Negation and Partial Axiomatizations of Dependence and Independence Logic Revisited

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Abstract. In this paper, we axiomatize the negatable consequences in dependence and independence logic by extending the natural deduction systems of the logics given in [10,20]. We give a characterization for negatable formulas in independence logic and negatable sentences in dependence logic, and identify an interesting class of formulas that are negatable in independence logic. Dependence and independence atoms, first-order formulas belong to this class.

1 Introduction

Negation and partial axiomatizations of dependence and independence logic have been studied in the literature. In this paper, we take a new look at these topics.

Dependence logic was introduced by Väänänen [23] as a development of *Henkin quantifier* [11] and *independence-friendly logic* [12]. Recently, Grädel and Väänänen [9] defined a variant of dependence logic, called *independence logic*. The two logics add to first-order logic new types of atomic formulas $=\!(\vec{x}, y)$ and $\vec{x} \perp_{\vec{z}} \vec{y}$, called *dependence atom* and *independence atom*, to explicitly specify the dependence and independence relations between variables. Intuitively, $=\!(\vec{x}, y)$ states that "the value of y is completely determined by the values of the variables in the tuple $\vec{x}$", and $\vec{x} \perp_{\vec{z}} \vec{y}$ states that "given the values of the variables $\vec{z}$, the values of $\vec{x}$ and the values of $\vec{y}$ are completely independent of each other". These properties cannot be meaningfully manifested in *single* assignments of the variables. Therefore unlike in the case of the usual Tarskian semantics, formulas of dependence and independence logic are evaluated on *sets* of assignments (called *teams*) instead. This semantics is called *team semantics* and was introduced by Hodges [13,14].

Dependence and independence logic are known to have the same expressive power as existential second-order logic Σ^1_1 (see [5,18]). This fact has two negative consequences: The logics are not closed under classical negation and are not axiomatizable. The aim of this paper is to shed some new light on these problems.

Regarding the first problem, "negation", which is usually a desirable connective for a logic, turns out to be a tricky connective in the context of team semantics. The negation that dependence and independence logic inherit from first-order logic (denoted by $\neg$) is a type of "syntactic negation", in the

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sense that in order to compute the meaning of the formula $\neg\phi$, the negation $\neg$ has to be brought to the very front of atomic formulas by applying De Morgen's laws and the double negation law. It was proved that this negation $\neg$ is actually not a semantic operator [19], meaning that ϕ and ψ are semantically equivalent does not necessarily imply that $\neg\phi$ and $\neg\psi$ are semantically equivalent. The *classical (contradictory) negation* (denoted by $\sim$ in the literature), on the other hand, is a semantic operator. Since the Σ_1^1 fragment of second-order logic is not closed under classical negation, neither dependence nor independence logic is closed under classical negation. Dependence logic extended with the classical negation $\sim$ is called *team logic* in the literature, and it has the same expressive power as full second-order logic (see [17,23]).

Since every formula of dependence and independence logic is satisfied on the empty team, the classical contradictory negation $\sim\phi$ of any formula will not be satisfied on the empty team, implying that $\sim\phi$ cannot possibly be definable in dependence or independence logic for any single formula ϕ. This technical subtlety makes the classical contradictory negation $\sim$ less interesting. In this paper, we will, instead, consider the *weak classical negation*, denoted by $\dot\sim$, which behaves exactly as the classical negation except that on the empty team $\dot\sim\phi$ is always satisfied. We will give a characterization for negatable formulas in independence logic and negatable sentences in dependence logic by generalizing an argument in [23]. We also identify an interesting class of formulas that are negatable in independence logic. First-order formulas, dependence and independence atoms belong to this class. Formulas of this class are closely related to the *dependency notions* considered in [6] and the *generalized dependence atoms* studied in [16,21].

As for the axiomatization problem, since Σ_1^1 is not axiomatizable, dependence and independence logic cannot possibly be axiomatized in full. Nevertheless, [10,20] defined natural deduction systems for the logics such that the equivalence

$$\Gamma \models \phi \iff \Gamma \vdash \phi \tag{1}$$

holds if Γ is a set of sentences of dependence or independence logic and ϕ is a first-order sentence. It was left open whether these partial axiomatizations can be generalized such that the above equivalence (1) holds if Γ is a set of formulas (that possibly contain free variables) and ϕ is a (possibly open) first-order formula. Kontinen [15] gave such a generalization by expanding the signature with an extra relation symbol so as to interpret the teams associated with the free variables. In this paper, we will generalize the partial axiomatization results in [10,20] via a different approach, an approach that makes use of the weak classical negation. We will define extensions of the systems given in [10,20] such that the equivalence (1) holds if Γ is a set of formulas and ϕ is a formula that is negatable in the logics.

2 Preliminaries

Let us start by recalling the syntax and semantics (i.e. team semantics) of dependence and independence logic.

Although team semantics is intended for extensions of first-order logic obtained by adding dependence or independence atoms, for the sake of comparison we will now introduce the team semantics for first-order logic too. First-order atomic formulas α for a given signature $\mathcal{L}$ are defined as usual. Well-formed formulas of first-order logic, also called *first-order formulas*, (in *negation normal form*) are defined by the following grammar:

$$\phi ::= \alpha \mid \neg\alpha \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi$$

Formulas will be evaluated on the usual first-order models over an appropriate signature $\mathcal{L}$. We will use the same notation M for both a model and its domain. Let R be a fresh k-ary relation symbol and R^M a k-ary relation on M. We write $\mathcal{L}(R)$ for the expanded signature and (M, R^M) denotes the $\mathcal{L}(R)$-expansion of M in which the relation symbol R is interpreted as R^M. We write $\phi(R)$ to emphasize that the relation symbol R occurs in the formula ϕ.

Definition 2.1. *Let M be a model and V a set of first-order variables. A* team *X of M over V is a set of assignments of M over V, i.e., a set of functions $s : V \to M$. The set V is called the* domain *of X, denoted by* $\mathrm{dom}(X)$.

There is one and only one assignment of M over the empty domain, namely the empty assignment $\emptyset$. The singleton of the empty assignment $\{\emptyset\}$ is a team of M, and the empty set $\emptyset$ is a team of M over any domain.

Let s be an assignment of M over V and $a \in M$. We write $s(a/x)$ for the assignment of M over $V \cup \{x\}$ defined as $s(a/x)(x) = a$ and $s(a/x)(y) = s(y)$ for all $y \in V \setminus \{x\}$. For any set $N \subseteq M$ and any function $F : X \to \wp(M) \setminus \{\emptyset\}$, define

$$X(N/x) = \{s(a/x) : a \in N,\ s \in X\} \text{ and } X[F/x] = \{s(a/x) : s \in X \text{ and } a \in F(s)\}$$

We write $\vec{x}$ for a sequence $x_1, \dots, x_n$ of variables and the length n will always be clear from the context or does not matter; similarly for a sequence $\vec{F}$ of functions and a sequence $\vec{s}$ of assignments. A team $X(M/x_1)\dots(M/x_n)$ will sometimes be abbreviated as $X(M/\vec{x})$, and $X[F_1/x_1]\dots[F_n/x_n]$ as $X[F_1/x_1, \dots, F_n/x_n]$ or $X[\vec{F}/\vec{x}]$.

We now define the team semantics for first-order formulas. Note that our version of the team semantics for disjunction and existential quantifier is known as the *lax semantics* in the literature.

Definition 2.2. *Define inductively the notion of a first-order formula ϕ being* satisfied *on a model M and a team X, denoted by $M \models_X \phi$, as follows:*

- $M \models_X \alpha$ *with* α *a first-order atomic formula iff for all* $s \in X$, $M \models_s \alpha$ *in the usual sense*
- $M \models_X \neg\alpha$ *with* α *a first-order atomic formula iff for all* $s \in X$, $M \models_s \neg\alpha$ *in the usual sense*
- $M \models_X \bot$ *iff* $X = \emptyset$
- $M \models_X \phi \wedge \psi$ *iff* $M \models_X \phi$ *and* $M \models_X \psi$
- $M \models_X \phi \vee \psi$ *iff there exist* $Y, Z \subseteq X$ *with* $X = Y \cup Z$ *such that* $M \models_Y \phi$ *and* $M \models_Z \psi$
- $M \models_X \exists x\phi$ *iff* $M \models_{X[F/x]} \phi$ *for some function* $F : X \to \wp(M) \setminus \{\emptyset\}$
- $M \models_X \forall x\phi$ *iff* $M \models_{X(M/x)} \phi$

A routine inductive proof shows that first-order formulas have the downward closure property and the union closure property:

(Downward Closure Property). $M \models_X \phi$ and $Y \subseteq X$ imply $M \models_Y \phi$
(Union Closure Property). $M \models_{X_i} \phi$ for all $i \in I$ implies $M \models_{\bigcup_{i\in I} X_i} \phi$

which combined are equivalent to the flatness property:

(Flatness Property). $M \models_X \phi \iff M \models_{\{s\}} \phi$ for all $s \in X$

It follows easily from the flatness property that the team semantics for first-order formulas coincides with the usual single-assignment semantics in the sense that

$$M \models_{\{s\}} \phi \iff M \models_s \phi \tag{2}$$

holds for any model M, any assignment s and any first-order formula ϕ. If ϕ is a first-order formula, then the string $\neg\phi$, called the *syntactic negation* of ϕ, can be viewed as a first-order formula in negation normal form obtained in the usual way (i.e. by applying De Morgan's laws, the double negation law, etc.), and we write $\phi \to \psi$ for the formula $\neg\phi \vee \psi$. Since first-order formulas satisfy the Law of Excluded Middle $\phi \vee \neg\phi$ under the usual single-assignment semantics, Expression (2) implies that $M \models_{\{s\}} \phi \vee \neg\phi$ always holds, which, together with the flatness property, implies that $M \models_X \phi \vee \neg\phi$ holds for all teams X and all models M, namely, the Law of Excluded Middle holds for first-order formulas also in the sense of team semantics.

We now turn to dependence and independence logic. Well-formed formulas of independence logic ($\mathcal{I}$) are defined by the following grammar:

$$\begin{aligned}\phi ::= \alpha \mid \neg\alpha \mid \bot \mid x_1 \dots x_n \perp_{z_1\dots z_k} y_1 \dots y_m \mid {=\!(}x_1, \dots, x_n, y) \mid x_1 \dots x_n \subseteq y_1 \dots y_n \mid \\ \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi\end{aligned}$$

where α ranges over first-order atomic formulas. The formulas ${=\!(}\vec{x}, y)$, $\vec{x} \perp_{\vec{z}} \vec{y}$ and $\vec{x} \subseteq \vec{y}$ are called *dependence atom*, *independence atom* and *inclusion atom*, respectively. We refer to any of these atoms as *atoms of dependence and independence*. For the convenience of our argument in the paper, the independence logic as defined has a richer syntax than the standard one in the literature, which has the same syntax as first-order logic extended with independence atoms only. The other atoms are definable in the standard independence logic; for a proof see e.g., [4]. *Dependence logic* ($\mathcal{D}$), which is a fragment of $\mathcal{I}$, is defined as first-order logic extended with dependence atoms, and first-order logic extended with inclusion atoms is called *inclusion logic*. In this paper we will only concentrate on dependence logic and independence logic.

The set $\mathrm{Fv}(\phi)$ of free variables of a formula ϕ of $\mathcal{I}$ is defined as usual and we also have the new cases for dependence and independence atoms:

- $\mathrm{Fv}(x_1 \dots x_n \perp_{z_1 \dots z_k} y_1 \dots y_m) = \{x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k\}$
- $\mathrm{Fv}(=\!\!(x_1, \dots, x_n, y)) = \{x_1, \dots, x_n, y\}$
- $\mathrm{Fv}(x_1, \dots, x_n \subseteq y_1, \dots, y_n) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$

We write $\phi(\vec{x})$ to indicate that the free variables occurring in ϕ are among $\vec{x}$. A formula ϕ is called a *sentence* if it has no free variable.

Definition 2.3. *Define inductively the notion of a formula ϕ of $\mathcal{I}$ being* satisfied *on a model M and a team X, denoted by $M \models_X \phi$. All the cases are identical to those defined in Definition 2.2 and additionally:*

- *$M \models_X \vec{x} \perp_{\vec{z}} \vec{y}$ iff for all $s, s' \in X$, $s(\vec{z}) = s'(\vec{z})$ implies that there exists $s'' \in X$ such that*

$$s''(\vec{z}) = s(\vec{z}) = s'(\vec{z}),\ s''(\vec{x}) = s(\vec{x}) \text{ and } s''(\vec{y}) = s'(\vec{y}).$$

- *$M \models_X =\!\!(\vec{x}, y)$ iff for all $s, s' \in X$, $s(\vec{x}) = s'(\vec{x})$ implies $s(y) = s'(y)$.*
- *$M \models_X \vec{x} \subseteq \vec{y}$ iff for all $s \in X$, there exists $s' \in X$ such that $s'(\vec{y}) = s(\vec{x})$.*

We write $\vec{x} \perp \vec{y}$ for $\vec{x} \perp_{\langle\rangle} \vec{y}$, and note that the semantic clause for $\vec{x} \perp \vec{y}$ reduces to

- $M \models_X \vec{x} \perp \vec{y}$ iff for all $s, s' \in X$, there exist $s'' \in X$ such that

$$s''(\vec{x}) = s(\vec{x}) \text{ and } s''(\vec{y}) = s'(\vec{y}).$$

A sentence ϕ is said to be *true* in M, written $M \models \phi$, if $M \models_{\{\emptyset\}} \phi$. We write $\Gamma \models \psi$ if for any model M and any team X, $M \models_X \phi$ for all $\phi \in \Gamma$ implies $M \models_X \psi$. We also write $\phi \models \psi$ for $\{\phi\} \models \psi$. If $\phi \models \psi$ and $\psi \models \phi$, then we write $\phi \equiv \psi$.

We leave it for the reader to verify that formulas of dependence logic have the downward closure property and formulas of independence logic have the empty team property and the locality property:

(Empty Team Property). $M \models_\emptyset \phi$
(Locality Property). If $\{s \upharpoonright \mathrm{Fv}(\phi) \mid s \in X\} = \{s \upharpoonright \mathrm{Fv}(\phi) \mid s \in Y\}$[1], then

$$M \models_X \phi \iff M \models_Y \phi.$$

[1] For an assignment $s : V \to M$ and a set $V' \subseteq V$ of variables, we write $s \upharpoonright V'$ for the restriction of s to the domain V'.

Recall that the existential second-order logic (Σ_1^1) consists of those formulas that are equivalent to some formulas of the form $\exists R_1 \dots \exists R_k \phi$, where ϕ is a first-order formula. An $\mathcal{L}(R)$-sentence $\phi(R)$ of Σ_1^1 is said to be *downward monotone* with respect to R if $(M, Q) \models \phi(R)$ and $Q' \subseteq Q$ imply $(M, Q') \models \phi(R)$. It is known that $\phi(R)$ is downward monotone with respect to R if and only if R occurs in $\phi(R)$ only negatively (see e.g., [18]). A team X of M over $\{x_1, \dots, x_n\}$ induces an n-ary relation

$$rel(X) := \{(s(x_1), \dots, s(x_n)) \mid s \in X\}$$

on M; conversely, an n-ary relation R on M induces a team

$$X_R := \{\{(x_1, a_1), \dots, (x_n, a_n)\} \mid (a_1, \dots, a_n) \in R\}.$$

Theorem 2.4 (see [5,18,23])

(i) *Every $\mathcal{L}$-sentence ϕ of $\mathcal{D}$ or $\mathcal{I}$ is equivalent to an $\mathcal{L}$-sentence τ_ϕ of Σ_1^1, i.e.,*

$$M \models \phi \iff M \models \tau_\phi$$

holds for any model M; and conversely, every $\mathcal{L}$-sentence of Σ_1^1 is equivalent to an $\mathcal{L}$-sentence $\rho(\psi)$ of $\mathcal{D}$ or $\mathcal{I}$.

(ii) *For every $\mathcal{L}$-formula ϕ of $\mathcal{I}$, there is an $\mathcal{L}(R)$-sentence $\tau_\phi(R)$ of Σ_1^1 such that for all models M and all teams X,*

$$M \models_X \phi \iff (M, rel(X)) \models \tau_\phi(R).$$

If, in particular, ϕ is a formula of $\mathcal{D}$, then the relation symbol R occurs in the sentence $\tau_\phi(R)$ only negatively.

(iii) *For every $\mathcal{L}(R)$-sentence $\psi(R)$ of Σ_1^1 that is downward monotone with respect to R, there is an $\mathcal{L}$-formula $\rho(\psi)$ of $\mathcal{D}$ such that for all models M and all teams X,*

$$M \models_X \rho(\psi) \iff (M, rel(X)) \models \psi(R) \vee \forall \vec{x} \neg R\vec{x}. \tag{3}$$

(iv) *For every $\mathcal{L}(R)$-sentence $\psi(R)$ of Σ_1^1, there is an $\mathcal{L}$-formula $\rho(\psi)$ of $\mathcal{I}$ such that (3) holds for all models M and all teams X.*

In the sequel, we will use the notations τ_ϕ and $\tau_\phi(R)$ to denote the (up to semantic equivalence) unique formulas obtained in the above theorem and refer to them as the Σ_1^1-*translations* of the formulas ϕ of $\mathcal{D}$ or $\mathcal{I}$.

3 First-Order Formulas and Negatable Formulas

Formulas of dependence and independence logic can be translated into Σ_1^1 (Theorem 2.4). Therefore in the environment of team semantics a first-order formula ϕ has two identities: It can be viewed either as a formula of $\mathcal{D}$ or $\mathcal{I}$ that is to be evaluated on teams, or as a usual formula of first-order logic that is to be evaluated on single assignments and is possibly (equivalent to) the Σ_1^1-translation τ_ψ of some formula ψ of $\mathcal{D}$ or $\mathcal{I}$. With the latter reading of a first-order formula ϕ, for all models M and all assignments s, $M \models_s \neg\phi$ iff $M \not\models_s \phi$ holds. In this sense, the formula $\neg\phi$ can be interpreted as the "classical (contradictory) negation" of ϕ. However, on the team semantics side, unless the team X is a singleton, $M \not\models_X \phi$ is in general not equivalent to $M \models_X \neg\phi$. To express the contradictory negation in the team semantics setting, let us define the *classical negation* $\sim$ and the *weak classical negation* $\dot\sim$ as follows:

- $M \models_X \sim\phi$ iff $M \not\models_X \phi$
- $M \models_X \dot\sim\phi$ iff either $M \not\models_X \phi$ or $X = \emptyset$

Since formulas of dependence and independence logic have the empty team property, the classical negation $\sim\phi$ of any formula ϕ is not definable in the logics and we are therefore not interested in the classical negation $\sim$ in this paper. On the other hand, the weak classical negation $\dot\sim\phi$ can be definable in the logics for some formulas ϕ. We say that a formula ϕ is *negatable* in $\mathcal{I}$ (or $\mathcal{D}$) if there is a formula ψ of $\mathcal{I}$ (or $\mathcal{D}$) such that $\dot\sim\phi \equiv \psi$. If a formula ϕ of $\mathcal{I}$ is negatable in $\mathcal{I}$, we also say that ϕ is a negatable formula in $\mathcal{I}$ or the formula ϕ of $\mathcal{I}$ is negatable; similarly for $\mathcal{D}$.

For any first-order sentence ϕ, we have $M \not\models_{\{\emptyset\}} \phi$ iff $M \models_{\{\emptyset\}} \neg\phi$ by the Law of Excluded Middle. Thus $\dot\sim\phi \equiv \neg\phi$, meaning that first-order sentences are negatable both in $\mathcal{D}$ and in $\mathcal{I}$. Next, we prove that negatable formulas in $\mathcal{D}$ are, actually, all flat.

Fact 3.1 *If a formula ϕ of $\mathcal{D}$ is negatable in $\mathcal{D}$, then it is upward closed (i.e. $M \models_X \phi$ and $\emptyset \neq X \subseteq Y$ imply $M \models_Y \phi$), and thus flat.*

Proof. Suppose ϕ is a formula of $\mathcal{D}$ that is not upward closed. Then, there exist a model M and two teams $X \neq \emptyset$ and $Y \supseteq X$ such that $M \models_X \phi$ and $M \not\models_Y \phi$. But this means that $\dot\sim\phi$ is not downward closed and thus not definable in $\mathcal{D}$.

We will see in the sequel that the above fact does not apply to independence logic. Also note that sentences are always upward closed (since to evaluate a sentence it is sufficient to consider the nonempty team $\{\emptyset\}$ only). Thus, the other direction of the above fact, if true, would imply that all sentences of $\mathcal{D}$ are negatable. But this is not the case, as we will see in the following characterization theorem for negatable sentences in $\mathcal{D}$ and negatable formulas in $\mathcal{I}$.

Theorem 3.2

(i) *An $\mathcal{L}$-formula ϕ of $\mathcal{I}$ is negatable in $\mathcal{I}$ if and only if its Σ_1^1-translation $\tau_\phi(R)$ is equivalent to a first-order sentence.*

(ii) *An $\mathcal{L}$-sentence ϕ of $\mathcal{D}$ is negatable in $\mathcal{D}$ if and only if its Σ_1^1-translation τ_ϕ is equivalent to a first-order sentence.*

The above theorem states that negatable formulas in $\mathcal{I}$ are exactly those formulas that have first-order translations, and negatable sentences in $\mathcal{D}$ are exactly those sentences that have first-order translations. Therefore the problem of determining whether a formula of $\mathcal{I}$ or a sentence of $\mathcal{D}$ is negatable reduces to the problem of determining whether a Σ_1^1-sentence (τ_ϕ) is equivalent to a first-order formula, or whether the second-order quantifiers in a Σ_1^1-sentence can be eliminated. This problem is known to be undecidable (this follows from e.g., [3]).

We devote the remainder of this section to the proof of Theorem 3.2. The item (ii) actually follows implicitly from the results in [23], and the item (i) can be proved by essentially the argument of Theorem 6.7 in [23]. To proceed, let us first direct our attention to the Σ_1^1 counterpart of dependence and independence logic and prove a general theorem for Σ_1^1. The proof below is inspired by Theorem 6.7 in [23].

Theorem 3.3

- **(i)** *Let $\phi(R)$ be an $\mathcal{L}(R)$-formula of Σ_1^1 such that $(M,\emptyset) \models \phi(R)$ for any $\mathcal{L}$-model M. The formula $\neg\phi\vee\forall\vec{x}\neg R\vec{x}$ belongs to Σ_1^1 if and only if ϕ is equivalent to a first-order formula.*
- **(ii)** *Let ϕ be an $\mathcal{L}$-formula of Σ_1^1. The $\mathcal{L}$-formula $\neg\phi$ belongs to Σ_1^1 if and only if ϕ is equivalent to a first-order formula.*

Proof. (i) It suffices to prove the direction "$\Longrightarrow$". Suppose both ϕ and $\neg\phi \vee \forall\vec{x}\neg R\vec{x}$ belong to Σ_1^1. We may assume without loss of generality that $\phi \equiv \exists S_1 \ldots \exists S_k\psi$ and $(\neg\phi \vee \forall\vec{x}\neg R\vec{x}) \equiv \exists T_1 \ldots \exists T_m\chi$ for some first-order formulas ψ and χ, and the relation variables $S_1, \ldots, S_k, T_1, \ldots, T_m$ are pairwise distinct. Assume also that $\phi(R)$ and $\exists S_1 \ldots \exists S_k\psi$ are $\mathcal{L}_1(R)$-formulas, and $\neg\phi(R)\vee\forall\vec{x}\neg R\vec{x}$ and $\exists T_1 \ldots \exists T_m\chi$ are $\mathcal{L}_2(R)$-formulas.

Claim 1: $\psi \models \neg\chi \vee \forall\vec{x}\neg R\vec{x}$.

Proof of Claim 1. Put $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{R, S_1, \ldots, S_k, T_1, \ldots, T_m\}$. For any $\mathcal{L}$-model M such that $M \models \psi$, we have $M \models \exists S_1 \ldots \exists S_k\psi$. If $R^M = \emptyset$, then $M \models \forall\vec{x}\neg R\vec{x}$, thereby $M \models \neg\chi \vee \forall\vec{x}\neg R\vec{x}$. If $R^M \neq \emptyset$, then we have $M \models \neg\forall\vec{x}\neg R\vec{x}$. By the assumption, we also have $M \models \phi$. Thus, we derive

$$\begin{aligned} M \models \neg\neg\phi \wedge \neg\forall\vec{x}\neg R\vec{x} &\Longrightarrow M \models \neg(\neg\phi \vee \forall\vec{x}\neg R\vec{x}) \Longrightarrow M \models \neg\exists T_1 \ldots \exists T_m\chi \\ &\Longrightarrow M \models \forall T_1 \ldots \forall T_m\neg\chi \Longrightarrow M \models \neg\chi \\ &\Longrightarrow M \models \neg\chi \vee \forall\vec{x}\neg R\vec{x} \end{aligned}$$

as required.

Now, by Craig's Interpolation Theorem of first-order logic, there exists a first-order $\mathcal{L}_1(R) \cap \mathcal{L}_2(R)$-formula θ such that $\psi \models \theta$ and $\theta \models \neg\chi \vee \forall\vec{x}\neg R\vec{x}$.

Claim 2: $\phi \equiv \theta$.

Proof of Claim 2. For any $\mathcal{L}_1(R)$-model M, if $M \models \phi$, then $(M, S_1^M, \ldots, S_k^M) \models \psi$ for some relations $S_1^M, \ldots, S_k^M$ on M. Hence, $M \models \theta$.

Conversely, for any $\mathcal{L}_1(R)$-model M such that $M \not\models \phi$, we have $R^M \neq \emptyset$ and $M \models \neg\phi \vee \forall\vec{x}\neg R\vec{x}$. The latter implies $(M, T_1^M, \ldots, T_m^M) \models \chi$ for some relations $T_1^M, \ldots, T_m^M$ on M. It then follows that $(M, T_1^M, \ldots, T_m^M) \not\models \neg\chi \vee \forall\vec{x}\neg R\vec{x}$. Hence, $M \not\models \theta$.

(ii) The nontrivial direction "$\Longrightarrow$" follows from a similar and simplified argument. Instead of proving Claim 1 as above, one proves $\psi \models \neg\chi$. □

Now, we are ready to give the proof of Theorem 3.2.

Proof (of Theorem 3.2). (i) Let ϕ be an $\mathcal{L}$-formula of $\mathcal{I}$. By Theorem 2.4(ii) there exists an $\mathcal{L}(R)$-sentence $\tau_\phi(R)$ of Σ_1^1 such that for any model M and any team X,

$$M \models_X \dot{\sim}\phi \iff M \not\models_X \phi \text{ or } X = \emptyset \iff (M, rel(X)) \models \neg\tau_\phi(R) \vee \forall\vec{x}\neg R\vec{x}. \tag{4}$$

Now, to prove the direction "$\Longleftarrow$", assume that $\tau_\phi(R)$ is equivalent to a first-order sentence. Then, the sentence $\neg\tau_\phi(R)$ is also equivalent to a first-order sentence, and thus by Theorem 2.4(iv) there exists a formula $\rho(\neg\tau_\phi)$ of $\mathcal{I}$ such that for all $\mathcal{L}$-models M and all teams X,

$$M \models_X \rho(\neg\tau_\phi) \iff (M, rel(X)) \models \neg\tau_\phi(R) \vee \forall\vec{x}\neg R\vec{x}.$$

It then follows from (4) that $\rho(\neg\tau_\phi) \equiv \dot{\sim}\phi$.

Finally, to prove the direction "$\Longrightarrow$", assume that $\dot{\sim}\phi \equiv \psi$ for some formula ψ of $\mathcal{I}$. By Theorem 2.4(ii) there exists an $\mathcal{L}(R)$-sentence $\tau_\psi(R)$ of Σ_1^1 such that for all models M and all teams X,

$$M \models_X \psi \iff (M, rel(X)) \models \tau_\psi(R).$$

By (4), $\tau_\psi(R) \equiv \neg\tau_\phi(R) \vee \forall\vec{x}\neg R\vec{x}$ and thereby the formula $\neg\tau_\phi(R) \vee \forall\vec{x}\neg R\vec{x}$ belongs to Σ_1^1. For any model M, since $M \models_\emptyset \phi$, we have $(M, \emptyset) \models \tau_\phi(R)$. Then, by Theorem 3.3(i), we conclude that $\tau_\phi(R)$ is equivalent to a first-order formula.

(ii) This item is proved by a similar argument that makes use of Theorem 2.4(i) and Theorem 3.3(ii). □

4 Axiomatizing Negatable Consequences in Dependence and Independence Logic

Dependence and independence logic are not axiomatizable, meaning that the consequence relation $\Gamma \models \phi$ cannot be effectively axiomatized. Nevertheless, if we restrict $\Gamma \cup \{\phi\}$ to a set of sentences and ϕ to a first-order sentence, the consequence relation $\Gamma \models \phi$ is axiomatizable and explicit axiomatizations for $\mathcal{D}$ and $\mathcal{I}$ are given in [10,20]. Throughout this section, let L denote one of the logics of $\mathcal{D}$ and $\mathcal{I}$, and $\vdash_{\mathsf{L}}$ denote the syntactic consequence relation associated with the deduction system of L defined in [20] or in [10].

Theorem 4.1 (see [10,20]). *Let Γ be a set of sentences of* L*, and ϕ a first-order formula. We have $\Gamma \models \phi \iff \Gamma \vdash_{\mathsf{L}} \phi$. In particular, $\Gamma \models \bot \iff \Gamma \vdash_{\mathsf{L}} \bot$.*

Kontinen [15] generalized the above axiomatization result to cover also the case when $\Gamma \cup \{\phi\}$ is a set of formulas (that possibly contain free variables) by adding a new relation symbol to interpret the teams. In this section, we will generalize Theorem 4.1 without expanding the signature to cover the case when $\Gamma \cup \{\phi\}$ is a set of formulas (that possibly contain free variables) and ϕ is negatable.

We first prove that under certain constraint the (possibly open) formula ψ in the entailment $\Delta, \psi \models \theta$ can be turned into a sentence without affecting the entailment relation.

Lemma 4.2. *Let $\Delta \cup \{\chi, \theta\}$ be a set of formulas of* L*. Let $\mathrm{Fv}(\chi) = \{x_1, \dots, x_n\}$ and $\mathrm{Fv}(\Delta) = \bigcup_{\delta \in \Delta} \mathrm{Fv}(\delta)$. Suppose that $\mathrm{Fv}(\chi) \cap \mathrm{Fv}(\Delta) = \emptyset$ and $\mathrm{Fv}(\chi) \cap \mathrm{Fv}(\theta) = \emptyset$. We have $\Delta, \chi \models \theta \iff \Delta, \exists x_1 \dots \exists x_n \chi \models \theta$.*

Proof. "$\Longrightarrow$": Suppose $\Delta, \chi \models \theta$. If $M \models_X \delta$ for all $\delta \in \Delta$ and $M \models_X \exists \vec{x} \chi$, then $M \models_{X[\vec{F}/\vec{x}]} \chi$ for some appropriate sequence of functions $\vec{F}$. Since $\mathrm{Fv}(\chi) \cap \mathrm{Fv}(\Delta) = \emptyset$, we derive $M \models_{X[\vec{F}/\vec{x}]} \delta$ for all $\delta \in \Delta$ by the locality property. Thus, by the assumption, we conclude that $M \models_{X[\vec{F}/\vec{x}]} \theta$, which implies $M \models_X \theta$ since $\mathrm{Fv}(\chi) \cap \mathrm{Fv}(\theta) = \emptyset$.

"$\Longleftarrow$": Suppose $\Delta, \exists \bar{x} \chi \models \theta$, and suppose $M \models_X \delta$ for all $\delta \in \Delta$ and $M \models_X \chi$. Then, we have $M \models_X \exists \vec{x} \chi$, which implies $M \models_X \theta$ by the assumption. □

To understand why Theorem 4.1 can be generalized, let us consider a set $\Gamma \cup \{\phi\}$ of formulas of L. Since Σ_1^1 admits the Compactness Theorem, we may assume that Γ is a finite set. Clearly, $\Gamma \models \phi$ is equivalent to $\Gamma, \dot{\sim} \phi \models \bot$, and further to $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi) \models \bot$ by Lemma 4.2, where $\mathrm{Fv}(\bigwedge \Gamma \wedge \dot{\sim} \phi) = \{x_1, \dots, x_n\}$. Adding appropriate rules to the deduction system to guarantee the equivalence of $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi) \vdash \bot$ and $\Gamma \vdash \phi$, the Completeness Theorem can be restated as $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi) \nvdash \bot \Longrightarrow \exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi) \not\models \bot$. Now, assuming that $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi)$ is deductively consistent, the problem reduces to the problem of constructing a model for the sentence $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi)$. If $\dot{\sim} \phi$ is definable in L, then the problem further reduces to the problem of constructing a model for the Σ_1^1 sentence $\tau_{\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi)}$, which can in principle be done in first-order logic. This argument shows that via the trick of weak classical negation Theorem 4.1 can, in principle, be generalized. Note that if Γ is a set of sentences and ϕ is a first-order sentence, then $\neg \phi \equiv \dot{\sim} \phi$ and the foregoing argument reduces to the argument given in [20].

Let us now make this idea precise. Given the Completeness Theorems in [10,20], it suffices to extend the natural deduction systems of [10,20] by adding the two rules below to ensure the equivalence of $\Gamma \vdash \phi$ and $\exists \vec{x} (\bigwedge \Gamma \wedge \dot{\sim} \phi) \vdash \bot$, where $\dot{\sim} \phi$ denotes the formula of L that is equivalent to the weak negation of ϕ.

RULES

Weak classical negation transition	Weak classical negation elimination
$$\dfrac{\begin{matrix} & [\exists\vec{x}(\psi\wedge\dot\sim\phi)] \\ D_1 & D_2 \\ \psi & \bot \end{matrix}}{\phi}\ (*)\quad \dot\sim\,\mathsf{Tr}$$	$$\dfrac{\begin{matrix} & [\psi] \\ D_1 & D_2 \\ \exists\vec{x}(\psi\wedge\dot\sim\phi) & \phi \end{matrix}}{\bot}\ (*)\quad \dot\sim\,\mathsf{E}$$
(∗) where the variables $x_1,\dots,x_n$ do not occur freely in any formula in the undischarged assumptions in the derivation D_2	

Let $\vdash^*_{\mathsf{L}}$ denote the syntactic consequence relation associated with the system of L extended with the rules $\dot\sim\,\mathsf{Tr}$ and $\dot\sim\,\mathsf{E}$. We now prove the Soundness and Completeness Theorem for this extended system.

Theorem 4.3. *Let $\Gamma\cup\{\phi\}$ be a set of formulas of L such that ϕ is negatable in L. We have $\Gamma\models\phi \iff \Gamma\vdash^*_{\mathsf{L}}\phi$.*

Proof. "$\Longleftarrow$": The Soundness of the system follows from Lemma 4.2; see Appendix I for the detailed proof.

"$\Longrightarrow$": Since L is compact, without loss of generality we may assume that Γ is finite. By Lemma 4.2 and the Completeness Theorem of L (Theorem 4.1), we derive

$$\Gamma\models\phi \iff \exists\bar{x}(\bigwedge\Gamma\wedge\dot\sim\phi)\models\bot \iff \exists\bar{x}(\bigwedge\Gamma\wedge\dot\sim\phi)\vdash_{\mathsf{L}}\bot \iff \Gamma\vdash^*_{\mathsf{L}}\phi$$

by applying the rules $\dot\sim\,\mathsf{Tr}$ and $\dot\sim\,\mathsf{E}$. □

A key issue in the application of the extended system is the issue of computing the weak negation of formulas in L, or, as the first step, deciding which formulas are negatable in L. As we already remarked, even if we have established in Theorem 3.2 a characterization for negatable formulas, the latter problem is undecidable. Nevertheless, it is possible to identify some interesting classes of negatable formulas. This is what we will pursue in the next section. Let us proceed now to prove that first-order formulas are negatable in $\mathcal{I}$. This will show that for independence logic Theorem 4.3 is indeed a generalization of Theorem 4.1 and also [15].

Given a first-order formula ϕ, consider its syntactic negation $\neg\phi$. By the flatness property and the Law of Excluded Middle, we have

$$M\models_X\neg\phi \iff M\not\models_{\{s\}}\phi \text{ for all } s\in X \tag{5}$$

for all models M and all nonempty teams X. This also shows that $\neg\phi$ is in general not equivalent to $\dot\sim\phi$, not even for atomic first-order formulas. Moreover, that the Σ^1_1-translation $\tau_\phi(R)$ of a first-order formula ϕ is equivalent to a first-order sentence is not a trivial consequence of Theorem 2.4 either, because, for instance, the translation of a first-order disjunction $\phi\vee\psi$, as given in [23], is $\tau_{\phi\vee\psi}(R)=\exists S\exists S'(\tau_\phi(S)\wedge\tau_\psi(S')\wedge\forall\vec{x}(R\vec{x}\to(S\vec{x}\vee S'\vec{x})))$.

Proposition 4.4. *If ϕ is a first-order formula, then $\dot\sim\phi(\vec{x}) \equiv \exists\vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg\phi(\vec{w}))$. In particular, first-order formulas are negatable in $\mathcal{I}$.*

Proof. For all models M and all teams X, since ϕ is flat,

$$M \models_X \dot\sim\phi \iff X = \emptyset \text{ or } M \not\models_X \phi \iff X = \emptyset \text{ or } \exists s \in X(M \not\models_{\{s\}} \phi(\vec{x})).$$

By the empty team property of independence logic, it suffices to show that

$$\exists s \in X(M \not\models_{\{s\}} \phi(\vec{x})) \iff M \models_X \exists\vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg\phi(\vec{w})).$$

for all models M and all nonempty teams X.

"$\Longrightarrow$": Assume $M \not\models_{\{s\}} \phi(\vec{x})$ for some $s \in X$ and $\vec{x} = x_1 \ldots x_n$. For each $1 \leq i \leq n$, inductively define a constant function F_i as follows:

- $F_1 : X \to \wp(M) \setminus \{\emptyset\}$ is defined as $F_1(t) = \{s(x_1)\}$;
- $F_i : X[F_1/w_1, \ldots, F_{i-1}/w_{i-1}] \to \wp(M) \setminus \{\emptyset\}$ is defined as $F_i(t) = \{s(x_i)\}$.

Consider the team $X[\vec{F}/\vec{w}]$ (see Fig. 1 in Appendix II for an example of such a team). Clearly, $M \models_{X[\vec{F}/\vec{w}]} \vec{w} \subseteq \vec{x}$. On the other hand, for any $t \in X[\vec{F}/\vec{w}]$, since $t(\vec{w}) = s(\vec{x})$ and $M \not\models_{\{s\}} \phi(\vec{x})$, we obtain $M \not\models_{\{t\}} \phi(\vec{w})$ by the locality property. Hence, $M \models_{X[\vec{F}/\vec{w}]} \neg\phi(\vec{w})$ by (5).

"$\Longleftarrow$": Conversely, suppose $M \models_X \exists\vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg\phi(\vec{w}))$. Then there are appropriate functions F_i for each $1 \leq i \leq n$ such that $M \models_{X[\vec{F}/\vec{w}]} \vec{w} \subseteq \vec{x}$ and $M \models_{X[\vec{F}/\vec{w}]} \neg\phi(\vec{w})$. By (5), the latter implies that $M \not\models_{\{t\}} \phi(\vec{w})$ for some $t \in X[\vec{F}/\vec{w}]$. By the former, there exists $s' \in X[\vec{F}/\vec{w}]$ such that $s'(\vec{x}) = t(\vec{w})$. This means, by the definition of $X[\vec{F}/\vec{w}]$, that there exists $s \in X$ such that $s(\vec{x}) = s'(\vec{x}) = t(\vec{w})$. Hence, $M \not\models_{\{s\}} \phi(\vec{x})$ by the locality property. □

We remarked that the Σ_1^1-translation of a disjunction $\phi \vee \psi$ of two negatable formulas ϕ and ψ is not itself a first-order formula. In the literature there is another disjunction $\veebar$, defined as follows, under which the set of negatable formulas is closed:

- $M \models_X \phi \veebar \psi$ iff $M \models_X \phi$ or $M \models_X \psi$

In the presence of the downward closure property this disjunction is called *intuitionistic disjunction*, and in the environment of $\mathcal{I}$ we shall call it *Boolean disjunction.* The disjunction is uniformly definable in $\mathcal{D}$ or $\mathcal{I}$ since

$$\phi \veebar \psi \equiv \exists w \exists u(=\!(w) \wedge =\!(u) \wedge ((w = u) \vee \phi) \wedge ((w \neq u) \vee \psi)),$$

and clearly $\dot\sim(\phi \wedge \psi) \equiv \dot\sim\phi \veebar \dot\sim\psi$ and $\dot\sim(\phi \veebar \psi) \equiv \dot\sim\phi \wedge \dot\sim\psi$.

Without going into detail we remark that the extended system can be applied to give a new formal proof of Arrow's Impossibility Theorem [2] in social choice theory. In [22] the theorem is formulated as an entailment $\Gamma_{\mathsf{Arrow}} \models \phi_{\mathsf{dictator}}$ in independence logic, where Γ_{Arrow} is a set of formulas expressing the conditions in Arrow's Impossibility Theorem and ϕ_{dictator} is a formula expressing the existence of a dictator. The formula ϕ_{dictator} is of the form $\veebar_{i=1}^{n}\phi_i$, where ϕ_i is a first-order formula expressing that voter i is a dictator (among n voters). By what we just obtained, the formula ϕ_{dictator} is negatable in $\mathcal{I}$ and the Completeness Theorem guarantees that $\Gamma_{\mathsf{Arrow}} \vdash_{\mathcal{I}}^{*} \phi_{\mathsf{dictator}}$ is derivable in our extended system.

5 A Hierarchy of Negatable Atoms

In this section, we define an interesting class of formulas that are negatable in $\mathcal{I}$. This class will be presented in the form of an alternating hierarchy of atoms that are definable in $\mathcal{I}$. These atoms are closely related to the *dependency notions* considered in [6], and the *generalized dependence atoms* studied in [16,21]. We will demonstrate that all first-order formulas, dependence atoms, independence atoms and inclusion atoms belong to this class. It then follows from the completeness result we obtained in the previous section that consequences of these types in $\mathcal{I}$ are derivable in the extended system.

Let us start by defining the notion of abstract relation. A k-ary *relation* R is a class of pairs (M, R^M) that is closed under taking isomorphic images, where M ranges over first-order models and $R^M \subseteq M^k$. For instance, the familiar equality $=$ is a binary relation defined by the class

$$\{(M, =^M) \mid M \text{ is a first-order model}\}, \text{ where } =^M := \{(a, a) \mid a \in M\}.$$

Every first-order formula $\phi(x_1, \dots, x_k)$ with k free variables is associated with a k-ary relation

$$\boldsymbol{\phi} := \{(M, \boldsymbol{\phi}^M) \mid M \text{ is a first-order model}\},$$

where $\boldsymbol{\phi}^M := \{(s(x_1), \dots, s(x_k)) \mid M \models_s \phi\}$. A k-ary relation R is said to be *(first-order) definable* if there exists a (first-order) formula $\phi_R(w_1, \dots, w_k)$ such that for all models M and all assignments s,

$$s(\vec{w}) \in R^M \iff M \models_s \phi_R(\vec{w}).$$

Clearly, the first-order formula $w = u$ defines the equality relation, and every first-order formula ϕ defines its associated relation $\boldsymbol{\phi}$.

If R is a k-ary relation, then we write $\overline{R}$ for the complement of R that is defined by letting $\overline{R}^M = M^k \setminus R^M$ for all models M. Clearly, if a first-order formula ϕ defines R, then its negation $\neg\phi$ defines $\overline{R}$.

If $\vec{s} = \langle s_1, \dots, s_k \rangle$, then we write $\vec{s}(\vec{x})$ for $\langle s_1(\vec{x}), \dots, s_k(\vec{x}) \rangle$. For every sequence $\mathsf{k} = \langle k_1, \dots, k_n \rangle$ of natural numbers and every $(k_1 + \dots + k_n)m$-ary relation R, we introduce two new atomic formulas $\Sigma^R_{n,\mathsf{k}}(x_1, \dots, x_m)$ and $\Pi^R_{n,\mathsf{k}}(x_1, \dots, x_m)$ with the semantics defined as follows:

- $M \models_\emptyset \Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $M \models_\emptyset \Pi^R_{n,\mathsf{k}}(\vec{x})$.
- If n is odd, then define for any model M and any *nonempty* team X
  - $M \models_X \Sigma^R_{n,\mathsf{k}}(\vec{x})$ iff there exist $s_{11}, \dots, s_{1k_1} \in X$ such that for all $s_{21}, \dots, s_{2k_2} \in X$, … there exist $s_{n1}, \dots, s_{nk_n} \in X$ such that $(\vec{s_1}(\vec{x}), \dots, \vec{s_n}(\vec{x})) \in R^M$;
  - $M \models_X \Pi^R_{n,\mathsf{k}}(\vec{x})$ iff for all $s_{11}, \dots, s_{1k_1} \in X$, there exist $s_{21}, \dots, s_{2k_2} \in X$ such that … for all $s_{n1}, \dots, s_{nk_n} \in X$, it holds that $(\vec{s_1}(\vec{x}), \dots, \vec{s_n}(\vec{x})) \in R^M$.
- Similarly if n is even.

Fact 5.1. $\dot\sim \Sigma^R_{n,\mathsf{k}}(\vec{x}) \equiv \Pi^{\overline{R}}_{n,\mathsf{k}}(\vec{x})$ *and* $\dot\sim \Pi^R_{n,\mathsf{k}}(\vec{x}) \equiv \Sigma^{\overline{R}}_{n,\mathsf{k}}(\vec{x})$.

Let us now give some examples of the $\Sigma^R_{n,\mathsf{k}}$ and $\Pi^R_{n,\mathsf{k}}$ atoms.

Example 5.2

(a) *The dependence atom* $=\!(x_1,\dots,x_k,y)$ *is a* $\Pi^{\mathbf{dep}_k}_{1,\langle 2\rangle}(x_1,\dots,x_k,y)$ *atom, where* $\mathbf{dep}_k$ *is a* $2(k+1)$*-ary relation defined as*

$$(a_1,\dots,a_k,b,a'_1,\dots,a'_k,b') \in (\mathbf{dep}_k)^M \quad \textit{iff} \quad [\,(a_1,\dots,a_k) = (a'_1,\dots,a'_k) \Longrightarrow b = b'\,].$$

The first-order formula $\big((w_1 = w'_1) \wedge \cdots \wedge (w_k = w'_k)\big) \to (u = u')$ *defines* $\mathbf{dep}_k$.

(b) *The independence atom* $x_1,\dots,x_k \perp_{z_1,\dots,z_n} y_1,\dots,y_m$ *is a*

$$\Pi^{\mathbf{ind}_{k,m,n}}_{2,\langle 2,1\rangle}(x_1,\dots,x_k,y_1,\dots,y_m,z_1,\dots,z_n)$$

atom, where $\mathbf{ind}_{k,m,n}$ *is a (first-order definable)* $(2{+}1)(k{+}m{+}n)$*-ary relation defined as* $(\vec{a},\vec{b},\vec{c},\vec{a'},\vec{b'},\vec{c'},\vec{a''},\vec{b''},\vec{c''}) \in (\mathbf{ind}_{k,m,n})^M$ *iff*

$$\begin{aligned}(c_n,\dots,c_n) =& (c'_1,\dots,c'_n) = (c''_1,\dots,c''_n)\\ &\Longrightarrow [\,(a''_1,\dots,a''_k) = (a_1,\dots,a_k) \textit{ and } (b''_1,\dots,b''_m) = (b'_1,\dots,b'_m)\,].\end{aligned}$$

(c) *The inclusion atom* $x_1,\dots,x_k \subseteq y_1,\dots,y_k$ *is a* $\Pi^{\mathbf{inc}_k}_{2,\langle 1,1\rangle}(x_1,\dots,x_k,y_1,\dots,y_k)$ *atom, where* $\mathbf{inc}_k$ *is a (first-order definable)* $(1+1)2k$*-ary relation defined as*

$$(a_1,\dots,a_k,b_1,\dots,b_k,a'_1,\dots,a'_k,b'_1,\dots,b'_k) \in (\mathbf{inc}_k)^M \textit{ iff } (a_1,\dots,a_k) = (b'_1,\dots,b'_k).$$

(d) *Every first-order formula* $\phi(x_1,\dots,x_k)$ *is a* $\Pi^{\boldsymbol{\phi}}_{1,\langle 1\rangle}(x_1,\dots,x_k)$ *atom, where* $\boldsymbol{\phi}$ *is a (first-order definable)* $1\cdot k$*-ary relation defined as*

$$(a_1,\dots,a_k) \in \boldsymbol{\phi}^M \quad \textit{iff} \quad M \models_{s_{\vec{a}}} \phi \textit{ where } s_{\vec{a}}(x_i) = a_i \textit{ for all } i.$$

In what follows, let $\mathsf{k} = \langle k_1,\dots,k_n\rangle$ be an arbitrary sequence of natural numbers, $\vec{x} = \langle x_1,\dots,x_m\rangle$ an arbitrary sequence of variables, and R an arbitrary $(k_1{+}\cdots{+}k_n)m$-ary relation. Suppose R is definable by a formula $\phi_R(\vec{\mathsf{w}_1},\dots,\vec{\mathsf{w}_n})$, where $\vec{\mathsf{w}_i} = \langle \mathsf{w}_{i,1},\dots,\mathsf{w}_{i,k_i}\rangle$ and $\mathsf{w}_{i,j} = \langle w_{i,j,1},\dots,w_{i,j,m}\rangle$. The $\Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $\Pi^R_{n,\mathsf{k}}(\vec{x})$ atoms can be translated into second-order logic in the same manner as in Theorem 2.4. For instance, if n is even, let S be a fresh m-ary relation symbol and let $\tau_{\Sigma^R_{n,\mathsf{k}}(\vec{x})}(S) :=$

$$\begin{aligned}\exists\vec{\mathsf{w}_1}\Big(S(\mathsf{w}_{1,1}) \wedge \cdots \wedge S(\mathsf{w}_{1,k_1}) \wedge \forall\vec{\mathsf{w}_2}\Big(S(\mathsf{w}_{2,1}) \wedge \cdots \wedge S(\mathsf{w}_{2,k_2}) \to \exists\vec{\mathsf{w}_3}\cdots\\ \cdots\exists\vec{\mathsf{w}_n}\Big(S(\mathsf{w}_{n,1}) \wedge \cdots \wedge S(\mathsf{w}_{n,k_n}) \wedge \phi_R(\vec{\mathsf{w}_1},\dots,\vec{\mathsf{w}_n})\underbrace{\Big)\cdots\Big)\Big)}_{n}.\end{aligned}$$

Then, we have $M \models_X \Sigma^R_{n,\mathsf{k}}(\vec{x}) \iff (M, rel(X)) \models \tau_{\Sigma^R_{n,\mathsf{k}}(\vec{x})}(S)$ for any model M and any team X. If $\phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})$ is a first-order formula, i.e., if R is first-order definable, then $\tau_{\Sigma^R_{n,\mathsf{k}}(\vec{x})}(S)$ is a first-order sentence. This shows, by Theorem 3.2(i), that $\Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $\Pi^R_{n,\mathsf{k}}(\vec{x})$ atoms are negatable in $\mathcal{I}$ as long as R is first-order definable.

Yet, in order to apply the rules of the extended deduction system defined in Sect. 4 to derive the $\Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $\Pi^R_{n,\mathsf{k}}(\vec{x})$ consequences in $\mathcal{I}$, one needs to compute the formulas that are equivalent to the weak classical negations of the $\Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $\Pi^R_{n,\mathsf{k}}(\vec{x})$ atoms in the original language of $\mathcal{I}$. This can be done by applying Fact 5.1 and going through the Σ^1_1-translation (i.e. applying Theorem 2.4(ii) and (iv)). However, as the Σ^1_1-translation creates a number of dummy symbols (see [5,23]), such an algorithm is inefficient. In the remainder of this section, we will give a direct definition of the atoms $\Sigma^R_{n,\mathsf{k}}(\vec{x})$ and $\Pi^R_{n,\mathsf{k}}(\vec{x})$ in the original language of $\mathcal{I}$.

For each $1 \leq i \leq n$, define

– $\mathsf{inc}(\mathsf{w_{i,1}}, \dots, \mathsf{w_{i,k_i}}; \vec{x}) := \bigwedge_{j=1}^{k_i} (\mathsf{w_{i,j}} \subseteq \vec{x})$

– $\mathsf{pro}(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_{i-1}}}; \vec{x}; \mathsf{w_{i,1}}, \dots, \mathsf{w_{i,k_i}}) :=$

$$\left(\bigwedge_{j=1}^{k_i} (\vec{x} \subseteq \mathsf{w_{i,j}})\right) \wedge \left(\bigwedge_{j=1}^{k_i} (\langle \mathsf{w_{i,j'}} \mid j' \neq j\rangle \perp \mathsf{w_{i,j}})\right) \wedge (\overrightarrow{\mathsf{w_1}} \dots \overrightarrow{\mathsf{w_{i-1}}} \perp \mathsf{w_{i,1}} \dots \mathsf{w_{i,k_i}})$$

and inductively define formulas σ_i and π_i as follows:

– $\sigma_1[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})] := \exists \overrightarrow{\mathsf{w_n}} \Big(\mathsf{inc}(\mathsf{w_{n,1}}, \dots, \mathsf{w_{n,k_n}}; \vec{x}) \wedge \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})\Big)$

– $\pi_1[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})] := \exists \overrightarrow{\mathsf{w_n}} \Big(\mathsf{pro}(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_{n-1}}}; \vec{x}; \mathsf{w_{n,1}}, \dots, \mathsf{w_{n,k_n}}) \wedge \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})\Big)$

– $\sigma_{i+1}[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})] := \exists \overrightarrow{\mathsf{w_{n-i}}} \Big(\mathsf{inc}(\mathsf{w_{n-i,1}}, \dots, \mathsf{w_{n-i,k_{n-i}}}; \vec{x}) \wedge \pi_i[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})]\Big)$

– $\pi_{i+1}[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})] := \exists \overrightarrow{\mathsf{w_{n-i}}} \Big(\mathsf{pro}(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_{n-i-1}}}; \vec{x}; \mathsf{w_{n-i,1}}, \dots, \mathsf{w_{n-i,k_{n-i}}}) \wedge \sigma_i[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})]\Big)$[2]

Theorem 5.3. *Let R and ϕ_R be as above. Then*

– $\Sigma^R_{n,\mathsf{k}}(x_1, \dots, x_m) \equiv \sigma_n[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})]$
– $\Pi^R_{n,\mathsf{k}}(x_1, \dots, x_m) \equiv \pi_n[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \dots, \overrightarrow{\mathsf{w_n}})]$

[2] If $i+1 = n$, then $\overrightarrow{\mathsf{w_1}} \dots \overrightarrow{\mathsf{w_{n-i-1}}}$ denotes the empty sequence $\langle\rangle$ and we stipulate $\langle\rangle \perp \vec{y} := \top$.

Proof. We only give the detailed proof for $\Sigma^R_{n,\mathsf{k}}(x_1,\dots,x_m)$ when n is odd. The other case and the other equivalence can be proved analogously.

Our proof makes use of Lemma A in Appendix III. First, note that

$$\sigma_n[\vec{x};\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})] := \exists\overrightarrow{\mathsf{w}_1}\Big(\mathsf{inc}(\mathsf{w}_{1,1},\dots,\mathsf{w}_{1,\mathsf{k}_1};\vec{x})\wedge \exists\overrightarrow{\mathsf{w}_2}\Big(\mathsf{pro}(\overrightarrow{\mathsf{w}_1};\vec{x};\mathsf{w}_{2,1},\dots,\mathsf{w}_{2,\mathsf{k}_2})\wedge\cdots\cdots\wedge \exists\overrightarrow{\mathsf{w}_\mathsf{n}}\Big(\mathsf{inc}(\mathsf{w}_{\mathsf{n},1},\dots,\mathsf{w}_{\mathsf{n},\mathsf{k}_\mathsf{n}};\vec{x})\wedge\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})\underbrace{\Big)\cdots\Big)\Big)}_{n}$$

Suppose $M\models_X \Sigma^R_{n,\mathsf{k}}(\vec{x})$ for some model M and some nonempty team X. Then

$$(\exists\overrightarrow{s_1}\in X^{k_1})(\forall\overrightarrow{s_2}\in X^{k_2})\cdots\cdots(\exists\overrightarrow{s_n}\in X^{k_n})(\overrightarrow{s_1}(\vec{x}),\dots,\overrightarrow{s_n}(\vec{x}))\in R^M. \tag{6}$$

Let $\Gamma_1=\langle\gamma_{1,1},\dots,\gamma_{1,k_1}\rangle$ be a sequence of constant choice functions $\gamma_{1,j}:X\to X$ defined as $\gamma_{1,j}(t)=s_{1,j}$. Let $\overrightarrow{F_{1,1}},\dots,\overrightarrow{F_{1,k_1}}$ be the group of simulating functions for $\Gamma_1[X]\upharpoonright\vec{x}$ on $\mathsf{w}_{1,1},\dots,\mathsf{w}_{1,\mathsf{k}_1}$ and Y_1 its associated team defined as in Lemma A(i) in Appendix III. Then, $M\models_{Y_1}\mathsf{inc}(\mathsf{w}_{1,1},\dots,\mathsf{w}_{1,\mathsf{k}_1};\vec{x})$. It then remains to show that $M\models_{Y_1}\pi_{n-1}[\vec{x};\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})]$.

Let $\vec{F_{2,1}},\dots,\vec{F_{2,k_2}}$ be the group of duplicating functions for $Y_1\upharpoonright\vec{x}$ on $\mathsf{w}_{2,1},\dots,\mathsf{w}_{2,\mathsf{k}_2}$, and Y_2 its associated team defined as in Lemma A(ii) in Appendix III. Then, $M\models_{Y_2}\mathsf{pro}(\overrightarrow{\mathsf{w}_1};\vec{x};\mathsf{w}_{2,1},\dots,\mathsf{w}_{2,\mathsf{k}_2})$.

It remains to show that $M\models_{Y_2}\sigma_{n-2}[\vec{x};\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})]$. By Lemma A(ii), for each $t\in Y_2$, there exists $\overrightarrow{s_{t,2}}=(s^t_{2,1},\dots,s^t_{2,k_2})\in X^{k_2}$ satisfying

$$s^t_{2,1}(\vec{x})=t(\mathsf{w}_{2,1}),\dots,s^t_{2,k_2}(\vec{x})=t(\mathsf{w}_{2,\mathsf{k}_2}).$$

Hence, by (6), there exists $\overrightarrow{s_{t,3}}=(s^t_{3,1},\dots,s^t_{3,k_3})\in X^{k_3}$ such that

$$(\forall\overrightarrow{s_4}\in X^{k_4})\cdots\cdots(\exists\overrightarrow{s_n}\in X^{k_n})(\overrightarrow{s_1}(\vec{x}),\overrightarrow{s_{t,2}}(\vec{x}),\overrightarrow{s_{t,3}}(\vec{x}),\overrightarrow{s_4}(\vec{x})\dots,\overrightarrow{s_n}(\vec{x}))\in R^M.$$

Let $\Gamma_3=\langle\gamma_{3,1},\dots,\gamma_{3,k_3}\rangle$ be a sequence of choice functions $\gamma_{3,j}:Y_2\to Y_2$ defined as $\gamma_{3,j}(t)=s^t_{3,j}$. Let $\overrightarrow{F_{3,1}},\dots,\overrightarrow{F_{3,k_3}}$ be the group of simulating functions for $\Gamma_3[Y_2]\upharpoonright\vec{x}$ on $\mathsf{w}_{3,1},\dots,\mathsf{w}_{3,\mathsf{k}_3}$, and Y_3 its associated team defined as in Lemma A(i) in Appendix III. Then, $M\models_{Y_3}\mathsf{inc}(\mathsf{w}_{3,1},\dots,\mathsf{w}_{3,\mathsf{k}_3};\vec{x})$ and it remains to show that $M\models_{Y_3}\pi_{n-3}[\vec{x};\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})]$.

Repeat the argument n times. In the last step we have Y_n and Γ_n defined and $M\models_{Y_n}\mathsf{inc}(\mathsf{w}_{\mathsf{n},1},\dots,\mathsf{w}_{\mathsf{n},\mathsf{k}_\mathsf{n}};\vec{x})$ by Lemma A(i). It then only remains to show that $M\models_{Y_n}\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})$. Since ϕ_R is flat, it suffices to show that $M\models_{\{t\}}\phi_R$ holds for all $t\in Y_n$. By the definition of Y_n and Lemma A(i)(ii), we have

$$(\overrightarrow{s_1}(\vec{x}),\overrightarrow{s_{t,2}}(\vec{x}),\overrightarrow{s_{t,3}}(\vec{x}),\overrightarrow{s_{t,4}}(\vec{x})\dots,\overrightarrow{s_{t,n}}(\vec{x}))\in R^M$$

$$\text{and } t(\overrightarrow{\mathsf{w}_1})=\overrightarrow{s_1}(\vec{x}),\ t(\overrightarrow{\mathsf{w}_2})=\overrightarrow{s_{t,2}}(\vec{x}),\ \dots,\ t(\overrightarrow{\mathsf{w}_\mathsf{n}})=\overrightarrow{s_{t,n}}(\vec{x}).$$

Thus, $M\models_{\{t\}}\phi_R(\overrightarrow{\mathsf{w}_1},\dots,\overrightarrow{\mathsf{w}_\mathsf{n}})$, as the first-order formula ϕ_R defines R.

Conversely, suppose $M \models_X \sigma_n[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \ldots, \overrightarrow{\mathsf{w_n}})]$ for some model M and some nonempty team X. Let Y be a team generated by the formula $\sigma_n[\vec{x}; \phi_R(\overrightarrow{\mathsf{w_1}}, \ldots, \overrightarrow{\mathsf{w_n}})]$ from X such that $M \models_Y \phi_R(\overrightarrow{\mathsf{w_1}}, \ldots, \overrightarrow{\mathsf{w_n}})$.

Pick any $t \in Y$. Since $M \models_Y \mathsf{inc}(\mathsf{w}_{1,1}, \ldots, \mathsf{w}_{1,\mathsf{k}_1}; \vec{x})$, there exist $s_{1,1}, \ldots, s_{1,k_1} \in X$ such that

$$s_{1,1}(\vec{x}) = t(\mathsf{w}_{1,1}), \ldots, s_{1,k_1}(\vec{x}) = t(\mathsf{w}_{1,k_1}).$$

Let $s_{2,1}, \ldots, s_{2,k_2} \in X$ be arbitrary. Since $M \models_Y \mathsf{pro}(\overrightarrow{\mathsf{w_1}}; \vec{x}; \mathsf{w}_{2,1}, \ldots, \mathsf{w}_{2,\mathsf{k}_2})$, it is not hard to see that there exist $t_2 \in Y$ such that

$$t_2(\overrightarrow{\mathsf{w_1}}) = t(\overrightarrow{\mathsf{w_1}}) = \overrightarrow{s_1}(\vec{x}) \text{ and } s_{2,1}(\vec{x}) = t_2(\mathsf{w}_{2,1}), \ldots, s_{2,k_2}(\vec{x}) = t_2(\mathsf{w}_{2,k_2}).$$

Repeat the argument n times to find in the same manner the corresponding assignments $\overrightarrow{s_3} \in X^{k_3}, \overrightarrow{s_5} \in X^{k_5}, \ldots, \overrightarrow{s_n} \in X^{k_n}$ and the corresponding assignments $t_4, t_6, \ldots, t_{n-1} \in Y$ for arbitrary $\overrightarrow{s_4} \in X^{k_4}, \overrightarrow{s_6} \in X^{k_6}, \ldots, \overrightarrow{s_{n-1}} \in X^{k_{n-1}}$. In the last step we have

$$t_{n-1}(\overrightarrow{\mathsf{w_1}}) = \overrightarrow{s_1}(\vec{x}), \ldots, t_{n-1}(\overrightarrow{\mathsf{w}_{n-1}}) = \overrightarrow{s_{n-1}}(\vec{x})$$

and there exist $s_{n,1}, \ldots, s_{n,k_n} \in X$ such that

$$s_{n,1}(\vec{x}) = t_{n-1}(\mathsf{w}_{n,1}), \ldots, s_{n,k_n}(\vec{x}) = t_{n-1}(\mathsf{w}_{n,k_n}).$$

Since $M \models_Y \phi_R(\overrightarrow{\mathsf{w_1}}, \ldots, \overrightarrow{\mathsf{w_n}})$, we have $M \models_{\{t_{n-1}\}} \phi_R(\overrightarrow{\mathsf{w_1}}, \ldots, \overrightarrow{\mathsf{w_n}})$ by the downward closure property. Since the first-order formula ϕ_R defines R, we conclude

$$(t_{n-1}(\overrightarrow{\mathsf{w_1}}), \ldots, t_{n-1}(\overrightarrow{\mathsf{w}_n})) \in R^M \text{ yielding } (\overrightarrow{s_1}(\vec{x}), \ldots, \overrightarrow{s_n}(\vec{x})) \in R^M.$$

□

6 Concluding Remarks

In this paper, we have extended the natural deduction systems of dependence and independence logic defined in [10, 20] and obtained complete axiomatizations of the negatable consequences in these logics. We also gave a characterization of negatable formulas in $\mathcal{I}$ and negatable sentences in $\mathcal{D}$. Determining whether a formula of $\mathcal{I}$ or $\mathcal{D}$ is negatable is an undecidable problem. Nevertheless, we identified an interesting class of negatable formulas. Formulas in this class are presented as $\Sigma^R_{n,\mathsf{k}}$ and $\Pi^R_{n,\mathsf{k}}$ atoms. First-order formulas, dependence and independence atoms belong to this class. Since the set of negatable formulas is closed under the Boolean connectives $\wedge$ and $\mathbin{\vee\!\!\!\vee}$, Boolean combinations of $\Sigma^R_{n,\mathsf{k}}$ and $\Pi^R_{n,\mathsf{k}}$ atoms are also negatable.

An interesting corollary of the paper is that Armstrong's Axioms [1] that characterize dependence atoms and the Geiger-Paz-Pearl axioms [8] that characterize independence atoms can be derived in our extended system of $\mathcal{I}$. We leave the derivations of these axioms for future work.

The results of this paper can be generalized in two directions. The first direction is to identify other negatable formulas than those in the set of the Boolean combinations of atoms from our hierarchy. The other direction is to analyze the $\Sigma^R_{n,\mathsf{k}}$ and $\Pi^R_{n,\mathsf{k}}$ atoms in more detail. As we saw in Example 5.2, first-order formulas and the atoms of dependence and independence situate only on the Π_1 or Π_2 level. Identifying interesting properties that situate on higher levels of the hierarchy and studying the logics that the higher level atoms induce would be an interesting topic for future research. For example, it is easy to verify that $\Pi^R_{1,\mathsf{k}}$ atoms (including first-order formulas and dependence atoms) are closed downward, and $\Sigma^R_{1,\mathsf{k}}$ atoms are closed upward. First-order logic extended with upward closed atoms is shown in [7] to be equivalent to first-order logic. Adding other such atoms to first-order logic results in many new logics that are expressively less than Σ^1_1 or independence logic and possibly stronger than first-order logic. These logics are potentially interesting, because, for instance, by the argument of this paper, the negatable consequences in these logics can in principle be axiomatized.

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Appendix I

Proof (of the direction "$\Longleftarrow$" of Theorem 4.3*).* It suffices to show by induction that $\Gamma \models \phi$ holds for each derivation D in the extended system with the conclusion ϕ and the hypotheses in Γ. We only give the proof for the induction step when the rule $\dot\sim\mathsf{Tr}$ is applied. The case when the rule $\dot\sim\mathsf{E}$ is applied can be proved similarly, and all the other cases follow from the arguments in [20] and in [10].

Assume that D_2 is a derivation for $\Delta, \exists\vec{x}(\psi\wedge\dot\sim\phi) \vdash^*_{\mathsf{L}} \bot$ and D_1 is a derivation for $\Pi \vdash^*_{\mathsf{L}} \psi$, where $\mathrm{Fv}(\Delta) \cap \{x_1, \ldots, x_n\} = \emptyset$. We show that $\Delta, \Pi \models \phi$. By the induction hypothesis, we have $\Delta, \exists\vec{x}(\psi\wedge\dot\sim\phi) \models \bot$ and $\Pi \models \psi$. From the former and Lemma 4.2 we obtain $\Delta, \psi \wedge \dot\sim\phi \models \bot$, which is equivalent to $\Delta, \psi \models \phi$. Since $\Pi \models \psi$, we conclude $\Delta, \Pi \models \phi$, as desired. □

Appendix II

	$x_1 \dots x_n$
	$a_1 \dots a_n$
s	$b_1 \dots b_n$
	$c_1 \dots c_n$

	$x_1 \dots x_n$	$w_1 \dots w_n$
	$a_1 \dots a_n$	$b_1 \dots b_n$
s'	$b_1 \dots b_n$	$b_1 \dots b_n$
	$c_1 \dots c_n$	$b_1 \dots b_n$

Fig. 1. (a) A team X. (b) A team $X[F/\vec{w}]$

	$\vec{x}$	$\vec{y}$
s_1	$\vec{a}$	$\vec{a}$
s_2	$\vec{a}$	$\vec{b}$
s_3	$\vec{a}$	$\vec{c}$
s_4	$\vec{b}$	$\vec{a}$
s_5	$\vec{b}$	$\vec{b}$
s_6	$\vec{b}$	$\vec{c}$

(a)

	$\vec{x}$	$\vec{y}$	$\vec{w_1}$
s'_1	$\vec{a}$	$\vec{a}$	$\vec{b}$
s'_2	$\vec{a}$	$\vec{b}$	$\vec{b}$
s'_3	$\vec{a}$	$\vec{c}$	$\vec{a}$
s'_4	$\vec{b}$	$\vec{a}$	$\vec{b}$
s'_5	$\vec{b}$	$\vec{b}$	$\vec{b}$
s'_6	$\vec{b}$	$\vec{c}$	$\vec{a}$

(b)

	$\vec{x}$	$\vec{y}$	$\vec{w_1}$	$\vec{w_2}$
s''_1	$\vec{a}$	$\vec{a}$	$\vec{b}$	$\vec{a}$
s''_2	$\vec{a}$	$\vec{b}$	$\vec{b}$	$\vec{b}$
s''_3	$\vec{a}$	$\vec{c}$	$\vec{a}$	$\vec{b}$
s''_4	$\vec{b}$	$\vec{a}$	$\vec{b}$	$\vec{a}$
s''_5	$\vec{b}$	$\vec{b}$	$\vec{b}$	$\vec{b}$
s''_6	$\vec{b}$	$\vec{c}$	$\vec{a}$	$\vec{b}$

(c)

Fig. 2. (a) A team X (b) A team $X[\vec{F_1}/\vec{w_1}]$ (c) A team $X[\vec{F_1}/\vec{w_1}, \vec{F_2}/\vec{w_2}]$

	$\vec{x}$	$\vec{y}$
s_1	$\vec{a}$	$\vec{b}$
s_2	$\vec{b}$	$\vec{a}$

(a)

	$\vec{x}$	$\vec{y}$	$\vec{w_1}$
s_{11}	$\vec{a}$	$\vec{b}$	$\vec{a}$
s_{12}	$\vec{a}$	$\vec{b}$	$\vec{b}$
s_{21}	$\vec{b}$	$\vec{a}$	$\vec{a}$
s_{22}	$\vec{b}$	$\vec{a}$	$\vec{b}$

(b)

	$\vec{x}$	$\vec{y}$	$\vec{w_1}$	$\vec{w_2}$
s_{111}	$\vec{a}$	$\vec{b}$	$\vec{a}$	$\vec{a}$
s_{112}	$\vec{a}$	$\vec{b}$	$\vec{a}$	$\vec{b}$
s_{121}	$\vec{a}$	$\vec{b}$	$\vec{b}$	$\vec{a}$
s_{122}	$\vec{a}$	$\vec{b}$	$\vec{b}$	$\vec{b}$
s_{211}	$\vec{b}$	$\vec{a}$	$\vec{a}$	$\vec{a}$
s_{212}	$\vec{b}$	$\vec{a}$	$\vec{a}$	$\vec{b}$
s_{221}	$\vec{b}$	$\vec{a}$	$\vec{b}$	$\vec{a}$
s_{222}	$\vec{b}$	$\vec{a}$	$\vec{b}$	$\vec{b}$

(c)

Fig. 3. (a) A team X (b) A team $X[\vec{F_1}/\vec{w_1}]$ (c) A team $X[\vec{F_1}/\vec{w_1}, \vec{F_2}/\vec{w_2}]$

Appendix III

Lemma A. *Let X be a nonempty team of a model M with $x_1, \dots, x_m \in \mathrm{dom}(X)$.*

(i) *Let $\gamma : X \to X$ be a choice function. Define inductively functions $F_1, \dots, F_m$ to simulate assignments in $\gamma[X]$ restricted to $\vec{x}$ on a sequence $\vec{w} = \langle w_1, \dots, w_m \rangle$ of new variables as follows:*
- *Define the function $F_1 : X \to \wp(M) \setminus \{\emptyset\}$ as $F_1(t) = \{\gamma(t)(x_1)\}$.*
- *For each $2 \le i \le m$, define the function $F_i : X[F_1/w_1, \dots, F_{i-1}/w_{i-1}] \to \wp(M) \setminus \{\emptyset\}$ as $F_i(t) = \{\gamma(t)(x_i)\}$.*

We call $\vec{F} = \langle F_1, \dots, F_m \rangle$ the sequence of simulating functions for $\gamma[X] \upharpoonright \vec{x}$ *on $\vec{w}$. Let $Y = X[\vec{F}/\vec{w}]$ (see Fig. 1 in Appendix II for an example of such a team with a constant choice function $\gamma(t) = s$ for all $t \in X$, or Fig. 2(b) for another example with an obvious choice function). Then, $t(\vec{w}) = \gamma(t)(\vec{x})$ for all $t \in Y$ and $M \models_Y \mathsf{inc}(\vec{w}; \vec{x})$.*
For a sequence $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ of choice functions $\gamma_i : X \to X$,
- *let $\vec{F_1}$ be the sequence of simulating functions for $\gamma_1[X] \upharpoonright \vec{x}$ on $\overrightarrow{w_1}$,*
- *and for each $2 \le i \le k$, let $\vec{F_i}$ be the sequence of simulating functions for $\gamma_i[X[\overrightarrow{F_1}/\overrightarrow{w_1}, \dots, \overrightarrow{F_{i-1}}/\overrightarrow{w_{i-1}}]] \upharpoonright \vec{x}$ on $\overrightarrow{w_i}$.*

We call $\vec{F_1}, \dots, \vec{F_k}$ the group of simulating functions for $\Gamma[X] \upharpoonright \vec{x}$ *on $\overrightarrow{w_1}, \dots, \overrightarrow{w_k}$, and the team $Y = X[\vec{F_1}/\overrightarrow{w_1}, \dots, \vec{F_k}/\overrightarrow{w_k}]$ its* associated team *(see Fig. 2 in Appendix II for examples of such teams). Then, $M \models_Y \mathsf{inc}(\overrightarrow{w_1}, \dots, \overrightarrow{w_k}; \vec{x})$.*

(ii) *Define inductively functions $F_1, \dots, F_m$ to duplicate assignments in X restricted to $\vec{x}$ on a sequence $\vec{w} = \langle w_1, \dots, w_m \rangle$ of new variables as follows:*
- *Define the function $F_1 : X \to \wp(M) \setminus \{\emptyset\}$ as $F_1(t) = \{s(x_1) \mid s \in X\}$.*
- *For each $2 \le i \le m$, define the function $F_i : X[F_1/w_1, \dots, F_{i-1}/w_{i-1}] \to \wp(M) \setminus \{\emptyset\}$ as*

$$F_i(t) = \{s(x_i) \mid s \in X \text{ and } s \upharpoonright \{x_1, \dots, x_{i-1}\} = t \upharpoonright \{w_1, \dots, w_{i-1}\}\}.$$

We call $\vec{F} = \langle F_1, \dots, F_m \rangle$ the sequence of duplicating functions for $X \upharpoonright \vec{x}$ *on $\vec{w}$. (see Fig. 3(b) in Appendix II for an example of a team $X[\vec{F}/\vec{w}]$).*
For a team X,
- *let $\vec{F_1}$ be the sequence of duplicating functions for $X \upharpoonright \vec{x}$ on $\overrightarrow{w_1}$,*
- *and for each $i = 2, \dots, k$, let $\vec{F_i}$ be the sequence of duplicating functions for $X[\overrightarrow{F_1}/\overrightarrow{w_1}, \dots, \overrightarrow{F_{i-1}}/\overrightarrow{w_{i-1}}] \upharpoonright \vec{x}$ on $\overrightarrow{w_i}$.*

We call $\vec{F_1}, \dots, \vec{F_k}$ the group of duplicating functions for $X \upharpoonright \vec{x}$ *on $\overrightarrow{w_1}, \dots, \overrightarrow{w_k}$. and the team $Y = X[\vec{F_1}/\overrightarrow{w_1}, \dots, \vec{F_k}/\overrightarrow{w_k}]$ its* associated team *(see Fig. 3 in Appendix II for examples of such teams). Then, $M \models_Y \mathsf{pro}(\vec{y}; \vec{x}; \overrightarrow{w_1}, \dots, \overrightarrow{w_k})$ for any sequence $\vec{y}$ of variables in* $\mathrm{dom}(X)$ *that has no variable in common with $\vec{x}$ and $\overrightarrow{w_1}, \dots, \overrightarrow{w_k}$, and for any $t \in Y$, there exist $s_1, \dots, s_k \in X$ such that $s_1(\vec{x}) = t(\overrightarrow{w_1}), \dots, s_k(\vec{x}) = t(\overrightarrow{w_k})$.*

Proof. We only give the detailed proof for $M \models_Y \mathsf{pro}(\vec{y}; \vec{x}; \overrightarrow{w_1}, \ldots, \overrightarrow{w_k})$ in the item (ii), i.e.,

$$M \models_Y \bigwedge_{i=1}^{k} (\vec{x} \subseteq \overrightarrow{w_i}) \wedge \left(\bigwedge_{i=1}^{k} (\langle \overrightarrow{w_j} \mid j \neq i \rangle \perp \overrightarrow{w_i}) \right) \wedge (\vec{y} \perp \overrightarrow{w_1} \ldots \overrightarrow{w_k}) \tag{7}$$

To show that Y satisfies the first conjunct of the formula in (7), it suffices to show that $M \models_{Y_i} \vec{x} \subseteq \overrightarrow{w_i}$ for each $1 \leq i \leq k$ and $Y_i = X[\vec{F_1}/\overrightarrow{w_1}, \ldots, \vec{F_i}/\overrightarrow{w_i}]$.

For any $t \in Y_i$, by the definition of $Y_i = Y_{i-1}[\vec{F_i}/\overrightarrow{w_i}]$, there exists $s \in X$ such that $s(\vec{x}) = t(\vec{x})$, and

$$t' = s \cup \{(w_{i,1}, s(x_1)), \ldots, (w_{i,m}, s(x_m))\} \in Y_{i-1}[F_{i,1}/w_{i,1}, \ldots, F_{i,m}/w_{i,m}]$$

Thus, $t'(\overrightarrow{w_i}) = s(\vec{x}) = t(\vec{x})$, as required.

To prove that Y satisfies the second and the third conjuncts of the formula in (7), we prove a more general property that $M \models_Y \overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}} \perp \overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}} v_1 \ldots v_c$ holds for any disjoint subsequences $\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}}$ and $\overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}}$ of $\overrightarrow{w_1} \ldots \overrightarrow{w_k}$ and any variables $v_1 \ldots v_c \in \mathrm{dom}(X)$. Assume that $\{\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}}, \overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}}\} = \{\overrightarrow{w_{l_1}} \ldots \overrightarrow{w_{l_d}}\}$ with $l_1 < \cdots < l_d$.

Let $s, s' \in Y$ be arbitrary. We need to find an $s'' \in Y$ such that $s''(\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}}) = s(\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}})$ and $s''(\overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}} v_1 \ldots v_c) = s'(\overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}} v_1 \ldots v_c)$. Let f be a function satisfying

$$f(\overrightarrow{w_{l_\xi}}) = \begin{cases} s(\overrightarrow{w_{l_\xi}}) & \text{if } l_\xi \in \{i_1, \ldots, i_a\} \\ s'(\overrightarrow{w_{l_\xi}}) & \text{if } l_\xi \in \{j_1, \ldots, j_b\} \end{cases}$$

There exists $s_1 \in X$ such that $s_1(\vec{x}) = f(\overrightarrow{w_{l_1}})$. Put $Y_{l_1-1} = X[\vec{F_1}/\overrightarrow{w_1}, \ldots, \overrightarrow{F_{l_1-1}}/\overrightarrow{w_{l_1-1}}]$ and $t = s' \upharpoonright \mathrm{dom}(Y_{l_1-1})$. By the construction,

$$t_{l_1} = t \cup \{(w_{l_1,1}, s_1(x_1)), \ldots, (w_{l_1,m}, s_1(x_m))\} \in Y_{l_1-1}[\overrightarrow{F_{l_1}}/\overrightarrow{w_{l_1}}] = Y_{l_1}.$$

Thus

$$t_{l_1}(\overrightarrow{w_{l_1}}) = s_1(\vec{x}) = f(\overrightarrow{w_{l_1}}) \text{ and } t_{l_1}(\vec{v}) = t(\vec{v}) = s'(\vec{v}).$$

Repeat the same argument for $f(\overrightarrow{w_{l_2}}), \ldots, f(\overrightarrow{w_{l_d}})$, we can find $t_{l_d} \in Y_{l_d}$ such that

$$t_{l_d}(\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}}) = s(\overrightarrow{w_{i_1}} \ldots \overrightarrow{w_{i_a}}) \text{ and } t_{l_d}(\overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}} v_1 \ldots v_c) = s'(\overrightarrow{w_{j_1}} \ldots \overrightarrow{w_{j_b}} v_1 \ldots v_c).$$

Finally, by the construction of Y, there exists $s'' \in Y$ such that $s'' \upharpoonright \mathrm{dom}(Y_{l_d}) = t_{l_d}$. Hence, s'' is the desired assignment. □

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