Geodetic Convexity Parameters for Graphs with Few Short Induced Paths

Mitre C. Dourado[1], Lucia D. Penso[2], and Dieter Rautenbach[2](✉)

[1] Departamento de Ciência da Computação, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
mitre@dcc.ufrj.br

[2] Institute of Optimization and Operations Research, Ulm University, Ulm, Germany
{lucia.penso,dieter.rautenbach}@uni-ulm.de

Abstract. We study several parameters of geodetic convexity for graph classes defined by restrictions concerning short induced paths. Partially answering a question posed by Araujo et al., we show that computing the geodetic hull number of a given P_9-free graph is NP-hard. Similarly, we show that computing the geodetic interval number of a given P_5-free graph is NP-hard. On the positive side, we identify several graph classes for which the geodetic hull number can be computed efficiently. Furthermore, following a suggestion of Campos et al., we show that the geodetic interval number, the geodetic convexity number, the geodetic Carathéodory number, and the geodetic Radon number can all be computed in polynomial time for $(q, q-4)$-graphs.

Keywords: Geodetic convexity · Hull number · Geodetic number · Interval number · Convexity number · Carathéodory number · Radon number · P_k-free graphs · $(q,\ q-4)$-graphs

1 Introduction

In the present paper we study five prominent graph parameters of geodetic convexity, the hull number, the interval number, the convexity number, the Carathéodory number, and the Radon number, for graph classes defined by restrictions concerning short induced paths. Our motivation mainly comes from two recent papers. In [7] Campos, Sampaio, Silva, and Szwarcfiter show that for the P_3-convexity, the above parameters can be determined in linear time for $(q, q-4)$-graphs. In their conclusion they suggest to consider the geodetic versions of the parameters for these graphs. In [3] Araujo, Morel, Sampaio, Soares, and Weber study the geodetic hull number of P_5-free graphs. They show that this number can be computed in polynomial time for triangle-free P_5-free graphs, and ask about the computational complexity of the geodetic hull number of P_k-free graphs in general.

Before we discuss further related work and our own contribution, we collect some relevant definitions. All graphs will be finite, simple, and undirected, and

P. Heggernes (Ed.): WG 2016, LNCS 9941, pp. 25–37, 2016.
DOI: 10.1007/978-3-662-53536-3_3

we use standard terminology and notation. A graph G is *F-free* for some graph F if G does not contain an induced subgraph that is isomorphic to F. For a positive integer n, let K_n, P_n, $K_{1,n-1}$, and C_n be the complete graph, the path, the star, and the cycle of order n, respectively. For an integer q at least 4, a graph G is a *$(q, q-4)$-graph* [5] if every set of q vertices of G induces at most $q-4$ distinct P_4s. The *clique number* $\omega(G)$ of a graph G is the maximum order of a clique in G, which is a set of pairwise adjacent vertices. The *independence number* $\alpha(G)$ of a graph G is the maximum order of an independent set in G, which is a set of pairwise non-adjacent vertices. A vertex of a graph is *simplicial* if its neighborhood is a clique. For an integer k, let $[k]$ be the set of all positive integers at most k.

For a set X of vertices of a graph G, the *interval* $I_G(X)$ of X in G is the set of vertices of G that contains X as well as all vertices of G that lie on shortest paths between vertices from X. If $I_G(X) = X$, then X is a *convex* set. The *hull* $H_G(X)$ of X in G is the smallest convex set that contains X. If $H_G(X) = V(G)$, then X is a *hull set* of G, and if $I_G(X) = V(G)$, then X is an *interval set* of G. The *hull number* $h(G)$ of G [24] is the smallest order of a hull set of G. Similarly, the *interval number* $i(G)$ of G, also known as the *geodetic number* [28], is the smallest order of an interval set of G. The *convexity number* $cx(G)$ of G [11] is the maximum cardinality of a convex set that is a proper subset of the vertex set of G. Inspired by a classical theorem of Carathéodory [6], the *Carathéodory number* $cth(G)$ of G [19] is the minimum integer k such that for every set X of vertices of G, and every vertex x in $H_G(X)$, there is a subset Y of X of order at most k such that x belongs to $H_G(Y)$. Similarly, inspired by a classical theorem by Radon [30], the *Radon number* $r(G)$ of G [13,14] is the minimum integer k such that for every set X of at least k vertices of G, there is a subset X_1 of X such that $H_G(X_1) \cap H_G(X \setminus X_1) \neq \emptyset$. A set A of vertices of G is *anti-Radon* if A has no subset A_1 with $H_G(A_1) \cap H_G(A \setminus A_1) \neq \emptyset$. It is easy to see that the Radon number is exactly one more than the maximum cardinality of an anti-Radon set. Note that a clique is anti-Radon.

For reduction arguments useful to prove Theorem 6 below, we consider a second kind of convexity. For a set X of vertices of a graph G, the *restricted interval* $I'_G(X)$ of X in G is the set of vertices of G that contains X as well as all vertices of G that lie on an induced P_3 between vertices from X, that is, we only consider shortest paths of order 3. This leads to a convexity that has recently been studied on its own right [4], and is different from the above-mentioned P_3-convexity. If $I'_G(X) = X$, then X is *restricted convex*. The *restricted hull* $H'_G(X)$ of X in G, a *restricted hull set* of G, a *restricted interval set* of G, the *restricted hull number* $h'(G)$ of G, the *restricted interval number* $i'(G)$ of G, the *restricted convexity number* $cx'(G)$ of G, the *restricted Carathéorody number* $cth'(G)$ of G, the *restricted Radon number* $r'(G)$ of G, and *restricted anti-Radon sets* are all defined in the obvious way. Note that $I'_G(X) \subseteq I_G(X)$, which implies $H'_G(X) \subseteq H_G(X)$. Hence, every anti-Radon set is a restricted anti-Radon set.
We briefly survey some known related results. The hull number is NP-hard for bipartite graphs [2] and even for partial cubes [1], but can be computed in

polynomial time for cographs [12], $(q, q-4)$-graphs [2], $\{C_3, P_5\}$-free graphs [3], distance-hereditary graphs [29], and chordal graphs [29]. Bounds on the hull number are given in [2,15,24]. The interval number is NP-hard for cobipartite graphs [22] and for chordal graphs as well as for chordal bipartite graphs [16], but can be computed in polynomial time for split graphs [16], proper interval graphs [23], block-cactus graphs [22], and monopolar chordal graphs [22]. The convexity number is NP-hard for bipartite graphs [17,27]. Finally, also the Carathéodory number [19] as well as the Radon number [14] are NP-hard. Next to the geodetic convexity and the P_3-convexity, further well-studied graphs convexities are the induced paths convexity, also known as the monophonic convexity [18,21,25], the all paths convexity [8], the triangle path convexity [9,10], and the convexity based on induced paths of order at least 4 [20].

Our contributions are as follows. Partially answering the question posed by Araujo et al. [3], we show that computing the hull number of a given P_ℓ-free graph is NP-hard for every $\ell \geq 9$. Similarly, we show that computing the interval number of a given P_ℓ-free graph is NP-hard for every $\ell \geq 5$. Furthermore, we extend the result of Araujo et al. [3] that the hull number can be computed in polynomial time for $\{C_3, P_5\}$-free graphs to $\{\text{paw}, P_5\}$-free graphs, to triangle-free graphs in which every six vertices induce at most one P_5, and to $\{C_3, \ldots, C_{k-2}, P_k\}$-free graphs for every integer k. Following the suggestion of Campos et al. [7], we show that the interval number, the convexity number, the Carathéodory number, and the Radon number as well as their restricted versions can all be computed in polynomial time for $(q, q-4)$-graphs. Section 2 contains our complexity results. In Sect. 3 we present the efficiently solvable cases, and in Sect. 4 we list some open problems.

2 Complexity Results

Theorem 1. *For a given P_9-free graph G, and a given integer k, it is NP-complete to decide whether $h(G) \leq k$.*

Proof. Since the hull of a set of vertices can be computed in polynomial time, the considered decision problem belongs to NP. In order to prove NP-completeness, we describe a polynomial reduction from a restricted version of SATISFIABILITY. Therefore, let $\mathcal{C}$ be an instance of SATISFIABILITY consisting of m clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$ such that every clause in $\mathcal{C}$ contains at most three literals, and, for every variable x_i, there are at most three clauses in $\mathcal{C}$ that contain either x_i or $\bar{x}_i$. Note that SATISFIABILITY is still NP-complete for such instances (cf. [LO1] in [26]).

Clearly, we may assume that no clause in $\mathcal{C}$ contains a variable as well as its negation, and that $n \geq 2$. If, for some variable x_i, no clause in $\mathcal{C}$ contains $\bar{x}_i$, then setting x_i to true, and removing all clauses from $\mathcal{C}$ that contain x_i, leads to an equivalent instance. Therefore, by symmetry, we may assume that, for every variable x_i, some clause in $\mathcal{C}$ contains x_i, and some clause in $\mathcal{C}$ contains $\bar{x}_i$. If, for some variable x_i, there is only one clause in $\mathcal{C}$ containing x_i, and only one clause in $\mathcal{C}$ containing $\bar{x}_i$, then introducing two new variables x_{n+1} and x_{n+2}, and adding the three clauses $x_i \vee x_{n+1} \vee x_{n+2}$, $\bar{x}_{n+1} \vee x_{n+2}$, and $x_{n+1} \vee \bar{x}_{n+2}$, leads to an equivalent instance. Therefore, by symmetry, we may assume that, for every variable x_i, there are exactly three clauses in $\mathcal{C}$ that contain either x_i or $\bar{x}_i$. If, for some variable x_i, there is one clause in $\mathcal{C}$ containing x_i, and two clauses in $\mathcal{C}$ containing $\bar{x}_i$, then exchanging x_i with $\bar{x}_i$ within $\mathcal{C}$, leads to an equivalent instance. Altogether, we may assume that, for every variable x_i, there are exactly two clauses in $\mathcal{C}$, say $C_{j_i^{(1)}}$ and $C_{j_i^{(2)}}$, that contain x_i, and exactly one clause in $\mathcal{C}$, say $C_{j_i^{(3)}}$, that contains $\bar{x}_i$. Furthermore, these three clauses are distinct.

Let the graph G be constructed as follows starting with the empty graph:

- For every $j \in [m]$, add a vertex c_j.
- For every $i \in [n]$, add a copy G_i of the graph in Fig. 1, and denote the vertices as indicated in the figure.
- Add two further vertices w_1 and w_2.
- Add further edges to turn the set

$$C = \{c_j : j \in [m]\} \cup \{y_i : i \in [n]\} \cup \{v_i : i \in [n]\}$$

  into a clique.
- For every i in $[n]$, add the three edges $x_i^{(1)} c_{j_i^{(1)}}$, $x_i^{(2)} c_{j_i^{(2)}}$, and $\bar{x}_i c_{j_i^{(3)}}$.
- For every i in $[n]$, add the two edges $v_i w_1$ and $v_i w_2$.

See Fig. 2 for a partial illustration.

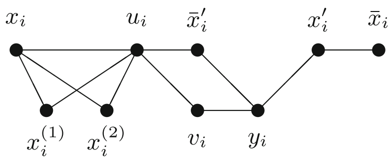

Fig. 1. The graph G_i.

Let $k = 2n + 2$. Note that the order of G is $9n + m + 2$. It remains to show that G is P_9-free, and that $\mathcal{C}$ is satisfiable if and only if $h(G) \leq k$.

Let P be an induced path in G. Since C is a clique, the subgraph $G[V(P) \cap C]$ of P induced by C is a (possibly empty) path of order at most 2. Note that all components of $G[V(G) \setminus C]$ have order at most 5, and only contain induced paths of order at most 3. This implies that P has order at most $3 + 2 + 3$, that is, G is P_9-free.

First, let $\mathcal{S}$ be a satisfying truth assignment for $\mathcal{C}$. Let

$$S = \{w_1, w_2\} \cup \bigcup_{i \in [n]:\, x_i \text{ true in } \mathcal{S}} \{x_i, x'_i\} \cup \bigcup_{i \in [n]:\, x_i \text{ false in } \mathcal{S}} \{\bar{x}_i, \bar{x}'_i\}.$$

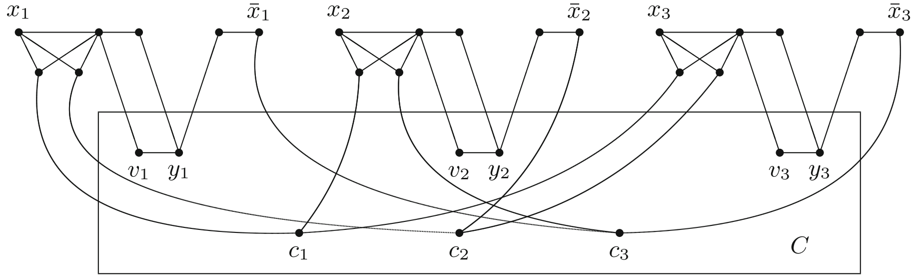

Fig. 2. Part of G, where $C_1 : x_1 \vee x_2 \vee x_3$, $C_2 : x_1 \vee \bar{x}_2 \vee x_3$, and $C_3 : \bar{x}_1 \vee x_2 \vee \bar{x}_3$. For the sake of visibility, the edges within C as well as the vertices w_1 and w_2 are not shown.

Clearly, $|S| = k$. Since $\{v_1, \ldots, v_n\} \subseteq H_G(\{w_1, w_2\})$, and $y_i \in H_G(\{x'_i, v_i\}) \cap H_G(\{\bar{x}'_i, v_i\})$ for $i \in [n]$, we obtain $\{v_1, \ldots, v_n\} \cup \{y_1, \ldots, y_n\} \subseteq H_G(S)$. If $i \in [n]$ is such that x_i is true in $\mathcal{S}$, and $\ell \in [n] \setminus \{i\}$, then $u_i \in H_G(\{x_i, v_i\})$, $\bar{x}'_i \in H_G(\{y_i, u_i\})$, and $\left\{c_{j_i^{(1)}}, c_{j_i^{(2)}}, x_i^{(1)}, x_i^{(2)}\right\} \subseteq H_G(\{x_i, v_\ell\})$. If $i \in [n]$ is such that x_i is false in $\mathcal{S}$, then $u_i \in H_G(\{\bar{x}'_i, v_i\})$, $x'_i \in H_G(\{\bar{x}_i, y_i\})$, and $c_{j_i^{(3)}} \in H_G(\{\bar{x}_i, v_i\})$. Since $\mathcal{S}$ is a satisfying truth assignment, this implies $\{x'_1, \ldots, x'_n\} \cup \{\bar{x}'_1, \ldots, \bar{x}'_n\} \cup \{u_1, \ldots, u_n\} \cup \{c_1, \ldots, c_m\} \subseteq H_G(S)$. For $i \in [n]$, we have $\left\{x_i^{(1)}, x_i^{(2)}\right\} \subseteq H_G\left(\left\{u_i, c_{j_i^{(1)}}, c_{j_i^{(2)}}\right\}\right)$, $x_i \in H_G\left(\left\{x_i^{(1)}, x_i^{(2)}\right\}\right)$, $\bar{x}_i \in H_G\left(\left\{x'_i, c_{j_i^{(3)}}\right\}\right)$. This implies

$$\bigcup_{i \in [n]} \left\{x_i, x_i^{(1)}, x_i^{(2)}, \bar{x}_i\right\} \subseteq H_G(S).$$

Altogether, it follows that S is a hull set, and, hence, $h(G) \leq |S| = k$.

Conversely, let S be a hull set of order at most $2n + 2$. Since w_1 and w_2 are simplicial, we have $w_1, w_2 \in S$. For $i \in [n]$, let

$$V_i^{(1)} = \left\{x_i, x_i^{(1)}, x_i^{(2)}, u_i, \bar{x}'_i\right\}, \quad V_i^{(2)} = \{x'_i, \bar{x}_i\}, \text{ and } \quad V_i^{(3)} = \{\bar{x}'_i, y_i, x'_i\}.$$

Since $N_G\left(V_i^{(1)}\right) \setminus V_i^{(1)} \subseteq C$, $N_G\left(V_i^{(2)}\right) \setminus V_i^{(2)} \subseteq C$, and C is a clique, the two sets $V(G) \setminus V_i^{(1)}$ and $V(G) \setminus V_i^{(2)}$ are convex, which implies that S intersects $V_i^{(1)}$ as well as $V_i^{(2)}$. Since $u_i v_i c_{j_i^{(3)}} \bar{x}_i$ is a path of order 4 between u_i and $\bar{x}_i$, no shortest path between two vertices in $V(G) \setminus V_i^{(3)}$ contains the two vertices $\bar{x}'_i$ and x'_i. Since $N_G(y_i) \setminus \{\bar{x}'_i, x'_i\} \subseteq C$, no shortest path between two vertices in $V(G) \setminus V_i^{(3)}$ intersects $V_i^{(3)}$ only in y_i. Since u_i has distance at most 2 from every vertex in C, no shortest path between two vertices in $V(G) \setminus V_i^{(3)}$ contains $\bar{x}'_i$ and y_i. Since $\bar{x}_i$ has distance at most 2 from every vertex in C, no shortest path between two vertices in $V(G) \setminus V_i^{(3)}$ contains y_i and x'_i. Altogether, it follows, that

$V(G) \setminus V_i^{(3)}$ is convex, which implies that S intersects $V_i^{(3)}$. Since $S \setminus \{w_1, w_2\}$ has order exactly $2n$, it follows that S contains exactly one vertex from $V_i^{(1)}$, exactly one vertex from $V_i^{(2)}$, and intersects $\{\bar{x}'_i, x'_i\}$. Since $y_i, x'_i \in H_G(\{\bar{x}'_i, \bar{x}_i\})$, we may assume that $\bar{x}'_i \in S$ implies $S \cap V(G_i) = \{\bar{x}_i, \bar{x}'_i\}$. Since $\{v_1, \ldots, v_n\} \subseteq H_G(\{w_1, w_2\})$ and $u_i, \bar{x}'_i, y_i, x_i^{(1)}, x_i^{(2)} \in H_G(\{x_i, x'_i, v_i, v_\ell\})$ for every $\ell \in [n] \setminus \{i\}$, we may assume that $x'_i \in S$ implies $S \cap V(G_i) = \{x_i, x'_i\}$. Altogether, for every $i \in [n]$, we obtain that

$$S \cap V(G_i) \in \{\{x_i, x'_i\}, \{\bar{x}_i, \bar{x}'_i\}\}. \tag{1}$$

Let $\mathcal{S}$ be the truth assignment, where we set x_i to be true exactly if $S \cap V(G_i) = \{x_i, x'_i\}$.

For $j \in [m]$, let

$$V_j = \{c_j\} \cup \bigcup_{i \in [n] : j = j_i^{(1)}} \left\{x_i, x_i^{(1)}\right\} \cup \bigcup_{i \in [n] : j = j_i^{(2)}} \left\{x_i, x_i^{(2)}\right\} \cup \bigcup_{i \in [n] : j = j_i^{(3)}} \{\bar{x}_i\}.$$

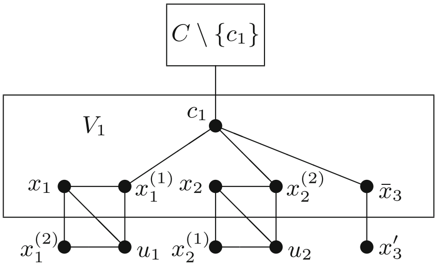

Fig. 3. The set V_1 for the clause $C_1 = x_1 \vee x_2 \vee \bar{x}_3$, where $j_1^{(1)} = j_2^{(2)} = 1$.

See Fig. 3 for an illustration. Note that $N_G(c_j) \setminus V_j = C \setminus \{c_j\}$. Furthermore, if $i \in [n]$ is such that $j = j_i^{(1)}$, then $N_G\left(\left\{x_i, x_i^{(1)}\right\}\right) \setminus V_j = \left\{u_i, x_i^{(2)}\right\}$, if $i \in [n]$ is such that $j = j_i^{(2)}$, then $N_G\left(\left\{x_i, x_i^{(2)}\right\}\right) \setminus V_j = \left\{u_i, x_i^{(1)}\right\}$, and, if $i \in [n]$ is such that $j = j_i^{(3)}$, then $N_G(\{\bar{x}_i\}) \setminus V_j = \{x'_i\}$. Since $C \setminus \{c_j\}$ is a clique, no shortest path between two vertices in $V(G) \setminus V_j$ intersects V_j only in c_j. If a shortest path P between two vertices in $V(G) \setminus V_j$ contains a vertex $x_i^{(r)}$ from V_j for some $r \in [2]$, then, possibly exchanging x_i with u_i on P, we may assume that P contains the vertex u_i. Since every two vertices in $\{u_1, \ldots, u_n\} \cup \{x'_1, \ldots, x'_n\}$ have distance at most three, no shortest path between two vertices in $V(G) \setminus V_j$ contains two vertices from

$$V_j \cap \left(\left\{x_1^{(1)}, \ldots, x_n^{(1)}\right\} \cup \left\{x_1^{(2)}, \ldots, x_n^{(2)}\right\} \cup \{\bar{x}_1, \ldots, \bar{x}_n\}\right).$$

This implies that, since u_i has distance at most two from each vertex in $C \setminus \{c_j\}$ for every $i \in [n]$, no shortest path between two vertices in $V(G) \setminus V_j$ contains a vertex from

$$V_j \cap \left(\left\{ x_1^{(1)}, \ldots, x_n^{(1)} \right\} \cup \left\{ x_1^{(2)}, \ldots, x_n^{(2)} \right\} \right).$$

Similarly, since x_i' has distance at most two from each vertex in $C \setminus \{c_j\}$ for every $i \in [n]$, no shortest path between two vertices in $V(G) \setminus V_j$ contains a vertex from

$$V_j \cap \{\bar{x}_1, \ldots, \bar{x}_n\}.$$

Altogether, it follows that $V(G) \setminus V_j$ is convex, which implies that S intersects

$$\bigcup_{i \in [n] : j = j_i^{(1)}} \left\{ x_i, x_i^{(1)} \right\} \cup \bigcup_{i \in [n] : j = j_i^{(2)}} \left\{ x_i, x_i^{(2)} \right\} \cup \bigcup_{i \in [n] : j = j_i^{(3)}} \{\bar{x}_i\}$$

for every $j \in [m]$. By (1) and the definition of $\mathcal{S}$, this implies that $\mathcal{S}$ is a satisfying truth assignment for $\mathcal{C}$, which completes the proof. □

Theorem 2. *For a given P_5-free graph G, and a given integer k, it is NP-complete to decide whether $i(G) \leq k$.*

3 Efficiently Solvable Cases

As observed in the introduction, Araujo et al. [3] show that the hull number can be computed in polynomial time for $\{C_3, P_5\}$-free graphs. We extend their result in several ways.

The *paw* is the unique graph with degree sequence $1, 2, 2, 3$. Note that the paw arises by attaching an endvertex to one vertex of a triangle.

Theorem 3. *The hull number of a given $\{\text{paw}, P_5\}$-free graph can be computed in polynomial time.*

Proof. Let G be a $\{\text{paw}, P_5\}$-free graph. Clearly, we may assume that G is connected. If G is P_4-free, then G is a cograph, and the statement follows from [12]. Hence, we may assume that G contains an induced path $P : u_1u_2u_3u_4$ of order 4. For a positive integer d, let $V_d = \{v \in V(G) : \text{dist}_G(v, V(P)) = d\}$, where $\text{dist}_G(v, V(P)) = \min\{\text{dist}_G(v, u) : u \in V(P)\}$. Let X be the union of $V(P)$ and the set of all simplicial vertices. We will show that adding at most one vertex to X yields a hull set of G, which implies that a minimum hull set can be found efficiently. In fact, every hull set contains all simplicial vertices, and considering the polynomially many extensions of the set of simplicial vertices by at most 5 vertices will yield a minimum hull set.

Let $v \in V_1$. First, we assume that v is adjacent to u_1. If v is adjacent to u_2, then, since G is paw-free, v is adjacent to all vertices of P. If v is not adjacent to u_2, then, since G is $\{\text{paw}, P_5\}$-free, $N_G(v) \cap V(P) \in \{\{u_1, u_3\}, \{u_1, u_4\}\}$.

Next, we assume that v is not adjacent to u_1 or u_4. Since G is paw-free, we obtain $N_G(v) \cap V(P) \in \{\{u_2\}, \{u_3\}\}$. Altogether, by symmetry, we obtain that $N_G(v) \cap V(P)$ is one of the sets $\{u_2\}$, $\{u_3\}$, $\{u_1, u_3\}$, $\{u_2, u_4\}$, $\{u_1, u_4\}$, and $V(P)$. Note that a vertex v in V_1 does not lie in $H_G(V(P)) \subseteq H_G(X)$ only if $N_G(v) \cap V(P) \in \{\{u_2\}, \{u_3\}\}$.

If $w \in V_2$, and v is a neighbor of w in V_1, then, since G is $\{\text{paw}, P_5\}$-free, $N_G(v) \cap V(P)$ is one of the sets $\{u_1, u_3\}$ and $\{u_2, u_4\}$. Since G is P_5-free, this implies $V_3 = \emptyset$, that is, $V(G) = V(P) \cup V_1 \cup V_2$. Suppose that w_1 and w_2 are adjacent vertices in V_2. Let $w_1 v u_i$ be a path between w_1 and $V(P)$. By symmetry, we may assume that $i < 4$. Since G is paw-free, the vertex w_2 is not adjacent to v. Now, $w_2 w_1 v u_i u_{i+1}$ is a P_5. Hence, V_2 is independent. Recall that every neighbor v in V_1 of a vertex in V_2 lies in $H_G(V(P))$. Therefore, every non-simplicial vertex in V_2 lies in $H_G(V(P)) \subseteq H_G(X)$, that is, $V(P) \cup V_2 \subseteq H_G(X)$.

Let V_1' be the set of non-simplicial vertices in V_1 that do not belong to $H_G(X)$. If $V_1' = \emptyset$, then $V_1 \subseteq H_G(X)$, and X is a hull set. Hence, we may assume that V_1' is not empty. If $A = \{v \in V_1' : N_G(v) \cap V(P) = \{u_2\}\}$, and $B = \{v \in V_1' : N_G(v) \cap V(P) = \{u_3\}\}$, then $V_1' = A \cup B$. Since G is paw-free, the sets A and B are independent. Since every vertex in V_1' is non-simplicial, it has two non-adjacent neighbors, at least one of which does not belong to $H_G(X)$. It follows that every vertex in A has a neighbor in B, and every vertex in B has a neighbor in A. Note that this implies that A and B are both not empty. Let H be the bipartite induced subgraph $G[A \cup B]$ of G with partite sets A and B. If $a_1, a_2 \in A$ and $b_1, b_2 \in B$ are such that $a_1 b_1, a_2 b_2 \in E(G)$ and $a_1 b_2, a_2 b_1 \notin E(G)$, then $b_1 a_1 u_2 a_2 b_2$ is a P_5. Hence, H is $2K_2$-free. Let a_1 be a vertex in A of maximum degree $d_H(a_1)$ in H. Suppose that a_1 is not adjacent to some vertex b_2 in B. Let a_2 be a neighbor of b_2 in A. Since $d_H(a_1) \geq d_H(a_2)$, there is a neighbor b_1 of a_1 in B that is not adjacent to a_2. Now, $a_1, a_2 \in A$ and $b_1, b_2 \in B$ are as above, which is a contradiction. Hence, $N_H(a_1) = B$, which implies that $B \subseteq H_G(\{a_1, u_3\}) \subseteq H_G(X \cup \{a_1\})$. Since $A \subseteq H_G(B \cup \{u_2\})$, it follows that $V_1' \subseteq H_G(X \cup \{a_1\})$, that is, $X \cup \{a_1\}$ is a hull set, which completes the proof. □

We proceed to our next generalization of the result of Araujo et al. [3].

Theorem 4. *Let k be a fixed positive integer.*
For a given $\{C_i : 3 \leq i \leq k-2\} \cup \{P_k\}$-free graph G, the hull number $h(G)$ can be computed in polynomial time.

Proof. The proof is by induction on k. For $k \leq 4$, the graph G is a cograph, and the statement follows from [12]. For $k = 5$, the statement follows from the result of Araujo et al. [3], or from Theorem 3. Now, let $k \geq 6$. The proof for $k = 6$ is similar to the proof of Theorem 3, and is given in the appendix. Hence, let $k \geq 7$.

Let G be a connected $\{C_i : 3 \leq i \leq k-2\} \cup \{P_k\}$-free graph. If G is P_{k-1}-free, then the result follows by induction. Hence, we may assume that G contains an induced path $P : u_1 \dots u_{k-1}$ of order $k-1$. Let X be the union of $V(P)$ and the set of all simplicial vertices. We will show that X is a hull set of G, which implies that a minimum hull set can be found efficiently. $\square$

Claim 1. *G has only cycles of orders $k-1$ and k, and every cycle of G is induced.*

Proof. Suppose that G has a cycle of order at least $k+1$. Let $C : x_0 \dots x_{\ell-1}x_0$ be a shortest cycle in G of order ℓ at least $k+1$. Since G is P_k-free, the cycle C is not induced. Let x_ix_j be an edge such that $j-i = \text{dist}_C(x_i, x_j)$ is minimum. By symmetry, we may assume that $i = 0$. Since $x_0x_1 \dots x_jx_0$ is an induced cycle of order $j+1$, we obtain $j \geq k-2$. Since $x_0x_1 \dots x_{j-1}$ is an induced path of order j, we obtain $j \leq k-1$, that is, $j \in \{k-2, k-1\}$. First, we assume that $j = k-1$. Since the path $x_1x_2 \dots x_k$ is not induced, the choice of x_ix_j implies that x_1x_k is an edge. Now, $x_1x_kx_{k-1}x_0x_1$ is a cycle of order 4, which is a contradiction. Hence $j = k-2$. Since the path $x_1x_2 \dots x_k$ is not induced, the choice of x_ix_j implies that there is an edge between $\{x_1, x_2\}$ and $\{x_{k-1}, x_k\}$. Since G is $\{C_4, C_5\}$-free, we obtain that x_2x_k is an edge. Since $x_0x_1x_2x_kx_{k-1}x_{k-2}x_0$ is an induced cycle of order 6, we obtain $k = 7$, which implies $j = 5$ and $\ell \geq 8$. Since $x_0x_5x_4x_3x_2x_7x_8 \dots x_0$ is a cycle of order $\ell-2$, the choice of C implies $\ell \leq 9$. Now, $x_0x_5x_6 \dots x_{\ell-1}x_0$ is a cycle of order $\ell-4 \leq 5$, which is a contradiction. Hence, G has no cycle of order at least $k+1$. In view of the forbidden induced subgraphs, this implies that G has only cycles of orders $k-1$ or k, and that every cycle of G is induced. $\square$

For a positive integer d, let $V_d = \{v \in V(G) : \text{dist}_G(v, V(P)) = d\}$.

Claim 2. *$V_d \neq \emptyset$ implies $d \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$.*

Proof. Suppose that V_d is non-empty for some $d \geq \left\lfloor \frac{k}{2} \right\rfloor$. Let $x_0 \dots x_d$ be a shortest path between a vertex x_0 in V_d and some vertex x_d of P. By symmetry, we may assume that $x_d = u_i$, where $i \geq \left\lceil \frac{k}{2} \right\rceil$. If x_{d-1} has a neighbor in $\left\{u_j : i - \left\lceil \frac{k-2}{2} \right\rceil \leq j \leq i-1\right\}$, then G has a cycle of order at most $\left\lceil \frac{k-2}{2} \right\rceil + 2 \leq k-2$, which is a contradiction. Hence, $x_0 \dots x_du_{i-1}u_{i-2} \dots u_{i-\left\lceil \frac{k-2}{2} \right\rceil}$ is an induced path of order $d+1+\left\lceil \frac{k-2}{2} \right\rceil \geq \left\lfloor \frac{k}{2} \right\rfloor + 1 + \left\lceil \frac{k-2}{2} \right\rceil = k$, which is a contradiction. $\square$

Claim 3. *For every d at least 2, every vertex in V_d has exactly one neighbor in V_{d-1}.*

Proof. Suppose that for some d at least 2, some vertex in V_d has two neighbors in V_{d-1}. This implies the existence of two distinct paths $Q : x_0x_1 \dots x_d$ and $Q' : x_0x'_1 \dots x'_d$ between some vertex x_0 in V_d and vertices x_d and x'_d of P. Since $2d \leq k-2$, we obtain that $V(Q) \cap V(Q') = \{x_0\}$. Let $x_d = u_i$ and $x'_d = u_j$, where $j > i$. The union of Q, Q', and the path $u_i \dots u_j$ is a cycle C. First, we assume that C has order k. Recall that, by Claim 1, all cycles of G are induced. Since $d \geq 2$, we obtain $j - i = k - 2d \leq k-4$. Hence, since P has order $k-1$, we may assume that $i \geq 2$. Since the path $u_{i-1} \dots u_jx'_{d-1} \dots x'_1x_0x_1 \dots x_{d-2}$ of order k

is not induced, we obtain that u_{i-1} is adjacent to x'_{d-1}. By Claim 1, this implies that $j-i \leq 2$. Now, $u_{i-1} \dots u_j x'_{d-1} u_{i-1}$ is a cycle of order at most 5, which is a contradiction. Hence, by Claim 1, the order of C is $k-1$. Since $d \geq 2$, we obtain $j-i \leq k-5$. Hence, since P has order $k-1$, we may assume that $i \geq 3$. Similarly as above, we obtain that x'_{d-1} is adjacent to u_{i-1} or u_{i-2}. If x'_{d-1} is adjacent to u_{i-1}, then $j-i=1$, and $u_{i-1}u_iu_jx'_{d-1}u_{i-1}$ is a cycle of order 4, which is a contradiction. Hence, x'_{d-1} is adjacent to u_{i-2}. By Claim 1, this implies that $j-i \leq 2$. Similarly as above, if $j-i=1$, then G contains a cycle of order 5, which is a contradiction. Hence $j-i=2$, and $u_{i-2} \dots u_j x'_{d-1} u_{i-2}$ is a cycle of order 6, which implies that $k=7$ and $d=2$. If $j \leq 5$, then $u_{i-1}x_2x_1x_0x'_1x'_2u_{j+1}$ is an induced path of order 7, which is a contradiction. Hence $j=6$, which implies that $i=j-2=4$. Now, $u_1u_2x'_1x_0x_1u_4u_5$ is an induced path of order 7, which is a contradiction. □

Claim 4. *For every d at least 2, the set V_d is independent.*

Proof. Suppose that for some d at least 2, the set V_d is not independent. This implies the existence of two distinct paths $Q : x_0x_1 \dots x_d$ and $Q' : x'_0x'_1 \dots x'_d$ between two adjacent vertices x_0 and x'_0 in V_d and vertices x_d and x'_d of P. If $V(Q) \cap V(Q') \neq \emptyset$, then Claims 1 and 2 imply that $x_d = x'_d$, $V(Q) \cap V(Q') = \{x_d\}$, and $d = \frac{k}{2} - 1$. By symmetry, we may assume that $x_d = u_i$, where $i \geq 3$. Now, $u_{i-2}u_{i-1}x_dx_{d-1} \dots x_1x_0x'_0x'_1 \dots x'_{d-2}$ is an induced path of order k, which is a contradiction. Hence, $V(Q) \cap V(Q') = \emptyset$. Let $x_d = u_i$ and $x'_d = u_j$, where $j > i$. The union of Q, Q', the edge $x_0x'_0$, and the path $u_i \dots u_j$ is a cycle C. First, we assume that C has order k. Since $d \geq 2$, we obtain $j-i = k-2d-1 \leq k-5$. Hence, since P has order $k-1$, we may assume that $i \geq 2$. Since the path $u_{i-1} \dots u_j x'_{d-1} \dots x'_1x'_0x_0x_1 \dots x_{d-2}$ of order k is not induced, we obtain that u_{i-1} is adjacent to x'_{d-1}. By Claim 1, this implies that $j-i \leq 2$. Now, $u_{i-1} \dots u_j x'_{d-1} u_{i-1}$ is a cycle of order at most 5, which is a contradiction. Hence, by Claim 1, the order of C is $k-1$. Since $d \geq 2$, we obtain $j-i = k-1-2d-1 \leq k-6$. Hence, since P has order $k-1$, we may assume that $i \geq 3$. Similarly as in the proof of Claim 3, we obtain that x'_{d-1} is adjacent to u_{i-2}. By Claim 1, this implies that $j-i \leq 2$. Similarly as above, if $j-i=1$, then G contains a cycle of order 5, which is a contradiction. Hence $j-i=2$, and $u_{i-2} \dots u_j x'_{d-1} u_{i-2}$ is a cycle of order 6, which implies that $k=7$. Now, $j-i \leq k-6=1$, which is a contradiction. □

Recall that X is the union of $V(P)$ and the set of all simplicial vertices.

Let $u \in V_d$ for some $d \geq 2$. Let $d' \geq d$ and $u' \in V_{d'}$ be such that u lies on a shortest path between u' and some vertex of P, and d' is maximum. Note that $d' = d$ and $u' = u$ is allowed. Since u' has no neighbor in $V_{d'+1}$, Claims 3 and 4 imply that u' is simplicial, and, hence, $u \in H_G(\{u'\} \cup V(P)) \subseteq H_G(X)$.

Let $u \in V_1 \setminus H_G(X)$. It follows that u does not have two neighbors on P, and also no neighbor in V_2. Since u is not simplicial, this implies that u has exactly one neighbor u_i on P as well as some neighbor u' in V_1. Let u_j be a neighbor of u' on P. Clearly, $i \neq j$. Note that $u_iuu'u_j$ is a path of order 4. Hence, since $u \notin H_G(V(P))$, the distance in G between u_i and u_j is at most 2, which implies the contradiction that G contains a cycle of order at most 5.

It follows that X is a hull set of G, which completes the proof. □

We present another generalization of the result of Araujo et al. [3], and consider triangle-free graphs in which every six vertices induce at most one P_5. Obviously, these graphs have been inspired by the $(q, q-4)$-graphs [5], and, consequently, we refer to them as (6,1)-*graphs*.

Theorem 5. *If G is a connected triangle-free $(6,1)$-graph, then either G is P_5-free or G arises from a star $K_{1,p}$ with $p \geq 2$ by subdividing two edges once and all remaining edges at most once.*

Corollary 1. *The hull number of a given triangle-free $(6,1)$-graph can be computed in polynomial time.*

Using the structural properties of $(q, q-4)$-graphs [5,7], and establishing suitable recursions for the convexity parameters, we obtain our final result.

Theorem 6. *Let q be a fixed integer at least 4.*
For a given $(q, q-4)$-graph G, all parameters $h(G)$, $h'(G)$, $i(G)$, $i'(G)$, $cx(G)$, $cx'(G)$, $cth(G)$, $cth'(G)$, $r(G)$, and $r'(G)$ can be computed in polynomial time.

4 Conclusion

We conclude with a number of questions. Can the considered parameters be determined in linear time for $(q, q-4)$-graphs? What is the complexity of the hull number for P_k-free graphs for $k \in \{5, 6, 7, 8\}$? What is the complexity of the other convexity parameters for $\{C_i : 3 \leq i \leq k-2\} \cup \{P_k\}$-free graphs or (triangle-free) $(6, 1)$-graphs?

For an integer q at least 5, let the $(q, q-5)$-graphs be those graphs in which every q vertices induce at most $q-5$ distinct P_5s. Do the (triangle-free) $(q, q-5)$-graphs allow a similar decomposition as the $(q, q-4)$-graphs [5,7]? Do these graphs have nice structural features?

References

1. Albenque, M., Knauer, K.: Convexity in partial cubes: the hull number. In: Pardo, A., Viola, A. (eds.) LATIN 2014. LNCS, vol. 8392, pp. 421–432. Springer, Heidelberg (2014)
2. Araujo, J., Campos, V., Giroire, F., Nisse, N., Sampaio, L., Soares, R.: On the hull number of some graph classes. Theoret. Comput. Sci. **475**, 1–12 (2013)
3. Araujo, J., Morel, G., Sampaio, L., Soares, R., Weber, V.: Hull number: P_5-free graphs and reduction rules. Discrete Appl. Math. **210**, 171–175 (2016)
4. Araujo, J., Sampaio, R., Santos, V., Szwarcfiter, J.L.: The convexity of induced paths of order three, applications: complexity aspects, manuscript
5. Babel, L., Olariu, S.: On the structure of graphs with few P_4's. Discrete Appl. Math. **84**, 1–13 (1998)

6. Carathéodory, C.: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rendiconti del Circolo Matematico di Palermo **32**, 193–217 (1911)
7. Campos, V., Sampaio, R.M., Silva, A., Szwarcfiter, J.L.: Graphs with few P_4's under the convexity of paths of order three. Discrete Appl. Math. **192**, 28–39 (2015)
8. Changat, M., Klavžar, S., Mulder, H.M.: The all-paths transit function of a graph. Czechoslovak Math. J. **51**, 439–448 (2001)
9. Changat, M., Mathew, J.: On triangle path convexity in graphs. Discrete Math. **206**, 91–95 (1999)
10. Changat, M., Prasanth, G.N., Mathews, J.: Triangle path transit functions, betweenness and pseudo-modular graphs. Discrete Math. **309**, 1575–1583 (2009)
11. Chartrand, G., Wall, C.E., Zhang, P.: The convexity number of a graph. Graphs Comb. **18**, 209–217 (2002)
12. Dourado, M.C., Gimbel, J.G., Kratochvíl, J., Protti, F., Szwarcfiter, J.L.: On the computation of the hull number of a graph. Discrete Math. **309**, 5668–5674 (2009)
13. Dourado, M.C., Pereira de Sá, V.G., Rautenbach, D., Szwarcfiter, J.L.: On the geodetic Radon number of grids. Discrete Math. **313**, 111–121 (2013)
14. Dourado, M.C., de Sá, V.G.P., Rautenbach, D., Szwarcfiter, J.L.: Near-linear-time algorithm for the geodetic Radon number of grids. Discrete Appl. Math. **210**, 277–283 (2016)
15. Dourado, M.C., Protti, F., Rautenbach, D., Szwarcfiter, J.L.: On the hull number of triangle-free graphs. SIAM J. Discrete Math. **23**, 2163–2172 (2010)
16. Dourado, M.C., Protti, F., Rautenbach, D., Szwarcfiter, J.L.: Some remarks on the geodetic number of a graph. Discrete Math. **310**, 832–837 (2010)
17. Dourado, M.C., Protti, F., Rautenbach, D., Szwarcfiter, J.L.: On the convexity number of graphs. Graphs Comb. **28**, 333–345 (2012)
18. Dourado, M.C., Protti, F., Szwarcfiter, J.L.: Complexity results related to monophonic convexity. Discrete Appl. Math. **158**, 1269–1274 (2010)
19. Dourado, M.C., Rautenbach, D., dos Santos, V., Schäfer, P.M., Szwarcfiter, J.L.: On the Carathéodory number of interval and graph convexities. Theoret. Comput. Sci. **510**, 127–135 (2013)
20. Dragan, F., Nicolai, F., Brandstädt, A.: Convexity and HHD-free graphs. SIAM J. Discrete Math. **12**, 119–135 (1999)
21. Duchet, P.: Convex sets in graphs II: minimal path convexity. J. Comb. Theory Ser. B **44**, 307–316 (1988)
22. Ekim, T., Erey, A.: Block decomposition approach to compute a minimum geodetic set. RAIRO Recherche Opérationnelle **48**, 497–507 (2014)
23. Ekim, T., Erey, A., Heggernes, P., van 't Hof, P., Meister, D.: Computing minimum geodetic sets of proper interval graphs. In: Fernández-Baca, D. (ed.) LATIN 2012. LNCS, vol. 7256, pp. 279–290. Springer, Heidelberg (2012)
24. Everett, M.G., Seidman, S.B.: The hull number of a graph. Discrete Math. **57**, 217–223 (1985)
25. Farber, M., Jamison, R.E.: Convexity in graphs and hypergraphs. SIAM J. Algebraic Discrete Methods **7**, 433–444 (1986)
26. Garey, M.R., Johnson, D.S., Computers, I.: A Guide to the Theory of NP-Completeness. W.H. Freeman & Co., New York (1979)
27. Gimbel, J.: Some remarks on the convexity number of a graph. Graphs Comb. **19**, 357–361 (2003)
28. Harary, F., Loukakis, E., Tsouros, C.: The geodetic number of a graph. Math. Comput. Model. **17**, 89–95 (1993)

29. Kanté, M.M., Nourine, L.: Polynomial time algorithms for computing a minimum hull set in distance-hereditary and chordal graphs. In: Emde Boas, P., Groen, F.C.A., Italiano, G.F., Nawrocki, J., Sack, H. (eds.) SOFSEM 2013. LNCS, vol. 7741, pp. 268–279. Springer, Heidelberg (2013)
30. Radon, J.: Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. Math. Ann. **83**, 113–115 (1921)