Chapter 3
Fuzzy Relations

Everything that exists in nature is due to chance and need.
(Democritus, C.460–C.370 BCE)

Abstract This chapter presents a short discussion of mathematical relations, basic concepts of fuzzy relations, and the composition between two fuzzy relations. Lastly, the chapter presents the rule for the composition of inferences, which is relevant to the modus ponens discussed in the next chapter.

Do the individuals of a species agree with Democritus: they relate to one another so as to construct the trajectories in the course of their lives only to survive apparently without any interest of optimizing anything? Or do they seek the maximum return for the minimum of effort, as was advocated by Leibniz when he said "that we live in the best of all worlds?" Maybe the difference between these two poles is just a matter of gradual truth.

Studies dealing with associations, relations or interactions between elements of many classes is of great interest in the analysis and understanding of many phenomena of real world problems and mathematics. Such studies are always concerned with establishing such relations. We will see in this chapter that fuzzy relations are a natural extension of classical mathematical relations and these fuzzy relations have wide applications.

3.1 Fuzzy Relations

The concept of relation in mathematics is formalized from the point of view of set theory. We will follow the same path. Intuitively, we say that the relation is *fuzzy* when we adopt a fuzzy set theory point of view, and is *crisp* when we use classical set theory to conceptualize the relation. The choice of the relation depends on the phenomenon that we are studying. However, fuzzy set theory is always more general

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than classical set theory, since fuzzy set theory includes classical set theory as a particular case (remember that a classical set—*crisp* set—is a particular fuzzy set). A classical relation indicates if there is or is not some association between two elements, while fuzzy relations indicate, in addition, the degree of this association.

Definition 3.1 A (classical) *relation* $\mathcal{R}$ over $U_1 \times U_2 \times \cdots \times U_n$ is any (classical) subset of the Cartesian product $U_1 \times U_2 \times \cdots \times U_n$. If the Cartesian product is formed by just two sets $U_1 \times U_2$, this relation is called a *binary relation* over $U_1 \times U_2$. If $U_1 = U_2 = \cdots = U_n = U$, we say that $\mathcal{R}$ is a *n-ary relation* over U.

A crisp relation $\mathcal{R}$ is a subset of the Cartesian product and can be represented by its characteristic function

$$\chi_{\mathcal{R}} : U_1 \times U_2 \times \ldots \times U_n \longrightarrow \{0, 1\},$$

with

$$\chi_{\mathcal{R}}(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in \mathcal{R} \\ 0 & \text{if } (x_1, x_2, \ldots, x_n) \notin \mathcal{R} \end{cases}. \tag{3.1}$$

The mathematical concept of fuzzy relation is formalized from the usual Cartesian product between sets, extending the characteristic function of a classical relation to a membership function.

Definition 3.2 A *fuzzy relation* $\mathcal{R}$ over $U_1 \times U_2 \times \ldots \times U_n$ is any fuzzy subset of $U_1 \times U_2 \times \ldots \times U_n$. Thus, a fuzzy relation $\mathcal{R}$ is defined by a membership function $\varphi_{\mathcal{R}} : U_1 \times U_2 \times \ldots \times U_n \longrightarrow [0, 1]$.

If the Cartesian product is formed by just two sets $U_1 \times U_2$, the relation is called a *binary fuzzy* relation over $U_1 \times U_2$. If the sets U_i, $i = 1, 2, \ldots, n$, are all equal to U, then we say that $\mathcal{R}$ is a *n-ary fuzzy relation* over U. For example, a binary fuzzy relation over U is a fuzzy relation $\mathcal{R}$ over $U \times U$. If a membership function of the fuzzy relation $\mathcal{R}$ is indicated by $\varphi_{\mathcal{R}}$, then the number

$$\varphi_{\mathcal{R}}(x_1, x_2, \ldots, x_n) \in [0, 1]$$

indicates the degree to which the elements x_i that compose the *n-tuple* $(x_1, x_2, \ldots, x_n)$ are related according to the relation $\mathcal{R}$.

The fuzzy inference relationships which are used in making decisions are of great importance, especially in the theory of the fuzzy controllers, as we shall see in Chap. 5. Technically, in fuzzy set theory this operation is similar to intersection, as seen in Chap. 1, Sect. 1.3. The great difference is in the associated universe from which the set comes. While in the intersection of fuzzy subsets, the same universal sets are the same, in the Cartesian product they can be different, as we shall see in the next definition.

Definition 3.3 The *fuzzy Cartesian product* of the fuzzy subsets $A_1, A_2, ..., A_n$ of $U_1, U_2, ..., U_n$, respectively, is the fuzzy relation $A_1 \times A_2 \times \cdots \times A_n$, whose membership function is given by

$$\varphi_{A_1 \times A_2 \times \cdots \times A_n}(x_1, x_2, \ldots, x_n) = \varphi_{A_1}(x_1) \wedge \varphi_{A_2}(x_2) \wedge \cdots \wedge \varphi_{A_n}(x_n),$$

where $\wedge$ represents the minimum.

Notice that if $A_1, A_2, \ldots, A_n$ are classical sets, then the classical Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ may be obtained by Definition 3.3, substituting the membership functions by the respective characteristic functions of the sets $A_1, A_2, \ldots, A_n$. The next example illustrates the application of the Cartesian product.

Example 3.1 Let us consider again the Table 1.1 of the Example 1.8 which relates the diagnostics of 5 patients with two symptoms: fever and myalgia.

Patient	F: Fever	M: Myalgia	D: Diagnosis
1	0.7	0.6	0.6
2	1.0	1.0	1.0
3	0.4	0.2	0.2
4	0.5	0.5	0.5
5	1.0	0.2	0.2

To diagnose a patient the doctor evaluates the symptoms that are specific to each disease. Many diseases can present symptoms like fever and myalgia with different intensities and measures. For example, for flu, the patient with fever and myalgia with intensities that, if represented by fuzzy subsets, must have distinct universal sets. The universe that indicates the fever can be given by the possible temperatures of a person, while the myalgia can be assessed by the numbers of painful areas.

The indication of how much an individual has flu can be taken as the degree of membership of the set of the fever symptoms and the set of myalgia. For example, the patient 1 in Table 1.1 has temperature x whose membership in the fever set F is $\varphi_F(x) = 0.7$ and the value y to myalgia is $\varphi_M(y) = 0.6$. The diagnosis of the patient 1 for the flu is then given by:

$$\text{Patient 1: } \varphi_{\text{flu}}(x, y) = \varphi_F(x) \wedge \varphi_M(y) = 0.7 \wedge 0.6 = 0.6.$$

Here we have used the fuzzy binary relationship "and" as "min". This means that the patient 1 is in the fuzzy subset of the ones who have fever and myalgia with membership degree 0.6 which coincides with the degree of its diagnosis for flu.

This number obtained can give support to a physician's decision of which adopted treatment is best for the patient. It is clear that from the theoretical point of view, the classical Cartesian product could also be employed for the diagnoses. In this case the information would be flu (degree one) or not flu (degree zero). Consequently, just patient 2 of Table 1.1 would be considered to have flu.

Chapter 6, Sect. 6.2.3, will present a more complete study about medical diagnoses. However, we want to note that in the example above we have used the "min" binary relationship because we have assumed that flu occurs with myalgia. However, if myalgia was not a strong correlated part of the diagnosis, we would use a different fuzzy relationship.

Exercise 3.1 Compare the Example 3.1 with the Example 1.8 and explain the difference between them.

Exercise 3.2 Investigate one more symptom that is typical for flu (coryza, for example) and add it as a fuzzy subset in Table 1.1 (create some values) and diagnose the patients who have flu using the "and" operator.

3.1.1 Forms of Representation and Properties of the Binary Relations

This text will just stress the forms of representation and some properties of the binary relations and fuzzy binary relations, which will be illustrated by some examples. The interested reader can consult other texts for more detailed discussion of relationships. The following example will help to illustrate the main representations that we will be using in this text.

Example 3.2 Let U be an ecosystem with the following populations: *eagles* (e), *snakes* (s), *insects* (i), *hare* (h) and *frogs* (f). A possible study between the individuals of these populations might be the predation process, that is, the relation *prey-predator*. To study the relation between two individuals from this ecosystem, this relation can be mathematically modeled by a *binary relation* $\mathcal{R}$ with $\varphi_{\mathcal{R}}(x, y) = 0$ if y is not a predator of x and $\varphi_R(x, y) \neq 0$ if y is a predator of x, where x and y are individuals from the set U.

Next, we will discuss two possible cases of the use of the classical relation and of the fuzzy relation for this example.

- If the interest regarding the relation is just to indicate who is the predator and who is the prey in U, then we can choose the classical theory and $\mathcal{R}$ will be a classical binary relation. In this case,

$$\varphi_{\mathcal{R}}(x, y) = \chi_{\mathcal{R}}(x, y) = \begin{cases} 1 & \text{if } y \text{ is a predator of } x \\ 0 & \text{if } y \text{ is not a predator of } x \end{cases}.$$

  A graphic representation for this relation is in Fig. 3.1, where we put the animals in alphabetical order on a pair of axes.
  The points that are highlighted in Fig. 3.1 indicate the pairs that belong to the relation $\mathcal{R}$, that is, the relation $\mathcal{R}$ reveals who is the predator of whom accordingly to some specialist.

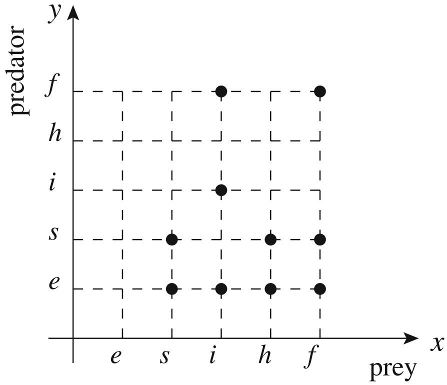

Fig. 3.1 Representation of the classical relation between the predators and their prey

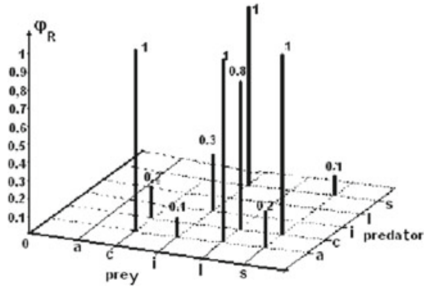

Fig. 3.2 Fuzzy relation and the many degrees of preference

- If there is interest in knowing, for example, the gradual preference of a predator for some prey in U, then a good option is to choose $\mathcal{R}$ as a fuzzy relation. In that case, $\varphi_{\mathcal{R}}(x, y)$ indicates the preference degree of y for prey x. Supposing that there is no difference in the predation degree in each species, one possibility for $\varphi_{\mathcal{R}}(x, y)$ for this example, might be measured according to a specialist as illustrated in Fig. 3.2, where in the third axis (vertical axis) this (fuzzy) measure is represented as degree $\varphi_{\mathcal{R}}(x, y)$.

When X and Y are finite, the most common forms to represent a binary fuzzy relation in $X \times Y$ are the tabular and the matrix forms. Let us define $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and the fuzzy relation $\mathcal{R}$ over $X \times Y$ with membership function given by $\varphi_{\mathcal{R}}(x_i, y_j) = r_{ij}$, for $1 \leq i \leq n$ and $1 \leq j \leq m$. The representations of $\mathcal{R}$ can be in table or in matrix form as it follows below.

$\mathcal{R}$	y_1	y_2	$\ldots$	y_n
x_1	r_{11}	r_{12}	$\ldots$	r_{1n}
x_2	r_{21}	r_{22}	$\ldots$	r_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_m	r_{m1}	r_{m2}	$\ldots$	r_{mn}

or $\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \ldots & r_{mn} \end{bmatrix}.$

To exemplify the representations in table and in matrix form for Example 3.2 we have, respectively,

		predator				
	$\mathcal{R}$	**e**	**s**	**i**	**h**	**f**
p	**e**	0.0	0.0	0.0	0.0	0.0
r	**s**	1.0	0.2	0.0	0.0	0.0
e	**i**	0.1	0.0	0.3	0.0	1.0
y	**h**	1.0	0.8	0.0	0.0	0.0
s	**f**	0.2	1.0	0.0	0.0	0.1

and

$$\mathcal{R} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.3 & 0.0 & 1.0 \\ 1.0 & 0.8 & 0.0 & 0.0 & 0.0 \\ 0.2 & 1.0 & 0.0 & 0.0 & 0.1 \end{bmatrix}.$$

The following definition will be used in subsequent analyses.

Definition 3.4 Let $\mathcal{R}$ be a binary fuzzy relation defined over $X \times Y$. The *inverse binary fuzzy relation*, $\mathcal{R}^{-1}$, defined over $Y \times X$, has the following membership function $\varphi_{\mathcal{R}^{-1}} : Y \times X \longrightarrow [0, 1]$, with $\varphi_{\mathcal{R}^{-1}}(y, x) = \varphi_{\mathcal{R}}(x, y)$.

Notice that the matrix of $\mathcal{R}^{-1}$ coincides with the transpose of $\mathcal{R}$, since $\varphi_{\mathcal{R}^{-1}}(y, x) = \varphi_{\mathcal{R}}(x, y)$. For this reason many texts of fuzzy logic adopt the term transpose relation instead of inverse (see Pedrycz and Gomide [1]). Thus, if $\mathcal{R}$ is the fuzzy relation of the Example 3.2, then the matrix representation of its inverse $\mathcal{R}^{-1}$ is given by its transpose

$$\mathcal{R}^{\top} = \begin{bmatrix} 0.0 & 1.0 & 0.1 & 1.0 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.8 & 1.0 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.1 \end{bmatrix}.$$

The transpose, $\mathcal{R}^{-1}$, for our prey-predator example, indicates that x is a prey of y, while by $\mathcal{R}$ we have that y is a predator of x.

3.2 Composition Between Binary Fuzzy Relations

The composition between relations is of great importance in many applications. This operation will be explored extensively in Chap. 6, where the main applications in medical diagnoses are developed. Also in Chap. 6 we will study many types of compositions between fuzzy relations. Here, in this section, we will present only the more traditional compositions of fuzzy logic.

Definition 3.5 Let $\mathcal{R}$ and $\mathcal{S}$ be two binary fuzzy relations in $U \times V$ and $V \times W$, respectively. The *composition* $\mathcal{R} \circ \mathcal{S}$ is a binary fuzzy relation in $U \times W$ whose membership function is given by

$$\varphi_{\mathcal{R}\circ\mathcal{S}}(x, z) = \sup_{y \in V} [\min(\varphi_{\mathcal{R}}(x, y), \varphi_{\mathcal{S}}(y, z))]. \tag{3.2}$$

Let the sets U, V and W be finite. Then the matrix form of the relation $\mathcal{R} \circ \mathcal{S}$, given by the composition [max–min], is obtained by a matrix multiplication, substituting the product by the minimum and the sum by the maximum. Indeed, suppose that

$$U = \{u_1, u_2, \ldots, u_m\}; V = \{v_1, v_2, \ldots, v_n\} \quad \text{and} \quad W = \{w_1, w_2, \ldots, w_p\}$$

and that

$$\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \ldots & r_{mn} \end{bmatrix}_{m\times n} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1p} \\ s_{21} & s_{22} & \ldots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \ldots & s_{np} \end{bmatrix}_{n\times p}.$$

According to Definition 3.5, the binary fuzzy relation given by the composition [max–min] has the following matrix form

$$\mathcal{T} = \mathcal{R} \circ \mathcal{S} = \begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1p} \\ t_{21} & t_{22} & \ldots & t_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \ldots & t_{mp} \end{bmatrix}_{m\times p},$$

where

$$t_{ij} = \sup_{1\le k\le n} [\min(\varphi_R(u_i, v_k), \varphi_S(v_k, w_j))] = \sup_{1\le k\le n} [\min(r_{ik}, s_{kj})]. \tag{3.3}$$

The special case of the composition [max–min], which will be presented next, will be used in a more general way in Chap. 6.

Definition 3.6 (*Rule of inference composition*) Let U and V be two sets with the respective classes of the fuzzy subsets $\mathcal{F}(U)$ and $\mathcal{F}(V)$ and $\mathcal{R}$ a binary relation over $U \times V$.

(i) The relation $\mathcal{R}$ defines a function of $\mathcal{F}(U)$ into $\mathcal{F}(V)$ such that for each $A \in \mathcal{F}(U)$ there is a corresponding element $B \in \mathcal{F}(V)$ whose membership function is given by

$$\varphi_B(y) = \varphi_{\mathcal{R}(A)}(y) = \sup_{x \in U}[\min(\varphi_{\mathcal{R}}(x, y), \varphi_A(x))]. \tag{3.4}$$

This composition is known as the *rule of inference composition* which will produce other rules as we shall see in Chaps. 4 and 5.

(ii) The relation $\mathcal{R}$ also defines a function of $\mathcal{F}(V)$ into $\mathcal{F}(U)$: for each $B \in \mathcal{F}(V)$ there is a corresponding element $A \in \mathcal{F}(U)$ whose membership function is given by

$$\varphi_A(x) = \varphi_{\mathcal{R}^{-1}(B)}(x) = \sup_{y \in V}[\min(\varphi_{\mathcal{R}^{-1}}(y, x), \varphi_B(y))]. \tag{3.5}$$

A is called the *inverse image* of B by $\mathcal{R}$.

Notice that formula (3.4) can be rewritten as

$$\varphi_B(y) = \varphi_{\mathcal{R}(A)}(y) = \sup_{x \in U}[\min(\varphi_A(x), \varphi_{\mathcal{R}}(x, y))].$$

Thus, according to (3.2),

$$B = \mathcal{R}(A) = A \circ \mathcal{R}.$$

In a similar way the inverse image is given by

$$A = B \circ \mathcal{R}^{-1}.$$

Exercise 3.3 Suppose that the universal sets U and V are finite so that A, B and $\mathcal{R}$ can be represented in matrix form. From the observation above verify that

$$B = A \circ \mathcal{R} \quad \text{and} \quad A = B \circ \mathcal{R}^{\top}$$

where A and B are the matrix forms of the respective fuzzy sets whose elements are obtained from (3.3).

We will present some important definitions to deepen our understanding of binary relations which will be made first for classical binary relations and next for the binary fuzzy relations. The definitions for the classical binary relations $\mathcal{R}$ will be made by the use of their characteristic functions $\chi_{\mathcal{R}} : U \times U \longrightarrow \{0, 1\}$, for a better understanding of the fuzzy case.

Definition 3.7 Let $\mathcal{R}$ be a (classical) binary relation over U. Then, for any x, y and z of U, the relation $\mathcal{R}$ is

(i) *reflexive* if $\chi_{\mathcal{R}}(x, x) = 1$;
(ii) *symmetric* if $\chi_{\mathcal{R}}(x, y) = 1$ implies $\chi_{\mathcal{R}}(y, x) = 1$;
(iii) *transitive* if $\chi_{\mathcal{R}}(x, y) = \chi_{\mathcal{R}}(y, z) = 1$ implies $\chi_{\mathcal{R}}(x, z) = 1$;
(iv) *anti-symmetric* if $\chi_{\mathcal{R}}(x, y) = \chi_{\mathcal{R}}(y, x) = 1$ implies $x = y$.

Observe that the definitions above represent exactly each one of the traditional definitions used in classical set theory. The use of the characteristic function was just an "artifice" to facilitate the understanding of those concepts in the fuzzy case. There are some little differences in the extensions of the concepts given in the Definition 3.7, when adapted to the fuzzy case, mainly the concept of transitivity (see [2, 3]).

Definition 3.8 Let $\mathcal{R}$ be a binary fuzzy relation over U, whose membership function is $\varphi_{\mathcal{R}}$. Then, for any $x,\ y$ and z of U, the fuzzy relation $\mathcal{R}$ is

(i) *reflexive* if $\varphi_{\mathcal{R}}(x, x) = 1$;
(ii) *symmetric* if $\varphi_{\mathcal{R}}(x, y) = \varphi_{\mathcal{R}}(y, x)$;
(iii) *transitive* if $\varphi_{\mathcal{R}}(x, z) \geq \varphi_{\mathcal{R}}(x, y) \wedge \varphi_{\mathcal{R}}(y, z)$, where $\wedge =$ minimum.
(iv) *anti-symmetric* if $\varphi_{\mathcal{R}}(x, y) > 0$ and $\varphi_{\mathcal{R}}(y, x) > 0$ implies $x = y$.

The reflexive relation is the relation in which all element have the maximum relation to themselves; the symmetric relation is characterized by the reciprocity between their elements; the transitive indicates that the relation between any two individuals can not be simultaneously less than the relation of each of them with the rest; and the last, the anti-symmetric, is the relation that does not admit any reciprocity between distinct elements. Relations that satisfy simultaneously the four properties above are, generally, very artificial. Typically when they are required to fulfill (ii) and (iv) the relation tends to be artificial. For example, if U has just one element x, the Cartesian product $U \times U = \{(x, x)\}$ satisfies the properties (i)–(iv) from the Definition 3.8. Relations that satisfy just the first three conditions are called *equivalence relations*. Concepts (i)–(iv) can be seen in the following example:

Example 3.3 Intuitively, the relation of the military hierarchy ($\mathcal{M}$): "is a higher rank than" is based on the rank of the individual, that is, x is related to y if the rank of x is higher than y. So, $\mathcal{M}$ is reflexive, transitive and anti-symmetric but not symmetric. On the other hand, the relation ($\mathcal{A}$): "*is friend of*" is reflexive, symmetric but not transitive.

The relations $\mathcal{M}$ and $\mathcal{A}$ are not necessarily fuzzy relations. However, if we want to indicate the degree that x *is higher than* y, based not on rank but in subjective factors, say status, then $\mathcal{M}$ can be considered a fuzzy relation. The same can be said about the relation $\mathcal{A}$.

References

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