Chapter 11
End Notes

"If you obey all the rules you miss all the fun".
(Katharine Hepburn)

Abstract This chapter presents the concept of joint possibility distribution from the point of view of fuzzy number membership. The concept of completely correlated fuzzy numbers is presented. Next, an interactive fuzzy number subtraction operator is discussed. Finally, two bio-mathematical models are studied using these concepts. The first models the risk of getting dengue fever and second is an epidemiological SI-model with completely correlated initial conditions.

This last chapter is a collection of various fun topics. So, if Katherine Hepburn is correct, we are not following all rules. However, we are following fuzzy rule. Let us start by presenting some more advanced topics associated with fuzzy number arithmetic. Then we will present biomathematical models that use some of these concepts, specifically, the concepts of t-norm, Takagi-Sugeno inference method, and interactivity between fuzzy numbers.

The models that we will present here do not properly fit in either demographic or environmental uncertainty as developed in Chaps. 9 and 10. So we opted to present them into new chapter. In the first model we use the concepts of t-norm to represent the interactivity between the involved individuals, but the equations are deterministic. In the second model we employ the Takagi-Sugeno inference method in order to obtain a partial differential equation that represent the evolution of an epidemic system in time and space. Finally, in the third model we use a fuzzy differential equation whose the variables are given by completely correlated fuzzy numbers.

Our first section starts with subtraction of fuzzy numbers of special type, interactive fuzzy numbers. The reason we start with subtraction is that subtraction is where additive inverses and change (derivatives) begin. The reason we start with interactivity is that it is not always the case that the entities modeled by fuzzy numbers are non-interactive.

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11.1 Subtration of Interactive Fuzzy Numbers

Intuitively, for the arithmetic presented in Chap. 2 there exists some type of independence (or non-interactive) between the α-levels of the two fuzzy numbers involved, since all elements of both intervals contribute to the result of operation in question. Such a fact is corroborated by the minimum t-norm in the united extension and Zadeh's extension for fuzzy numbers. However, it is possible to define an arithmetic for fuzzy numbers that resembles the arithmetic for random variables by means of joint distributions.

The standard difference between two fuzzy numbers based on the difference between intervals (see Proposition 2.5) and takes into account all the possible combinations between two elements, one at each α-level. Consequently, the result is always greater (in diameter) than any of the sets involved in the operation. In fact, the width of the result of subtracting of two intervals is the sum of the widths of these two intervals. Thus, the difference between two non-crisp fuzzy numbers is always a non-crisp number and subtracting a non-crisp number from itself is never zero.

Using the Hukuhara Difference (2.9), the result of subtracting a non-crisp number A from itself ($A \ominus_H A$) is, in fact, non-zero. However, for this case, a necessary condition for the subtraction between two different fuzzy numbers A and B to exist is that the first term must have a bigger diameter than the second one. A proposed remedy to some of the issues involving interval subtraction is the Generalized Hukuhara difference, which also satisfies $A -_{gH} A = 0$ [1, 2] and is defined for a bigger class of fuzzy numbers than the Hukuhara difference. An extension of the generalized Hukuhara difference is the generalized difference [1, 3], which has the same results of the generalized Hukuhara operator (when it exists), but is defined for a larger class of fuzzy numbers. Another possibility is CIA (Constraint Interval Arithmetic) [4]. In this case, the diameter of the difference between two fuzzy numbers using CIA is often smaller than using standard difference.

All differences mentioned above make use of the interval arithmetic on α-levels. Extensions to fuzzy numbers are computed via Negoita and Ralescu's Representation Theorem (Theorem 1.4) over the resulting α-levels.

Another way to subtract is similar to the arithmetic for random variables, that is, the subtractions between fuzzy numbers are obtained using the joint possibility (or membership) distribution between the involved fuzzy numbers [5]. The comparison between the results obtained from the two approaches (interval-valued and joint possibility distribution) is made via a kind of Nguyen extension theorem [5].

The following concepts are from possibility theory and will be used to define the interactive difference between fuzzy numbers [5]. Some these concepts have been presented in Chap. 4.

Definition 11.1 Let A and B be fuzzy numbers and $J \in \mathcal{F}_C(\mathbb{R}^2)$, the class of fuzzy normal subsets of $\mathbb{R}^2$. Then J is a joint possibility distribution of A and B if

$$\max_{y \in \mathbb{R}} \varphi_J(x, y) = \varphi_A(x) \quad \text{and} \quad \max_{x \in \mathbb{R}} \varphi_J(x, y) = \varphi_B(y).$$

Moreover, φ_A and φ_B are called marginal distributions of J.

If the joint possibility distribution is given by a t-norm $\triangle$, then

$$\varphi_J(x, y) = (\varphi_A(x)\triangle\varphi_B(y)).$$

When $\triangle = \min$, A and B are called non-interactive fuzzy numbers. The results of this section may be considered as a generalization of Zadeh's extension for fuzzy arithmetic which is a particular case of what is presented when the t-norm is the minimum norm. The next definition describes a joint possibility distribution which is not given by means of t-norm.

Definition 11.2 Two fuzzy numbers A and B are said to be completely correlated if there are $q, r \in \mathbb{R}$, with $q \neq 0$, such that their joint possibility distribution C is defined by

$$\begin{aligned}\varphi_C(x, y) &= \varphi_A(x)\mathcal{X}_{\{qx+r=y\}}(x, y)\\ &= \varphi_B(y)\mathcal{X}_{\{qx+r=y\}}(x, y)\end{aligned} \tag{11.1}$$

where

$$\mathcal{X}_{\{qx+r=y\}}(x, y) = \begin{cases} 1 & \text{if } qx + r = y \\ 0 & \text{if } qx + r \neq y \end{cases}$$

is the membership function on the real line $\{(x, y) \in \mathbb{R}^2 : qx + r = y\}$.

We have, in this case:

$$[C]^\alpha = \{(x, qx + r) \in \mathbb{R}^2 : x = (1 - s)a_1^\alpha + sa_2^\alpha, s \in [0, 1]\}$$

where $[A]^\alpha = [a_1^\alpha, a_2^\alpha]; [B]^\alpha = q[A]^\alpha + r$, for any $\alpha \in [0, 1]$. Moreover, if as $q \neq 0$,

$$\varphi_B(x) = \varphi_A\left(\frac{x - r}{q}\right), \forall x \in \mathbb{R}.$$

It is important to observe that some pairs of fuzzy numbers cannot be completely correlated. For example, a triangular fuzzy number can not be completely correlated with a trapezoidal fuzzy number. Recently, we presented a family of parametrized joint possibility distributions which extend the properties of completely fuzzy number [6, 7].

In Definition 11.2, if q is positive (negative), the fuzzy numbers A and B are said to be completely positively (negatively) correlated. If $[B]^\alpha = q[A]^\alpha + r$, the correlated addition of A and B is the fuzzy number $A + B$ with α-cuts

$$[A + B]^\alpha = (q + 1)[A]^\alpha + r. \tag{11.2}$$

We will formulate the extension principle in what follows for the joint possibility distribution of fuzzy numbers [5].

Definition 11.3 Let J be a joint possibility distribution with marginal possibility distributions φ_A and φ_B, and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function. Then the extension f_J of f by J at pair (A, B) is the fuzzy set $f_J(A, B)$ whose membership function is given by

$$\varphi_{f_J(A,B)}(z) = \begin{cases} \sup\limits_{z=f(x,y)} \varphi_J(x, y) & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases}$$

where $f^{-1}(z) = \{(x, y) : f(x, y) = z\}$.

The next result can be viewed as a generalization of Nguyen's theorem [8].

Theorem 11.1 ([5]) *Let $A, B \in \mathcal{F}(\mathbb{R})$ be completely correlated fuzzy numbers, C its joint possibility distribution and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a continuous function. Then,*

$$[f_C(A, B)]^\alpha = f([C]^\alpha), \forall\alpha \in [0, 1].$$

11.1.1 Difference Between Fuzzy Numbers

Next we present different ways, as found in literature, to obtain the difference between fuzzy numbers.

Difference Via Interval Analytic Theory

Initially we present the fuzzy differences arising from the interval analysis theory.

Definition 11.4 (*Standard difference*) Let A, B be fuzzy numbers with α-levels given by $[a_1^\alpha, a_2^\alpha]$ and $[b_1^\alpha, b_2^\alpha]$, respectively. The α-levels of the standard difference, $A - B$, are defined by

$$[A - B]^\alpha = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha].$$

This standard difference can also be called Minkowski difference and it coincides to the one introduced in Proposition 2.5.

Lodwick [4, 9] proposed constraint interval arithmetic (CIA) and in particular subtraction is defined as follows.

Definition 11.5 (*CIA*) The subtraction between two fuzzy numbers A and B is defined level-wise by

$$[A -_{\text{CIA}} B]^\alpha = \{[(1 - \lambda_A)a_1^\alpha + \lambda_A a_2^\alpha] - [(1 - \lambda_B)b_1^\alpha + \lambda_B b_2^\alpha], \\ 0 \leq \lambda_A \leq 1, 0 \leq \lambda_B \leq 1\}.$$

Remark 11.2 Using CIA, we have

$$[A -_{\text{CIA}} A]^\alpha =$$

$$\{[(1-\lambda_A)a_1^\alpha + \lambda_A a_2^\alpha] - [(1-\lambda_A)a_1^\alpha + \lambda_A a_2^\alpha]\} = \{0\}$$

where $0 \leq \lambda_A \leq 1$. Therefore, $A -_{\text{CIA}} A = \{0\}$.

Definition 11.6 Given two fuzzy numbers A, B the Hukuhara difference (H-difference) $A \ominus_H B = C$ is the fuzzy number C such that $A = B + C$, if it exists.

Note that the above definition was presented in Chap. 2 (see Definition 2.9).

Definition 11.7 ([1, 2]) Given two fuzzy numbers A, B the generalized Hukuhara difference (gH-difference) $A \ominus_{gH} B$ is the fuzzy number C (if it exists) such that in this case we write $A \ominus_{gH} B = C$.

$$\begin{cases} (i) \;\; A = B + C \quad \text{or} \\ (ii) \; B = A - C. \end{cases}$$

Definition 11.8 ([1, 3]) Given two fuzzy numbers A, B the generalized difference (g-difference) $A \ominus_g B = C$ is the fuzzy number C with α-levels

$$[A \ominus_g B]^\alpha = \text{ cl } \bigcup_{\beta \geq \alpha} ([A]^\beta \ominus_{gH} [B]^\beta), \forall \alpha \in [0, 1],$$

where the gH-difference ($\ominus_{gH}$) is related to the intervals $[A]^\beta$ and $[B]^\beta$.

Bede and Stefanini [1, 3] proposed the generalized difference between fuzzy numbers as a difference that always exists and results in a fuzzy number. But for this, as observed in [10], a convexification is required in order that the difference is always a fuzzy number. For each of the differences presented in this subsection, $A - B$ is a fuzzy number according to Theorem 1.4.

Differences Via Joint Possibility Distribution

Differences via joint possibility distribution are obtained with the help of Definition 11.3. Note that this form of dealing with fuzzy numbers is inspired by the arithmetic of random variables, which considers the joint probability distribution.

Definition 11.9 Suppose A and B are two fuzzy numbers. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x - y$, that is, the subtraction operator for real numbers. The difference using the joint distribution J is the fuzzy number $A -_J B$, whose membership function is defined by

$$\varphi_{(A -_J B)}(z) = \sup_{(x,y) \in f^{-1}(z)} \varphi_J(x, y), \tag{11.3}$$

where $f^{-1}(z) = \{(x, y) : f(x, y) = x - y = z\}$.

Next the difference using the joint possibility distribution is given via t-norms.

Definition 11.10 (Differences via the t-norm) Let A, B be fuzzy numbers and $f(x, y) = x - y$ the subtraction operator, then the extension sup $-T$ of the fuzzy number $A -_\triangle B$ is obtained by the following membership function

$$\varphi_{A-_\triangle B}(z) = \sup_{(x,y)\in f^{-1}(z)} (\varphi_A(x) \triangle \varphi_B(y)), z \in \mathbb{R}.$$

where $f^{-1}(z) = \{(x, y) : f(x, y) = x - y = z\}$.

Note that Definition 2.9 (c) arises when $\triangle$ is the minimum t-norm. Moreover, the difference using joint possibility distributions may not necessarily be given by a t-norm.

Definition 11.11 ([5]) The subtraction of two completely correlated fuzzy numbers A and B is defined by

$$\varphi_{A-_C B}(z) = \sup_{(x,y)\in f^{-1}(z)} \varphi_C(x, y).$$

That is, $\varphi_{A-_C B}(z) = \sup_{z=x-y} \varphi_B(y)\mathcal{X}_{\{qx+r=y\}}(x, y)$.

From Theorem 11.1, [5], we have that, for all $\alpha \in [0, 1]$,

$$[A -_C B]^\alpha = (q - 1)[B]^\alpha + r.$$

Remark 11.3 The sum of two completely correlated fuzzy numbers A and B is the fuzzy number $A +_C B$ which α-levels are given by

$$[A +_C B]^\alpha = (q + 1)[B]^\alpha + r.$$

Definition 11.12 Let C be a joint possibility distribution with marginal possibility distributions A and B, and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a function. If A and B are completely correlated fuzzy numbers, then the extension of f applied to (A, B) is the fuzzy set $f_c(A, B)$ whose membership function is defined by

$$\varphi_{f_C(A,B)}(u, v) = \begin{cases} \sup_{(x,y)\in f^{-1}(u,v)} \varphi_c(x, y), & \text{if } f^{-1}(u, v) \neq \emptyset \\ 0 & , \text{if } f^{-1}(u, v) = \emptyset, \end{cases}$$

where $f^{-1}(u, v) = \{(x, y) : f(x, y) = (u, v)\}$.

The next theorem will be used to study the solution of an epidemiological model (SI) where S and I is considered completely correlated.

Theorem 11.4 *Let $A, B \in \mathcal{F}(\mathbb{R})$ be completely correlated fuzzy numbers, let C be their joint possibility distribution, and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous function. Then,*

$$[f_C(A, B)]^\alpha = f([C]^\alpha).$$

The proof of the Theorem 11.4 is found in [11]. We end this section by recalling that the interactive difference can be used to define derivative for autocorrelated fuzzy process (see [12, 13]).

11.2 Prey-Predator

The study presented in this section is based on [14, 15]. The classical models of interaction between species of prey-predator type uses the hypothesis that the predation rates are related to the probability of encounters between a predator and a prey. A typical case is in the Lotka-Volterra model below, already seen in Sect. 9.3,

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + dxy \end{cases}, \tag{11.4}$$

where x and y are, respectively, the number (or density) of prey and predators, $a > 0$ is the growth rate of prey, $c > 0$ is the mortality rate of predators, $b > 0$ is the proportion of successful attacks of the predators and $d > 0$ is the biomass conversion rate of the prey to predators.

This model supposes that both of species are uniformly distributed in the habitat and this is implicit in the terms bxy and dxy which are proportional to the probability of the number of encounters between prey and predators. In other words, we can say that the rate of encounters is derived from the"mass action law" that in the context of physicochemical establishes that the rates of molecular collisions of two chemicals and is proportional to the product of their concentrations [16].

On the other hand, we know that if the habitat where the prey and predators are living together is small, that is the area in which the two populations exist is small, the predation happens immediately, because there are enough prey. Therefore it is possible that if the number of prey is bigger than the number of predators, the predation rate is proportional to just the number of predators. Now, if the number of predators is bigger than the prey, then the predation rate is given by the number of prey. So, in both cases the predation rate is proportional to the minimum between the populations of prey and predators [17, 18]. These observations translate into various t-norms operations, more specifically, other t-norms besides the product t-norm which is commonly used in models to represent the interaction between species. The more detailed analysis of this situation is found in Sect. 9.3.

11.2.1 Prey-Predator with the Minimum t-Norm

These observations and those found in [17] lead us to consider the predation rate as proportional to the minimum between the populations of prey and predators. To model this situation, we change the t-norm of the product to the minimum $t - norm$ to represent the interaction between species. Thus, we have the following model [14, 15]:

$$\begin{cases} \frac{dx}{dt} = \ ax - b(x \wedge y) \\ \frac{dy}{dt} = -cy + d(x \wedge y) \end{cases}, \tag{11.5}$$

where a, b, c and d are the same as in (11.4).

The phase plane of the model (11.5) can be seen in the Fig. 11.1. From (11.5) it is possible to see that the only equilibrium point is the trivial one (0, 0). But the qualitative aspect of phase plane of (11.5) is very different to that of the one in Sect. 9.3.

11.2.2 Prey-Predator with the Hamacher t-Norm

The Hamacher t-norm (see Chap. 4), given by

$$\nabla_H (x, y) = \frac{xy}{p + (1 - p)(x + y - xy)}, \ p \geq 0 \tag{11.6}$$

and it can replace the product t-norm in the prey-predator model (11.4). The parameter p can be tunned in order to fit the specific character of the population under consideration. The various t-norms are obtained by a parameter p where the product t-norm occurs for $p = 1$. Thus, we have the following model

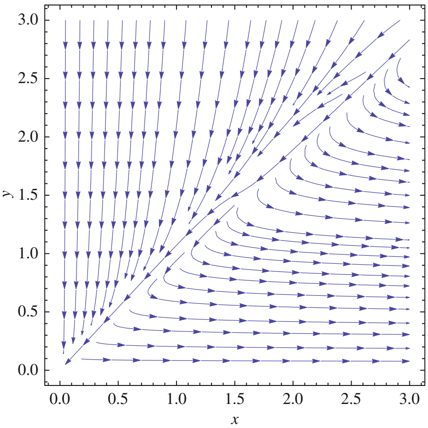
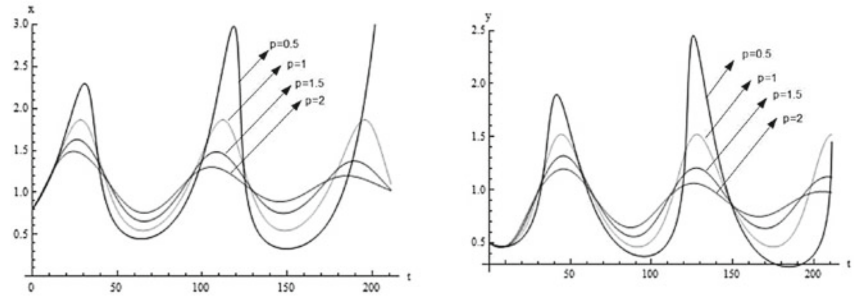


Fig. 11.1 Phase plane of the model (11.5) where $a = 0.08$, $b = 0.09$, $c = 0.075$ and $d = 0.07$



(a) with $a = 0.08$, $b = 0.09$, $c = 0.075$ and $d = 0.07$, for $p = 0.5, 1, 1.5$, and 2.

(b) with $a = 0.08$, $b = 0.09$, $c = 0.075$ and $d = 0.07$, for $p = 0.5, 1, 1.5$, and 2.

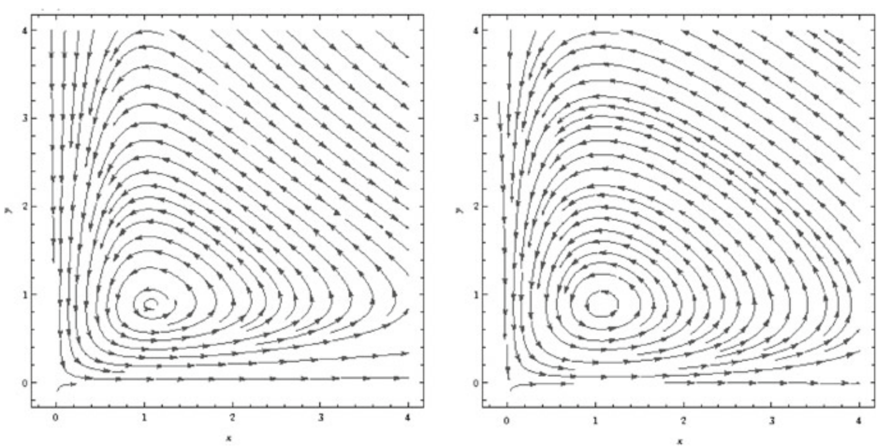
Fig. 11.2 Solution *x* and *y* for time *t*

$$\begin{cases} \frac{dx}{dt} = ax - \frac{bxy}{p+(1-p)(x+y-xy)} \\ \frac{dy}{dt} = -cy + \frac{dxy}{p+(1-p)(x+y-xy)} \end{cases}, \tag{11.7}$$

where a, b, c and d are the same as in (11.4). Figure 11.2a, b illustrate solutions of model (11.7) for some values of p.

It is interesting to note that the peaks of the solutions in Fig. 11.2a, b increase as p gets bigger. In cases where $p > 1$, the solutions go together to an equilibrium point, which differs from the classical case ($p = 1$), where there exists a periodic curve [15, 16].

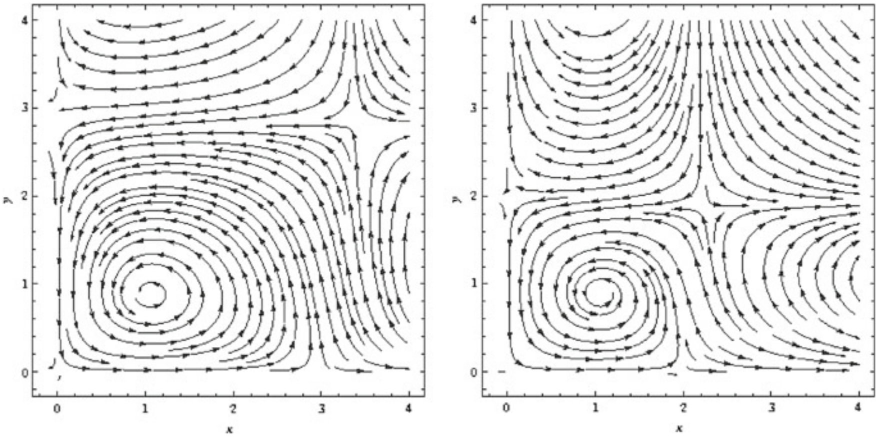
Figures 11.3a up to 11.4b show the phase plane of the solutions of the model (11.7) for values of $p = 0.5, 1.0, 1.5, 2.0$. It is possible to observe that for $0 \leq p < 1$ we



(a) with $p = 0.5$.

(b) with $p = 1$.

Fig. 11.3 Phase plane of the solution with parameters like Fig. 11.2a



(a) with $p = 1.5$. (b) with $p = 2$.

Fig. 11.4 Phase plane of the solution with parameters like Fig. 11.2a

have repelling equilibrium points, for $p = 1$ we have closed orbits around the equilibrium point and for $p > 1$ we have attractors equilibrium points and saddle points. Therefore, the choice of the parameter p will determine the stability of the system equilibrium (11.7).

11.3 Epidemiological Model

The more common epidemiological models that describe the dynamics of diseases spread by direct contact are SI, SIS and SIR where S is susceptible, I is infected and R is recovered. In these models the change of the state of a susceptible individual to the class of infected ones occurs as a result of the contact between individuals with infectious pathogens and healthy ones, that is, the transmission rate is proportional to the encounter rate of those individuals which is traditionally modeled by the product between the densities (quantities) [16, 19].

We have studied in Sect. 10.2 the simplest classical model that describes the dynamics of the diseases transmitted by direct contact without vital dynamics, that is, without birth/death. This is the SI model, given by

$$\begin{cases} \frac{dx}{dt} = -\beta xy; \ \ x(0) = x_0 > 0 \\ \frac{dy}{dt} = \ \ \beta xy; \ \ y(0) = y_0 > 0, \end{cases} \tag{11.8}$$

where $x(t)$ and $y(t)$ are, respectively, the fractions of susceptible and infected individuals at time t and the parameter $\beta > 0$ is the disease transmission rate. From (11.8) we have $x(t) + y(t) = 1$ and, as we saw in Sect. 10.2, the solution of (11.8) is given by

$$y(t) = \frac{y_0 e^{\beta t}}{x_0 + y_0 e^{\beta t}} \quad \text{and} \quad x(t) = 1 - y(t) = \frac{x_0}{x_0 + y_0 e^{\beta t}}. \tag{11.9}$$

In some epidemiological SI models, an individual who is infected can not recover. A typical case of this model is HIV, where the virus attacks the immune system which is responsible for protecting the body against diseases. For more details the reader might want to see [20].

One of the first models developed for populations of individuals that already have HIV is due to Anderson et al. [21]. The Anderson model studies the transfer between the individuals that are asymptomatic to the symptomatic ones, that is, it is not a direct transmission model [22]. This model is given by

$$\begin{cases} \dfrac{dx}{dt} = -\lambda x; \quad x(0) = x_0 > 0 \\ \dfrac{dy}{dt} = \lambda x; \quad y(0) = y_0 > 0, \end{cases} \tag{11.10}$$

where λ is the transfer rate from the asymptomatic to symptomatic phase (AIDS), $x(t)$ and $y(t)$ are, respectively, the fractions of infected individuals who did not develop AIDS and the others who did develop the disease. According to the model above we have the constraint

$$x(t) + y(t) = 1, \qquad \forall t \geq 0.$$

The solution of (11.10) is given by

$$x(t) = x_0 e^{-\lambda t} \quad \text{and} \quad y(t) = 1 - x_0 e^{-\lambda t}. \tag{11.11}$$

In (11.10) the dynamics of the disease is not modeled by the product operation. Since all individuals are already infected, the transmission does not depend on the encounter between them. Epidemiological modelers frequently discuss how to handle these types of models in which infections are not always transmitted by are interaction between a product-like accumulation of two populations (infected and susceptible).

Apparently the models (11.8) and (11.10) are not linked. However we will see that this is not one hundred percent true when different t-norms are used instead of the product.

11.3.1 SI Model with Minimum t-Norm

We base this section on [17, 23] where the model we have chosen uses the minimum t-norm, instead of product operation. This is because, when the susceptive population is small, its variation rate is proportional to susceptive population. On the other hand, the variation rate of infected population is proportional to this population when it is small. Thus, in both cases, the variation rate is proportional to the minimum between susceptive and infected population.

The model is given by the following differential equations [14, 15],

$$\begin{cases} \frac{dx}{dt} = -\lambda(x \wedge y) \\ \frac{dy}{dt} = \lambda(x \wedge y) \end{cases}. \tag{11.12}$$

The solution of (11.12) is given by

$$x(t) = \begin{cases} 1 - y_0 e^{\lambda t} & \text{if } t \leq \bar{t} \\ 0.5 e^{-\lambda(t-\bar{t})} & \text{if } t > \bar{t} \end{cases} \tag{11.13}$$

and

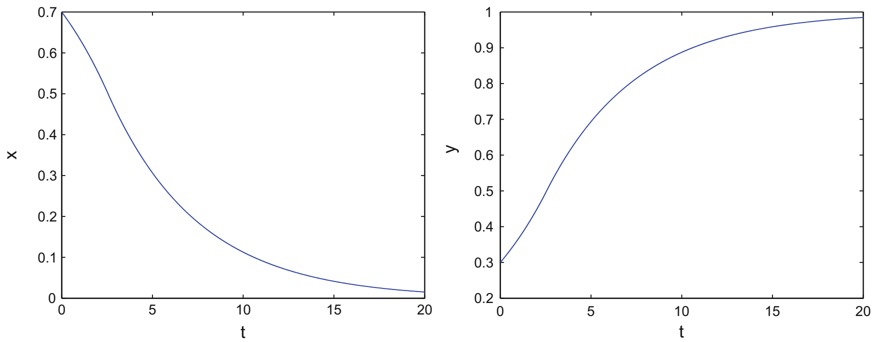
$$y(t) = \begin{cases} y_0 e^{\lambda t} & \text{if } t \leq \bar{t} \\ 1 - 0.5 e^{-\lambda(t-\bar{t})} & \text{if } t > \bar{t} \end{cases}, \tag{11.14}$$

where $\bar{t} = \frac{1}{\lambda} \ln \frac{0.5}{y_0}$. The model (11.10) coincides with (11.12) when the individuals of population interact and the majority is already symptomatic, that is, $(x \wedge y) = x$.

Illustrations of the solutions (11.13) and (11.14) can be seen in Fig. 11.5a and b, respectively. It is possible to verify the similarity between the curves that represent the proportion of infected individuals in this section to those in Sect. 10.2.

11.3.2 SI Model with Hamacher t-Norm

We will next use the t-norm of Hamacher, which is given by



(a) $\lambda = 0.2$ and $x(0) = 0.7$. (b) $\lambda = 0.2$ and $y(0) = 0.3$.

Fig. 11.5 Proportion of susceptible and infected individuals versus time

$$\nabla_H(x, y) = \frac{xy}{p + (1-p)(x + y - xy)}, \tag{11.15}$$

to model the interaction between the individuals. Thus, we have

$$\begin{cases} \frac{dx}{dt} = \frac{-\lambda xy}{p+(1-p)(x+y-xy)} \\ \frac{dy}{dt} = \frac{\lambda xy}{p+(1-p)(x+y-xy)} \end{cases}. \tag{11.16}$$

Since $x + y = 1$, that is, there is no vital dynamics, we have

$$\nabla_H(x, 1-x) = \frac{x(1-x)}{p + (1-p)(1 - x(1-x))}.$$

The implicit solution of (11.16) for the susceptible is given by

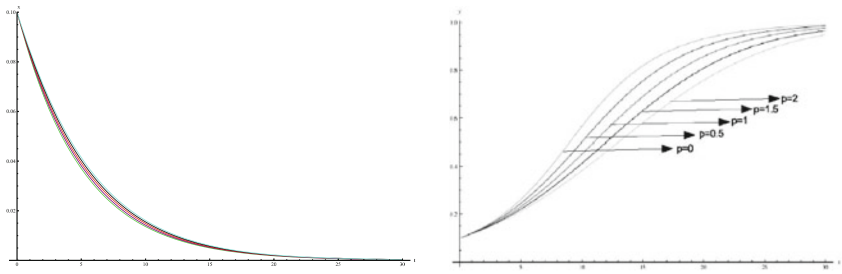
$$c_1 + \lambda t = (1-p)x(t) - \ln(1 - x(t)) + \ln x(t)$$

while the infected ones follows from the equation

$$(p - 1 - c_1) + \lambda t = (p-1)y(t) - \ln(1 - y(t)) + \ln y(t).$$

We next do a comparative study of the models (11.8) and (11.16). Let us suppose that in (11.16) the interaction is proportional to the product (xy). Thus, in (11.16) we can interpret $\beta = \frac{\lambda}{p+(1-p)(x+y-xy)}$ as the transfer rate from susceptible to infected class of individuals. In this case, the rate $\beta = \beta(x, y)$ depends on both concentrations of x and y. Therefore, the number of susceptible individuals is provided by (11.16) and is inferior to the one that is provided by (11.8) for $p < 1$, and superior for $p > 1$. The illustration of the solutions for some values of p can be seen in Fig. 11.6a and b.

We finish this section by observing that all model presented here are in fact classical ones, in the sense that both are differential equation and their solutions are deterministic.



(a) with $\lambda = 0.2$ and $x(0) = 0.9$. (b) with $\lambda = 0.2$ and $y(0) = 0.1$.

Fig. 11.6 Number of susceptible **a** and infected **b** versus time of the model (11.16)

11.4 Takagi–Sugeno Method to Study the Risk of Dengue

This section discusses concepts related to the formulation of models of Takagi–Sugeno and an application will be presented for the analysis of the risk of dengue[1] in the southern region of Campinas. The city of Campinas, located in southeast region of Brazil in the state of São Paulo, experienced the largest epidemic of dengue in 2007, with 1089.4 registered cases per 100, 000 inhabitants. High rates of incidence of dengue have been reported in the south region of the city, leading researchers from the Faculty of Medical Sciences of the University of Campinas to initiate a study on the phenomenon and its possible causes [24, 25].

11.4.1 Takagi–Sugeno Model

Chapter 5 developed the theory of fuzzy inference processes of the Takagi–Sugeno-Kang type where the consequent of each rule is explicitly given by a function of the input values of this rule. Currently, researchers are using this idea to construct fuzzy rules whose consequences are differential equations, applied to different problems [26–28].

The formulation of Takagi–Sugeno model for the risk of dengue in the southern region of Campinas, that will be presented, is based on references [24, 25]. The following is a summary of the main concepts used by these authors.

Consider the nonlinear partial differential equation (PDE) problem,

$$\frac{\partial y(x,t)}{\partial t} = \kappa(y(x,t))\frac{\partial^2 y(x,t)}{\partial x^2} + f(y(x,t)) + g(x)u(t) \tag{11.17}$$

for $0 \leq x \leq L$, $t > 0$ where $y(x,t)$ is the displacement, $\kappa(y(x,t)) \geq 0$ and $f(x,t)$ are nonlinear functions satisfying $\kappa(0) = 0$ and $f(0,0) = 0$, $u(t)$ is the distribution of the control force and $g(x)$ is an influence function. The initial and boundary conditions are given by

$$y(0,t) = y(L,t) = 0, \quad y_x(0,t) = y_x(L,t) = 0 \ \text{ and } \ y(x,0) = y_0(x) \tag{11.18}$$

In the fuzzy formulation of systems (11.17)–(11.18) we will set $u(t) = 0$, i.e.,

$$\frac{\partial y(x,t)}{\partial t} = \kappa(y(x,t))\frac{\partial^2 y(x,t)}{\partial x^2} + f(y(x,t)). \tag{11.19}$$

A model of type Takagi–Sugeno is used to approximate equation (11.19) and it has the following fuzzy rules:

[1] Dengue is a mosquito borne disease that causes fever and in some cases death.

$$\text{Rule } i : \text{If } \; y(x,t) \; \text{is} \; F_i, \; \text{then}$$
$$\frac{\partial y(x,t)}{\partial t} = \kappa_i \frac{\partial^2 y(x,t)}{\partial x^2} + a_i y(x,t) \tag{11.20}$$

where F_i are fuzzy sets, $\kappa_i \geq 0$, a_i are known constants for $i = 1, 2, \ldots, M$; and M is the number of rules.

Fuzzy rule i means that if the input variable $y(x,t)$ is locally represented by the fuzzy set F_i, then the non-linear partial differential equation (11.19) can be represented by the linear equation (11.20). The process of fuzzy inference is done as follows

$$\frac{\partial y(x,t)}{\partial t} = \sum_{i=1}^{M} \varphi_i(y(x,t)) \left[\kappa_i \frac{\partial^2 y(x,t)}{\partial x^2} + a_i y(x,t) \right], \tag{11.21}$$

where

$$\varphi_i(y(x,t)) = \varphi_{F_i}(y(x,t)) / (\sum_{k=1}^{M} \varphi_{F_k}(y(x,t)))$$

is the membership degree of $y(x,t)$ belonging to F_i. The denominator of $\varphi_i(y(x,t))$ is only for normalization so that the total sum is

$$\sum_{k=1}^{M} \varphi_i(y(x,t))) = 1.$$

11.4.2 Dengue Risk Model

This section develops a model of the risk of dengue epidemic from the point of view spacial temporal dynamics for the southeast part of the city of Campinas, Brazil. With the collaboration of the Laboratory for Spacial Analysis of Epidemiological Data (epiGeo) researchers of University of Campinas [29], we obtained data that generated the initial risk map of dengue in the region studied as shown in Fig. 11.7.

The mesh adopted corresponds to a 40×40 grid which covers approximately $20\,\text{km} \times 20\,\text{km}$ of the region. Note that higher risks are associated with warm colors, that is, with reddish hues.

The researchers of epiGeo determined, from the initial data that gave rise to the map, the relative risk. The relative risk is given as the quotient between the probabilities of the exposed individual and of the control (not exposed individual) [25]. For example, if the risk is 2, then at that geographic location, the individuals have twice the risk of contracting the dengue disease than individuals not exposed to risk factors. From Fig. 11.7, the risk of dengue was classified as *Low*, *Medium* and *High* and membership functions were constructed as illustrated in Fig. 11.8.

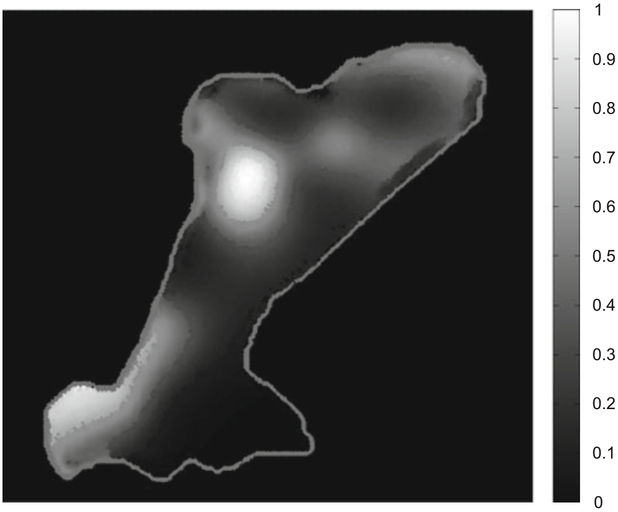
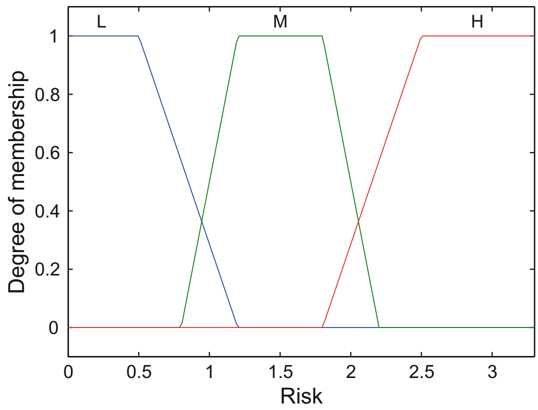


Fig. 11.7 Map of dengue risk developed by the epiGeo [25]

Fig. 11.8 Membership functions for the risk of dengue [25]



Let $r(x, y, t)$ be the risk of dengue. The fuzzy rules that were developed, using the ideas presented by [26], were the following.

Rule 1 : If $r(x, y, t)$ is Low (L), then

$$\frac{\partial r(x, y, t)}{\partial t} = \kappa_B \left[\frac{\partial^2 r(x, y, t)}{\partial x^2} + \frac{\partial^2 r(x, y, t)}{\partial y^2} \right] + a_B r(x, y, t).$$

Rule 2 : If $r(x, y, t)$ is Medium (M), then

$$\frac{\partial r(x,y,t)}{\partial t} = \kappa_M \left[\frac{\partial^2 r(x,y,t)}{\partial x^2} + \frac{\partial^2 r(x,y,t)}{\partial y^2}\right] + a_M r(x,y,t).$$

Rule 3 : If $r(x, y, t)$ is High (H), then

$$\frac{\partial r(x,y,t)}{\partial t} = \kappa_A \left[\frac{\partial^2 r(x,y,t)}{\partial x^2} + \frac{\partial^2 r(x,y,t)}{\partial y^2}\right] + a_A r(x,y,t).$$

In this case ($M = 3$), and from (11.21) we have

$$\frac{\partial r(x,y,t)}{\partial t} = \sum_{i=1}^{M} \varphi_i(r(x,y,t)) \left[\kappa_i \left(\frac{\partial^2 r(x,y,t)}{\partial x^2} + \frac{\partial^2 r(x,y,t)}{\partial y^2}\right) + a_i r(x,t)\right]. \tag{11.22}$$

The parameters κ_i in (11.22) represent the spatial distribution of risk for the given domain, that is, the associated geographical region.

A system based on fuzzy rules was constructed to find κ_i taking into account environmental factors that influence the dynamics of *Aedes aegypti* and the affect they have on the dispersion of risk. From this point of view, one might consider the dynamics of *Aedes aegypti* as environment fuzziness (see Chaps. 9 and 10). The input variables that affect the dynamics of *Aedes aegypti* are *rainfall*, *human inhabitants* and *mosquito breeding containers*. The membership functions adopted for the input variables can be found in [24, 25]. The membership functions constructed for the output variable κ_i are shown in Fig. 11.9. A stochastic model was constructed to determine the amount of rain, taking into account historical records provided by the Agronomic Institute of Campinas.
References [24, 25] contain the details of the procedures used to calculate the parameters a_i and the numerical methods to implement the equations.

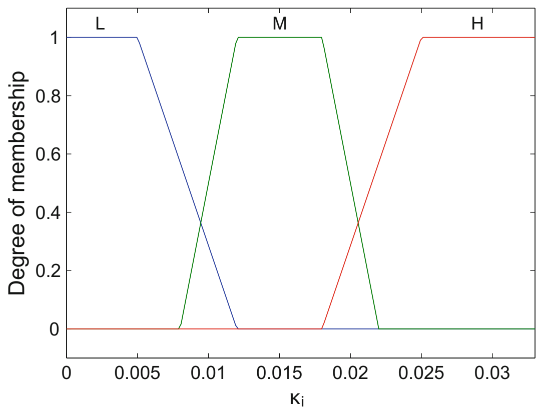


Fig. 11.9 Membership functions for κ_i [25]

11.4.3 Simulations

We used MATLAB to implement computationally the PDE given in (11.22) and we coupled the MATLAB toolkit for stochastic systems and fuzzy logic to determine the parameters followed by their numerical methods solvers. The interested reader can find a detailed development in [24, 25].

The spatial discretization we chose was WENO-5 (weighted essentially non-oscillatory schemes) for the non-smooth regions of the map and CFDS-4 (centered finite difference scheme of fourth order) for the smooth regions of the map. The time discretization used Runge–Kutta TVD (Total Variation Diminishing). Figure 11.10 shows a representative scheme of our coupling.

Case 1

Simulations of the evolution of dengue risk over time were implemented for the months of December, January and February, corresponding to summer in the region. Figure 11.11 shows the results obtained.

These results show, in general, that there was a spread of the disease risk over the region. It is observed that there was a higher risk (red) of dengue that occurred in our simulation during the three months. We conclude that, for the summer months, the estimated values for the parameters κ_i and a_i in this simulation favored the spread of the risk of dengue in the southern region of Campinas.

Public health officials usually adopt measures to combat the *Aedes aegypti* mosquito breeding to decrease the incidence of dengue. So, the next simulation assumes a reduction of potential mosquito breeding sites available in the region and this is used in the fuzzy rules for κ_i, to see whether or not the model/simulation obtained a decreased risk of dengue.

The fuzzy rules were developed according to [26] and it was assumed $u(t) = 0$ in (11.17). The measure of control, which was the reduction of mosquito breeding sites, was inserted into the parameter estimation procedure κ_i through the system based on fuzzy rules. For a more comprehensive study, one could try to obtain a function $u(.)$ that takes into account other possible disease controls.

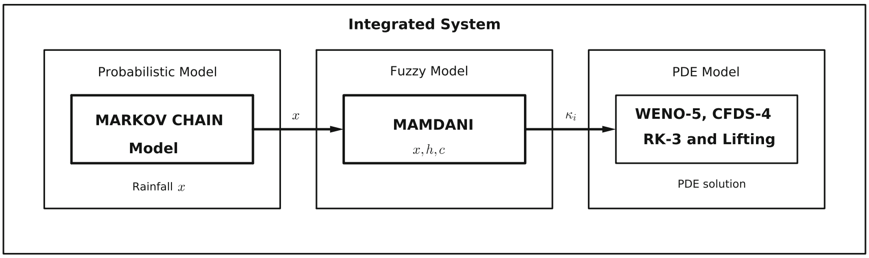
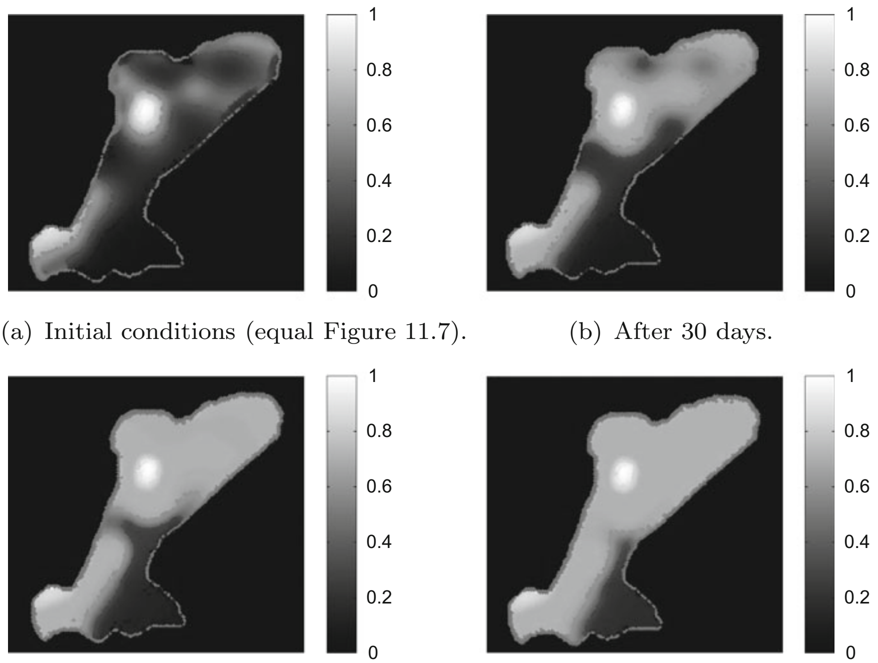


Fig. 11.10 Representative scheme of the coupling mathematical tools



(a) Initial conditions (equal Figure 11.7).

(b) After 30 days.

(c) After 60 days.

(d) After 90 days.

Fig. 11.11 Evolution of the risk of dengue [25]

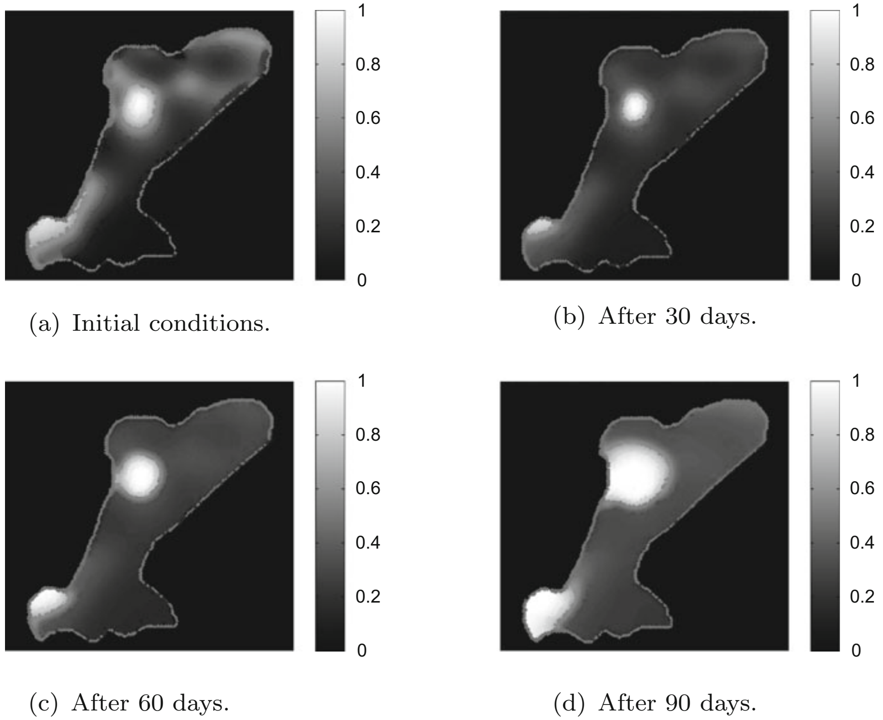
Case 2

Suppose we have a reduction of 80 % in the number of mosquito breeding sites for *Aedes aegypti* in the region. Starting from the given initial conditions, we have the results illustrated in Fig. 11.12.

The figures show that after the first 30 days, there was reduction in the risk of dengue in virtually the entire region considered, indicating that a relatively important measure is to invest in a large reduction of sites available for the breeding of the dengue vector. However note that after 60 and 90 days, in the vicinity of the red regions, there was a growth and the spread of risk. This indicates that a public health policy needs to be one of continuing reduction of mosquito breeding sites.

11.4.4 Final Considerations

This section proposed a Takagi–Sugeno model to assess the risk of dengue in the southern region of Campinas. Preliminary studies regarding the risk of dengue in this region were conducted by researchers at EpiGeo Laboratory of the University



(a) Initial conditions. (b) After 30 days. (c) After 60 days. (d) After 90 days.

Fig. 11.12 Evolution of the dengue risk [25]

of Campinas. Such information was the starting point for the proposed model. The model was comprised of rules where the consequences are PDEs. The inference combines such rules and the resulting equations are solved numerically by means of methods developed in [24].

We observe from the situation, that an effective measure to reduce the risk of dengue is to aggressively reduce the potential mosquito's, *Aedes aegypti*, breeding sites. For a more comprehensive study, one could try to obtain a function $u(t)$ that takes into account various possible disease controls such as genetically modified mosquitoes, house screens, and/or mosquito eating animals such as bats and swallows (birds).

11.5 The SI-model with Completely Correlated Initial Conditions

This section presents the SI epidemiological model by consedering uncertain parameters modeled by completely correlated fuzzy numbers. More specifically we will analyze the model, via the extension principle that will account for the correlation among the variables. The initial conditions are given by interactive fuzzy numbers [11].

The SI-model, as we have already seen, is described by the system of differential equations

$$\begin{cases} \dfrac{dS}{dt} = & -\beta SI, \quad S(0) = S_0 \\ \\ \dfrac{dI}{dt} = \beta SI, & I(0) = I_0 > 0, \end{cases} \tag{11.23}$$

where $S(t)$ and $I(t)$ are, respectively, the fractions of susceptible and infected individuals at the time t. The parameter β is a positive constant representing the rate of contact of the disease.

Suppose that there is no variation in the total number of the population, that is, consider the model without vital dynamics,

$$S(t) + I(t) = 1, \qquad \forall t \geq 0. \tag{11.24}$$

Thus, we get for each $t \geq 0$, the deterministic solution of problem (11.23) given by

$$L_t(S_0, I_0) = \left(\frac{S_0}{S_0 + I_0 e^{\beta t}}, \frac{I_0 e^{\beta t}}{S_0 + I_0 e^{\beta t}} \right). \tag{11.25}$$

Now, consider system (11.23) where the initial conditions are uncertain and modeled by fuzzy numbers. Since $S_0 + I_0 = 1$, we are dealing with completely correlated fuzzy numbers where $r = 1$ and $q = -1$ which we explain next. According to Definition 11.2, $S_0 + I_0 = 1$ means that the joint possibility distribution C of S_0 and I_0 is such that

$$\varphi_C(s_0, i_0) = \varphi_{S_0}(s_0)\mathcal{X}_{\{s_0+i_0=1\}}(s_0, i_0) = \varphi_{I_0}(i_0)\mathcal{X}_{\{s_0+i_0=1\}}(s_0, i_0). \tag{11.26}$$

In this case, for each $\alpha \in [0, 1]$, we have

$$\varphi_{I_0}(i_0) = \varphi_{S_0}(1 - i_0), \quad [I_0]^\alpha = [a_1^\alpha, a_2^\alpha], \quad [S_0]^\alpha = (-1)[I_0]^\alpha + 1$$

and

$$[C]^\alpha = \{(1 - i_0, i_0) \in \mathbb{R}^2 : i_0 = (1 - \gamma)a_1^\alpha + \gamma a_2^\alpha, \gamma \in [0, 1]\}. \tag{11.27}$$

Thus, $S_0 + I_0 = 1$ implies $q = -1$ and $r = 1$.

Taking into consideration (11.27), and remembering the notions found in Sect. 8.1.3, Eq. (11.23) becomes

$$\begin{cases} \left(\dfrac{dS}{dt}, \dfrac{dI}{dt} \right) = \left(-\beta SI, \beta SI \right) \\ \\ (S_0, I_0) \quad \in \quad C \end{cases}. \tag{11.28}$$

Solution of the Fuzzy SI-model Via Differential Inclusion

The solution to the Eq. (11.28) via fuzzy differential inclusion requires that we apply the method described in Sect. 8.1.3 of Chap. 8, so that the solution of the problem (11.28), using differential inclusion, is obtained from the solution of the auxiliary problem

$$\begin{cases} (\frac{dS}{dt}, \frac{dI}{dt}) = (-\beta SI, \beta SI) \\ \\ (S_0, I_0) \in \quad [C]^\alpha, \end{cases} \tag{11.29}$$

where $[C]^\alpha$ is given by Eq. (11.27). The attainable sets of the problem (11.29) are given by

$$\begin{aligned} \mathcal{A}_t([C]^\alpha) &= \left\{ x(t, S_0, I_0) : x(., S_0, I_0) \text{ is solution of } (11.29) \right\} \\ &= \left\{ x(t, S_0, I_0) : x'(t, S_0, I_0) = (-\beta SI, \beta SI),\ (S_0, I_0) \in [C]^\alpha \right\} \\ &= \left\{ \left(\frac{s_0}{s_0 + i_0 e^{\beta t}}, \frac{i_0 e^{\beta t}}{s_0 + i_0 e^{\beta t}} \right) : (s_0, i_0) \in [C]^\alpha \right\} \\ &= \left\{ \left(\frac{1-i_0}{(1-i_0) + i_0 e^{\beta t}}, \frac{i_0 e^{\beta t}}{(1-i_0) + i_0 e^{\beta t}} \right) : \right. \\ &\qquad \left. i_0 = (1-\gamma) a_1^\alpha + \gamma a_2^\alpha, \gamma \in [0, 1] \right\}. \end{aligned}$$

Solution of the Fuzzy SI-model Via Extension Principle

We will study the fuzzy SI-model given by (11.28) using as a tool the extension principle via Definition 11.12 in (11.25). According to Theorem 11.4, the α-levels of the solution obtained by the extension principle of the problem (11.28) are given by the expression

$$[(L_t)_C(S_0, I_0)]^\alpha = L_t([C]^\alpha).$$

Therefore, the α-levels of the solution of the problem (11.28) are

$$\begin{aligned} [(L_t)_C(S_0, I_0)]^\alpha &= L_t([C]^\alpha) \\ &= \left\{ L_t(s_0, i_0) : (s_0, i_0) \in [C]^\alpha \right\} \\ &= \left\{ L_t(1 - i_0, i_0) : i_0 = (1-\gamma) a_1^\alpha + \gamma a_2^\alpha, \gamma \in [0, 1] \right\} \\ &= \left\{ \left(\frac{1-i_0}{(1-i_0) + i_0 e^{\beta t}}, \frac{i_0 e^{\beta t}}{(1-i_0) + i_0 e^{\beta t}} \right) : \right. \\ &\qquad \left. i_0 = (1-\gamma) a_1^\alpha + \gamma a_2^\alpha, \gamma \in [0, 1] \right\}. \end{aligned}$$

That is, as predicted by Theorem 3.2 in [11], for every $t \geq 0$, the sets $(L_t)_C(S_0, I_0)$ and $\mathcal{A}_t(C)$ are identical.

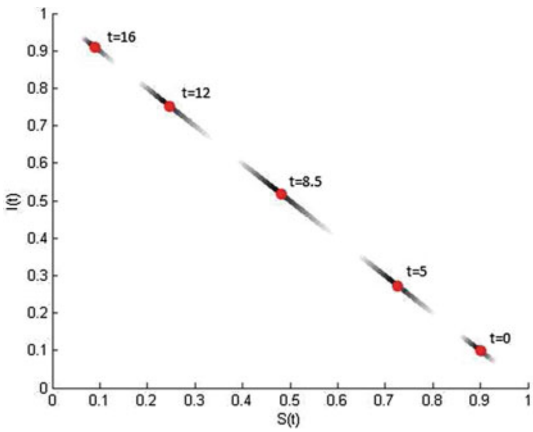


Fig. 11.13 Fuzzy solution to the problem (11.28) in the phase-portrait. The *dots* correspond to the deterministic solution for different values of time t. The initial conditions are the completely correlated triangular fuzzy numbers $I_0 = (0.05; 0.08; 0.11)$ and $S_0 = (0.89; 0.92; 0.95)$, with $S_0 + I_0 = 1$, and the contact rate is $\beta = 0.3$. Darker regions (for each $t \geq 0$) mean greater possibility (membership) of the number of susceptible and infected to the solution of the problem. The deterministic solution has membership degree equal to 1 in the fuzzy solution [11, 30]

Figure 11.13 represents the solution to the problem (11.28) employing the completely correlated triangular fuzzy numbers $I_0 = (0.05; 0.08; 0.11)$ and $S_0 = (0.89; 0.92; 0.95)$ as initial conditions, with $S_0 + I_0 = 1$ (according to Formula (11.3)) and contact rate $\beta = 0.3$. Note that the fact that the fuzzy numbers are completely correlated forces the solution to problem (11.28) to be a curve contained within the line $S + I = 1$, for each $t \geq 0$.

We conclude this study by commenting that the solutions $S(t)$ and $I(t)$, of a general epidemiological model are trajectories out of line $x + y = 1$ (see [16]). The deterministic solution of problem (11.23), belong to line $x + y = 1$ because we admit the correlation $S(t) + I(t) = 1$ for each $t \geq 0$. If bise this we admit that $S(t)$ and $I(t)$ are fuzzy numbers, then due $S(t) + I(t) = 1$, we have fuzzy numbers negatively completely correlated.

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