# Chapter 1
# Fuzzy Sets Theory and Uncertainty in Mathematical Modeling

*Man is the measure of all things: of things which are, that they are, and of things which are not, that they are not.*
(Protagoras – 5th Century BCE)

**Abstract** This chapter presents a brief discussion about uncertainty based on philosophical principles, mainly from the point of view of the pre-Socratic philosophers. Next, the notions of fuzzy sets and operations on fuzzy sets are presented. Lastly, the concepts of alpha-level and the statement of the well-known Negoita-Ralescu Representation Theorem, the representation of a fuzzy set by its alpha-levels, are discussed.

## 1.1 Uncertainty in Modeling and Analysis

The fundamental entity of analysis for this book is *set*, a collection of objects. A second fundamental entity for this book is *variable*. The variable represents what one wishes to investigate by a mathematical modeling process that aims to quantify it. In this context, the variable is a symbolic receptacle of what one wishes to know. The quantification process involves a set of values which is ascribed a-priori. Thus, when one talks about a variable being fuzzy, a real-number, a random number, and so on, one is ascribing to the variable its attribution.

A set also has an existence or context. That is, when one is in the process of creating a mathematical model, one ascribes to sets attributions associated with the model or problem at hand. One speaks of a set being a classical set, a fuzzy set, a set of distributions, a random sets, and so on. Given that models of existent problems or conditions are far from ideal deterministic mathematical entities, we are interested in dealing directly with associated inexactitudes and so ascribe to our fundamental objects of modeling and analysis properties of determinism (exactness) and non-determinism (inexactness).

This book is about processes in which uncertainty both in the input or data side and in the relational structure is inherent to the problem at hand. Social and biological



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the modeling are characterized by such uncertainties. The mathematical theory on which we focus to enable modeling with uncertainty occurring in biological and social systems is fuzzy set theory first developed by L. Zadeh [1].

Uncertainty has long been a concern of researchers and philosophers alike, throughout the ages as it is to us in this present book. The pursuit of the truth, of what is, of what exists, which is one aspect of uncertainty if we characterize truth or existence certainty, has been debated since the dawn of thinking. In ancient Greece individuals and schools explicitly asked the question: “What exists? Is everything in transformation or is there permanence?” These are two dimensions of thought and can be considered completely separate issues and even contradictory issues.

The pre-Socratic philosophers tried to make statements summarizing their thoughts about the Universe in an attempt to explain what is existent in the universe. In the words of Heraclitus of Ephesus (6th to 5th Century BCE), “panta hei”, which means “everything flows, everything changes”. By way of illustration, consider a situation in which a river is never the same, one cannot bathe in the same river twice. Cratylus, his disciple, took Heraclitus’ thoughts to the extreme by saying that we cannot bathe in the river even once, because if we assign an identity to things or give them names, we are also giving stability to these things which, in his view, are undergoing constant change.

The Eleatic school, in contrast to Heraclitus, questions the existence of motion or change itself. According to Parmenides of Elea (6th to 5th Century BCE): “the only thing that exists is the being - which is the same as thinking”. Zeno, his main follower, denies that there is motion as this was understood at the time by giving his famous paradox of Achilles and the turtle [2].

The Sophists interpret what Parmenides said as the impossibility of false rhetoric. According to Protagoras (5th Century BCE): “Man is the measure of all things”. There is no absolute truth or falsehood. In the Sophists’ view, humankind must seek solutions in the practical. The criterion of true or false is related to the theoretical and must therefore be replaced by (more practical) patterns related to the concepts of better or worse. Rhetoric is the way to find such patterns.

Most of the pre-Socratic philosophers with the exception of Heraclitus, believed there was something eternal and unchanging behind the coming-to-be (that which is in the process of being, of becoming), that was the eternal source, the foundation of all beings. According to Thales, it was water; in the opinion of Anaximenes, the air; Pythagoras thought it was numbers; and, Democritus believed that this source lay in the atoms and in the void. This eternal something which was unchangeable and which held all things was called by the Greeks *arche*.

Certainty and uncertainty were widely discussed by Greek philosophers. The Sophists (a term derived from sophistes, sages) were known to teach the art of rhetoric. Protagoras, the most important Sophist along with Górgias, taught students how to turn weaknesses of argument into strengths. Rhetoric, for the Sophists, is a posture or attitude with respect to knowledge that has a total skepticism in relation to any kind of absolute knowledge. This, no matter how things are, is because everything is relative and also depends on who gives judgement about them. Górgias said that rhetoric surpasses all other arts, being the best because it makes all things submit to spontaneity rather than to violence. As is well known, Socrates confronted the Sophists of his day with the question: "What is?" That is, if everything is relative, what exists?

Plato, a disciple of Socrates, initially shared the ideas of Heraclitus that everything is changing, the flow of coming-to-be, everything was in process. However, if everything was in motion then knowledge would not be possible. To avoid falling back into skepticism, Plato thought of a "world of ideas". Around this world, there would be changes, and things would be eternal beyond the space-time dimension. The so-called "sensory world", which is the world as perceived by the five senses, would then come into being. It would be true that the "world of ideas" would be behind the coming-to-be of this "sensory world". For Plato the most important thing was not the final concept, but the path taken to reach it. The "world of ideas" is not accessible by the senses but rather just by intuition, while intellectual dialectics is the movement of asceticism in pursuit of the truth. Therefore, Plato promotes a synthesis between Heraclitus and Parmenides.

On the other hand, for Aristotle, the world of ideas and essences is not contained in things themselves. Universal knowledge is linked to its underlying logic (the Logos, the same reason, the principle of order and study of the consequences) and also the syllogism, which is the formal mechanism for deduction. Based on certain general assumptions, knowledge must strictly follow an order using the concept of the demonstrative syllogism. In short, and perhaps naively, we think that the most important difference between Aristotle and the Sophists is the fact that, for Aristotle, there is an eternal, an immutable, independent of human beings, while the Sophists consider that there is no eternal and absolute truth, but rather just the knowledge obtained from our senses. For Plato and Aristotle, respectively, dialectics and syllogisms are to be used in the quest for the truth. The Sophists consider that rhetoric, the art of persuasion, is convincing in relation to the search for the truth, because truth does not exist as an absolute.

Understanding that subjectivity, imprecision, uncertainty, are inherent to certain terms of language, Górgias denied the existence of absolute truth: even if absolute truth existed, it would be incomprehensible to man, even if it were comprehensible to one man, it would not be communicable to others. In order to stimulate our thought about this aspect of the uncertainty of language, we will try to reach a compromise between the positions of the Sophists, on the one hand, and Plato and Aristotle, on the other, by means of a simple example.

It is common practice to propose a meeting with another person by saying something like "Let's meet at four o'clock". Well, the abstract concept of "four o'clock", indicating a measurement of time, shows a need to establish communication (in the abstract) and also enable the holding of the event, our meeting. If this were not the case, how should we then communicate our meeting? - A point for Plato. On the other hand, if we take this at face value, the meeting would never take place as our respective clocks would never reach four o'clock simultaneously, even if they had been synchronized, as we could not get to the point marked in hours, minutes, seconds, and millionths of seconds. A point for Górgias. Admitting that we often carry out our commitments at the appointed time and place, it looks like we equally need abstract truths and practical standards of a sensible world.

We articulated the thoughts above in order to point out the difficulty of talking about certainty or uncertainty and of fuzzy or determinism. If we look in a dictionary for terms synonymous with uncertainty, we find, for example: subjectivity, inaccuracy, randomness, doubt, ambiguity, and unpredictably, among others. Historically, researchers, from what we have noticed, have, in their quantitative treatment, made distinctions between the different types of uncertainty. The uncertainty arising from the randomness of events has been well documented, and now occupies a prominent position in the gallery of mathematics, in probability theory. Quantum physics has used stochastic theories, and a series of formulae now try to explain the "relationships of uncertainty". One of the most widely known of these is the Uncertainty Principle devised by the physicist W. Heisenberg (1927), which relates the position and the velocity (momentum) of a particle. In a nutshell, Heisenberg's Uncertainty Principle says that one cannot know simultaneous for certain the exact position and speed (momentum) of a subatomic particle. One can know one or the other, but not both.

Unlike randomness, some variables used in our daily lives, and which are perfectly understood when transmitted linguistically between partners, have always remained outside the scope of traditional mathematical treatment. This is the case of some linguistic variables that have arisen from the need to distinguish between qualifications through a grading system. To describe certain phenomena within the sensible world, we have used degrees that represent qualities or partial truths, or "better standards" to use Sophist language. This is the case, for example, with such concepts as tall, heavy smoker, or infections. This kind ambiguity in language which is a type of uncertainty in terms of its precise meaning since these terms are by their very nature, imprecise, is from a linguistic point of view, a *flexibility* regarding what elements belong to the category/set (tall, heavy smoker). Moreover, the main contribution that fuzzy logic made and is making, is to the mathematical analysis of fuzzy sets, these vague, flexible, open concepts. Fuzzy logic gives precision to imprecise (linguistic) terms so that mathematical analysis of these flexible categories is meaningful. In language usage, we could refer to the sets of tall people, smokers or infections. These are typical examples of "sets" whose boundaries can be considered transitional, flexible, vague, since they are defined through subjective or flexible properties or attributes.

Let's consider the example of tall people. To make a formal mathematical representation of this set, we could approach it in at least two different ways. The first is the classical approach, establishing a height above which a person could be considered tall. In this case, the set is well-defined. The second and less conventional approach to this issue would be that of considering all people as being tall with greater or less extent, that is, there are people who are more or less tall or not tall at all. This means that the less tall the individual, the lower the degree of relevance to this class. We can therefore say that all people belong to the set of tall people, with greater or less extent. This latter approach is what we intend to discuss in our book. It was from such notions, where the defining characteristics or properties of the set is flexible, transitional, open, that fuzzy theory appeared. Fuzzy set theory has grown considerably since it was introduced in 1965, both theoretically and in diverse applications especially in the field of technology - microchips.

The word "fuzzy" is of English origin and means (see *Concise Oxford English Dictionary*, $11^{th}$ edition) indistinct or vague. Other meanings include blurred, having the nature or characteristic of fuzzy. Fuzzy set theory was introduced in 1965 by Lotfi Asker Zadeh [1] (an electrical engineer and researcher in mathematics, computer science, artificial intelligence), who initially intended to impart a mathematical treatment on certain subjective terms of language, such as "about" and "around", among others. This would be the first step in working towards programming and storing concepts that are vague on computers, making it possible to perform calculations on vague or flexible entities, as do human beings. For example, we are all unanimous in agreeing that the doubling of a quantity "around 3" results in another "around 6 ".

The formal mathematical representation of a fuzzy set is based on the fact that any classic subset can be characterized by a function, its characteristic function, as follows.

**Definition 1.1** Let $U$ be a non-empty set and $A$ a subset of $U$. The characteristic function of $A$ is given by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $x \in U$.

In this context, a classical subset $A$ of $U$ can uniquely be associated with its characteristic function. So, in the classical case, we may opt for using the language from either "set theory" or "function theory", depending on the problem at hand.

Note that the characteristic function $\chi_A : U \to \{0, 1\}$ of the subset $A$ shows which elements of the universal set $U$ are also elements of $A$, where $\chi_A(x) = 1$ meaning that the element $x \in A$, while $\chi_A(x) = 0$ means that $x$ is not element of $A$. However, there are cases where an element is partially in a set which means we cannot always say that an element completely belongs to a given set or not. For example, consider the subset of real numbers "near 2":

$$A = \{x \in \mathbb{R} : x \text{ is near } 2\}.$$

**Question**. Does the number 7 and the number 2.001 belong to $A$? The answer to this question is not no/yes (so is uncertain from this point of view) because we do not know to what extent we can objectively say when a number is near 2. The only reasonable information in this case is that 2.001 is nearer 2 than 7.

We now start the mathematical formalization of fuzzy set theory that shall be addressed in this text, starting with the concept of fuzzy subsets.

## 1.2 Fuzzy Subset

Allowing leeway in the image or range set of the characteristic function of a set from the Boolean set {0, 1} to the interval [0, 1], Zadeh suggested the formalization of the mathematics behind vague concepts, such as the case of "near 2," using fuzzy subsets.

**Definition 1.2** Let $U$ be a (classic universal) set. A fuzzy subset $F$ of $U$ is defined by a function $\varphi_F$, called the **membership function** (of $F$)

$$\varphi_F : U \longrightarrow [0, 1].$$

The subscript $F$ on $\varphi$ identifies the subset ($F$ in this case) and the function $\varphi_F$ is the analogue of the characteristic function of the classical subset as defined in Definition 1.1 above. The value of $\varphi_F(x) \in [0, 1]$ indicates the degree to which the element $x$ of $U$ belongs to the fuzzy set $F$; $\varphi_F(x) = 0$ and $\varphi_F(x) = 1$, respectively, mean $x$ for sure does not belong to fuzzy subset $F$ and $x$ for sure belongs to the fuzzy subset $F$. From a formal point of view, the definition of a fuzzy subset is obtained simply by increasing the range of the characteristic function from {0, 1} to the whole interval [0, 1]. We can therefore say that a classical set is a special case of a fuzzy set when the range of the membership function $\varphi_F$ is restricted to $\{0, 1\} \subseteq [0, 1]$, that is, the membership function $\varphi_F$ retracts to the characteristic function $\chi_F$. In fuzzy language, a subset in the classic sense is usually called a **crisp subset**.

A fuzzy subset $F$ of $U$ can be seen as a standard (classic) subset of the Cartesian product $U \times [0, 1]$. Moreover, we can identify a fuzzy subset $F$ of $U$ with the set of ordered pairs (i.e., the graph of $\varphi_F$):

$$\{(x, \varphi_F(x)) : \text{ with } x \in U\}.$$

The classic subset of $U$ defined below

$$\text{supp } F = \{x \in U : \varphi_F(x) > 0\}$$

is called the **support** of $F$ and has a fundamental role in the interrelation between classical and fuzzy set theory. Interestingly, unlike fuzzy subsets, a support is a crisp set. Figure 1.1 illustrates this fact.

It is common to denote a fuzzy subset, say $F$, in fuzzy set literature, not by its membership function $\varphi_F$ but simply by the letter $F$. In this text we have decided to distinguish between $F$ and $\varphi_F$. In classical set theory, whenever we refer to a particular set $A$ we are actually considering a subset of a universal set $U$ but for the sake of simplicity or convenience, we say set $A$ even though set $A$ is actually a subset. The fuzzy set literature also uses of these terms. This text will use both terms interchangeably.

We now present some examples of fuzzy subsets.

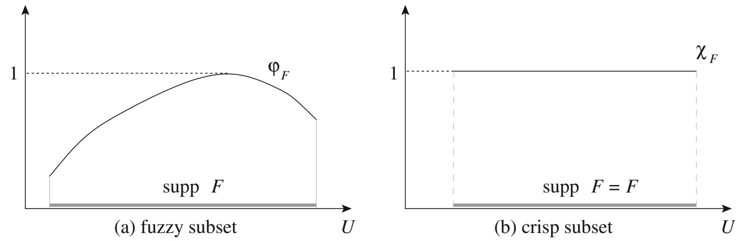


**Fig. 1.1** Illustration of subsets fuzzy and crisp

*Example 1.1* (*Even numbers*) Consider the set of natural even numbers:

$$E = \{n \in \mathbb{N} \, : \, n \text{ is even}\}.$$

This set $E$ has characteristic function which assigns to any natural number n the value $\chi_E(n) = 1$ if $n$ is even and $\chi_E(n) = 0$ if $n$ odd. This means that the set of even numbers is a particular fuzzy set of the set of natural numbers $\mathbb{N}$, since $\chi_E(n) \in [0, 1]$, in particular

$$\chi_E(n) = \varphi_E(n) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 0 \text{ otherwise.} \end{cases}$$

In this case, it was possible to determine all the elements of $E$, in the domain of the universal set $\mathbb{N}$ of natural numbers, because every natural number is either even or odd. However, this is not the case for other sets with imprecise boundaries.

*Example 1.2* ($Numbers\ near$ 2) Consider the following subset $F$ of the real numbers near of 2:

$$F = \{x \in \mathbb{R} : x \text{ is near } 2\}.$$

We can define the function $\varphi_F : \mathbb{R} \longrightarrow [0, 1]$, which associates each real value $x$ proximity to point 2 using the expression

$$\varphi_F(x) = \begin{cases} (1 - |x - 2|) & \text{if } 1 \leq x \leq 3 \\ 0 & \text{if } x \notin [1, 3] \end{cases}, \; x \in \mathbb{R}.$$

In this case the fuzzy subset $F$ of points near 2, characterized in $\varphi_F$, is such that $\varphi_F(2.001) = 0.999$ and $\varphi_F(7) = 0$. We say that $x = 2.001$ is near to 2 with proximity degree 0.999; $x = 7$ is not near 2.

On the other hand, in the example above, one may suggest a different membership function to show proximity to the value 2. For example, if the closeness proximity function were defined by

$$\nu_F(x) = exp\left[-(x-2)^2\right],$$

with $x \in \mathbb{R}$, then the elements of the fuzzy set $F$, characterized by the function $\nu_F$, as above, have different degrees of belonging from $\varphi_F$: $\nu_F(2.001) = 0.99999$ and $\nu_F(7) = 1.388 \times 10^{-11}$.

We can see that the notion of proximity is subjective and also depends on the membership function which can be expressed in countless different ways, depending on how we wish to evaluate the idea of a "nearness". Note that we could also define the concept "numbers near 2" by a classic set with membership function $\varphi_{F_\epsilon}$, considering, for example, a sufficiently small value of $\epsilon$ and the characteristic function for the interval $(2-\epsilon, 2+\epsilon)$, by following expression:

$$\varphi_{\epsilon F}(x) = \begin{cases} 1 & \text{if } |x-2| < \epsilon \\ 0 & \text{if } |x-2| \geq \epsilon. \end{cases}$$

Note that being close to 2 means being within a preset neighborhood of 2. The element of subjectivity lies in the choice of the radius of the neighborhood considered.In this specific case, all the values within the neighborhood are close to 2 with the same degree of belonging, which is 1.

*Example 1.3* (*Small natural numbers*) Consider the fuzzy subset $F$ containing the small natural numbers,

$$F = \{n \in \mathbb{N} : n \text{ is small}\}.$$

Does the number 0 (zero) belong to this set? What about the number 1.000? In the spirit of fuzzy set theory, it could be said that both do indeed belong to $F$, but with different degrees depending on the membership function $\varphi_F$ with respect to the fuzzy set $F$. The membership function associated with $F$ must be built in a way that is consistent with the term "small", the context of the problem and the application (mathematical model). One possibility for the membership function of $F$ would be

$$\varphi_F(n) = \frac{1}{n+1}, n \in \mathbb{N}.$$

Therefore, we could say that the number 0 (zero) belongs to $F$ with a degree of belonging equal to $\varphi_F(0) = 1$, while 999 also belongs to $F$, albeit with a degree of belonging equal to $\varphi_F(999) = 0.001$.

It is clear that in this case the choice of the function $\varphi_F$ was made in a somewhat arbitrary fashion, only taking into account the meaning of "small". To make a mathematical model of the "small natural number" notion, and thus to link $F$ to a membership function, we could, for example, choose any monotonically decreasing sequence, starting at 1 (one) and converging to 0 (zero) as

$$\{\varphi_n\}_{n\in\mathbb{N}};\ \text{with } \varphi_0 = 1.$$

For example,

$$\varphi_F(n) = e^{-n};$$
$$\varphi_F(n) = \frac{1}{n^2+1};$$
$$\varphi_F(n) = \frac{1}{ln\,(n+e)}.$$

The function to be selected to represent the fuzzy set considered depends on several factors related to the context of the problem under study. From the standpoint of strict fuzzy set theory, any of the previous membership functions can represent the subjective concept in question. However, what should indeed be noted is that each of the above functions produces a different fuzzy set, according to Definition 1.2.

The examples we have presented above possess a universal set $U$ for each fuzzy set that is clearly articulated. However, this is not always the case. In most cases of interest for mathematical modeling, the universal set needs to be delineated and in most instances the support set as well. Let's illustrate this point with a few more examples.

*Example 1.4* (*Fuzzy set of young people, Y*) Consider the inhabitants of a specific city. Each individual in this population can be associated to a real number corresponding to their age. Consider the whole universe as the ages within the interval $U = [0, 120]$ where $x \in U$ is interpreted as the age of a given individual. A fuzzy subset $Y$, of young people of this city, could be characterized by the following two membership functions for young, $Y_1$, $Y_2$ according different experts:

$$\varphi_{Y_1}(x) = \begin{cases} 1 & \text{if } x \le 10 \\ \dfrac{80-x}{70} & \text{if } 10 < x \le 80 \\ 0 & \text{if } x > 80 \end{cases},$$

$$\varphi_{Y_2}(x) = \begin{cases} \left(\dfrac{40-x}{40}\right)^2 & \text{if } 0 \le x \le 40 \\ 0 & \text{if } 40 < x \le 120 \end{cases}.$$

In the first case $Y_1$, the support is the interval $[0, 80]$ and in the second case $Y_2$, the support is $[0, 40]$. The choice of which function to be used to represent the concept of young people relies heavily on the context or analysis. Undoubtedly, about to retire professors would choose $Y_1$. Note that the choice of $U = [0, 120]$ as the interval for the universal set is linked to the fact that we have chosen to show how much an individual is young and our knowledge that statistically in the world, no one has lived beyond 120. If another characteristic were to be adopted, such as the number of grey hairs, to indicate the degree of youth, the universe would be different as well as the support.

The next example shows a bit more about fuzzy set theory in the mathematical modeling of "fuzzy concepts". In this example, we shall present a mathematical modeling treatment that allows the quantification and exploration of a theme of important social concern, poverty. This concept could be modeled based on a variety of appropriate variables, calorie intake, consumption of vitamins, iron intake, the volume of waste produced, or even the income of every individual, among many other features that are possible. However, we have chosen to represent poverty assuming that the only variable is income level. A possible mathematical model for poverty is shown below.

*Example 1.5* (*Fuzzy subset of the poor*) Consider that the concept of poor is based on the income level $r$. Hence, it is reasonable to assume that when you lower the income level, you raise the level of poverty of the individual. This means that the fuzzy subset $A_k$, of the poor in a given location, can be given by the following membership function:

$$\varphi_{A_k}(r) = \begin{cases} \left\{1 - \left[\left(\frac{r}{r_0}\right)^2\right]\right\}^k & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases}.$$

The parameter $k$ indicates a characteristic of the group we are considering. This parameter might indicate such things as the environment in which the individual people are situated. The parameter value, $r_0$, is the minimum income level believed to be required to be out of poverty.

As illustrated in Fig. 1.2 above, we have that if $k_1 \geq k_2$, then $\varphi_{A_{k_1}}(\bar{r}) \leq \varphi_{A_{k_2}}(\bar{r})$ which means that an individual group in $k_1$, with an income level $\bar{r}$, would be poorer for this income level were the individual in group $k_2$. We can also say that in terms of income, it is easier to live in the places where $k$ is greatest. So, intuitively, $k$ shows

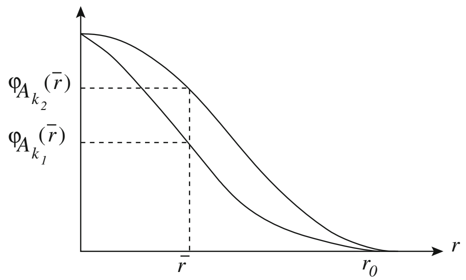


**Fig. 1.2** The membership function of the fuzzy subset of "poor"

whether the environment where the group lives is less or more favorable to life. The parameter $k$ may give an idea of the degree of saturation that a group has on the environment and, therefore, can be considered as an environmental parameter.

## 1.3 Operations with Fuzzy Subsets

This section presents the typical operations on fuzzy sets such as union, intersection and complementation. Each one of these operations is obtained from membership functions. Let $A$ and $B$ be two fuzzy subsets of $U$, with their respective membership functions $\varphi_A$ and $\varphi_B$. We say that $A$ is a fuzzy subset of $B$, and write $A \subset B$ if $\varphi_A(x) \leq \varphi_B(x)$ for all $x \in U$. Remember that the membership function of the empty set (∅) is given by $\varphi_\emptyset(x) = 0$, while the universal set $U$ has membership function $\varphi_U(x) = 1$ for all $x \in U$. Hence we can say that $\emptyset \subset A$ and $A \subset U$ for all $A$.

**Definition 1.3** (*Union*) The union between $A$ and $B$ is the fuzzy subset of $U$ whose membership function is given by:

$$\varphi_{A\cup B}(x) = max\{\varphi_A(x), \varphi_B(x)\}, x \in U.$$

We note that this definition is an extension of the classic case. In fact, when $A$ and $B$ are classics subsets of $U$ have:

$$\begin{aligned} max\{\chi_A(x), \chi_B(x)\} &= \begin{cases} 1 & \text{if } x \in A \text{ or } x \in B \\ 0 & \text{if } x \notin A \text{ and } x \notin B \end{cases} \\ &= \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{if } x \notin A \cup B \end{cases} \\ &= \chi_{A\cup B}(x),\ x \in U. \end{aligned}$$

**Definition 1.4** (*Intersection*) The intersection between $A$ and $B$ is the fuzzy subset of $U$ whose membership function is given by the following equation:

$$\varphi_{A\cap B}(x) = min\{\varphi_A(x), \varphi_B(x)\}, x \in U.$$

**Definition 1.5** (*Complement*) The complement of $A$ is the fuzzy subset $A^{'}$ in $U$ whose membership function is given by:

$$\varphi_{A'}(x) = 1 - \varphi_A(x),\ x \in U.$$

**Exercise 1.1** Suppose that $A$ and $B$ are classic subsets of $U$.

1. Check that

$$\min\{\chi_A(x), \chi_B(x)\} = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \notin A \cap B. \end{cases}$$

2. Check that $\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x)$. Note that this identity is does not hold in cases where $A$ and $B$ are fuzzy subsets.
3. Check that $\chi_{A\cap A'}(x) = 0$ $\left(A \cap A' = \emptyset\right)$ and that $\chi_{A\cup A'}(x) = 1$ $\left(A \cup A' = U\right)$ for all $x \in U$.

Unlike the classical situation, in the fuzzy context (see Fig. 1.3) we can have:

- $\varphi_{A\cap A'}(x) \neq 0 = \varphi_{\emptyset}(x)$ which means that we may not have $A \cap A' = \emptyset$;
- $\varphi_{A\cup A'}(x) \neq 1 = \varphi_U(x)$ which means that we may not have $A \cup A' = U$.

In the following example, we intend to exploit the special features presented by the concept of the complement of a fuzzy set.

*Example 1.6* (*Fuzzy set of the elderly*) The fuzzy set $O$ of the elderly (the old) should reflect a situation opposite of young people given above, when considering the ages that belong to $O$. While youth membership functions should decrease with age, the elderly should be increase with age. One possibility for the membership function of $O$ is:

$$\varphi_O(x) = 1 - \varphi_Y(x),$$

where $\varphi_Y$ is the membership function of the fuzzy subset "young". Therefore, the fuzzy set $O$ is the complement of fuzzy $Y$. In this example, if we take the set of young people $Y_1$ as having the membership function mentioned in the first part of Example 1.4, then:

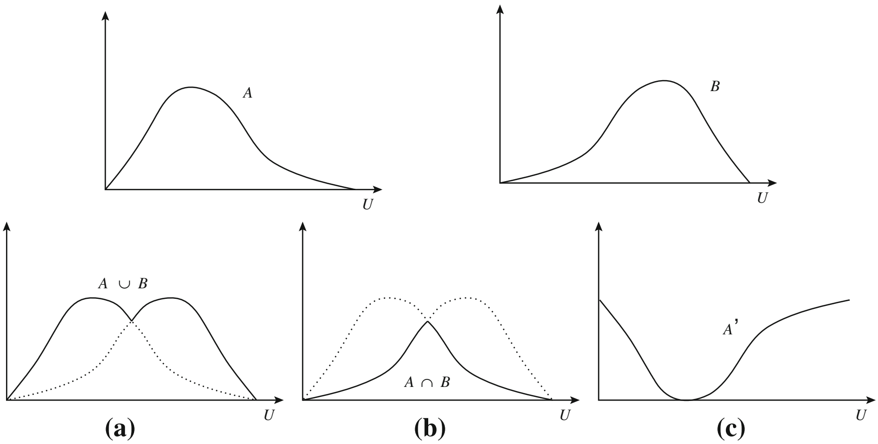


**Fig. 1.3** Operations with fuzzy subsets: **a** union, **b** intersection, and **c** complement


**Fig. 1.4** Fuzzy subsets of young and elderly

$$\varphi_O(x) = 1 - \varphi_{Y_1}(x) = \begin{cases} 0 & \text{if } x \leq 10 \\ \dfrac{x-10}{70} & \text{if } 10 < x \leq 80 \\ 1 & \text{if } x > 80 \end{cases} .$$

A graphical representation for $O$ and $Y_1$ is shown in Fig. 1.4.

Note that this operation, complement, exchanges degrees of belonging for the fuzzy subsets of $O$ and $Y_1$. This property characterizes the fuzzy complement, which means that while $\varphi_A(x)$ represents the degree of compatibility of $x$ with the linguistic concept in question, $\varphi_{A'}(x)$ shows the incompatibility of $x$ with the same concept.

One consequence of the imprecision of fuzzy sets is that there is a certain overlap of a fuzzy set with its complement. In Example 1.6, an individual who belongs to the of fuzzy set young with grade 0.8, also belongs to its complement $O$ with grade 0.2. Also note that it is quite possible for a member to belong to one set and also its complementary set with the same degree of belonging (in Fig. 1.4 this value is 45), showing that the more doubt we have about an element belonging to the set, the nearer to 0.5 is the degree of belong to this set. That is, the closer to a 0.5 membership value an element is, the greater the doubt of whether or not this element belongs to the set. The degree 0.5 is the maximum doubt (greatest entropy). This is a major difference from classical set theory in which an element either belongs to a set or to its complement, these being mutually exclusive, and there is absolutely no doubt.

Here it must also be noted that we have defined young and elderly (old), which are admittedly linguistic terms of opposite meanings, through the use of fuzzy sets that are not necessarily complementary. For example, we could have used $\varphi_{Y_1}$:

$$\varphi_{Y_2}(x) = \begin{cases} \left(\dfrac{40-x}{40}\right)^2 & \text{if } 0 \leq x \leq 40 \\ 0 & \text{if } 40 < x \leq 120, \end{cases}$$

in which case we could have obtained

$$\varphi_O(x) = \begin{cases} \left(\dfrac{x-40}{80}\right)^2 & \text{if } 40 < x \leq 120 \\ 0 & \text{if } x \leq 40. \end{cases}$$

**Exercise 1.2** Assume that the fuzzy set for young people, $Y$, is given by

$$\varphi_Y(x) = \begin{cases} \left[1 - \left(\dfrac{x}{120}\right)^2\right]^4 & \text{if } x \in [10, 120] \\ 1 & \text{if } x \notin [10, 120] \end{cases}.$$

1. Define a fuzzy set for the elderly.
2. Determine the age of an individual considered of middle age, which means grade 0.5, both in terms of youth and of elderly (old) age, assuming that the fuzzy set of the elderly is the complement to that of the young.
3. Draw the graph of the young and elderly (old) for part 2, and then compare it with Example 1.6.

We will next extend the concept to the complement for $A \subseteq B$ where $A$ is a fuzzy subset of fuzzy set $B$ and both in relation to the universe $U$. In this case, the complement of $A$ in relation to $B$ is denoted by the fuzzy set $A^{'}_B$ which has the following membership function:

$$\varphi_{A^{'}_B}(x) = \varphi_B(x) - \varphi_A(x), \ x \in U.$$

Note also that the complement of $A$ in relation to $U$ is a particular case of the complement of $A$ in $B$ since $\varphi_U(x) = 1$.

In the following example, we shall try to further exploit the concept of the ideas of complements with fuzzy subsets as defined in Example 1.5.

*Example 1.7* (*Fuzzy set of the poor revisited*) If the environment in which a group lives suffers any kind of degradation, from what we saw in Example 1.5, this results in a decreased environmental parameter, declining from $k_1$ to a lower value $k_2$, so that the individual having income level $r$ in $k_1$ has degree of poverty $\varphi_{A_{k_1}}(r)$ less than that of another $\varphi_{A_{k_2}}(r)$ with the same income $r$ in $k_2$. That is,

$$\varphi_{A_{k_1}}(r) < \varphi_{A_{k_2}}(r) \Leftrightarrow A_{k_1} \subset A_{k_2}.$$

Such a change could lead to the poverty level of a pauper, represented by $A_{k_2}$. The fuzzy complement of $A_{k_1}$ in $A_{k_2}$ is the fuzzy subset given by

$$\left(A^{'}\right)_{A_{k_2}}.$$

This set is not empty, and its membership function is given by

$$\varphi_{(A')_{A_{k_2}}} = \varphi_{A_{k_2}}(r) - \varphi_{A_{k_1}}(r), \; r \in U.$$

A recompense to the group that has suffered such a fall should be that of the same status of poverty as before. That is, given an income of $r_1$, the group should have an income of $r_2$ (after the fall) which means that

$$\varphi_{A_{k_2}}(r_2) - \varphi_{A_{k_1}}(r_1) = 0.$$

Therefore $r_2 - r_1 > 0$ and the recompense should be $r_2 - r_1$ (see Fig. 1.5).

We shall now make some brief comments and also look at the consequences of the major operations between fuzzy sets.

If $A$ and $B$ are sets in the classical sense, then the characteristic functions of their operations also satisfy the definitions defined for the fuzzy case, showing coherence between such concepts. For example, if $A$ is a (classic) subset of $U$, then the characteristic function $\chi_{A'}(x)$ of its complement is such that

$$\begin{cases} \chi_{A'}(x) = 0 & \text{if } \chi_A(x) = 1 \Leftrightarrow x \in A; \\ \chi_{A'}(x) = 1 & \text{if } \chi_A(x) = 0 \Leftrightarrow x \notin A. \end{cases}$$

In this case, either $x \in A$ or $x \notin A$, while the theory of fuzzy sets does not necessarily have this dichotomy. As seen in the Example 1.6, it is not always true that $A \cap A' = \emptyset$ for fuzzy sets, and it may not even true that $A \cup A' = U$. The following example reinforces these facts.

*Example 1.8* (*Fuzzy sets of fever and/or myalgia, muscular rheumatism*) Let's suppose that the universal set $U$ is the set of all patients within a clinic, identified by numbers 1, 2, 3, 4 and 5. Let $A$ and $B$ be fuzzy subsets that represent patients with fever and myalgia, respectively. Table 1.1 shows the operations union, intersection and complement.

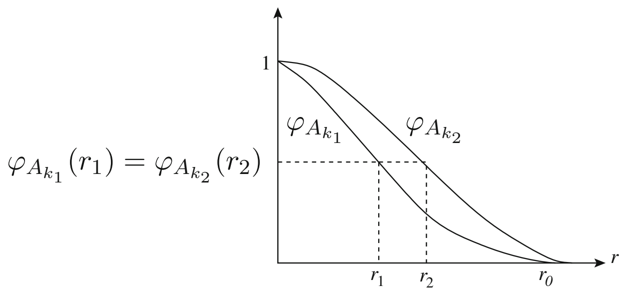


**Fig. 1.5** Recompense for changing in environment

**Table 1.1** Illustration of operations between fuzzy subsets

| Patient | Fever: $A$ | Myalgia: $B$ | $A \cup B$ | $A \cap B$ | $A'$ | $A \cap A'$ | $A \cup A'$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.7 | 0.6 | 0.7 | 0.6 | 0.3 | 0.3 | 0.7 |
| 2 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| 3 | 0.4 | 0.2 | 0.4 | 0.2 | 0.6 | 0.4 | 0.6 |
| 4 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| 5 | 1.0 | 0.2 | 1.0 | 0.2 | 0.0 | 0.0 | 1.0 |

The values in all columns except the first, show the degree to which each patient belongs to the fuzzy sets $A$, $B$, $A \cup B$, $A \cap B$, $A'$, $A \cap A'$, $A \cup A'$; respectively, where $A$ and $B$ are hypothetical data. In the column $A \cap A'$, the value of 0.3 shows that patient number 1 is both in the first group of patients with a fever as well as in the group with non-fever. As we have seen, this is a fact that would not be possible in classical set theory in which there is the exclusion law by which any set and its complement are mutually exclusive, $A \cap A' = \emptyset$.

The fuzzy subsets $A$ and $B$ of $U$ are equal if their membership functions are identical, that is, if $\varphi_A(x) = \varphi_B(x)$ for all $x \in U$. Below is listed the main properties of the operations as defined in this section.

**Proposition 1.1** *The operations between fuzzy subsets satisfying the following properties:*

- $A \cup B = B \cup A$,
- $A \cap B = B \cap A$,
- $A \cup (B \cup C) = (A \cup B) \cup C$,
- $A \cap (B \cap C) = (A \cap B) \cap C$,
- $A \cup A = A$,
- $A \cap A = A$,
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
- $A \cap \emptyset = \emptyset$ *and* $A \cup \emptyset = A$,
- $A \cap U = A$ *and* $A \cup U = U$,
- $(A \cup B)' = A' \cap B'$ *and* $(A \cap B)' = A' \cup B'$ $\left(DeMorgan's\ Law\right)$.

*Proof* The proof of each property is an immediate application of the properties between maximum and minimum functions, which means

$$\begin{cases} max\left[\varphi(x), \psi(x)\right] = \dfrac{1}{2}\left[\varphi(x) + \psi(x) + |\varphi(x) - \psi(x)|\right] \\ min\left[\varphi(x), \psi(x)\right] = \dfrac{1}{2}\left[\varphi(x) + \psi(x) - |\varphi(x) - \psi(x)|\right]. \end{cases}$$

where $\varphi$ and $\psi$ are functions with image in $\mathbb{R}$. We will only prove one of De Morgan's laws, because the other properties have similar proofs. If we consider that $\varphi_A$ is the membership function associated with the subset $A$, then we have:

$$\begin{aligned}
\varphi_{A'\cup B'}(u) &= max\,[1-\varphi_A(u)\,,\,1-\varphi_B(u)] \\
&= \frac{1}{2}\,[(1-\varphi_A(u)) + (1-\varphi_B(u)) + |\varphi_A(u)-\varphi_B(u)\,|] \\
&= \frac{1}{2}\,[2-(\varphi_A(u)+\varphi_B(u)) - |\varphi_A(u)-\varphi_B(u)\,|] \\
&= 1-\frac{1}{2}\,[\varphi_A(u)+\varphi_B(u) - |\varphi_A(u)-\varphi_B(u)\,|] \\
&= 1-min\,[\varphi_A(u)\,,\,\varphi_B(u)] = 1-\varphi_{A\cap B}(u) = \varphi_{(A\cap B)'}(u)\,,
\end{aligned}$$

for all $u \in U$. ■

**Exercise 1.3** Consider the fuzzy subset of tall people (in meters) in Brazil as defined by,

$$\varphi_A(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1.4 \\ \frac{1}{0.4}(x-1.4) & \text{if } 1.4 < x \le 1.8 \\ 1 & \text{if } x > 1.8 \end{cases}.$$

And people of average height $x$ (in meters) as follows,

$$\varphi_B(x) = \begin{cases} 0 & \text{if } x \le 1.4 \\ \frac{1}{0.2}(x-1.4) & \text{if } 1.4 < x \le 1.6 \\ 1 & \text{if } 1.6 < x \le 1.7 \\ \frac{1}{0.1}(1.8-x) & \text{if } 1.7 < x \le 1.8 \\ 0 & \text{if } x > 1.8 \end{cases}.$$

Obtain $(A \cup B)'$ and $A' \cup B'$ and give an interpretation for such operations.

To end this chapter, we look, in the next section, at a special class of crisp sets which are closely related to each fuzzy subset. These crisp sets can be interpreted as representing the level of vagueness represented by each fuzzy set.

## 1.4 Concept of $\alpha$-Level

A fuzzy subset $A$ of $U$ is "formed" by elements of $U$ with an order (hierarchy) that is given by the membership degrees. An element $x$ of $U$ will be in an "order class" $\alpha$ if its degree of belonging (its membership value) is at least the threshold level $\alpha \in [0, 1]$ that defines that class. The classic set of such elements is called an $\alpha$-level of $A$, denoted $[A]^{\alpha}$.

**Definition 1.6** ($\alpha$*-level*) Let $A$ be a fuzzy subset of $U$ and $\alpha \in [0, 1]$. The $\alpha$-level of the subset $A$ is classical set $[A]^\alpha$ of $U$ defined by

$$[A]^\alpha = \{x \in U : \varphi_A(x) \geq \alpha\} \text{ for } 0 < \alpha \leq 1.$$

When $U$ is a topological space, the zero $\alpha$-level of the fuzzy subset $A$ is defined as the smallest closed subset (in the classic sense) in $U$ containing the support set of $A$. In mathematical terms, $[A]^0$ is the closure of the support of $A$ and is also denoted by $\overline{\text{suppA}}$. This consideration becomes essential in theoretical situations appearing in this text. Note also that the set $\{x \in U : \varphi_A(x) \geq 0\} = U$ is not necessarily equal to $[A]^0 = \overline{\text{suppA}}$.

*Example 1.9* Let $U = \mathbb{R}$ be the set of real numbers and let $A$ be a fuzzy subset of $\mathbb{R}$ with the following function membership function:

$$\varphi_A(x) = \begin{cases} x - 1 & \text{if } 1 \leq x \leq 2 \\ 3 - x & \text{if } 2 < x < 3 \\ 0 & \text{if } x \notin [1, 3) \end{cases}.$$

In this case we have:

$$[A]^\alpha = [\alpha + 1, 3 - \alpha] \text{ for } 0 < \alpha \leq 1 \text{ and } [A]^0 = \overline{]1, 3[} = [1, 3] \text{ (Fig. 1.6)}.$$

*Example 1.10* Let $U = [0, 1]$ and $A$ be the fuzzy subset of $U$ whose membership function is given by $\varphi_A(x) = 4\left(x - x^2\right)$. Then,

$$[A]^\alpha = \left[\frac{1}{2}\left(1 - \sqrt{1 - \alpha}\right), \frac{1}{2}\left(1 + \sqrt{1 - \alpha}\right)\right]$$

for all $\alpha \in [0, 1]$ (Fig. 1.7).

We observed that if $x$ is an element of $[A]^\alpha$, then $x$ belongs to the fuzzy set $A$ with at least membership function degree $\alpha$. We have also that if $\alpha \leq \beta$ then $[A]^\beta \subseteq [A]^\alpha$.

The following theorem shows that a fuzzy set is uniquely determined by its $\alpha$-cuts.

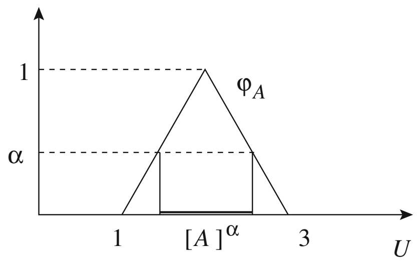


**Fig. 1.6** $\alpha$-level of the fuzzy subset $A$

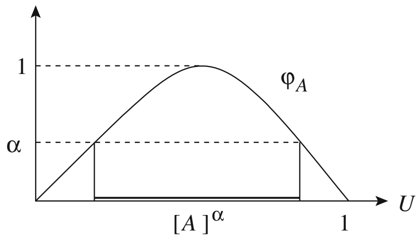


**Fig. 1.7** $\alpha$-level of the fuzzy subset $A$

**Theorem 1.2** *Let A and B be fuzzy subset of U. A necessary and sufficient condition for $A = B$ to hold is that $[A]^\alpha = [B]^\alpha$, for all $\alpha \in [0, 1]$.*

*Proof* Of course $A = B \Longrightarrow [A]^\alpha = [B]^\alpha$ for all $\alpha \in [0, 1]$. Let's now suppose that $[A]^\alpha = [B]^\alpha$ for all $\alpha \in [0, 1]$. If $A \neq B$ then there is an $x \in U$ such that $\varphi_A(x) \neq \varphi_B(x)$. Therefore we have $\varphi_A(x) < \varphi_B(x)$ or that, conversely, $\varphi_A(x) > \varphi_B(x)$. If we imagine that $\varphi_A(x) > \varphi_B(x)$, then we come to the conclusion that $x \in [A]^{\varphi_A(x)}$ and $x \notin [B]^{\varphi_A(x)}$ and therefore $[A]^{\varphi_A(x)} \neq [B]^{\varphi_A(x)}$, which contradicts the hypothesis that $[A]^\alpha = [B]^\alpha$ for all $\alpha \in [0, 1]$. Similar contradiction is reached if we assume that $\varphi_A(x) < \varphi_B(x)$. ■

One consequence of this theorem is that we now have a relationship between the membership function of a fuzzy subset and the characteristic functions of its $\alpha$-levels.

**Corollary 1.3** *The membership function $\varphi_A$ of a fuzzy set A can be expressed in terms of the characteristic function of their $\alpha$-levels, as follows:*

$$\varphi_A(x) = sup\{min\left[\alpha, \chi_{[A]^\alpha}(x)\right]\}, \text{ where } \chi_{[A]^\alpha}(x) = \begin{cases} 1 & \text{if } x \in [A]^\alpha \\ 0 & \text{if } x \notin [A]^\alpha \end{cases}.$$

The following theorem is of extreme importance in the study of fuzzy set theory and shows a condition which is sufficient for a family of subsets, in the classical sense, of $U$ can be formed by different $\alpha$-levels of a fuzzy subset.

**Theorem 1.4** (Negoita and Ralescu's Theorem of Representation [3]) *Let $A_\alpha$, $\alpha \in [0, 1]$, be a family of classical subsets of U, such that the following conditions hold:*

1. $\bigcup A_\alpha \subseteq A_0$ *with* $\alpha \in [0, 1]$*;*
2. $A_\alpha \subseteq A_\beta$ *if* $\beta \leq \alpha$*;*
3. $A_\alpha = \bigcap_{k \geq 0} A_{\alpha_k}$ *if* $\alpha_k$ *converges to* $\alpha$ *with* $\alpha_k \leq \alpha$.

*Under these conditions, there is one single fuzzy subset of A in U whose $\alpha$-levels are exactly the classic subsets $A_\alpha$, in other words,*

$$[A]^\alpha = A_\alpha.$$

The idea of the proof is to construct, for each $x \in U$, the membership function of $A$, as following,

$$\varphi_A(x) = sup\{\alpha \in [0, 1] : x \in A_\alpha\}.$$

For a complete proof see Negoita and Ralescu [3].

Using the definition of $\alpha$-levels, we have the following properties:

1. $[A \cup B]^\alpha = [A]^\alpha \cup [B]^\alpha$,
2. $[A \cap B]^\alpha = [A]^\alpha \cap [B]^\alpha$.

On the other hand, since in general $[A]^\alpha \cup \left[A'\right]^\alpha \neq U$, we have that $\left[A'\right]^\alpha \neq ([A]^\alpha)'$.

**Definition 1.7** A fuzzy set is said to be normal when all its $\alpha$-levels are not empty or in other words, if $[A]^1 \neq \emptyset$.

Recalling that the support of the fuzzy subset $A$ is the classic set

$$suppA = \{x \in U : \varphi_A(x) > 0\},$$

it is common to describe $A$ using the following notation, when it has a denumerable number of elements in its support, using the following notation,

$$A = \frac{\varphi_A(x_1)}{x_1} + \frac{\varphi_A(x_2)}{x_2} + .... = \sum_{i=1}^{\infty} \frac{\varphi_A(x_i)}{x_i},$$

and

$$A = \frac{\varphi_A(x_1)}{x_1} + \frac{\varphi_A(x_2)}{x_2} + .... + \frac{\varphi_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\varphi_A(x_i)}{x_i}.$$

when $A$ has finite discrete support. That is, $suppA = \{x_1, x_2, \ldots, x_n\}$. It is worth noting that the notation $\frac{\varphi_A(x_i)}{x_i}$, does not indicate division. It's just a way to visualize an element $x_i$ and its respective degree of belonging, its membership value, $\varphi_A(x_i)$. Also here the "+" symbol in the notation does not mean addition and $\sum$ does not mean summation. It's just a way to connect the elements of $U$ that are in $A$ with their respective degrees.

*Example 1.11* (*Finite fuzzy set*) Let $A$ be the fuzzy set of real numbers represented by

$$A = \sum_{i=1}^{n} \frac{\varphi_A(x_i)}{x_i} = \frac{0.1}{1} + \frac{0.2}{2} + \frac{0.25}{3} + \frac{0.7}{5} + \frac{0.9}{8} + \frac{1.0}{10}.$$

So,

$$A' = \sum_{i=1}^{n} \left[\frac{1 - \varphi_A(x_i)}{x_i}\right] = \frac{0.9}{1} + \frac{0.8}{2} + \frac{0.75}{3} + \frac{0.3}{5} + \frac{0.1}{8} + \frac{0.0}{10}.$$

In this case we have for example, 0.15-level of $A$ and its complement $A'$ are respectively, $[A]^{0.15} = \{2, 3, 5, 8, 10\}$ and $\left[A'\right]^{0.15} = \{1, 2, 3, 5\}$.

*Example 1.12* (*Fuzzy set of wolves*) Let $A$ be a pack of $n$ wolves. The degree of predation for each wolf may be associated with their age $x \in ]0, 15]$, assuming that the maximum age of a wolf is 15 years. The finite number of wolves means that one has only a finite number of wolves for each wolf ages. We will denote the set of these ages, as $A = \{x_1, x_2, \ldots, x_n\}$ and let us define the degree of predation of a wolf as $\varphi_P(x)$, considering that many young wolves prey less than adults, and that old wolves have reduced their ability for predation. Hence, the fuzzy subset of predators in the pack can be given by the membership function

$$\varphi_P(x) = \begin{cases} 0.5 & \text{if } 0 \le x \le 2 \\ 1.0 & \text{if } 2 < x < 10 \\ 0.2(15 - x) & \text{if } 10 \le x \le 15 \end{cases}.$$

With the above notation, the fuzzy finite subset $P$ is conveniently denoted by

$$P = \frac{\varphi_P(x_1)}{x_1} + \frac{\varphi_P(x_2)}{x_2} + \cdots + \frac{\varphi_P(x_n)}{x_n},$$

meaning that $\varphi_P\left(x_j\right)$ is the predation capacity of an individual of age $x_j$.

## 1.5 Summary

This chapter has discussed the differences of fuzzy set and uncertainty along with a brief philosophical discussion of the difficulties associated with these concepts. Our main interest is in fuzzy sets and their use in mathematical models of fuzzy logic and fuzzy dynamical systems. Secondly, we defined and illustrated the basic operations of fuzzy sets. More will be introduced in the context of the topics that follow. Lastly, the classical sets, called $\alpha$-levels, were discussed since they are central to the analysis in the ensuing chapters.

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