Weighted Efficient Domination for P_6-Free and for P_5-Free Graphs

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Abstract. In a finite undirected graph $G = (V, E)$, a vertex $v \in V$ *dominates* itself and its neighbors in G. A vertex set $D \subseteq V$ is an *efficient dominating set* (*e.d.s.* for short) of G if every $v \in V$ is dominated in G by exactly one vertex of D. The *Efficient Domination* (ED) problem, which asks for the existence of an e.d.s. in G, is known to be $\mathbb{NP}$-complete for P_7-free graphs and solvable in polynomial time for P_5-free graphs. The P_6-free case was the last open question for the complexity of ED on F-free graphs.

Recently, Lokshtanov, Pilipczuk and van Leeuwen showed that weighted ED is solvable in polynomial time for P_6-free graphs, based on their quasi-polynomial algorithm for the Maximum Weight Independent Set problem for P_6-free graphs. Independently, by a direct approach which is simpler and faster, we found an $\mathcal{O}(n^5 m)$ time solution for weighted ED on P_6-free graphs. Moreover, we showed that weighted ED is solvable in linear time for P_5-free graphs which solves another open question for the complexity of (weighted) ED.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph without loops and multiple edges; let $|V| = n$ and $|E| = m$. A vertex $v \in V$ *dominates* itself and its neighbors. A vertex subset $D \subseteq V$ is an *efficient dominating set* (*e.d.s.* for short) of G if every vertex of G is dominated by exactly one vertex in D.

The notion of efficient domination was introduced by Biggs [1] under the name *perfect code*. Note that not every graph has an e.d.s.; the Efficient Dominating Set (ED) problem asks for the existence of an e.d.s. in a given graph G. We can assume that G is connected; if not then the ED problem for G can be splitted into ED for each of its connected components. If a graph G with vertex weight function $w : V \to \mathbb{Z} \cup \{\infty, -\infty\}$ and an integer k is given, the Minimum Weight Efficient Dominating Set (WED) problem asks whether G has an e.d.s. D of total weight $w(D) := \Sigma_{x \in D} w(x) \leq k$.

As mentioned in [4], the maximization version of WED can be defined analogously, replacing the condition $w(D) \leq k$ with $w(D) \geq k$. Since negative weights

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are allowed, the maximization version of WED is equivalent to its minimization version; subsequently we restrict the problem to the minimization version WED. The vertex weight ∞ plays a special role; vertices which are definitely not in any e.d.s. get weight ∞, and thus, in the WED problem we are asking for an e.d.s. of finite minimum weight. Let $\gamma_{ed}(G, w)$ denote the minimum weight of an e.d.s. in G. We call a vertex $x \in V$ *forced* if x is contained in every e.d.s. of finite weight in G.

The importance of the ED problem for graphs mostly results from the fact that it is a special case of the Exact Cover problem for hypergraphs (problem [SP2] of [9]); ED is the Exact Cover problem for the closed neighborhood hypergraph of G.

For a subset $U \subseteq V$, let $G[U]$ denote the *induced subgraph* of G with vertex set U. For a graph F, a graph G is F*-free* if G does not contain any induced subgraph isomorphic to F. Let P_k denote a chordless path with k vertices. $F + F'$ denotes the disjoint union of graphs F and F'; for example, $2P_3$ denotes $P_3 + P_3$.

Many papers have studied the complexity of ED on special graph classes - see e.g. [3–6,12] for references. In particular, a standard reduction from the Exact Cover problem shows that ED remains $\mathbb{NP}$-complete for $2P_3$-free (and thus, for P_7-free) chordal graphs. For P_6-free graphs, the question whether ED can be solved in polynomial time was the last open case for F-free graphs [5]; it was the main open question in [6]. As a first step towards a dichotomy, it was shown in [3] that for P_6-free chordal graphs, WED is solvable in polynomial time.

Recently, it has been shown by Lokshtanov et al. [10] that WED is solvable in polynomial time for P_6-free graphs (the time bound is more than $\mathcal{O}(n^{500})$). Their result for WED is based on their quasi-polynomial algorithm for the Maximum Weight Independent Set problem for P_6-free graphs. Independently, in [7] we found a polynomial time solution for WED on P_6-free graphs using a direct approach which is simpler than the one in [10] and leads to the much better time bound $\mathcal{O}(n^5 m)$. According to [5], the results of [7,10] finally lead to a dichotomy for the WED problem on P_k-free graphs and moreover on F-free graphs.

In our approach, we need the following notion: A graph $G = (V, E)$ is *unipolar* if there is a partition of V into sets A and B such that $G[A]$ is P_3-free (i.e., the disjoint union of cliques) and $G[B]$ is a complete graph. See e.g. [8,11] for recent work on unipolar graphs. Note that ED remains $\mathbb{NP}$-complete for unipolar graphs [8] (which can also be seen by the standard reduction from Exact Cover; there, every clique in $G[A]$ has only two vertices). Clearly, every unipolar graph is $2P_3$-free and thus P_7-free. It follows that for each $k \geq 7$, WED is NP-complete for P_k-free unipolar graphs.

The main results of this paper are the following:

1. In Sect. 2, we give a polynomial time reduction of the WED problem for P_6-free graphs to WED for P_6-free unipolar graphs.
2. In Sect. 3, we solve WED for P_6-free unipolar graphs in polynomial time.
3. In Sect. 4, we describe the polynomial time algorithm for the WED problem on P_6-free graphs. Thus, we obtain a dichotomy for the WED problem on F-free graphs, and in particular on P_k-free graphs and on P_k-free unipolar graphs.

In the full version of this paper, we describe a linear time algorithm for WED on P_5-free graphs based on modular decomposition (see [2] for details); this answers another open question in [6].

Due to space limitation, most of the proofs and the linear time algorithm for WED on P_5-free graphs cannot be given here.

2 Reducing WED on P_6-Free Graphs to WED on P_6-Free Unipolar Graphs

Throughout this section, let $G = (V, E)$ be a connected P_6-free graph. Subsequently we consider the distance levels of $v \in V$ according to the usual approach which is used already in various papers such as [6]. For $v \in V$, let $N_i(v)$ denote the i-th distance level of v, that is $N_i(v) = \{u \in V \mid d_G(u, v) = i\}$. Then, since G is P_6-free, we have $N_i(v) = \emptyset$ for each $i \geq 5$. If $v \in D$ for an e.d.s. D of G then, by the e.d.s. property, we have

$$(N_1(v) \cup N_2(v)) \cap D = \emptyset. \tag{1}$$

Let $G_v := (N_2(v) \cup N_3(v) \cup N_4(v), E_v)$ such that $N_2(v)$ is turned into a clique by correspondingly adding edges, i.e., $E_v = E' \cup F$ where E' is the set of the original edges in $G[N_2(v) \cup N_3(v) \cup N_4(v)]$ and F is the set of new edges turning $N_2(v)$ into a clique, and let $w(x) := \infty$ for every $x \in N_2(v)$. We first claim:

Proposition 1. *G_v is P_6-free.*

Proof. Suppose to the contrary that there is a P_6 P in G_v, say with vertices a, b, c, d, e, f and edges ab, bc, cd, de, ef. If $\{ab, bc, cd, de, ef\} \cap F = \emptyset$ then P would be a P_6 in G which is a contradiction. Thus, $\{ab, bc, cd, de, ef\} \cap F \neq \emptyset$. Then clearly, $|\{ab, bc, cd, de, ef\} \cap F| = 1$ since $N_2(v)$ is a clique in G_v. Now in any case, we get a P_6 in G by adding $N[v]$ and the corresponding edges in $N[v]$ and between $N_1(v)$ and $N_2(v)$ which is a contradiction. □

Obviously, the following holds:

Proposition 2.

(i) *For vertex $v \in V$ with $w(v) < \infty$, D is a finite weight e.d.s. in G with $v \in D$ if and only if $D \setminus \{v\}$ is a finite weight e.d.s. in G_v.*

(ii) *Thus, if for every $v \in V$, WED is solvable in time T for G_v then WED is solvable in time $n \cdot T$ for G.*

From now on, let $D(v)$ denote an e.d.s. of finite weight of G_v. We call a vertex x *v-forced* if $x \in D(v)$ for every e.d.s. $D(v)$ of finite weight of G_v and *v-excluded* if $x \notin D(v)$ for every such e.d.s. $D(v)$ of G_v. Clearly, if x is v-excluded then we can set $w(x) = \infty$, e.g., for all $x \in N_2(v)$, $w(x) = \infty$ as above.

Let $Q_1, \ldots, Q_r$ denote the connected components of $G_v[N_3(v) \cup N_4(v)]$ (i.e., of $G[N_3(v) \cup N_4(v)]$). By (1), we have:

$$\text{For each}\, i \in \{1, \ldots, r\}, \text{ we have}\, |Q_i \cap D(v)| \geq 1. \tag{2}$$

Clearly, the $D(v)$-candidates in Q_i must have finite weight.

A component Q_i is *trivial* if $|Q_i| = 1$. Obviously, by (2), the vertices of the trivial components are v-forced.

Clearly, since $D(v)$ is an e.d.s. of finite weight, every $x \in N_2(v)$ must contact a component Q_i for some $i \in \{1, \ldots, r\}$.

2.1 Join-Reduction

Now we consider a graph $G = (A \cup B, E)$ such that $A_1, \ldots, A_k$ are the components of $G[A]$, and a vertex weight function w with $w(b) = \infty$ for all $b \in B$. Assume that G has an e.d.s. D of finite weight. As above, we can assume that every component A_i is nontrivial since any trivial component A_i consists of a forced D-vertex.

By the e.d.s. property of D, we have (analogously to condition (2)):

$$\text{For every}\, x \in B, x \textcircled{1} A_i \,\text{for at most one}\, i \in \{1, \ldots, k\}. \tag{3}$$

Thus, from now on, we can assume that every vertex $x \in B$ has a join to at most one component A_i. Moreover, if $x \textcircled{1} A_i$ for some $i \in \{1, \ldots, k\}$ then for every neighbor $y \in A_j$ of x, $j \neq i$, $y \notin D$, i.e., we can set $w(y) = \infty$, and thus, $y \notin D$ for any e.d.s. D of finite weight of G.

For any vertex $x \in B$ with $x \textcircled{1} A_i$ for exactly one $i \in \{1, \ldots, k\}$, by the e.d.s. property of D, $|D \cap A_i| \geq 2$ is impossible. Thus, x is correctly dominated if $|D \cap A_i| = 1$, that is, the D-candidates in A_i are universal for A_i; let U_i denote the set of universal vertices in A_i (note that U_i is a clique). Clearly, for $x \textcircled{1} A_i$ we have:

$$\text{If}\, U_i = \emptyset \,\text{then}\, G \,\text{has no finite weight e.d.s.} \tag{4}$$

Thus, for every A_i such that there is a vertex $x \in B$ with $x \textcircled{1} A_i$, we can reduce A_i to the clique U_i, we can omit x in B, and for every neighbor $y \in A_j$ of x, $j \neq i$, we set $w(y) = \infty$. The following algorithm is needed twice in this manuscript:

Join-Reduction Algorithm

Given: A graph $G = (A \cup B, E)$ such that $A_1, \ldots, A_k$ are the components of $G[A]$, and a vertex weight function w with $w(b) = \infty$ for all $b \in B$.

Task: Reduce G in time $\mathcal{O}(n^3)$ to an induced subgraph $G' = (A' \cup B', E')$ with weight function w' and components $A'_1, \ldots, A'_k$ of $G[A']$ such that we have:

- (i) For every $b \in B'$ and every $i \in \{1, \ldots, k\}$, if b contacts (nontrivial) component A'_i then b distinguishes A'_i.
- (ii) $\gamma_{ed}(G, w) = \gamma_{ed}(G', w') < \infty$ or state that G has no such e.d.s.

begin

(a) Determine the sets
$B_{join} := \{b \in B \mid \text{there is an } i \in \{1, \dots, k\} \text{ with } b (1) A_i\}$ and
$A_{join} := \{A_i \mid i \in \{1, \dots, k\} \text{ and there is a } b \in B \text{ with } b (1) A_i\}$.
(b) **If** there is a vertex $b \in B_{join}$ and there are $i \neq j$ with $b (1) A_i$ and $b (1) A_j$ **then** STOP $-$ G does not have an e.d.s. of finite weight.
{From now on, every $b \in B_{join}$ has a join to exactly one $A_i \in A_{join}$.}
(c) **For all** $b \in B_{join}$ and $A_i \in A_{join}$ such that $b (1) A_i$ **do**
begin
(c.1) Determine the set U_i of universal vertices in A_i. **If** $U_i = \emptyset$ **then** STOP $-$ G does not have an e.d.s. of finite weight **else** set $A'_i := U_i$.
(c.2) For every neighbor $y \in A \setminus A_i$ of b, set $w'(y) := \infty$.
end
(d) For every $A_i \notin A_{join}$, set $A'_i := A_i$, and finally set $A' := A'_1 \cup \ldots \cup A'_k$, $B' := B \setminus B_{join}$ and $G' := G[A' \cup B']$.

end

Lemma 1. *The Join-Reduction Algorithm is correct and can be done in time* $\mathcal{O}(n^3)$.

For applying the Join-Reduction Algorithm to G_v, we set $B := N_2(v)$ and $A := N_3(v) \cup N_4(v)$. For reducing WED on G to WED on a unipolar graph G', this is a first step which, by condition (i) of the Task, leads to the fact that finally, for every nontrivial component Q_j of $G[N_3(v) \cup N_4(v)]$, every vertex in $N_2(v)$ which contacts Q_j also distinguishes Q_j.

2.2 Component-Reduction

Let $G'_v = (A' \cup B', E')$ be the result of applying the Join-Reduction algorithm to G_v; let B' be the corresponding subset of $N_2(v)$ and let A' be the corresponding subset of $N_3(v) \cup N_4(v)$. Recall that in G'_v, B' is a clique. In the next step, we reduce WED for G_v to WED for unipolar graphs.

We consider the components Q_i of $G'_v[A']$ which are not yet a clique; as already mentioned, we can assume that if $x \in B'$ has a neighbor in Q_i then it has a neighbor and a non-neighbor in Q_i. For $1 \leq i \leq r$, let $Q_i^+(x) := Q_i \cap N(x)$ and $Q_i^-(x) := Q_i \setminus N(x)$. Since Q_i is connected, we have: If x distinguishes Q_i then it distinguishes an edge in Q_i.

For $x, x' \in B'$ and edges y_1z_1 in Q_i, y_2z_2 in Q_j, $i \neq j$, let $xy_1 \in E, xz_1 \notin E$ and $x'y_2 \in E, x'z_2 \notin E$. Then, since G and G_v are P_6-free, we have:

$$xy_2 \in E \text{ or } xz_2 \in E \text{ or } x'y_1 \in E \text{ or } x'z_1 \in E. \quad (5)$$

Another useful P_6-freeness argument is the following:

$$\text{For } x \in B' \text{ and } y \in Q_i^+(x), y \text{ does not distinguish any edge in } Q_i^-(x). \quad (6)$$

We claim:

$$\text{There is a vertex } b^* \in B' \text{ which contacts } Q_i \text{ for every } i \in \{1, \ldots, r\}. \tag{7}$$

Let $q^* \in D(v)$ be the vertex dominating b^*; without loss of generality assume that $q^* \in Q_1$, and let $D(v, q^*)$ denote a finite weight e.d.s. of G_v containing q^*. Q_1 is partitioned into

$Z := N[q^*] \cap Q_1$,
$W := Q_1 \cap N(b^*) \setminus Z$, and
$Y := Q_1 \setminus (Z \cup W)$.

Then clearly, the following properties hold:

Lemma 2.

(i) $Z \cap D(v, q^*) = \{q^*\}$.
(ii) $W \cap D(v, q^*) = \emptyset$.
(iii) $Z \circledcirc Y$.
(iv) *For every component K of $G[Y]$, the set of $D(v, q^*)$-candidate vertices in K is a clique.*

For the algorithmic approach, we set $w(y) = \infty$ for every $y \in W$ and for every non-universal vertex $y \in K$ in any component K of $G_v[Y]$.

For $i \geq 2$, let $Q_i^+ := Q_i \cap N(b^*)$ and $Q_i^- := Q_i \setminus N(b^*)$. Clearly, by the e.d.s. property, for every $i \geq 2$, $Q_i^+ \cap D(v, q^*) = \emptyset$; set $w(y) = \infty$ for every $y \in Q_i^+$. Thus, the components of $G[Q_i^-]$ must contain the corresponding $D(v, q^*)$-vertices.

Again, as in Lemma 2 *(iv)*, for each such component K, the $D(v, q^*)$-candidates must be universal vertices for K since by (6), two such $D(v, q^*)$-candidates in K would have a common neighbor in Q_i^+, i.e., only the universal vertices of component K are the $D(v, q^*)$-candidate vertices; set $w(y) = \infty$ for every non-universal vertex $y \in K$.

Let $I := \{a \in A' : w(a) = \infty\}$. Then I admits a partition $\{I_1, I_2, I_3\}$ as defined below:

- I_1 is formed by those vertices of I which are either in Z, or in Y, or in Q_i^- for $i \geq 2$.
- I_2 is formed by those vertices of I which are either in W and contact exactly one component of $G[Y]$ or in Q_i^+ and contact exactly one component of $G[Q_i^-]$ for $i \geq 2$.
- I_3 is formed by those vertices of I which are either in $W \setminus I_2$ or in $Q_i^+ \setminus I_2$ for $i \geq 2$.

Note that we have:

(a) By construction and by the e.d.s. property, if $I_3 \neq \emptyset$ then $D(v, q^*)$ does not exist (in fact each vertex of I_3 either would not be dominated by any $D(v, q^*)$-candidate or would be dominated by more than one $D(v, q^*)$-candidate).

(*b*) By construction, if $D(v, q^*)$ exists then $D(v, q^*)$ is an e.d.s. of $G'_v[(A' \cup B') \setminus (I_1 \cup I_2)]$ as well; in particular, by construction and by (6), each vertex of $I_1 \cup I_2$ is dominated in G'_v by exactly one vertex of $D(v, q^*)$; then vertices of $I_1 \cup I_2$ can be removed.
(*c*) $G'_v[(A' \cup B') \setminus (I_1 \cup I_2)]$ is unipolar (once assuming that $I_3 = \emptyset$).

Then for every potential $D(v)$-neighbor q^* of b^*, we can reduce the WED problem for G'_v to the WED problem for $G'_v(q^*)$ consisting of B' and the P_3-free subgraph induced by $\{q^*\}$ and by the corresponding cliques of universal vertices in components K as described above with respect to $D(v, q^*)$. Clearly, the $D(v, q^*)$-candidates in the cliques of the P_3-free subgraph can be chosen corresponding to optimal weights.

Summarizing, we can do the following:

Component-Reduction Algorithm

Given: The result $H = G'_v = (A' \cup B', E')$ with vertex weight function w of applying the Join-Reduction algorithm to G_v such that $K_1, \ldots, K_s$ denote the clique components and $Q_1, \ldots, Q_r$ denote the non-clique components of $G'_v[A']$.

Task: Reduce H in time $\mathcal{O}(n^3)$ to (less than n) unipolar graphs $H_\ell = G'(q^*)$ with weight function w_ℓ, $1 \leq \ell < n$, such that $\gamma_{ed}(H, w) = \min_\ell \gamma_{ed}(H_\ell, w_\ell)$ or state that H has no e.d.s. of finite weight.

begin

(*a*) Determine a vertex $b^* \in B'$ contacting every Q_i, $i \in \{1, \ldots, r\}$.
(*b*) For every $q^* \in N(b^*) \cap A'$ with $w(q^*) < \infty$, say $q^* \in Q_i$, reduce Q_i according to Lemma 2 and for all j, $j \neq i$, reduce Q_j according to the paragraph after the proof of Lemma 2 such that finally, the resulting subgraph $G'(q^*)$ is unipolar.

end

Lemma 3. *The Component-Reduction Algorithm is correct and can be done in time* $\mathcal{O}(n^3)$.

Corollary 1. *If WED is solvable in polynomial time on* P_6*-free unipolar graphs then it is solvable in polynomial time on* P_6*-free graphs.*

3 Solving WED on P_6-Free Unipolar Graphs in Polynomial Time

Throughout this section, let $G = (V, E)$ be a connected P_6-free unipolar graph with partition $V = A \cup B$ such that $G[A]$ is the disjoint union of cliques $A_1, \ldots, A_k$, and $G[B]$ is a complete subgraph. Clearly, if $k \leq 3$ then every e.d.s. of G contains at most four vertices. Thus, from now on, we can assume that $k \geq 4$. In particular, for any e.d.s. D of G, $|D \cap B| \leq 1$. Thus, WED for such graphs is solvable in polynomial time if and only if WED is solvable in polynomial time for e.d.s. D with $B \cap D = \emptyset$.

Clearly, for a P_6-free unipolar graph, the following holds (recall (5)):

Claim 1. *If for distinct $b_1, b_2 \in B$, b_1 distinguishes an edge x_1x_2 in A_i and b_2 distinguishes an edge y_1y_2 in A_j, $i \neq j$, then either b_2 contacts $\{x_1, x_2\}$ or b_1 contacts $\{y_1, y_2\}$.*

The key result of this section is the following:

Lemma 4. *For connected unipolar graphs fulfilling Claim 1, it can be checked in polynomial time whether G has a finite weight e.d.s. D with $B \cap D = \emptyset$.*

Lemma 4 is based on various propositions described subsequently. As a first step, we again reduce G corresponding to the Join-Reduction Algorithm of Sect. 2: Since $B \cap D = \emptyset$, clearly, $|D \cap A_i| = 1$ for every $i \in \{1, \ldots, k\}$. Thus, if $A_i = \{a_i\}$ then a_i is a forced D-vertex; from now on, we can assume that every A_i is nontrivial.

Moreover, every $b \in B$ must contact at least one A_i, and if b has a join to two components A_i, A_j, $i \neq j$, then G does not have an e.d.s. Thus, by (3) and the subsequent paragraph in Sect. 2, from now on, we can assume that no vertex $b \in B$ has a join to any A_i, i.e., if b contacts A_i then it distinguishes A_i.

Again, as by (7), there is a vertex $b^* \in B$ which contacts every A_i. However, we need a stronger property. For this, we define the following notions:

Definition 1. *For vertices $b_1, b_2 \in B$ and a nontrivial component $K = A_i$ of A, we say:*

- (i) b_2 overtakes b_1 for K if b_2 *distinguishes an edge in* $K \setminus N(b_1)$.
- (ii) b_2 includes b_1 for K *if* $N(b_2) \cap K \supseteq N(b_1) \cap K$.
- (iii) b_2 strictly includes b_1 for K *if* $N(b_2) \cap K \supset N(b_1) \cap K$.
- (iv) b_1 and b_2 cover K *if* $N(b_1) \cup N(b_2) = K$.
- (v) $b_1 \to b_2$ *if* b_2 *overtakes* b_1 *for at least three distinct nontrivial components of* A.
- (vi) $b^* \in B$ *is* a good vertex of B *if for none of the vertices* $b \in B \setminus \{b^*\}$, $b^* \to b$ *holds.*

Assume that G has an e.d.s. D of finite weight.

Claim 2. *For vertices $b_1, b_2 \in B$, we have:*

- (i) *b_1 and b_2 cover at most two A_i, A_j, $i, j \in \{1, \ldots, k\}$, $i \neq j$.*
- (ii) *If b_2 overtakes b_1 for A_i then for any A_j, $j \neq i$, b_1 does not overtake b_2.*
- (iii) *If b_2 overtakes b_1 for some A_i, A_j, $i \neq j$, then b_2 strictly includes b_1 for A_i, A_j.*
- (iv) *If b_2 overtakes b_1 for some A_i, A_j, $i \neq j$, then b_2 includes b_1 for all but at most two A_ℓ, $1 \leq \ell \leq k$.*
- (v) *If b_2 strictly includes b_1 for some A_i then b_2 includes b_1 for all but at most two A_ℓ, $1 \leq \ell \leq k$.*

Let $H = (B, F)$ denote the directed graph with vertex set B and edges $b \to b' \in F$ as in Definition 1 (v). Thus, a good vertex of B is one with outdegree 0 with respect to H. As usual, H is a *directed acyclic graph* (*dag* for short) if there is no directed cycle in H.

Claim 3. *H is a dag.*

It is well known that any dag has a vertex with outdegree 0. Thus, Claim 3 implies:

Claim 4. *There is a good vertex $b^* \in B$.*

Let b^* be such a good vertex. Then, since by the condition in Lemma 4, $B \cap D = \emptyset$, b^* must have a D-neighbor $a^* \in A \cap N(b^*) \cap D$; the algorithm tries all possible vertices in $A \cap N(b^*)$. Let $D(a^*)$ denote an e.d.s. with $a^* \in D(a^*)$; without loss of generality, assume that $a^* \in A_1$. Clearly, $(A_1 \setminus \{a^*\}) \cap D(a^*) = \emptyset$. Without loss of generality, let us assume that $A_1 = \{a^*\}$. Since a^* dominates b^*, each neighbor of b^* in A_i, $i \geq 2$, is not in $D(a^*)$. For $i \in \{2, \ldots, k\}$, let $A'_i := A_i \setminus N(b^*)$, and let $A' = \{a^*\} \cup A'_2 \cup \ldots \cup A'_k$. Obviously, we have:

(a) For each A'_i, $|A'_i \cap D(a^*)| = 1$.
Moreover, as before, we can assume:
(b) For each vertex $b \in B$, b does not have a join to two distinct A'_i, A'_j, $i \neq j$.
(c) If vertex $b \in B$ has a join to exactly one A'_i then it does not contact the remaining components A'_j, $j \neq i$.
Thus, again by (3) and the subsequent paragraph in Sect. 2, from now on, we can assume that no vertex $b \in B$ has a join to any A_i, i.e., if b contacts A_i then it distinguishes A_i. Next we claim:
(d) At most two distinct components A'_i, A'_j are distinguished by some vertex of $B \setminus \{b^*\}$.

Summarizing, by the above, $D(a^*)$ exists if and only if

(i) the above properties hold and
(ii) $G[A' \cup B]$ has a (weighted) e.d.s. $D(a^*)$ with $B \cap D(a^*) = \emptyset$.

Checking (i) can be done in polynomial time (actually one should just check if some of the above properties hold). Checking (ii) can be done in polynomial time as shown below: For the components of $G[A']$, let

– $C_1(A')$ be the set of those components of $G[A']$ which are not distinguished by any vertex of B, and
– $C_2(A')$ be the set of those components of $G[A']$ which are distinguished by some vertex of B.

For each member K of $C_1(A')$, any vertex of K (of minimum weight, for WED) can be assumed to be the only vertex in $D(a^*) \cap K$, without loss of generality since such vertices form a clique and have respectively the same neighbors in $G[(A' \cup B) \setminus K]$ (for WED, one can select a vertex of minimum weight).

Concerning $C_2(A')$, we have $|C_2(A')| \leq 2$ by property (d). Then the set $\{(a^*, a_2, \ldots, a_k) \mid a_i \in A'_i, i \in \{2, \ldots, k\}\}$ of k-tuples of candidate vertices in $D(a^*)$ contains $\mathcal{O}(n^2)$ members by property (d). Thus one can check in polynomial time if $D(a^*)$ exists.

Algorithm WED for P_6-free unipolar graphs

Given: A connected P_6-free unipolar graph $G = (A \cup B, E)$ such that B is a clique and $G[A]$ is the disjoint union of cliques $A_1, \ldots, A_k$.

Task: Determine an e.d.s. of G with minimum finite weight if there is one or state that G does not have such an e.d.s.

(a) Reduce G to G' by the Join-Reduction Algorithm. {From now on, we can assume that for every $b \in B$ and every $i \in \{1, \ldots, k\}$, b distinguishes A_i if b contacts A_i.}
(b) Construct the dag H according to Definition 1 (v), and determine a good vertex $b^* \in B$ in H.
(c) For every neighbor $a^* \in A'$ of b^*, determine the $\mathcal{O}(n^2)$ possible tuples of $D(a^*)$-candidates and check whether they are an e.d.s. of finite weight.
(d) Finally, choose an e.d.s. of minimum finite weight or state that G' does not have such an e.d.s.

Theorem 1. *Algorithm WED for P_6-free unipolar graphs is correct and can be done in time $\mathcal{O}(n^3m)$.*

4 The Algorithm for WED on P_6-Free Graphs

By combining the principles described above (and in particular by Corollary 1, Lemma 4, and Theorem 1) we obtain:

Algorithm WED for P_6-free graphs

Given: A P_6-free graph $G = (V, E)$.

Task: Determine an e.d.s. of G with minimum finite weight if there is one or state that G does not have such an e.d.s.
For every $v \in V$ **do**
begin

(a) Determine the distance levels $N_i(v)$, $1 \leq i \leq 4$.
(b) For G_v as defined in Sect. 2, with $B = N_2(v)$ and $A = N_3(v) \cup N_4(v)$, reduce G_v to G'_v by the Join-Reduction Algorithm. {From now on, we can assume that for every $b \in B$ and every $i \in \{1, \ldots, k\}$, b distinguishes A_i if b contacts A_i.}
(c) According to the Component-Reduction Algorithm, determine a vertex $b^* \in B$ contacting every component in $G[A]$ which is not a clique, and for every neighbor $q^* \in N(b^*) \cap A$, do:
(c.1) Reduce G'_v to $G'(v, q^*)$ by the Component-Reduction Algorithm. {Now, $G'(v, q^*)$ is P_6-free unipolar.}
(c.2) Carry out the Algorithm WED for P_6-free unipolar graphs for input $G'(v, q^*)$ with its weight function.

(*d*) Finally, for every resulting candidate e.d.s., check whether it is indeed a finite weight e.d.s. of G, choose an e.d.s. of minimum finite weight of G or state that G does not have such an e.d.s.

end

Theorem 2. *Algorithm WED for P_6-free graphs is correct and can be done in time $\mathcal{O}(n^5 m)$.*

5 Conclusion

As mentioned, the direct approach for solving WED on P_6-free graphs gives a dichotomy result for the complexity of WED on F-free graphs. In [3], using an approach via G^2, it was shown that WED can be solved in polynomial time for P_6-free chordal graphs, and a conjecture in [3] says that for P_6-free graphs with e.d.s., the square is perfect which would also lead to a polynomial time algorithm for WED on P_6-free graphs but anyway, the time bound of our direct approach is better than in the case when the conjecture would be true.

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References

1. Biggs, N.: Perfect codes in graphs. J. Comb. Theory (B) **15**, 289–296 (1973)
2. Brandstädt, A.: Weighted efficient domination for P_5-free graphs in linear time. CoRR arXiv:1507.06765v1 (2015)
3. Brandstädt, A., Eschen, E.M., Friese, E.: Efficient domination for some subclasses of P_6-free graphs in polynomial time. In: Extended Abstract to Appear in the Conference Proceedings of WG 2015. Full version: CoRR arXiv:1503.00091v1 (2015)
4. Brandstädt, A., Fičur, P., Leitert, A., Milanič, M.: Polynomial-time algorithms for weighted efficient domination problems in AT-free graphs and dually chordal graphs. Inf. Process. Lett. **115**, 256–262 (2015)
5. Brandstädt, A., Giakoumakis, V.: Weighted efficient domination for (P_5+kP_2)-free graphs in polynomial time. CoRR arXiv:1407.4593v1 (2014)
6. Brandstädt, A., Milanič, M., Nevries, R.: New polynomial cases of the weighted efficient domination problem. In: Chatterjee, K., Sgall, J. (eds.) MFCS 2013. LNCS, vol. 8087, pp. 195–206. Springer, Heidelberg (2013)
7. Brandstädt, A., Mosca, R.: Weighted efficient domination for P_6-free graphs in polynomial time. CoRR arXiv:1508.07733v1 (2015). (based on a manuscript by R. Mosca, Weighted Efficient Domination for P_6-Free Graphs, July 2015)
8. Eschen, E., Wang, X.: Algorithms for unipolar and generalized split graphs. Discrete Appl. Math. **162**, 195–201 (2014)
9. Garey, M.R., Johnson, D.S.: Computers and Intractability - A Guide to the Theory of NP-Completeness. Freeman, San Francisco (1979)

10. Lokshtanov, D., Pilipczuk, M., van Leeuwen, E.J.: Independence and efficient domination on P_6-free graphs. In: Conference Proceedings SODA 2016, pp. 1784–1803. CoRR arXiv:1507.02163v2 (2015)
11. McDiarmid, C., Yolov, N.: Recognition of unipolar and generalized split graphs. Algorithms **8**, 46–59 (2015)
12. Milanič, M.: Hereditary efficiently dominatable graphs. J. Graph Theory **73**, 400–424 (2013)