frac{p_0}{E}}}\,, \qquad \underline{\underline{\varepsilon_z^b = -(1-\nu^2)\,\frac{p_0}{E}}}\,, \qquad \underline{\underline{\sigma_y^b = -\nu\,p_0}}\,.$$

In case **c)**, both $\sigma_x^c = \sigma_y^c = 0$, because the deformations in these directions are not restrained. Therefore Hooke's law reduces to

$$E\varepsilon_x^c = \nu\,p_0\,, \qquad E\varepsilon_y^c = \nu\,p_0\,, \qquad E\varepsilon_z^c = -p_0\,,$$

and we have

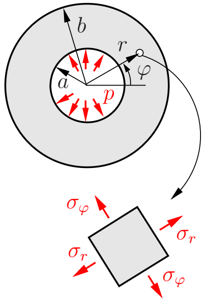
$$\underline{\underline{\varepsilon_x^c = \varepsilon_y^c = \nu\,\frac{p_0}{E}}}\,, \qquad \underline{\underline{\varepsilon_z^c = -\frac{p_0}{E}}}\,.$$

Note: For $\nu > 0$ we have $|\varepsilon_z^a| < |\varepsilon_z^b| < |\varepsilon_z^c|$. Especially for $\nu = 1/3$ it follows

$$\varepsilon_z^a = -\,6p_0/(9E)\,, \qquad \varepsilon_z^b = -\,8p_0/(9E)\,, \qquad \varepsilon_z^c = -\,9p_0/(9E)\,.$$

Due to the deformation constraints in x- and y-direction the plate behaves rather *stiff* in case a)!

Problem 1.19 In a thick-walled cylinder with a restrained deformation in longitudinal direction (plane strain state) the following stresses are present due to loading by an internal pressure p: **P1.19**



$$\sigma_r = -p\,\frac{a^2}{b^2-a^2}\left(\frac{b^2}{r^2}-1\right)\,,$$

$$\sigma_\varphi = p\,\frac{a^2}{b^2-a^2}\left(\frac{b^2}{r^2}+1\right)\,.$$

Determine the stress σ_z and the resulting force F_z in axial direction of the cylinder.

Where does the maximum normal stress occur, and what is its value?
Given: $p = 50$ MPa, $a = 100\,$mm, $b = 200\,$mm, $\nu = 1/3$.

Solution As the deformation in axial direction of the cylinder is restrained, we have $\varepsilon_z = 0$. Hooke's law in this direction provides

$$E\varepsilon_z = 0 = \sigma_z - \nu(\sigma_r + \sigma_\varphi)\,.$$

Inserting the known relations yields the stress

$$\underline{\underline{\sigma_z}} = \nu(\sigma_r+\sigma_\varphi) = 2\nu\,p\,\frac{a^2}{b^2-a^2} = \frac{2}{9}\,p = \underline{\underline{11.1\text{ MPa}}}\,.$$

As σ_z is constant across the section, the resulting force is computed by multiplication of σ_z with the cross section area:

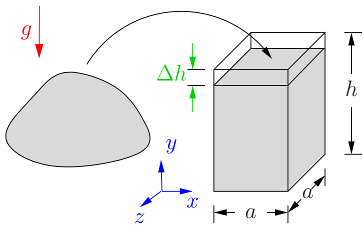
$$\underline{\underline{F_z}} = \sigma_z\pi(b^2-a^2) = 2\pi\nu\,p\,a^2 = \underline{\underline{1.05\cdot 10^6\text{ N}}}\,.$$

The absolute values of the stresses σ_r and σ_φ are maximum at the inner boundary of the cylinder ($r = a$). There we have

$$\sigma_r(a) = -p\,,\qquad \sigma_\varphi(a) = \frac{5}{3}\,p\,,\qquad \sigma_z = \frac{2}{9}\,p\,.$$

Thus the *hoop* stress σ_φ on the inside is the largest normal stress.

P1.20 **Problem 1.20** A rigid box with quadratic cross section is filled with clay (volume $V = a^2 h$, density ρ). The material behavior of the clay is approximated by Hooke's law (Young's modulus E, Poisson's ratio ν).



Determine the settlement Δh of the clay as a consequence of the weight of the clay and the horizontal pressure distribution at the box walls as a function of y.

Solution Due to the given loading situation only normal stresses σ_x, σ_y, and σ_z are present in the three coordinate directions x, y, and z. Except for the strain ε_y no other strains occur. For σ_y it holds according to Hooke's law with $\varepsilon_x = \varepsilon_z = 0$

$$\sigma_y = \frac{E}{1+\nu}\left(\varepsilon_y + \frac{\nu}{1-2\nu}\,\varepsilon_y\right) = \frac{E}{1+\nu}\,\frac{1-\nu}{1-2\nu}\,\varepsilon_y \,.$$

With the stress distribution

$$\sigma_y = -\rho g(h-y)$$

the settlement Δh is computed by

$$\varepsilon_y = \frac{\mathrm{d}v}{\mathrm{d}y} \,.$$

By integration we obtain Δh:

$$\underline{\underline{\Delta h}} = v(h) = \int_0^h \varepsilon_y \,\mathrm{d}y = -\int_0^h \rho g(h-y)\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\,\mathrm{d}y$$

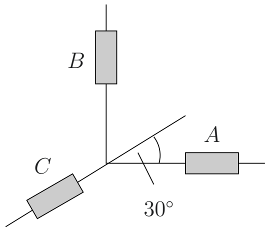
$$= -\left[\rho g \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\left(hy - \frac{y^2}{2}\right)\right]_0^h = \underline{\underline{-\frac{1}{2}\,\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho g h^2}}.$$

The horizontal pressure distribution as a function of y follows from Hooke's law:

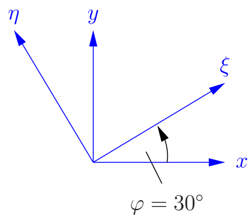
$$\sigma_x = \sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)}\,\varepsilon_y \,, \qquad \varepsilon_y = -\rho g(h-y)\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}$$

$$\leadsto \quad \underline{\underline{\sigma_x(y) = \sigma_z(y) = \frac{-\nu}{1-\nu}\rho g(h-y)}} \,.$$

Problem 1.21 In a sheet metal (Young's modulus E and Poisson's ratio ν) the three strains $\varepsilon_A = \bar{\varepsilon}$, $\varepsilon_B = 3\,\bar{\varepsilon}$ und $\varepsilon_C = 2\,\bar{\varepsilon}$ are measured by strain gauges in the sketched directions. **P1.21**



a) Determine the principal strains ε_1 and ε_2.
b) Compute the principal stresses σ_1 and σ_2 under the assumption of a plane state of stress.
c) Calculate the principal directions.



Solution to a) We introduce a x, y- and a ξ, η- coordinate system in direction of the strain gauges. Then it holds for the measured strains

$$\varepsilon_x = \bar{\varepsilon}\,, \qquad \varepsilon_y = 3\bar{\varepsilon}\,, \qquad \varepsilon_\xi = 2\bar{\varepsilon}\,.$$

To compute the principal strains we have to determine the shear strain γ_{xy}. According to the transformation relations for $\varphi = 30°$ we have

$$\begin{aligned}
\varepsilon_\xi &= \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\varphi + \frac{1}{2}\gamma_{xy}\sin 2\varphi \\
&= \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{4}(\varepsilon_x - \varepsilon_y) + \frac{\sqrt{3}}{4}\gamma_{xy}\,, \\
2\bar{\varepsilon} &= 2\bar{\varepsilon} + \left(-\frac{1}{2}\right)\bar{\varepsilon} + \frac{\sqrt{3}}{4}\gamma_{xy}\,.
\end{aligned}$$

This yields the result

$$\gamma_{xy} = \frac{2}{\sqrt{3}}\bar{\varepsilon}\,.$$

With this at hand, the principal strains can be calculated via

$$\varepsilon_{1/2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}$$

to be

$$\underline{\underline{\varepsilon_1 = 2\left(1 + \frac{1}{\sqrt{3}}\right)\bar{\varepsilon}}}\,, \qquad \underline{\underline{\varepsilon_2 = 2\left(1 - \frac{1}{\sqrt{3}}\right)\bar{\varepsilon}}}\,.$$

to b) Using the assumption of a plane stress state, Hooke's law formulated in principal directions provides the principal stresses

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) \quad \rightsquigarrow \quad \underline{\underline{\sigma_1 = \frac{2\,E\,\bar{\varepsilon}}{1-\nu^2}\left(1+\nu+\frac{1-\nu}{\sqrt{3}}\right)}}\,,$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \quad \rightsquigarrow \quad \underline{\underline{\sigma_2 = \frac{2\,E\,\bar{\varepsilon}}{1-\nu^2}\left(1+\nu-\frac{1-\nu}{\sqrt{3}}\right)}}\,.$$

to c) The principal directions follow either from the stress or from the strain components. Here we use the strain components to obtain from the general formula

$$\tan 2\varphi^* = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{\frac{2}{\sqrt{3}}}{-2} = -\frac{1}{\sqrt{3}}$$

the solutions

$$\underline{\underline{\varphi^* = -15^\circ}} \qquad \text{und} \qquad \underline{\underline{\varphi^* = 75^\circ}}\,.$$

In order to decide, which direction corresponds to the principal strain ε_1 or ε_2, respectively, we use the angle $\varphi^* = -15^\circ$ in the coordinate transformation. This yields with the given strain components

$$\begin{aligned}\varepsilon_\xi &= \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos(-30^\circ) + \frac{1}{2}\gamma_{xy}\sin(-30^\circ)\\ &= 2\bar{\varepsilon} - \bar{\varepsilon}\frac{\sqrt{3}}{2} - \frac{\bar{\varepsilon}}{\sqrt{3}}\frac{1}{2} = 2\left(1 - \frac{1}{\sqrt{3}}\right)\bar{\varepsilon} = \varepsilon_2\,.\end{aligned}$$

The smallest principal strain ε_2 occurs at the angle $\varphi^* = -15^\circ$, while the largest principal strain ε_1 is related to the direction of $\varphi^* = 75^\circ$.

Chapter 2

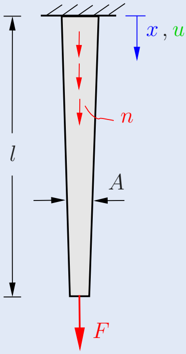
2

Tension and Compression in Bars

Tensile or compressive loading in bars

Assumptions:

- Length l of the bar is large compared to characteristic dimensions of the cross section $A(x)$.
- Axis of the bar (line connecting centroids of the cross sections) is a straight line.
- Common line of action (external loads F and $n(x)$ are aligned with the axis of the bar).
- Cross section $A(x)$ can only vary slightly.



Stress: Assuming a constant stress σ across the section A the following relation with the normal force N holds:

$$\boxed{\sigma(x) = \frac{N(x)}{A(x)}\,.}$$

Basic equations of a deformable bar:

equilibrium condition	$\dfrac{\mathrm{d}N}{\mathrm{d}x} = -n\,,$
Hooke's law	$\varepsilon = \dfrac{\sigma}{E} + \alpha_T\,\Delta T\,,$
Kinematic relation	$\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

E = Young's modulus,

α_T = coefficient of thermal expansion,

ΔT = temperature difference with respect to a reference state,

$u(x)$ = displacement of a point x within the bar.

The basic equations lead to a *single* differential equation for the displacements ($\{\cdot\}' := \mathrm{d}\{\cdot\}/\mathrm{d}x$):

$$(EAu')' = -n + (EA\alpha_T\Delta T)'\,.$$

Elongation of a bar: $$\Delta l = u(l) - u(0) = \int_0^l \varepsilon \, \mathrm{d}x \, .$$

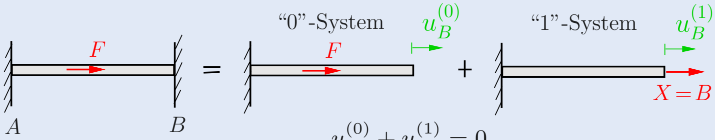
special cases:

$$\Delta l = \int_0^l \frac{N}{EA} \mathrm{d}x \qquad (\Delta T = 0) \, ,$$

$$\Delta l = \frac{Fl}{EA} \qquad (N = F = \text{const}, \; EA = \text{const}, \; \Delta T = 0),$$

$$\Delta l = \alpha_T \Delta T \, l \qquad (N = 0, \; EA = \text{const}, \; \alpha_T \Delta T = \text{const}).$$

Superposition: The solution of a statically indeterminate problem can be achieved by *superposition* of solutions of associated statically determinate problems considering the compatibility conditions.

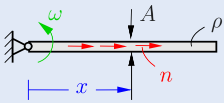


$$u_B^{(0)} + u_B^{(1)} = 0 \, .$$

Rotating bar: A bar rotating with the angluar velocity ω experiences an axial loading per unit length of

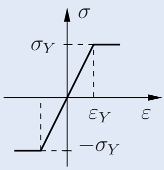
$$n = \rho A \, x\omega^2 \, .$$

Here ρ is the density and x represents the distance of the cross section A from the center of rotation.



Elastic-plastic bar: For an elastic-ideal-plastic material behavior, Hooke's law is valid only until a certain *yield limit* σ_Y :

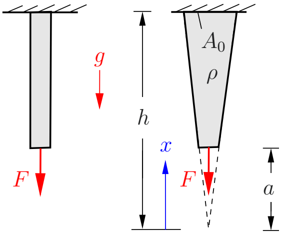
$$\sigma = \begin{cases} E\,\varepsilon \, , & |\varepsilon| \le \varepsilon_Y \, , \\ \sigma_Y \, \mathrm{sign}(\varepsilon) \, , & |\varepsilon| \ge \varepsilon_Y \, . \end{cases}$$



System of bars: The displacements are obtained by "disconnecting" and "reconnecting" of the bars from the nodes using a *displacement diagram*.

Note: In areas with rapidly changing cross sections (notches, holes) the above theory for bars is not applicable.

P2.1 **Problem 2.1** Determine the stresses distribution $\sigma(x)$ in the homogeneous bar due to its weight. The bar has constant thickness and a linear varying width. Furthermore, identify the location and value of the smallest stress.



It is reasonable to introduce the x-coordinate at the intersection of the extended edges of the trapeziod. The x dependent cross section area follows then as

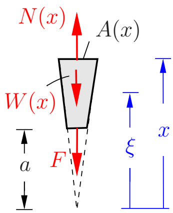
$$A(x) = A_0 x/l\,.$$

With the weight

$$W(x) = \rho g V(x) = \rho g \int_a^x A(\xi)\mathrm{d}\xi = \rho g A_0 \frac{x^2 - a^2}{2l}$$

of the lower part equilibrium provides

$$N(x) = F + W(x) = F + \rho g A_0 \frac{x^2 - a^2}{2h}\,.$$



This leads to the stress

$$\underline{\underline{\sigma(x)}} = \frac{N(x)}{A(x)} = \underline{\underline{\frac{Fh + \rho g \frac{A_0}{2}\left(x^2 - a^2\right)}{A_0 x}}}\,.$$

The location x^* of the minimum is determined by condition $\sigma' = 0$:

$$\sigma' = -\frac{Fh}{A_0}\frac{1}{x^2} + \frac{\rho g}{2}\left(1 + \frac{a^2}{x^2}\right) = 0 \quad \leadsto \quad \underline{\underline{x^* = \sqrt{\frac{2Fh}{\rho g A_0} - a^2}}}\,.$$

The value of the minimum stress is

$$\underline{\underline{\sigma_{\min}}} = \sigma(x^*) = \rho g \sqrt{\frac{2Fh}{\rho g A_0} - a^2} = \underline{\underline{\rho g x^*}}\,.$$

Note:

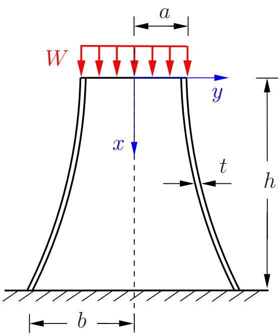
- For $\rho g = 0$ ("weight-less bar") no minimum exists. The largest stress occurs at $x = a$.
- The minimum will be located within the bar, only if $a < x^* < h$ or $\rho g A_0 a^2/(2h) < F < \rho g A_0 (h^2 + a^2)/(2h)$ holds.

Problem 2.2 The contour of a lighthouse with circular thin-walled cross section follows a hyperbolic equation

$$y^2 - \frac{b^2 - a^2}{h^2}\,x^2 = a^2\,.$$

Determine the stress distribution as a consequence of weight W of the lighthouse head (the weight of the structure can be neglected).

Given: $b = 2a$, $t \ll a$.

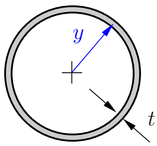


Solution As the weight W is the only acting external load, the normal force N is constant (compression):

$$N = -W\,.$$

The cross section area A is changing. It can be approximated by (thin-walled structure with $t \ll y$)

$$\begin{aligned} A(x) = 2\pi y t &= 2\pi t\sqrt{a^2 + \frac{b^2 - a^2}{h^2}\,x^2} \\ &= 2\pi t\sqrt{a^2 + 3\frac{a^2}{h^2}\,x^2} \\ &= 2\pi a t\sqrt{1 + 3\,\frac{x^2}{h^2}}\,. \end{aligned}$$



The stress follows now as

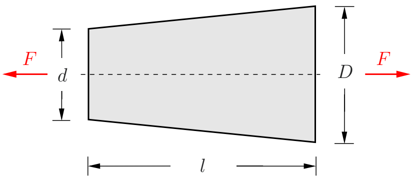
$$\underline{\underline{\sigma(x) = \frac{N}{A} = -\frac{W}{2\pi a t\sqrt{1 + 3\,\frac{x^2}{h^2}}}}}\,.$$

Especially at the top and bottom position we get

$$\sigma(x=0) = -\frac{W}{2\pi a t} \qquad \text{bzw.} \qquad \sigma(x=h) = -\frac{W}{4\pi a t}\,.$$

Note: The stress at the top is twice as large as the stress at the bottom, which is a inefficient use of material. This situation changes if the weight of the thin-walled structure is included in the analysis.

P2.3 **Problem 2.3** Determine the elongation Δl of the conical shaft (Young's modulus E) under the application of a tensile force F.



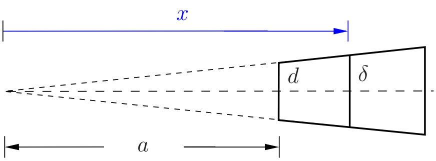
Solution The normal force $N = F$ is constant, while the cross section area A varies. With $\sigma = N/A$ the elongation is computed by

$$\Delta l = \int_0^l \varepsilon \, dx = \frac{1}{E}\int_0^l \sigma \, dx = \frac{1}{E}\int_0^l \frac{N dx}{A} = \frac{F}{E}\int_0^l \frac{dx}{A(x)} \,.$$

To describe the change of the cross section area $A(x)$ we start the x-axis at the peak of the frustum. Using the intercept theorem and the auxiliary variable a we obtain for the diameter

$$\delta(x) = d\,\frac{x}{a}$$

and for the area



$$A(x) = \frac{\pi}{4}\,\delta^2(x) = \frac{\pi}{4}\,d^2\,\frac{x^2}{a^2} \,.$$

Introducing this in the relation for the elongation, then integration provides (integration limits!):

$$\Delta l = \frac{F}{E}\int_a^{a+l} \frac{dx}{\frac{\pi}{4}\,d^2\,\frac{x^2}{a^2}} = \frac{4Fa^2}{\pi E\,d^2}\left(-\frac{1}{x}\right)\Bigg|_a^{a+l} \,.$$

With

$$\frac{a+l}{D} = \frac{a}{d} \quad \leadsto \quad a = \frac{d}{D}\,\frac{l}{1-\frac{d}{D}}$$

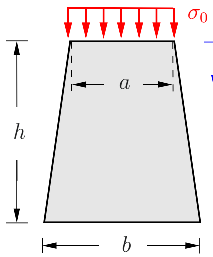
the elongation is

$$\underline{\underline{\Delta l = \frac{4Fl}{\pi E D d}}} \,.$$

Test: For $D = d$ (constant cross section) we obtain $\Delta l = \dfrac{4Fl}{\pi E d^2} = \dfrac{Fl}{EA}$
.

Problem 2.4 A homogeneous frustum of a pyramid (Young's modulus E) with a square cross section is loaded on its top surface by a stress σ_0. **P2.4**

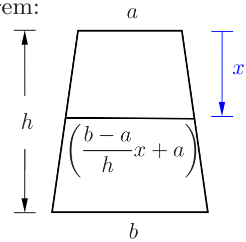
Determine the displacement field $u(x)$ of a cross section at position x.



Solution The normal force $N = -\sigma_0 a^2$ is constant. From the kinematic relation $\varepsilon = \mathrm{d}u/\mathrm{d}x$ and Hooke's law $\varepsilon = \sigma/E = N/EA$ we obtain a differential equation for the displacement u

$$EA(x)\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\sigma_0 a^2 .$$

The area $A(x)$ follows from the intercept theorem:



$$A(x) = [a + (b-a)\frac{x}{h}]^2 .$$

Thus we have

$$E\big(a + \frac{b-a}{h}\,x\big)^2\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\sigma_0 a^2 .$$

Separation of variables yields

$$\mathrm{d}u = -\frac{\sigma_0 a^2}{E}\,\frac{\mathrm{d}x}{\left(\frac{b-a}{h}\,x + a\right)^2} \quad \rightsquigarrow \quad \int\limits_{u(0)}^{u(x)} \mathrm{d}u = -\frac{\sigma_0 a^2}{E}\int\limits_0^x \frac{\mathrm{d}\xi}{\left(\frac{b-a}{h}\,\xi + a\right)^2} .$$

Using the substitution $z = a + (b-a)\,\xi/h, \quad \mathrm{d}z = (b-a)\,\mathrm{d}\xi/h$ leads to

$$u(x) - u(0) = -\frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(-\frac{1}{z}\right)\Bigg|_a^{\frac{b-a}{h}x+a} = -\frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\Bigg(\frac{1}{a} - \frac{1}{\frac{b-a}{h}\,x + a}\Bigg) .$$

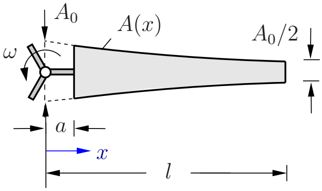
The displacement $u(0)$ of the top surface follows from the boundary condition that the displacement has to vanish on the bottom edge $x = h$:

$$u(h) = 0 \quad \rightsquigarrow \quad u(0) = \frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\sigma_0 a h}{Eb} .$$

From this relation the displacement follows

$$\underline{\underline{u(x) = \frac{\sigma_0 a^2}{E}\,\frac{h}{b-a}\Bigg(-\frac{1}{b} + \frac{1}{\frac{b-a}{h}\,x + a}\Bigg)}} .$$

P2.5 **Problem 2.5** The cross section of a solid helicopter blade (density ρ, Young's modulus E) is described by the equation $A(x) = A_0 e^{-\alpha x/l}$.



Determine the stress distribution $\sigma(x)$, if the blade is rotating with a constant angular velocity ω. Compute the elongation Δl under the assumption $a = 0$.

Solution First, the sketched geometry $A(l) = A_0/2$ yields

$$A_0 e^{-\alpha} = A_0/2 \quad \rightsquigarrow \quad e^{\alpha} = 2 \quad \rightsquigarrow \quad \alpha = \ln 2 = 0.693\,.$$

The rotation causes a distributed load per unit length

$$n = \rho\omega^2 x A(x) = \rho\omega^2 A_0 x e^{-\alpha x/l}\,.$$

The equilibrium condition $N' = -n$ provides the normal force by integration

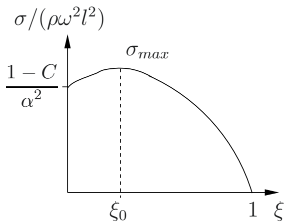
$$N = -\int n\,\mathrm{d}x = -\frac{\rho\,\omega^2 A_0 l^2}{\alpha^2}\left[-\frac{\alpha x}{l}e^{-\alpha x/l} - e^{-\alpha x/l} + C\right]\,.$$

The integration constant C is determined by the boundary condition:

$$N(l) = 0 \quad \rightsquigarrow \quad C = (1+\alpha)e^{-\alpha} = 0.847\,.$$

Introducing the dimensionless coordinate $\xi = x/l$ yields

$$N(\xi) = \frac{\rho\,\omega^2 A_0 l^2}{\alpha^2}[(1+\alpha\xi)e^{-\alpha\xi} - C]\,,$$



and for the stress distribution

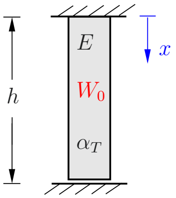
$$\underline{\underline{\sigma(\xi) = \frac{N}{A} = \frac{\rho\,\omega^2 l^2}{\alpha^2}[1+\alpha\xi - C e^{\alpha\xi}]}}\,.$$

The elongation is calculated from

$$\underline{\underline{\Delta l}} = \int_0^l \varepsilon\,\mathrm{d}x = \frac{l}{E}\int_0^1 \sigma\,\mathrm{d}\xi = \frac{\rho\,\omega^2 l^3}{\alpha^2 E}\left[\xi + \frac{\alpha\xi^2}{2} - \frac{C}{\alpha}e^{\alpha\xi}\right]_0^1$$
$$\underline{\underline{= \frac{\rho\,\omega^2 l^3}{E\alpha^2}\left[1 + \frac{\alpha}{2} - \frac{C}{\alpha}e^{\alpha} + \frac{C}{\alpha}\right] = 0.258\,\frac{\rho\omega^2 l^3}{E}}}\,.$$

Note: Due to the varying cross section the maximum stress occurs at the position $\xi_0 = -(\ln C)/\alpha = 0.24$ and attains the maximum value $\sigma_{\max} = -(\rho\omega^2 l^2 \ln C)/\alpha^2 = 0.347\,\rho\omega^2 l^2$.

Problem 2.6 A massive bar (weight W_0, cross section area A, thermal expansion coefficient α_T) is fixed at $x = 0$ and just touches the ground in a stress-free manner. **P2.6**

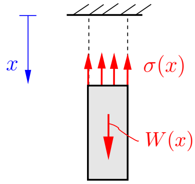


Determine the stress distribution $\sigma(x)$ in the bar after a uniform heating by ΔT.

Which ΔT causes compression everywhere in the bar?

Solution We investigate the "two load cases", weight und heating. The weight causes a a normal force

$$N(x) = W(x) = W_0 \frac{h - x}{h} = W_0 \left(1 - \frac{x}{h}\right)$$



which is related to the stress distribution

$$\sigma_1(x) = \frac{N(x)}{A} = \frac{W_0}{A}\left(1 - \frac{x}{h}\right)$$

The *heating* produces an additional strain, which is blocked by the support on the bottom. The relation

$$\varepsilon = \frac{\sigma_2(x)}{E} + \alpha_T \Delta T = 0$$

yields

$$\sigma_2(x) = -E\alpha_T \Delta T\,.$$

Thus the total stress is computed by

$$\underline{\underline{\sigma(x)}} = \sigma_1 + \sigma_2 = \underline{\underline{\frac{W_0}{A}\left(1 - \frac{x}{h}\right) - E\alpha_T \Delta T}}\,.$$

Due to the blocked temperature strain, there exists a compressive stress at the end of the bar ($x = h$) at all times. As the stress distribution is linear, the stress will be compressive everywhere, if compression is present at the top edge. Thus the relation

$$\sigma(x = 0) < 0 \qquad \text{bzw.} \qquad \frac{W_0}{A} - E\alpha_T \Delta T < 0$$

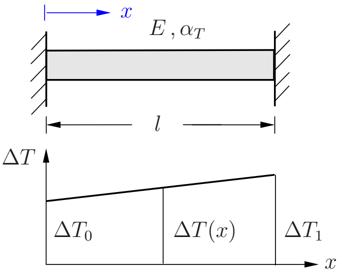
provides the necessary temperature difference

$$\underline{\underline{\Delta T > \frac{W_0}{E A \alpha_T}}}\,.$$

P2.7

Problem 2.7 An initially stress-free fixed bar (cross section area A) experiences a temperature increase varying linearly in x.

Determine the stress and strain distribution.



Solution The bar is supported in a statically indeterminate way. Thus we use equilibrium, kinematics and Hooke's law for the solution of the problem. With $n = 0$ and $\sigma = N/A$ these equations read

$$\sigma' = 0\,, \qquad \varepsilon = u'\,, \qquad \varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T(x)$$

with

$$\Delta T(x) = \Delta T_0 + (\Delta T_1 - \Delta T_0)\frac{x}{l}\,.$$

Combining the above relations renders the differential equation for the displacements

$$u'' = \alpha_T \Delta T' = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,.$$

Integrating twice yields

$$u' = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,x + C_1\,,$$

$$u = \frac{\alpha_T}{l}(\Delta T_1 - \Delta T_0)\,\frac{x^2}{2} + C_1 x + C_2\,.$$

The two integration constants follow from the boundary conditions:

$$u(0) = 0 \;\rightsquigarrow\; C_2 = 0\,, \qquad u(l) = 0 \;\rightsquigarrow\; C_1 = -\frac{\alpha_T}{2}(\Delta T_1 - \Delta T_0)\,.$$

We obtain the displacement field

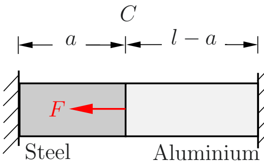
$$\underline{\underline{u(x) = \frac{\alpha_T l}{2}(\Delta T_1 - \Delta T_0)\left(\frac{x^2}{l^2} - \frac{x}{l}\right)}}$$

together with the (constant) stress

$$\underline{\underline{\sigma}} = E(u' - \alpha_T \Delta T) = \underline{\underline{-\frac{\alpha_T}{2}(\Delta T_1 + \Delta T_0)E}}\,.$$

Note: With constant heating $\Delta T_1 = \Delta T_0$ the displacement $u(x)$ vanishes. In this situation the stress is $\sigma = -\alpha_T \Delta T_0 E$.

Problem 2.8 A bar with a constant cross section A is fixed at both ends. The bar is made of two different materials, that are joint together at point C. **P2.8**



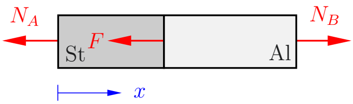
a) What are the reaction forces, if an external force F is applied at point C ?

b) Determine the normal force that is caused by a pure heating by ΔT?

Given: $E_{St}/E_{Al} = 3, \;\; \alpha_{St}/\alpha_{Al} = 1/2$.

Solution We treat the system as two joint bars with constant normal forces.

to a)



equilibrium: $-N_A + N_B = F$,

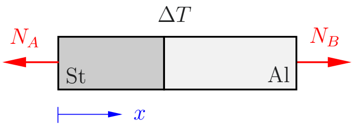
kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \dfrac{N_A a}{E_{St} A}$, $\quad \Delta l_{Al} = \dfrac{N_B(l-a)}{E_{Al} A}$.

The 4 equations for the 4 unknowns (N_A, N_B, Δl_{St}, Δl_{Al}) yield with the given numerical values

$$\underline{\underline{N_A = -F\,\frac{3(l-a)}{3l-2a}}}\ , \qquad \underline{\underline{N_B = F\,\frac{a}{3l-2a}}}\ .$$

to b)



equilibrium: $N_A = N_B = N$,

kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \dfrac{N\,a}{E_{St} A} + \alpha_{St} \Delta T\, a$,

$$\Delta l_{Al} = \frac{N(l-a)}{E_{Al} A} + \alpha_{Al} \Delta T\,(l-a)\,.$$

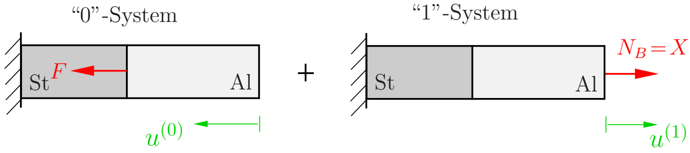
Solving the system of equations for the normal force N yields with the given numerical values

$$\underline{\underline{N = -\frac{2l-a}{3l-2a}\, E_{St}\, \alpha_{St}\, A\, \Delta T\,.}}$$

P2.9

Problem 2.9 Solve Problem 2.8 by superposition.

Solution to a) We choose the reaction force N_B as statically redundant quantity.



Hooke’s law provides

$$u^{(0)} = \frac{Fa}{E_{St}A}\,, \qquad u^{(1)} = \frac{X(l-a)}{E_{Al}A} + \frac{Xa}{E_{St}A}\,.$$

As the right edge is fixed compatibility requires

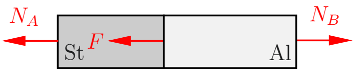
$$u^{(0)} = u^{(1)}\,.$$

This condition yields

$$\underline{\underline{N_B}} = X = \frac{Fa}{a + (l-a)\dfrac{E_{St}A}{E_{Al}A}} = \underline{\underline{F\,\frac{a}{3l-2a}}}\,.$$

From equilibrium we have

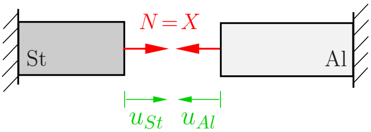
$$\underline{\underline{N_A = N_B - F = -F\,\frac{3(l-a)}{3l-2a}}}\,.$$



to b) In the free body diagram we choose the normal force N as statically redundant quantity X. From Hooke’s law

$$u_{St} = \frac{Xa}{E_{St}A} + \alpha_{St}\Delta T a\,,$$

$$u_{Al} = \frac{X(l-a)}{E_{Al}A} + \alpha_{Al}\Delta T (l-a)$$



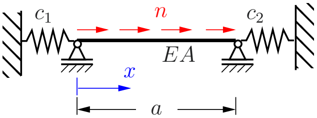
and the compatibility

$$u_{St} + u_{Al} = 0$$

we obtain

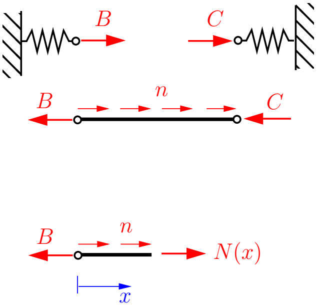
$$\underline{\underline{N}} = X = -\frac{\alpha_{St}a + \alpha_{Al}(l-a)}{\dfrac{a}{E_{St}A} + \dfrac{(l-a)}{E_{Al}A}} = \underline{\underline{-\frac{2l-a}{3l-2a}\,E_{St}\,\alpha_{St}\,A\,\Delta T}}\,.$$

Problem 2.10 An elastically supported bar ($c_1 = 2c_2 = EA/2a$) is loaded by a constant axial load n. **P2.10**



Compute the distribution of the normal force $N(x)$ in the bar.

Solution Using the free body diagram with the forces B and C at the ends of the bar, the equilibrium conditions can be formulated



$$B+C = na\,, \qquad N(x) = B - nx\,.$$

The elongation/shortening of the springs is given by

$$\Delta u_1 = \frac{B}{c_1}\,, \qquad \Delta u_2 = \frac{C}{c_2}\,.$$

The elongation of the bar is computed from

$$\Delta u_{\mathrm{St}} = \int_0^a \varepsilon\,\mathrm{d}x = \int_0^a \frac{N}{EA}\,\mathrm{d}x\,.$$

With $N = B - nx$ we obtain

$$\Delta u_{\mathrm{St}} = \frac{Ba}{EA} - \frac{na^2}{2\,EA}\,.$$

Finally, the kinematic relation

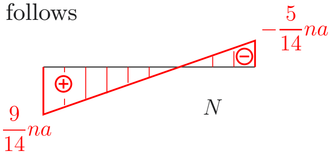
$$\Delta u_1 + \Delta u_{\mathrm{St}} = \Delta u_2 \qquad \leadsto \qquad \frac{B}{c_1} + \frac{Ba}{EA} - \frac{na^2}{2\,EA} = \frac{C}{c_2}$$

with $C = -B + na$ and the given value for c_1 and c_2 yields

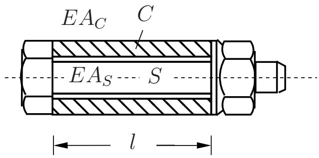
$$B\left(\frac{2a}{EA} + \frac{4a}{EA} + \frac{a}{EA}\right) = na\left(\frac{a}{2\,EA} + \frac{4a}{EA}\right) \quad \leadsto \quad B = \frac{9}{14}\,na$$

and the distribution of the normal force follows

$$\underline{\underline{N(x) = \frac{9}{14}\,na - nx\,.}}$$

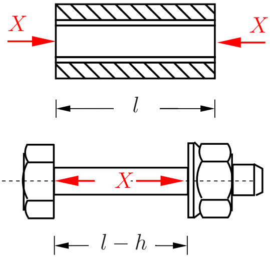


P2.11 **Problem 2.11** Determine the compression Δl_C of a casing C of length l, if the nut of screw S (lead h) is turned by one revolution.



Given: $\dfrac{EA_C}{EA_S} = \dfrac{4}{3}$.

Solution After the revolution of the nut we cut the system of screw and casing and introduce the statically indeterminate force F between the two parts.



The casing experiences a compression

$$\Delta l_C = \frac{Xl}{EA_C} \, .$$

For the screw we obtain an elongation

$$\Delta l_S = \frac{Xl}{EA_S} \, .$$

The length changes have to be adjusted in such a way that casing and screw have the same length. Therefore compatibility can be written as

$$h = \Delta l_C + \Delta l_S \, .$$

Inserting the length changes yields the force

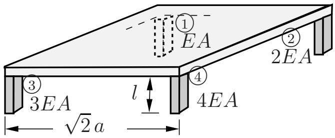
$$X = \frac{h}{l} \frac{1}{\dfrac{1}{EA_C} + \dfrac{1}{EA_S}}$$

and the compression of the casing

$$\underline{\underline{\Delta l_C}} = \frac{Xl}{EA_C} = h \frac{1}{1 + \dfrac{EA_C}{EA_S}} = h \frac{1}{1 + \frac{4}{3}} = \underline{\underline{\frac{3}{7}}} \, h \, .$$

Note: As the axial rigidity of the casing is larger than the one of the screw, the compression is only $3/7$ of the lead. If equal axial rigidities are present $EA_C = EA_S$, the length change of both parts will be equal, i. e. $\Delta l_C = \Delta l_S = h/2$.

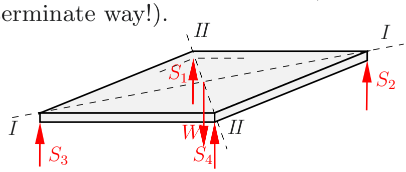
Problem 2.12 A *rigid* quadratic plate (weight W, edge length $\sqrt{2}\,a$) is supported on 4 elastic posts. The posts are of equal length l, but possess different axial rigidities. **P2.12**



Determine the weight distribution on the 4 posts?

Determine the displacement f in the middle of the plate.

Solution The system is statically indeterminate of degree one (a table on 3 posts rests in a statically determinate way!).
Equilibrium yields

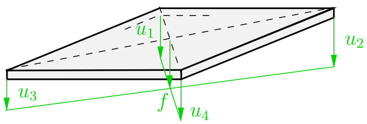


$$\uparrow: \quad S_1 + S_2 + S_3 + S_4 = W\,,$$

$$\overset{\curvearrowright}{I}: \quad aS_4 = aS_1\,,$$

$$\overset{\curvearrowright}{II}: \quad aS_2 = aS_3\,.$$

The displacement f in the middle is obtained from the average value of the displacements u_i (= length change of the posts) at opposite corners (rigid plate). Accordingly the compatibility reads:



$$f = \frac{1}{2}(u_1 + u_4) = \frac{1}{2}(u_2 + u_3)\,.$$

With Hooke's law

$$u_i = \frac{S_i l}{EA_i}$$

and $S_1 = S_4$, $S_2 = S_3$ we obtain as intermediate result

$$\frac{S_1 l}{EA} + \frac{S_1 l}{4EA} = \frac{S_2 l}{2EA} + \frac{S_2 l}{3EA} \qquad \rightsquigarrow \qquad \frac{5}{4}S_1 = \frac{5}{6}S_2\,.$$

Inserting this into the first equilibrium condition yields

$$S_1 + \frac{3}{2}S_1 + \frac{3}{2}S_1 + S_1 = G \quad \rightsquigarrow \quad \underline{\underline{S_1 = S_4 = \frac{1}{5}G}}\,, \qquad \underline{\underline{S_2 = S_3 = \frac{3}{10}G}}\,.$$

form which the displacement follows:

$$\underline{\underline{f = \frac{1}{2}\left(\frac{S_1 l}{EA} + \frac{S_1 l}{4EA}\right) = \frac{1}{8}\frac{Gl}{EA}}}\,.$$

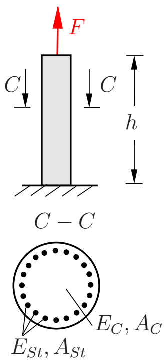
P2.13

Problem 2.13 A column of steel reinforced concrete is loaded by a tensile force F.

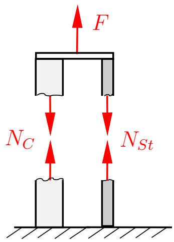
What are the stresses in the concrete and the steel as well as the height change Δh of the column, if we assume

a) a perfect bonding between steel and concrete?
b) the concrete is cracked and does not carry any load?

Given: $E_{St}/E_C = 6, \;\; A_{St}/A_C = 1/9$.



Solution **to a)** We consider the composite as a system of two "bars" of different materials, which experience under load F the same length change Δl. With this the basic equations of the system are:



equilibrium: $N_{St} + N_C = F$,

kinematics: $\Delta h_{St} = \Delta h_C = \Delta h$,

Hooke's law: $\Delta h_{St} = \dfrac{N_{St}h}{EA_{St}}, \quad \Delta h_C = \dfrac{N_C h}{EA_C}$.

Solution of the system of equation yields – with the stiffness ratio $EA_C/EA_{St} = 3/2$ – the normal forces

$$N_{St} = F\,\frac{1}{1+\dfrac{EA_C}{EA_{St}}} = \frac{2}{5}\,F\,, \qquad N_C = F\,\frac{\dfrac{EA_C}{EA_{St}}}{1+\dfrac{EA_C}{EA_{St}}} = \frac{3}{5}\,F$$

and the height change

$$\underline{\underline{\Delta h = \frac{Fh}{EA_{St}+EA_C}}} \qquad \text{i. e.} \qquad \underline{\underline{\Delta h}} = \frac{Fh}{EA_{St}}\,\frac{1}{1+\dfrac{EA_C}{EA_{St}}} = \underline{\underline{\frac{2}{5}\,\frac{Fl}{EA_{St}}}}\,.$$

The stresses result from $A = A_C + A_{St}$ and $A_{St} = A/10$ and $A_C = 9A/10$

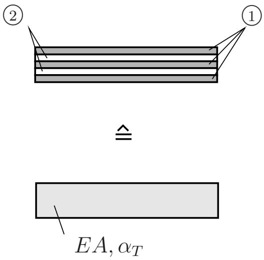
$$\underline{\underline{\sigma_{St}}} = \frac{N_{St}}{A_{St}} = \underline{\underline{4\,\frac{F}{A}}}\,, \qquad \underline{\underline{\sigma_C}} = \frac{N_C}{A_C} = \underline{\underline{\frac{2}{3}\,\frac{F}{A}}}\,.$$

to b) If only the steel carries load, we will obtain with $N_{St} = F$

$$\underline{\underline{\sigma_{St}}} = \frac{F}{A_{St}} = \underline{\underline{10\,\frac{F}{A}}}\,, \qquad \underline{\underline{\Delta h = \frac{Fh}{EA_{St}}}}\,.$$

Problem 2.14 A laminated bar made of bonded layers of two different materials (respective axial rigidities EA_1, EA_2 and coefficients of thermal expansion α_{T1}, α_{T2}) is to be replaced by a bar made of a homogeneous material. **P2.14**

Determine EA and α_T such that the homogeneous bar experiences the same elongation as the laminated bar under application of a force and a temperature change ?



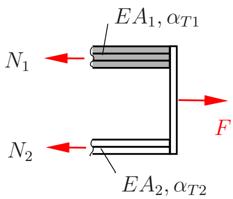
Solution For the laminated bar, subjected to a force F and a temperature increase ΔT, the basic equation yield

equilibrium: $N_1 + N_2 = F$,

kinematics: $\Delta l_1 = \Delta l_2 = \Delta l_{\text{lam}}$,

Hooke's law: $\Delta l_1 = \dfrac{N_1 l}{EA_1} + \alpha_{T1}\Delta T\, l$,

$$\Delta l_2 = \frac{N_2 l}{EA_2} + \alpha_{T2}\Delta T\, l\,.$$



This yields

$$\Delta l_{\text{lam}} = \frac{Fl}{EA_1 + EA_2} + \frac{EA_1\alpha_{T1} + EA_2\alpha_{T2}}{EA_1 + EA_2}\Delta T\, l\,.$$

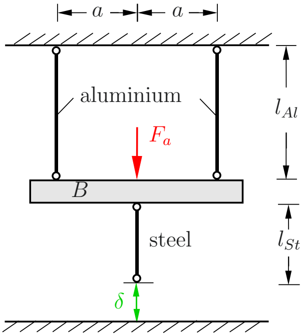
For a homogeneous bar under identical loading conditions, we have

$$\Delta l_{\text{hom}} = \frac{Fl}{EA} + \alpha_T \Delta T\, l\,.$$

The length changes Δl_{lam} and Δl_{hom} agree for arbitrary F and ΔT only, if

$$\underline{\underline{EA = EA_1 + EA_2}}\,, \qquad \underline{\underline{\alpha_T = \frac{EA_1\alpha_{T1} + EA_2\alpha_{T2}}{EA_1 + EA_2}}}\,.$$

P2.15 **Problem 2.15** In the depicted support construction for the *rigid* body B the lower support bar is too short by the length δ. In order to assemble the structure a force F_a is applied, such that the end of the bar just touches the ground. After assembly the force F_a is removed. The diameters of all bars d_i are identical.



a) Compute the required assembly force F_a.

b) Determine the displacement v_B of the body and the forces in the bars after assembly.

Given: $l_{Al} = 1\,\text{m}$, $d_{Al} = 2\,\text{mm}$, $E_{Al} = 0.7 \cdot 10^5\,\text{MPa}$, $l_{St} = 1.5\,\text{m}$, $d_{St} = 2\,\text{mm}$, $E_{St} = 2.1 \cdot 10^5\,\text{MPa}$, $\delta = 5\,\text{mm}$.

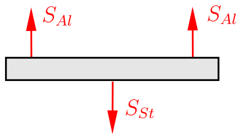
Solution to a) Each aluminium bar carries half of the assembly force (equilibrium) and elongates by the amount delta δ. This yields

$$S_{Al} = \frac{F_a}{2}\,, \qquad \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \frac{F_a l_{Al}}{2EA_{Al}} = \delta\,,$$

$$\leadsto \quad \underline{\underline{F_a}} = 2\frac{\delta}{l_{Al}} EA_{Al} = 2 \cdot \frac{5}{1000} \cdot 0,7 \cdot 10^5 \cdot \pi \cdot 1^2 = \underline{\underline{2200\,\text{N}}}\,.$$

to b) After removal of the force F_a new forces S_{Al} and S_{St} are present. This leads to the equilibrium condition

$$S_{St} = 2 S_{Al}\,,$$



Hooke's law

$$\Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}}\,, \qquad \Delta l_{St} = \frac{S_{St} l_{St}}{EA_{St}}$$

and the compatibility condition

$$\Delta l_{Al} + \Delta l_{St} = \delta\,.$$

Solving the 4 equations yields

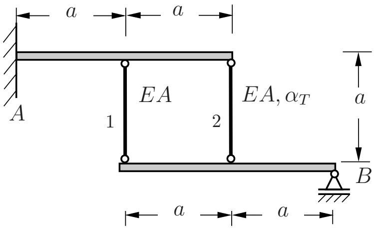
$$\underline{\underline{S_{Al}}} = \frac{\delta}{l_{Al}} \frac{EA_{Al}}{1 + 2\,\frac{l_{St}}{l_{Al}}\,\frac{EA_{Al}}{EA_{St}}} = \frac{5}{1000}\,\frac{0,7 \cdot 10^5 \cdot \pi \cdot 1^2}{1 + 2 \cdot \frac{3}{2} \cdot \frac{1}{3}} = \underline{\underline{550\,\text{N}}}\,,$$

$$\underline{\underline{S_{St}}} = 2 S_{Al} = \underline{\underline{1100\,\text{N}}}\,, \qquad \underline{\underline{v_K}} = \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \underline{\underline{2.5\,\text{mm}}}\,.$$

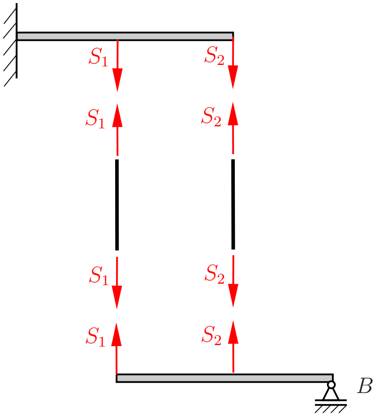
Problem 2.16 Two *rigid* beams are connected by two elastic bars. The first beams is fixed at point A, while the second is simply supported at point B. Bar 2 is heated by a temperature ΔT.

P2.16

Compute the forces in the two bars.



Solution We cut the system and use the following free body diagram to formulate the equilibrium conditions



$$\overset{\curvearrowright}{B} : \quad 2aS_1 + aS_2 = 0\,,$$

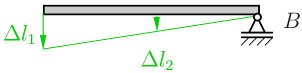
Hooke's law

$$\Delta l_1 = \frac{S_1 a}{EA}\,,$$

$$\Delta l_2 = \frac{S_2 a}{EA} + \alpha_T \Delta T \cdot a$$

and the compatibility condition

$$\Delta l_1 = 2\Delta l_2\,.$$



Solving for the unknown forces in the bars yields

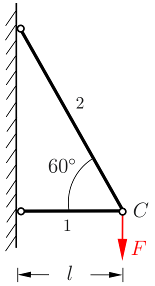
$$\underline{\underline{S_1 = \frac{2}{5}\, EA\, \alpha_T \Delta T\,,}} \qquad \underline{\underline{S_2 = -\frac{4}{5}\, EA\, \alpha_T \Delta T\,.}}$$

Note: In the heated bar compressive forces are generated due to the constrained deformations.

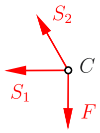
P2.17

Problem 2.17 In the depicted two bar system both bars have the same axial rigidity EA.

Determine the displacement of point C where the load is applied.



Solution From equilibrium we have



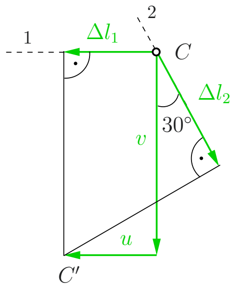
$$\uparrow: \quad S_2 \sin 60^\circ = F \qquad \rightsquigarrow \qquad S_2 = \frac{2}{3}\sqrt{3}\,F\,,$$

$$\rightarrow: \quad -S_1 - S_2 \cos 60^\circ = 0 \rightsquigarrow \qquad S_1 = -\frac{1}{3}\sqrt{3}\,F\,.$$

Thus the elongation and shrinking of the bars follow as

$$\Delta l_2 = \frac{S_2 l_2}{EA} = \frac{\frac{2}{3}\sqrt{3}\frac{l}{\cos 60^\circ}F}{EA} = \frac{4\sqrt{3}}{3}\,\frac{Fl}{EA}\,, \quad \Delta l_1 = \frac{S_1 l_1}{EA} = -\frac{\sqrt{3}}{3}\,\frac{Fl}{EA}\,.$$

To determine the displacements of point C we construct the displacement diagram. In this diagram the length changes are introduced. As the length changes are small $\Delta l_i \ll l$ they are not drawn to the scale. In this example Δl_1 is a shrinkage (to the left) and Δl_2 an elongation. Considering that the bars can only rotate around the hinge points we introduce the right angles and read off the displacement diagram:



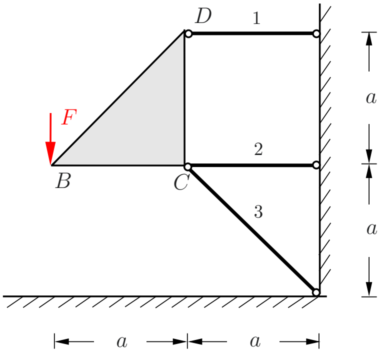
$$\underline{\underline{u}} = |\Delta l_1| = \underline{\underline{\frac{\sqrt{3}}{3}\,\frac{Fl}{EA}}}\,,$$

$$\underline{\underline{v}} = \frac{\Delta l_2}{\cos 30^\circ} + \frac{u}{\tan 60^\circ} = \frac{4\sqrt{3}}{3}\,\frac{Fl}{EA}\,\frac{1}{\frac{1}{2}\sqrt{3}} + \frac{\sqrt{3}}{3}\,\frac{Fl}{EA}\,\frac{1}{\sqrt{3}} = \underline{\underline{3\,\frac{Fl}{EA}}}\,.$$

Problem 2.18 A rigid weightless triangle is supported by 3 bars with the axial rigidity EA. The triangle is loaded in point B by the force F. **P2.18**

a) Determine the forces S_i in the 3 bars and their elongations Δl_i.

b) Compute the displacement of point C.

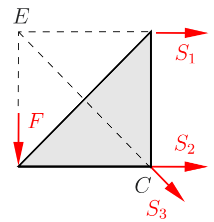


Solution to a) The system is statically determinately supported. The forces in the bars follow immediately from the equilibrium conditions:

$$\overset{\curvearrowleft}{C}: \; aS_1 = aF \qquad \rightsquigarrow \quad \underline{\underline{S_1 = F}}\,,$$

$$\overset{\curvearrowleft}{E}: \; aS_2 = 0 \qquad \rightsquigarrow \quad \underline{\underline{S_2 = 0}}\,,$$

$$\uparrow: \; S_3 \sin 45^\circ + F = 0 \rightsquigarrow \quad \underline{\underline{S_3 = -\sqrt{2}\,F}}\,.$$

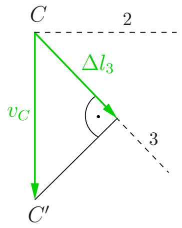


Related to these forces are the following elongations

$$\underline{\underline{\Delta l_1}} = \frac{S_1 l_1}{EA} = \underline{\underline{\frac{Fa}{EA}}}\,, \qquad \underline{\underline{\Delta l_2 = 0}}\,,$$

$$\underline{\underline{\Delta l_3}} = \frac{S_3 l_3}{EA} = -\frac{\sqrt{2}\,F \cdot \sqrt{2}\,a}{EA} = \underline{\underline{-2\frac{Fa}{EA}}}\,.$$

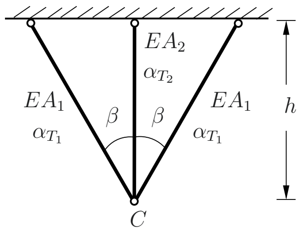
to b) The displacement of point C is sketched in the displacement diagram. As bar 2 experiences no force and thus no length change, the horizontal displacement vanishes. From the displacement diagram we obtain for the vertical displacement v_C:



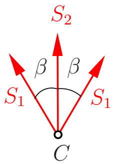
$$\underline{\underline{v_C}} = \sqrt{2}\,|\Delta l_3| = \underline{\underline{2\sqrt{2}\,\frac{Fa}{EA}}}\,.$$

P 2.19 **Problem 2.19** In the depicted truss the members have the axial rigidities EA_1, EA_2 and the coefficients of thermal expansion α_{T1}, α_{T2}.

Determine the axial forces in the trusses, if the system is heated by ΔT?



Solution As the system is statically indeterminate, we have to use all basic equations. We start with the equilibrium



$$2S_1 \cos\beta + S_2 = 0$$

and continue with Hooke's law

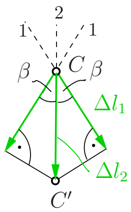
$$\Delta l_1 = \frac{S_1 l_1}{EA_1} + l_1 \alpha_{T1} \Delta T \,,$$

$$\Delta l_2 = \frac{S_2 l_2}{EA_2} + l_2 \alpha_{T2} \Delta T \,,$$

where

$$l_1 = \frac{h}{\cos\beta} \,, \qquad l_2 = h \,.$$

The compatibility of the displacements is according to the the displacement diagram:



$$\Delta l_1 = \Delta l_2 \cos\beta \,.$$

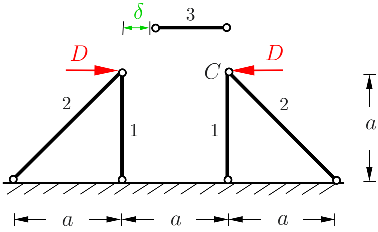
Solving the 4 equations for the two truss forces and the two elongations yields

$$\underline{\underline{S_1 = EA_1 \frac{\alpha_{T2}\cos^2\beta - \alpha_{T1}}{1 + 2\cos^3\beta \dfrac{EA_1}{EA_2}} \Delta T}} \,, \qquad \underline{\underline{S_2 = -2\cos\beta \; S_1}} \,.$$

Note: For $\cos\beta = \sqrt{\alpha_{T1}/\alpha_{T2}}$ we obtain $S_1 = S_2 = 0$: the trusses can than expand without causing forces! (special case $\alpha_{T1} = \alpha_{T2}$ $\rightsquigarrow$ $\beta = 0$)

Problem 2.20 Truss member 3 was produced too short to be assembled between two identical trusses. **P2.20**

a) Determine the required assembly force D?
b) Calculate the normal force S_3 after the assembly ($D = 0$)?

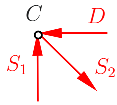


Given: $EA_1 = EA_3 = EA$, $EA_2 = \sqrt{2}\,EA$.

Solution to a) The force D has to move point C by $\delta/2$ in horizontal direction during assembly. From equilibrium

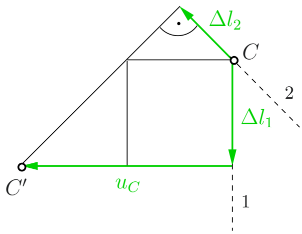
$$\rightarrow: \quad S_2 \cos 45° = D\,,$$

$$\uparrow: \quad S_1 = S_2 \cos 45°\,,$$



kinematics (S_1 was positively introduced as compressive force!!) with the prescribed displacement

$$u_C = \Delta l_1 + \Delta l_2 \sqrt{2}\,, \qquad u_C = \frac{\delta}{2}\,,$$



and Hooke's law

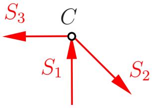
$$\Delta l_1 = \frac{S_1 a}{EA}\,, \qquad \Delta l_2 = \frac{S_2 a \sqrt{2}}{\sqrt{2}\,EA}$$

we obtain

$$\underline{\underline{D = \frac{1}{6}\,\frac{\delta}{a}\,EA\,.}}$$

to b) Equlibrium, kinematics and Hooke's law are as in a), but D has to be replaced by S_3. With the known compatibility condition

$$2u_C + \Delta l_3 = \delta \qquad \text{and} \qquad \Delta l_3 = \frac{S_3 a}{EA}$$

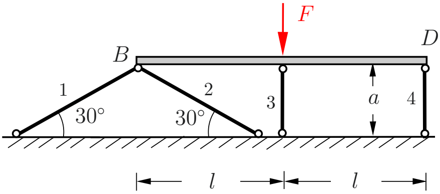


it follows

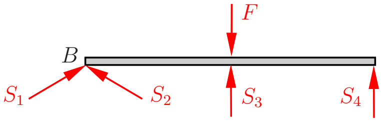
$$\underline{\underline{S_3 = \frac{1}{7}\,\frac{\delta}{a}\,EA\,.}}$$

P2.21 **Problem 2.21** A centric loaded *rigid* beam is supported by 4 elastic bars of equal axial rigidity EA.

Determine the forces in the bars?



Solution **a)** First, we solve the statically indeterminate system by applying all basic equations simultaneously. Using equilibrium



$$\rightarrow: \quad S_1 = S_2 \ ,$$

$$\uparrow: \quad (S_1 + S_2)\sin 30^\circ + S_3 + S_4 = F \ ,$$

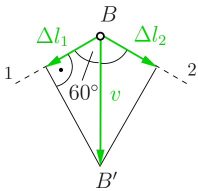
$$\overset{\curvearrowleft}{B}: \quad lS_3 + 2lS_4 = lF \ ,$$

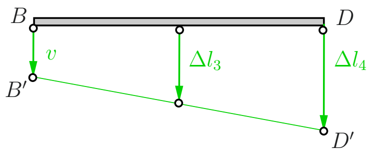
Hooke's laws

$$\Delta l_1 = \Delta l_2 = \frac{S_1 2a}{EA} \ ,$$

$$\Delta l_3 = \frac{S_3 a}{EA} \ , \qquad \Delta l_4 = \frac{S_4 a}{EA}$$

and the geometry of the deformation



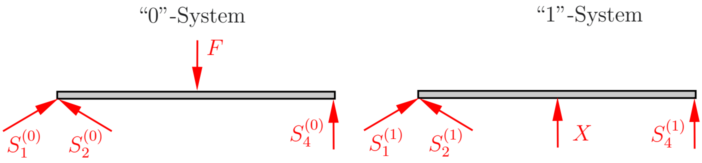


$$v = \frac{\Delta l_1}{\cos 60^\circ} \qquad\qquad \Delta l_3 = \frac{1}{2}(v + \Delta l_4)$$

we obtain as solution

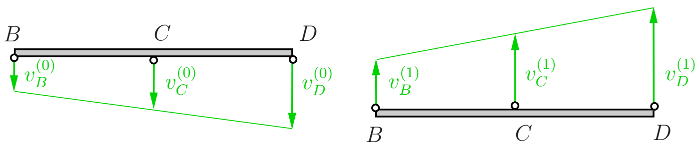
$$\underline{\underline{S_1 = S_2 = S_4 = \frac{2}{9}\,F}} \ , \qquad \underline{\underline{S_3 = \frac{5}{9}F}} \ .$$

b) Now, we solve the problem by superposition. The system is divided into two statically determinate basic systems:



Equilibrium yields

$$S_1^{(0)} = S_2^{(0)} = S_4^{(0)} = \frac{F}{2}\,, \qquad S_1^{(1)} = S_2^{(1)} = S_4^{(1)} = \frac{X}{2}\,.$$



From geometry and Hooke's laws it follows

$$v_B^{(0)} = \frac{\Delta l_1^{(0)}}{\cos 60^\circ} = \frac{F\,2a}{EA}\,, \qquad v_B^{(1)} = \frac{X\,2a}{EA}\,,$$

$$v_D^{(0)} = \Delta l_4^{(0)} = \frac{Fa}{2EA}\,, \qquad v_D^{(1)} = \frac{Xa}{2EA}\,,$$

$$v_C^{(0)} = \frac{1}{2}\left(v_B^{(0)} + v_D^{(0)}\right) = \frac{5}{4}\frac{Fa}{EA}\,, \qquad v_C^{(1)} = \frac{5}{4}\frac{Xa}{EA}\,,$$

$$\Delta l_3^{(1)} = \frac{Xa}{EA}\,.$$

The kinematic compatibility requires the total displacement of point C to coincide with the shortening of truss 3:

$$v_C^{(0)} - v_C^{(1)} = \Delta l_3^{(1)}\,.$$

Inserting the displacements yields

$$\underline{\underline{X = S_3 = \frac{5}{9}\,F}}$$

and

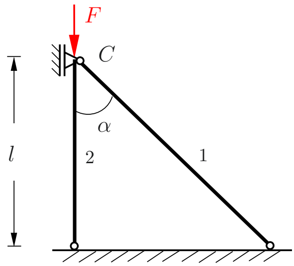
$$\underline{\underline{S_1}} = S_1^{(0)} - S_1^{(1)} = \underline{\underline{\frac{2}{9}\,F}}\,, \qquad \underline{\underline{S_4}} = S_4^{(0)} - S_4^{(1)} = \underline{\underline{\frac{2}{9}\,F}}\,.$$

P2.22

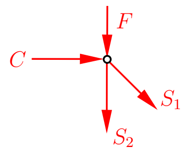
Problem 2.22 The depicted truss system (axial rigidity EA) is loaded by the external force F and additionally pinned at point C.

a) Determine the reaction force at point C.

b) Calculate the vertical displacement of point C.



Solution **to a)** Using equilibrium



$$\downarrow: \quad F + S_2 + S_1 \cos\alpha = 0\,,$$

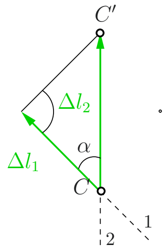
$$\rightarrow: \quad C + S_1 \sin\alpha = 0\,,$$

Hooke's laws

$$\Delta l_1 = \frac{S_1 l_1}{EA}\,, \qquad \Delta l_2 = \frac{S_2 l_2}{EA}\,,$$

and kinematics

$$\Delta l_1 = \Delta l_2 \cos\alpha$$



yields

$$\underline{\underline{C = \frac{\sin\alpha\cos^2\alpha}{1+\cos^3\alpha}\,F}}\,, \qquad \underline{\underline{S_1 = -\frac{\cos^2\alpha}{1+\cos^3\alpha}\,F}}\,, \qquad \underline{\underline{S_2 = -\frac{1}{1+\cos^3\alpha}\,F}}\,.$$

to b) Knowing S_2 the vertical displacement of point C follows as

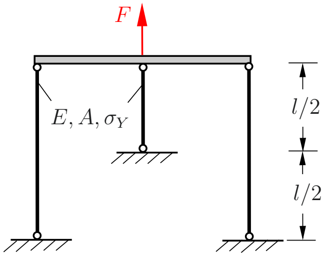
$$\underline{\underline{v_C}} = \Delta l_2 = \frac{S_2 l}{EA} = \underline{\underline{-\frac{1}{1+\cos^3\alpha}\,\frac{Fl}{EA}}}\,.$$

In contrast to the displacement diagram, in which tensile forces (elongations) are assumed, compressive force occur in the system. Due to shortening point C moves in downwards direction.

Test: $\alpha = \pi/2$ yields $S_1 = 0$ and $S_2 = -F$.

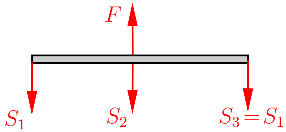
$\alpha = 0$ yields $S_1 = S_2 = -F/2$.

Problem 2.23 A *rigid* beam is supported by three bars of elastic-ideal-plastic material. P2.23



a) At what force F_{max}^{el} and at which location in the bars is the yield stress σ_Y reached at first?
b) At what force F_{max}^{pl} occurs plastic yielding in all bars of the system?

Solution to a) The system is statically indeterminate. Using symmetry equilibrium provides



$$2S_1 + S_2 = F$$

Kinematics is expressed by

$$\Delta l_1 = \Delta l_2 \, .$$

Until plastic yielding Hooke's law can be used

$$\Delta l_1 = \frac{S_1 l}{EA} \, , \qquad \Delta l_2 = \frac{S_2 l}{2EA} \, .$$

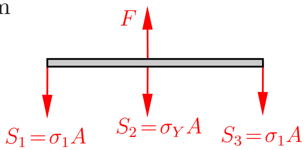
The solution provides forces and stresses in the bars

$$S_1 = \frac{F}{4} \, , \qquad S_2 = \frac{F}{2} \qquad \rightsquigarrow \qquad \sigma_1 = \frac{F}{4A} \, , \qquad \sigma_2 = \frac{F}{2A} \, .$$

As the stress in bar 2 is the highest, the yield limit is reached first there during load increase:

$$\sigma_2 = \sigma_Y \qquad \rightsquigarrow \qquad \underline{\underline{F_{max}^{el} = 2\,\sigma_Y A}} \, .$$

to b) For a load increase above F_{max}^{el} bar 1 and bar 3 still respond elastically, while bar two undergoes plastic deformation: $\sigma_2 = \sigma_Y$. Thus with $S_i = \sigma_i A$ it follows from equilibrium



$$2\sigma_1 A + \sigma_Y A = F$$

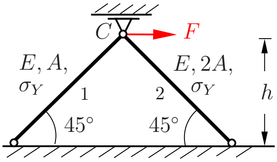
$$\rightsquigarrow \quad \sigma_1 = \frac{F}{2A} - \frac{\sigma_Y}{2} \, .$$

All bars are undego plastic deformation if

$$\sigma_1 = \sigma_Y \quad \rightsquigarrow \quad \frac{F}{2A} - \frac{\sigma_Y}{2} = \sigma_Y \qquad \rightsquigarrow \qquad \underline{\underline{F_{max}^{pl} = 3\,\sigma_Y A}} \, .$$

P2.24

Problem 2.24 In the depicted symmetric system all bars are made of the same elastic-ideal-plastic material, but have different cross sections.



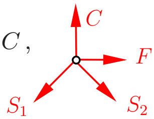
a) At what force F_{max}^{el} and at which location in the bars is the yield stress σ_Y reached at first? Determine the reaction force at C for this situation.

b) Determine the force F_{max}^{pl} when both bars deform plastically?

c) Calculate the displacement u_{max}^{el} of point C for case a)?

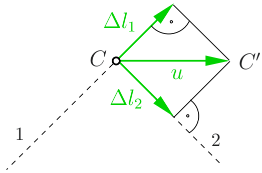
Solution **to a)** Until reaching the force F_{max}^{el} the system responds elastically. Therefore the equilibrium conditions are given by

$$\rightarrow: \ \frac{\sqrt{2}}{2} S_1 - \frac{\sqrt{2}}{2} S_2 = F\,, \qquad \uparrow: \ \frac{\sqrt{2}}{2} S_1 + \frac{\sqrt{2}}{2} S_2 = C\,,$$



together with Hooke's law

$$\Delta l_1 = \frac{S_1 \sqrt{2}\, h}{EA}\,, \qquad \Delta l_2 = \frac{S_2 \sqrt{2}\, h}{2EA}$$



and the kinematics (bar 2 will shorten)

$$\Delta l_1 = -\Delta l_2\,.$$

From the above relation we obtain

$$S_1 = \frac{\sqrt{2}}{3} F\,, \quad S_2 = -\frac{2\sqrt{2}}{3} F\,, \qquad C = -\frac{F}{3}\,, \quad \Delta l_1 = -\Delta l_2 = \frac{2Fh}{3EA}$$

$$\leadsto \qquad \sigma_1 = \frac{S_1}{A} = \frac{\sqrt{2}}{3}\frac{F}{A}\,, \qquad \sigma_2 = \frac{S_2}{2A} = -\frac{\sqrt{2}}{3}\frac{F}{A}\,.$$

The absolute value of the stresses is identical in both bars. Yielding will occur if

$$\sigma_1 = |\sigma_2| = \sigma_Y \quad \leadsto \quad \underline{\underline{F_{max}^{el} = \frac{3}{2}\sqrt{2}\,\sigma_Y A}}, \quad \leadsto \quad \underline{\underline{C_{max}^{el} = -\frac{\sqrt{2}}{2}\,\sigma_Y A.}}$$

to b) As at F_{max}^{el} plastic yielding occurs in both bars, we have

$$\underline{\underline{F_{max}^{el} = F_{max}^{pl}}}.$$

to c) Until the yield limit is reached the displacement of C is given by

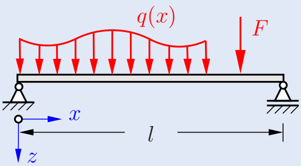
$$u = \sqrt{2}\,\Delta l_1 = \frac{2\sqrt{2}}{3}\frac{Fh}{EA}\,, \qquad \leadsto \qquad \underline{\underline{u_{max}^{el}}} = u(F_{max}^{el}) = \underline{\underline{2\,\frac{\sigma_Y}{E}\,h}}\,.$$

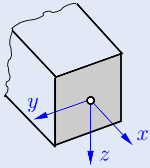
Chapter 3

Bending of Beams

3

Beam = straight structural element, length l large compared to dimensions of the cross section, perpendicular loads.





3.1

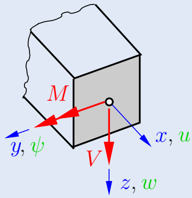
3.1 Ordinary bending

nomenclature and assumptions:

- x = axis of cross section centroids; y, z = principal axis of the second moment of area (moment of inertia).
- kinematic assumption: *plane cross sections remain plane*

$$w = w(x)\,, \qquad u = z\,\psi(x)\,,$$

w = displacement in z-direction,
u = displacement in x-direction,
ψ = rotation angle of cross section.

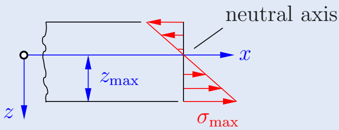


- stress resultants:

V = V_z = shear force,
M = M_y = bending moment.

Normal stress

$$\boxed{\sigma(z) = \frac{M}{I}\,z}$$



I = moment of inertia with respect to y-axis,
z = distance to *neutral axis* (= axis of centroids).

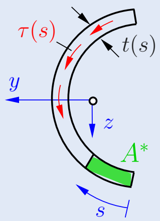
The largest absolute value of the stress occurs in the extreme fibre:

$$\boxed{\sigma_{\max} = \frac{M}{W}\,,} \qquad W = \frac{I}{|z_{\max}|} \quad = \quad \text{section modulus.}$$

Shear stress

a) thin-walled, open profile

$$\tau(s) = \frac{V\ S(s)}{I\ t(s)}\ ,$$

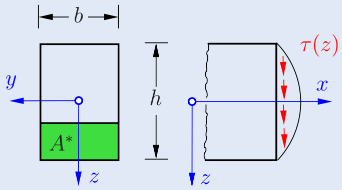


$S(s)$ = static moment of A^* with regard to y-axis,

$t(s)$ = thickness of profile at position s.

b) compact cross section

$$\tau(z) = \frac{V\ S(z)}{I\ b(z)}\ .$$



special case: rectangle

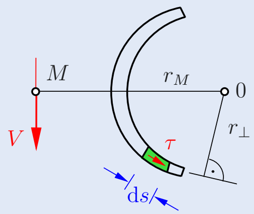
$$\tau = \frac{3}{2}\frac{Q}{A}\left(1 - \frac{4z^2}{h^2}\right)\ .$$

Note: $\tau_{\text{max}} = \tau(z=0) = \frac{3}{2}\frac{Q}{bh}$ is 50% larger than $\tau_{\text{mean}} = \frac{Q}{bh}$.

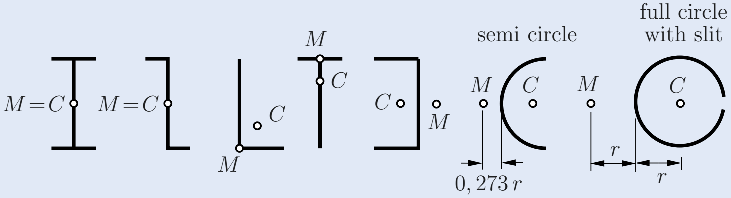
Shear center M of singly symmetrical cross sections.

moment of V with regard to 0 = moment of distributed shear stresses with regard to 0:

$$r_M Q = \int \tau(s)\, r_\perp(s)\, t(s)\, \mathrm{d}s$$



Position of centriod C und shear center M for selected profiles:



Basic equations

equilibrium conditions	$\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q\,, \quad \dfrac{\mathrm{d}M}{\mathrm{d}x} = V\,,$
Hooke's law, kinematics	$M = EI\psi'$
	$V = GA_S(\psi + w')\,,$

EI	$=$	bending stiffness,
GA_S	$=$	shear stiffness,
A_S	$=$	κA = shear area (κ = shear correction factor).

Rigid with respect to shear (Bernoulli beam): If we additionally assume, that cross sections perpendicular to the undeformed beam axis remain perpendicular to the deflection curve during the deformation, it follows from Hooke's law for the shear force ($GA_S \to \infty$)

$$\psi = -w'\,.$$

Differential equation of the deflection curve for the Bernoulli beam: Inserting into Hooke's law for M yields

$$\boxed{EIw'' = -M\,.}$$

This leads with the equilibrium conditions to

$$(EIw'')'' = q\,,$$

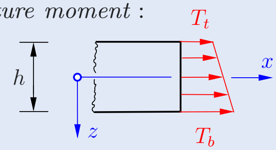
or for $EI = \text{const}$

$$\boxed{EIw^{IV} = q\,.}$$

Temperature induced moment

A linearly, across the height h, varying temperature field (= temperature gradient) can be treated by a *temperature moment* :

$$M_T = EI\alpha_T\,\frac{T_b - T_t}{h}\,,$$

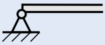
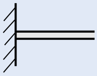
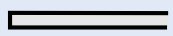
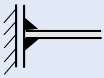


α_T = coefficient of thermal expansion.

In this case, the differential equation for the deflection curve yields

$$\boxed{EIw'' = -(M + M_T)\,.}$$

Table of boundary conditions

support	w	w'	M	V
	0	$\neq 0$	0	$\neq 0$
	0	0	$\neq 0$	$\neq 0$
 free end	$\neq 0$	$\neq 0$	0	0
	$\neq 0$	0	$\neq 0$	0

Solution methods

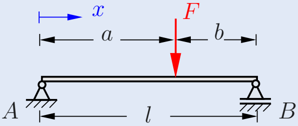
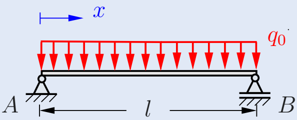
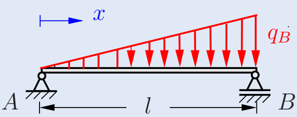
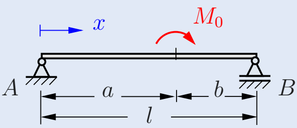
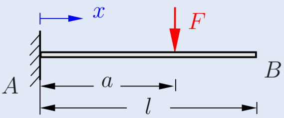
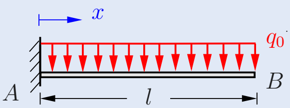
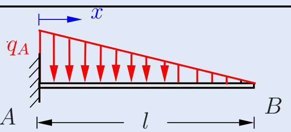
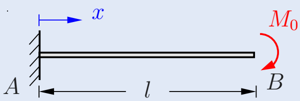
1. For continuous functions of $q(x)$ or $M(x)$, four or two times integration of the corresponding differential equation yields the deflection curve $w(x)$. The four or two integration constants are obtained by the boundary conditions (see table of boundary conditions).

2. For several regions (discontinuities in the loads, deformation, concentrated forces or concentrated moments), the integration has to be performed piecewise. The integration constants are determined from boundary and matching (continuity) conditions. The computation can by simplified by using the Macauley bracket (see Engineering Mechanics 1):

$$< x - a >^n = \begin{cases} 0 & \text{für } x < a\,, \\ (x-a)^n & \text{für } x > a\,. \end{cases}$$

3. Statically indeterminate problems can be solved by using *superposition* of known deflections and rotations. For this purpose, deflection and rotations of the most frequent load cases and support situations can be found in the table on page 62/63.

4. Statically indeterminate problems can also be solved by using the *principle of virtual forces (energy method)* (see chapter 5).

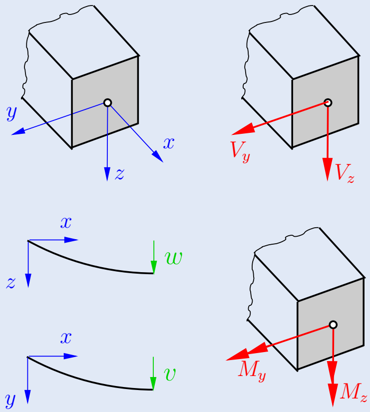
no.	load case	EIw'_A	EIw'_B
1		$\dfrac{Fl^2}{6}(\beta-\beta^3)$	$-\dfrac{Fl^2}{6}(\alpha-\alpha^3)$
2		$\dfrac{q_0 l^3}{24}$	$-\dfrac{q_0 l^3}{24}$
3		$\dfrac{7}{360} q_B l^3$	$-\dfrac{1}{45} q_B l^3$
4		$\dfrac{M_0 l}{6}(3\beta^2-1)$	$\dfrac{M_0 l}{6}(3\alpha^2-1)$
5		0	$\dfrac{Fa^2}{2}$
6		0	$\dfrac{q_0 l^3}{6}$
7		0	$\dfrac{q_A l^3}{24}$
8		0	$M_0 l$

explanations:
$$\xi = \frac{x}{l}\,, \quad \alpha = \frac{a}{l}\,, \quad \beta = \frac{b}{l}\,, \quad (\,)' \mathrel{\widehat{=}} \frac{\mathrm{d}}{\mathrm{d}x}(\,) = \frac{1}{l}\,\frac{\mathrm{d}}{\mathrm{d}\xi}(\,)\,,$$

$EIw(x)$	$EIw_{\max}$
$\frac{Fl^3}{6}[\beta\xi(1-\beta^2-\xi^2)+<\xi-\alpha>^3]$	$\frac{Fl^3}{48}$ for $\alpha=\beta=1/2$
$\frac{q_0 l^4}{24}(\xi-2\xi^3+\xi^4)$	$\frac{5}{384}q_0 l^4$
$\frac{q_B l^4}{360}(7\xi-10\xi^3+3\xi^5)$	see problem 3.13
$\frac{M_0 l^2}{6}[\xi(3\beta^2-1)+\xi^3-3<\xi-\alpha>^2]$	$\frac{M_0 l^2}{27}\sqrt{3}$ for $a=0$
$\frac{Fl^3}{6}[3\xi^2\alpha-\xi^3+<\xi-\alpha>^3]$	$\frac{Fl^3}{3}$ for $a=l$
$\frac{q_0 l^4}{24}(6\xi^2-4\xi^3+\xi^4)$	$\frac{q_0 l^4}{8}$
$\frac{q_A l^4}{120}(10\xi^2-10\xi^3+5\xi^4-\xi^5)$	$\frac{q_A l^4}{30}$
$M_0\frac{x^2}{2}$	$M_0\frac{l^2}{2}$

$<\xi-\alpha>^n \;\widehat{=}\;$ Macauley bracket

3.2 Biaxial bending



x = axis of centroids,
y, z = arbitrary orthogonal axis.

shear forces V_y , V_z

and

bending moments M_y , M_z (positive when positive right-hand screw at positive intersection).

Differential equation of the deflection for *shear rigid* beams:

$$Ew'' = \frac{1}{\Delta}(-M_y I_z + M_z I_{yz})$$

$$Ev'' = \frac{1}{\Delta}(M_z I_y - M_y I_{yz})$$

$$\Delta = I_y I_z - I_{yz}^2 ,$$

I_y, I_z, I_{yz} = second order area moments.

Normal stress

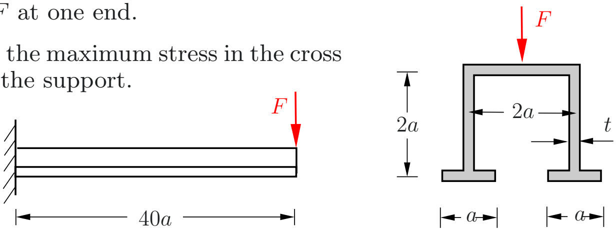
$$\sigma = \frac{1}{\Delta}[(M_y I_z - M_z I_{yz})z - (M_z I_y - M_y I_{yz})y] .$$

Special case: If y, z are *principal axis* ($I_{yz} = 0$), then

$$EI_y w'' = -M_y , \quad EI_z v'' = M_z , \quad \sigma = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y .$$

Problem 3.1 A cantilever beam with the depicted cross section (constant wall thickness t, $t \ll a$ is subjected to a concentrated force F at one end. **P3.1**

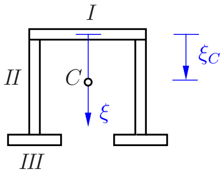
Determine the maximum stress in the cross section at the support.



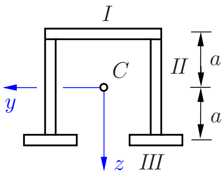
Solution The distance of the centroid ξ_C from the top surface is obtained from the sub-areas by using $t \ll a$

$$\xi_C = \frac{\Sigma \xi_i A_i}{\Sigma A_i} = \frac{2\overbrace{(2at \cdot a)}^{II} + 2\overbrace{(at \cdot 2a)}^{III}}{\underbrace{2at}_{I} + 2 \cdot \underbrace{2at}_{II} + 2 \cdot \underbrace{at}_{III}} = \frac{8a^2 t}{8at}$$

$$= a\,.$$



The second moment of area with regard to the y-axis is computed by using the parallel-axis theorem.



$$I_y = \overbrace{a^2 \cdot 2at}^{I} + 2\overbrace{\left\{\frac{t(2a)^3}{12}\right\}}^{II} + 2\overbrace{\left\{a^2 \cdot at\right\}}^{III} = \frac{16}{3} t a^3 ,$$

Thus we obtain for the section modulus

$$W = \frac{I_y}{z_{\max}} = \frac{\frac{16}{3} t a^3}{a} = \frac{16}{3} t a^2 \,.$$

The stress in the cross section at the support is calculated using the bending moment at this position

$$M = -40\, aF$$

to be

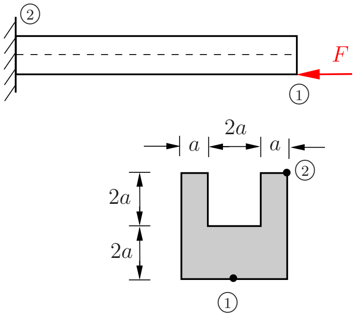
$$\underline{\underline{\sigma_{\max}}} = \frac{|M|}{W} = \frac{40aF}{\frac{16}{3} t a^2} = \underline{\underline{\frac{30}{4}\,\frac{F}{at}}}$$

(the upper fibre is in tension, the lower under compression).

P3.2

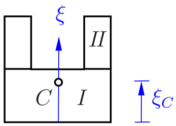
Problem 3.2 A cantilever beam with the sketched cross section is loaded by the force F at point ① .

Determine the normal stresses at point ② at the support.



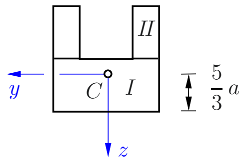
Solution As the neutral axis is passing trough the centroids of the cross sections, we first determine the position of the centroid:

$$\xi_C = \frac{\Sigma A_i \xi_i}{\Sigma A_i} = \frac{\overbrace{8a^2\cdot a}^{I} + 2\overbrace{\left\{2a^2\cdot 3a\right\}}^{II}}{8a^2+4a^2} = \frac{5}{3}a\,.$$



The second moment of area with respect to the y-axis is computed by summing up the contributions of the sub-areas:

$$I_y = \left[\frac{4a(2a)^3}{12} + \left(\frac{2}{3}\,a\right)^2 8a^2\right] + \\ +2\left[\frac{a(2a)^3}{12} + \left(\frac{4}{3}a\right)^2 2a^2\right] = \frac{44}{3}a^4\,.$$



The following stress resultants are present in the cross section at the support

$$N = -F \qquad \text{and} \qquad M_y = -\frac{5}{3}aF\,.$$

The associated stresses are (σ_N due to normal force, σ_M due to bending moment)

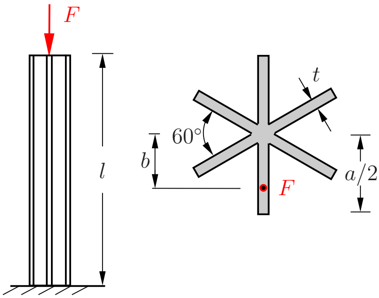
$$\sigma_N = \frac{N}{A} = -\frac{F}{12a^2} \qquad \text{and} \qquad \sigma_M = \frac{M_y}{I_y}z = -\frac{5}{3}\,\frac{aFz}{\frac{44}{3}a^4} = -\frac{5}{44}\,\frac{Fz}{a^3}\,.$$

At point ② superposition with $z_2 = -\frac{7}{3}a$ yields

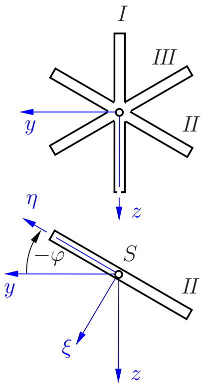
$$\underline{\underline{\sigma}} = \sigma_N + \sigma_M(z_2) = -\frac{F}{12a^2} + \frac{5}{44}\,\frac{F}{a^3}\,\frac{7}{3}a = \underline{\underline{\frac{2}{11}\,\frac{F}{a^2}}}\,.$$

Problem 3.3 The column with a star-shaped cross section ($t \ll a$) is loaded by a force F, applied off center. P3.3

Determine
a) the maximum absolute value of the stress,
b) the maximal value of b such that nowhere in the cross section tensile stresses occur.



Solution **to a)** Due to the load and the symmetry of the cross section it is convenient to introduce the following y, z-coordinate system. This yields



$$I_{yI} = \frac{ta^3}{12}\,.$$

The second moments of area for the sub-areas *II* and *III* with respect to the y-axis are determined by the transformation equations

$$I_\eta = \frac{at^3}{12}\,, \quad I_\zeta = \frac{ta^3}{12}\,, \quad I_{\eta\zeta} = 0\,, \quad \varphi = -30^\circ\,.$$

Using $t \ll a$ we obtain

$$I_{yII} = I_{yIII} = \frac{I_\eta + I_\zeta}{2} + \frac{I_\eta - I_\zeta}{2}\cos 2\varphi + I_{\eta\zeta}\sin 2\varphi = \frac{ta^3}{24} - \frac{ta^3}{24}\,\frac{1}{2} = \frac{ta^3}{48}\,.$$

This leads to

$$I_y = I_{yI} + 2I_{yII} = \frac{ta^3}{12} + 2\frac{ta^3}{48} = \frac{ta^3}{8}\,.$$

Together with the stress resultants $N = -F$ and $M_y = -bF$ it follows

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y}z = -\frac{F}{3at} - \frac{8bF}{ta^3}z\,.$$

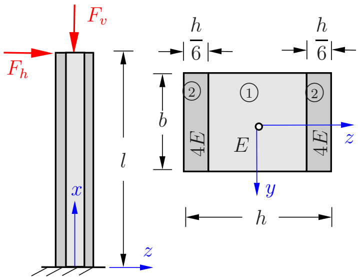
The largest stress (compression) occurs at $z = a/2$:

$$\underline{\underline{\sigma_{\max} = -\frac{F}{at}\left(\frac{1}{3} + 4\frac{b}{a}\right)}}\,.$$

to b) Tensile stress occurs first at $z = -a/2$:

$$\sigma(-\frac{a}{2}) = 0 \quad \rightsquigarrow \quad -\frac{F}{3at} + 4\frac{Fb}{ta^2} = 0 \quad \rightsquigarrow \quad \underline{\underline{b = \frac{a}{12}}}\,.$$

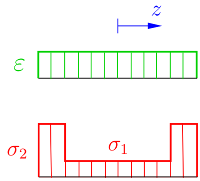
P3.4 **Problem 3.4** A column is clamped at the bottom and is carrying a vertical load F_v at the center of the top cross section and a horizontal load F_h in the middle of edge b. The column is made of 3 layers with different Young's moduli.



Determine the normal stress distribution in the cross section at the clamping.

Solution We consider the different load cases independently.

to a) With the vertical load F_v, we obtain from



equilibrium $\quad \sigma_1 A_1 + \sigma_2 A_2 = -F_v\,,$

Hooke's law $\quad \sigma_i = E_i \varepsilon_i$

and geometry $\quad \varepsilon_1 = \varepsilon_2 = \varepsilon$

the strain

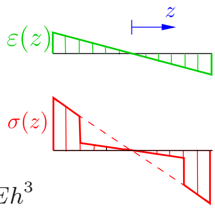
$$E_1\varepsilon_1 A_1 + E_2\varepsilon_2 A_2 = E\varepsilon\frac{2}{3}bh + 4E\varepsilon\frac{1}{3}bh = -F_v \quad \rightsquigarrow \quad \varepsilon = -\frac{F_v}{2Ebh}$$

and the associated stresses

$$\underline{\underline{\sigma_1 = -\frac{F_v}{2bh}}}\,, \qquad \underline{\underline{\sigma_2 = -2\frac{F_v}{bh}}}\,.$$

to b) F_h causes a moment $M_S = -F_h l$ at the support. Then geometry (assume: cross sections remain plane)

$$u = \psi \cdot z \quad \rightsquigarrow \quad \varepsilon = \psi' \cdot z\,,$$



Hooke's law $\quad \sigma(z) = E(z)\varepsilon(z)$

and

$$M = \int \sigma z \mathrm{d}A = 2b\psi'[E_1 \int_0^{h/3} z^2 \mathrm{d}z + E_2 \int_{h/3}^{h/2} z^2 \mathrm{d}z]$$

$$= 2b\psi' E[\frac{1}{3}(\frac{h}{3})^3 + \frac{4}{3}((\frac{h}{2})^3 - (\frac{h}{3})^3)] = \frac{7}{27} b\psi' E h^3$$

lead to (using $M = M_S$)

$$\psi' = -\frac{27}{7}\frac{F_h l}{Ebh^3}\,.$$

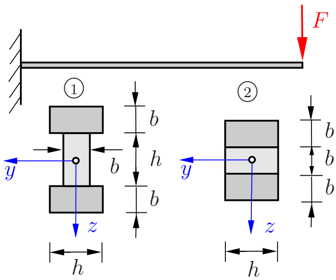
Finally, the stresses follow as

$$\sigma_1 = E_1\psi' z = E\frac{27}{7}\frac{M}{Ebh^3}z \quad \rightsquigarrow \quad \underline{\underline{\sigma_1(\frac{h}{3}) = -\frac{9F_h l}{7bh^2}}}\,,$$

$$\sigma_2 = E_2\psi' z = 4E\frac{27}{7}\frac{M}{Ebh^3}z \quad \rightsquigarrow \quad \underline{\underline{\sigma_2(\frac{h}{2}) = -\frac{54F_h l}{7bh^2}}}\,.$$

Problem 3.5 A wooden cantilever can be assembled from 3 beams (dimensions of the cross section $b = a$ and $h = 2a$) in different ways. **P3.5**

What is the maximal force F for the two variants ① and ② , if the maximal allowed shear stress in the bonding layer is given by τ_{allow}?



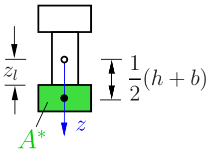
Solution With $V = F$ the shear stress in the bonding layer becomes in general ($z = z_l$)

$$\tau(z_l) = \frac{F S(z_l)}{I\, b(z_l)} \,.$$

This yields with $\tau(z_l) = \tau_{\text{allow}}$ the maximal load $F_{\max}$

$$F_{\max} = \frac{\tau_{\text{allow}} I\, b(z_l)}{S(z_l)} \,.$$

For variant ① we obtain



$$I = \frac{bh^3}{12} + 2\Big[\frac{hb^3}{12} + \Big(\frac{h}{2} + \frac{b}{2}\Big)^2 bh\Big] = 10\, a^4 \,,$$

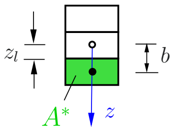
$$b(z_l) = b = a \,,$$

$$S(z_l) = \int_{A^*} z \mathrm{d}A = \frac{1}{2}(h+b)bh = 3\, a^3 \,.$$

which leads to the force

$$\underline{\underline{F_{1\max}}} = \tau_{\text{allow}} \frac{10a^4 \cdot a}{3a^3} = \underline{\underline{\frac{10}{3}\,\tau_{\text{allow}}\, a^2}} \,.$$

Analogously we obtain for variant ②



$$I = \frac{h(3b)^3}{12} = \frac{9}{2} a^4 \,, \qquad b(z_l) = h = 2a \,,$$

$$S(z_l) = \int_{A^*} z \mathrm{d}A = b \cdot bh = 2a^3$$

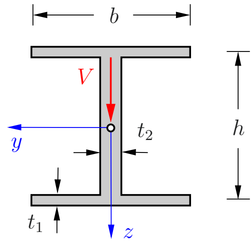
and the force

$$\underline{\underline{F_{2\max}}} = \tau_{\text{allow}} \frac{9a^4 \cdot 2a}{2 \cdot 2a^3} = \underline{\underline{\frac{9}{2}\,\tau_{\text{allow}}\, a^2}} \,.$$

Note: The shear stresses in the cross section at $z = z_l$ and in the corresponding perpendicular bonding interface are equal (*associated shear stresses!*).

P3.6

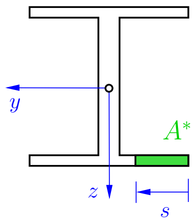
Problem 3.6 Determine the shear stress due to an applied shear resultant force V in the depicted thin- walled **I**-profile.



Solution The shear stresses are computed from

$$\tau = \frac{V\ S(s)}{I\ t(s)}$$

Thus we need to determine the second moment of area I with regard to the y-axis. With $t_1 \ll b$ and $t_2 \ll h$ we obtain



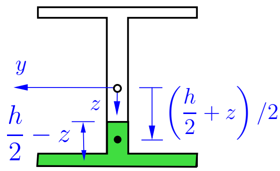
$$I = I_1 + I_2 = 2\,t_1 b\left(\frac{h}{2}\right)^2 + t_2\frac{h^3}{12}$$

$$= \frac{h^2}{12}(t_2 h + 6 t_1 b) = \frac{h^2}{12}(A_1 + 6A_2)\,.$$

The static moment of sub-area A^* for a position s in the lower sub-area is given by

$$S(s) = \frac{h}{2}\,t_1 s$$

and for a position z in the second sub-area it follows



$$S(z) = 2\left(\frac{h}{2}\,t_1\,\frac{b}{2}\right) + \frac{\frac{h}{2}+z}{2}\left(\frac{h}{2} - z\right)t_2$$

$$= A_1\frac{h}{2} + \frac{t_2}{8}(h^2 - 4z^2)\,.$$

These relations yield the shear stress in the upper sub-area

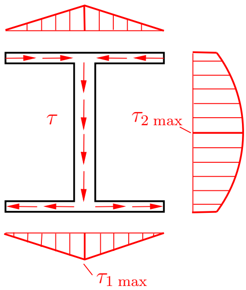
$$\tau_1(s) = \frac{V\,\dfrac{h}{2}\,t_1 s}{\dfrac{h^2}{12}(A_2+6A_1)t_1} = \frac{V}{A_2}\,\frac{\dfrac{A_2}{A_1}}{1+\dfrac{A_2}{6A_1}}\,\frac{s}{h}$$

and in the second sub-area

$$\tau_2(z) = \frac{V\left[A_1\dfrac{h}{2}+\dfrac{t_2}{8}(h^2-4z^2)\right]}{\dfrac{h^2}{12}(A_2+6A_1)\,t_2} = \frac{V}{A_2}\,\frac{1+\dfrac{A_2}{4A_1}\left[1-\left(\dfrac{2z}{h}\right)^2\right]}{1+\dfrac{A_2}{6A_1}}\,.$$

The maximum shear stress occurs at the center of the profile,

$$\tau_{2\ \max} = \tau_2(z=0) = \frac{V}{A_2}\,\frac{1+\dfrac{A_2}{4A_1}}{1+\dfrac{A_2}{6A_1}}\,,$$



it depends on the area ratio A_2/A_1. The maximum shear stress in the first sub-area is given by

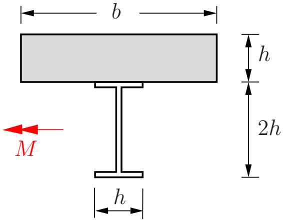
$$\tau_{1\ \max} = \tau_G(s=b/2) = \frac{V}{A_2}\,\frac{\dfrac{A_2}{A_1}}{1+\dfrac{A_2}{6A_1}}\,\frac{b}{2h}\,.$$

For example $A_1 = A_2$ and $b = h$ yields $\tau_{2\ \max} = \dfrac{15}{14}\dfrac{V}{A_2}$ at the center and $\tau_{1\ \max} = \dfrac{6}{14}\dfrac{V}{A_2}$. For this situation the smallest value in the vertical sub-area

$$\tau_{2\ \min} = \tau_2(z=h/2) = \frac{V}{A_2}\,\frac{1}{1+\dfrac{A_2}{6A_1}} = \frac{12}{14}\,\frac{V}{A_2}\,,$$

is only 20% smaller than $\tau_{2\ \max}$. As a rough estimate we can use the average shear stress $\tau_{\text{ave}} = V/A_2$ in the central sub-area.

P3.7 **Problem 3.7** A composite beam consists of an upper concrete slab and a steel I beam. The structure is loaded by a bending moment M.

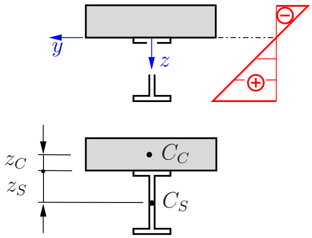


Given : $M = 1000$ kNm
$E_C = 3.5 \cdot 10^4$ N/mm^2
$E_S = 2.1 \cdot 10^5$ N/mm^2
$h = 40$ cm
$A_S = h^2 / 6$
$I_S = h^4 / 18$

a) Determine the width b of the concrete slab, such that compressive stresses occur only in the concrete part, while the tension is present in the steel part.

b) For this case compute the stresses in the extreme fibres of the two materials.

Solution to a) For the case that compression occurs only in the concrete and tension only in the steel sub-area the strain in the bonding layer has do be zero (=neutral fibre). With the chosen coordinate system we have



$$\varepsilon = a z \,,$$

where a is not yet determined. The stresses in steel and concrete are

$$\sigma_S = E_S\,\varepsilon = a\,E_S\,z\,, \qquad \sigma_C = E_C\,\varepsilon = a\,E_C\,z\,.$$

As the beam is loaded only by a bending moment, the normal force N has to vanish:

$$N = \int_{A_S} \sigma_S \,\mathrm{d}A + \int_{A_C} \sigma_C \,\mathrm{d}A = 0 \quad \rightsquigarrow \quad E_S \int_{A_S} z \,\mathrm{d}A + E_C \int_{A_C} z\,\mathrm{d}A = 0\,.$$

With

$$\int_{A_S} z\,\mathrm{d}A = z_S A_S = h\,\frac{h^2}{6} = \frac{h^3}{6}\,, \qquad \int_{A_C} z\,\mathrm{d}A = z_C A_C = -\,\frac{h}{2} hb = -\,\frac{h^2 b}{2}$$

and $E_S/E_C = 6$ the required width b is obtained:

$$6\,\frac{h^3}{6} - \frac{h^2 b}{2} = 0 \qquad \rightsquigarrow \qquad \underline{\underline{b = 2h = 80\,\mathrm{cm}}}\,.$$

to b) The unknown factor a follows from the prescribed bending moment.

From the definitions

$$M = \int\limits_{A_S} z\,\sigma_S\,\mathrm{d}A + \int\limits_{A_C} z\,\sigma_C\,\mathrm{d}A = a\,E_S \int\limits_{A_S} z^2 \mathrm{d}A + a\,E_C \int\limits_{A_C} z^2 \mathrm{d}A\,.$$

and the evaluation of the integrals

$$\int\limits_{A_S} z^2 \mathrm{d}A = I_S + h^2 A_S = \frac{h^4}{18} + \frac{h^4}{6} = \frac{2}{9}\,h^4$$

$$\int\limits_{A_C} z^2 \mathrm{d}A = \frac{bh^3}{3} = \frac{2}{3}\,h^4$$

it follows

$$M = \frac{a h^4 E_C}{9}\left[2\,\frac{E_S}{E_C} + 6\right] = 2 a h^4 E_C \qquad \rightsquigarrow \qquad a = \frac{M}{2h^4 E_C}\,.$$

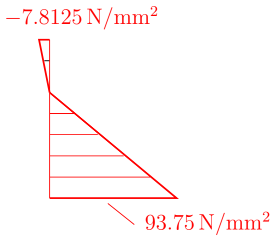
With this result the stresses in the steel and concrete are

$$\sigma_S = \frac{E_S M}{2 E_C h^4}\,z = 3\,\frac{M}{h^4}\,z\,, \qquad\qquad \sigma_C = \frac{M}{2h^4}\,z\,.$$

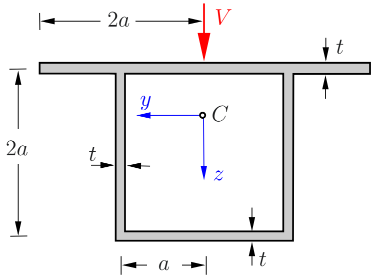
For the top extreme fibre in concrete ($z^t = -h$) and the bottom extrem fibre in steel ($z^b = 2h$) we obtain

$$\underline{\underline{\sigma_C^t}} = -\,\frac{M}{2h^3} = \underline{\underline{-7.8125\,\mathrm{N/mm^2}}}\,,$$

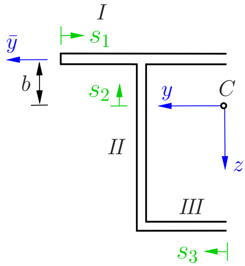
$$\underline{\underline{\sigma_S^b}} = 6\,\frac{M}{h^3} = \underline{\underline{93.75\,\mathrm{N/mm^2}}}\,,$$



P3.8 **Problem 3.8** Determine the shear stresses due to a shear force V for the depicted thin-walled beam cross section ($t \ll a$).



Solution At first we compute the cross section area, the location of the centroid and the second moment of area:



$$A = 4at + 2 \cdot 2at + 2at = 10\,at\,,$$

$$bA = 2a \cdot 2at + 2a \cdot 2at \quad \rightsquigarrow \quad b = \frac{4}{5}\,a\,,$$

$$I_{\bar{y}} = (2a)^2 2at + 2\,\frac{t(2a)^3}{3} = \frac{40}{3}ta^3\,,$$

$$I = I_y = I_{\bar{y}} - b^2 A = \frac{104}{15}ta^3\,.$$

Due to symmetry of the cross section the shear stress is symmetric to the z-axis.
Thus only half of the cross section has to be considered. With the coordinantes s_1 to s_3 we obtain for the static moments in the sub-areas *I* to *III*

$$S_I = b\,s_1 t = \frac{4}{5}\,at\,s_1\,,$$

$$S_{II} = b\,2at + \left(s_2 + \frac{b - s_2}{2}\right)(b - s_2)\,t = \frac{48}{25}a^2 t - \frac{1}{2}t\,s_2^2\,,$$

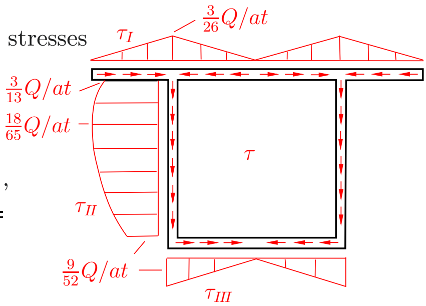
$$S_{III} = (2a - b)t\,s_3 = \frac{6}{5}\,at\,s_3\,.$$

These relations result in the shear stresses

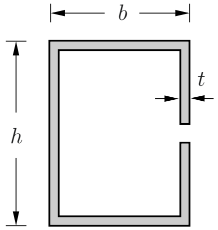
$$\underline{\underline{\tau_I}} = \frac{Q\,S_I}{I\,t} = \underline{\underline{\frac{3}{26}\,\frac{Q}{at}\,\frac{s_1}{a}}}\,,$$

$$\underline{\underline{\tau_{II}}} = \frac{Q\,S_{II}}{I\,t} = \underline{\underline{\frac{Q}{at}\left(\frac{18}{65} - \frac{15}{208}\,\frac{s_2^2}{a^2}\right)}},$$

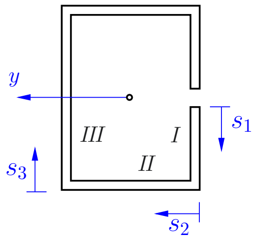
$$\underline{\underline{\tau_{III}}} = \frac{Q\,S_{III}}{I\,t} = \underline{\underline{\frac{9}{52}\,\frac{Q}{at}\,\frac{s_3}{a}}}\,.$$



Problem 3.9 Locate the shear center for the depicted thin-walled ($t \ll b, h$) box profile with a slit. P3.9

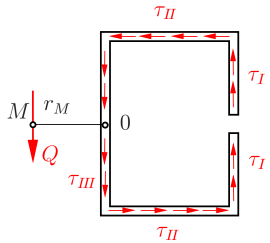


Solution We start by computing the static moments with respect to the y-axis of the three sub-areas:



$$S_I = t\frac{s_1^2}{2}\,, \qquad S_{II} = t\frac{h^2}{8} + \frac{h}{2}ts_2\,,$$
$$S_{III} = t\frac{h^2}{8} + \frac{h}{2}bt + s_3 t\left(\frac{h}{2} - \frac{s_3}{2}\right)\,.$$

Thus the shear stresses become



$$\tau_I = \frac{Q}{I}\frac{s_1^2}{2}\,,$$
$$\tau_{II} = \frac{Q}{I}\left(\frac{h^2}{8} + \frac{h}{2}s_2\right)\,,$$
$$\tau_{III} = \frac{Q}{I}\left(\frac{h^2}{8} + \frac{h}{2}b + \frac{s_3}{2}(h - s_3)\right)\,.$$

The equivalency of moments with respect to 0 provides

$$Q\,r_M = 2\int_0^{h/2} \tau_I bt\,\mathrm{d}s_1 + 2\int_0^b \tau_{II}\frac{h}{2}t\,\mathrm{d}s_2 = \frac{Qt}{I}\left(b\frac{h^3}{24} + \frac{1}{8}bh^3 + \frac{1}{4}h^2b^2\right)$$
$$= \frac{Qtbh^2}{I}\left(\frac{1}{6}h + \frac{1}{4}b\right)\,.$$

With the second moment of area for the thin-walled profile

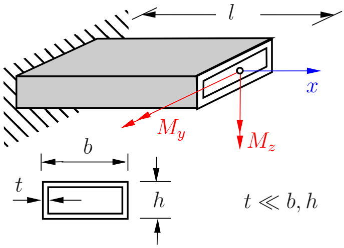
$$I = 2\Big[\frac{th^3}{12} + bt\Big(\frac{h}{2}\Big)^2\Big] = th^2\left(\frac{h}{6} + \frac{b}{2}\right)$$

we obtain the distance r_M of the shear center M to the reference point 0

$$\underline{\underline{r_M}} = \frac{tbh^2}{th^2}\,\frac{\frac{1}{6}h + \frac{1}{4}b}{\frac{1}{6}h + \frac{1}{2}b} = \underline{\underline{b\,\frac{2h+3b}{2h+6b}}}\,.$$

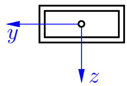
P3.10 **Problem 3.10** The cantilever with thin-walled box cross section is loaded by two bending moments $M_y = Fl$ and $M_z = 2Fl$.

Determine the distribution of the normal stresses in the cross section for $b = 2h$.



Solution Because of symmetry y and z are principal axes. The stress distribution is computed from

$$\sigma = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y \, .$$



With

$$I_y = 2 \cdot \frac{th^3}{12} + 2 \cdot \left(\frac{h}{2}\right)^2 tb = \frac{1}{6} th^2 (h + 3b) \, ,$$

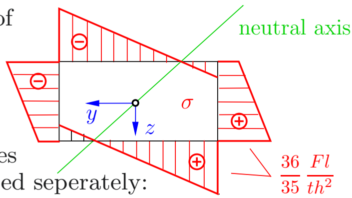
$$I_z = 2 \cdot \frac{tb^3}{12} + 2 \cdot \left(\frac{b}{2}\right)^2 ht = \frac{1}{6} tb^2 (b + 3h)$$

and the given bending moments we find

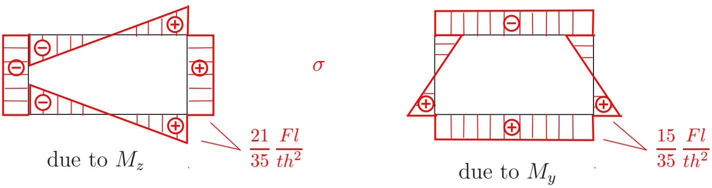
$$\underline{\underline{\sigma}} = \frac{Fl}{\frac{1}{6} th^2 \cdot 7h} z - \frac{2Fl}{\frac{1}{6} t\, 4h^2 \cdot 5h} y = \underline{\underline{\frac{6Fl}{th^3} \left(\frac{z}{7} - \frac{y}{10}\right)}} \, .$$

The equation of the neutral axis (line of zero stress) is computed from $\sigma = 0$

$$z = \frac{7}{10} y \, .$$

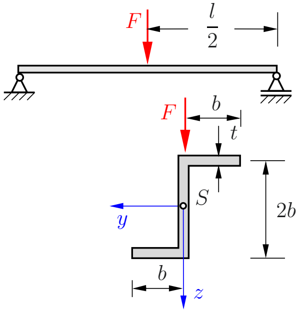


To clarify the representation the stresses due to the two loading cases are depicted seperately:



Problem 3.11 A beam, simply supported at both ends, with a thin-walled profile ($t \ll b$) is loaded by a force F in the middle. **P3.11**

Determine the stress distribution under the load as well as the location and value of the maximum stress.



Solution For the unsymmetrical profile the principal axes are not known. We have to use the equations for biaxial bending. Thus we obtain for the stresses with $M_z = 0$

$$\sigma = \frac{M_y}{\Delta}(I_z z + I_{yz} y)\,.$$

The moment due to the load is given by

$$M_y = M_{\max} = \frac{Fl}{4}\,.$$

Together with the geometric quantities of the cross section

$$I_y = \frac{t(2b)^3}{12} + 2\cdot b^2(bt) = \frac{8}{3}\,tb^3\,, \quad I_z = 2\Big[\frac{tb^3}{12} + \Big(\frac{b}{2}\Big)^2 bt\Big] = \frac{2}{3}\,t\,b^3\,,$$

$$I_{yz} = -2\cdot b\cdot\frac{b}{2}\cdot bt = -tb^3\,,$$

$$\Delta = I_y I_z - I_{yz}^2 = \frac{16}{9}t^2b^6 - t^2b^6 = \frac{7}{9}\,t^2b^6$$

we obtain the stress

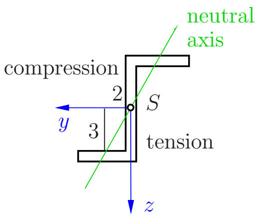
$$\underline{\underline{\sigma}} = \frac{Fl}{4\cdot\frac{7}{9}t^2b^6}\left(\frac{2}{3}t\,b^3 z - t\,b^3 y\right) = \underline{\underline{\frac{3}{28}\,\frac{Fl}{t\,b^3}\,(2z-3y)}}\,.$$

The neutral axis follows from the condition

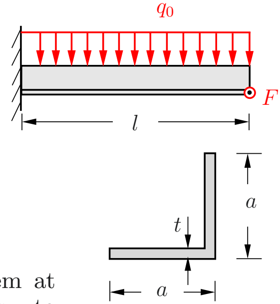
$$\sigma = 0 \quad \leadsto \quad z = \frac{3}{2}\,y\,.$$

The maximal stresses occur at points with the largest distance to the neutral axis ($y = 0$, $z = \pm b$):

$$\underline{\underline{\sigma_{\max} = \pm\frac{3}{14}\,\frac{Fl}{t\,b^2}}}\,.$$



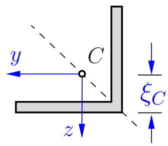
P3.12 **Problem 3.12** A cantilever beam with thin-walled profile ($t \ll a$) is subjected to a constant line load q_0 and a concentrated force F.



Determine the distribution of the normal stress in the cross section at the support.

Given: $F = 2q_0 l$.

Solution We place a y, z-coordinate system at the not yet known centroid. By symmetry to the 45°-axis the distance ξ_C to both sub-areas is identical. As the static moment vanishes with regard to the symmetry axis, we have



$$\xi_C \, at = \left(\frac{a}{2} - \xi_C\right) a\, t \quad \rightsquigarrow \quad \xi_C = \frac{a}{4} .$$

With regard to the symmetry axis we find

$$I_y = I_z = \frac{ta^3}{12} + \left(\frac{a}{4}\right)^2 a\, t + \left(\frac{a}{4}\right)^2 a\, t = \frac{5}{24} ta^3 ,$$

$$I_{yz} = -\frac{a}{4}\,\frac{a}{4} a\, t - \left(-\frac{a}{4}\right)\left(-\frac{a}{4}\right) a\, t = -\frac{1}{8} ta^3 .$$

This yields

$$\Delta = I_y I_z - I_{yz}^2 = \left(\frac{5}{24}\right)^2 t^2 a^6 - \frac{1}{64} t^2 a^6 = \frac{1}{36} t^2 a^6 .$$

The internal moments at the support are given by

$$M_y = -\frac{q_0 l^2}{2} \qquad \text{and} \qquad M_z = Fl = +2q_0 l^2 .$$

Finally we obtain for the stress

$$\underline{\underline{\sigma}} = \frac{1}{\Delta} \left\{ [M_y I_z - M_z I_{yz}]\, z - [M_z I_y - M_y I_{yz}]\, y \right\}$$

$$= \frac{36}{t^2 a^6} \left\{ \left[-\frac{q_0 l^2}{2}\,\frac{5}{24} ta^3 - 2q_0 l^2 \left(-\frac{ta^3}{8}\right)\right] z \right.$$

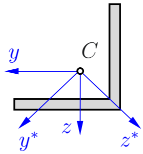
$$\left. - \left[2q_0 l^2 \frac{5}{24} ta^3 + \frac{q_0 l^2}{2}\left(-\frac{ta^3}{8}\right)\right] y \right\}$$

$$= \underline{\underline{\frac{3}{4}\,\frac{q_0 l^2}{ta^3}\,(7z - 17y)}} .$$

Alternatively we can describe the stress distribution with respect to the principal axes y^*, z^*, which we know from symmetry considerations. The principal values of the second moments of area follow with $I_y = I_z$ and $\varphi = 45°$

$$I_y^* = \frac{I_y + I_z}{2} + I_{yz} = \frac{5}{24}ta^3 - \frac{1}{8}ta^3 = \frac{1}{12}ta^3 \,,$$

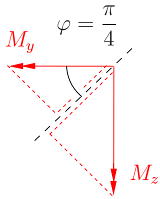
$$I_z^* = \frac{I_y + I_z}{2} - I_{yz} = \frac{5}{24}ta^3 + \frac{1}{8}ta^3 = \frac{1}{3}ta^3 \,.$$



Decomposition of the loading in the principal directions yields

$$M_y^* = -\frac{q_0 l^2}{2}\cos\varphi + Fl\sin\varphi$$
$$= q_0 l^2 \left(2 - \frac{1}{2}\right)\frac{1}{2}\sqrt{2}\,,$$

$$M_z^* = \frac{q_0 l^2}{2}\sin\varphi + Fl\cos\varphi$$
$$= q_0 l^2 \left(\frac{1}{2} + 2\right)\frac{1}{2}\sqrt{2}\,,$$



which leads to the stresses in the principal directions

$$\sigma = \frac{M_y^*}{I_y^*}z^* - \frac{M_z^*}{I_z^*}y^* = \frac{3\sqrt{2}}{4}\frac{q_0 l^2}{ta^3}(12z^* - 5y^*)\,.$$

To check the result we transform with

$$z^* = -y\sin\varphi + z\cos\varphi = (z-y)\frac{1}{2}\sqrt{2}\,,$$

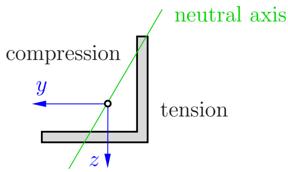
$$y^* = y\cos\varphi + z\sin\varphi = (z+y)\frac{1}{2}\sqrt{2}$$

back and find by re-substitution

$$\sigma = \frac{3}{4}\frac{q_0 l^2}{ta^3}[12(z-y) - 5(z+y)] = \frac{3}{4}\frac{q_0 l^2}{ta^3}(7z - 17y)\,.$$

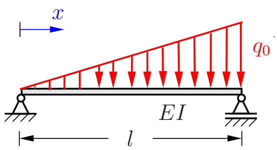
The neutral axis satisfies the equation

$$z = \frac{17}{7}y\,.$$



P3.13

Problem 3.13 The beam is simply supported at both ends. Determine
a) location and value of maximal moment,
b) location and value of maximal deflection,
c) the slope of the deflection curve at both supports.



Solution Bending moment and deflection curve can be computed independently, because the beam is statically determinate.

to a) The given loading provides

$$q = q_0\,\frac{x}{l}$$

by twice integration

$$V = -q_0\,\frac{x^2}{2l} + C_1\,,$$
$$M = -q_0\,\frac{x^3}{6l} + C_1 x + C_2\,.$$

With the *static* boundary conditions

$$M(0) = 0 \quad \leadsto \quad C_2 = 0\,, \qquad M(l) = 0 \quad \leadsto \quad C_1 = \frac{q_0 l}{6}$$

we obtain

$$V = \frac{q_0 l}{6}\Big[1 - 3\big(\frac{x}{l}\big)^2\Big]\,, \qquad M = \frac{q_0 l^2}{6}\Big[\frac{x}{l} - \big(\frac{x}{l}\big)^3\Big]\,.$$

Location and value of the maximal moment are determined by the condition $M' = 0$:

$$M' = V = 0 \quad \leadsto \quad 1 - 3\big(\frac{x^*}{l}\big)^2 = 0 \quad \leadsto \quad \underline{\underline{x^* = \frac{1}{3}\sqrt{3}\,l = 0,577\,l}}\,,$$

$$\underline{\underline{M_{\max}}} = M(x^*) = \frac{1}{18}\sqrt{3}\;q_0 l^2\,(1 - \frac{1}{3}) = \underline{\underline{\frac{1}{27}\sqrt{3}\;q_0 l^2}}\,.$$

to b) With the known function of the moment

$$M = \frac{q_0 l^2}{6}\left[\frac{x}{l} - \left(\frac{x}{l}\right)^3\right]$$

we derive from $EI\,w'' = -M$ by twice integration

$$EI\,w' = -\frac{q_0 l^2}{6}\left(\frac{x^2}{2l} - \frac{1}{4}\frac{x^4}{l^3}\right) + C_3\,,$$
$$EI\,w = -\frac{q_0 l^2}{6}\left(\frac{x^3}{6l} - \frac{1}{20}\frac{x^5}{l^3}\right) + C_3 x + C_4\,.$$

The new integration constants are determined from the *geometric* boundary conditions

$$w(0)=0 \quad \rightsquigarrow \quad C_4=0\,,$$

$$w(l)=0 \quad \rightsquigarrow \quad C_3=\frac{q_0 l^3}{6}\left(\frac{1}{6}-\frac{1}{20}\right)=\frac{7}{360}q_0 l^3\,.$$

Finally we obtain (cf. table on page 62, load case no. 3)

$$EI\,w=\frac{q_0 l^4}{360}\left[7\,\frac{x}{l}-10\left(\frac{x}{l}\right)^3+3\left(\frac{x}{l}\right)^5\right].$$

The maximal deflection is computed by using the condition $w'=0$:

$$EI\,w'=0 \quad \rightsquigarrow \quad 7-30\left(\frac{x^{**}}{l}\right)^2+15\left(\frac{x^{**}}{l}\right)^4=0$$

$$\rightsquigarrow \quad \left(\frac{x^{**}}{l}\right)^4-2\left(\frac{x^{**}}{l}\right)^2+\frac{7}{15}=0\,,$$

$$\rightsquigarrow \quad \underline{\underline{x^{**}}}=\underline{\underline{\sqrt{1\overset{(+)}{-}\sqrt{\frac{8}{15}}}\,l}}=0,519\,l\,.$$

(The (+)-sign provides an x-value outside of the range of validity.) Thus we have

$$\underline{\underline{w_{\max}}}=w(x^{**})=\frac{q_0 l^4}{360EI}\sqrt{1-\sqrt{\frac{8}{15}}}\left[7-10\left(1-\sqrt{\frac{8}{15}}\right)+3\left(1-\sqrt{\frac{8}{15}}\right)^2\right]$$

$$=\underline{\underline{0,0065\,\frac{q_0 l^4}{EI}}}\,.$$

to c) The slope of the deflection curve follows as

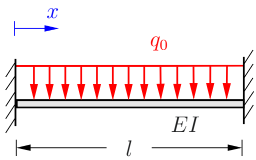
$$\underline{\underline{w'(0)}}=\frac{C_3}{EI}=\underline{\underline{\frac{7}{360}\,\frac{q_0 l^3}{EI}}}\,,$$

$$\underline{\underline{w'(l)}}=-\frac{q_0 l^2}{6EI}\left(\frac{l}{2}-\frac{l}{4}\right)+\frac{7}{360}\,\frac{q_0 l^3}{EI}=\underline{\underline{-\frac{8}{360}\,\frac{q_0 l^3}{EI}}}\,.$$

Note: Maximal moment and maximal deflection occur at different locations: $x^* \neq x^{**}$.

P3.14

Problem 3.14 Determine the function of the bending moment for the depicted beam.



Solution The beam is statically *indeterminate*. Thus the function of the moment needs to be computed with help of the deflection curve. From the differential equation we derive by integration

$$EI\,w^{IV} = q = q_0\,,$$

$$-EI\,w''' = Q = -q_0 x + C_1\,,$$

$$-EI\,w'' = M = -q_0\frac{x^2}{2} + C_1 x + C_2\,,$$

$$EI\,w' = q_0\frac{x^3}{6} - C_1\frac{x^2}{2} - C_2 x + C_3\,,$$

$$EI\,w = q_0\frac{x^4}{24} - C_1\frac{x^3}{6} - C_2\frac{x^2}{2} + C_3 x + C_4\,.$$

The 4 integration constants follow from the 4 geometric boundary conditions:

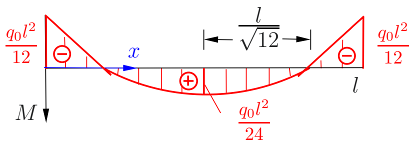
$$w'(0) = 0 \quad\rightsquigarrow\quad C_3 = 0\,,$$

$$w(0) = 0 \quad\rightsquigarrow\quad C_4 = 0\,,$$

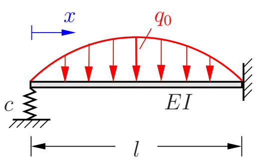
$$\left.\begin{aligned} w'(l) = 0 \quad&\rightsquigarrow\quad \frac{q_0 l^3}{6} - C_1\frac{l^2}{2} - C_2 l = 0 \\ w(l) = 0 \quad&\rightsquigarrow\quad \frac{q_0 l^4}{24} - C_1\frac{l^3}{6} - C_2\frac{l^2}{2} = 0 \end{aligned}\right\} \quad\rightsquigarrow\quad \begin{aligned} C_1 &= \frac{q_0 l}{2} \\ C_2 &= -\frac{q_0 l^2}{12}\,. \end{aligned}$$

This yields

$$\underline{\underline{M = -\frac{q_0 l^2}{12}\left[1 - 6\,\frac{x}{l} + 6\left(\frac{x}{l}\right)^2\right]}}\,.$$



Problem 3.15 Determine the deflection of the depicted beam. The left end of the beam is elastically supported by a spring, the right end is clamped, and the load has the shape of a quadratic parabola. **P3.15**



Solution We start by computing the quadratic equation for the line load. From the general equation $q = A + Bx + Cx^2$ and

$$\begin{aligned} q(0) &= 0 \quad \rightsquigarrow \quad A = 0\,, \\ \left.\begin{aligned} q(l) &= 0 \quad \rightsquigarrow \quad Bl + Cl^2 = 0\,, \\ q(\tfrac{l}{2}) &= q_0 \quad \rightsquigarrow \quad B\frac{l}{2} + C\frac{l^2}{4} = q_0\,, \end{aligned}\right\} &\rightsquigarrow \quad C = -\frac{B}{l}\,, \quad B = 4\frac{q_0}{l} \end{aligned}$$

it follows $\quad q(x) = 4q_0\left[\frac{x}{l} - (\frac{x}{l})^2\right]\,.$

Four times integration of $EI\,w^{IV} = q$ yields

$$\begin{aligned} -EI\,w''' = V &= -4q_0\Big(\frac{x^2}{2l} - \frac{x^3}{3l^2}\Big) + C_1\,, \\ -EI\,w'' = M &= -4q_0\Big(\frac{x^3}{6l} - \frac{x^4}{12l^2}\Big) + C_1 x + C_2\,, \\ EI\,w' &= 4q_0\Big(\frac{x^4}{24l} - \frac{x^5}{60l^2}\Big) - C_1\frac{x^2}{2} - C_2 x + C_3\,, \\ EI\,w &= 4q_0\Big(\frac{x^5}{120l} - \frac{x^6}{360l^2}\Big) - C_1\frac{x^3}{6} - C_2\frac{x^2}{2} + C_3 x + C_4\,. \end{aligned}$$

The boundary conditions provide

$$\begin{aligned} M(0) &= 0 \quad \rightsquigarrow C_2 = 0\,, \\ V(0) &= c\cdot w(0) \rightsquigarrow C_1 = c\,\frac{C_4}{EI}\,, \\ w'(l) &= 0 \quad \rightsquigarrow \frac{q_0 l^3}{10} - C_1\frac{l^2}{2} + C_3 = 0\,, \\ w(l) &= 0 \quad \rightsquigarrow \frac{q_0 l^4}{45} - C_1\frac{l^3}{6} + C_3 l + C_4 = 0\,. \end{aligned}$$

The 3 equations for C_1, C_3, and C_4 yield with the abbreviation $\Delta = 1 + cl^3/3EI$

$$C_1 = \frac{7}{90}\frac{c}{\Delta}\frac{q_0 l^4}{EI}\,, \quad C_3 = -\frac{q_0 l^3}{10\Delta}\Big(1 - \frac{1}{18}\frac{cl^3}{EI}\Big)\,, \quad C_4 = \frac{7}{90}\frac{q_0 l^4}{\Delta}$$

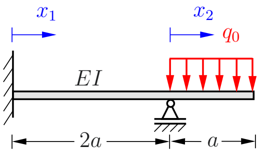
which leads to the final result

$$\underline{\underline{w = \frac{q_0 l^4}{10EI}\left[\frac{1}{3}(\frac{x}{l})^5 - \frac{1}{9}(\frac{x}{l})^6 - \frac{7}{54}\frac{cl^3}{\Delta EI}(\frac{x}{l})^3 - \Big(1 - \frac{1}{18}\frac{cl^3}{EI}\Big)\frac{1}{\Delta}(\frac{x}{l}) + \frac{7}{9\Delta}\right]}}.$$

P3.16

Problem 3.16 A cantilever beam is subjected to a constant distributed load q_0.

Determine the deflection at the free end.



Solution We solve the problem in two different ways.
1st solution: Due to the discontinuity of $q(x)$ we have to consider two domains:

$$
\begin{aligned}
0 \le x_1 < 2a \qquad q_1 &= 0\,,\\
V_1 &= C_1\,,\\
M_1 &= C_1 x_1 + C_2\,,\\
EI\,w_1' &= -C_1\frac{x_1^2}{2} - C_2 x_1 + C_3\,,\\
EI\,w_1 &= -C_1\frac{x_1^3}{6} - C_2\frac{x_1^2}{2} + C_3 x_1 + C_4\,,\\
0 < x_2 \le a \qquad q_2 &= q_0\,,\\
V_2 &= -q_0 x_2 + C_5\,,\\
M_2 &= -q_0\frac{x_2^2}{2} + C_5 x_2 + C_6\,,\\
EI\,w_2' &= q_0\frac{x_2^3}{6} - C_5\frac{x_2^2}{2} - C_6 x_2 + C_7\,,\\
EI\,w_2 &= q_0\frac{x_2^4}{24} - C_5\frac{x_2^3}{6} - C_6\frac{x_2^2}{2} + C_7 x_2 + C_8\,.
\end{aligned}
$$

The 8 integration constants C_i follow from:

$$
\begin{array}{l}
\text{4 boun–}\\ \text{dary}\\ \text{conditons}
\end{array}
\left\{
\begin{array}{llll}
w_1'(0) = 0 \leadsto & C_3 = 0\,, & w_1(0) = 0 \leadsto & C_4 = 0\,,\\
Q_2(a) = 0 \leadsto & C_5 = q_0 a\,, & M_2(a) = 0 \leadsto & C_6 = -\dfrac{q_0 a^2}{2}
\end{array}
\right.
$$

$$
\begin{array}{l}
\text{and 4}\\ \text{contin–}\\ \text{uity}\\ \text{condi–}\\ \text{tions}
\end{array}
\left\{
\begin{array}{ll}
M_1(2a) = M_2(0) & \leadsto C_1 2a + C_2 = C_6\,,\\
w_1'(2a) = w_2'(0) & \leadsto -C_1\dfrac{(2a)^2}{2} - C_2 2a + C_3 = C_7\,,\\
w_1(2a) = w_2(0) = 0 & \leadsto -C_1\dfrac{(2a)^3}{6} - C_2\dfrac{(2a)^2}{2}\\
& \qquad +C_3 2a + C_4 = C_8 = 0
\end{array}
\right.
$$

$$
\leadsto \quad C_1 = -\frac{3}{8}q_0 a\,, \quad C_2 = \frac{1}{4}q_0 a^2\,, \quad C_7 = \frac{1}{4}q_0 a^3\,, \quad C_8 = 0\,.
$$

(For the shear force no continuity condition is available because it experiences a jump related to the unknown reaction force B). The deflection at the free end yields

$$\underline{\underline{w_2(a)}} = \frac{q_0}{EI}\left\{\frac{a^4}{24} - \frac{a^4}{6} + \frac{a^4}{4} + \frac{a^4}{4}\right\} = \underline{\underline{\frac{3}{8}\frac{q_0 a^4}{EI}}}\,.$$

2nd solution: Using the Macauley bracket we can describe both domains by a *single* equation. We introduce x from the left end and have to consider the jump in the shear resultant at B (assumed to be positive in upward direction):

$$\begin{aligned}
q &= q_0 <x-2a>^0\,,\\
V &= -q_0 <x-2a>^1 + B <x-2a>^0 + C_1\,,\\
M &= -\frac{1}{2}q_0 <x-2a>^2 + B <x-2a>^1 + C_1 x + C_2\,,\\
EI\,w' &= \frac{1}{6}q_0 <x-2a>^3 - \frac{1}{2}B <x-2a>^2 - \frac{1}{2}C_1 x^2 - C_2 x + C_3\,,\\
EI\,w &= \frac{1}{24}q_0 <x-2a>^4 - \frac{1}{6}B <x-2a>^3 - \frac{1}{6}C_1 x^3 - \frac{1}{2}C_2 x^2 + C_3 x + C_4.
\end{aligned}$$

The 5 unknowns C_i and B follow from

$$\text{4 boundary conditions and}\left\{\begin{aligned}
&w'(0) = 0 \;\rightsquigarrow C_3 = 0\,,\\
&w(0) = 0 \;\rightsquigarrow C_4 = 0\,,\\
&Q(3a) = 0 \;\rightsquigarrow -q_0 a + B + C_1 = 0\,,\\
&M(3a) = 0 \rightsquigarrow -q_0\frac{a^2}{2} + Ba + C_1 3a + C_2 = 0
\end{aligned}\right.$$

$$\text{1 reaction condition}\left\{ w(2a) = 0 \rightsquigarrow -C_1\frac{(2a)^3}{6} - C_2\frac{(2a)^2}{2} + C_3 2a + C_4 = 0\,.\right.$$

Solving yields:

$$C_1 = -\frac{3}{8}q_0 a\,,\quad C_2 = \frac{1}{4}q_0 a^2\,,\quad C_3 = 0\,,\quad C_4 = 0\,,\quad B = \frac{11}{8}q_0 a\,.$$

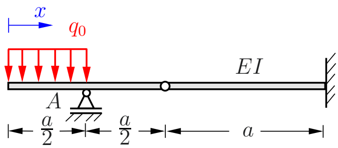
Thus the deflection at the free end is given by

$$\underline{\underline{w(3a)}} = \frac{q_0}{EI}\left[\frac{a^4}{24} - \frac{11}{8}a\frac{a^3}{6} + \frac{3}{8}a\frac{(3a)^3}{6} - \frac{1}{4}a^2\frac{(3a)^2}{2}\right] = \underline{\underline{\frac{3}{8}\frac{q_0 a^4}{EI}}}\,.$$

Note: The computation of displacements at designated locations is less complex with methods discussed in chapter 5.

P3.17 **Problem 3.17** The depicted beam is loaded on its cantilever part by a constant line load.

Compute the deflection at the hinge and determine the slope difference at the hinge.



Solution With the help of the Macauley bracket the entire domain can be descibed by a *single* equation. During integration the jump in the slope $\Delta\varphi$ at the hinge has to be considered separately.

$$q = q_0 - q_0 < x - \frac{a}{2} >^0 ,$$

$$V = -q_0 x + q_0 < x - \frac{a}{2} >^1 + A < x - \frac{a}{2} >^0 + C_1 ,$$

$$M = -q_0 \frac{x^2}{2} + \frac{q_0}{2} < x - \frac{a}{2} >^2 + A < x - \frac{a}{2} >^1 + C_1 x + C_2 ,$$

$$EI\,w' = q_0 \frac{x^3}{6} - \frac{q_0}{6} < x - \frac{a}{2} >^3 - \frac{A}{2} < x - \frac{a}{2} >^2 - C_1 \frac{x^2}{2} - C_2 x$$
$$+ EI\Delta\varphi < x - a >^0 + C_3 ,$$

$$EI\,w = q_0 \frac{x^4}{24} - \frac{q_0}{24} < x - \frac{a}{2} >^4 - \frac{A}{6} < x - \frac{a}{2} >^3 - C_1 \frac{x^3}{6} - C_2 \frac{x^2}{2}$$
$$+ EI\Delta\varphi < x - a >^1 + C_3 x + C_4 .$$

The 4 integration constants C_i, the unknown reaction force A, and the slope difference $\Delta\varphi$ at the hinge are determined from the following 6 conditions

$$V(0) = 0 \quad \leadsto \quad C_1 = 0 , \qquad M(0) = 0 \quad \leadsto \quad C_2 = 0 ,$$

$$M(a) = 0 \quad \leadsto \quad A = \frac{3}{4} q_0 a , \qquad w(\frac{a}{2}) = 0 \quad \leadsto \quad \frac{1}{384} q_0 a^4 + C_3 \frac{a}{2} + C_4 = 0 ,$$

$$w'(2a) = 0 \leadsto \frac{4}{3} q_0 a^3 - \frac{27}{48} q_0 a^3 - \frac{27}{32} q_0 a^3 + EI\Delta\varphi + C_3 = 0 ,$$

$$w(2a) = 0 \quad \leadsto \quad \frac{2}{3} q_0 a^4 - \frac{81}{384} q_0 a^4 - \frac{81}{192} q_0 a^4 + EI\Delta\varphi\, a + C_3 2a + C_4 = 0.$$

This yields the solution

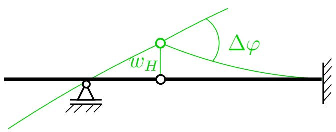
$$C_3 = -\frac{5}{24} q_0 a^3 , \quad C_4 = \frac{39}{384} q_0 a^4 , \quad EI\Delta\varphi = \frac{9}{32} q_0 a^3 .$$

Thus we obtain for the deflection at the hinge

$$\underline{\underline{w_H = w(a) = -\frac{1}{12} \frac{q_0 a^4}{EI}}}$$

and for the slope difference

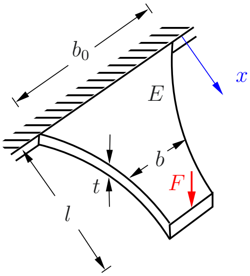
$$\underline{\underline{\Delta\varphi = \frac{9}{32} \frac{q_0 a^3}{EI}}} .$$



P3.18

Problem 3.18 A leaf spring with constant thickness t and variable width $b = b_0 l/(l + x)$ is fixed at one side and loaded at one edge by F.

Determine the deflection at the position of the load.



Solution The system is statically determinate. Hence the function of the moment follows from equilibrium considerations:

$$V = F = \text{const}\,, \qquad M = Fx + C\,.$$

The condition $M(l) = 0$ yields $C = -Fl$ and thus

$$M = -F(l - x)\,.$$

Use of the differential equation $EI\,w'' = -M$ yields with

$$I(x) = b(x)\frac{t^3}{12} = \frac{b_0 t^3}{12}\,\frac{l}{l + x}$$

and the abbreviation $I_0 = b_0 t^3/12$:

$$w'' = \frac{F(l - x)(l + x)}{EI_0 l} = \frac{F}{EI_0 l}(l^2 - x^2)\,.$$

By integration we obtain

$$w' = \frac{F}{EI_0 l}\left(l^2 x - \frac{x^3}{3} + C_1\right),$$

$$w = \frac{F}{EI_0 l}\left(l^2\frac{x^2}{2} - \frac{x^4}{12} + C_1 x + C_2\right).$$

The boundary conditions

$$w'(0) = 0 \quad \rightsquigarrow \quad C_1 = 0\,, \qquad w(0) = 0 \quad \rightsquigarrow \quad C_2 = 0$$

render the solution

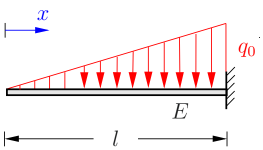
$$\underline{\underline{w(l) = w_{\max} = \frac{5}{12}\,\frac{Fl^3}{EI_0}}}\,.$$

Note: For a beam with *constant* width b_0 the same load results in a smaller deflection

$$w(l) = \frac{Fl^3}{3EI_0} = \frac{4}{12}\,\frac{Fl^3}{EI_0}\,.$$

P3.19 **Problem 3.19** A cantilever beam with rectangular cross section (width b, height $h(x)$) is subjected to a linear varying load such that the extreme fibre experiences a stress σ_0.

Determine the deflection of the left end.



Solution First we have to compute the unknown cross section height. Using

$$\sigma_{\max} = \frac{|M|}{W} = \sigma_0$$

together with

$$M = -\frac{q_0 x^3}{6l}\,, \quad I = \frac{b\,h^3(x)}{12}\,, \quad W(x) = \frac{I}{h/2} = \frac{b\,h^2(x)}{6}$$

yields $h(x)$

$$h(x) = \sqrt{\frac{q_0}{\sigma_0 b l}}\, x^{3/2}\,.$$

This leads to

$$I(x) = \frac{q_0}{12\sigma_0 l}\sqrt{\frac{q_0}{b\sigma_0 l}}\, x^{9/2}\,.$$

Integration of $EI\,w'' = -M$ provides together with the boundary conditions $w'(l) = w(l) = 0$:

$$w'' = -\frac{M}{EI} = \frac{q_0 x^3 12\sigma_0 l}{6lEq_0}\sqrt{\frac{b\sigma_0 l}{q_0}}\, x^{-9/2} = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\, x^{-3/2}\,,$$

$$w' = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\left(-2x^{-1/2} + 2l^{-1/2}\right)\,,$$

$$w = 2\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l}{q_0}}\left(-4x^{1/2} + 2l^{-1/2}x + 2l^{1/2}\right)\,.$$

Evaluation at $x = 0$ yields the deflection at the left end

$$\underline{\underline{w(0) = 4\frac{\sigma_0}{E}\sqrt{\frac{b\sigma_0 l^2}{q_0}}\,.}}$$

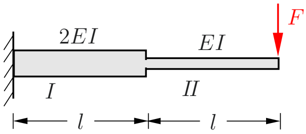
As a test we check the physical dimensions ($F\widehat{=}$force, $L\widehat{=}$length):

$$[w] = \frac{FL^{-2}}{FL^{-2}}\sqrt{\frac{LFL^{-2}L^2}{FL^{-1}}} = L\,.$$

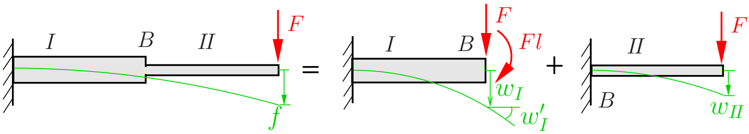
Problem 3.20 The depicted beam is assembled from two parts with different bending stiffness.

Determine the deflection at the free end.

P3.20



Solution We use superposition together with the tabulated results on page 62. First we assume that beam II is fixed at point B and compute the defection w_{II}. To this we have to add the deflection w_I of the left beam I due to F and $M = Fl$. Finally we have to consider the slope w_I', that appears at the left beam. This slop has to be multiplied by the length l and added as an additional deflection at the right end:



$$f = w_{II} + w_I + w_I' l = w_{II} + (w_{I_F} + w_{I_M}) + (w'_{I_F} + w'_{I_M}) l \,.$$

According to load case no. 5

$$w_{II} = \frac{Fl^3}{3EI}\,, \qquad w_{I_F} = \frac{Fl^3}{3(2EI)}\,, \qquad w'_{I_F} = \frac{Fl^2}{2(2EI)}$$

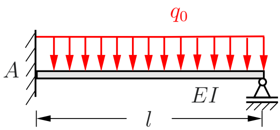
and load case no. 8

$$w_{I_M} = \frac{(Fl)l^2}{2(2EI)}\,, \quad w'_{I_M} = \frac{(Fl)l}{(2EI)}\,.$$

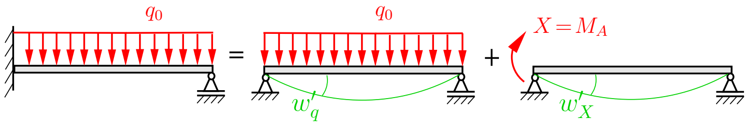
superposition yields the deflection at the end

$$\underline{\underline{f}} = \frac{Fl^3}{3EI}\left\{1 + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{2}\right\} = \underline{\underline{\frac{3}{2}\,\frac{Fl^3}{EI}}}\,.$$

P3.21 **Problem 3.21** Determine the deflection curve for the depicted beam.



Solution The beam is statically indeterminate. We free the support moment at the left end and introduce the unknown moment X:



From the table on page 62 we obtain for the slope:

load case no. 2 $\qquad w'_q = \dfrac{q_0 l^3}{24EI}\,,$

load case no. 4 (with $\beta = 1$) $\qquad w'_X = \dfrac{Xl}{3EI}\,.$

The total slope at the left support has to vanish. Thus compatibility provides

$$w'_q + w'_X = 0 \quad \leadsto \quad X = M_A = -\frac{1}{8} q_0 l^2\,.$$

Superposition of the deflection curves in table on page 62 yields the deflection curve of the system

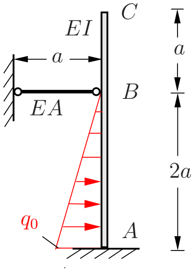
$$\underline{\underline{EI\,w}} = EI(w_q + w_X)$$

$$= \frac{q_0 l^4}{24}(\xi - 2\xi^3 + \xi^4) - \frac{1}{8} q_0 l^2 \frac{l^2}{6}(2\xi + \xi^3 - 3\xi^2)$$

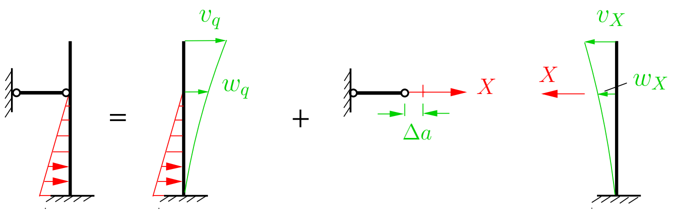
$$= \underline{\underline{\frac{q_0 l^4}{48}(3\xi^2 - 5\xi^3 + 2\xi^4)}}\,.$$

Problem 3.22 A pole is clamped at A and supported at B by an elastic rope. The pole is subjected to a horizontal linearily varying load. **P3.22**

Compute the horizontal displacement v at point C for $\dfrac{EI}{a^2EA} = \dfrac{1}{3}$.



Solution We disconnect rope and pole:



Compatibility at the connection of the rope requires

$$w_q - w_X = \Delta a\,, \qquad \text{where} \qquad \Delta a = \frac{Xa}{EA} \quad \text{(see chapter 2)}\,.$$

With the table on page 62 we obtain:

$$\text{load case no. 7} \qquad w_q = \frac{q_0(2a)^4}{30EI} = \frac{8}{15}\frac{q_0a^4}{EI}\,,$$

$$\text{load case no. 5} \qquad w_X = \frac{X(2a)^3}{3EI} = \frac{8}{3}\frac{Xa^3}{EI}\,.$$

Using these values in the compatibility condition provides

$$\frac{8}{15}\frac{q_0a^4}{EI} - \frac{8}{3}\frac{Xa^3}{EI} = \frac{Xa}{EA} \quad \leadsto \quad X = \frac{\frac{1}{5}q_0a}{1+\frac{3}{8}\frac{EI}{a^2EA}} = \frac{8}{45}q_0a\,.$$

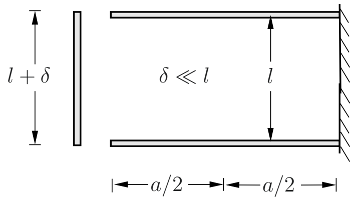
The displacement v results from superposition (for the linear varying load we have to consider the displacement w_q and the slope w_q': $v_q = w_q + w_q'a$):

$$\underline{\underline{EI\,v}} = EI(v_q + v_X) = \frac{q_0(2a)^4}{30} + \frac{q_0(2a)^3}{24}\,a - \underbrace{\frac{X(3a)^3}{6}\left[3\cdot\frac{2}{3} - 1 + \left(\frac{1}{3}\right)^3\right]}_{\text{load case no. 5 with } \alpha = 2/3}$$

$$= \frac{13}{15}q_0a^4 - \frac{14}{3}Xa^3 = \underline{\underline{\frac{q_0a^4}{27}}}\,.$$

P3.23

Problem 3.23 Two parallel beams (bending stiffness EI, length a) have a distance of l and are clamped at the left support. An elastic bar (axial rigidity EA) of length $l+\delta$ is force fitted at $a/2$ between the two beams.



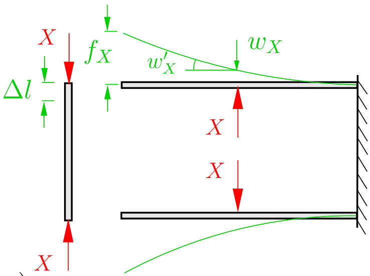
a) Determine the force in the bar?

b) Compute the change e by which the distance l at the beam ends is changed.

Solution to a) From geometry (compatibility)

$$l + 2w_X = (l+\delta) - \Delta l$$

$$\leadsto \quad 2w_X + \Delta l = \delta$$



we obtain (see table on page 62, load case no. 5)

$$w_X = \frac{X\left(\frac{a}{2}\right)^3}{3EI} \quad \text{und} \quad \Delta l = \frac{Xl}{EA}$$

and the force in the bar (compression)

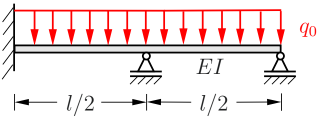
$$\underline{\underline{S}} = X = \frac{\delta}{\frac{l}{EA} + \frac{a^3}{12EI}} = \underline{\underline{\delta\,\frac{EA}{l}\,\frac{1}{1+\frac{a^3 EA}{12\,l\,EI}}}}\,.$$

to b) The opening e is computed with help of the table on page 62 from load case no. 5

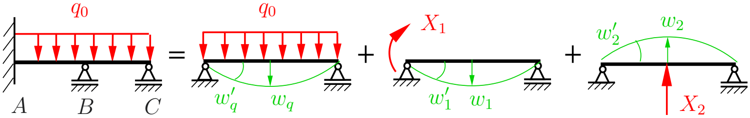
$$\underline{\underline{e}} = 2\,f_X = 2\,\frac{Xa^3}{6\,EI}\left[3\cdot 1\cdot\frac{1}{2} - 1 + \left(\frac{1}{2}\right)^3\right] = \underline{\underline{\frac{5}{24}\,\frac{a^3 EA}{l\,EI}\,\frac{\delta}{1+\frac{a^3 EA}{12\,l\,EI}}}}\,.$$

Note: In the limit case $EI \to \infty$ one obtains $S = \delta\,\dfrac{EA}{l}$ and $e = 0$.

Problem 3.24 Compute the reaction forces for the depicted beam. **P3.24**



Solution The system is *twice* statically indeterminate. We treat the support moment $M_A = X_1$ and the reaction force $B = X_2$ as static redundant quantities and use superposition:



Considering the (arbitrary chosen) directions yields for the compatibility

$$w_q' + w_1' - w_2' = 0\,,$$

$$w_q + w_1 - w_2 = 0\,.$$

From the table on page 62 (no. 2, 4 and 1) we obtain

$$\frac{q_0 l^3}{24} + \frac{X_1 l}{3} - \frac{X_2 l^2}{16} = 0\,,$$

$$\frac{5}{384} q_0 l^4 + \frac{1}{16} X_1 l^2 - \frac{X_2 l^3}{48} = 0\,,$$

which yields

$$X_1 = -\frac{1}{56}\, q_0 l^2\,, \qquad X_2 = \frac{4}{7}\, q_0 l\,.$$

The support reactions are determined by superposition of the 3 load cases

$$\underline{\underline{A}} = \frac{q_0}{2} - \frac{X_1}{l} - \frac{X_2}{2} = \underline{\underline{\frac{13}{56}\, q_0 l}}\,,$$

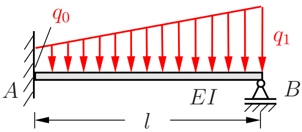
$$\underline{\underline{B}} = X_2 = \underline{\underline{\frac{4}{7}\, q_0 l}}\,,$$

$$\underline{\underline{C}} = \frac{q_0 l}{2} + \frac{X_1}{l} - \frac{X_2}{2} = \underline{\underline{\frac{11}{56}\, q_0 l}}\,,$$

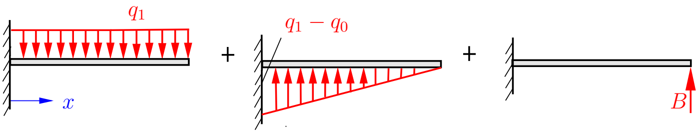
$$\underline{\underline{M_A}} = X_1 = \underline{\underline{-\frac{1}{56}\, q_0 l^2}}\,.$$

P3.25

Problem 3.25 Determine the deflection curve for the depicted beam subjected to a trapezoidal load.



Solution The beam is statically indeterminate. We choose B as the static redundant quantity and use superposition of 3 load cases (the trapezoidal load is replaced by an equivalent constant and linearly varying load)



The table on page 62 (load case no. 6, 7 and 5) provides

$$EI\,w(x) = \frac{q_1 l^4}{24}(6\xi^2 - 4\xi^3 + \xi^4) - \frac{(q_1 - q_0)l^4}{120}(10\xi^2 - 10\xi^3 + 5\xi^4 - \xi^5) - \frac{Bl^3}{6}(3\xi^2 - \xi^3)\,.$$

The support condition at B yields the reaction force B

$$w(l) = 0 \qquad \rightsquigarrow \qquad B = \frac{3}{8}q_1 l - \frac{(q_1 - q_0)l}{10}\,.$$

By recasting the above equations

$$\frac{q_1 l^4}{24} = \frac{(q_1 - q_0)l^4}{24} + \frac{q_0 l^4}{24}$$

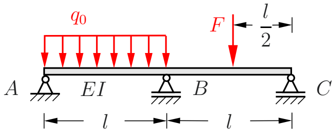
we determine the deflection curve

$$\underline{\underline{EI\,w(x) = \frac{q_0 l^4}{24}\left\{\xi^4 - \frac{5}{2}\xi^3 + \frac{3}{2}\xi^2\right\} + \frac{(q_1 - q_0)l^4}{120}\left\{\xi^5 - \frac{9}{2}\xi^3 + \frac{7}{2}\xi^2\right\}}}\,.$$

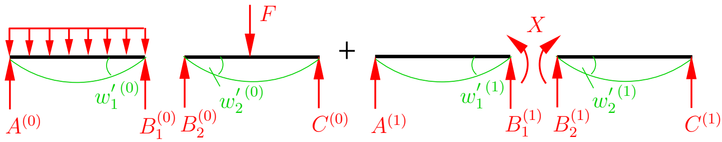
Problem 3.26 For the beam with two domains determine the support reactions and the deflection at the center of each domain.

Given: $F = 2q_0 l$.

P3.26



Solution We divide the beam into 2 separate (hinged at both ends) beams and introduce the moment at the central support as statically redundant quantity:



Equilibrium yields

$$A^{(0)} = B_1^{(0)} = \frac{1}{2} q_0 l \,, \qquad B_2^{(0)} = C^{(0)} = \frac{F}{2} \,,$$
$$A^{(1)} = C^{(1)} = -B_1^{(1)} = -B_2^{(1)} = \frac{X}{l} \,.$$

The table on page 62 provides

$$w_1'^{(0)} = -\frac{q_0 l^3}{24EI} \,, \quad w_2'^{(0)} = \frac{F l^2}{16EI} \,, \quad w_1'^{(1)} = -w_2'^{(1)} = -\frac{Xl}{3EI} \,.$$

Compatibility can be formulated as

$$w_1'^{(0)} + w_1'^{(1)} = w_2'^{(0)} + w_2'^{(1)}$$

which yields together with the tabulated results

$$X = -\frac{1}{16} q_0 l^2 - \frac{3}{32} F l = -\frac{1}{4} q_0 l^2 = M_B \,.$$

The support reactions are computed by superposition

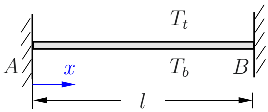
$$\underline{\underline{A}} = A^{(0)} + A^{(1)} = \frac{1}{2} q_0 l - \frac{1}{4} q_0 l = \underline{\underline{\frac{1}{4} q_0 l}} \,,$$
$$\underline{\underline{B}} = B_1^{(0)} + B_1^{(1)} + B_2^{(0)} + B_2^{(1)} = \underline{\underline{2 q_0 l}} \,,$$
$$\underline{\underline{C}} = C^{(0)} + C^{(1)} = \frac{F}{2} - \frac{1}{4} q_0 l = \underline{\underline{\frac{3}{4} q_0 l}} \,.$$

For the deflections at the center of the domains we compute

$$\underline{\underline{f_1}} = f_1^{(0)} + f_1^{(1)} = \frac{5}{384} \frac{q_0 l^4}{EI} + \frac{X l^2}{6EI} \left(\frac{1}{2} - \frac{1}{8} \right) = \underline{\underline{-\frac{q_0 l^4}{384\, EI}}} \,,$$

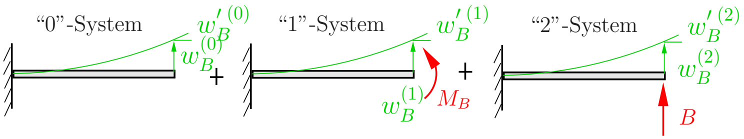
$$\underline{\underline{f_2}} = f_2^{(0)} + f_2^{(1)} = \frac{F l^3}{48\, EI} + \frac{X l^2}{6\, EI} \left(\frac{1}{2} - \frac{1}{8} \right) = \underline{\underline{\frac{5\, q_0 l^4}{192\, EI}}} \,.$$

P3.27 **Problem 3.27** A beam (rectangular cross section, width b, height h) that is clamped at both ends is subjected along its length l to a constant temperature difference $T_t - T_b$.



Determine the defection of the beam and the maximum stresses.

Solution The beam is twice statically indeterimante. We choose as statically redundant quantities the reaction moment $X_1 = M_B$ and the reaction force $X_2 = B$. We use superpostion of the three (statically determinate) systems:



The deflection in the "0"-System is computed by the temperature moment

$$M_{\Delta T} = EI\alpha_T(T_b - T_t)/h$$

using the differential equation $w''^{(0)} = -M_{\Delta T}/EI$ and considering the boundary conditions $w^{(0)}(0) = 0$, $w'^{(0)}(0) = 0$:

$$w'^{(0)}(x) = -\frac{M_{\Delta T}}{EI}\,x\,, \qquad w^{(0)}(x) = -\frac{M_{\Delta T}}{EI}\,\frac{x^2}{2}\,.$$

Due to the clamping at B compatibility requires

$$w_B = w_B^{(0)} + w_B^{(1)} + w_B^{(2)} = 0\,, \qquad w_B' = w_B'^{(0)} + w_B'^{(1)} + w_B'^{(2)} = 0\,.$$

From the table on page 62 we obtain

$$-\frac{M_{\Delta T}}{EI}\,l - \frac{M_B l}{EI} - \frac{Bl^2}{2EI} = 0\,, \qquad -\frac{M_{\Delta T}}{EI}\,\frac{l^2}{2} - \frac{M_B l^2}{2EI} - \frac{Bl^3}{3EI} = 0\,,$$

with the solution

$$B = 0\,, \qquad M_B = -M_{\Delta T}\,.$$

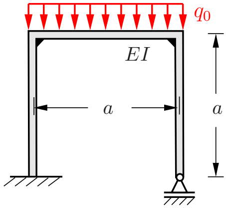
As $M_B = M$ is constant along the entire length of the beam the deflection becomes

$$w'' = -\frac{M + M_{\Delta T}}{EI} = 0 \qquad \text{i. e.} \qquad \underline{\underline{w \equiv 0}}\,.$$

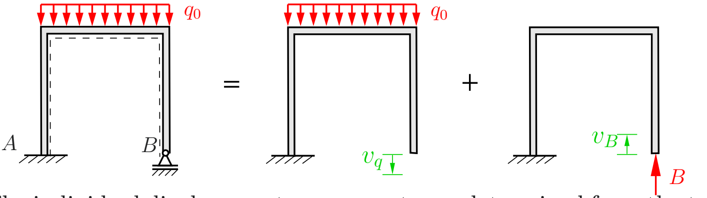
The maximum stress is computed with the section modulus $W = bh^2/6$

$$\underline{\underline{|\sigma_{\max}|}} = \frac{|M|}{W} = \underline{\underline{6\,\frac{M_{\Delta T}}{bh^2}}}\,.$$

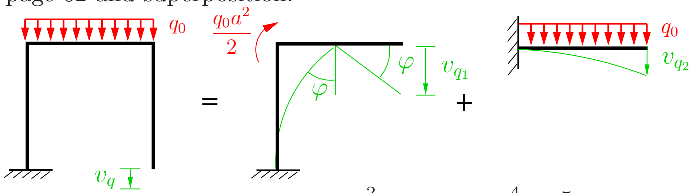
Problem 3.28 Determine the support reactions for the depicted frame. **P3.28**



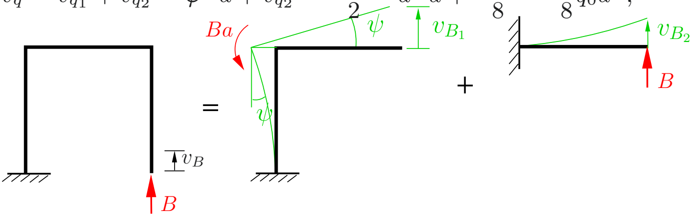
Solution We free the right support and use B as static redundant quantity



The individual displacement components are determined from the table on page 62 and superposition:



$$v_q = v_{q_1} + v_{q_2} = \varphi \cdot a + v_{q_2} = \frac{q_0 a^2}{2} \cdot a \cdot a + \frac{q_0 a^4}{8} = \frac{5}{8} q_0 a^4 ,$$



$$v_B = v_{B_1} + v_{B_2} = \psi \cdot a + v_{B_2} = Ba \cdot a \cdot a + B\frac{a^3}{3} = \frac{4}{3} Ba^3 .$$

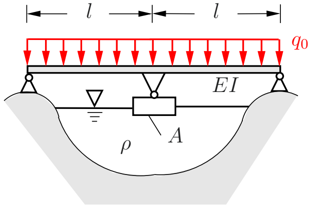
The compatibility at B provides the reaction force B:

$$v_q = v_B \quad \rightsquigarrow \quad \underline{\underline{B = \frac{15}{32} q_0 a}} .$$

The other support reactions follow from equilibrium

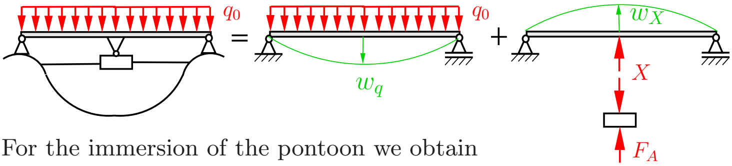
$$\underline{\underline{A = \frac{17}{32} q_0 a}} \quad \text{and} \quad \underline{\underline{M_A = -\frac{1}{32} q_0 a^2}} .$$

P3.29 **Problem 3.29** An auxiliary bridge, that is resting on the river banks, is supported in the middle by an additional pontoon (block with cross section A at the water line). The bridge is subjected to a constant load q_0.
Given: water density ρ, $EI/Al^3\rho g = 1/24$.



Determine the immersion depth f of the pontoon due to q_0.

Solution The system is statically *indeterminately* supported. We use the pontoon force as statically redundant force and apply superposition:



For the immersion of the pontoon we obtain

$$f = w_q - w_X \,.$$

Archimedes' principle yields the buoyant force F_A that is equal to the weight of displaced fluid (see also chapter 7), i. e. we have

$$X = F_A = \rho g f A \qquad \rightsquigarrow \qquad f = \frac{X}{\rho g A} \,.$$

The table on page 62 provides

$$\text{no. 2}: \quad w_q = \frac{5}{384}\frac{q_0 (2l)^4}{EI} \,, \qquad \text{no. 1}: \quad w_X = \frac{X (2l)^3}{48EI} \,.$$

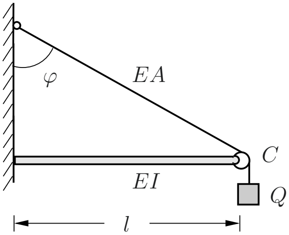
Using the above results

$$\frac{X}{\rho g A} = \frac{5}{384}\frac{q_0 16 l^4}{EI} - \frac{X 8 l^3}{48 EI} \quad \rightsquigarrow \quad X = \frac{\dfrac{5}{24}\dfrac{q_0 l^4}{EI}}{\dfrac{1}{6}\dfrac{l^3}{EI} + \dfrac{1}{\rho g A}} = q_0 l \,.$$

the immersion depth is given by

$$\underline{\underline{f}} = \frac{X}{\rho g A} = \frac{q_0 l}{\rho g A}\frac{EI}{EI}\frac{l^3}{l^3} = \underline{\underline{\frac{1}{24}\frac{q_0 l^4}{EI}}} \,.$$

Problem 3.30 An elastic rope (length s) is fixed to the wall and in C frictionless redirected by a pulley. The pulley is attached to a beam (axial rigidity $\to \infty$), **P3.30**



Determine the displacement of the load Q.

Solution The displacement of Q is computed by the length change

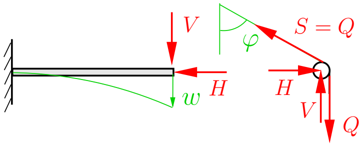
$$\Delta s = \frac{Q\,s}{EA}$$

of the rope and a contributions δ of the deflection of the pulley. The deflection is calculated by the vertical load on the beam

$$V = Q - S\cos\varphi = Q(1 - \cos\varphi)$$

to be

$$w = \frac{Vl^3}{3EI} = \frac{Q(1-\cos\varphi)l^3}{3EI}\,.$$

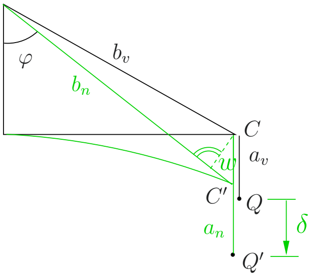


The deflection δ of the load Q follows from

$$\begin{aligned} \delta &= w + a_n - a_v \\ &= w + (s - b_n) - (s - b_v) \\ &= w + b_v - b_n \end{aligned}$$

with

$$b_n - b_v = w\cos\varphi \qquad (\text{for } w \ll b_v)\,.$$

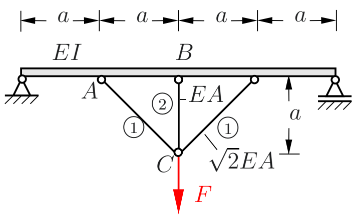


This leads to the deflection of Q

$$\underline{\underline{v_Q}} = \delta + \Delta s = w(1-\cos\varphi) + \frac{Qs}{EA} = \underline{\underline{Q\left[\frac{s}{EA} + \frac{l^3(1-\cos\varphi)^2}{3EI}\right]}}\,.$$

P3.31

Problem 3.31 The depicted structure consists of a beam and bars with stiffness ratio $\alpha = EI/a^2EA$. The structure is loaded by the force F.

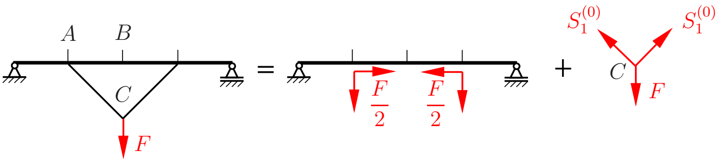


a) Determine the forces in the bars for $\alpha = 1/8$

b) For which value of α vanishes the force S_2?

c) For which α follows $M_B = 0$?

Solution The system is statically indeterminate in the interior. We free the middle bar (basic system):



Equilibrium in C yields $S_1^{(0)} = \sqrt{2}F/2$. The beam is loaded by the components $F/2$. With the table on page 62 (load case no. 1) the displacement at A is given by

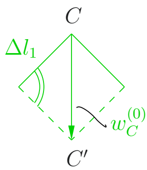
$$EI\, w_A^{(0)} = \frac{F}{2}\frac{(4a)^3}{6}\left[\frac{3}{4}\cdot\frac{1}{4}\left(1-\frac{9}{16}-\frac{1}{16}\right)+\frac{1}{4}\cdot\frac{1}{4}\left(1-\frac{1}{16}-\frac{1}{16}\right)\right] = \frac{2}{3}Fa^3 ,$$

and at location B

$$EI\, w_B^{(0)} = 2\cdot\frac{F}{2}\frac{(4a)^3}{6}\frac{1}{4}\cdot\frac{1}{2}\left(1-\frac{1}{16}-\frac{1}{4}\right) = \frac{11}{12}Fa^3 .$$

Due to the truss elongation Δl_1 point C experiences the displacement

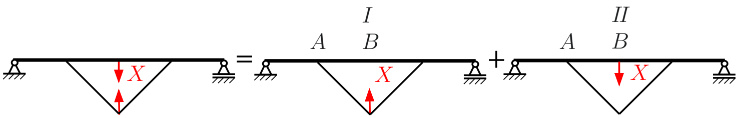
$$w_C^{(0)} = \Delta l_1\sqrt{2} = \frac{S_1 l_1}{\sqrt{2}EA}\sqrt{2} = \frac{\frac{1}{2}\sqrt{2}\,Fa\sqrt{2}}{\sqrt{2}\,EA}\sqrt{2} = \frac{Fa}{EA} .$$



Hence the total displacement of C is given by

$$v_C^{(0)} = w_B^{(0)} + w_C^{(0)} = \frac{2}{3}\frac{Fa^3}{EI} + \frac{Fa}{EA} .$$

Now we load the system by the unknown normal force $S_2 = X$ and consider the two load cases independently:



In sub-system I the deformation is analogous to the basic system, if F is replaced by $-X$, i. e.

$$v_C^{(I)} = -\frac{2}{3}\frac{Xa^3}{EI} - \frac{Xa}{EA}\,, \qquad w_B^{(I)} = -\frac{11}{12}\frac{Xa^3}{EI}\,.$$

The displacement in sub-system II is again determined from the table on page 62

$$w_B^{(II)} = \frac{X(4a)^3}{48EI} = \frac{4}{3}\frac{Xa^3}{EI}\,,$$

$$v_C^{(II)} = w_A^{(II)} = \frac{X(4a)^3}{6EI}\left\{\frac{1}{2}\frac{1}{4}\left(1-\frac{1}{4}-\frac{1}{16}\right)\right\} = \frac{11}{12}\frac{Xa^3}{EI}\,.$$

Compatibility requires that the difference in the total displacement at points C und B are equal to the elongation of bar 2:

$$v_C^{(0)} + v_C^{(I)} + v_C^{(II)} - \left[w_B^{(0)} + w_B^{(I)} + w_B^{(II)}\right] = \frac{Xa}{EA}$$

or

$$\frac{2Fa^3}{3EI} + \frac{Fa}{EA} - \frac{2Xa^3}{3EI} - \frac{Xa}{EA} + \frac{11Xa^3}{12EI} - \left(\frac{11Fa^3}{12EI} - \frac{11Xa^3}{12EI} + \frac{4Xa^3}{3EI}\right) = \frac{Xa}{EA}$$

$$\leadsto \quad \underline{\underline{X = \frac{\alpha - \frac{1}{4}}{2\alpha + \frac{1}{6}}F}}\,.$$

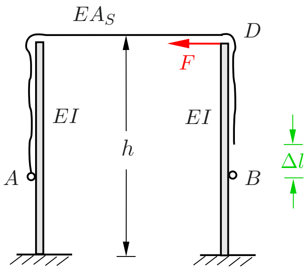
With this result the answers to the questions are:

to a) $X = \underline{\underline{S_2}} = \dfrac{\frac{1}{8}-\frac{1}{4}}{\frac{1}{4}+\frac{1}{6}}F = \underline{\underline{-\dfrac{3}{10}F}}, \quad \underline{\underline{S_1}} = \dfrac{1}{2}\sqrt{2}\,(F-X) = \underline{\underline{\dfrac{13}{20}\sqrt{2}\,F}},$

to b) $S_2 = X = 0 \quad \leadsto \quad \underline{\underline{\alpha = \dfrac{1}{4}}},$

to c) $M_B = \dfrac{F}{2}2a - \left(\dfrac{F}{2} - \dfrac{X}{2}\right)a = 0 \quad \leadsto \quad X = -F\,,$

$$\leadsto \quad \frac{\alpha - \frac{1}{4}}{2\alpha + \frac{1}{6}}F = -F \quad \leadsto \quad \underline{\underline{\alpha = \frac{1}{36}}}\,.$$



P3.32 **Problem 3.32** The two depicted posts have to be connected by a rope. The rope has to be fixed at points A and B. The rope is too short by Δl.

a) Determine the horizontal force F at the right post that is required to fix the rope stress-free.

b) The force F is removed after assembly. Determine the force in the rope and the moments at both supports.

Solution **to a)** The force F has to bend the post by Δl to the left. From the table on page 62 (load case no. 5) we obtain

$$\Delta l = \frac{Fh^3}{3EI} \quad \rightsquigarrow \quad \underline{\underline{F = \frac{3EI}{h^3}\Delta l}}\,.$$

to b) The length Δl follows from the extension Δl_S of the rope due to a yet unknown force S in the rope and the deflection f_S of both posts due to the same unknown force S. Compatibility states

$$\Delta l = \Delta l_S + f_S + f_S$$

which yields

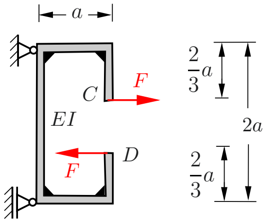
$$\Delta l = \frac{Sl}{EA_S} + \frac{Sh^3}{3EI} + \frac{Sh^3}{3EI} \qquad \rightsquigarrow \qquad \underline{\underline{S = \frac{\Delta l}{l} EA_S \frac{1}{1 + \frac{2}{3}\frac{h^3 EA_S}{lEI}}}}\,.$$

Finally the moments at the support follow from equilibrium

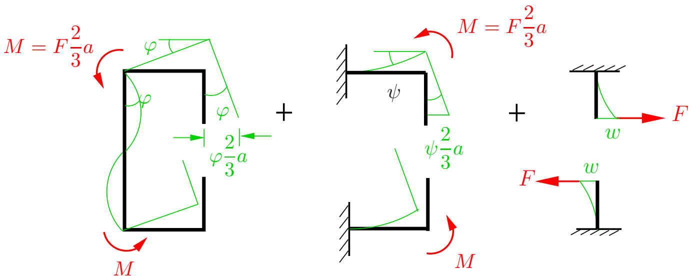
$$\underline{\underline{M}} = hS = \underline{\underline{\frac{\Delta l}{l} EA_S h \frac{1}{1 + \frac{2}{3}\frac{h^3 EA_S}{lEI}}}}\,.$$

Problem 3.33 A plane frame is loaded in C and D by two forces. **P3.33**

Determine the reciprocative horizontal displacement Δu of C und D.



Solution To apply the table on page 62 we have to separate the deformation of the individual beams and use superposition.



$$C \text{ is moved by} \quad \varphi \cdot \frac{2}{3}\,a + \psi \cdot \frac{2}{3}\,a + w \quad \text{to the right,}$$

$$D \text{ is moved by} \quad \varphi \cdot \frac{2}{3}\,a + \psi \cdot \frac{2}{3}\,a + w \quad \text{to the left.}$$

Thus, the reciprocative displacement follows

$$\Delta u = 2\left[\varphi \cdot \frac{2}{3}a + \psi \cdot \frac{2}{3}a + w\right] .$$

With the table on page 62 it follows:

$$\text{load case no. 2} \quad EI\,\varphi = \left(\frac{2}{3}\,Fa\right)\frac{2a}{3} - \left(\frac{2}{3}\,Fa\right)\frac{2a}{6} = \frac{2}{9}\,Fa^2\,,$$

$$\text{load case no. 8} \quad EI\,\psi = \left(\frac{2}{3}\,Fa\right)a = \frac{2}{3}\,Fa^2\,,$$

$$\text{load case no. 5} \quad EI\,w = \frac{F\left(\frac{2}{3}a\right)^3}{3} = \frac{8}{81}\,Fa^3\,,$$

which yields

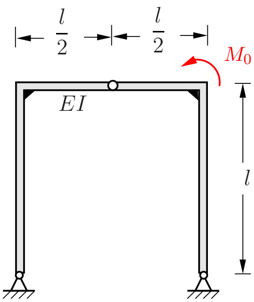
$$\underline{\underline{\Delta u}} = 2\left(\frac{4}{27} + \frac{4}{9} + \frac{8}{81}\right)\frac{Fa^3}{EI} = \underline{\underline{\frac{112}{81}\,\frac{Fa^3}{EI}}}\,.$$

Note: Due to the antisymmetry of the system the vertical displacements of C and D are the same.

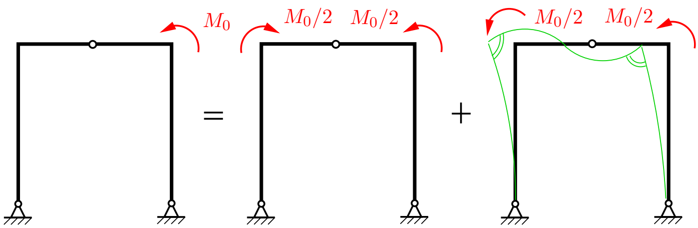
P3.34

Problem 3.34 The depicted frame is loaded by a moment M_0.

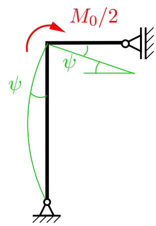
Determine the reciprocative rotation $\Delta\varphi_H$ at the hinge.



Solution It is reasonable to split the loading into a symmetric and antisymmetric contribution:



The *antisymmetric* loading causes *no* reciprocative rotation at the hinge. For the *symmetric* loadign it suffices to consider half of the frame structure. The rotation ψ results solely from the bending of the vertical post (only a normal force occurs in the horizontal beam). Thus from the table on page 62 (load case no. 4 with $\beta = 1$ and $\alpha = 0$) we obtain

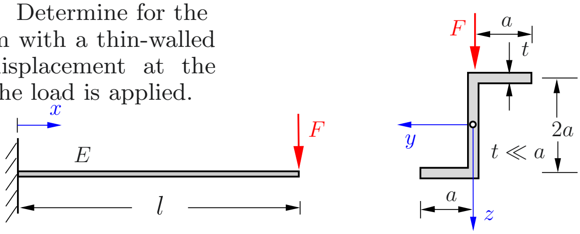


$$\psi = \frac{\frac{M_0}{2}l}{3EI} = \frac{M_0 l}{6EI}\,.$$

Hence the reciprocative rotation follows

$$\underline{\underline{\Delta\varphi_H}} = 2\psi = \underline{\underline{\frac{M_0 l}{3EI}}}\,.$$

Problem 3.35 Determine for the depicted beam with a thin-walled profile the displacement at the point where the load is applied. **P3.35**



Solution Due to the unsymmetrical profile oblique bending occurs. The displacements are computed using the two related differential equations. The bending moments are given by

$$M_y = -F(l-x)\,, \qquad M_z = 0\,,$$

and the second moments of area for the thin-walled profile follow from

$$I_y = \frac{t(2a)^3}{12} + 2(at)a^2 = \frac{8}{3}ta^3, \qquad I_z = \frac{2}{3}ta^3,$$
$$I_{yz} = -2(ta)a\frac{a}{2} = -ta^3, \qquad \Delta = I_y I_z - I_{yz}^2 = \frac{7}{9}t^2a^6.$$

Thus the two differential equations can be integrated for the z-direction

$$Ew'' = -\frac{M_y I_z}{\Delta} = \frac{6}{7}\frac{F}{ta^3}(l-x)\,,$$
$$Ew' = -\frac{3}{7}\frac{F}{ta^3}(l-x)^2 + C_1\,,$$
$$Ew = \frac{1}{7}\frac{F}{ta^3}(l-x)^3 + C_1x + C_2$$

and the y-direction

$$Ev'' = -\frac{M_y I_{yz}}{\Delta} = -\frac{9}{7}\frac{F}{ta^3}(l-x)\,,$$
$$Ev' = \frac{9}{14}\frac{F}{ta^3}(l-x)^2 + C_3\,,$$
$$Ev = -\frac{3}{14}\frac{F}{ta^3}(l-x)^3 + C_3x + C_4\,.$$

The boundary conditions at the support yield

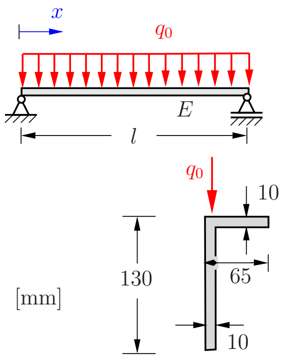
$$v'(0) = 0 \;\rightsquigarrow\; C_3 = -\frac{9}{14}\frac{Fl^2}{ta^3}\,, \quad w'(0) = 0 \;\rightsquigarrow\; C_1 = \frac{3}{7}\frac{Fl^2}{ta^3}\,,$$
$$v(0) = 0 \;\rightsquigarrow\; C_4 = \frac{3}{14}\frac{Fl^3}{ta^3}\,, \quad w(0) = 0 \;\rightsquigarrow\; C_2 = -\frac{1}{7}\frac{Fl^3}{ta^3}\,.$$

Thus the displacements at the point, where the load is applied $x = l$, are

$$\underline{\underline{w(l) = \frac{2}{7}\frac{Fl^3}{Eta^3}}}\,, \qquad \underline{\underline{v(l) = -\frac{3}{7}\frac{Fl^3}{Eta^3}}}\,.$$

Note: Although the load is acting in *vertical* direction a displacement in *horizontal* direction occurs. The profile preferably deforms in the direction which is related to the smaller second moment of area.

P3.36 **Problem 3.36** The simply supported beam is loaded by a constant distributed load.

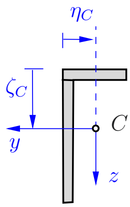


Determine the displacement of the centroid of the cross section in the middle of the beam (only deformation due to bending).

Given: $l = 2$ m ,

$$E = 2.1 \cdot 10^5 \text{ MPa} ,$$
$$q_0 = 10^4 \text{ N/m} .$$

Solution We compute the geometric quantities of the cross section:



$$A = 65 \cdot 10 + 120 \cdot 10 = 1850 \text{ mm} ,$$

$$\zeta_C = \frac{(65 \cdot 10) \cdot 5 + (120 \cdot 10) \cdot 70}{1850} = 47.16 \text{ mm} ,$$

$$\eta_C = \frac{(65 \cdot 10) \cdot 32.5 + (120 \cdot 10) \cdot 5}{1850} = 14.66 \text{ mm} ,$$

$$I_y = \frac{65 \cdot 10^3}{12} + (42.16)^2(65 \cdot 10) + \frac{10 \cdot 120^3}{12} + (22.84)^2(10 \cdot 120)$$
$$= 322.7 \text{ cm}^4 ,$$

$$I_z = \frac{10 \cdot 65^3}{12} + (17.84)^2(65 \cdot 10) + \frac{120 \cdot 10^3}{12} + (9.66)^2(10 \cdot 120)$$
$$= 55.8 \text{ cm}^4 ,$$

$$I_{yz} = -(-17.84)(-42.16)(65 \cdot 10) - (22.84)(9.66)(10 \cdot 120)$$
$$= -75.4 \text{ cm}^4 ,$$

$$\Delta = I_y I_z - I_{yz}^2 = 12321.5 \text{ cm}^8 .$$

The loading causes only a moment along the y-axis:

$$M_y(x) = \frac{q_0 l}{2} x - q_0 \frac{x^2}{2} .$$

The basic equations simplify to

$$Ew'' = -\frac{M_y I_z}{\Delta}\,, \qquad Ev'' = -\frac{M_y I_{yz}}{\Delta}\,.$$

Integrating twice yields

$$Ew' = -\frac{I_z}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^2}{2} - \frac{x^3}{3} + C_1\right)\,,$$

$$Ew = -\frac{I_z}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^3}{6} - \frac{x^4}{12} + C_1 x + C_2\right)\,,$$

$$Ev' = -\frac{I_{yz}}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^2}{2} - \frac{x^3}{3} + C_3\right)\,,$$

$$Ev = -\frac{I_{yz}}{\Delta}\,\frac{q_0}{2}\left(l\frac{x^3}{6} - \frac{x^4}{12} + C_3 x + C_4\right)\,.$$

The boundary conditions

$$w(0) = 0 \quad\rightsquigarrow\quad C_2 = 0\,, \qquad v(0) = 0 \quad\rightsquigarrow\quad C_4 = 0\,,$$

$$w(l) = 0 \quad\rightsquigarrow\quad C_1 = -\frac{l^3}{12}\,, \quad v(l) = 0 \quad\rightsquigarrow\quad C_3 = -\frac{l^3}{12}$$

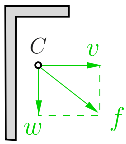
together with the abbreviation $\xi = \dfrac{x}{l}$ yield

$$Ew = \frac{q_0 l^4}{24}\left\{\xi^4 - 2\xi^3 + \xi\right\}\frac{I_z}{\Delta}\,,$$

$$Ev = \frac{q_0 l^4}{24}\left\{\xi^4 - 2\xi^3 + \xi\right\}\frac{I_{yz}}{\Delta}\,.$$

In the middle of the beam ($\xi = 1/2$) the curly brackets attain the value $5/16$ which leads with the given numerical values (converted to cm) to

$$\underline{\underline{w}} = 10^2 \cdot 200^4\,\frac{5}{384}\,\frac{55.8}{12321.5}\cdot\frac{1}{2.1\cdot 10^7} = \underline{\underline{0.45\ \text{cm}}}\,,$$

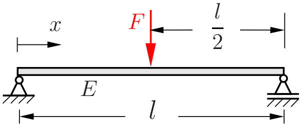


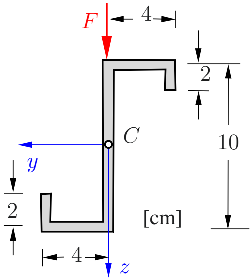
$$\underline{\underline{v}} = 10^2 \cdot 200^4\,\frac{5}{384}\,\frac{-75.4}{12321.5}\cdot\frac{1}{2.1\cdot 10^7} = \underline{\underline{-0.61\ \text{cm}}}\,,$$

$$\underline{\underline{f}} = \sqrt{w^2 + v^2} = \underline{\underline{0.76\ \text{cm}}}\,.$$

P3.37

Problem 3.37 In the middle of a beam the force F is applied. The thin-walled profile is produced from an aluminium sheet of 2 mm thickness.





Compute the deformation at the point where the force is applied.

Given: $l = 2$ m ,

$$E = 7 \cdot 10^4 \text{ MPa} ,$$

$$F = 1200 \text{ N} .$$

Solution The displacement can be determined with regards to the y, z-axes, or with regard to the principal axes. We want to consider both possibilities.

1st solution: The position of the centroid is known. With regard to the y, z-axes we find

$$I_y = \frac{0.2 \cdot 10^3}{12} + \left(\frac{0.2 \cdot 10^3}{12} - \frac{0.2 \cdot 6^3}{12}\right) + 2 \cdot 5^2 \cdot 0.2 \cdot 4 = 69.73 \text{ cm}^4 ,$$

$$I_z = \frac{0.2 \cdot 8^3}{12} + 2 \cdot 4^2 \cdot 0.2 \cdot 2 = 21.33 \text{ cm}^4 ,$$

$$I_{yz} = -2\{5 \cdot 2 \cdot 0.2 \cdot 4 + 4 \cdot 4 \cdot 0.2 \cdot 2\} = -28.8 \text{ cm}^4 ,$$

$$\Delta = I_y I_z - I_{yz}^2 = 657.9 \text{ cm}^8 .$$

With the bending moments $M_y = \frac{F}{2} x , \quad M_z = 0 \quad$ für $0 \le x \le l/2$ (symmetry) the differential equations are given by

$$E w'' = -\frac{F\, I_z}{2\,\Delta}\, x , \qquad E v'' = -\frac{F\, I_{yz}}{2\,\Delta}\, x .$$

After integration and incorporation of the boundary conditions we obtain in the middle of the beam (see also table on page 62):

$$\underline{\underline{w}} = \frac{F l^3}{48 E} \frac{I_z}{\Delta} = \frac{1200 \cdot 200^3}{48 \cdot 7 \cdot 10^6} \cdot \frac{21.33}{657.9} = \underline{\underline{0.93 \text{ cm}}} ,$$

$$\underline{\underline{v}} = \frac{F l^3}{48 E} \frac{I_{yz}}{\Delta} = \frac{1200 \cdot 200^3}{48 \cdot 7 \cdot 10^6} \cdot \frac{(-28.8)}{657.9} = \underline{\underline{-1.25 \text{ cm}}} ,$$

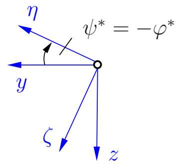
$$\underline{\underline{f}} = \sqrt{w^2 + v^2} = \underline{\underline{1.56 \text{ cm}}} .$$

2nd solution: We refer to the principal axes. The principal directions and values of the second moment of area are given by

$$\tan 2\varphi^* = \frac{2I_{yz}}{I_y - I_z} = -1.19 \quad \rightsquigarrow \quad \varphi^* = -24.98^\circ$$

$$I_{1,2} = \frac{91.06}{2} \pm \sqrt{24.2^2 + 28.8^2}$$

$$\rightsquigarrow \quad I_1 = I_\eta = 83.15\ \text{cm}^4\ , \quad I_2 = I_\zeta = 7.91\ \text{cm}^4\ .$$



Decomposition of the load into principal directions yields

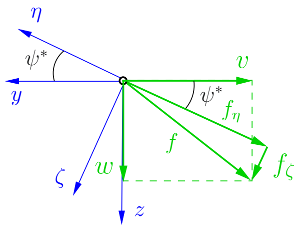
$$F_\zeta = F\cos\psi^* = 0.906\ F\ , \qquad F_\eta = -F\sin\psi^* = 0.422\ F\ ,$$

and the displacements follow from the table on page 62 (load case no. 1)

$$f_\eta = \frac{F_\eta l^3}{48EI_\zeta} = -\frac{1200\cdot 0.422\cdot 200^3}{48\cdot 7\cdot 10^6\cdot 7.91} = -1.52\ \text{cm}\ ,$$

$$f_\zeta = \frac{F_\zeta l^3}{48EI_\eta} = \frac{1200\cdot 0.906\cdot 200^3}{48\cdot 7\cdot 10^6\cdot 83.15} = 0.31\ \text{cm}\ ,$$

$$\underline{\underline{f}} = \sqrt{f_\eta^2 + f_\zeta^2} = \underline{\underline{1.55\ \text{cm}}}\ .$$



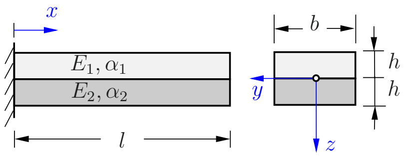
For comparison with the 1st solution we transfer the displacements into the y,z-coordinate system:

$$\underline{\underline{|v|}} = |f_\eta|\cos\psi^* - f_\zeta\sin\psi^* = \underline{\underline{1.25\ \text{cm}}}\,,$$
$$\underline{\underline{w}} = |f_\eta|\sin\psi^* + f_\zeta\cos\psi^* = \underline{\underline{0.93\ \text{cm}}}\,.$$

Note: We used in the computations numerical values up to the second digit. Thus the numerical value for the total displacement f differs in the second digit.

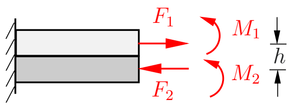
P3.38

Problem 3.38 A beam composed of two different materials (a bi-metal beam to measure temperature) is heated uniformly by a temperature difference ΔT.



Determine the deformation at the free end.

Solution We assume a linear stress distribution in each material and replace the stresses by a resultant force F_i and a resulting moment M_i. If we suppose $\alpha_2 > \alpha_1$ the lower part wants expand more. As this is prevented by the upper part, the lower part is under compression, while tension prevails in the upper part. F_1 and F_2 cause a moment in the composite beam which is in equilibrium with M_1 and M_2 (no external loads). Thus the following equations hold:



$$\begin{aligned} &\text{statics} && N = 0 \quad \rightsquigarrow \quad F_1 = F_2 = F\,, \\ & && M = 0 \quad \rightsquigarrow \quad Fh = M_1 + M_2\,, \\ &\text{Hooke's law} && w_1'' = -\frac{M_1}{E_1}\,\frac{12}{bh^3}\,, \quad w_2'' = -\frac{M_2}{E_2}\,\frac{12}{bh^3}\,. \end{aligned}$$

Kinematic compatibility demands

$$w_1'' = w_2'' = w''\,.$$

Additionally the strains have to match at the interface. They consist of three contributions: temperature $\alpha_i \Delta T$, normal force F/EA and bending M/EW. Considering tension and compression we formulate

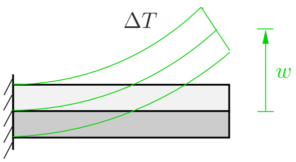
$$\alpha_1 \Delta T + \frac{F}{bhE_1} + \frac{M_1 6}{E_1 bh^2} = \alpha_2 \Delta T - \frac{F}{bhE_2} - \frac{M_2 6}{E_2 bh^2}\,.$$

Eliminating the moments M_i and rearrangement to get w'' yields

$$w'' = -\frac{12 E_1 E_2 (\alpha_2 - \alpha_1)\Delta T}{h(E_1^2 + 14 E_1 E_2 + E_2^2)} = -C\,.$$

Integration, by incorporating the boundary conditions at the left end, provides the displacement at the free end

$$\underline{\underline{w = -C\,\frac{l^2}{2}}}\,.$$



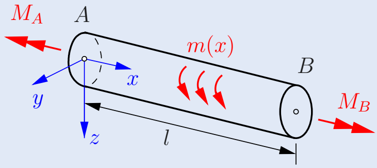
Chapter 4

Torsion

4

Torsion

If an external load causes an internal moment M_x along the longitudinal axis, the bar is loaded by *torsion* (twisting). In the following we refer to the moment M_x as *torque* or *torsional moment* M_T.



Prerequisites, assumptions:

- Warping of the cross sections is not constrained (*pure torsion*),
- The shape of the cross sections does not change during rotation.

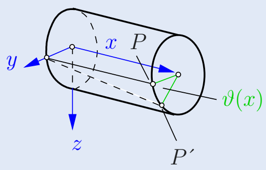
Equilibrium conditions

$$\frac{\mathrm{d}M_T}{\mathrm{d}x} = -m\,, \qquad m(x) = \text{external moment per unit length.}$$

Differential equation for the angle of twist

$$\boxed{GI_T\frac{\mathrm{d}\vartheta}{\mathrm{d}x} = M_T\,,}$$

ϑ = angle of twist,
GI_T = torsional rigidity,
G = shear modulus,
I_T = torsional constant.



Twist of end sections

$$\boxed{\Delta\vartheta = \vartheta(l) - \vartheta(0) = \int_0^l \vartheta'(x)\mathrm{d}x = \int_0^l \frac{M_T}{GI_T}\mathrm{d}x\,.}$$

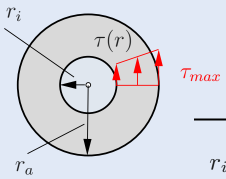
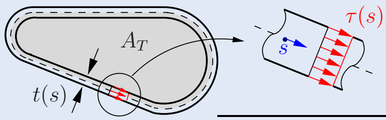
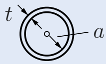
Special case: $GI_T = \text{const},\ M_T = \text{const}$

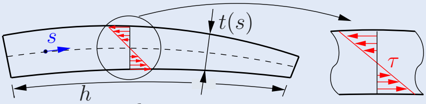
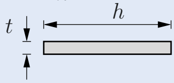
$$\boxed{\Delta\vartheta = \frac{M_T l}{GI_T}\,.}$$

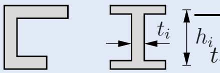
Maximum shear stress

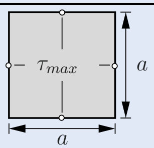
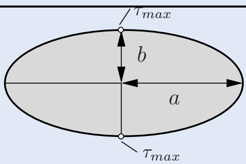
$$\boxed{\tau_{\max} = \frac{M_T}{W_T}\,,} \qquad W_T = \text{sectional moment of torsion.}$$

The location of the maximum shear stress is provided in the following table.

<table>
<tr><th>Cross section</th><th>I_T</th></tr>
<tr><td>
</td><td>$I_T = I_p = \dfrac{\pi}{2}(r_a^4 - r_i^4)$</td></tr>
<tr><td>$r_i = 0$ (full circle)</td><td>$I_T = \dfrac{\pi}{2} r_a^4$</td></tr>
<tr><td>thin-walled, closed profile

</td><td>$I_T = \dfrac{4A_T^2}{\displaystyle\oint \frac{\mathrm{d}s}{t(s)}}$</td></tr>
<tr><td>

$a = \text{const}$
$t = \text{const}$</td><td>$I_T = 2\pi a^3 t$</td></tr>
<tr><td>thin-walled, open profile

</td><td>$I_T = \dfrac{1}{3}\displaystyle\int_0^h t^3(s)\,\mathrm{d}s$</td></tr>
<tr><td>

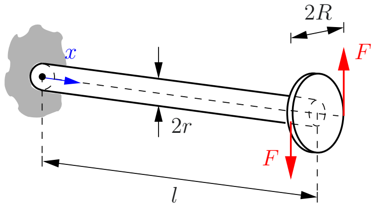
$t = \text{const}$</td><td>$I_T = \dfrac{1}{3} h t^3$</td></tr>
<tr><td>

$t_i = \text{const}$</td><td>$I_T = \dfrac{1}{3}\sum h_i t_i^3$</td></tr>
<tr><td>square

</td><td>$I_T = 0{,}141 a^4$</td></tr>
<tr><td>ellipse

</td><td>$I_T = \pi \dfrac{a^3 b^3}{a^2 + b^2}$</td></tr>
</table>

<table>
<tr><th>W_T</th><th>Remarks</th></tr>
<tr><td>$W_T = \frac{I_T}{r_a} = \frac{\pi}{2}\frac{r_a^4 - r_i^4}{r_a}$</td><td rowspan="2">The shear stresses are distributed linearly across the cross section: $$\tau(r) = \frac{M_T}{I_T} r \ .$$ Cross sections remain plane during deformation.</td></tr>
<tr><td>$W_T = \frac{\pi}{2} r_a^3$</td></tr>
<tr><td>$W_T = 2A_T\, t_{\min}$</td><td rowspan="2">τ is constant across the wall-thickness t. The shear flow $$T = \tau t = \frac{M_T}{2A_T}$$ is constant.
$\tau_{\max}$ occurs at the smallest wall-thickness $t_{\min}$.
A_T is the area encircled by the central line of the profile.</td></tr>
<tr><td>$W_T = 2\pi a^2 t$</td></tr>
<tr><td>$W_T = \frac{I_T}{t_{\max}}$</td><td rowspan="3">$\tau$ is linearly distributed across the wall-thickness.
$\tau_{\max}$ occurs at the largest wall-thickness $t_{\max}$.</td></tr>
<tr><td>$W_T = \frac{1}{3} h t^2$</td></tr>
<tr><td>$W_T = \frac{I_T}{t_{\max}}$</td></tr>
<tr><td>$W_T = 0.208\ a^3$</td><td>$\tau_{\max}$ occurs at in the middle of the lateral lengths.</td></tr>
<tr><td>$W_T = \frac{\pi}{2}\, a b^2$</td><td>$\tau_{\max}$ occurs at the ends of the smaller semi-axis.</td></tr>
</table>

P4.1

Problem 4.1 A shaft with circular cross section is clamped at one end and loaded by a pair of forces.



Determine F such that the admissible shear stress τ_{admis} is not exceeded. Compute for this case the twist of the end section.

Given: $R = 200$ mm, $r = 20$ mm, $l = 5$ m, $\tau_{\text{zul}} = 150$ MPa,
$G = 0.8 \cdot 10^5$ MPa.

Solution The torque (torsional moment)

$$M_T = 2RF$$

is constant along the bar. The maximum shear stress in the cross section is given with

$$W_T = \frac{\pi}{2} r^3$$

by

$$\tau_{\max} = \frac{M_T}{W_T} = \frac{4RF}{\pi r^3} \; .$$

In order not to exceed the admissible shear stress,

$$\tau_{\max} \le \tau_{\text{admis}} \qquad \leadsto \qquad F \le \frac{\pi r^3}{4R} \tau_{\text{admis}} \; .$$

must hold and we obtain

$$\underline{\underline{F_{\max}}} = \frac{\pi r^3}{4R} \tau_{\text{allow}} = \frac{\pi \cdot 8000 \cdot 150}{4 \cdot 200} = \underline{\underline{4712 \text{ N}}} \; .$$

For this load the twist (in radians) can be computed using

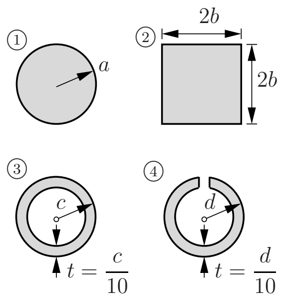
$$I_T = \frac{\pi}{2} r^4 \qquad \text{and} \qquad M_T = 2RF_{\max} \; .$$

Inserting yields

$$\underline{\underline{\Delta\vartheta}} = \frac{M_T l}{G I_T} = \frac{\tau_{\text{zul}} l}{G r} = \frac{150 \cdot 5000}{0.8 \cdot 10^5 \cdot 20} = \underline{\underline{0.47}} \; .$$

This value is equivalent to an angle of 27°.

Problem 4.2 A shaft has to carry the torque $M_T = 12 \cdot 10^3$ Nm. Select a cross section from the depicted group. P4.2



Dimension the cross sections such that the admissible shear stress $\tau_{\text{admis}} = 50$ MPa is not exceeded. Which cross section is the most efficient in terms of material usage?

Solution The admissble shear stress is reached for

$$\tau_{\max} = \frac{M_T}{W_T} = \tau_{\text{admis}} \, .$$

With the section moment for torsion

$$W_{T_1} = \frac{\pi}{2}\, a^3 \, , \qquad W_{T_2} = 0.208 \cdot 8\, b^3 = 1.664\, b^3 \, ,$$

$$W_{T_3} = 2\pi c^2 t = \frac{\pi}{5}\, c^3 \, , \qquad W_{T_4} = \frac{2\pi}{3}\, d\, t^2 = \frac{\pi}{150}\, d^3$$

we determine with the given numerical values

$$\underline{\underline{a}} = \sqrt[3]{\frac{2M_T}{\pi\tau_{\text{zul}}}} = \underline{\underline{53.5 \text{ mm}}} \, , \qquad \underline{\underline{b}} = \sqrt[3]{\frac{M_T}{1.664\, \tau_{\text{zul}}}} = \underline{\underline{52.4 \text{ mm}}} \, ,$$

$$\underline{\underline{c}} = \sqrt[3]{\frac{5\, M_T}{\pi\, \tau_{zul}}} = \underline{\underline{72,6 \text{ mm}}} \, , \qquad \underline{\underline{d}} = \sqrt[3]{\frac{150\, M_T}{\pi\, \tau_{zul}}} = \underline{\underline{225.5 \text{ mm}}} \, .$$

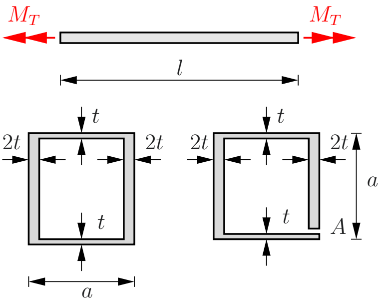
The cross section areas are

$$A_1 = \pi a^2 = 89.8 \text{ cm}^2 \, , \qquad A_2 = 4b^2 = 110.0 \text{ cm}^2 \, ,$$

$$\underline{\underline{A_3}} = \frac{\pi}{5} c^2 = \underline{\underline{33.1 \text{ cm}^2}} \, , \qquad A_4 = \frac{\pi}{5} d^2 = 319.4 \text{ cm}^2 \, .$$

Therefore, the third cross section (i. e. the thin-walled *closed* profile) is the most material efficient profile.

P4.3 **Problem 4.3** Determine the maximum admissible torque (torsional moment) and the corresponding admissible twist for the closed profile and the profile that is slit at A.



Given: $a = 10$ cm, $t = 2$ mm,
$\tau_{\text{admis}} = 20$ MPa,
$l = 5$ m,
$G = 0.8 \cdot 10^5$ MPa.

Solution The admissible torque and the admissible twist are computed for both profiles via

$$M_{T_{\text{admis}}} = \tau_{\text{admis}} W_T \,, \qquad \Delta\vartheta_{\text{admis}} = \frac{M_{T_{\text{admis}}} l}{G I_T} = \frac{\tau_{\text{admis}} W_T l}{G I_T} \,.$$

In the case of the *closed* profile with $t \ll a$ it holds

$$A_T = a^2 \,, \qquad \oint \frac{\mathrm{d}s}{t(s)} = 2\left(\frac{a}{2t} + \frac{a}{t}\right) = 3\frac{a}{t} \,,$$

$$I_T = \frac{4A_T^2}{\oint \frac{\mathrm{d}s}{t(s)}} = \frac{4}{3} t a^3 \,, \qquad W_T = 2 A_T t_{\min} = 2a^2 t$$

and we obtain

$$\underline{\underline{M_{T_{\text{admis}}}}} = \tau_{\text{admis}} 2a^2 t = \underline{\underline{800 \text{ Nm}}} \,,$$

$$\underline{\underline{\Delta\vartheta_{\text{allow}}}} = \frac{3\tau_{\text{admis}} l}{2Ga} = \underline{\underline{0.01875}} \quad (\widehat{=} 1,07^\circ) \,.$$

If the profile is *open* (slit at position A), we compute with

$$I_T = \frac{1}{3} \sum_i t_i^3 h_i = 6t^3 a \,, \qquad W_T = \frac{I_T}{t_{\max}} = 3t^2 a$$

the torque and twist

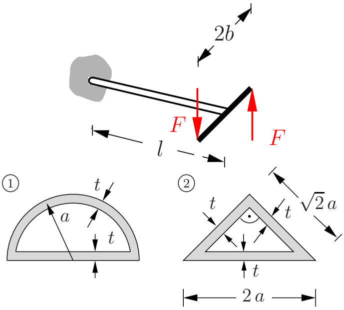
$$\underline{\underline{M_{T_{\text{admis}}}}} = \tau_{\text{admis}} 3t^2 a = \underline{\underline{24 \text{ Nm}}} \,,$$

$$\underline{\underline{\Delta\vartheta_{\text{admis}}}} = \frac{\tau_{\text{admis}} l}{2Gt} = \underline{\underline{0.3125}} \quad (\widehat{=} 17.9^\circ) .$$

Note: The closed profile is much stiffer with respect to torsion than the open profile.

Problem 4.4 A shaft is loaded by a pair of forces. The shaft is assmbled from two different thin-walled cross sections ($t \ll a$) of the same material (shear modulus G). **P4.4**

Determine in both cases the admissible forces and the corresponding twist such that the shear stress τ_{admis} is not exceeded.



Solution The torque $M_T = 2bF$ is constant along the length of the shaft. Stress and twist are determined from

$$\tau = \frac{M_T}{W_T} = \frac{2bF}{W_T}\,, \qquad \Delta\vartheta = \frac{M_T l}{GI_T} = \frac{2bFl}{GI_T}\,.$$

The admissible shear stress will not be exeeded for

$$\tau \leq \tau_{\text{admis}} \quad \rightsquigarrow \quad F \leq \frac{W_T \tau_{\text{admis}}}{2b} \quad \rightsquigarrow \quad F_{\text{admis}} = \frac{W_T \tau_{\text{admis}}}{2b}\,,$$

$$\Delta\vartheta_{\text{admis}} = \frac{2blF_{\text{admis}}}{GI_T} = \frac{\tau_{\text{admis}} W_T l}{GI_T}\,.$$

With the values for the two different cross sections

$$① \; A_T = \frac{\pi}{2}a^2\,, \quad \oint \frac{\mathrm{d}s}{t} = \frac{a}{t}(2+\pi)\,, \qquad W_T = \pi a^2 t\,, \quad I_T = \frac{\pi^2}{2+\pi}a^3 t\,,$$

$$② \; A_T = a^2\,, \quad \oint \frac{\mathrm{d}s}{t} = \frac{a}{t}(2+2\sqrt{2})\,, \quad W_T = 2a^2 t\,, \quad I_T = \frac{2}{1+\sqrt{2}}a^3 t$$

we obtain

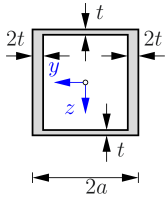
$$\underline{\underline{F_{\text{admis}_1} = \frac{\pi}{2}\frac{a^2 t}{b}\tau_{\text{admis}}}}\,, \qquad \underline{\underline{F_{\text{admis}_2} = \frac{a^2 t}{b}\tau_{\text{admis}}}}\,,$$

$$\underline{\underline{\Delta\vartheta_{\text{admis}_1} = \frac{2+\pi}{\pi}\frac{l\tau_{\text{admis}}}{aG}}}\,, \qquad \underline{\underline{\Delta\vartheta_{\text{admis}_2} = (1+\sqrt{2})\frac{l\tau_{\text{admis}}}{aG}}}\,.$$

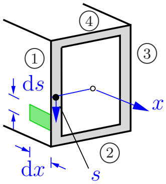
Note: The admissible force is larger for the first profile, while the admissible twist is larger for the second profile.

P4.5 **Problem 4.5** The thin-walled box girder is loaded by a torque M_T.

Determine the warping of the cross section.



Solution The warping $u(s)$ (displacement in longitudinal direction) is computed from the shear strain



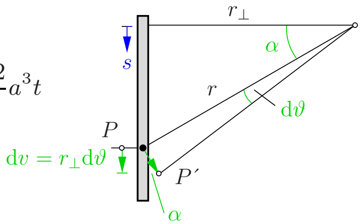
$$\gamma = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}$$

of the wall segments. With

$$\gamma = \frac{\tau}{G} = \frac{M_T}{G2A_T t(s)} ,$$

$$\frac{\partial v}{\partial x} = r_\perp \frac{\mathrm{d}\vartheta}{\mathrm{d}x} = r_\perp(s)\frac{M_T}{GI_T} ,$$

$$A_T = 4a^2 , \qquad I_T = \frac{4 \cdot 16a^4}{\frac{4a}{t} + \frac{4a}{2t}} = \frac{32}{3}a^3 t$$



we obtain

$$\frac{\partial u}{\partial s} = \frac{M_T}{8Ga^2 t}\left[\frac{t}{t(s)} - \frac{3r_\perp(s)}{4a}\right] .$$

Integration in region ① provides ($t(s) = 2t$, $r = a$) with $u(s=0) = 0$ (then u vanishes on average)

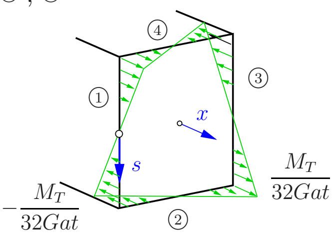
$$\underline{\underline{u_1(s)}} = \frac{M_T}{8Ga^2 t}\left[\frac{1}{2} - \frac{3}{4}\right] s = \underline{\underline{-\frac{M_T}{32Ga^2 t}}}\, s .$$

Analogously, we obtain in regions ② , ③ , ④

$$\underline{\underline{u_2(s) = \frac{M_T}{32Ga^2 t}[s - 2a]}} ,$$

$$\underline{\underline{u_3(s) = -\frac{M_T}{32Ga^2 t}[s - 4a]}} ,$$

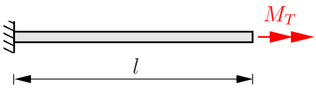
$$\underline{\underline{u_4(s) = \frac{M_T}{32Ga^2 t}[s - 6a]}} .$$

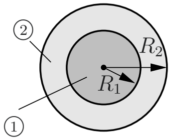


Problem 4.6 A tube ② is mounted by heat shrinking on a shaft ① with circular cross section of different material.

P4.6

Determine the maximum shear stresses in ① and ② as well as the twist under the application of a torque M_T.





Solution First we consider shaft ① and pipe ② independently. For the angle of twist and the stress it yields

$$\vartheta_1 = \frac{M_{T_1} l}{G_1 I_{p_1}}\,, \qquad \tau_{\max_1} = \frac{M_{T_1}}{W_{T_1}}\,,$$

$$\vartheta_2 = \frac{M_{T_2} l}{G_2 I_{p_2}}\,, \qquad \tau_{\max_2} = \frac{M_{T_2}}{W_{T_2}}$$

with

$$I_{p_1} = \frac{\pi}{2} R_1^4\,, \quad I_{p_2} = \frac{\pi}{2}\left(R_2^4 - R_1^4\right)\,, \quad W_{T_1} = \frac{I_{p_1}}{R_1}\,, \quad W_{T_2} = \frac{I_{p_2}}{R_2}\,.$$

Together with equilibrium

$$M_T = M_{T_1} + M_{T_2}$$

and geometric compatibilty

$$\vartheta_1 = \vartheta_2 = \vartheta$$

we obtain

$$M_{T_1} = M_T \frac{G_1 I_{p_1}}{G_1 I_{p_1} + G_2 I_{p_2}}\,, \qquad M_{T_2} = M_T \frac{G_2 I_{p_2}}{G_1 I_{p_1} + G_2 I_{p_2}}$$

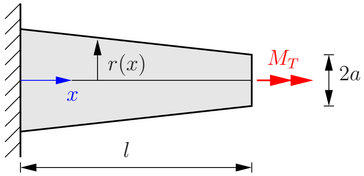
and

$$\underline{\underline{\tau_{\max_1} = \frac{M_T G_1 r_1}{G_1 I_{p_1} + G_2 I_{p_2}}}}\,, \qquad \underline{\underline{\tau_{\max_2} = \frac{M_T G_2 r_2}{G_1 I_{p_1} + G_2 I_{p_2}}}}\,,$$

$$\underline{\underline{\vartheta = \frac{M_T l}{G_1 I_{p_1} + G_2 I_{p_2}}}}\,.$$

P4.7 **Problem 4.7** A conical shaft with varying radius is loaded by a torque M_T.

Determine the twist and the peripheral stress as a function of x.



Solution The differential equation for the twist angle is given with

$$r(x) = a\left(2 - \frac{x}{l}\right) , \qquad I_p(x) = \frac{\pi}{2} r^4 = \frac{\pi}{2} a^4 \left(2 - \frac{x}{l}\right)^4$$

by

$$\vartheta' = \frac{M_T}{G I_p} = \frac{2M_T}{\pi G a^4} \frac{1}{\left(2 - \frac{x}{l}\right)^4} .$$

Integration with respect to x yields

$$\vartheta(x) = \frac{2M_T l}{3\pi G a^4} \frac{1}{\left(2 - \frac{x}{l}\right)^3} + C .$$

The integration constants are determined from the boundary conditions

$$\vartheta(0) = 0 \quad \leadsto \quad C = -\frac{2M_T l}{3\pi G a^4} \frac{1}{8} .$$

Thus the twist results in

$$\underline{\underline{\vartheta(x) = \frac{M_T l}{12\pi G a^4} \left\{ \frac{1}{\left(1 - \frac{x}{2l}\right)^3} - 1 \right\}}} .$$

The peripheral shear stress is computed with

$$W_T(x) = \frac{I_p}{r} = \frac{\pi}{2} a^3 \left(1 - \frac{x}{l}\right)^3$$

as

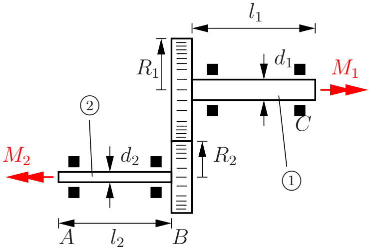
$$\underline{\underline{\tau_P(x) = \frac{M_T}{W_T} = \frac{2M_T}{\pi a^3 \left(2 - \frac{x}{l}\right)^3}}} .$$

Twist and stress have a maximum at $x = l$:

$$\vartheta(l) = \frac{7M_T l}{12\pi G a^4} , \qquad \tau_P(l) = \frac{2M_T}{\pi a^3} .$$

P4.8

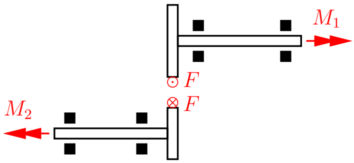
Problem 4.8 The depicted gear-system consists of two shafts (lengths l_1, l_2) of identical material, that are connected by two gear wheels (radii R_1, R_2). The shaft ① is loaded by an external torque M_1.



a) Determine M_2 such that equilibrium is fulfilled.
b) Choose the diameters d_1 and d_2 such that the admissible shear stress τ_{admis} is not exceeded?
c) Compute the angle of twist at position C, if shaft ② is fixed at position A.

Solution **to a)** Equilibrium of moments

$$M_1 = R_1 F \,, \qquad M_2 = -R_2 F$$



yields

$$\underline{\underline{M_2 = -\frac{R_2}{R_1} M_1}} \,.$$

to b) The critical value of the shear stress is reached in each shaft for:

$$\tau_{\max_1} = \frac{|M_1|}{W_1} = \frac{16 M_1}{\pi d_1^3} = \tau_{\text{admis}} \quad \leadsto \quad \underline{\underline{d_1 = \sqrt[3]{\frac{16 M_1}{\pi \tau_{\text{admis}}}}}} \,,$$

$$\tau_{\max_2} = \frac{|M_2|}{W_2} = \frac{R_2}{R_1} \frac{16 M_1}{\pi d_2^3} = \tau_{\text{admis}} \quad \leadsto \quad \underline{\underline{d_2 = \sqrt[3]{\frac{R_2}{R_1}}\, d_1}} \,.$$

to c) For the twist angle in ① and ② we obtain

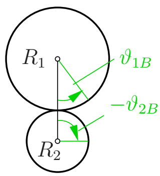
$$\Delta\vartheta_1 = \frac{l_1 M_1}{G I_{T_1}} = \frac{32 M_1 l_1}{\pi G d_1^4} \,, \qquad \Delta\vartheta_2 = \vartheta_{2B} = \frac{32 M_2 l_2}{\pi G d_2^4} \,.$$

With the continuity of the rotations

$$\vartheta_{1B} R_1 = -\vartheta_{2B} R_2$$

and

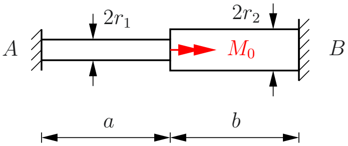
$$\vartheta_C = \vartheta_{1B} + \Delta\vartheta_1$$



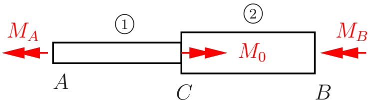
we compute

$$\underline{\underline{\vartheta_C = \frac{32 M_1}{G \pi d_1^4} \left\{ l_1 + \left(\frac{R_2}{R_1} \right)^{\frac{2}{3}} l_2 \right\}}} \,.$$

P4.9 **Problem 4.9** A homogeneous, graded shaft with circular cross section is clamped at both ends and loaded by the torque M_0.



Compute the torques at the support positions A and B as well as the twist at the point where M_0 is applied.



Solution The system is statically indeterminate because the support torques M_A and M_B cannot be computed solely from the equilibrium conditions.

$$M_A + M_B = M_0$$

By cutting the shaft at C constant torques are obtained in the regions ① and ② . This results in the following twists

$$\vartheta_1 = \frac{M_A a}{GI_{p_1}} , \qquad \vartheta_2 = \frac{M_B b}{GI_{p_2}} .$$

Geometric compatibility requires that the two angles of twist are identical:

$$\vartheta_C = \vartheta_1 = \vartheta_2 .$$

Together with

$$I_{p_1} = \frac{\pi}{2} r_1^4 , \qquad I_{p_2} = \frac{\pi}{2} r_2^4$$

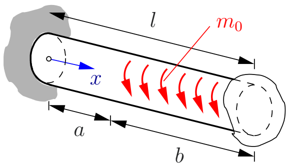
we obtain

$$\underline{\underline{M_A = M_0 \frac{1}{1 + \frac{r_2^4 a}{r_1^4 b}}}} , \qquad \underline{\underline{M_B = M_0 \frac{1}{1 + \frac{r_1^4 b}{r_2^4 a}}}} ,$$

$$\underline{\underline{\vartheta_C = \frac{2M_0 ab}{\pi G \left(br_1^4 + ar_2^4\right)}}} .$$

Problem 4.10 A shaft is clamped at both ends and loaded along part b of its length l by a constant distributed torque m_0. **P4.10**

Determine the function of twist angle and torque.



Solution The external torque $m(x)$ has a jump at position $x = a$. We use the Macauley bracket to incorparate the discontinuous function. With

$$m(x) = m_0 < x - a >^0$$

the differential equation for the twist angle follows

$$GI_T\vartheta'' = -m(x) = -m_0 < x - a >^0 \quad .$$

Integrating twice yields

$$GI_T\vartheta' = M_T = -m_0 < x - a >^1 + C_1$$

$$GI_T\vartheta = -\tfrac{1}{2}m_0 < x - a >^2 + C_1 x + C_2 \; .$$

The constants folllow from the boundary conditions

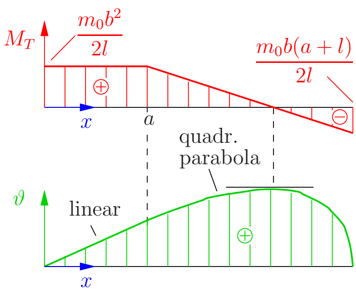
$$\vartheta(0) = 0 \quad \rightsquigarrow \quad C_2 = 0 \; ,$$

$$\vartheta(l) = 0 \quad \rightsquigarrow \quad C_1 = \frac{1}{2}\frac{m_0 b^2}{l} \; .$$

Finally we obtain

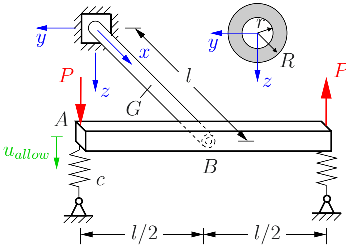
$$\underline{\underline{M_T(x) = m_0 b\left\{\frac{b}{2l} - \frac{< x - a >^1}{b}\right\}}} \; ,$$

$$\underline{\underline{\vartheta(x) = \frac{1}{2}\frac{m_0 b^2}{GI_T}\left\{\frac{x}{l} - \frac{< x - a >^2}{b^2}\right\}}} \; .$$



P4.11 **Problem 4.11** The depicted shaft with ring-shaped cross section is clamped at one end. At the other end a rigid beam is attached. The beam is supported by two springs and loaded by the forces P. Determine

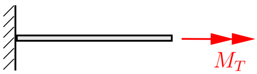
a) the maximum force $P_{\max}$ for a prescribed admissible displacement u_{admis} (in z-direction) at point A,
b) position and value of the maximum shear stress in the cross section of the truss for $P = P_{\max}$.



Given : $u_{\mathrm{admis}} = 2$ cm, $l = 2$ m
$r = 5$ cm, $R = 10$ cm
$c = 10^6$ N/m
$G = 8 \cdot 10^{10}$ N/m^2

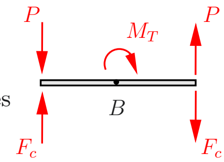
Solution **to a)** The system is statically indeterminate. We free the system at point B leading to the twist of the shaft

$$\Delta\varphi = \frac{M_T\, l}{GI_p} \quad \rightsquigarrow \quad M_T = \frac{GI_p}{l}\,\Delta\varphi$$



with (small twist angles)

$$\Delta\varphi = \frac{u_{\mathrm{admis}}}{l/2} = 0.2\ .$$



Equilibrium of moments for the beam provides

$$\overset{\curvearrowleft}{B}:\ M_T = lP_{\max} - lF_c\,, \text{ where } F_c = c\,u_{\mathrm{admis}}\ .$$

Eliminating $\Delta\varphi$, M_T and F_c yields

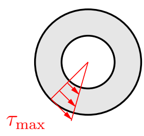
$$\underline{\underline{P_{\max} = \Big(2\,\frac{GI_p}{l^3} + c\Big)u_{\mathrm{admis}}}}\ .$$

With $I_p = \pi(R^4 - r^4)/2 = 1.47 \cdot 10^{-4}$ m^4 and the given numerical values we obtain

$$\underline{\underline{P_{\max}}} = \left(\frac{2 \cdot 8 \cdot 10^{10} \cdot 1.47}{10^4 \cdot 8} + 10^6\right) 2 \cdot 10^{-2} = \underline{\underline{78.7\,\mathrm{kN}}}$$

to b) The shear stress assumes its maximum value at the outer perimeter of the cross section. The absolute value is computed by

$$\begin{aligned} M_T &= P_{\max}\, l - c\,u_{\mathrm{admis}}\, l \\ &= (78.7 - 10^3 \cdot 0.02)\,2 = 117.4\,\mathrm{kNm} \end{aligned}$$

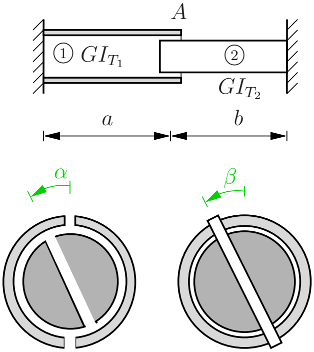


and

$$\underline{\underline{\tau_{\max}}} = \frac{M_T R}{I_p} = \frac{117.4 \cdot 0.1}{1.47 \cdot 10^{-4}} = \underline{\underline{79.8\ \mathrm{MN/m^2}.}}$$

Problem 4.12 The hollow shaft ① and the solid shaft ② are joint by a bolt at A.

P4.12

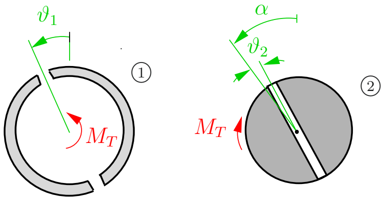


Determine the torque M_T and the twist angle β of the bolt after assembly for the case that the ends of the shafts have an angular difference of α in the stress-free state.

Solution In the assembled state both shafts are loaded by the torque M_T. We cut the system at position A and determine the angle of twist of ① and ② separately:

$$\vartheta_1 = \frac{M_T a}{GI_{T_1}}, \qquad \vartheta_2 = \frac{M_T b}{GI_{T_2}}.$$

From the geometric compatibility in the assembled state



$$\alpha - \vartheta_2 = \vartheta_1$$

and

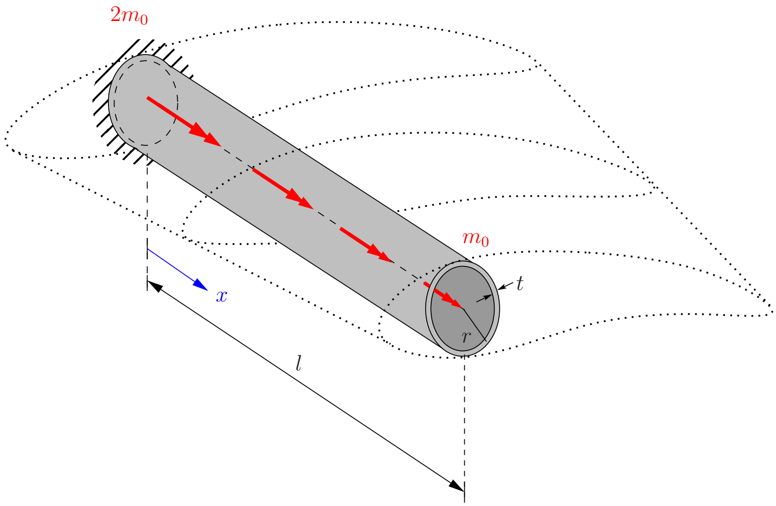
$$\beta = \vartheta_1$$

we obtain for M_T and β

$$\underline{\underline{M_T = GI_{T_1}\frac{\alpha}{a}\frac{1}{1+\frac{b}{a}\frac{I_{T_1}}{I_{T_2}}}}},$$

$$\underline{\underline{\beta = \vartheta_1 = \frac{\alpha}{1+\frac{b}{a}\frac{I_{T_1}}{I_{T_2}}}}}.$$

P4.13 **Problem 4.13** The thin-walled spar with ring-shaped cross section (length l, shear modulus G, radius r, thickness $t \ll r$) is located in the interior of an airplane wing. It is loaded by a distributed torque $m_T(x)$ with $m_T(0) = 2m_0$ and $m_T(l) = m_0$. The spar is clamped at the fuselage.



Determine
a) the torque $M_T(x)$ in the spar,
b) the distribution of the shear stress $\tau(x)$ and the maximum shear stress $\tau_{\max}$ due to torsion,
c) the angle ϑ_l, by which the end of the wing at $x = l$ rotates with regard to the fuselage.

Solution **to a)** The distributed torque is given by

$$m_T(x) = \left(2 - \frac{x}{l}\right) m_0 \,.$$

The torque follows by integration

$$M_T(x) = -\int m_T(x)\,\mathrm{d}x + C_1 = \left(\frac{x^2}{2l} - 2x\right) m_0 + C_1$$

which leads with the boundary condition

$$M_T(l) = 0$$

$$\rightsquigarrow \qquad \left(\frac{l}{2} - 2l\right) m_0 + C_1 = 0 \qquad \rightsquigarrow \qquad C_1 = \frac{3}{2} m_0 l$$

to

$$\underline{\underline{M_T(x) = \left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right) m_0 l}}\ .$$

to b) For the thin-walled spar cross section the shear stresses are computed using the second moment of area for torsion $I_T = 2\pi r^3 t$:

$$\underline{\underline{\tau(x)}} = \frac{M_T}{I_T} r = \underline{\underline{\frac{m_0 l}{2\pi r^2 t}\left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right)}}\ .$$

The maximum shear stress occurs at position $x = 0$ and its value is given by

$$\underline{\underline{\tau_{\max} = \frac{3}{4}\frac{m_0 l}{\pi r^2 t}}}\ .$$

to c) With the second moment of area for torsion I_T and the shear modulus G we obtain for the twist

$$\vartheta'(x) = \frac{M_T(x)}{G I_T} = \frac{m_0 l}{2G\pi r^3 t}\left(\frac{x^2}{2l^2} - 2\frac{x}{l} + \frac{3}{2}\right)$$

as well as for the edge rotation

$$\vartheta(x) = \frac{m_o l}{2G\pi r^3 t}\left(\frac{x^3}{6l^2} - \frac{x^2}{l} + \frac{3}{2}x\right) + C_2\ .$$

The integration constant is determined from the boundary condition $\vartheta(0) = 0$ to be $C_2 = 0$. Thus the edge rotation ϑ_l at the end of the wing yields ($x = l$):

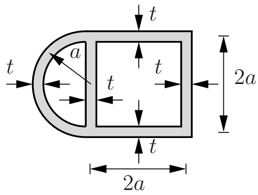
$$\vartheta_l = \vartheta(l) = \frac{m_o l^2}{2G\pi r^3 t}\left(\frac{1}{6} - 1 + \frac{3}{2}\right) \qquad \rightsquigarrow \qquad \underline{\underline{\vartheta_l = \frac{m_0 l^2}{3G\pi r^3 t}}}\ .$$

P4.14

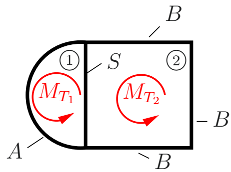
Problem 4.14 A shaft with the depicted thin-walled profile is loaded by a torque M_T.

a) Determine the shear stress in different sections of the profile.

b) Compute the maximum admissible torque, such that the admissible shear stress τ_{admis} is not exceeded.



Solution The profile consists of two parts. For each part the following holds:



$$T = \tau(s) \cdot t(s) = \frac{M_{T_i}}{2A_{T_i}} ,$$

$$\vartheta_i' = \frac{M_{T_i}}{GI_{T_i}} = \frac{1}{2GA_{T_i}} \oint_i \frac{T}{t} \mathrm{d}s .$$

With the given values

$$A_{T_1} = \frac{\pi}{2} a^2 , \quad A_{T_2} = 4a^2$$

we obtain by considerating that the shear flux in section S is composed of the contributions from the torques M_{T_1} and M_{T_2}:

$$\vartheta_1' = \frac{1}{\pi a^2 G} \left\{ \frac{M_{T_1}}{\pi a^2} \frac{\pi a}{t} + \left[\frac{M_{T_1}}{\pi a^2} - \frac{M_{T_2}}{8a^2} \right] \frac{2a}{t} \right\} ,$$

$$\vartheta_2' = \frac{1}{8a^2 G} \left\{ \frac{M_{T_2}}{8a^2} \frac{6a}{t} + \left[\frac{M_{T_2}}{8a^2} - \frac{M_{T_1}}{\pi a^2} \right] \frac{2a}{t} \right\} .$$

Inserting this result into the geometric compatibility

$$\vartheta' = \vartheta_1' = \vartheta_2'$$

yields

$$\frac{M_{T_1}}{M_{T_2}} = \frac{2+\pi}{10+\frac{16}{\pi}}$$

with

$$M_T = M_{T_1} + M_{T_2}$$

the torques

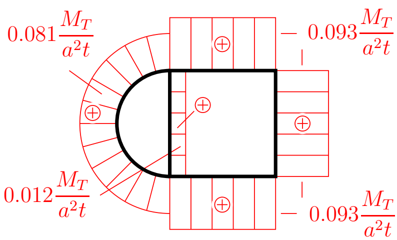
$$M_{T_1} = \frac{2+\pi}{12+\pi+\frac{16}{\pi}} M_T = 0.254\; M_T \ , \quad M_{T_2} = 0.746\; M_T \ .$$

Now the stresses in the sections A, B and S follow

$$\underline{\underline{\tau_A}} = \frac{M_{T_1}}{2A_{T_1}t} = \underline{\underline{0.081\frac{M_T}{a^2t}}} \ ,$$

$$\underline{\underline{\tau_B}} = \frac{M_{T_2}}{2A_{T_2}t} = \underline{\underline{0.093\frac{M_T}{a^2t}}} \ ,$$

$$\underline{\underline{\tau_S}} = \tau_B - \tau_A = \underline{\underline{0.012\frac{M_T}{a^2t}}} \ .$$



Equalizing the maximum shear stress with the admissible shear stress

$$\tau_{\max} = \tau_B = 0.093\ \frac{M_T}{a^2t} = \tau_{\text{admis}} \ ,$$

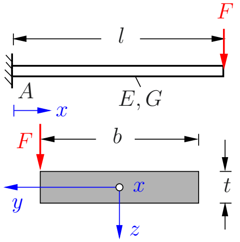
provides the maximum admissible torque

$$\underline{\underline{M_{T_{\text{admis}}} = 10-75\ \frac{\tau_{\text{admis}}a^2t}{M_T}}} \ .$$

Note: Inserting M_{T_1} and M_{T_2} in ϑ' determines the second moment of area for torsion $I_T = 13.7a^3t$. Neglecting the section S, we obtain $I_T = 13.6\ a^3t$. Thus section S only contributes a small amount to the torsional rigidity.

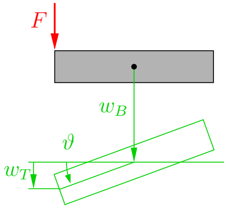
P4.15 **Problem 4.15** The fixed leaf spring ($t \ll b$) is eccentrically loaded by a force F.

Compute the deflection at the point loading. Determine the maximum normal and shear stress.



Solution The leaf spring is subjected to a bending and a torsion load. Due to bending the deflection is given by the table on page 62.

$$w_B = \frac{Fl^3}{3EI} \quad \text{with} \quad I = \frac{bt^3}{12}\,.$$



The constant torque

$$M_T = Fb/2$$

causes a rotation at the end of the spring

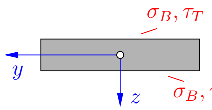
$$\vartheta = \frac{M_T l}{G I_T} \quad \text{with} \quad I_T = \frac{1}{3} b t^3$$

and the corresponding displacement $w_T = \frac{b}{2}\vartheta$. The total deflection is thus obtained by

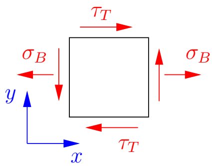
$$\underline{\underline{w}} = w_B + w_T = \underline{\underline{\frac{4Fl^3}{Ebt^3}\left(1 + \frac{3Eb^2}{16Gl^2}\right)}}\,.$$

Bending and torsion cause stress in the extreme fibre of the fixed cross section

$$\sigma_B = \frac{M}{W} = \frac{6lF}{bt^2}\,, \qquad \tau_T = \frac{M_T}{W_T} = \frac{3bF}{2bt^2}\,.$$



An area element at the top surface ($z = -t/2$) is loaded as sketched. Thus the maximum normal and shear stress follow

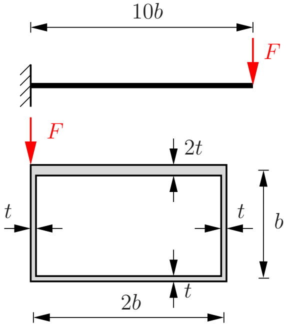


$$\underline{\underline{\sigma_1}} = \frac{\sigma_B}{2} + \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau_T^2} = \underline{\underline{\frac{3Fl}{bt^2}\left(1 + \sqrt{1 + \frac{b^2}{4l^2}}\right)}}\,,$$

$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau_T^2} = \underline{\underline{\frac{3Fl}{bt^2}\sqrt{1 + \frac{b^2}{4l^2}}}}\,.$$

Problem 4.16 An element of a bridge is constructed as a thin-walled ($t \ll b$) box girder. During construction the box girder is eccentrically loaded.

P4.16



Determine the location and value of the maximum normal and shear stress.

Solution Section properties of the profile are

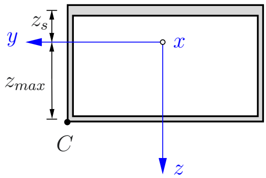
$$z_s = \frac{2b^2t + 2\cdot\frac{b}{2}(b\cdot t)}{8bt} = \frac{3}{8}b\,, \qquad S_y(z_{max}) = b\,t\,\frac{5}{8}\,b = \frac{5}{8}\,b^2\,t$$

$$I_y = 2\left(\frac{tb^3}{12} + \frac{tb^3}{64}\right) + 4bt\left(\frac{3}{8}b\right)^2 + 2bt\left(\frac{5}{8}b\right)^2$$

$$= \frac{37}{24}tb^3\,,$$

$$W = \frac{I_y}{z_{\max}} = \frac{37}{15}tb^2\,,$$

$$W_T = 2A_T t_{\min} = 4b^2 t\,.$$



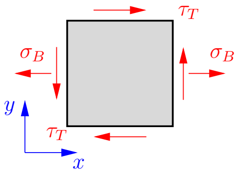
Using bending moment, torque, shear force in the clamped support

$$M_B = -10\,b\,F\,, \qquad M_T = b\,F\,, \qquad V_z = F$$

yields for the lower section

$$\sigma_B = \frac{M_B}{W} = -\frac{150}{37}\,\frac{F}{bt}\,,$$

$$\tau_T = \frac{M_T}{W_T} = \frac{1}{4}\,\frac{F}{bt}\,, \qquad \tau_Q = \frac{V_z\,S_y}{I_y\,t} = \frac{15}{37}\,\frac{F}{bt}\,.$$



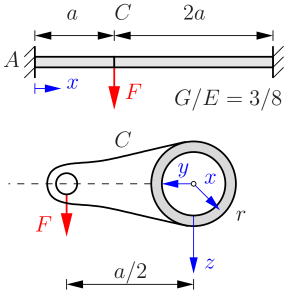
The largest absolute value for the normal stress and the shear stress are obtained by $\tau = \tau_T + \tau_Q$ at location C

$$\underline{\underline{\sigma_2}} = \frac{\sigma_B}{2} - \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau^2} = \underline{\underline{-4.16\,\frac{F}{bt}}}\,,$$

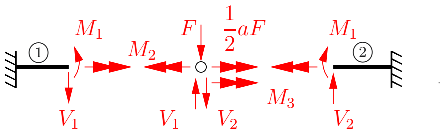
$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_B}{2}\right)^2 + \tau^2} = \underline{\underline{2.13\,\frac{F}{bt}}}\,.$$

P4.17 **Problem 4.17** The depicted cantilever with thin-walled circular cross section is clamped at both ends and loaded eccentrically at point C.

Determine the deflection at the point where the load is applied and compute the normal stress and the shear stresses due to torsion.



Solution The cantilever is cut at point C. Equilibrium yields



$$M_2 = M_3 + \frac{1}{2}aF\,, \quad V_1 = V_2 + F\,.$$

The deflection, the angle of bending, and the angle of twist are given at point C by (see table on page 62):

$$w_{C_1} = \frac{V_1 a^3}{3EI} - \frac{M_1 a^2}{2EI}\,, \quad w_{C_2} = -\frac{8V_2 a^3}{3EI} - \frac{4M_1 a^2}{2EI}\,,$$

$$w'_{C_1} = \frac{V_1 a^2}{2EI} - \frac{M_1 a}{EI}\,, \quad w'_{C_2} = +\frac{4V_2 a^2}{2EI} + \frac{2M_1 a}{EI}\,,$$

$$\vartheta_{C_1} = \frac{M_2 a}{GI_T}\,, \quad \vartheta_{C_2} = -\frac{2M_3 a}{GI_T}\,.$$

Compatibility demands

$$w_{C_1} = w_{C_2}\,, \quad w'_{C_1} = w'_{C_2}\,, \quad \vartheta_{C_1} = \vartheta_{C_2}$$

which renders

$$V_1 = \frac{20}{27}F\,, \quad {}_2 = -\frac{7}{27}F\,, \quad M_1 = \frac{8}{27}aF\,,$$

$$M_2 = \frac{1}{3}aF\,, \quad M_3 = -\frac{1}{6}aF\,.$$

The second moments of area and the elasticity constants

$$I_T = 2I = 2\pi r^3 t \qquad \text{und} \qquad \frac{G}{E} = \frac{3}{8}$$

yield the deflection at the point of loading

$$\underline{\underline{w_F}} = w_{C_1} + \frac{a}{2}\vartheta_{C_1} = \underline{\underline{\frac{26Fa^3}{81EI}}} .$$

To compute the stresses, we need the bending moments at A and B:

$$M_A = M_1 - V_1 a = -\frac{4}{9}aF ,$$

$$M_B = M_1 + V_2 2a = -\frac{2}{9}aF .$$

The maximum normal stresses due to bending in A, B and C are given with the section modulus $W = I\,/\,r$

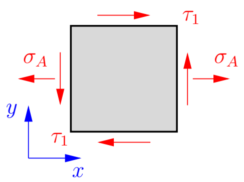
$$\sigma_A = \frac{|M_A|}{W} = \frac{4arF}{9\ I} , \quad \sigma_B = \frac{2arF}{9\ I} ,$$

$$\sigma_C = \frac{|M_1|}{W} = \frac{8arF}{27\ I} .$$

The shear stresse in secion ① or ② are calculated with $W_T = 2W = \frac{2I}{r}$:

$$\tau_1 = \frac{M_2}{W_T} = \frac{arF}{6\ I} , \quad \tau_2 = \frac{M_3}{W_T} = \frac{arF}{12\ I} .$$

The largest stresses occur at the point A. An area element at the top surface (analogously on the bottom surface) is loaded as sketched. For the principal stress and the maximum shear stress we obtain



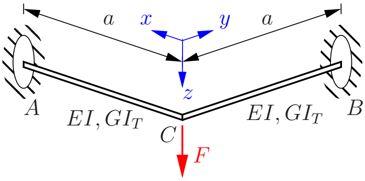
$$\underline{\underline{\sigma_1}} = \frac{\sigma_A}{2} + \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2} = \underline{\underline{\frac{arF}{2\ I}}} ,$$

$$\underline{\underline{\tau_{\max}}} = \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2} = \underline{\underline{\frac{5arF}{18\ I}}} .$$

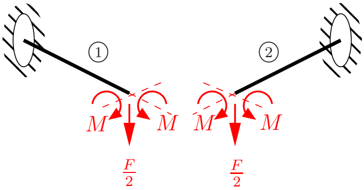
P4.18

Problem 4.18 The depicted cantilever is fixed at both ends and bent by 90°. The cantilever is loaded at point C by the force F.

Compute the deflection at the point C.



Solution To solve the problem we use superposition. We cut the system at point C and apply symmetry arguments for the depicted loading with respect to bending and torsion. At this stage the moment M is unknown. From the table on page 62 we deduce



$$w'_C = \frac{Fa^2}{4EI} - \frac{Ma}{EI}\,, \qquad w_C = \frac{Fa^3}{6EI} - \frac{Ma^2}{2EI}\,.$$

The angle of twist due to torsion at C is given by

$$\vartheta_C = \frac{Ma}{GI_T}\,.$$

The geometric compatibility

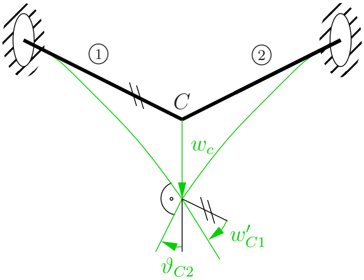
$$w'_{C1} = \vartheta_{C2}$$

yields

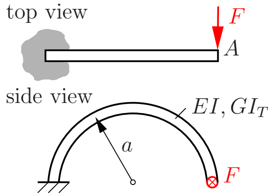
$$M = \frac{Fa}{4}\,\frac{GI_T}{EI + GI_T}$$

and the final result

$$\underline{\underline{w_C = \frac{Fa^3}{24EI}\,\frac{4EI + GI_T}{EI + GI_T}}}\,.$$

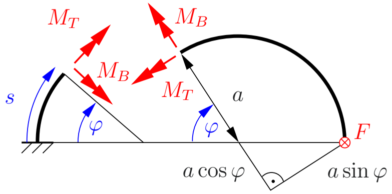


Problem 4.19 The depicted semi-circular support is loaded at point A by a force F. **P4.19**



Determine the deflection at the point A.

Solution Equilibrium of moments provides the bending moment M_B and the torque M_T



$$M_B(\varphi) = -aF\sin\varphi \ ,$$

$$M_T(\varphi) = a(1+\cos\varphi)F \ .$$

The angle of twist is given by

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}s} = \frac{M_T}{GI_T} \qquad \text{mit} \qquad \mathrm{d}s = a\mathrm{d}\varphi \ .$$

The twist $\mathrm{d}\vartheta$ at position φ causes the deflection at A

$$\mathrm{d}w_{TA} = a\sin\varphi\,\mathrm{d}\vartheta \ .$$

Combining the previous results and integration yields the deflection due to torsion

$$w_{TA} = \int \mathrm{d}w_{TA} = \frac{Fa^3}{GI_T}\int_0^\pi \sin\varphi(1+\cos\varphi)\mathrm{d}\varphi = \frac{2Fa^3}{GI_T} \ .$$

The deflection due to bending is follows from

$$EI\frac{\mathrm{d}^2w_B}{\mathrm{d}s^2} = -M_B \quad \rightsquigarrow \quad \frac{\mathrm{d}^2w_B}{\mathrm{d}\varphi^2} = \frac{Fa^3}{EI}\sin\varphi \ ,$$

$$\frac{\mathrm{d}w_B}{\mathrm{d}\varphi} = \frac{Fa^3}{EI}(-\cos\varphi + C_1) \ , \qquad w_B(\varphi) = \frac{Fa^3}{EI}(-\sin\varphi + C_1\varphi + C_2)$$

and the boundary conditions

$$w_B'(0) = 0 \quad \rightsquigarrow \quad C_1 = 1 \ , \qquad w_B(0) = 0 \quad \rightsquigarrow \quad C_2 = 0 \, .$$

Using these constants yields

$$w_B(\varphi) = \frac{Fa^3}{EI}(\varphi - \sin\varphi) \ .$$

Finally the total deflection at A is given at position $\varphi = \pi$

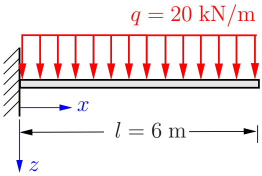
$$\underline{\underline{w_A}} = w_{TA} + w_B(\pi) = \underline{\underline{\frac{Fa^3}{EI}\left(\pi + 2\frac{EI}{GI_T}\right)}} \ .$$

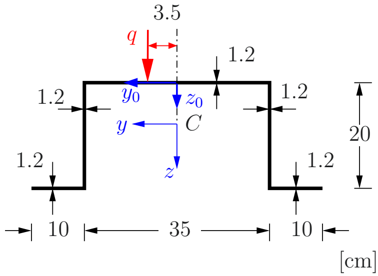
P4.20 **Problem 4.20** A cantilever beam with the depicted profile is subjected to an eccentric line load q. Determine at the clamped support

a) the largest shear stress due to the shear force and its position,

b) the shear stress due to torsion.

c) the distribution of the shear stresses due to shear force and torsion across the profile. Determine position and value of the largest shear stress.





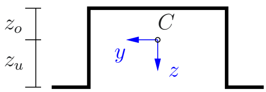
Solution We start by computing the stress resultants at the clamped support:

$$
\begin{aligned}
V_z &= q\,l &&= 20\cdot 6 &&= 120 \text{ kN}\,,\\
M_y &= -\frac{q\,l^2}{2} &&= -20\cdot\frac{6^2}{2} &&= -360 \text{ kNm}\,,\\
M_T &= q\,l\cdot 3.5\,\text{cm} &&= 20\cdot 6\cdot 0.035 &&= 4.2 \text{ kNm}\,.
\end{aligned}
$$

With the geometric data of the profile we calculate the position of the centroid C and the second moment of area I_y:

$$z_o = \frac{\sum z_i A_i}{\sum A_i} = \frac{2\cdot(20\cdot 1.2)\cdot 10 + 2\cdot(10\cdot 1.2)\cdot 20}{35\cdot 1.2 + 2\cdot 20\cdot 1.2 + 2\cdot 10\cdot 1.2} = 8.42 \text{ cm}\,,$$

$$z_u = 20 - z_o = 11.58 \text{ cm}\,,$$



$$
\begin{aligned}
I_y &= \sum \frac{b_i h_i^3}{12} + \sum A_i \bar{z}_i^2\\
&= (35\cdot 1.2)\cdot 8.42^2 + 2\cdot\frac{20^3\cdot 1.2}{12}\\
&\quad +2\cdot(20\cdot 1,2)\cdot 1.58^2 + 2\cdot(10\cdot 1.2)\cdot 11.58^2\\
&= 7915.8 \text{ cm}^4\,.
\end{aligned}
$$

to a) The shear stress due to the shear force is obtained by

$$\tau = \frac{V_z\, S_y}{I_y\, h} = \frac{120}{7915.8 \cdot 1.2}\, S_y = 0.01263\, S_y\,.$$

The static moment S_y reaches its maximum at $z = 0$:

$$S_{y\,\max} = S(z=0) = 8.4 \cdot 1.2 \cdot \frac{35}{2} + \frac{1}{2}\, 8.4^2 \cdot 1.2 = 218.7\ \text{cm}^3\,.$$

From this result the maximum shear stress due to shear force follows

$$\tau_{V\,\max} = 0.01263 \cdot 218.7$$

$$\leadsto \quad \underline{\underline{\tau_{V\,\max} = 2.76\ \text{kN/cm}^2 = 27.6\ \text{N/mm}^2}}\,.$$

to b) The shear stress due to torsion is calculated using the second moment of area for torsion respectively the torsion modulus of the profile:

$$I_T = \frac{1}{3} \sum h_i t_i^3 = \frac{1}{3}(35 + 2 \cdot 20 + 2 \cdot 10) \cdot 1.2^3 = 54.7\ ,\text{cm}^4$$

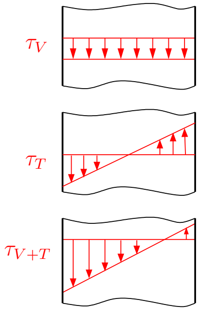
$$W_T = \frac{1}{3} \frac{\sum h_i t_i^3}{t_{\max}} = \frac{54.7}{1.2} = 45.6\ \text{cm}^3\,.$$

With the already calculated torque M_T we obtain

$$\tau_T = \frac{M_T}{W_T} = \frac{4.2 \cdot 10^2}{45.6}$$

$$\leadsto \quad \underline{\underline{\tau_T = 9.21\ \text{kN/cm}^2 = 92.1\ \text{N/mm}^2}}\,.$$

to c) The largest shear stress occurs at the position $z = 0$. It is distributed linearly across the wall thickness with the following extreme values:



$$\tau_{\text{inside}} = 27.6 - 92.1 = -64.5\ \text{N/mm}^2,$$

$$\tau_{\text{outside}} = 27.6 + 92.1 = 119.7\ \text{N/mm}^2$$

$$\leadsto \quad \underline{\underline{\tau_{\max} = 119.9\ \text{N/mm}^2}}\,.$$

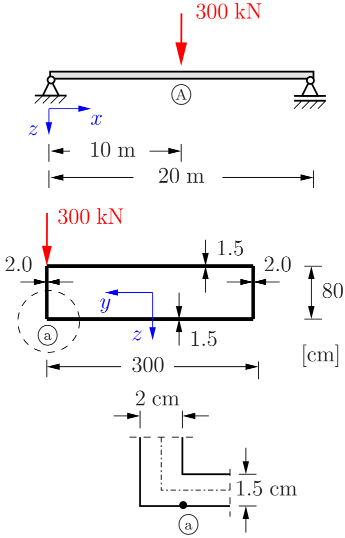
P4.21 **Problem 4.21** A thin-walled box girder is loaded by a force of 300 kN. Determine for the cross section at position Ⓐ

a) the stress distribution (normal and shear stresses) due to shear force and torsion,

b) the position of the maximum principal stress and

c) the value and direction of the principal stress at the vertex ⓐ of the profile.

Remark: Assume for the torsional load case a fork bearing at the left end.



Solution The second moment of area is given by

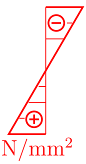
$$I_y = \sum_i \frac{b_i h_i^3}{12} + \sum_i A_i \bar{z}_i^2 = 2 \cdot \frac{2 \cdot 80^3}{12} + 2 \cdot (1.5 \cdot 300) \cdot 40^2 = 1.611 \cdot 10^6 \text{ cm}^4 .$$

The stress resultants at position Ⓐ (or directly left of it) are

$$V_z = \frac{300}{2} = 150\,\text{kN}, \qquad M_y = \frac{300 \cdot 20}{4} = 1500 \text{ kNm},$$

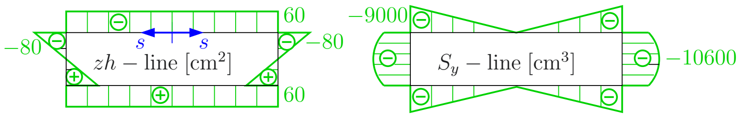
$$M_T = 300 \cdot 1.5 = 450\,\text{kNm}.$$

to a) The normal stress is linear across the height of the cross section and reaches in point ⓐ the value



$$\sigma_x = \frac{M_y}{I_y} z_a = \frac{1500 \cdot 1000 \cdot 1000}{1.611 \cdot 10^6 \cdot 10^4} \cdot 40 \cdot 10 = 37.25 \text{ N/mm}^2 .$$

The shear stresses due to V_z are determined by the zh-line and S_y-line.



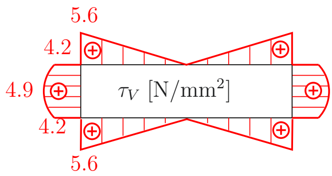
By using the S_y-line we obtain

$$\tau_V = \frac{V_z S_y}{I_y h} = \frac{150}{1.611 \cdot 10^6} \frac{S_y}{h} = 9.3 \cdot 10^{-5} \frac{S_y}{h} \ \mathrm{kN/cm^2}\,.$$

At position ⓐ they assume the value

$$\tau_{Va} = \frac{150 \cdot 9000}{1.611 \cdot 10^6 \cdot 1.5}$$

$$= 0.56 \ \mathrm{kN/cm^2} = 5.6 \ \mathrm{N/mm^2}\,.$$

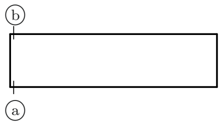


The shear stresses due to torsion are given by

$$\tau_T = \frac{M_T}{2 A_T h}\,, \qquad A_T = 300 \cdot 80 = 24000 \ \mathrm{cm^2}$$

$$\rightsquigarrow \underline{\underline{\tau_{Ta}}} = \frac{450 \cdot 10^3 \cdot 10^3}{2 \cdot 24000 \cdot 1.5 \cdot 10^3} = \underline{\underline{6.25 \ \mathrm{N/mm^2}}}\,.$$

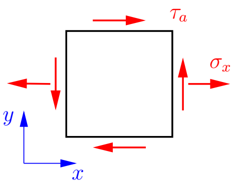
to b) The maximum shear stresses occur at points ⓐ and ⓑ, the maximum normal stresses at point ⓐ. Thus the principal stresses assume the largest value at ⓐ.



to c) In point ⓐ the shear and normal stresses are:

$$\tau_a = \tau_{Va} + \tau_{Ta} = 5.6 + 6.25 = 11.85 \ \mathrm{N/mm^2}\,,$$

$$\sigma_x = 37.25 \ \mathrm{N/mm^2}\,.$$



The principal stresses are given by

$$\underline{\underline{\sigma_1}} = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_a^2} = \underline{\underline{40.7 \ \mathrm{N/mm^2}}},$$

$$\underline{\underline{\sigma_2}} = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_a^2} = \underline{\underline{-3.45 \ \mathrm{N/mm^2}}}.$$

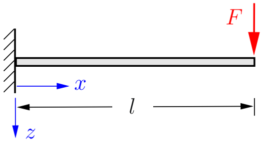
For the direction of the principal stress σ_1 we compute

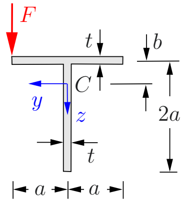
$$\tan 2\alpha_0 = \frac{2\tau}{\sigma_x} = 0.636 \qquad \rightsquigarrow \qquad \underline{\underline{\alpha_0 = 16.23^\circ}}\,.$$

P4.22 **Problem 4.22** A cantilever beam with thin-walled T-profile ($t \ll a$) is eccentrically loaded by a force F. The clamped support is designed such that warping is allowed.

Determine the maximum stresses due to bending, shear force and torsion. At which position do they occur?

Given: $t = a/10$, $l = 20\,a$





Solution We start by determining the following geometric properties of the profile:

$$b = \frac{a}{2}\,,$$

$$I = b^2 2at + \left[\frac{t(2a)^3}{12} + b^2 2at\right] = \frac{1}{6}\,a^4\,, \quad W = \frac{I}{3a/2} = \frac{1}{9}\,a^3\,,$$

$$S_C = b\,2at + \frac{b}{2}\frac{at}{2} = \frac{9}{80}\,a^3\,,$$

$$I_T = \frac{1}{3}\,2(2a)t^3 = \frac{4}{3000}\,a^4\,, \qquad W_T = \frac{I_T}{t} = \frac{4}{300}\,a^3\,.$$

The bending moment reaches its maximum at the clamped support ($x = 0$), while shear force and torque are constant along the beam:

$$M_{\max} = -lF = -20aF\,, \qquad V = F\,, \qquad M_T = aF\,.$$

We compute the maximum bending stress (compression, at the lower surface, at $x = 0$), the maximum shear stress due to shear force (at the centroid C), and the shear stress due to torsion (at the outer boundary of the flanges):

$$\underline{\underline{\sigma_{\max}}} = \frac{|M_{\max}|}{W} = \frac{20aF}{\frac{1}{9}a^3} = \underline{\underline{180\,\frac{F}{a^2}}}\,,$$

$$\underline{\underline{\tau_V^C}} = \frac{C\,S_C}{I\,t} = \frac{F\,\frac{9}{80}a^3}{\frac{1}{6}a^4\,\frac{1}{10}a} = \underline{\underline{\frac{27}{4}\,\frac{F}{a^2}}}\,,$$

$$\underline{\underline{\tau_{M_T}}} = \frac{M_T}{W_T} = \frac{aF}{\frac{4}{300}a^3} = \underline{\underline{75\,\frac{F}{a^2}}}\,.$$

Note: The shear stress due to shear force is small compared to the shear stress due to torsion.

Chapter 5

Energy Methods

5

Energy theorem

The work W done by the external forces (moments) during loading of an elastic body is equal to the strain energy Π stored in the body:

$$\boxed{W = \Pi\,.}$$

The specific strain energy can be written in index notation for three-dimensional problems of elastostatics:

$$\Pi^* = \frac{E}{2(1+\nu)}\left[\varepsilon_{ik}\,\varepsilon_{ik} + \frac{\nu}{1-2\nu}\,\varepsilon_{ii}^2\right] = \frac{1}{2E}\left[(1+\nu)\,\sigma_{ik}\,\sigma_{ik} - \nu\,\sigma_{ii}^2\right]$$

with $\varepsilon_{ik}\,\varepsilon_{ik} := \sum_{i=1}^{3}\sum_{k=1}^{3}\varepsilon_{ik}\,\varepsilon_{ik}$ and $\varepsilon_{ii} := \sum_{i=1}^{3}\varepsilon_{ii}$.

The following expressions hold for bars and beams:

loading	strain energy per length unit	strain-energy
tension / compression	$\Pi^* = \frac{1}{2}\frac{N^2}{EA}$	$\Pi = \frac{1}{2}\int_l \frac{N^2}{EA}\,\mathrm{d}x$
bending	$\Pi^* = \frac{1}{2}\frac{M^2}{EI}$	$\Pi = \frac{1}{2}\int_l \frac{M^2}{EI}\,\mathrm{d}x$
shear	$\Pi^* = \frac{1}{2}\frac{V^2}{GA_S}$	$\Pi = \frac{1}{2}\int_l \frac{V^2}{GA_S}\,\mathrm{d}x$
torsion	$\Pi^* = \frac{1}{2}\frac{M_T^2}{GI_T}$	$\Pi = \frac{1}{2}\int_l \frac{M_T^2}{GI_T}\,\mathrm{d}x$

Total strain energy (tension + bending + shear + torsion):

$$\Pi = \int_l \frac{N^2}{2EA}\,\mathrm{d}x + \int_l \frac{M^2}{2EI}\,\mathrm{d}x + \int_l \frac{V^2}{2GA_S}\,\mathrm{d}x + \int_l \frac{M_T^2}{2GI_T}\,\mathrm{d}x\,.$$

special case: bar ($N = \text{const},\ EA = \text{const}$): $\quad \Pi = \dfrac{N^2 l}{2EA}\,.$

special case: truss system $\quad \Pi = \sum_i \dfrac{S_i^2\,l_i}{2EA_i}\,.$

Remark: For slender beams, the shear contribution can be neglected compared to the bending contribution.

Principle of virtual forces

The displacement of a point due to tension, bending, shear, and torsion can be computed from

$$f_i = \int_l \frac{N\overline{N}}{EA}\,\mathrm{d}x + \int_l \frac{M\overline{M}}{EI}\,\mathrm{d}x + \int_l \frac{V\overline{V}}{GA_S}\,\mathrm{d}x + \int_l \frac{M_T\overline{M}_T}{GI_T}\,\mathrm{d}x \,.$$

where

f_i	$=$	displacement (rotation) at position i,
$N,\ M,\ V,\ M_T$	$=$	stress resultants due to the external loads,
$\overline{N},\ \overline{M},\ \overline{V},\ \overline{M}_T$	$=$	stress resultants due to a *virtual* force (moment) "1"at position i in direction of f_i.

Since the shear contributions are usually small compared to the other contributions, they will be neglected in the following problems.

Special case truss:

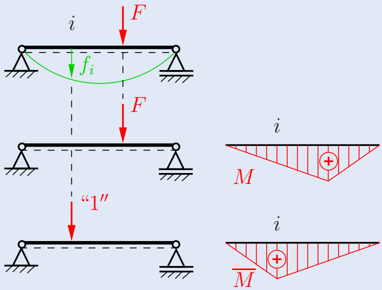
$$f_i = \sum_k \frac{S_k\overline{S}_k}{EA_k}\, l_k \;,$$

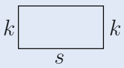
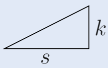
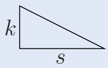
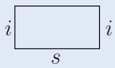
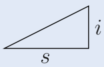
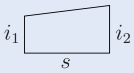
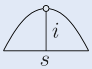
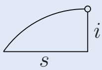
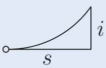
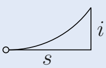
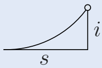
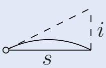
Special case bending of beams:

$$f_i = \int_l \frac{M\overline{M}}{EI}\,\mathrm{d}x \,.$$

Application to statically determinate problems

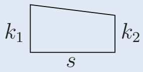
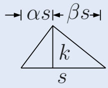
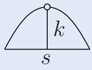
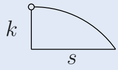
To compute the displacement f_i at an arbitrary position i, the bending moment due to the external loads (M) and due to the *virtual* load ($\overline{M}$) have to be determined.
The integral $\int M\overline{M}\mathrm{d}x$ can be evaluated by resorting to the tabulated values on page 146.



M_i \ M_k			
	sik	$\frac{1}{2}sik$	$\frac{1}{2}sik$
	$\frac{1}{2}sik$	$\frac{1}{3}sik$	$\frac{1}{6}sik$
	$\frac{s}{2}(i_1+i_2)k$	$\frac{s}{6}(i_1+2i_2)k$	$\frac{s}{6}(2i_1+i_2)k$
quadratic parabola 	$\frac{2}{3}sik$	$\frac{1}{3}sik$	$\frac{1}{3}sik$
quadratic parabola 	$\frac{2}{3}sik$	$\frac{5}{12}sik$	$\frac{1}{4}sik$
quadratic parabola 	$\frac{1}{3}sik$	$\frac{1}{4}sik$	$\frac{1}{12}sik$
cubic parabola 	$\frac{1}{4}sik$	$\frac{1}{5}sik$	$\frac{1}{20}sik$
cubic parabola 	$\frac{3}{8}sik$	$\frac{11}{40}sik$	$\frac{1}{10}sik$
cubic parabola 	$\frac{1}{4}sik$	$\frac{2}{15}sik$	$\frac{7}{60}sik$

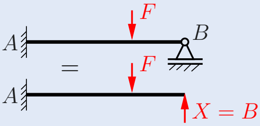
Quadratic parabola: –○– $\widehat{=}$ apex of the parabola,

Cubic parabola: –○– $\widehat{=}$ root of the linear load $q(x)$.

		quadratic parabola	
			
$\frac{si}{2}(k_1 + k_2)$	$\frac{1}{2} sik$	$\frac{2}{3} sik$	$\frac{2}{3} sik$
$\frac{si}{6}(k_1 + 2k_2)$	$\frac{1}{6} sik(1+\alpha)$	$\frac{1}{3} sik$	$\frac{1}{4} sik$
$\frac{s}{6}(2i_1k_1 + i_1k_2$ $+2i_2k_2 + i_2k_1)$	$\frac{sk}{6}[(1+\beta)i_1$ $+(1+\alpha)i_2]$	$\frac{sk}{3}(i_1 + i_2)$	$\frac{sk}{12}(5i_1 + 3i_2)$
$\frac{si}{3}(k_1 + k_2)$	$\frac{1}{3} sik(1+\alpha\beta)$	$\frac{8}{15} sik$	$\frac{7}{15} sik$
$\frac{si}{12}(3k_1 + 5k_2)$	$\frac{sik}{12}(5-\beta-\beta^2)$	$\frac{7}{15} sik$	$\frac{11}{30} sik$
$\frac{si}{12}(k_1 + 3k_2)$	$\frac{sik}{12}(1+\alpha+\alpha^2)$	$\frac{1}{5} sik$	$\frac{2}{15} sik$
$\frac{si}{20}(k_1 + 4k_2)$	$\frac{sik}{20}(1+\alpha)(1+\alpha^2)$	$\frac{2}{15} sik$	$\frac{1}{12} sik$
$\frac{si}{40}(4k_1 + 11k_2)$	$\frac{sik}{10}(1+\alpha+\alpha^2-\frac{\alpha^3}{4})$	$\frac{11}{15} sik$	$\frac{29}{120} sik$
$\frac{si}{60}(7k_1 + 8k_2)$	$\frac{sik}{20}(1+\alpha)(\frac{7}{3}-\alpha^2)$	$\frac{1}{5} sik$	$\frac{1}{6} sik$

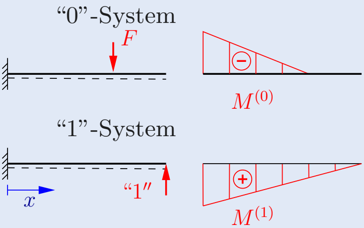
Trapezoids: individual i- or k-values can be negative.

Application to statically indeterminate problems



The *statically redundant* (unknown) force $X = B$ is computed from the kinematic constraint, that the displacement has to vanish at point i of X (support):

$$f_i = 0 \, .$$



The relation $\quad f_i = \int\limits_l \frac{M\overline{M}}{EI}\,\mathrm{d}x$

yields with

$$M = M^{(0)} + X\ M^{(1)} \ , \qquad \overline{M} = M^{(1)}$$

the statical redundant force

$$\boxed{X = B = -\frac{\int M^{(0)} M^{(1)} \mathrm{d}x}{\int M^{(1)} M^{(1)} \mathrm{d}x} \, .}$$

The integrals can be evaluated by resorting to the values tabulated on page 146.

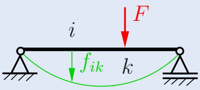
Remark: In n-fold statically indeterminate problems, n *statically redundant* (unknown) forces/moments X_i occur. They are determined from n kinematic constraints (e. g. $f_i = 0$).

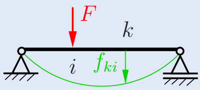
Method of Castigliano

The derivative of the strain energy with respect to the external force (moment) F_i is equal to the displacement (rotation) f_i in the direction of the force (moment) at the point where the force (moment) is applied.

$$\boxed{f_i = \frac{\partial \Pi}{\partial F_i} \, .}$$

Reciprocity theorem of Maxwell and Betti

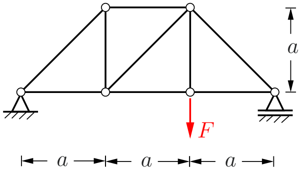




$$\boxed{f_{ik} = f_{ki}}$$

Problem 5.1 The depicted system is made of trusses with identical axial rigidity EA. **P5.1**

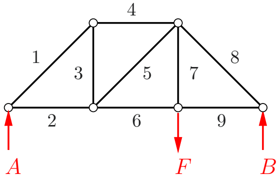
Determine the vertical displacement f of the force F.



Solution The problem is solved using conservation of energy $W = \Pi$. To assume the value f the force has to do the work $W = \frac{1}{2} F f$. The strain energy Π is calculated by

$$\Pi = \frac{1}{2} \sum \frac{S_i^2 \, l_i}{EA_i} = \frac{1}{2EA} \sum S_i^2 \, l_i \; .$$

Knowing the reaction forces $A = F/3$ and $B = 2F/3$ the normal forces in the truss system can be tabulated



i	l_i	S_i	$\frac{S_i^2 \, l_i}{F^2 a}$
1	$\sqrt{2}a$	$-\sqrt{2}F/3$	$2\sqrt{2}/9$
2	a	$F/3$	$1/9$
3	a	$F/3$	$1/9$
4	a	$-F/3$	$1/9$
5	$\sqrt{2}a$	$-\sqrt{2}F/3$	$2\sqrt{2}/9$
6	a	$2F/3$	$4/9$
7	a	F	$9/9$
8	$\sqrt{2}a$	$-2\sqrt{2}F/3$	$8\sqrt{2}/9$
9	a	$2F/3$	$4/9$

$$\sum S_i^2 \, l_i = \frac{4}{9}(5 + 3\sqrt{2}) F^2 a$$

Thus we compute

$$\underline{\underline{f = \frac{4}{9} \frac{(5 + 3\sqrt{2})}{EA} F a}} \; .$$

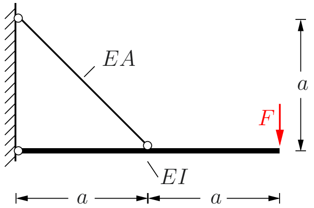
Alternatively the method of Castigliano can be applied. Using the strain energy

$$\Pi = \frac{1}{2} \sum \frac{S_i^2 \, l_i}{EA_i} = \frac{2}{9} \frac{(5 + 3\sqrt{2})}{EA} F^2 a$$

and the condition $f = \partial \Pi / \partial F$ we get

$$\underline{\underline{f}} = \frac{\partial \Pi}{\partial F} = \underline{\underline{\frac{4}{9} \frac{(5 + 3\sqrt{2})}{EA} F a}} \; .$$

P5.2 **Problem 5.2** A beam (flexural rigidity EI, axial rgidity $EA \to \infty$) is loaded by the force F and supported by the inclined rope (axial rigidity EA).



Compute the vertical displacement f in force direction.

Solution The problem can be solved by using the energy theorem

$$W = \Pi\,.$$

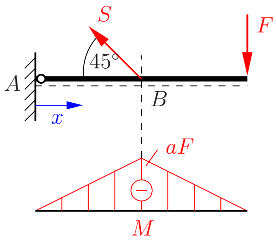
The work of the external force F is given by

$$W = \frac{1}{2}\,F\,f\,.$$

The strain energy consists of beam bending and tension in the rope:

$$\Pi = \Pi_S + \Pi_B\,.$$

With



$$\overset{\curvearrowleft}{A}:\; 2aF - \frac{\sqrt{2}}{2}aS = 0 \quad \rightsquigarrow \quad S = \frac{4}{\sqrt{2}}F$$

$$\uparrow:\; A_V + S\frac{\sqrt{2}}{2} - F = 0 \;\rightsquigarrow\; A_V = -F$$

and

$$M(x) = -Fx \qquad (0 \le x \le a)$$

we obtain for the rope

$$\Pi_S = \frac{S^2\,l}{2EA} = 4\sqrt{2}\,\frac{F^2 a}{EA}$$

and for the beam (using the symmetry of $M(x)$)

$$\Pi_B = \int \frac{M^2}{2EI}\,\mathrm{d}x = 2\int_0^a \frac{F^2 x^2}{2EI}\,\mathrm{d}x = \frac{1}{3}\,\frac{F^2 a^3}{EI}\,.$$

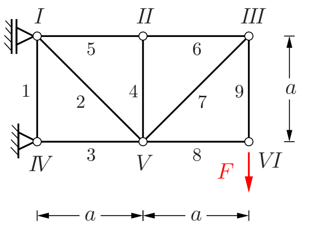
Finally, the energy theorem yields

$$\underline{\underline{f = \frac{2}{3}\,\frac{Fa^3}{EI} + 8\sqrt{2}\,\frac{Fa}{EA}}}\,.$$

Note: In section AB of the beam exists a compressive normal force $N = -2F$. The corresponding contribution to the strain energy is zero, because the beam is assumed to have infinite axial rigidity.

Problem 5.3 In the depicted truss is loaded by the force F. All trusses possess the same axial rigidity EA. **P5.3**

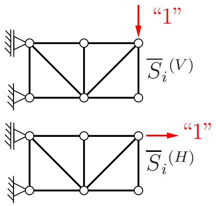
Compute the vertical and horizontal displacement of node III.



Solution Using the principle of virtual forces the displacements follow from

$$f = \sum \frac{S_i \overline{S}_i}{EA_i} l_i = \frac{1}{EA} \sum S_i \overline{S}_i \, l_i \, .$$

As the system is statically determinate, we can compute the forces in the truss members S_i due to the load F from equilibrium considerations.
Loading node III by virtual forces "1" in vertical or horizontal direction, provides the forces $\overline{S}_i{}^{(V)}$ or $\overline{S}_i{}^{(H)}$ in the truss members, respectively.



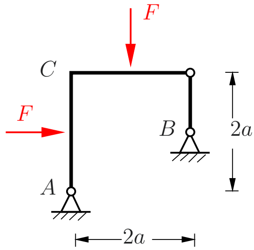
i	l_i	S_i	$\overline{S}_i{}^{(V)}$	$S_i\overline{S}_i{}^{(V)}l_i$	$\overline{S}_i{}^{(H)}$	$S_i\overline{S}_i{}^{(H)}l_i$
1	a	$-F$	-1	Fa	0	0
2	$\sqrt{2}a$	$\sqrt{2}F$	$\sqrt{2}$	$2\sqrt{2}Fa$	0	0
3	a	$-2F$	-2	$4Fa$	0	0
4	a	0	0	0	0	0
5	a	F	1	Fa	1	Fa
6	a	F	1	Fa	1	Fa
7	$\sqrt{2}a$	$-\sqrt{2}F$	$-\sqrt{2}$	$2\sqrt{2}Fa$	0	0
8	a	0	0	0	0	0
9	a	F	0	0	0	0
		$\sum S_i\overline{S}_i{}^{(V)}l_i = (7+4\sqrt{2})Fa$			$\sum S_i\overline{S}_i{}^{(H)}l_i = 2Fa$	

This leads to the vertical and horizontal displacements:

$$\underline{\underline{f_V = (7+4\sqrt{2})\frac{Fa}{EA}}} \, , \qquad \underline{\underline{f_H = 2\frac{Fa}{EA}}} \, .$$

P5.4 **Problem 5.4** The depicted plane frame (flexural rigidity EI) is subjected to two point forces F.

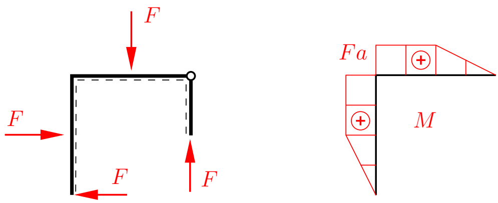
Compute for the rigid corner C
a) the horizontal displacement,
b) the vertical displacement,
c) the rotation.



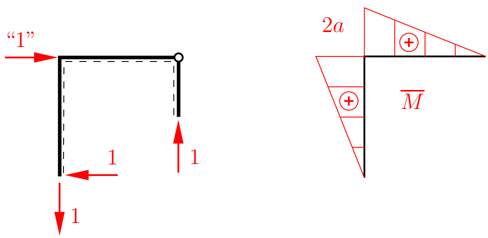
Solution We use the principle of virtual forces, neglecting shear, tension and torsion contributions:

$$f_i = \int_l \frac{M\overline{M}}{EI}\,\mathrm{d}x$$

The bending moment M due to the external forces F is sketched below



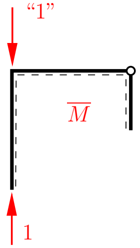
to a) Horizontal displacement of the corner: We apply a horizontal virtual force "1" at the corner C and determine the associated bending moment.



Using integration in sections together with the tabulated values on page 146 yields:

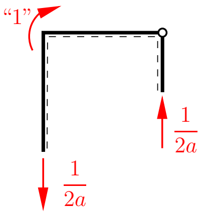
$$\underline{\underline{f_H}} = \frac{1}{EI}\int M\overline{M}\,\mathrm{d}x = \frac{2}{EI}\left(\int\limits_0^a M\overline{M}\,\mathrm{d}x + \int\limits_a^{2a} M\overline{M}\,\mathrm{d}x\right)$$
$$= \frac{2}{EI}\left(\frac{1}{3}(a)(a)(Fa) + \frac{a}{2}(a+2a)Fa\right)$$
$$= \underline{\underline{\frac{11}{3}\frac{Fa^3}{EI}}}\,.$$

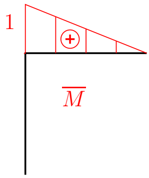
to b) Vertical displacement of the corner: Application of a vertical virtual force "1" yields no loading due to bending and thus no displacement:



$$\overline{M} = 0\,, \qquad \underline{\underline{f_V = 0}}\,.$$

to c) Rotation of the corner: The virtual moment "1", applied at corner C, yields the following bending moment $\overline{M}$:





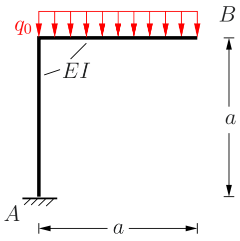
For the rotation ψ of the corner we obtain by use of the table on page 146:

$$\underline{\underline{\psi}} = \frac{1}{EI}\left(\int\limits_0^a M\overline{M}\,\mathrm{d}x + \int\limits_a^{2a} M\overline{M}\,\mathrm{d}x\right)$$
$$= \frac{1}{EI}\left(\frac{Fa^2}{2}\,(1+\frac{1}{2}) + \frac{1}{3}\,a\,Fa\,\frac{1}{2}\right)$$
$$= \underline{\underline{\frac{11}{12}\frac{Fa^2}{EI}}}\,.$$

P5.5

Problem 5.5 Determine the vertical displacement f_B and the rotation ψ_B in point B at the end of the frame.

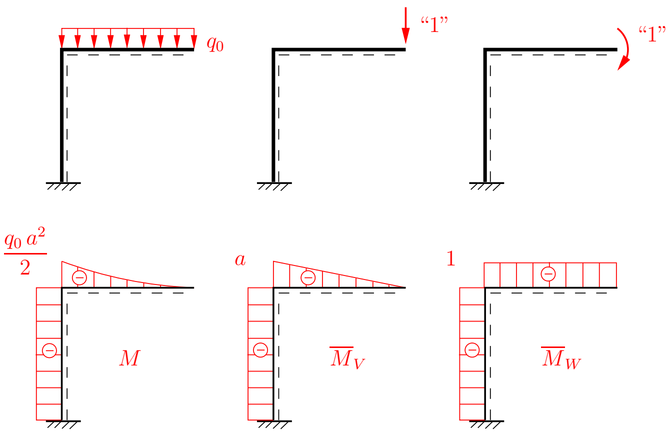
All beams in the frame are rigid in axial direction $EA \to \infty$.



Solution Using the principle of virtual forces we can compute the displacement and the rotation from

$$f = \int \frac{M\overline{M}}{EI}\,\mathrm{d}x \, .$$

For the original and the auxiliary system we get:



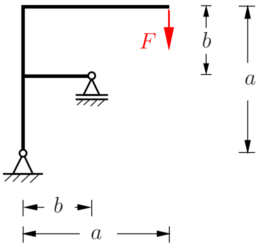
Using the table on page 146 we compute

$$\underline{\underline{f_B}} = \frac{1}{EI}\int M\overline{M}_V\,\mathrm{d}x = \frac{1}{EI}\left[\frac{a}{4}\cdot\frac{q_0 a^2}{2}\cdot a + a\cdot\frac{q_0 a^2}{2}\cdot a\right] = \underline{\underline{\frac{5}{8}\frac{q_0 a^4}{EI}}} \, ,$$

$$\underline{\underline{\psi_B}} = \frac{1}{EI}\int M\overline{M}_W\,\mathrm{d}x = \frac{1}{EI}\left[\frac{a}{3}\cdot\frac{q_0 a^2}{2}\cdot 1 + a\cdot\frac{q_0 a^2}{2}\cdot 1\right] = \underline{\underline{\frac{2}{3}\frac{q_0 a^3}{EI}}} \, .$$

Problem 5.6 The depicted frame is constructed with beams having identical flexural rigidity EI. P5.6

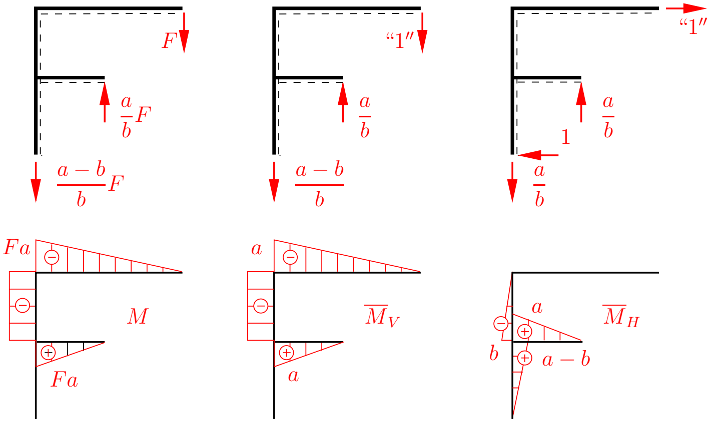
Compute the vertical and horizontal displacement at the loading point.



Solution According to the principle of virtual forces the displacements are obtained from

$$f = \int \frac{M\overline{M}}{EI}\mathrm{d}x\,.$$

The bending moments M due to the load and $\overline{M}_V$ and $\overline{M}_H$ due to the auxiliary loads are given below:



Using the table on page 146 we compute

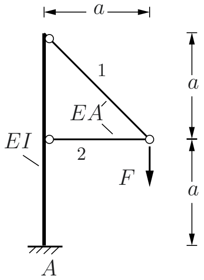
$$\underline{\underline{f_V}} = \frac{1}{EI}\left\{\frac{1}{3}a(-Fa)(-a) + b(-Fa)(-a) + \frac{1}{3}b(Fa)a\right\} = \underline{\underline{\frac{Fa^3}{3EI}\left(1+\frac{4b}{a}\right)}},$$

$$\underline{\underline{f_H}} = \frac{1}{EI}\left\{\frac{1}{2}b(-Fa)(-b) + \frac{1}{3}b(Fa)a\right\} = \underline{\underline{\frac{Fa^2b}{3EI}\left(1+\frac{3b}{2a}\right)}}\,.$$

P5.7

Problem 5.7 The sketched system consists of a clamped beam ($EA \to \infty$) with flexural rigidity EI and two bars of identical axial rigidity EA.

Compute the vertical and horizontal displacement at the point of load application.



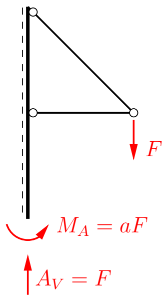
Solution The beam is subjected to bending, while the bars experience tension or compression. We compute the displacements based on the principle of virtual forces

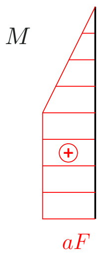
$$f = \int \frac{M\overline{M}}{EI}\,\mathrm{d}x + \sum_i \frac{S_i \overline{S}_i}{EA_i}\, l_i \ .$$

As the system is statically determinate, we obtain M and S_i from equilibrium conditions:

$$S_1 = \sqrt{2}\, F\ ,$$

$$S_2 = -F\ .$$

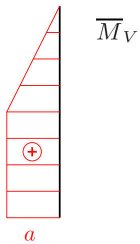




The vertical displacement can be calculated by loading the system with a force “1” in vertical direction. By replacing F by “1”the above results can be used:

$$\overline{S}_{1_V} = \sqrt{2}\ ,$$

$$\overline{S}_{2_V} = -1\ .$$



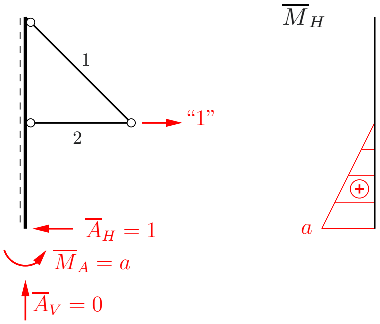
Using the table on page 146 yields

$$\underline{\underline{f_V}} = \frac{1}{EI}\left\{a(aF)a + \frac{1}{3}a(aF)a\right\}$$
$$+\frac{1}{EA}\left\{\sqrt{2}F\cdot\sqrt{2}\cdot\sqrt{2}a + (-F)(-1)a\right\}$$
$$= \underline{\underline{\frac{4}{3}\frac{Fa^3}{EI} + \frac{(1+2\sqrt{2})Fa}{EA}}}\ .$$

To compute the horizontal displacement we use the following auxiliary system:

$\overline{S}_{1_H} = 0\ ,$

$\overline{S}_{2_H} = 1\ .$

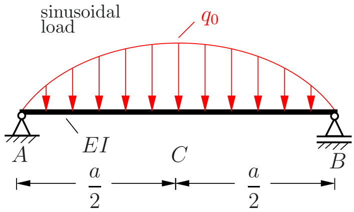


The displacement f_H follows from evaluation

$$\underline{\underline{f_H}} = \frac{1}{EI}\left\{\frac{1}{2}a(aF)a + 0\right\} + \frac{1}{EA}\left\{0 + (-F)\cdot 1\cdot a\right\}$$
$$= \underline{\underline{\frac{Fa^3}{2EI} - \frac{Fa}{EA}}}\ .$$

Note: For $\dfrac{EI}{a^2EA} = \dfrac{1}{2}$ it holds $f_H = 0$. For a rigid beam ($EI \to \infty$) the load application point moves to the left ($f_H < 0$).

P5.8 **Problem 5.8** Determine the vertical displacement f in the center of the beam.



Solution The principle of virtual forces yields the vertical displacement

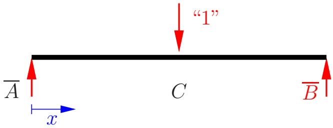
$$f = \int \frac{M\overline{M}}{EI} \mathrm{d}x \, .$$

The bending moment M due to the given loading is computed via integration:

$$q(x) = q_0 \sin\left(\frac{\pi}{a}x\right), \quad V = q_0 \, \frac{a}{\pi} \, \cos\left(\frac{\pi}{a}x\right), \quad M = q_0 \, \frac{a^2}{\pi^2} \sin\left(\frac{\pi}{a}x\right).$$

For the virtual load “1” it follows:

$$\overline{M} = \begin{cases} \overline{A}\, x = \dfrac{1}{2}x, & x \leq a/2 \\ \overline{B}\,(a-x) = \dfrac{a-x}{2}, & x \geq a/2 \end{cases}$$



The vertical displacement ist then obtained

$$f = \int \frac{M\overline{M}}{EI} \mathrm{d}x = \frac{1}{EI}\left\{ \int\limits_0^{a/2} \frac{x}{2}\, M \,\mathrm{d}x + \int\limits_{a/2}^{a} \left(\frac{a-x}{2}\right) M \mathrm{d}x \right\}.$$

Integration with

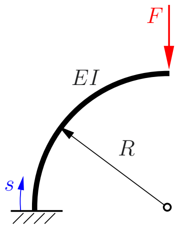
$$\int x \, \sin cx \, \mathrm{d}x = \frac{\sin cx}{c^2} - \frac{x \, \cos cx}{c}$$

renders the result

$$\underline{\underline{f}} = \frac{q_0 \, a^2}{2\, EI\, \pi^2} \left\{ \left[\frac{\sin(\frac{\pi}{a}x)}{\frac{\pi^2}{a^2}} - \frac{x \, \cos(\frac{\pi}{a}x)}{\frac{\pi}{a}} \right]_0^{a/2} \right.$$

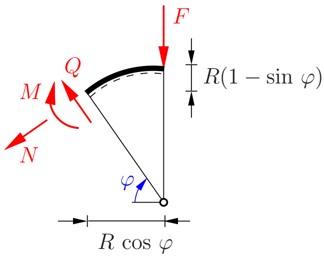
$$\left. + \left[\frac{-a^2}{\pi} \cos(\frac{\pi}{a}x) - \frac{\sin(\frac{\pi}{a}x)}{\frac{\pi^2}{a^2}} + \frac{x \, \cos(\frac{\pi}{a}x)}{\frac{\pi}{a}} \right]_{a/2}^{a} \right\} = \underline{\underline{\frac{a^4}{\pi^4} \frac{q_0}{EI}}} \, .$$

Problem 5.9 The sketched arch is clamped and subjected to a vertical force F. **P5.9**



Compute the vertical and horizontal displacement at the point of loading. Only deformations due to bending shall be considered.

Solution We use the principle of virtual forces. The displacements are computed from

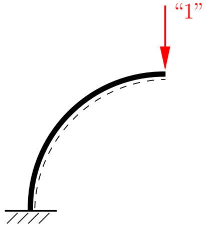


$$f = \int \frac{M\overline{M}}{EI}\, \mathrm{d}s\ .$$

The bending moment M is obtained from equilibrium considerations

$$M = -FR\cos\varphi\ .$$

To determine the vertical displacement we apply a force “1” in vertical direction. This yields

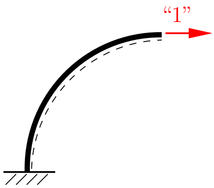


$$\overline{M}_V = -R\cos\varphi$$

together with $\mathrm{d}s = R\,\mathrm{d}\varphi$ the displacement

$$\underline{\underline{f_V}} = \frac{R}{EI}\int_0^{\pi/2} M\overline{M}_V\ \mathrm{d}\varphi = \frac{FR^3}{EI}\int_0^{\pi/2} \cos^2\varphi\ \mathrm{d}\varphi = \underline{\underline{\frac{\pi FR^3}{4EI}}}\ .$$

The auxiliary force in horizontal direction causes the bending moment



$$\overline{M}_H = -R(1-\sin\varphi)$$

and the displacement

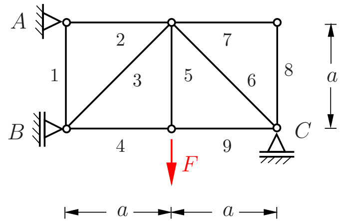
$$\underline{\underline{f_H}} = \frac{R}{EI}\int_0^{\pi/2} M\overline{M}_H \mathrm{d}\varphi = \frac{R^3F}{EI}\int_0^{\pi/2} (\cos\varphi - \sin\varphi\cos\varphi)\ \mathrm{d}\varphi = \underline{\underline{\frac{FR^3}{4EI}}}\ .$$

Note: For the integration the two relations $\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$ and $\sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi$ were used.

P5.10

Problem 5.10 The depicted truss is made of members with identical axial rgidity EA.

Determine the normal forces in the truss members and the vertical displacement under the load.

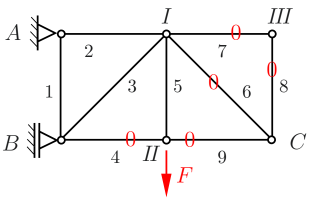


Solution The support of the truss is statically indeterminate. We use the reaction force C as statically redundant force and compute its value from the support constraint

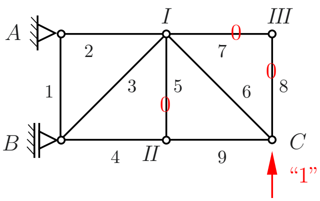
$$f_C = \sum \frac{S_i\,\overline{S}_i\,l_i}{EA_i} = \frac{1}{EA}\sum S_i\,\overline{S}_i\,l_i = 0\,.$$

Here we only compute the normal forces in the "0"-system. The computation of these forces in the "1"- and "2"-system follow by an analogous procedure.

"0"-system:



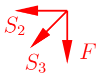
"1"-system:



For example at node I ("0"-system):

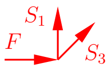
$$\uparrow:\quad S_3 = -\sqrt{2}S_5 = -\sqrt{2}F$$

$$\rightarrow:\quad S_2 = F$$



For example at node B ("0"-System):

$$\uparrow:\quad S_1 = -\frac{\sqrt{2}}{2}\,S_3 = F$$



With $S_i = S_i^{(0)} + C\cdot S_i^{(1)}$ and $\overline{S}_i = S_i^{(1)}$ it follows

$$\underline{\underline{C}} = -\frac{\sum S_i^{(0)}S_i^{(1)}l_i}{\sum S_i^{(1)}S_i^{(1)}l_i} = \underline{\underline{\frac{3+2\sqrt{2}}{7+4\sqrt{2}}}}\,F\,.$$

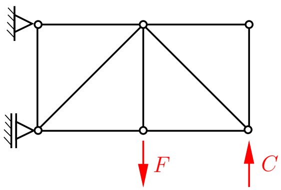
i	l_i	$S_i^{(0)}$	$S_i^{(1)}$	$S_i^{(0)} S_i^{(1)} l_i$	$S_i^{(1)} S_i^{(1)} l_i$	$S_i^{(2)}$
1	a	F	-1	$-Fa$	a	1
2	a	F	-2	$-2Fa$	$4a$	1
3	$\sqrt{2}a$	$-\sqrt{2}F$	$\sqrt{2}$	$-2\sqrt{2}Fa$	$2\sqrt{2}a$	$-\sqrt{2}$
4	a	0	1	0	a	0
5	a	F	0	0	0	1
6	$\sqrt{2}a$	0	$-\sqrt{2}$	0	$2\sqrt{2}a$	0
7	a	0	0	0	0	0
8	a	0	0	0	0	0
9	a	0	1	0	a	0
			$\sum =$	$\left(-3-2\sqrt{2}\right) Fa$	$\left(7+4\sqrt{2}\right) a$	

From this table we deduct the forces in the truss members

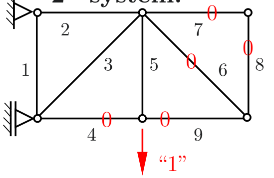
$$\underline{\underline{S_1 = \frac{4+2\sqrt{2}}{7+4\sqrt{2}}\, F}}, \quad \underline{\underline{S_2 = \frac{1}{7+4\sqrt{2}}\, F}}, \quad \underline{\underline{S_3 = -\frac{4+4\sqrt{2}}{7+4\sqrt{2}}\, F}},$$

$$\underline{\underline{S_4 = S_9 = \frac{3+2\sqrt{2}}{7+4\sqrt{2}}\, F}}, \quad \underline{\underline{S_5 = F}}, \quad \underline{\underline{S_6 = -\frac{4+3\sqrt{2}}{7+4\sqrt{2}}\, F}}, \quad \underline{\underline{S_7 = S_8 = 0}}.$$

To compute the vertical displacement at the loading point we consider the system as a statically determinate system loaded by F and C, which fulfills the support constraint $f_C = 0$. For this situation we know the forces S_i.



“2”-system:



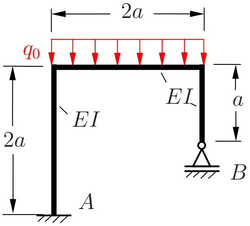
With the forces $\overline{S}_i = S_i^{(2)}$ of the auxiliary system “2” we obtain

$$\underline{\underline{f_F}} = \frac{1}{EA} \sum S_i \overline{S}_i\, l_i$$

$$= \frac{Fa}{EA\left(7+4\sqrt{2}\right)} \left[(4+2\sqrt{2}) + 1 - (4+4\sqrt{2})(-\sqrt{2})\sqrt{2} + (7+4\sqrt{2})\right]$$

$$= \underline{\underline{\frac{20+14\sqrt{2}}{7+4\sqrt{2}}\, \frac{Fa}{EA}}}.$$

P5.11 **Problem 5.11** Determine the bending moment and the horizontal displacement f_H of the support B of the depicted frame structure



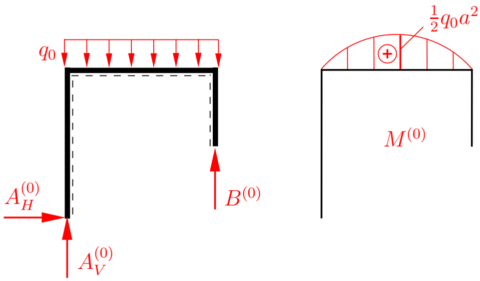
Solution The system is statically indeterminate. To determine support reactions we use the principle of virtual forces, where we consider the moment $X = M_A$ as statical redundant reaction. Thus we obtain the following bending moments and support reaction in the "0"- and "1"-system:

"0"-system:

$$A_H^{(0)} = 0 \,,$$

$$A_V^{(0)} = q_0 a \,,$$

$$B^{(0)} = q_0 a \,.$$

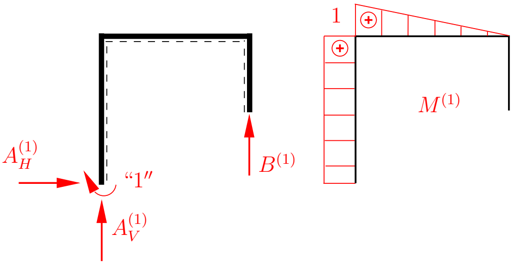


"1"-system:

$$A_H^{(1)} = 0 \,,$$

$$A_V^{(1)} = -\frac{1}{2a} \,,$$

$$B^{(1)} = \frac{1}{2a} \,.$$



The condition that the rotation at position A has to vanish

$$\varphi_A = 0 = \int \frac{M\overline{M}}{EI}\,\mathrm{d}x \,,$$

yields with

$$M = M^{(0)} + X\, M^{(1)} \qquad \text{and} \qquad \overline{M} = M^{(1)}$$

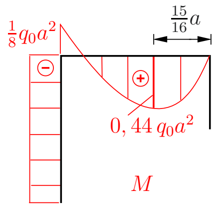
$$\underline{\underline{X = M_A}} = -\frac{\int M^{(0)} M^{(1)} \mathrm{d}x}{\int M^{(1)} M^{(1)} \mathrm{d}x} = -\frac{\frac{1}{3} \cdot 2a \left(\frac{1}{2} q_0 a^2\right) \cdot 1}{\frac{1}{3} \cdot 2a \cdot 1 \cdot 1 + 2a \cdot 1 \cdot 1} = \underline{\underline{-\frac{q_0 a^2}{8}}}.$$

The support reactions and the bending moment follow as

$$\underline{\underline{A_H}} = A_H^{(0)} + X \cdot A_H^{(1)} = \underline{\underline{0}}\,,$$

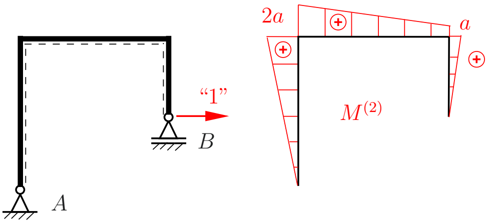
$$\underline{\underline{A_V}} = A_V^{(0)} + X \cdot A_V^{(1)} = \underline{\underline{\frac{17}{16}\, q_0 a}}\,,$$

$$\underline{\underline{B}} = B^{(0)} + X \cdot B^{(1)} = \underline{\underline{\frac{15}{16}\, q_0 a}}\,.$$



To compute the horizontal displacement at B, the frame is considered as a statically determinate system loaded by q_0 *and* $X = M_A$. At this system an auxiliary force is applied ("2"-system) rendering the following bending moment:

"2"-system:



Using

$$M = M^{(0)} + X \cdot M^{(1)} \qquad \text{and} \qquad \overline{M} = M^{(2)}$$

we compute with the table on page 146

$$\underline{\underline{f_H}} = \frac{1}{EI} \int M \overline{M} \mathrm{d}x = \frac{1}{EI} \left\{ \int M^{(0)} M^{(2)} \mathrm{d}x + X \int M^{(1)} M^{(2)} \mathrm{d}x \right\}$$

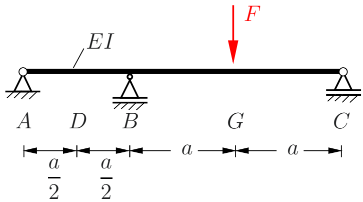
$$= \frac{1}{EI} \left\{ \frac{2a}{3} \frac{q_0 a^2}{2} (2a + a) - \frac{q_0 a^2}{8} \left[\frac{1}{6} \cdot 2a \cdot 1 \cdot (2 \cdot 2a + a) + \frac{1}{2} \cdot 2a \cdot 1 \cdot 2a \right] \right\}$$

$$= \underline{\underline{\frac{13}{24} \frac{q_0 a^4}{EI}}}\,.$$

P5.12

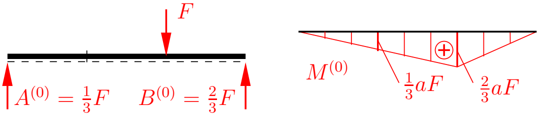
Problem 5.12 Compute the reaction forces and the deflection at points D and G.

Now an additional force of $2F$ is applied at D. How does the deflection at G change?

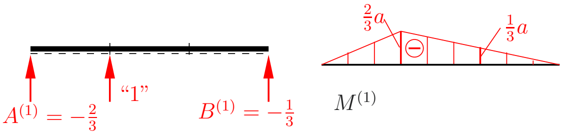


Solution We apply the principle of virtual forces and use the reaction force B as statically redundant force. Together with the bending moments in the "0"- and "1"-System

"0"-system:



"1"-system:



the kinematic constraint $f_B = 0$ yields the reaction force B:

$$\underline{\underline{X = B}} = -\frac{\dfrac{1}{EI}\displaystyle\int M^{(0)} M^{(1)} \mathrm{d}x}{\dfrac{1}{EI}\displaystyle\int M^{(1)} M^{(1)} \mathrm{d}x} = -\frac{\dfrac{a}{3}\dfrac{aF}{3}\left(-\dfrac{2a}{3}\right) + \dfrac{a}{6}\left[2\,\dfrac{aF}{3}\left(-\dfrac{2a}{3}\right)\right.}{\dfrac{a}{3}\,\dfrac{2a}{3}\,\dfrac{2a}{3} + \dfrac{2a}{3}\,\dfrac{2a}{3}\,\dfrac{2a}{3}}$$

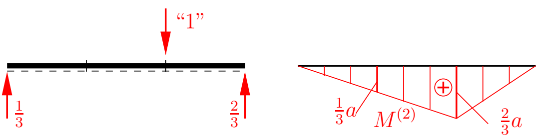
$$+\frac{\left.\dfrac{aF}{3}\left(-\dfrac{a}{3}\right) + 2\,\dfrac{2aF}{3}\left(-\dfrac{a}{3}\right) + \dfrac{2aF}{3}\left(-\dfrac{2a}{3}\right)\right] + \dfrac{a}{3}\,\dfrac{2aF}{3}\left(-\dfrac{a}{3}\right)}{\dfrac{a}{3}\,\dfrac{2a}{3}\,\dfrac{2a}{3} + \dfrac{2a}{3}\,\dfrac{2a}{3}\,\dfrac{2a}{3}} = \underline{\underline{\frac{7}{8}F}}\,.$$

Furthermore, we compute

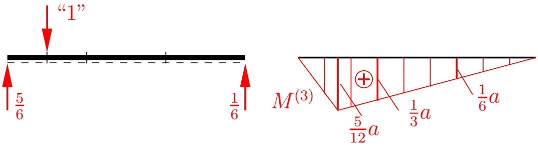
$$\underline{\underline{A}} = A^{(0)} + X \cdot A^{(1)} = \frac{1}{3}F - \frac{7}{8}F \cdot \frac{2}{3} = \underline{\underline{-\frac{F}{4}}}\,, \qquad \underline{\underline{C = \frac{3}{8}F}}\,.$$

To determine the deflection we consider the beam as a statically determinate system loaded by F and B. From the two auxiliary systems

"2"-system:



"3"-system:



we obtain

$$\underline{\underline{f_G}} = \frac{1}{EI}\int [M^{(0)} + X \cdot M^{(1)}]M^{(2)}\mathrm{d}x$$

$$= \frac{1}{EI}\left\{\int M^{(0)}M^{(2)}\mathrm{d}x + X\int M^{(1)}M^{(2)}\mathrm{d}x\right\}$$

$$= \frac{1}{EI}\left\{\frac{2a}{3}\,\frac{2aF}{3}\,\frac{2a}{3} + \frac{7}{8}F\left[\frac{a}{3}\left(-\frac{2a}{3}\right)\frac{a}{3}\right.\right.$$

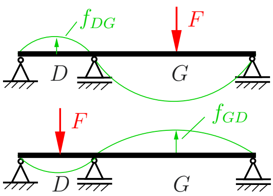
$$\left.\left. + \frac{a}{6}\left(-\frac{4a}{3}\,\frac{a}{3} - \frac{2a}{3}\,\frac{2a}{3} - \frac{2a}{3}\,\frac{2a}{3} - \frac{a}{3}\,\frac{a}{3}\right)\right] + \frac{a}{3}\,\frac{2aF}{3}\,\frac{2a}{3}\right\}$$

$$= \underline{\underline{\frac{5}{48}\,\frac{Fa^3}{EI}}}\,,$$

$$\underline{\underline{f_D}} = f_{DG} = \frac{1}{EI}\int [M^{(0)} + X \cdot M^{(1)}]M^{(3)}\mathrm{d}x$$

$$= \frac{1}{EI}\left\{\int M^{(0)}M^{(3)}\mathrm{d}x + X\int M^{(1)}M^{(3)}\mathrm{d}x\right\} = \underline{\underline{-\frac{1}{64}\,\frac{Fa^3}{EI}}}\,.$$

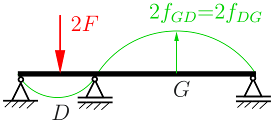
The deflection at G due to the additional load $2F$ is computed from the reciprocity theorem of Maxwell-Betti. Based on this theorem the deflection f_{DG} at D due to the force F in G is equal to the deflection f_{GD} at G due to the force F in D. As a consequence of the force $2F$ at D we obtain at G the deflection $2f_{GD}$. Thus, the total deflection at G is given by



$$\boxed{f_{GD} = f_{DG}}$$

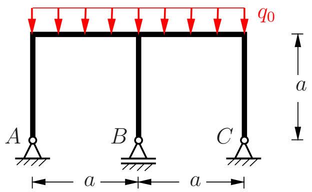
$$\underline{\underline{f}} = f_G + 2f_{DG}$$

$$= \left(\frac{5}{48} - 2\frac{1}{64}\right)\frac{Fa^3}{EI} = \underline{\underline{\frac{7}{96}\,\frac{Fa^3}{EI}}}\,.$$

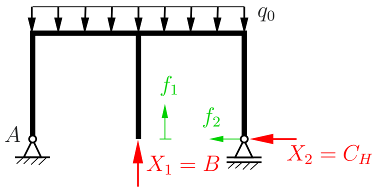


P5.13 **Problem 5.13** The depicted frame is loaded by a constant line load q_0. The frame is made of beams with identical flexural rigidity EI.

Determine the reaction forces in the supports.

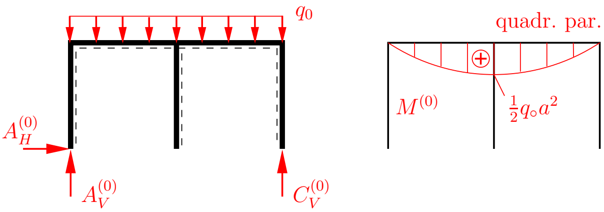


Solution The frame is two times statically indeterminate. We consider the reaction force B and the horizontal force C_H as statically redundant forces to obtain the depicted system. The unknown forces $X_1 = B$ and $X_2 = C_H$ are computed from the kinematic constraints $f_1 = 0$ and $f_2 = 0$.



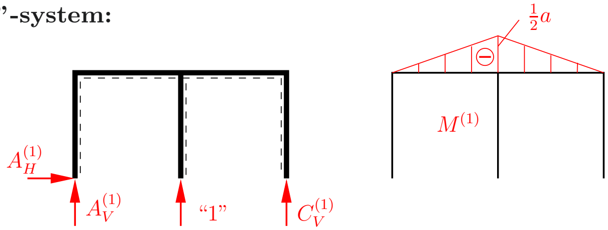
Using the principle of virtual forces we construct the following basic and auxiliary systems :

“0”-system:



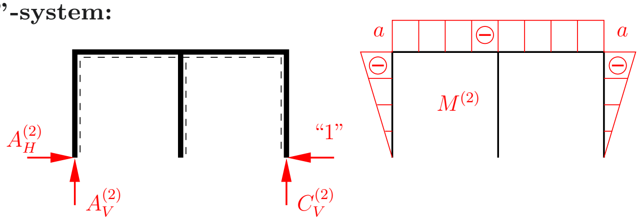
$$A_V^{(0)} = q_0 a\,, \qquad A_H^{(0)} = 0\,, \qquad C_V^{(0)} = q_0 a\,.$$

“1”-system:



$$A_V^{(1)} = C_V^{(1)} = -\frac{1}{2}\,, \qquad A_H^{(1)} = 0\,.$$

“2”-system:



$$A_V^{(2)} = C_V^{(2)} = 0 \,, \qquad A_H^{(2)} = 1 \,.$$

From conditions

$$f_1 = \frac{1}{EI}\int [M^{(0)} + X_1 M^{(1)} + X_2 M^{(2)}] M^{(1)} \mathrm{d}x = 0 \,,$$

$$f_2 = \frac{1}{EI}\int [M^{(0)} + X_1 M^{(1)} + X_2 M^{(2)}] M^{(2)} \mathrm{d}x = 0$$

we obtain by using the table on page 146

$$\int M^{(0)} M^{(1)} \mathrm{d}x = -2\,\frac{5a}{12}\,\frac{q_0 a^2}{2}\,\frac{a}{2} = -\frac{5 q_0 a^4}{24} \,, \qquad \int M^{(1)} M^{(1)} \mathrm{d}x = \frac{a^3}{6} \,,$$

$$\int M^{(1)} M^{(2)} \mathrm{d}x = \frac{a^3}{2} \,, \qquad \int M^{(0)} M^{(2)} \mathrm{d}x = -\frac{2}{3}\, q_0 a^4 \,,$$

$$\int M^{(2)} M^{(2)} \mathrm{d}x = 2\,\frac{a}{3}(-a)(-a) + 2a(-a)(-a) = \frac{8}{3}\, a^3$$

the following two equations

$$-\frac{5 q_0 a^4}{24} + X_1 \frac{a^3}{6} + X_2 \frac{a^3}{2} = 0 \,, \qquad -\frac{2 q_0 a^4}{3} + X_1 \frac{a^3}{2} + X_2 \frac{8a^3}{3} = 0 \,.$$

The solution is given by

$$\underline{\underline{X_1 = B = \frac{8}{7}\, q_0 a}} \,, \qquad \underline{\underline{X_2 = C_H = \frac{1}{28}\, q_0 a}}$$

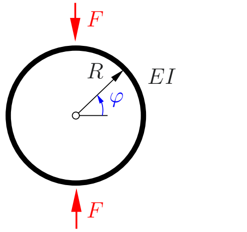
and the remaining support reactions follow

$$\underline{\underline{A_V}} = A_V^{(0)} + X_1 A_V^{(1)} + X_2 A_V^{(2)} = q_0 a - \frac{8}{7}\, q_0 a \cdot \frac{1}{2} + 0 = \underline{\underline{\frac{3}{7}\, q_0 a}} \,,$$

$$\underline{\underline{A_H}} = A_H^{(0)} + X_1 A_H^{(1)} + X_2 A_H^{(2)} = \underline{\underline{\frac{1}{28}\, q_0 a}} \,,$$

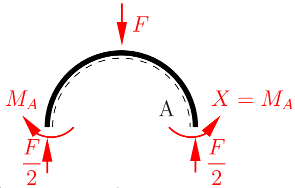
$$\underline{\underline{C_V}} = q_0 a - \frac{8}{7}\, q_0 a \cdot \frac{1}{2} = \underline{\underline{\frac{3}{7}\, q_0 a}} \,.$$

P5.14 **Problem 5.14** An elastic circular arc is loaded by two opposing forces F.

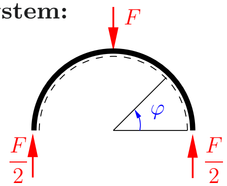


Determine the bending moment and the compression in the circular arc. Assume that the arc is *rigid* with respect to an axial deformation.

Solution We cut the arc at the mid plane (at $\varphi = 0,\ \pi$) and realize that the system is internally statically indeterminate (the stress resultants cannot be determined from equilibrium conditions). The unknown bending moment $X = M_A$ is computed from the fact that the slope at A has to vanish (symmetry!). Use of the principle of virtual forces yields:

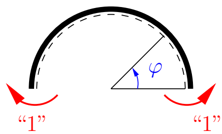


"0"-system:



$$M^{(0)} = \frac{1}{2}FR(1 - \cos\varphi)\ ,$$

"1"-system:



$$M^{(1)} = 1\ .$$

From

$$\psi_A = \frac{1}{EI}\int M\overline{M}\mathrm{d}s = 0$$

we obtain with

$$M = M^{(0)} + X \cdot M^{(1)}\ , \qquad \overline{M} = M^{(1)}\ , \qquad \mathrm{d}s = R\,\mathrm{d}\varphi$$

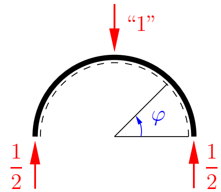
for M_A:

$$\underline{\underline{X = M_A}} = -\frac{\int M^{(0)}M^{(1)}\mathrm{d}s}{\int M^{(1)}M^{(1)}\mathrm{d}s} = -\frac{2\int\limits_0^{\pi/2}\frac{FR}{2}(1-\cos\varphi)R\,\mathrm{d}\varphi}{2\int\limits_0^{\pi/2}R\,\mathrm{d}\varphi} = \underline{\underline{-FR\left(\frac{1}{2} - \frac{1}{\pi}\right)}}\ .$$

These intermediate results yield the bending moment in the range $0 \le \varphi \le \pi/2$

$$\underline{\underline{M}} = M^{(0)} + X \cdot M^{(1)} = \underline{\underline{\frac{1}{2}FR\left(\frac{2}{\pi} - \cos\varphi\right)}}\ .$$

To compute the vertical displacement at the loading point we consider the semicircular arc as being loaded by the force F and the moment M_A and to be simply supported (statically determinate). For this system the bending moment M is known. Form the related auxiliary system we obtain



$$\overline{M} = \frac{1}{2}R(1-\cos\varphi)\,.$$

With this result the displacement follows

$$f_F = 2\,\frac{1}{EI}\int\limits_0^{\pi/2} M\overline{M}R\,\mathrm{d}\varphi = \frac{FR^3}{2EI}\int\limits_0^{\pi/2}\left(\frac{2}{\pi}-\cos\varphi\right)(1-\cos\varphi)\mathrm{d}\varphi$$

$$= \frac{FR^3}{2EI}\left[\frac{2}{\pi}\varphi-\left(\frac{2}{\pi}+1\right)\sin\varphi+\frac{\varphi}{2}+\frac{1}{4}\sin 2\varphi\right]_0^{\pi/2} = \frac{FR^3}{8EI}\left(\pi-\frac{8}{\pi}\right).$$

The compression of the circular arc yields

$$\underline{\underline{\Delta v = 2f_F = \frac{FR^3}{EI}\,\frac{\pi^2-8}{4\pi}}}\,.$$

Using the theorem of Castigliano to solve the problem, we derive with

$$M = \frac{1}{2}FR(1-\cos\varphi)+M_A \qquad \text{and} \qquad \Pi = \int\frac{M^2}{2EI}\mathrm{d}s$$

and by using the fact that the slope at A has to vanish

$$\psi_A = \frac{\partial\Pi}{\partial M_A} = 0$$

the result

$$\int M\frac{\partial M}{\partial M_A}\mathrm{d}s = 0 \quad \rightsquigarrow \quad 2\int\limits_0^{\pi/2}\left[\frac{1}{2}FR(1-\cos\varphi)+M_A\right]\cdot 1\cdot R\,\mathrm{d}\varphi = 0$$

$$\rightsquigarrow \quad \underline{\underline{M_A = -FR\left[\frac{1}{2}-\frac{1}{\pi}\right]}} \qquad \text{and} \qquad \underline{\underline{M = \frac{1}{2}FR\left(\frac{2}{\pi}-\cos\varphi\right)}}\,.$$

The displacement f_F is computed from

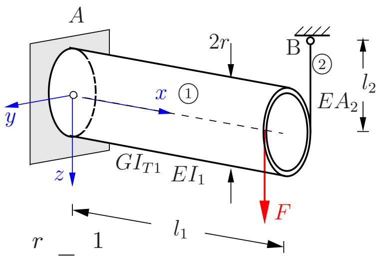
$$\underline{\underline{f_F}} = \frac{\partial\Pi}{\partial F} = \frac{1}{EI}\int M\frac{\partial M}{\partial F}\mathrm{d}s$$

$$= \frac{2}{EI}\int\limits_0^{\pi/2}\left[\frac{FR}{2}\left(\frac{2}{\pi}-\cos\varphi\right)\right]\left[\frac{R}{2}\left(\frac{2}{\pi}-\cos\varphi\right)\right]R\,\mathrm{d}\varphi = \underline{\underline{\frac{FR^3}{8EI}\left(\pi-\frac{8}{\pi}\right)}}\,.$$

P5.15

Problem 5.15 The depicted pipe ① is clamped on one side and supported by the additional rope ② .

Determine the support reactions in A and B, if the pipe is loaded by the force F.



Given: $\dfrac{G_1}{E_1} = \dfrac{3}{8}\,, \quad \dfrac{EI_1 l_2}{EA_2 l_1^3} = \dfrac{1}{100}\,, \quad \dfrac{r}{l_1} = \dfrac{1}{10}\,.$

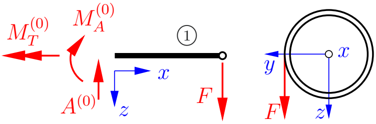
Solution The system is statically indeterminate. We choose the reaction force in B as statically redundant force. This leads to the following "0"- and "1"-system:

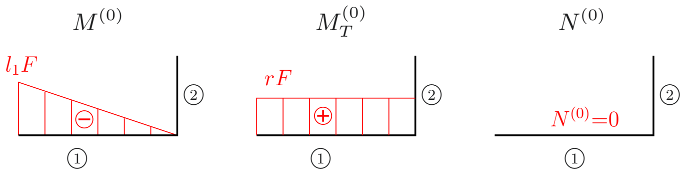
"0"-System:

$$A^{(0)} = F\,,$$

$$M_A^{(0)} = -l_1 F\,,$$

$$M_T^{(0)} = rF\,.$$



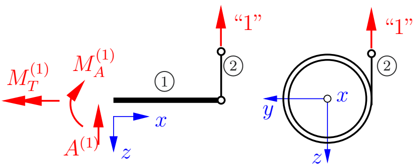


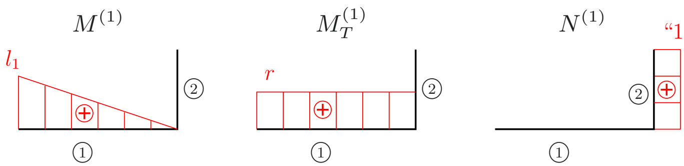
"1"-System:

$$A^{(1)} = -1\,,$$

$$M_A^{(1)} = l_1\,,$$

$$M_T^{(1)} = r\,.$$





From the constraint, that the displacement at B has to vanish,

$$f_B = 0 = \int \frac{M\overline{M}}{EI}\,\mathrm{d}x + \int \frac{M_T\overline{M}_T}{GI_T}\,\mathrm{d}x + \int \frac{N\overline{N}}{EA}\,\mathrm{d}x\ ,$$

we obtain with

$$M = M^{(0)} + X \cdot M^{(1)}\ , \qquad M_T = M_T^{(0)} + X \cdot M_T^{(1)}\ ,$$

$$N = N^{(0)} + X \cdot N^{(1)}\ , \quad \overline{M} = M^{(1)}\ , \quad \overline{M}_T = M_T^{(1)}\ , \quad \overline{N} = N^{(1)}$$

the unknown force $X = B$

$$X = B = -\frac{\displaystyle\int \frac{M^{(0)}M^{(1)}}{EI_1}\,\mathrm{d}x + \int \frac{M_T^{(0)}M_T^{(1)}}{GI_{T1}}\,\mathrm{d}x + \int \frac{N^{(0)}N^{(1)}}{EA_2}\,\mathrm{d}x}{\displaystyle\int \frac{M^{(1)}M^{(1)}}{EI_1}\,\mathrm{d}x + \int \frac{M_T^{(1)}M_T^{(1)}}{GI_{T1}}\,\mathrm{d}x + \int \frac{N^{(1)}N^{(1)}}{EA_2}\,\mathrm{d}x}$$

$$= -\frac{\dfrac{1}{EI_1}\dfrac{1}{3}l_1(-l_1F)l_1 + \dfrac{1}{GI_{T1}}l_1(rF)r + 0}{\dfrac{1}{EI_1}\dfrac{1}{3}\,l_1\,l_1\,l_1 + \dfrac{1}{GI_{T1}}l_1 r\,r + \dfrac{1}{EA_2}l_2 \cdot 1 \cdot 1}\ .$$

Using $I_{T1} = 2I_1$ (circular cross section!) and the given relations leads to

$$\underline{\underline{X = B = \frac{96}{107}\,F}}\ .$$

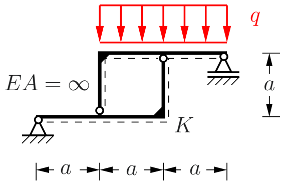
The support reactions at A are given by

$$\underline{\underline{A}} = A^{(0)} + X \cdot A^{(1)} = F - \frac{96}{107}\,F = \underline{\underline{\frac{11}{107}\,F}}\ ,$$

$$\underline{\underline{M_A}} = M_A^{(0)} + X \cdot M_A^{(1)} = -l_1F + \frac{96}{107}\,l_1F = \underline{\underline{-\frac{11}{107}\,l_1F}}\ ,$$

$$\underline{\underline{M_{TA}}} = M_T^{(0)} + X \cdot M_T(1) = rF - \frac{96}{107}\,rF = \underline{\underline{\frac{11}{107}\,rF}}\ .$$

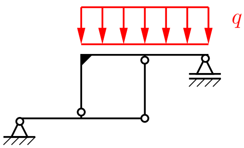
P5.16 **Problem 5.16** Determine the second moment of area I_y for the depicted statically indeterminate structure, such that the vertical displacement at point K is exactly $w_K = 1\,\text{cm}$.

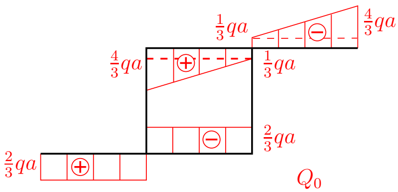


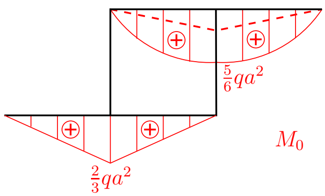
Given: $E = 21 \cdot 10^7$ kN/m^2,
$a = 3$ m,
$q = 5$ kN/m.

Solution To determine the displacement at point K we first have to compute the stress resultants in the statically indeterminate system. For this a hinge is introduced at K.

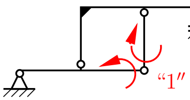
"0"-system:

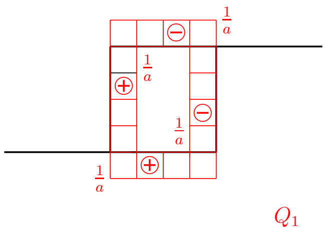


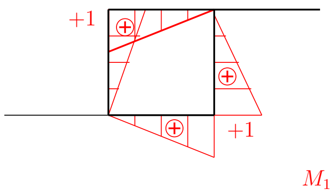




"1"-system:







The rotations at point K in the "0"- and "1"-system are given by

$$EI_y\delta_{10} = \frac{1}{6}a\cdot 1\cdot\frac{2}{3}qa^2 + \frac{1}{6}a\cdot 1\cdot\frac{5}{6}qa^2 + \frac{1}{3}a\cdot 1\cdot\frac{1}{8}qa^2 = \frac{7}{24}qa^3\ ,$$

$$EI_y\delta_{11} = 4\cdot\left(\frac{1}{3}\cdot a\cdot 1^2\right) = \frac{4}{3}a\ .$$

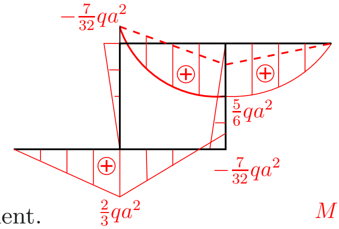
With

$$\delta_{10} + X_1\delta_{11} = 0$$

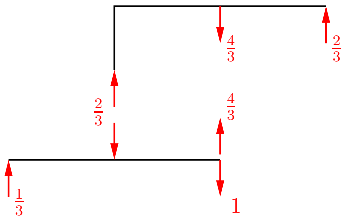
we determine the statically redundant quantity (bending moment at K)

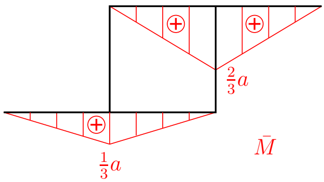
$$X_1 = -\frac{7}{32}\,qa^2\ ,$$

which leads to the total bending moment.



To compute the displacement of point K we apply in the statically determinate "0"-system a force "1"and compute the bending moments.





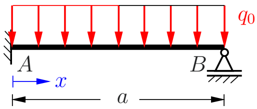
$$\begin{aligned}EI_y\delta_{1K} &= 2\cdot\left(\frac{1}{3}a\frac{a}{3}\frac{2}{3}qa^2\right) - \frac{1}{6}a\frac{a}{3}\left(-\frac{7}{32}qa^2\right)\\ &\quad+\frac{1}{6}a\frac{2a}{3}\left(-\frac{7}{32}qa^2\right) + 2\left(\frac{1}{3}a\frac{2a}{3}\frac{5}{6}qa^2\right) + 2\left(\frac{1}{3}a\frac{2a}{3}\frac{1}{8}qa^2\right)\\ &= qa^4\left(\frac{4}{27} - \frac{7}{576} - \frac{14}{576} + \frac{10}{27} + \frac{1}{18}\right)\\ &= \frac{929}{1728}qa^4.\end{aligned}$$

The required second moment of area results from condition $\delta_{1K} = w_K$

$$\underline{\underline{I_y}} = \frac{1}{Ew_K}\,\frac{929}{1728}\,qa^4 = \frac{1}{21\cdot 10^3}\,\frac{929}{1728}\,\frac{5}{100}\,300^4 = \underline{\underline{10368\ \text{cm}^4}}\ .$$

P 5.17 **Problem 5.17** The depicted beam with flexural rigidity EI is statically indeterminately supported.

Compute the deflection at the center of the beam.

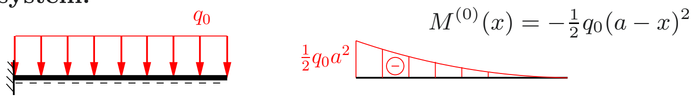


Solution We regard the reaction force in B as statical redundant force and use the principle of virtual forces to determine B from the constraint

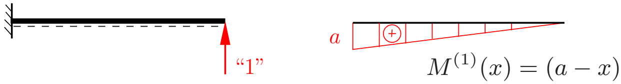
$$f_B = \int \frac{M\overline{M}}{EI}\,\mathrm{d}x = 0\,.$$

For the "0"- and "1"-system we obtain:

"0"-system:



"1"-system:

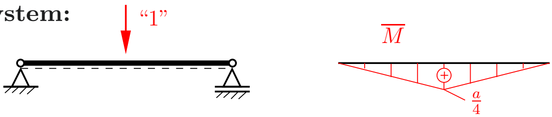


With the help of the table on page 146 we deduct

$$X = B = -\frac{\int M^{(0)}M^{(1)}\mathrm{d}x}{\int M^{(1)}M^{(1)}\mathrm{d}x} = \frac{3}{8}\,q_0\,a\,.$$

To determine the vertical displacement, the beam is considered as a simply supported beam on two supports. For this situation we compute the bending moment due to the ("2"-system) under a virtual load.

"2"-system:

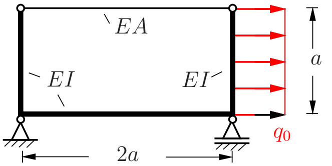


With $\overline{M} = M^{(2)}$ and $M = M^{(0)} + XM^{(1)}$ it follows

$$\begin{aligned}
\underline{\underline{f_V}} &= \frac{1}{EI}\int (M^{(0)} + XM^{(1)})M^{(2)}\mathrm{d}x \\
&= \frac{1}{EI}\int (M^{(0)}M^{(2)})\mathrm{d}x + \frac{X}{EI}\int (M^{(1)}M^{(2)})\mathrm{d}x \\
&= \frac{1}{EI}\left(-\frac{7}{384}\,q_0\,a^4 + \frac{X}{16}\,a^3\right) = \underline{\underline{\frac{qa^4}{192\,EI}}}\,.
\end{aligned}$$

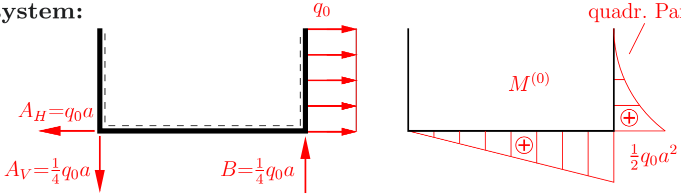
Problem 5.18 The sketched frame (axial rigidity $EA \to \infty$, flexural rigidity EI) is closed by an elastic truss (axial rigidity EA). The system is subjected to a constant line load q_0 **P5.18**

Compute the force in the truss.

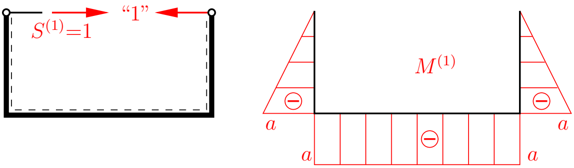


Solution The system is internally statically indeterminate. We choose the force in the truss as statically redundant force. From the principle of virtual forces follow the basic and auxiliary system:

"0"-system:



"1"-System:



From the condition, that difference in the displacement of the frame and the end of the truss has to vanish,

$$\Delta f = \frac{1}{EI} \int M\overline{M}\, \mathrm{d}x + \frac{S\overline{S}2a}{EA} = 0\,,$$

follows together with

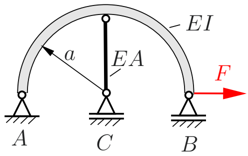
$$M = M^{(0)} + X \cdot M^{(1)}\,, \quad \overline{M} = M^{(1)}\,, \quad S = X\,, \quad \overline{S} = S^{(1)} = 1$$

the force in the truss

$$\underline{\underline{X}} = \frac{-\frac{1}{EI}\int M^{(0)} M^{(1)} \mathrm{d}x}{\frac{1}{EI}\int M^{(1)} M^{(1)} \mathrm{d}x + \frac{2a}{EA}} = -\frac{\frac{1}{2} 2a\left(\frac{1}{2} q_0 a^2\right)(-a) + \frac{1}{4} a \left(\frac{1}{2} q_0 a^2\right)(-a)}{2\left[\frac{1}{3} a(-a)(-a)\right] + 2a(-a)(-a) + \frac{2aEI}{EA}}$$

$$= \underline{\underline{\frac{15}{64} \frac{1}{1 + \frac{3EI}{4EAa^2}} q_0 a}}\,.$$

P5.19 **Problem 5.19** The semicircular arc (flexural rigidity EI) is supported by a bar (axial rigidity EA) and loaded by a force F.



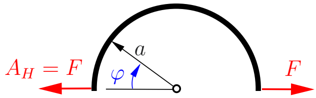
Compute the force in the bar and the deflection at the connection point of bar and arc.

Solution The system is statically indeterminate. We use the principle of virtual forces and the force in the bar as statically redundant force. This leads to the following "0"- and "1"-system:

"0"-system:

$$M^{(0)}(\varphi) = Fa\sin\varphi\,,$$

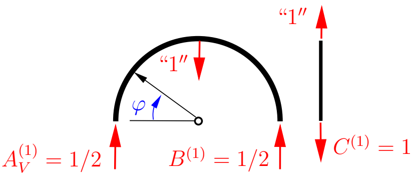
$$S^{(0)} = 0\,.$$



"1"-system:

$$M^{(1)}(\varphi) = \frac{1}{2}\,a(1-\cos\varphi)\,,$$

$$S^{(1)} = 1\,.$$



The difference in displacements of arc and bar has to vanish:

$$\Delta f = \frac{1}{EI}\int M\overline{M}\,\mathrm{d}x + \frac{S\,\overline{S}a}{EA} = 0\,, \quad \text{with}$$

$$M = M^{(0)} + X\cdot M^{(1)}\,, \quad \overline{M} = M^{(1)}\,, \quad S = X\,, \quad \overline{S} = S^{(1)} = 1\,.$$

This condition provides the force in the bar

$$\underline{\underline{S}} = \frac{-\dfrac{1}{EI}\displaystyle\int M^{(0)}M^{(1)}\mathrm{d}x}{\dfrac{1}{EI}\displaystyle\int M^{(1)}M^{(1)}\mathrm{d}x + \dfrac{a}{EA}} = -\frac{2\dfrac{Fa}{2}\displaystyle\int_0^{\pi/2}\sin\varphi(1-\cos\varphi)\mathrm{d}x}{2\dfrac{a^2}{4}\displaystyle\int_0^{\pi/2}(1-\cos\varphi)^2\mathrm{d}x + \dfrac{aEI}{EA}}$$

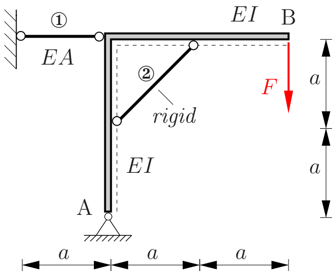
$$= \underline{\underline{-\frac{4\,F}{(3\pi-8) + 8\,\dfrac{EI}{EAa^2}}}}\,.$$

The deflection f of the arc is given by the deformation of the bar:

$$\underline{\underline{f}} = -\frac{S\,a}{EA} = \underline{\underline{\frac{4\,Fa}{(3\pi-8)EA + 8\,\dfrac{EI}{a^2}}}}\,.$$

Problem 5.20 The depicted frame (axial rigidity $EA \to \infty$, flexural rigidity EI) is loaded by a force F. The bar ① has the axial rigidity EA, while bar ② is considered to be *rigid*.

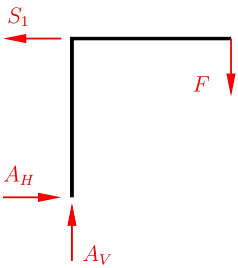
P5.20



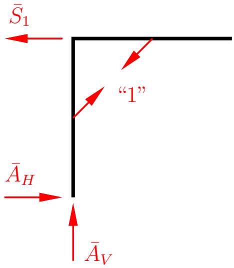
Determine the force S_2 in bar ② and the vertical displacement v_B at point B.

Solution The system is statically determinate supported, but due to the rigid truss internally statically indeterminate. To compute the force S_2 in bar ② we use the following "0"- and "1"-systems. The reaction forces and the force in bar ① can be determined from equilibrium conditions.

"0"-system:



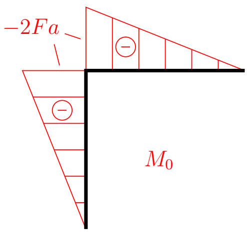
"1"-system:



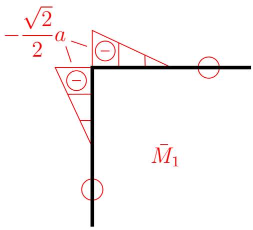
$A_V = F$, $A_H = F$, $S_1 = F$ $\qquad$ $\bar{A}_V = \bar{A}_H = \bar{S}_1 = 0\,,\ \bar{S}_2 = 1$

The bending moments M_0 and $\bar{M}_1$ of both systems are sketched below:

"0"-System:



"1"-System:



When evaluating the principle of virtual forces we have to consider that the bar is *rigid*. This yields

$$\alpha_{10} = \frac{1}{EI}\int M_0\bar{M}_1 \mathrm{d}x = \frac{1}{EI}\cdot\frac{1}{6}a(-Fa-4Fa)\frac{\sqrt{2}}{2}a(-1)\cdot 2 = \frac{5\sqrt{2}}{6EI}Fa^3\,,$$

$$\alpha_{11} = \frac{1}{EI}\int \bar{M}_1^2 \mathrm{d}x = \frac{1}{EI}\cdot\frac{1}{3}a\cdot\frac{1}{2}a^2\cdot 2 = \frac{a^3}{3EI}\,,$$

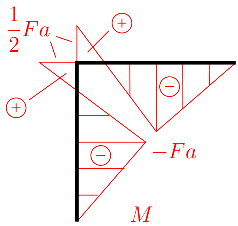
and for the force in bar ② we obtain

$$\underline{\underline{S_2}} = X = -\frac{\alpha_{10}}{\alpha_{11}} = -\frac{5\sqrt{2}}{6EI}Fa^3\cdot\frac{3EI}{a^3} = \underline{\underline{-\frac{5\sqrt{2}}{2}F}}\,.$$

At point B the vertical displacement v_B and the force F have the same direction, hence we can use the energy theorem:

$$\frac{1}{2}Fv_B = \frac{1}{2}\int\frac{M^2}{EI}\mathrm{d}x + \frac{1}{2}\sum_i\frac{S_i^2 l_i}{EA}\,.$$

Its application is based on the bending moment in the total system $M = M_0 + X\bar{M}_1$:



Evaluation the integrals using the bending moment M yields

$$\int\frac{M^2}{EI}\mathrm{d}x = \frac{1}{EI}\left(\frac{1}{3}aF^2a^2 + \frac{1}{3}\cdot\frac{2}{3}aF^2a^2 + \frac{1}{3}\cdot\frac{1}{3}a\frac{1}{4}F^2a^2\right)\cdot 2 = \frac{7}{6EI}F^2a^3$$

Furthermore, with the force S_1 in the bar (note, bar ② is *rigid*)

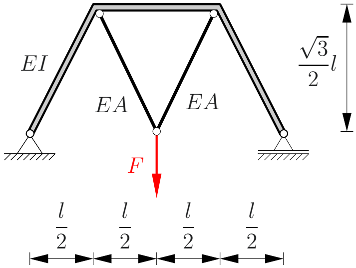
$$\sum_i\frac{S_i^2 l_i}{EA} = \frac{F^2 a}{EA}$$

we compute the vertical displacement

$$\underline{\underline{v_B = \left(\frac{7a^2}{6EI} + \frac{1}{EA}\right)Fa\,.}}$$

Problem 5.21 A trapezoidal, frame (axial rigidity EA, flexural rigidity EI) with two bars (axial rigidity EA) is loaded by a force F. **P5.21**

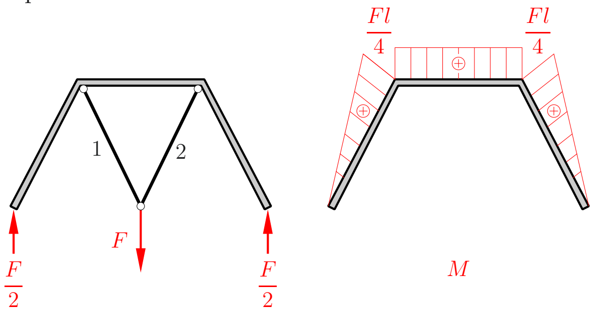
Compute the horizontal and vertical displacement at the loading point.



Solution Since the vertical displacement v_F of the load has the same direction as load F, we can determine v_F by the conservation of energy:

$$\frac{1}{2}Fv_F = \frac{1}{2}\int \frac{M^2}{EI}\,\mathrm{d}x + \frac{1}{2}\sum_i \frac{S_i^2 l_i}{EA_i}\,.$$

The structure is statically determinate supported. Thus reaction forces, stress resultants, and the forces in the bars can be determined from equilibrium conditions.



$$S_1 = S_2 = \frac{F}{\sqrt{3}}$$

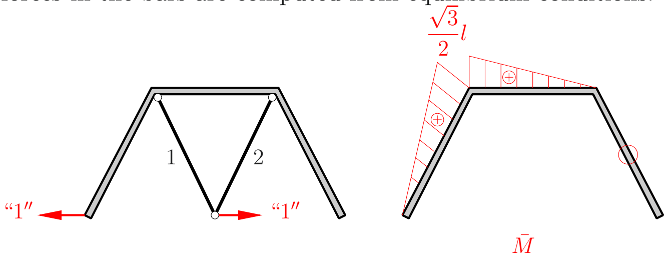
Using the bending moment and the bar forces yields

$$Fv_F = \frac{1}{EI}\left[2\frac{1}{3}\left(\frac{Fl}{4}\right)^2 l + \left(\frac{Fl}{4}\right)^2 l\right] + \frac{1}{EA}\left[2\left(\frac{F}{\sqrt{3}}\right)^2 l\right]\,.$$

Thus we compute for the vertical displacement

$$\underline{\underline{v_F = \frac{11}{12}\frac{Fl^3}{EI} + \frac{Fl}{3\,EA}}}\,.$$

The horizontal displacement at the loading point follows by loading with a virtual "1" force in horizontal direction. Bending moment and forces in the bars are computed from equilibrium conditions.



$$\bar{S}_1 = 1$$
$$\bar{S}_2 = -1$$

Using the "0"- and "1"-system we determine the horizontal displacement

$$u_F = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x + \sum_i \frac{S_i\bar{S}_i l_i}{EA_i} = \frac{1}{EI}\left(\frac{1}{3}\frac{Fl}{4}\frac{\sqrt{3}}{2}l\cdot l + \frac{1}{2}\frac{Fl}{4}\frac{\sqrt{3}}{2}l\cdot l\right)$$

with the final result

$$\underline{\underline{u_F = \frac{5\sqrt{3}}{48}\frac{Fl^3}{EI}}}\,.$$

Note: The deformation of the bars and the right part of the frame do not contribute to the horizontal displacement.

Chapter 6

Buckling of Bars

6

The total potential of elastic systems loaded by conservative forces consists of an external potential $\Pi^{(e)}$ of the applied forces and the potential (strain energy) $\Pi^{(i)}$ of the internal forces:

$$\Pi = \Pi^{(e)} + \Pi^{(i)} \,.$$

In an **equilibrium** state,

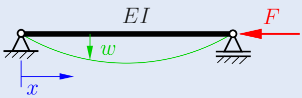
$$\boxed{\delta\Pi = 0}$$

holds.
Formal application of the **stability conditions** for rigid bodies (see book 1, chapter 7) to elastic systems yields

$$\delta^2\Pi = \delta^2\Pi^{(e)} + \delta^2\Pi^{(i)} \begin{cases} > 0 & \text{stable equilibrium,} \\ = 0 & \text{indifferent equilibrium,} \\ < 0 & \text{unstable equilibrium.} \end{cases}$$

The *critical load* of an elastic system is reached, if the equilibrium is *indifferent*. Besides the original equilibrium state, neighboring equilibrium states exist related to deformation ("buckling"). Critical loads and associated equilibirum states can be determined from equilibrium conditions in the deformed state or by investigating $\delta^2\Pi$.

For an elastic bar under compression equilibrium conditions in the deformed state provide the **differential equation for Euler's column**

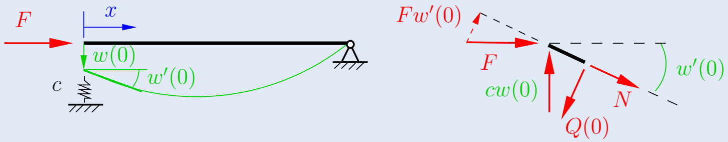


$$\boxed{w^{IV} + \lambda^2 w'' = 0\,,} \qquad \lambda^2 = \frac{F}{EI}$$

with the general solution

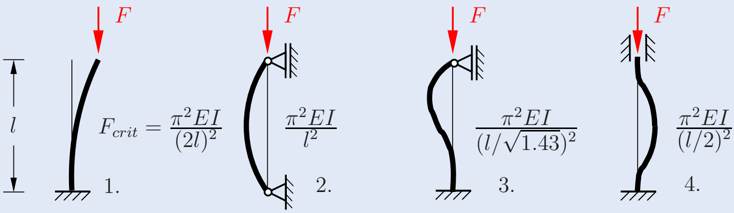
$$w = A\cos\lambda x + B\sin\lambda x + C\lambda x + D \,.$$

The constants A, B, C and D are determined from the *boundary conditions* for the kinematic and static quantities. Note that these conditions have to be formulated in the deformed state. For example, under the assumption of small rotations an elastic support at position $x = 0$ is described by



$$
\begin{aligned}
Q(0) &= c\,w(0) - Fw'(0) &&\leadsto\quad EIw'''(0) + c\,w(0) - Fw'(0) = 0\,,\\
M(0) &= 0 &&\leadsto\quad EIw''(0) = 0\,.
\end{aligned}
$$

Four characteristic boundary conditions establish the **Euler buckling cases**:

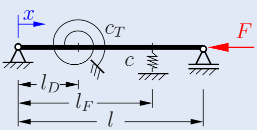


Approximate solutions for the critical load can be obtained by using the ansatz $\tilde{w}(x)$ in the energy functional for buckling (**Rayleigh-quotient**):

$$
\Pi = \frac{1}{2}\int_0^l \left(EI\tilde{w}''^2\,\mathrm{d}x - \tilde{F}_{\text{crit}}\tilde{w}'^2 \right)\mathrm{d}x \;= 0 \quad\leadsto\quad \boxed{\tilde{F}_{\text{crit}} = \frac{\int_0^l EI\tilde{w}''^2\,\mathrm{d}x}{\int_0^l \tilde{w}'^2\,\mathrm{d}x}}\;.
$$

To determine $\tilde{F}_{\text{crit}}$, the ansatz $\tilde{w}(x)$ has to satisfy the essential (kinematic) boundary conditions (note that the result for $\tilde{F}_{\text{crit}}$ improves, if $\tilde{w}(x)$ satisfies also the static boundary conditions). The approximate solution is in general on the unsafe side, because the inequality $\tilde{F}_{\text{crit}} \geq F_{\text{crit}}$ holds.

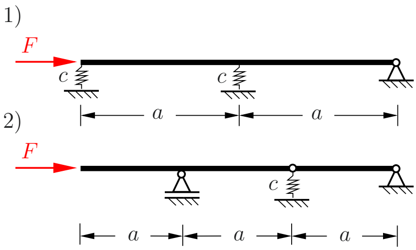
Individual springs at position x_i are included in the nominator by $c[\tilde{w}(x_i)]^2$, while torsion springs are incorporated by $c_T[\tilde{w}'(x_i)]^2$:



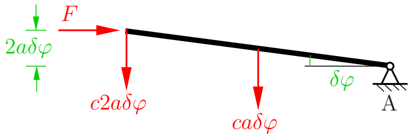
$$
\tilde{F}_{\text{crit}} = \frac{\int_0^l EI\tilde{w}''^2\,\mathrm{d}x + c_T[\tilde{w}'(l_D)]^2 + c[\tilde{w}(l_F)]^2}{\int_0^l \tilde{w}'^2\,\mathrm{d}x}.
$$

P6.1 **Problem 6.1** Both depicted systems consist of rigid bars supported by elastic springs.

Determine the critical loads F_{crit}.



Solution **to 1)** We consider the system in the deflected state. From equilibrium



$$\overset{\curvearrowleft}{A}: \quad a(ca\delta\varphi) + 2a(2ca\delta\varphi) - 2a\delta\varphi F = 0$$

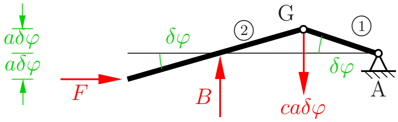
we obtain

$$\delta\varphi(5ca - 2F) = 0 \,.$$

Thus a neighboring equilibrium state ($\delta\varphi \neq 0$) related to the equilibrium state $\varphi = 0$ is only possible for

$$\underline{\underline{F_{\text{crit}} = \frac{5}{2}\, ca}} \,.$$

to 2) Equilibrium conditions for the defelcted state



$$\overset{\curvearrowleft}{A}: \quad a(ca\delta\varphi) - 2aB + a\delta\varphi F = 0 \,,$$

$$② \quad \overset{\curvearrowleft}{G}: \quad 2a\delta\varphi\, F - aB = 0$$

provide after elimination of B

$$\delta\varphi(ca - 3F) = 0 \,.$$

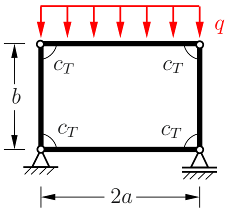
This results in the critical force

$$\underline{\underline{F_{\text{crit}} = \frac{ca}{3}}} \,.$$

Problem 6.2 The depicted frame consists of four rigid bars connected by hinges and torsional springs with stiffness c_T.

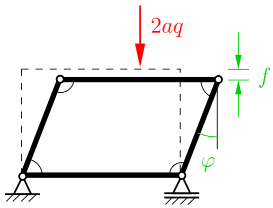
P6.2

Determine the critical load q_{crit}.



Solution From the sketched deflected state follows the geometric relation:

$$f = b\,(1 - \cos\varphi)\ .$$



Hence the potential energy is given by

$$\begin{aligned}\Pi &= \Pi^{(i)} + \Pi^{(a)}\\ &= 4\,\frac{1}{2}c_T\,\varphi^2 - 2qaf\\ &= 2c_T\,\varphi^2 - 2qab(1-\cos\varphi)\ .\end{aligned}$$

The system is in equilibrium, if

$$\delta\Pi = \frac{d\Pi}{d\varphi}\delta\varphi = (4c_T\varphi - 2qab\sin\varphi)\delta\varphi = 0\ .$$

Thus for equilibrium in the deflected state with $\delta\varphi \neq 0$, we must have

$$4c_T\varphi - 2qab\sin\varphi = 0/;.$$

The trivial equilibrium state is related to $\varphi = 0$.

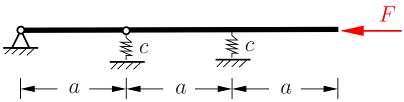
Using the second variation of the potential energy we can determine the type of the equilibrium

$$\delta^2\Pi = \frac{d^2\Pi}{d\varphi^2}(\delta\varphi)^2 = (4c_T - 2qab\cos\varphi)(\delta\varphi)^2 \quad \begin{cases} > 0 & \text{stable,}\\ = 0 & \text{indifferent,}\\ < 0 & \text{unstable}\end{cases}$$

At the trivial equilibrium state ($\varphi = 0$) the system is indifferent for the critical load

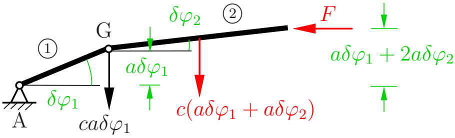
$$\underline{\underline{q_{\text{crit}} = \frac{2c_T}{a\,b}}}\ .$$

P6.3 **Problem 6.3** The depicted system consists of rigid bars that are elastically supported.



Determine the critical loads and sketch the associated buckling figures.

Solution The system has two degrees of freedom. The equilibrium conditions in the deflected state are



$$\overset{\curvearrowright}{A}: \quad ca^2\delta\varphi_1 + 2ca^2(\delta\varphi_1 + \delta\varphi_2) - a(\delta\varphi_1 + 2\delta\varphi_2)F = 0\ ,$$

$$② \ \overset{\curvearrowright}{G}: \quad ca^2(\delta\varphi_1 + \delta\varphi_2) - 2a\delta\varphi_2 F = 0\ .$$

Using $\lambda = F/ca$ we obtain the homogeneous system of equations

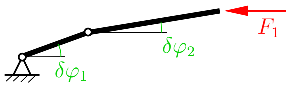
$$(3-\lambda)\delta\varphi_1 + 2(1-\lambda)\delta\varphi_2 = 0\ ,$$
$$1\cdot\delta\varphi_1 + (1-2\lambda)\delta\varphi_2 = 0\ .$$

For a non-trivial solution the determinant of the coefficient matrix has to vanish:

$$\begin{vmatrix} (3-\lambda) & 2(1-\lambda) \\ 1 & (1-2\lambda) \end{vmatrix} = 0 \quad \leadsto \quad \lambda^2 - \frac{5}{2}\lambda + \frac{1}{2} = 0 \quad \leadsto \quad \begin{aligned} \lambda_1 &= \frac{5+\sqrt{17}}{4}\ , \\ \lambda_2 &= \frac{5-\sqrt{17}}{4}\ . \end{aligned}$$

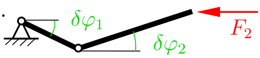
Re-substituting provides

$$\underline{\underline{F_1 = \frac{5+\sqrt{17}}{4}\, c\, a\ ,}} \qquad \delta\varphi_1 = \frac{3+\sqrt{17}}{2}\,\delta\varphi_2$$



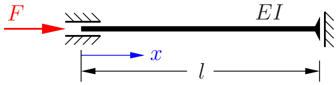
and

$$\underline{\underline{F_2 = \frac{5-\sqrt{17}}{4}\, c\, a\ ,}} \qquad \delta\varphi_1 = -\frac{\sqrt{17}-3}{2}\,\delta\varphi_2 .$$



The originally straight equilibrium state can buckle into two neighboring states, because the system has two degrees of freedom. Since $F_2 < F_1$, force F_2 is the critical load: $F_{\text{crit}} = F_2$.

Problem 6.4 Determine for the depicted elastic bar the conditions for buckling and the critical load. P6.4



Solution From the general solution of the Euler coloumn

$$
\begin{aligned}
w &= A\cos\lambda x + B\sin\lambda x + C\lambda x + D\,, \qquad \lambda^2 = \frac{F}{EI}\,,\\
w' &= -A\lambda\sin\lambda x + B\lambda\cos\lambda x + C\lambda\,,\\
w'' &= -M/EI = -A\lambda^2\cos\lambda x - B\lambda^2\sin\lambda x\,,\\
w''' &= -Q/EI = A\lambda^3\sin\lambda x - B\lambda^3\cos\lambda x
\end{aligned}
$$

in conjunction with the boundary conditions, we derive

$$
\begin{aligned}
w(0) = 0 &: \quad \leadsto \quad A + D = 0 \quad \leadsto \quad D = -A\,,\\
w'(0) = 0 &: \quad \leadsto \quad B + C = 0 \quad \leadsto \quad C = -B\,,\\
w'(l) = 0 &: \quad \leadsto \quad -A\sin\lambda l + B\cos\lambda l + C = 0\,,\\
Q(l) = 0 &: \quad \leadsto \quad A\sin\lambda l - B\cos\lambda l = 0\,.
\end{aligned}
$$

Inserting $C = -B$ yields for the last two equations

$$A\sin\lambda l - B(\cos\lambda l - 1) = 0\,,$$

$$A\sin\lambda l - B\cos\lambda l = 0\,.$$

This leads to $B = 0$, and the condition of buckling is given by

$$\underline{\underline{\sin\lambda l = 0}} \qquad \leadsto \qquad \underline{\underline{\lambda_n l = n\pi}} \qquad (n = 1, 2, 3, \ldots)\,.$$

The smallest eigenvalue $\lambda_1 l = \pi$ provides the critical load

$$\underline{\underline{F_{\text{crit}} = \pi^2\frac{EI}{l^2}}}\,.$$

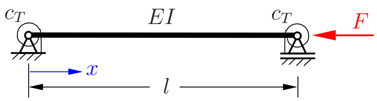
Inserting the constants and the eigenvalue yields the buckling shape

$$w = A(\cos\frac{\pi x}{l} - 1)\,,$$

where A remains undetermined.

Note: The critical load for the considered case is identical to the 2nd Euler buckling case.

P6.5 **Problem 6.5** The depicted beam is subjected to axial compression and is supported at both ends elastically by torsional springs.



Given.: $EI = l\,c_T$.

a) Determine the critical load.

b) Compute with the ansatz $\tilde{w}_1(x) = a(l-x)x$ and $\tilde{w}_2(x) = a\sin(\pi x/l)$ the approximate solution for the critical load via the Rayleigh-quotient.

Solution to a) The general solution of the buckling problem

$$w = A\cos\lambda x + B\sin\lambda x + C\lambda x + D\,, \qquad \lambda^2 = \frac{F}{EI}$$

yields with the boundary conditions

$$w(0) = 0\,, \quad M(0) = -EIw''(0) = -c_T w'(0)\,,$$
$$w(l) = 0\,, \quad M(l) = -EIw''(l) = +c_T w'(l)$$

and with the abbreviation $\kappa = EI/lc_T$ the following system of equations

$$A + D = 0\,,$$
$$\kappa A\lambda^2 l = -B\lambda - C\lambda\,,$$
$$A\cos\lambda l + B\sin\lambda l + C\lambda l + D = 0\,,$$
$$\kappa A\lambda^2 l\cos\lambda l + \kappa B\lambda^2 l\sin\lambda l = -A\lambda\sin\lambda l + B\lambda\cos\lambda l + C\lambda\,.$$

Elimination of the constants yields an equation for the eigenvalues

$$2 - 2(1 + \kappa\lambda^2 l^2)\cos\lambda l - \lambda l[1 - (\kappa\lambda l)^2 - 2\kappa]\sin\lambda l = 0\,.$$

For $\kappa = 1$ we deduce from this equation (e. g. by a graphical solution method) the first eigenvalue and the associated critical load

$$\lambda_1 l = 3.67 \qquad \leadsto \qquad \underline{\underline{F_{\text{crit}}}} = \lambda_1^2 EI = \underline{\underline{13.49\,\frac{EI}{l^2}}}\,.$$

Note: We obtain the eigenvalue equations for a beam clamped on both ends as special case ($\kappa = 0$ or $c_T \to \infty$)

$$2 - 2\cos\lambda l - \lambda l\sin\lambda l = 0 \quad \leadsto \quad \lambda l = 2\pi\,,$$

and similarily for the simply supported beam $\kappa \to \infty$ or $c_T \to 0$)

$$\sin\lambda l = 0 \quad \leadsto \quad \lambda l = \pi\,.$$

to b) To determine the critical load with the first ansatz we need to compute the derivatives:

$$\tilde{w}_1(x) = a\,(lx - x^2), \quad \tilde{w}_1'(x) = a\,(l - 2x), \quad \tilde{w}_1''(x) = -2a\ .$$

Substituting this into the formula of the Rayleigh-quotient yields:

$$\tilde{F}_{\text{crit }1} = \frac{\int\limits_0^l EI \cdot (-2a)^2 \,\mathrm{d}x + c_T\,[a\,(l-0)]^2 + c_T\,[a\,(l-2l)]^2}{\int\limits_0^l [a\,(l-2x)]^2\,\mathrm{d}x}\ .$$

Integration and rearrangement yields

$$\tilde{F}_{\text{crit }1} = \frac{\Big[4a^2EIx\Big]_0^l + c_T a^2 l^2 + c_T a 2 l^2}{\Big[a^2l^2x - 2a^2lx^2 + \frac{4}{3}a^2x^3\Big]_0^l} = \frac{4a^2lEI + c_Ta^2l^2 + c_Ta^2l^2}{a^2l^3 - 2a^2l^3 + \frac{4}{3}a^2l^3}\ .$$

Inserting $lc_T = EI$ leads to the final result

$$\underline{\underline{\tilde{F}_{\text{crit }1} = 18\,\frac{EI}{l^2}}}\ .$$

Analogously the second ansatz renders step by step

$$\tilde{w}_2'(x) = \frac{\pi}{l}a\cos\left(\frac{\pi}{l}x\right), \quad \tilde{w}_2''(x) = -\left(\frac{\pi}{l}\right)^2 a\sin\left(\frac{\pi}{l}x\right),$$

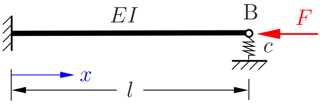
$$\tilde{F}_{\text{crit }2} = \frac{\int\limits_0^l EI\big[-\big(\frac{\pi}{l}\big)^2 a\sin\big(\frac{\pi}{l}x\big)\big]^2\mathrm{d}x + c_T\big(\frac{\pi}{l}a\big)^2\big[\cos^2\big(\frac{\pi}{l}0\big) + \cos^2\big(\frac{\pi}{l}l\big)\big]}{\int\limits_0^l \big[\frac{\pi}{l}a\cos\big(\frac{\pi}{l}x\big)\big]^2\,\mathrm{d}x}$$

$$= \frac{EI\left(\frac{\pi}{l}\right)^2\int\limits_0^l \sin^2\left(\frac{\pi}{l}x\right)\,\mathrm{d}x + c_T\left[\cos^2(0) + \cos^2(\pi)\right]}{\int\limits_0^l \cos^2\left(\frac{\pi}{l}x\right)\,\mathrm{d}x}$$

$$= \frac{EI\big(\frac{\pi}{l}\big)^2\left[\frac{1}{2} - \frac{1}{4}\frac{l}{\pi}\sin\big(2\frac{\pi}{l}x\big)\right]_0^l + 2c_T}{\left[\frac{1}{2} + \frac{1}{4}\frac{l}{\pi}\sin\big(2\frac{\pi}{l}x\big)\right]_0^l} = \frac{EI\big(\frac{\pi}{l}\big)^2\left(\frac{1}{2}l - 0\right) + 2c_T}{\frac{1}{2}l - 0},$$

$$\leadsto \qquad \underline{\underline{\tilde{F}_{\text{crit }2}}} = \frac{EI\,(\pi^2+4)}{l^2} = \underline{\underline{13.87\,\frac{EI}{l^2}}}\ .$$

P6.6 **Problem 6.6** A beam is clamped at the left end and is elastically supported at B by a spring (spring constant c).



Deduce the condition for buckling.

Solution The general solution for the buckling problem is given by

$$w = A\cos\lambda x + B\sin\lambda x + C\lambda x + D\,, \qquad \lambda^2 = \frac{F}{EI}\,,$$

$$w' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x + C\lambda\,,$$

$$w'' = -M/EI = -A\lambda^2\cos\lambda x - B\lambda^2\sin\lambda x\,,$$

$$w''' = -Q/EI = A\lambda^3\sin\lambda x - B\lambda^3\cos\lambda x\,.$$

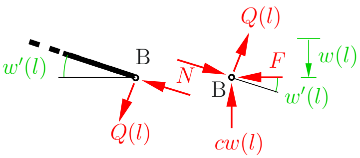
The boundary conditions

$$w(0) = 0\,,$$

$$w'(0) = 0\,,$$

$$M(l) = 0\,,$$

$$Q(l) = -c\,w(l) + F\,w'(l)$$



lead to the homogeneous system of equations

$$A + D = 0\,,$$

$$B + C = 0\,,$$

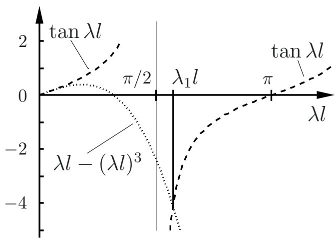
$$-A\cos\lambda l - B\sin\lambda l = 0\,,$$

$$A\cos\lambda l + B\sin\lambda l + C(\lambda l - EI\lambda^3/c) + D = 0\,.$$

Eliminating the constants yields the equation for the eigenvalues (buckling condition)

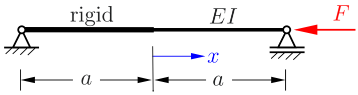
$$\underline{\underline{\tan\lambda l = \lambda l - (\lambda l)^3\frac{EI}{cl^3}}}\,.$$

The solution of this transcendental equation can by obtained graphically. The special case $EI/cl^3 = 1$ yields the first eigenvalue



$$\lambda_1 l \cong 1.81 \quad \leadsto \quad \underline{\underline{F_{\text{crit}} \cong 3.27\,\frac{EI}{l^2}}}\,.$$

Problem 6.7 The depicted beam consists of a rigid and an elastic part (flexural rigidity EI). **P6.7**



Determine the bucking condition and the critical load.

Solution The general solution of the buckling problem is given by

$$w = A\cos\lambda x + B\sin\lambda x + C\lambda x + D\,, \qquad \lambda^2 = \frac{F}{EI}\,.$$

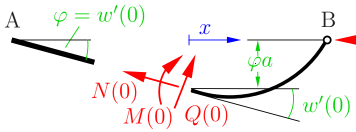
From the boundary and transmission conditions

$$w(a) = 0\,,$$

$$M(a) = -EIw''(a) = 0\,,$$

$$w(0) = \varphi\, a = w'(0)\, a\,,$$

$$Q(0) = Fw'(0)$$



we derive

$$A\cos\lambda a + B\sin\lambda a + C\lambda a + D = 0\,,$$

$$A\cos\lambda a + B\sin\lambda a = 0\,,$$

$$A + D = B\lambda a + C\lambda a\,,$$

$$EI\ B\lambda^3 = F(B\lambda + C\lambda)\,.$$

This yields the constants $C = 0$, $D = 0$, $A = B\lambda a$ and the buckling condition

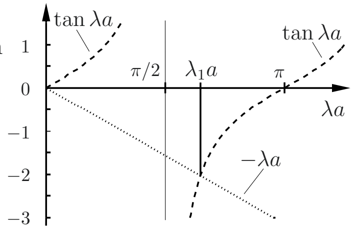
$$\underline{\underline{\tan\lambda a = -\lambda a}}\,.$$

The graphical (or numerical) solution renders the first eigenvalue

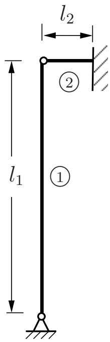
$$\lambda_1 a \cong 2.03$$

and hence the critical load

$$\underline{\underline{F_{\text{crit}} \cong 4.12\,\frac{EI}{a^2}}}\,.$$



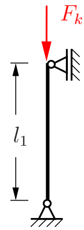
P6.8 **Problem 6.8** Consider the sketched half-frame with different cross section data in the regions ① and ② .



Given:
$$
\begin{aligned}
l_1 &= 5.0 \text{ m} ,\\
l_2 &= 1.0 \text{ m} ,\\
E &= 2.1 \cdot 10^4 \text{ kN/cm}^2 ,\\
\alpha_T &= 1.2 \cdot 10^{-5} \text{ 1/K} ,\\
A_1 &= 50.0 \text{ cm}^2 ,\\
I_1 &= 500 \text{ cm}^4 ,\\
I_2 &= 10000 \text{ cm}^4 .
\end{aligned}
$$

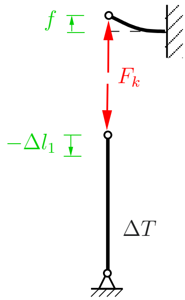
How much can region ① be heated until buckling occurs?

Solution We choose a substitute system for region ① according to Euler case 2 with length l_1. For this case the buckling load is:



$$
F_b = \pi^2 \frac{EI_1}{l_1^2} = \pi^2 \frac{2.1 \cdot 10^4 \cdot 500}{500^2}
$$
$$
= 414.52 \text{ kN} .
$$

The displacements of shaft ① and beam ② are given by:



$$
f = \frac{F_b\, l_2^3}{3EI_2} = \frac{414.52 \cdot 100^3}{3 \cdot 2.1 \cdot 10^4 \cdot 10^4}
$$
$$
= 0.658 \text{ cm} ,
$$

$$
\Delta l_1 = \varepsilon_1 l_1 = -\frac{F_k\, l_1}{EA_1} + \alpha_T \Delta T\, l_1
$$
$$
= -0.1974 + 6 \cdot 10^{-3} \Delta T .
$$

Using compatibility

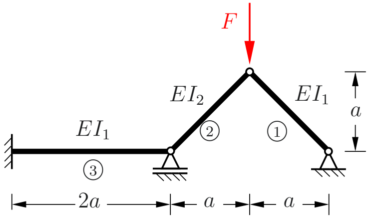
$$
f = \Delta l_1 \quad \rightsquigarrow \quad 0.658 = -0.1974 + 6 \cdot 10^{-3} \Delta T
$$

we can determine the required temperature difference

$$
\underline{\underline{\Delta T = 142.5 \text{ K}}} .
$$

Problem 6.9 The depicted system consists of bars with different flexural rigidity. **P6.9**

Assign the individual bars to the corresponding Euler cases and determine which bar is first buckling for the ratio $EI_2 = 2EI_1$.



Solution The Euler cases are determined from the table on page 183:

Bar ① and bar ② correspond to the second Euler buckling mode, because these bars are hinged at both ends.

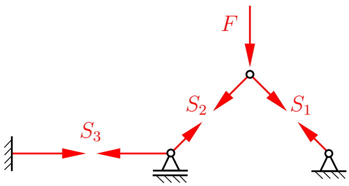
Bar ③ is clamped at the right side and simply supported at the left side. This corresponds to the third Euler case.

The forces due to the load F are given by

$$S_1 = -\frac{F}{\sqrt{2}}\,,$$

$$S_2 = -\frac{F}{\sqrt{2}}\,,$$

$$S_3 = -\frac{F}{2}\,.$$



Thus we obtain the following critical forces

$$\frac{F_{1\,\text{crit}}}{\sqrt{2}} = \frac{\pi^2\,EI_1}{2a^2} \quad\rightsquigarrow\quad F_{1\,\text{crit}} = \frac{1}{\sqrt{2}}\,\frac{\pi^2\,EI_1}{a^2}\,,$$

$$\frac{F_{2\,\text{crit}}}{\sqrt{2}} = \frac{\pi^2\,EI_2}{2a^2} \quad\rightsquigarrow\quad F_{2\,\text{crit}} = \sqrt{2}\,\frac{\pi^2\,EI_1}{a^2}\,,$$

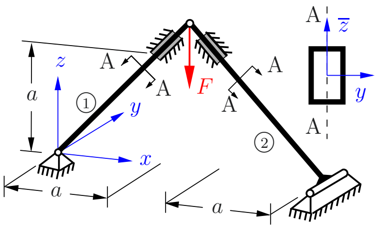
$$\frac{F_{3\,\text{crit}}}{2} = 2.04\,\frac{\pi^2\,EI_1}{4a^2} \quad\rightsquigarrow\quad F_{3\,\text{crit}} = 1.02\,\frac{\pi^2\,EI_1}{a^2}\,.$$

Because $F_{1\,\text{crit}} < F_{3\,\text{crit}} < F_{2\,\text{crit}}$, bar ① buckles first. Hence force $F_{1\,\text{crit}}$ is crucial for the failure of the entire system.

P6.10

Problem 6.10 The depicted construction is assmbled from two bars with double symmetric profile ($I_y = 2I_{\overline{z}}$ for both bars).

Determine the maximal height a, such that no buckling occurs.



Solution Due to the symmetry of the structure and the load the following compressive normal forces appear in the two bars

$$S_1 = S_2 = \frac{F}{\sqrt{2}} .$$

To investigate the stability behaviour we consider the different support conditions and the different flexural rigidities. Bar ① is simply supported at the lower end. The upper end is fixed by a rigid slider and connected to bar ② by a hinge. This corresponds to the Euler case no. 3. For buckling along the local y-axis of the profile we compute

$$S_1 = \frac{F}{\sqrt{2}} = 2.04 \, \frac{\pi^2 \, EI_y}{2a^2} \quad \rightsquigarrow \quad a_{1\,y} = 1.20 \, \pi \sqrt{\frac{EI_y}{F}}$$

and for buckling along the local $\overline{z}$-axis with $EI_{\overline{z}} = 0.5 \, EI_y$

$$S_1 = \frac{F}{\sqrt{2}} = 2.04 \, \frac{\pi^2 \, EI_{\overline{z}}}{2a^2} \quad \rightsquigarrow \quad a_{1\,\overline{z}} = 0.85 \, \pi \sqrt{\frac{EI_y}{F}} .$$

follows. Bar ② is hinged with one rotation axis in y-direction at the lower end. With regard to rotation along the x-axis the support is rigid. The support at the upper end is analogous to bar ① . Buckling along the local y-axis corresponds to the third Euler buckling mode. With $S_2 = S_1$ we obtain

$$a_{2\,y} = a_{1\,y} .$$

Buckling along the local $\overline{z}$-axis is equivalent to the Euler case no. 4 and yields with $EI_{\overline{z}} = 0.5 \, EI_y$

$$S_2 = \frac{F}{\sqrt{2}} = 2.04 \, \frac{\pi^2 \, EI_y}{2a^2} \quad \rightsquigarrow \quad a_{2\,\overline{z}} = 1.19 \, \pi \sqrt{\frac{EI_y}{F}} .$$

Since $a_{1\,\overline{z}}$ is the smallest value, the critical length is given by

$$\underline{\underline{a_{\text{crit}} = 0.85 \, \pi \sqrt{\frac{EI_y}{F}}}} .$$

Chapter 7

Hydrostatics

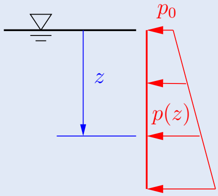
7

Prerequisite: The density ρ (unit: $\mathrm{kg/m^3}$) of the fluid is constant.

Pressure: The pressure p (unit: $\mathrm{Pa} \equiv \mathrm{N/m^2}$) is a force per area, that is identical for all cross sections and always acts normal to the cross section (hydrostatic stress state).

Pressure in a fluid under the action of gravity and a surface pressure p_0 is given by:

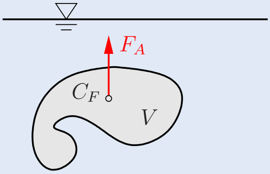
$$p\,(z) = p_0 + \varrho\, g\, z\,.$$



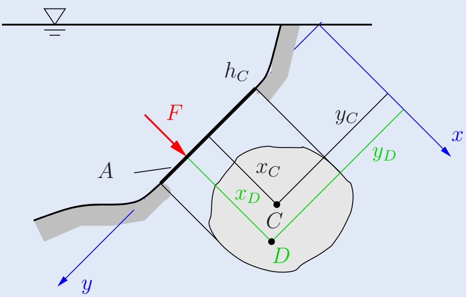
The **buoyancy** force acting on a body (volume V) immersed in a fluid is equal to the weight of the displaced fluid volume.
Buoyancy force:

$$F_A = \rho\, g\, V\,.$$

The line of action related to the buoyancy force passes through the center of gravity C_F of the displaced fluid volume.



Fluid pressure on plane surfaces



Resulting force

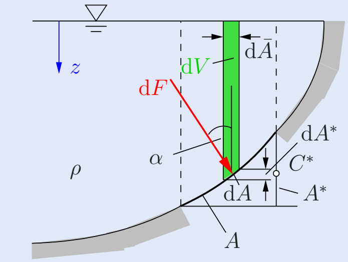
$$F = p\,(y_C)\, A = \rho\, g\, h_C\, A\,.$$

Center of pressure D

$$y_D = \frac{I_x}{S_x}\,,$$

$$x_D = -\frac{I_{xy}}{S_x}\,.$$

Fluid pressure on curved surfaces



$$dF_V = p\,dA\cos\alpha = \rho\,g\,dV$$

$$dF_H = p\,dA\sin\alpha = p\,dA^*$$

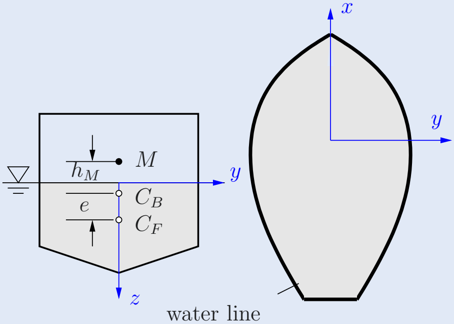
Integration yields

$$F_V = \rho\,g\,V\,,$$

$$F_H = p_{C^*}\,A^*\,.$$

The resulting horizontal component of the fluid pressure F_H is equal to the product of the vertically projected area A^* and the pressure p_{C^*} in the centroid of the projected area.

Stability of a floating body: The equilibrium state is stable if the meta center M is above the centroid C_B of the body:



$$h_M = \begin{cases} > 0 & : \text{stable} \\ < 0 & : \text{unstable} \end{cases}$$

with the position of the meta center

$$h_M = \frac{I_x}{V} - e\,.$$

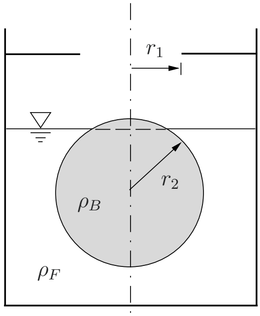
Here the following data are used

I_x : second moment of area defined by the water line,

V : volume of the displaced fluid,

e : distance of the centroid of the body centroid C_B from the centroid of the displaced fluid C_F.

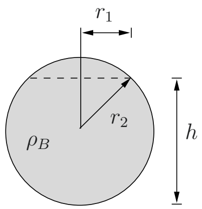
P7.1 **Problem 7.1** A container is closed during filling by a ball valve.



Determine the density ρ_B of the ball, such that no air remains in the container when the ball closes the valve.

Given.: ρ_F, r_1, r_2.

Solution The ball has to submerge to a depth that just closes the opening when the container is full. The buoyancy force is than $\rho_F\, g\, V_1$, where V_1 is the volume of the displaced fluid (spherical segment). The buoyancy force has to be equal to the weight of the ball



$$\rho_F\, g\, V_1 = \rho_B\, g\, V.$$

With the volume of a sphere

$$V = \frac{4}{3}\,\pi\, r_2^3$$

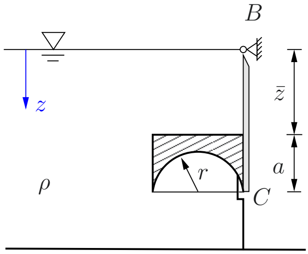
and the spherical section

$$V_1 = \pi\, h^2\,(r_2 - \frac{h}{3})\,, \qquad h = r_2 + \sqrt{r_2^2 - r_1^2}$$

we compute for the density of the ball

$$\underline{\underline{\rho_B}} = \rho_F\,\frac{V_1}{V} = \rho_F\,\frac{\pi\, h^2\,(r_2 - \frac{h}{3})}{\frac{4}{3}\,\pi\, r_2^3} = \underline{\underline{\rho_F\,\frac{3}{4}\left(\frac{h}{r_2}\right)^2\left(1 - \frac{h}{3\, r_2}\right)}}\,.$$

Problem 7.2 The design of the depicted valve of a water basin ensures that the valve opens if the water level reaches the hinge at point B. The flap valve is assumed to be massless. **P7.2**

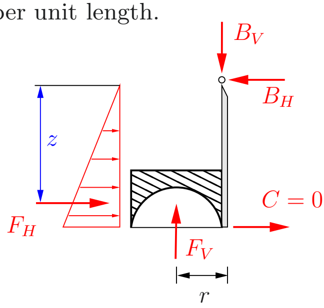


Determine $\bar{z}$ for the valve to function in the described way.

Given: ρ, a, r.

Solution The thickness of the flap valve is irrelevant for the following considerations, as all forces are assumed per unit length.

We compute the resulting horizontal force from the linear pressure distribution:



$$F_H = \frac{1}{2}\,\rho\, g\,(\bar{z} + a)^2$$

with

$$z = \frac{2}{3}\,(\bar{z} + a)\;.$$

The vertical buoyancy force can be computed from the weight of the displaced water by using the area of the dashed region:

$$F_V = \rho\, g\,\left(2\,a\,r - \frac{\pi}{2}\,r^2\right)\;.$$

The flap valve just opens if the reaction force in C vanishes. Equilibrium of moments with regard to B provides:

$$\overset{\curvearrowleft}{B}\;:\; -rF_V + zF_H = 0$$

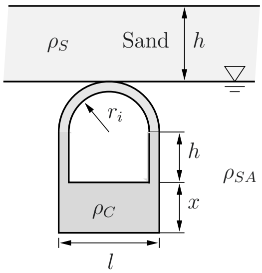
$$\leadsto \quad -\rho\, g\,\left(2\,a\,r - \frac{\pi}{2}\,r^2\right) r + \frac{1}{2}\,\rho\, g\,(\bar{z} + a)^2\,\frac{2}{3}\,(a + \bar{z}) = 0\;.$$

The solution of this equation with respect to $\bar{z}$ yields the water level

$$\underline{\underline{\bar{z} = \sqrt[3]{3\,(2\,a\,r - \frac{\pi}{2}\,r^2\,)\,r} - a\;.}}$$

P7.3 **Problem 7.3** The depicted cross section of a tunnel is immersed in water saturated "liquid" sand (density ρ_{SA}). Above resides a layer of dry sand (density ρ_S).

Determine the thickness x of the concrete base (density ρ_C), such that a safety factor $\eta = 2$ against lifting is reached. It is assumed that the weight of the dry sand is acting on the cross section of the tunnel.



Given: $\rho_B = 2.5 \cdot 10^3$ kg/m^3, $\rho_S = 2.0 \cdot 10^3$ kg/m^3,
$\rho_{SA} = 1.0 \cdot 10^3$ kg/m^3, $l = 10$ m, $r_i = 4$ m, $h = 7$ m.

Solution The weight (per unit length) of the tunnel cross section and sand load is given by

$$G = \rho_C\, g \left[x\, l + (\frac{l}{2} - r_i)\, 2\, h + \frac{\pi}{2}(\frac{l^2}{4} - r_i^2) \right] + \rho_S\, g\, l\, h\,.$$

With the buoyancy force (per unit length)

$$B = \rho_{SA}\, g \left[(h + x)\, l + \frac{\pi}{2}\frac{l^2}{4} \right]$$

we can determine the height of the concrete base, such that a safety factor against lifting

$$\eta = 2 = \frac{G}{B}$$

is achieved. Solving for x yields:

$$(2\rho_{SA}\, l - \rho_B l)x = \rho_S\, lh + \rho_B \left[(\frac{l}{2} - r_i)2h + \frac{\pi}{2}(\frac{l^2}{4} - r_i^2) \right]$$
$$-2\,\rho_{SA}\big(hl + \frac{\pi}{2}\frac{l^2}{4}\big).$$

With the given data we get

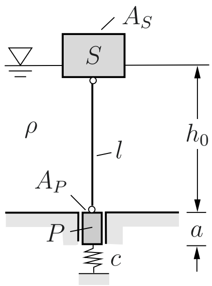
$$(20 - 25)\, x = 2 \cdot 70 + 2.5\, \left[14 + \frac{\pi}{2}(25 - 16) \right] - 2\, \left(70 + \frac{\pi}{2}\, 25 \right)$$

$$\leadsto \quad -5\, x = 210.34 - 218.54$$

$$\leadsto \quad \underline{\underline{x \;=\; 1.64\,\mathrm{m}}}\,.$$

Problem 7.4 A cylindrical plug P (cross section A_P, length a) is elastically supported and closes straight with the bottom of a basin for the water line h_0. In this situation the force vanishes in the rope (length l) to which a floater S is attached (cross section $A_S > A_P$).

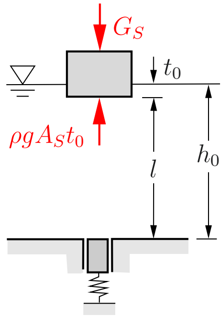
P7.4



a) Determine the weight G_S of the floater.

b) Which maximal water height h_1 can be reached before leaking occurs?

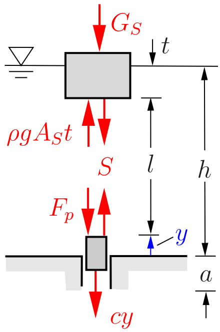
Solution **to a)** The weight G_S of the floater is computed from equilibrium and geometry in the reference situation:



$$\left.\begin{aligned} \rho g A_S\, t_0 &= G_S \\ h_0 &= l + t_0 \end{aligned}\right\} \quad \rightsquigarrow \quad \underline{\underline{G_S = (h_0 - l)\rho g A_S}}\,.$$

to b) For a water line h the plug is elevated by a distance y due to the force in the rope S. The equilibrium conditions for the floater, for the plug, and the geometric conditions are

$$\rho g A_S\, t = G_S + S\,, \qquad S - F_p = cy\,,$$
$$h = l + t + y\,.$$



In the equilibrium expression, F_p is the difference in the pressure force in the displaced and the reference situation (the forces due to lateral pressure are in equilibrium):

$$F_p = \rho g(h-y)A_p - \rho g h_0 A_p = \rho g(h-y-h_0)A_p\,.$$

Eliminating G_S, S, F_p, and t yields

$$h - h_0 = y\left[1 + \frac{c}{\rho g(A_S - A_P)}\right].$$

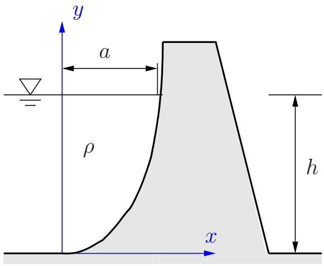
The maximal height$h = h_1$ is reached, if $y = a$ is attained:

$$\underline{\underline{h_1 = h_0 + a\left[1 + \frac{c}{\rho g(A_S - A_P)}\right]}}\,.$$

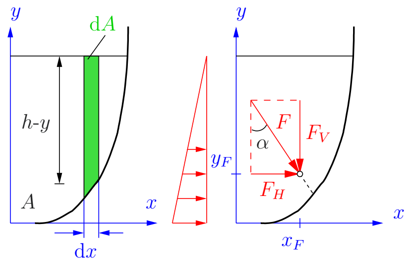
P7.5 **Problem 7.5** A dam of length l has a surface of parabolic shape with a horizontal tangent at the bottom of the water basin.

Determine the force resulting from the pressure, the position of the point of action, and the line of action for a water height h.

Given: h, l, $a = h/4$, ρ.



Solution The vertical component of the force is $F_V = \rho\, g\, V$ with the volume $V = l\, A$. The area is determined by the function $y(x) = 16\, x^2/h$ of the parabola



$$A = \int_0^a (h - y)\,\mathrm{d}x$$

$$= \int_0^a (h - \frac{16}{h}\, x^2)\,\mathrm{d}x = \Big[\, h\, x - \frac{16}{3\, h}\, x^3 \Big]_0^a = \frac{h^2}{6}\,.$$

Thus the vertical component of the pressure force becomes:

$$F_V = \frac{1}{6}\,\rho\, g\, h^2\, l.$$

The vertical force acts at the centroid C of the area

$$\underline{\underline{x_F}} = \frac{1}{A} \int_0^a x\,(h - \frac{16}{h}\, x^2)\,\mathrm{d}x = \Big[\, h\, \frac{x^2}{2} - \frac{16}{h}\, \frac{x^4}{4} \Big]_0^a = \underline{\underline{\frac{3}{32}\, h}}\,.$$

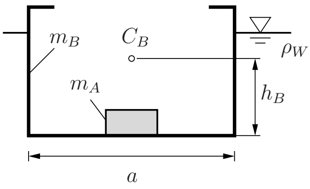
The horizontal component of the fluid pressure is computed by the projected area $A^* = h\, l$ and the pressure $p_{S^*} = \frac{1}{2}\,\rho\, g\, h$ in the centroid of the projected area:

$$F_H = \frac{1}{2}\,\rho\, g\, h^2\, l \qquad \text{with} \quad y_F = \frac{1}{3}\, h\,.$$

By the theorem of Pythagoras, we obtain the resulting force, its line of action passes through the point $(x_F\,, y_F)$ and forms an angle α to the y-axis:

$$\underline{\underline{F}} = \sqrt{F_H^2 + F_V^2} = \underline{\underline{\frac{1}{6}\sqrt{10}\,\rho\, g\, h^2\, l}}, \quad \underline{\underline{\alpha}} = \arctan\frac{F_H}{F_V} = \arctan 3 = \underline{\underline{71.5^o}}.$$

Problem 7.6 A prismatic body with the mass m_B, width a, and length l is floating in the water. Its centroid C_B is in the height h_B. **P7.6**



Determine the additional point mass m_A, such that the body floats in a stable manner.

Given: ρ_W, m_B, h_{SB}, l, a.

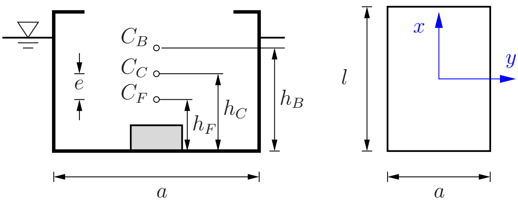
Solution Stable floating of the body is defined by the position of the meta center $h_M = I_x \,/\, V - e > 0$. For $h_M = 0$ the limit of the stable state is reached.

The volume V of the displaced fluid is obtained by equilibrium (buoyancy = weight of the body and added mass):

$$\rho_W \, g \, V = (m_B + m_A) \, g \quad \rightsquigarrow \quad V = \frac{1}{\rho_W} \, (m_B + m_A)$$

The second moment of area is

$$I_x = \frac{l \, a^3}{12} \, .$$



For $e = h_C - h_F$ we need the center of gravity h_C of the floating construction and h_F of the displaced fluid. They are determined by

$$h_C \, (m_B + m_A) = h_B \, m_B \quad \rightsquigarrow \quad h_C = h_B \, \frac{m_B}{m_B + m_A} \, ,$$

$$V = a \, l \, (2 \, h_F) \quad \rightsquigarrow \quad h_F = \frac{m_B + m_A}{2 \, a \, l \, \rho_W} \, .$$

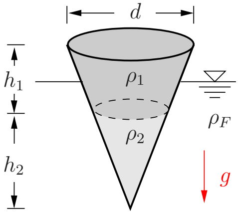
The limit for stable floating is reached if $h_M = 0$:

$$1 - 12 \, h_C \, \frac{m_B}{l \, a^3 \, \rho_W} + \frac{12 \, (m_B + m_A)^2}{2 \, l^2 \, a^4 \, {\rho_W}^2} = 0 \, .$$

Solving for the required additional mass m_A yields

$$\underline{\underline{m_A = \frac{l \, a^2 \, \rho_W}{\sqrt{6}} \sqrt{12 \, h_{CB} \, \frac{m_B}{l \, a^3 \, \rho_W} - 1} - m_B \, .}}$$

P7.7 **Problem 7.7** A cone-shaped floating device is made of two materials with densities ρ_1 and ρ_2.



Determine the diameter d of the cone, such that it floats stable in a fluid of density ρ_F.

Given:

$$\rho_1 = \frac{2}{3}\,\rho_F\,, \quad \rho_2 = \frac{1}{3}\,\rho_F\,,$$

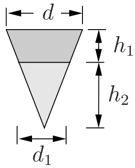
$$h_1 = 2\,h\,, \qquad h_2 = 4\,h\,.$$

Solution The cone has a stable floating position, if the following conditions are met:

$$(1):\quad G = A,$$

$$(2):\quad h_M = \frac{I_x}{V} - e > 0.$$

(1) Floating condition:



$$\frac{d}{h_1+h_2} = \frac{d_1}{h_2} \quad \rightsquigarrow \quad d_1 = d\,\frac{h_2}{h_1+h_2} = \frac{2}{3}\,d.$$

The force due to weight is

$$G = V_1\,\rho_1\,g + V_2\,\rho_2\,g$$

$$= \frac{1}{12}\pi\,h_1\,(d^2 + dd_1 + d_1^2)\,\rho_1\,g + \frac{1}{12}\,\pi\,h_2\,d_1^2\,\rho_2\,g$$

$$= \frac{23}{81}\,\pi\,h\,d^2\,\rho_F\,g = 0.892\,h\,d^2\,\rho_F g\,.$$

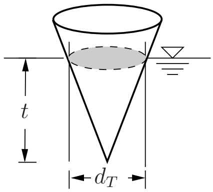
The immersion depth t and the diameter $d_T = d\,t/(h_1+h_2)$ of the cone at the water line of the fluid follows the buoyancy force

$$A = \frac{1}{12}\,\pi\,t\,d_T^2\,\rho_F\,g$$

$$= \frac{1}{432}\,\pi\,\frac{d^2}{h^2}\,\rho_F\,g\,t^3\,.$$

For $G = A$ we obtain

$$t^3 = \frac{368}{3}\, h^3 \quad \rightsquigarrow \quad t = 4.969\, h\,.$$



(2) Stability condition:

The volume of the displaced fluid is given by

$$V = \frac{1}{432}\, \pi\, \frac{d^2}{h^2}\, t^3 = \frac{23}{81}\, \pi\, h\, d^2 = 0.892\, h\, d^2\,,$$

and the second moment of area I_x is

$$I_x = \frac{d_T^4\, \pi}{64} = \frac{(0.828\, d)^4\, \pi}{64} = 0.023\, d^4\,.$$

The distance of the centroid of the body from the centroid of the displaced fluid is provided by

$$e = x_S - \frac{3}{4}\, t$$

with

$$x_S = \frac{\frac{3}{4}\,(h_1 + h_2)\, \rho_1\, \frac{1}{16}\, \pi\, d^2\, (h_1 + h_2) + \frac{3}{4}\, h_2\, (\rho_2 - \rho_1)\, \frac{1}{16}\, \pi\, d_1^2\, h_2}{\rho_1\, \frac{1}{16}\, \pi\, d^2\, (h_1 + h_2) + (\rho_2 - \rho_1)\, \frac{1}{16}\, \pi\, d_1^2\, h_2}$$

$$= \frac{18\, h - \frac{16}{9}\, h}{4 - \frac{16}{27}} = 4.761\, h$$

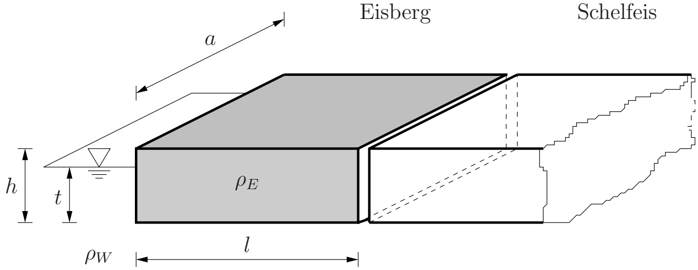
$$\rightsquigarrow \quad e = 4.761\, h - \frac{3}{4} \cdot 4.969\, h = 1.034\, h\,.$$

For the diameter of the cone we finally obtain

$$h_M = \frac{0.023\, d^4}{0.892\, h\, d^2} - 1.034 > 0 \qquad \rightsquigarrow \qquad \underline{\underline{d > 6.333\, h}}\,.$$

P7.8

Problem 7.8 A block-shaped iceberg of dimensions $a \times h \times l$ calves of a floating ice shelf. It is assumed that $a \gg h$. The density of the water is ρ_W, the density of the ice $\rho_I = \frac{9}{10}\rho_W$.



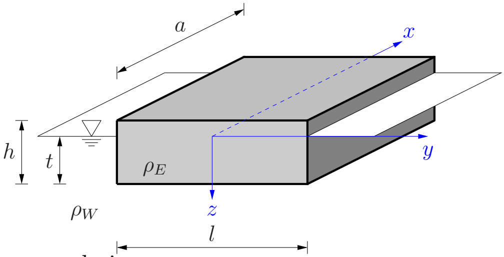
For which length l does the iceberg float in a stable way?

Solution We start by determining the immersion depth t of the iceberg. Equilibrium between iceberg and buoyancy force renders for the given density ratio the immersion depth

$$\rho_I g h l a = \rho_W g t l a \qquad \leadsto \qquad t = \frac{9}{10} h \; .$$

To analyze the floating stability we consider the position h_M of the meta center:

$$\begin{aligned} h_M &= \frac{I_x}{V} - e \; , \\ I_x &= \frac{a l^3}{12} \; , \\ V &= a l t = \frac{9}{10} a l h \; , \\ e &= \frac{h}{2} - \frac{t}{2} = \frac{h}{20} \; . \end{aligned}$$



By combining all relations we derive

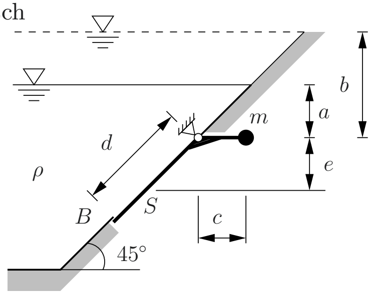
$$h_M = \frac{5}{54}\frac{l^2}{h} - \frac{h}{20} \; .$$

We consider the limit of floating stability ($h_M = 0$). This determines the length l_0 :

$$l_0^2 = \frac{27}{50} h^2 \qquad \leadsto \qquad l_0 = \sqrt{\frac{27}{50}} h \approx 0.735 h \; .$$

In a stable floating state, we must have $h_M > 0$. Thus, the iceberg floats stable for $\underline{\underline{l > l_0}}$. For $l < l_0$ the iceberg tips over.

Problem 7.9 A circular shaped hatch closes the outflow of a tank. **P7.9**

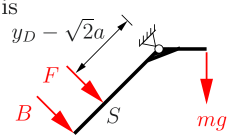


a) Determine the mass m, such that the hatch opens if m is attached in the distance c from the hinge point.

b) Determine the distance by which the mass m has to be shifted, for the hatch to open when the water level reaches the height b.

Given: a, b, c, d, e, m, ρ.

Solution **zu a)** The force acting on the hatch is



$$F = \rho\, g\, A\, h_S = \rho\, g\, \frac{\pi\, d^2}{4}\, (a+e)\,.$$

The point of action of F is determined by

$$y_D = y_S + \frac{I_\xi}{y_S\, A} = \sqrt{2}\,(a+e) + \frac{d^2}{16\sqrt{2}\,(a+e)}\,.$$

The hatch opens, if $B = 0$. Equilibrium of moments provides

$$F\,(\, y_D - \sqrt{2}\, a\,) - m\, g\, c = 0\,.$$

From this we compute the required mass

$$\underline{\underline{m = \rho\, \frac{\pi\, d^2}{4\, c}\, (a+e)\, \left[\sqrt{2}\, e + \frac{d^2}{16\,\sqrt{2}\,(a+e)}\right]}}\,.$$

to b) For the water level b the force acting on the hatch is

$$F = \rho\, g\, A\, h_s = \rho\, g\, \frac{\pi\, d^2}{4}\, (b+e)\,.$$

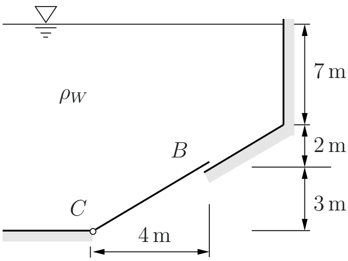
With the point of action

$$y_D = \sqrt{2}\,(b+e) + \frac{d^2}{16\,\sqrt{2}\,(b+e)}$$

of F the equilibrium condition $F\,(\, y_D - \sqrt{2}\, b\,) - m\, g\, c = 0$ yields the distance c:

$$\underline{\underline{c = \rho\, \frac{\pi\, d^2}{4}\, (b+e)\, \left[\sqrt{2}\, e + \frac{d^2}{16\,\sqrt{2}\,(b+e)}\right]\, \frac{1}{m}}}\,.$$

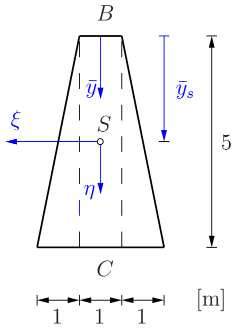
P7.10 **Problem 7.10** A trapezoidal hatch closes the outflow of the depicted basin.



Determine the resulting force on the hatch together with the support reactions in point B.

Given: $\rho_W = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$, $g = 9.81\,\dfrac{\text{m}}{\text{s}^2}$.

Solution The area $A = 10\,\text{m}^2$, the centroid of the hatch



$$\bar{y}_s = \left(5{\cdot}2,5{+}5{\cdot}\frac{2}{3}{\cdot}5\right)\frac{1}{10} = \frac{35}{12}\,\text{m}$$

and the pressure

$$p\,(\bar{y}_s) = \rho\, g\left[9 + \frac{3}{5}\cdot\frac{35}{12}\right] = \frac{43}{4}\,\rho\, g$$

are used to compute the resulting force

$$\underline{\underline{F}} = \rho\, g\, A\, p\,(\bar{y}_s) = 10^3{\cdot}9,81{\cdot}10{\cdot}\frac{43}{4} = \underline{\underline{1.05\,\text{MN}}}\,.$$

The position of the line of action follows from

$$I_\xi = \frac{5^3\cdot 1}{12} + 5{\cdot}1\left(\frac{35}{12} - 2,5\right)^2 + 2\,\frac{5^3\cdot 1}{36} + 5{\cdot}1\left(\frac{35}{12} - \frac{10}{3}\right)^2 = 19.1\,\text{m}^4\,,$$

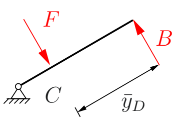
$y_s = \bar{y}_s + 15\,\text{m}$ and $y_D = \bar{y}_D + 15\,\text{m}$ to be

$$y_D = \frac{I_x}{S_x} = \frac{y_s^2\,A + I_\xi}{y_s\,A} \quad\leadsto\quad \bar{y}_D = \bar{y}_s + \frac{I_\xi}{y_s\,A} = \frac{35}{12} + \frac{19,1}{(\frac{35}{12} + 15)\,10}$$

$$= 3.02\,\text{m}\,.$$

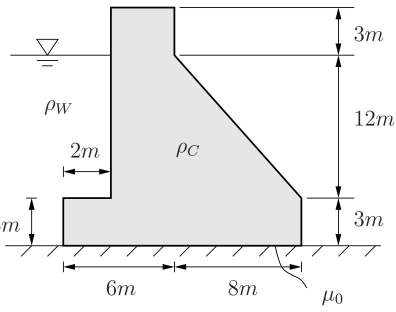
The support reaction is determined by equilibrium of moments with regard to the hinge point C of the hatch

$$\overset{\curvearrowleft}{C}:\quad B\cdot 5 - F\,(5 - 3,02) = 0$$



$$\leadsto\quad \underline{\underline{B}} = 1,05\,\frac{5-3,02}{5} = \underline{\underline{0.415\,\text{MN}\,.}}$$

Problem 7.11 A concrete dam (density ρ_C) closes a basin that is filled up to the height $h = 15$ m. **P7.11**



Determine

a) the safety factor against sliding at the bottom (adhesion coefficient μ_0),
b) the safety against tilting,
c) the stress distribution at the bottom, if it assumed to be a linear distribution.

Given: $\rho_C = 2.5 \cdot 10^3$ kg/m^3, $\rho_W = 10^3$ kg/m^3, $\mu_0 = 0.5$, $g = 10$ m/s^2

Solution **to a)** To determine the safety factor against sliding we compute the horizontal forces due to the water pressure and compare them to the adhesion forces acting at the bottom. The horizontal force due to water pressure is computed from

$$F_H = \frac{1}{2}\,\rho_W\, g\, h\, A = \frac{1}{2}\,10^3 \cdot 10 \cdot 15 \cdot 15 \cdot 1 = 1125 \text{ kN/m}\,.$$

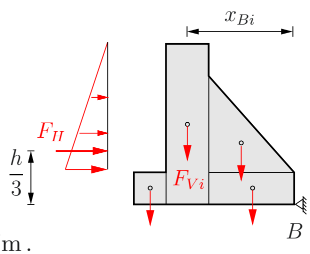
The resulting force due to the weight of the concrete and the water pressure is

$$F_V = 2.5 \cdot 10^3\,(3 \cdot 2 + 4 \cdot 18 + 3 \cdot 8 + \frac{1}{2} \cdot 12 \cdot 8) + 10^3\,(2 \cdot 12) = 3990\,\text{kN/m}\,.$$

Using Coulomb's friction law we determine the safety factor η_S against the onset of sliding

$$\underline{\underline{\eta_S}} = \frac{\mu_0\, F_V}{F_H} = \frac{0.5 \cdot 3990}{1125} = \underline{\underline{1.77}}\,.$$

to b) The dam can tilt around point B. The safety against tilting is determined by comparing the moment of forces. The moment of the water pressure is given by



$$M_{BW} = F_H\,\frac{h}{3} = 1125 \cdot \frac{15}{3} = 5625\,\text{kNm}\,.$$

The moment related to the weights is

$$M_{BG} = \sum_i F_{Vi}\, x_{Bi}$$
$$= 2.5 \cdot 10^3\,(3 \cdot 2 \cdot 13 + 4 \cdot 18 \cdot 10 + 3 \cdot 8 \cdot 4$$
$$+\frac{1}{2} \cdot 12 \cdot 8 \cdot \frac{2}{3} \cdot 8) + 10^3\,(2 \cdot 12 \cdot 13) = 31870\,\text{kNm}\,.$$

This results in a safety factor against tilting

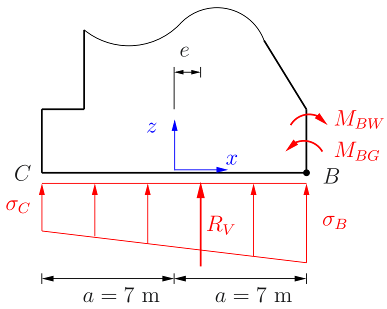
$$\underline{\underline{\eta_T}} = \frac{M_{BG}}{M_{BW}} = \frac{31780}{5625} = \underline{\underline{5.67}}\,.$$

to c) To compute the stress distribution in the bottom gap of the dam we determine the excentricity of the resulting force $R_V = \sum_i F_{Vi}$. The vertical component of the force acting in the gap yields, according to the sketch below,

$$R_V\,(a - e) = M_{BG} - M_{BW}$$
$$\leadsto \quad e = a - \frac{M_{BG} - M_{BW}}{R_V} = 7 - \frac{31870 - 5625}{3990} = 0.422\,\text{m}\,.$$

With the introduced coordinate-system we compute the normal stresses in the bottom gap (like in a beam cross section)

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y}\,x\,.$$



Here we have to insert the following data: $A = 14\,\text{m}^2$, $I_y = 1 \cdot 14^3\,/\,12 = 288.67\,\text{m}^4$, $N = -R_V = -3990$ kN, $M_y = N \cdot e = -1685$ kNm. As a result we obtain for the stress distribution

$$\underline{\underline{\sigma}} = \frac{-3990}{14} + \frac{-1685}{228.67}\,x = \underline{\underline{-285 - 7.37\,x\ \text{kN/m}^2}}\,.$$

For the selected points C and B evaluation yields

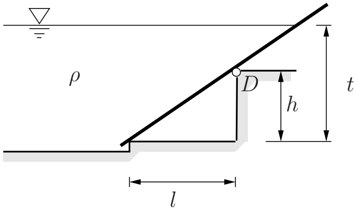
$$\sigma_C = -0.23\,\text{MPa} \quad \text{and} \quad \sigma_B = -0.34\,\text{MPa}\,.$$

Problem 7.12 A rectangular plate of width b closes the outlet of a basin. It is hinged at point D. **P7.12**

a) Determine the water height t, for which the plate starts to rotate around point D.

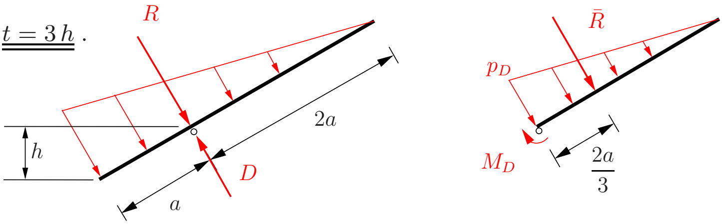
b) Compute the bending moment at point D for this situation.

Given: b, l, h, ρ.



to a) The plate starts to rotate, if the resulting force R of the water pressure is above point D. In the limit case the resulting force of the water pressure passes through point D. From this we can determine the water height

$$\underline{\underline{t = 3\,h}}\,.$$



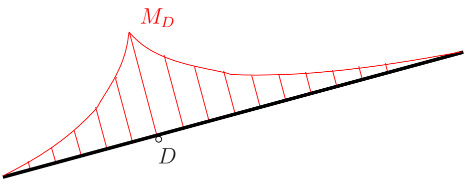
to b) To compute the bending moment in the plate we start with the moment at point D. With the resultant $\bar{R}$ of the upper plate and the pressure at point D,

$$\bar{R} = \frac{1}{2}\,p_D\,2\,a\,b\,, \quad p_D = \rho\,g\,2\,h\,,$$

we obtain

$$\underline{\underline{M_D}} = -\bar{R}\,\frac{2}{3}\,a = -\frac{2}{3}\,p_D\,b\,a^2 = \underline{\underline{-\frac{4}{3}\,\rho\,g\,(l^2 + h^2)\,h\,b}}\,.$$

The distribution of the bending moment is cubic for a linearly varying load. The maximum occurs at the hinge point D.

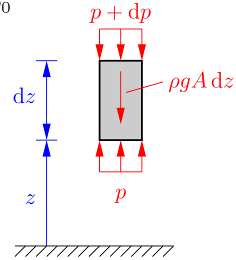


P7.13 **Problem 7.13** The pressure p in gases depends on the density ρ. The relation between the two state variables is provided by the universal gas equation $p = \rho RT$ (universal gas constant R, temperature T). E. g. for air at sea level and at $T = 0°$ it holds: $p_0 = 101325\,\mathrm{Pa}$ and $\rho_0 = 1.293\,\mathrm{kg/m^3}$.

Determine the dependency of air pressure on height for the case of a constant temperature (barometric height relation).

Solution First, we apply the universal gas law at sea level. This yields

$$p_0 = \rho_0 RT \qquad \text{or} \qquad RT = \frac{p_0}{\rho_0}\,.$$



Equilibrium of an infinitesimal air column with cross section A and height $\mathrm{d}z$

$$\uparrow: \quad pA - \rho g A\,\mathrm{d}z - (p + \mathrm{d}p)\,A = 0$$

leads to

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g\,.$$

Using the universal gas equation yields

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{pg}{RT}\,.$$

By separation of variables and integration we obtain:

$$\frac{\mathrm{d}p}{p} = -\frac{g}{RT}\mathrm{d}z \;\rightsquigarrow\; \int_{p_0}^{p} \frac{\mathrm{d}\bar{p}}{\bar{p}} = -\int_0^z \frac{g}{RT}\mathrm{d}\bar{z} \;\rightsquigarrow\; \ln\frac{p}{p_0} = -\frac{g}{RT}z\,.$$

This renders the air pressure as a function of the height

$$\underline{\underline{p = p_0\,\mathrm{e}^{-\dfrac{gz}{RT}}}}\,.$$

The air pressure decreases exponentially with the height. From the relation $RT = p_0/\rho_0$ and the gravity constant $g = 9.80665\,\mathrm{m/s^2}$ we deduce

$$p = 101325\,\mathrm{Pa}\,\mathrm{e}^{-\dfrac{z}{7991\,\mathrm{m}}}\,.$$

Note: In a height of $5,5\,\mathrm{km}$ the pressure has dropped to one half of its original value.