Chapter 4
Quantification of Prior Knowledge Through Subjective Probability Assessment

4.1 Introduction

Prior distribution is an essential component of the Bayesian framework developed in the previous chapter, and it reflects the prior knowledge (including the existing information collected from various sources and engineers' expertise) obtained during preliminary stages (e.g., desk study or site reconnaissance) of geotechnical site characterization. When only a typical range of a soil parameter concerned is available as the prior knowledge, a uniform prior distribution of the soil parameter that covers the typical range can be used in the Bayesian framework. As the information provided by prior knowledge improves, a more sophisticated and informative prior distribution can be estimated from prior knowledge. Based on the prior knowledge obtained from desk study and/or site reconnaissance, a subjective probability assessment framework (SPAF) is proposed in this chapter to assist engineers in quantifying the information provided by prior knowledge and estimating the prior distribution from prior knowledge.

This chapter starts with brief description of uncertainties in prior knowledge, followed by development of the SPAF based on a stage cognitive model of engineers' cognitive process. Each stage of the cognitive process is implemented in the proposed SPAF, and several suggestions are provided for each stage to assist engineers in utilizing prior knowledge in a relatively rational way and reducing effects of cognitive biases and limitations mentioned in Chap. 2. The proposed SPAF is applied to characterize probabilistically the sand effective friction angle at a US National Geotechnical Experimentation Site (NGES) at Texas A&M University, and it is illustrated under two scenarios: one with sparse prior knowledge and the other with a reasonable amount of prior knowledge.

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4.2 Uncertainties in Prior Knowledge

Prior knowledge includes the existing information (e.g., geological information, geotechnical problems and properties, and groundwater conditions) about a specific site collected from various sources (see Table 2.1 in Chap. 2) during desk study and site reconnaissance and engineers' expertise (Clayton et al. 1995; Mayne et al. 2002). The collected information contains various uncertainties, such as inherent variability of soil properties, measurement errors, statistical uncertainty incorporated in historical data, and transformation uncertainty associated with regression models used to interpret the historical data. In addition, the quantity and quality of the expertise of an individual engineer depend on various external factors (e.g., educational background and career experience) and internal factors (e.g., personal attributes and individual cognitive process) (Vick 2002). Variations of such external and internal factors lead to uncertainties in engineers' expertise. Because of uncertainties in the existing information and engineers' expertise, estimates of soil properties and their statistics from prior knowledge are uncertain results rather than cut-and-dried conclusions. Such uncertain estimates are, therefore, referred to as "prior uncertain estimates" in this book.

The plausibility of prior uncertain estimates reflects the confidence level (or degrees-of-belief) of prior knowledge on such estimates, and it can be evaluated intuitively and qualitatively through engineering judgments (including various cognitive heuristics discussed in Chap. 2, such as availability heuristic, representative heuristic, and anchoring and adjustment heuristic). Because of various cognitive biases and limitations (see Chap. 2), outcomes from such intuitive and qualitative evaluations might deviate from the actual beliefs of engineers and be inconsistent with basic probability axioms (Vick 2002). The next section presents a subjective probability assessment framework (SPAF), in which subjective probability is applied to quantify the plausibility of prior uncertain estimates of statistics $\Theta = [\theta_1, \theta_2, \ldots, \theta_{n_m}]$ (e.g., the mean, standard deviation, and correlation length) of the soil property x concerned and to express engineering judgments on x and its statistics Θ in a probabilistic manner. By this means, the plausibility of prior uncertain estimates of Θ is quantified by the probability distribution of Θ, which can be taken as the prior distribution of Θ in the Bayesian framework developed in Chap. 3.

4.3 Subjective Probability Assessment Framework (SPAF)

Engineers formulate subjective probability through a series of internal cognitive activities (i.e., cognitive process). These cognitive activities can be divided into several stages and be described by a stage cognitive model. Consider, for example, the stage cognitive model presented by Vick (2002), as shown in Fig. 4.1. Based on the stage cognitive model, a subjective probability assessment framework (SPAF) is developed in this section, which is shown in Fig. 4.1.

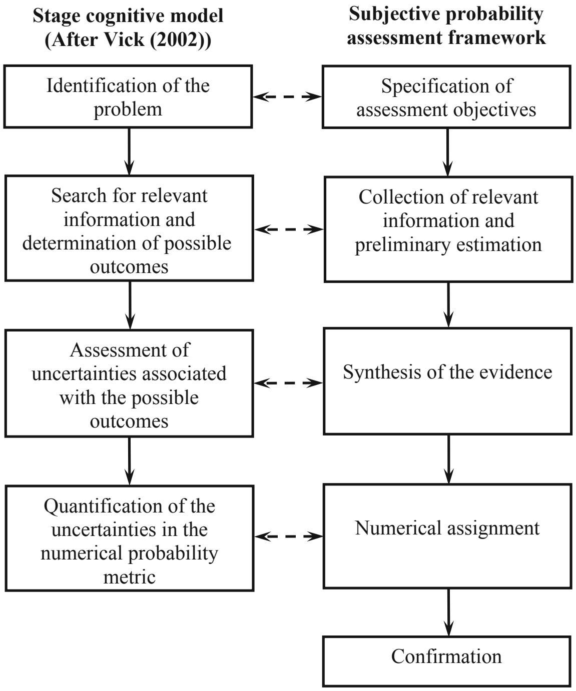


Fig. 4.1 Subjective probability assessment framework (SPAF) (After Cao et al. 2016)

The SPAF starts with specification of assessment objectives (e.g., determining the soil property x and its statistics Θ of interest), followed by collection of relevant information and making prior uncertain estimates on the assessment objectives. A piece of relevant information, a prior uncertain estimate obtained from this piece of relevant information, and the correlations between the relevant information and the prior uncertain estimate are collectively referred to as "a piece of evidence" in this chapter, as shown in Fig. 4.2. The third step (i.e., synthesis of the evidence) deals cautiously with the evidence collected in the second step. In this step, uncertainties associated with each evidence are evaluated, and engineering judgments are formulated internally based on the evidence. After that, the fourth step (i.e., numerical assignment) is to express the engineering judgments through numerical values (i.e., subjective probability values and probability distributions of Θ). The final step (i.e., confirmation) aims to check whether or not the outcomes (e.g., probability distributions of Θ) obtained from the SPAF are consistent with probability axioms and reality and reflect engineers' actual beliefs on assessment objectives.

It is worthwhile to point out that the first four steps of the proposed SPAF correspond to the four stages of the cognitive model of engineers' cognitive process (see Fig. 4.1), respectively. This allows engineers to formulate the subjective probability (or engineering judgments) on assessment objectives by following their

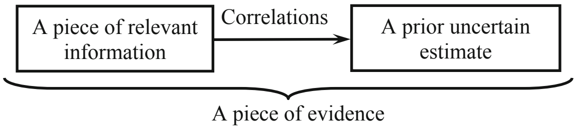


Fig. 4.2 Illustration of the evidence (After Cao et al. 2016)

cognitive process naturally. These five steps of the SPAF are introduced in detail in the following five sections, respectively. Several suggestions are also provided for each step of the SPAF to assist engineers in utilizing prior knowledge in a relatively rational way and reducing effects of cognitive biases and limitations (e.g., availability bias, representativeness bias, insufficient adjustment, and limited capacity of processing information) (see Chap. 2).

4.4 Specification of Assessment Objectives

If assessment objectives are misunderstood, inappropriate information might be collected and utilized in subjective probability assessment. This subsequently results in various cognitive biases (e.g., availability bias and insufficient adjustment) in subjective probability assessment. Therefore, it is of great significance to define and understand assessment objectives clearly at the beginning of the subjective probability assessment. Several suggestions are provided herein to assist engineers in specifying and understanding the assessment objectives properly, such as

(1) Write down the soil property x of interest and define a general assessment objective. For example, the general assessment objective can be “probabilistic characterization of the soil property x”;
(2) Decompose the general objective into several sub-objectives. Each sub-objectives is corresponding to a statistic $\theta_i, i = 1, 2, \ldots, n_m$, of x. The statistics $\Theta = [\theta_1, \theta_2, \ldots, \theta_{n_m}]$ of interest depend on the probability theory that is applied to model the inherent variability of x in the Bayesian framework developed in Chap. 3. For example, when using random field theory to model inherent spatial variability of x within a statistically homogenous soil layer, the statistics of interest are model parameters of the random field, i.e., mean μ, standard deviation σ, and correlation length λ of x. In other words, Θ consists of three random variables: μ, σ, and λ, i.e., $\Theta = [\mu, \sigma, \lambda]$. Hence, the sub-objectives can be defined as “evaluating probability of the mean μ of x,” “evaluating probability of the standard deviation σ of x,” and “evaluating probability of the correlation length λ of x”;
(3) Identify probability terms (including statistics of x) that engineers are not familiar with. For engineers, training in probability theory and statistics is usually limited to basic information during their early years of education (El-Ramly et al. 2002). They might be unfamiliar with some probability terms, e.g., correlation length λ. These probability terms should be written down;

(4) Try to understand physically probability terms that engineers are not familiar with. Physical interpretation of probability terms helps engineers understand these terms with relative ease. For example, the correlation length λ of x is a separate distance in which the soil property x shows relatively strong correlation from point to point (Vanmarcke 1977, 1983). By this definition, the correlation length is understood with relative ease.

Decomposition of the general assessment objective and physical interpretations of probabilistic terms assist engineers in clearly understanding the assessment objectives. This helps engineers collect properly information related to the assessment objectives (including the general assessment objective and sub-objectives), as discussed in the next section.

4.5 Collection of Relevant Information and Preliminary Estimation

The next step is to assemble the relevant information on assessment objectives from the prior knowledge (i.e., the collected existing information and engineers' expertise). A piece of relevant information might result in several prior uncertain estimates of the soil property x and/or its statistics Θ using available correlations (e.g., empirical regressions or theoretical correlations) or intuitive inference. Subsequently, it provides several pieces of evidence on assessment objectives. Evidence can be divided into two types: supportive evidence and disconfirming evidence (Vick 2002). Supportive evidence provides information that is consistent with the preconceived view of the assessor (e.g., engineers) about the soil property x, while disconfirming evidence contradicts with the preconceived view of the assessor. Examination of relevant information in prior knowledge eventually provides an evidence list. A relatively comprehensive evidence list includes both supportive evidence and disconfirming evidence collected from the existing information and engineers' expertise. It helps engineers reduce the availability bias arising from missing some useful evidence, the representativeness bias resulted from overemphasis on one particular type of information, and the confirmation bias due to overlooking disconfirming evidence (Vick 2002). Several suggestions are provided to assist engineers in acquiring a relatively comprehensive evidence list from prior knowledge.

As shown in Table 2.1 in Chap. 2, there are seven types of existing information, i.e., geological information, geotechnical problems and properties, site topography, groundwater conditions, meteorological conditions, existing construction and services, and previous land use. Engineers are suggested to cautiously search for relevant information and/or evidence in each type of existing information and the corresponding expertise. For each type of existing information, this can be performed through the following three steps:

(1) Identify the possible sources (including sources of the existing information (see Table 2.1) and engineers' expertise) of relevant information pertaining to this type of existing information;
(2) Assemble the relevant information from each possible source, and write it down;
(3) Evaluate the correlations between each piece of relevant information and the soil property x and its statistics $\Theta = [\mu, \sigma, \lambda]$, and write down the possible outcomes (i.e., prior uncertain estimates). Prior uncertain estimates of x and its statistics Θ can be obtained from the relevant information by correlations (including empirical and theoretical relationships and/or intuitive inference) and/or by conventional statistical equations. The mean and standard deviation of a random variable X (e.g., x or $\theta_i, i = 1, 2, \ldots, n_m$) can be calculated as (e.g., Baecher and Christian 2003)

$$\overline{X} = \frac{\sum_{i=1}^{n_X} X_i}{n_X} \tag{4.1}$$

$$w_X = \frac{\sum_{i=1}^{n_X} (X_i - \overline{X})^2}{n_X - 1} \tag{4.2}$$

in which $\overline{X}$ = the mean of X; w_X = the standard deviation of X; X_i, i = 1, 2, … n_x, are samples of X; n_x is the number of samples of X. When only a range of X from the minimum $X_{\min}$ to the maximum $X_{\max}$ is available, the mean value $\overline{X}$ and standard deviation w_X can be estimated as (e.g., Duncan 2000; Baecher and Christian 2003)

$$\overline{X} = \frac{X_{\max} + X_{\min}}{2} \tag{4.3}$$

$$w_X = \frac{X_{\max} - X_{\min}}{6} \tag{4.4}$$

Outcomes of the three steps are suggested to be written down clearly, including the types, sources, contents of relevant information, the prior uncertain estimates of x and Θ, and the correlations used to obtain the prior uncertain estimates. After that, an evidence list is obtained. Diligent attempts to search for relevant information and/or evidence in prior knowledge lead to a relatively comprehensive evidence list, which helps engineers reduce effects of cognitive biases (Vick 2002). In addition, writing down the relevant information and/or evidence in a list allows engineers to carefully think about each piece of relevant information and/or evidence. This helps engineers overcome the limitation of human information-processing capacity (Vick 2002).

4.6 Synthesis of the Evidence

Engineers utilize the collected evidence to formulate internally engineering judgments on the soil property x and its statistics Θ. The evidence has two essential cognitive properties: strength and weight (Griffin and Tversky 1992; Vick 2002). The strength means the forcefulness or extremeness (i.e., how convincingly or persuasively the evidence argues for a proposition) of the evidence; and the weight indicates the quality (i.e., how reliable it is) and quantity (i.e., how much of it) of the evidence (Griffin and Tversky 1992; Vick 2002). Because of the limitation of information-processing capacity, engineers consider the two properties of the evidence separately and tend to focus more on the strength of the evidence and to underestimate the effect of the weight (Vick 2002). This, sometimes, makes engineers overly confident of strong but unreliable evidence and underemphasize (or ignore) the relatively weak evidence with high weight (e.g., good quality and large quantity), and subsequently leads to overconfidence bias, representativeness bias, and insufficient adjustment (Vick 2002). To reduce effects of the cognitive biases, there is a need to properly balance the effects of strength and weight of the evidence and to synthesize the evidence further for subjective probability assessment. This can, for example, be carried out through the following four steps: (1) evaluating the strength of the evidence; (2) evaluating the weight of the evidence; (3) assembling the evidence and statistical analysis; and (4) reassembling the relevant evidence for each sub-objective, as discussed in the following four subsections, respectively.

4.6.1 Evaluation of the Strength of Evidence

A piece of evidence consists of a piece of relevant information, a prior uncertain estimate, and correlations between the relevant information and the estimate (see Fig. 4.2). The prior uncertain estimate in the evidence can be obtained from theoretical and/or empirical correlations or be inferred intuitively according to the expertise of engineers. Correlations in the evidence can be, therefore, categorized into two types: theoretical/empirical correlation and intuitive inference. Based on the type of correlations used in the evidence, the strength of the evidence can be divided into three levels: weak, moderate, and strong. If only the intuitive inference is used in the evidence, the strength of the evidence is weak. If only theoretical/empirical correlations (e.g., theories of probability and soil mechanics, empirical regressions) are used in the evidence, the strength of the evidence is strong. When both intuitive inference and theoretical/empirical correlations are required to obtain the prior uncertain estimate in the evidence and they are used sequentially, the strength of the evidence is moderate. In addition, when the relevant information is completely equivalent to the prior uncertain estimate and there is no need of correlations, the relevant information totally supports the prior uncertain estimate in the evidence. In such cases, the strength of the evidence is strong.

After the strength of all the evidence in the evidence list is obtained, engineers are suggested to check the strength intuitively. This can be implemented by intuitively evaluating how convincingly the relevant information supports the corresponding prior uncertain estimate according to the correlations used in the evidence. The intuitive evaluation outcomes can also be categorized into three possible levels: highly, moderately, and lowly persuasive, which are corresponding to strong, moderate, and weak strength, respectively. If the strength of a piece of evidence obtained from intuitive judgment is inconsistent with the strength of the evidence obtained previously, engineers are suggested to cautiously think about the inconsistency and try to find out the reasons for the inconsistency. The strength of evidence, sometimes, needs to be properly adjusted according to the causes that lead to the inconsistency.

4.6.2 Evaluation of the Weight of Evidence

The weight of the evidence depends on several factors, including the source of the relevant information, quantity of the relevant information (e.g., the number of existing in-situ test data), and accuracy of the analysis method used to obtain the prior uncertain estimate in the evidence. The weight of the evidence can be evaluated according to the three factors. This can, for example, be performed in two steps:

(1) Evaluate the weight of the relevant information based on its source. The relevant information might have been collected from four types of sources: official publications (e.g., geotechnical reports, peer-reviewed academic journals, textbooks, and geological maps) on the site concerned, official publications on another site, informal sources on the site concerned, and informal sources on another site. By the source of the relevant information, the weight of relevant information can be divided into three levels: high, moderate, and low. Relevant information obtained from official publications on the site concerned has high weight. Relevant information obtained from informal sources on the site concerned or official publications on another site has moderate weight. Information collected from informal sources on another site has low weight.

(2) Adjust the weight of the relevant information to the weight of the evidence according to the accuracy of the analysis method used in the evidence. The analysis method used to obtain the prior uncertain estimate in the evidence can be categorized into two types: qualitative analysis and quantitative analysis. The accuracy of qualitative analysis is considered relatively poor compared with that of quantitative analysis. When qualitative analysis is used in the evidence, the weight of the evidence is obtained by decreasing the weight of the corresponding relevant information by one level. When quantitative analysis is used in the evidence, adjustment of the weight of the relevant information depends on the quantity of data used in the analysis. If there is a

relatively large number of data used in the analysis, the weight of the evidence is obtained by increasing the weight of the corresponding relevant information by one level. If there are relatively limited data, the weight of the evidence is obtained by decreasing the weight of the corresponding relevant information by one level.

For the evidence in which the relevant information is completely equivalent to the prior uncertain estimate, the weight of the evidence is determined by adjusting the weight of the relevant information according to the quantity of data contained in the relevant information. Note that when the weight of the relevant information is already high and there exists a need of increasing it to obtain the weight of the evidence, the weight of the evidence is still high. Similarly, when weight of the relevant information is already low and there exists a need of decreasing it to obtain the weight of the evidence, the weight of the evidence is still low.

When the weight of all the evidence in the evidence list is determined, engineers are suggested to intuitively check the weight of all the evidence in the list. This can be implemented by intuitively thinking about how reliable the evidence is. The outcomes of intuitively weighing the evidence can be divided into three levels: highly, moderately, and lowly reliable, which are corresponding to high, moderate, and low weight, respectively. If the weight of the evidence obtained from intuitively weighing the evidence is inconsistent with the weight of the evidence obtained previously, engineers are suggested to cautiously examine the inconsistency and try to find out the reasons resulting in the inconsistency. The weight of the evidence, sometimes, needs to be properly adjusted according to the causes that lead to the inconsistency.

4.6.3 Assembling the Evidence and Statistical Analysis

After the strength and weight of all the evidence are obtained, the next step is to assemble the evidence about the same variable X (i.e., x or $\theta_i, i = 1, 2, \ldots, n_m$) together. For each variable X, the evidence can be categorized into several groups by strength and weight, and the evidence in each group has the same strength and weight. Because both strength and weight have three possible levels (see Sects. 4.6.1 and 4.6.2), there are 9 possible evidence groups: (1) group with weak strength and low weight; (2) group with weak strength and moderate weight; (3) group with weak strength and high weight; (4) group with moderate strength and low weight; (5) group with moderate strength and moderate weight; (6) group with moderate strength and high weight; (7) group with strong strength and low weight; (8) group with strong strength and moderate weight; and (9) group with strong strength and high weight. For each evidence group, conventional statistical equations (e.g., Eqs. (4.1)–(4.4)) can be used to analyze the information on the variable X (i.e., x or $\theta_i, i = 1, 2, \ldots, n_m$) provided by the evidence. By this means, some estimates of statistics of X are obtained. The procedure described above is repeatedly performed for each variable involved in the subjective probability assessment objectives, including x in the general assessment objective and $\theta_i, i = 1, 2, \ldots, n_m$, in sub-objectives.

4.6.4 Reassembling the Relevant Evidence for Each Sub-objective

Evidence with regard to the soil property x provides information on different statistics $\theta_i, i = 1, 2, \ldots, n_m$, of x. In other words, estimates from the evidence on x might be related to different sub-objectives since each sub-objective involves only one statistic of x. For the convenience of subjective probability assessment, the relevant evidence (or evidence groups) about the same sub-objective shall be assembled together and be written down with the strength and weight of the evidence. After that, engineers can examine cautiously the relevant evidence on each sub-objective and make their engineering judgments on the sub-objective internally based on the evidence.

It is worthwhile to point out that the evidence group with limited evidence shall be used with caution when formulating engineering judgments. In addition, when using the relevant evidence to make engineering judgments on a sub-objective, the strength and weight of the evidence need to be considered. The relevant evidence with strong strength and high weight is more persuasive and reliable than that with relatively weak strength and relatively low weight. Convincingness and reliability of the evidence decrease as the levels of strength and weight decrease.

Based on the relevant evidence, engineers have formulated their engineering judgments on statistics $\theta_i, i = 1, 2, \ldots, n_m$, in sub-objectives internally. The next step is to elicit engineering judgments on $\theta_i, i = 1, 2, \ldots, n_m$, from engineers and to express the engineering judgments through numerical values. Engineers are, however, not used to thinking in terms of probability due to relatively limited training in probability theory and statistics (El-Ramly et al. 2002; Vick 2002; Baecher and Christian 2003). In the next section, the equivalent lottery method and verbal descriptors of the likelihood (or plausibility) are used to assist engineers in assigning numerical values (i.e., subjective probability) to their engineering judgments on $\theta_i, i = 1, 2, \ldots, n_m$, in sub-objectives.

4.7 Numerical Assignment

4.7.1 Equivalent Lottery Method

Equivalent lottery method (e.g., Clemen 1996; Vick 2002) assists engineers in making decisions by comparing two lotteries. One of the two lotteries involves the event of interest, and the other one is designed as reference lottery in which the probability information is contained as a reference. The plausibility of the event concerned is equal to the probability in the reference lottery when indifference between two lotteries is achieved by adjusting one of them. For example, two lotteries are used to determine the median value of the statistic $\theta_i, i = 1, 2, \ldots, n_m$, for a given range from the minimum $\theta_{i,\min}$ to the maximum $\theta_{i,\max}$, which are given by

Lottery 1:

Win a prize if $\theta_{i,\min} \leq \theta_i \leq a_s$ occurs, where a_s is a possible value of θ_i falling within the range $[\theta_{i,\min}, \theta_{i,\max}]$.
Win nothing if $a_s < \theta_i \leq \theta_{i,\max}$ occurs.

Lottery 2:

Win a prize with known probability $p = 0.5$.
Win nothing with probability $1 - p = 0.5$.

The second lottery (i.e., lottery 2) is the reference lottery. Engineers can adjust the value of a_s between the minimum and maximum (i.e., $\theta_{i,\min}$ and $\theta_{i,\max}$) of θ_i until they are indifferent between the two lotteries according to the previously obtained relevant evidence on θ_i. The indifference indicates that engineers believe that the two lotteries are equivalent to each other. Since occurrence probabilities of the two choices in lottery 2 are fixed at 0.5, engineers believe that occurrence probabilities of the two choices in lottery 1 are also 0.5 after the indifference is reached. Therefore, when the indifference is reached, engineers believe that the probability of $\theta_{i,\min} \leq \theta_i \leq a_s$ for a given range from $\theta_{i,\min}$ to $\theta_{i,\max}$ is equal to that of $a_s < \theta_i \leq \theta_{i,\max}$, and both of them are equal to 0.5. In other words, a_s is the median value of θ_i for the given range from $\theta_{i,\min}$ to $\theta_{i,\max}$ after the indifference is reached.

The equivalent lottery method described above requires a range of θ_i, i.e., $[\theta_{i,\min}, \theta_{i,\max}]$, as input. Using different ranges of θ_i in the equivalent lottery method leads to different median values. For example, using the range from 1 % percentile (i.e., $\theta_{i,0.01}$) of θ_i to 99 % percentile (i.e., $\theta_{i,0.99}$) of θ_i in the equivalent lottery method leads to a median value of θ_i equivalent to its 50 % percentile (i.e., $\theta_{i,0.5}$). Subsequently, using the range from $\theta_{i,0.01}$ to $\theta_{i,0.5}$ (i.e., $[\theta_{i,0.01}, \theta_{i,0.5}]$) in the equivalent lottery method results in a median value of θ_i equivalent to its 25 % percentile (i.e., $\theta_{i,0.25}$). Similarly, using the range from $\theta_{i,0.5}$ to $\theta_{i,0.99}$ (i.e., $[\theta_{i,0.5}, \theta_{i,0.99}]$) in the equivalent lottery method results in a median value of θ_i equivalent to its 75 % percentile (i.e., $\theta_{i,0.75}$). Then, using ranges of $[\theta_{i,0.01}, \theta_{i,0.25}]$, $[\theta_{i,0.25}, \theta_{i,0.5}]$, $[\theta_{i,0.5}, \theta_{i,0.75}]$, and $[\theta_{i,0.75}, \theta_{i,0.99}]$ in the equivalent lottery method leads to the median values equivalent to its 12.5 % (i.e., $\theta_{i,0.125}$), 37.5 % (i.e., $\theta_{i,0.375}$), 62.5 % (i.e., $\theta_{i,0.625}$), and 87.5 % (i.e., $\theta_{i,0.875}$) percentiles, respectively.

4.7.2 Verbal Descriptors of the Likelihood

To start the equivalent lottery method described above, 1 % and 99 % percentiles (i.e., $\theta_{i,0.01}$ and $\theta_{i,0.99}$) of θ_i should be determined first. Direct elicitation of numerical values of probability from engineers might lead to unstable and incoherent results because engineers are not used to thinking in terms of numerical values of probability (e.g., Baecher and Christian 2003). On the other hand, engineers prefer to express their engineering judgments using words that indicate the likelihood, namely verbal descriptors of the likelihood (Vick 2002). Verbal descriptors can be mapped to numerical values of probability by transformation conventions. For example, Table 4.1 shows a transformation convention between verbal descriptors and numerical values of probability (Vick 2002). By this convention, the words "virtually impossible," "very unlikely," "equally likely," "very likely," and "virtually certain" are equivalent to the probability of 0.01, 0.1, 0.5, 0.9, and 0.99, respectively. Note that in this convention the probability value ranges from 0.01 to 0.99 (see Table 4.1), which happens to be the valid cognitive discrimination range (i.e., from 0.01 to 0.99) of subjective probability (Fischhoff et al. 1977; Hogarth 1975; Vick 1997, 2002).

The probability value in the transformation convention increases from 0.01 to 0.99 monotonically (see Table 4.1). Therefore, the transformation convention corresponds to the cumulative distribution function (CDF) of the variable θ_i concerned, and the words "virtually impossible" and "virtually certain" can be used to determine the 1 % and 99 % percentiles of θ_i, respectively. 1 % and 99 % percentiles are located at the lower and upper tails of probability density function (PDF), respectively. They can be considered as the minimum and maximum possible values of θ_i, respectively. Therefore, in the convention shown in Table 4.1, the words "virtually impossible" and "virtually certain" are actually used to determine the minimum and maximum possible values of θ_i in terms of PDF, respectively. In this chapter, the words "minimum" and "maximum" are directly used to determine the 1 % and 99 % percentiles, respectively. The 1 % and 99 % percentiles of θ_i are then determined by asking "What is the minimum possible value of θ_i?" and "What is the maximum possible value of θ_i?", respectively.

It is also worthwhile to note that the verbal descriptor "equally likely" is equivalent to the probability of 0.5 (see Table 4.1). By this convention, the two lotteries proposed in the previous subsection can be rewritten as follows:

Table 4.1 Verbal descriptors and their probability equivalents (After Vick 2002)

Verbal descriptor	Probability equivalent	Percentile
Virtually impossible	0.01	$\theta_{i,0.01}$
Very unlikely	0.10	$\theta_{i,0.1}$
Equally likely	0.50	$\theta_{i,0.5}$
Very likely	0.90	$\theta_{i,0.9}$
Virtually certain	0.99	$\theta_{i,0.99}$

Lottery 1:

Win a prize if $\theta_{i,\min} \le \theta_i \le a_s$ occurs, where a_s is a possible value of θ_i falling within the range $[\theta_{i,\min}, \theta_{i,\max}]$.
Win nothing if $a_s < \theta_i \le \theta_{i,\max}$ occurs.

Lottery 2:

Win a prize or nothing equally likely.

4.7.3 Implementation of the Equivalent Lottery Method

Using the two lotteries and verbal descriptors, the percentiles of θ_i in each sub-objective can be determined accordingly. A questionnaire is designed in this chapter to implement the equivalent lottery method, as shown in Appendix 4.1. When answering the questions in the questionnaire, engineers might revisit the relevant evidence on θ_i collected before.

The questionnaire starts with a question (i.e., Q1) that is used to determine a reference prize for the lottery 1, followed by the second and third questions (i.e., Q2 and Q3) for determining $\theta_{i,0.01}$ and $\theta_{i,0.99}$, respectively. Then, the equivalent lottery method can be used to estimate the median value (i.e., $\theta_{i,0.5}$) for the given range from $\theta_{i,0.01}$ and $\theta_{i,0.99}$ in Q4 if there is sufficient information provided by the relevant evidence on θ_i. Such a procedure can be repeatedly performed to determine the percentiles of θ_i progressively using different ranges of θ_i in the equivalent lottery method, as described in Sect. 4.7.1. The questionnaire shall be stopped when engineers believe that there is no sufficient information on θ_i to balance the two lotteries in the equivalent lottery method for a given range of θ_i.

For example, if the information on θ_i is very sparse, it might be too difficult for engineers to estimate 50 %, 25 %, 75 %, 12.5 %, 37.5 %, 62.5 %, and 87.5 % percentiles (i.e., $\theta_{i,0.5}, \theta_{i,0.25}, \theta_{i,0.75}, \theta_{i,0.125}, \theta_{i,0.375}, \theta_{i,0.625}$, and $\theta_{i,0.875}$) of θ_i. In such cases, the questionnaire is stopped after $\theta_{i,0.01}$ and $\theta_{i,0.99}$ are obtained from Q2 and Q3. On the other hand, if there is a large number of information on θ_i, the questionnaire can be continued to obtain more percentiles of θ_i after 12.5 %, 37.5 %, 62.5 %, and 87.5 % percentiles are obtained.

The questionnaire is repeated n_m times for the n_m sub-objectives. After that, the percentiles of the statistics $\theta_i, i = 1, 2, \ldots, n_m$, of x are obtained. The prior distribution of θ_i is then estimated from its percentiles, as discussed in the next subsection.

4.7.4 Prior Distribution

After the percentiles of θ_i are obtained, its cumulative distribution function (CDF) and probability density function (PDF) can be estimated from its percentiles through two methods: a simplified method and a least squares regression method.

4.7.4.1 A Simplified Method

The range of θ_i from $\theta_{i,0.01}$ to $\theta_{i,0.99}$ is divided into several intervals by its percentiles. Consider, for example, that θ_i is uniformly distributed within each interval. Then, an empirical CDF of θ_i is obtained by plotting a line through the points at percentiles (i.e., data pairs of the percentiles of θ_i and their respective cumulative probability levels, such as ($\theta_{i,0.01}$, 1 %), ($\theta_{i,0.5}$, 50 %), and ($\theta_{i,0.99}$, 99 %)). The PDF of θ_i is estimated by constructing a histogram with bins equal to the intervals of θ_i between adjacent percentiles. In each bin, the PDF value of θ_i is calculated as the ratio of the increase in cumulative probability level in this bin over the length of the bin. For example, the PDF value of θ_i in the bin from $\theta_{i,0.5}$ to $\theta_{i,0.625}$ is calculated as $(0.625 - 0.5)/(\theta_{i,0.625} - \theta_{i,0.5})$. The PDF of θ_i is then taken as the prior distribution of θ_i in the Bayesian framework formulated in Chap. 3.

4.7.4.2 A Least Squares Regression Method

The CDF of θ_i can also be obtained by fitting a probability distribution with assessment results (i.e., data pairs of the percentiles of θ_i and their respective cumulative probability levels) using the least squares regression method (e.g., Baecher and Christian 2003; Ang and Tang 2007). The least squares regression method requires a probability distribution as the model function for data fitting. Consider, for example, the Gaussian CDF as the model function. The least squares regression method provides a Gaussian CDF of θ_i as the best fit of the assessment results and, simultaneously, gives the values of the mean and standard deviation of the Gaussian distribution. Using the mean and standard deviation, the PDF of θ_i is determined, which is then taken as the prior distribution of θ_i in the Bayesian framework formulated in Chap. 3. Note that the least squares regression method can be achieved using commercial software packages. For example, MATLAB (Mathworks Inc. 2010) provides a built-in function "nlinfit" for the least squares regression method.

4.8 Confirmation of Assessment Outcomes

Because of cognitive biases and limitations, the assessment outcomes (e.g., the percentiles and probability distributions of $\theta_i, i = 1, 2, \ldots, n_m$) might violate the basic probability axioms and deviate from the actual beliefs of engineers (Vick

2002). Several suggestions are provided herein to help engineers check the assessment outcomes, such as

(1) Check the coherence between the assessment outcomes and the basic probability axioms (e.g., probability falls within the range from 0 to 1, and integration on a PDF is equal to unity) (e.g., Ang and Tang 2007; Ross 2007). The assessment outcomes (the percentiles and probability distributions of $\theta_i, i = 1, 2, \ldots, n_m$) obtained from the proposed SPAF have to conform to the basic probability axioms.
(2) Examine biases arising from cognitive heuristics. This can be carried out by reviewing all the evidence carefully and checking that "is there any evidence that is overlooked or underemphasized" and "is there any evidence that is overemphasized". The careful examination of the evidence reduces the overconfidence bias arising from overemphasis on the supportive evidence and ignorance of disconfirming evidence (Vick 2002). In addition, the attempts to find out the evidence that is overlooked or underemphasized reduce the availability bias.
(3) Engineers are suggested to interpret the assessment outcomes to check whether or not the outcomes are reasonable in reality according to their expertise and reflect their actual beliefs.

If there is any inconsistency or any evidence that is misused, engineers need to adjust properly the percentiles obtained from the SPAF and to reevaluate the prior distributions accordingly, and they are suggested to write down the reasons for the adjustment. This provides an opportunity to examine the adjustment and to reduce the hindsight bias (Vick 2002). The confirmation-reevaluation process might be iterated several times until engineers believe that the assessment outcomes are reasonable in reality and reflect their actual beliefs according to the prior knowledge, and all the evidence has been taken into account properly. After the final confirmation of the assessment outcomes, probability distributions of $\theta_i, i = 1, 2, \ldots, n_m$, quantify their respective plausibility according to the prior knowledge. In the next section, the proposed SPAF is applied to characterize probabilistically soil properties at the sand site of US NGES at Texas A&M University (TAMU) (Briaud 2000), and it is illustrated under two scenarios: scenario I with uninformative prior knowledge and scenario II with a reasonable amount of prior knowledge, as discussed in the following two sections.

4.9 Scenario I: Uninformative Prior Knowledge

4.9.1 Assessment Objectives

The sand site of US National Geotechnical Experimentation Site (NGES) at Texas A&M University is comprised of a top layer of sands to about 12.5 m deep and a stiff clay layer thereafter (see Fig. 2.5 in Chap. 2). Consider, for example, the sand effective friction angle ϕ' of interest, i.e., $x = \phi'$. The general assessment objective is, therefore, defined as "probabilistic characterization of effective friction angle ϕ' at the sand site." The sand effective friction angle ϕ' within a statistically homogenous soil layer can be probabilistically characterized by the random field theory (Vanmarcke 1977, 1983), in which three model parameters are required, i.e., mean μ, standard deviation σ, and correlation length λ of ϕ'. The statistics of interest are μ, σ, and λ, i.e., $\Theta = [\mu, \sigma, \lambda]$. The general assessment objective is then decomposed into three sub-objectives: "evaluating probability of μ," "evaluating probability of σ," and "evaluating probability of λ."

4.9.2 *Relevant Information and Prior Uncertain Estimates*

For illustration, suppose that only one piece of relevant information is obtained according to previous engineering experience at this site (e.g., Briaud 2000), and it indicates that the site is underlain by sand layers. The piece of relevant information leads to three pieces of evidence, as shown in Table 4.2. For sands, the typical value of ϕ' falls within the range from 27.5° to 50.0° by Terzaghi and Peck (1967) and Kulhawy and Mayne (1990) (see Table 2.11), i.e., evidence (1). The respective typical ranges of σ and λ are from 3.7° to 5.5° (i.e., evidence (2)) and from 2.0 m to 6.0 m (i.e., evidence (3)) by Phoon and Kulhawy (1999a, 1999b).

4.9.3 *Strength and Weight of the Evidence and Statistical Analysis*

4.9.3.1 Strength and Weight of the Evidence

Table 4.3 summarizes the strength (i.e., Column 6) and weight (i.e., Column 7) of the 3 pieces of evidence and the procedure of evaluating their strength and weight, including source of the information (i.e., Column 2), procedure of estimation (i.e., Column 3), type of analysis (i.e., Column 4), and type of correlation (i.e., Column 5).

Table 4.2 Summary of relevant information and prior uncertain estimates for scenario I

Type	Relevant information	Correlations	Prior uncertain estimates		No. of evidence
Geotechnical properties	Sands (Briaud 2000)	Table 2.11 after Kulhawy and Mayne (1990)	ϕ'	27.5°–50.0°	(1)
		Phoon and Kulhawy (1999b)	σ	3.7°–5.5°	(2)
		Phoon and Kulhawy (1999a)	λ	2.0–6.0 m	(3)

Table 4.3 Summary of strength and weight of the evidence for scenario I

No. of evidence	Source of information	Procedure of estimation	Type of analysis	Type of correlation	Strength	Weight
(1)	An official report of the sand site	Soil type —range of ϕ'	Qualitative analysis	Empirical correlation	Strong	Moderate
(2)	An official report of the sand site	Soil type —range of σ	Qualitative analysis	Empirical correlation	Strong	Moderate
(3)	An official report of the sand site	Soil type —range of λ	Qualitative analysis	Empirical correlation and Intuitive inference	Moderate	Low

In evidence (1), the range of ϕ' is estimated from the relevant information (i.e., the site is underlain by sands) by an empirical correlation (Kulhawy and Mayne 1990). Thus, evidence (1) has a strong strength. The relevant information is obtained from an official report (Briaud 2000). Thus, the weight of the information is high. The range of ϕ' in evidence (1) is qualitatively estimated from the relevant information, so that the weight of the evidence is obtained through decreasing the level of the weight of the relevant information by one level, i.e., moderate.

In evidence (2), the relevant information is related to the range of σ by an empirical correlation (Phoon and Kulhawy 1999b). Thus, evidence (2) has a strong strength. The information is obtained from an official report of the sand site (Briaud 2000). Thus, the weight of the relevant information is high. The range of σ is qualitatively estimated from the relevant information, so that the weight of the evidence is obtained through decreasing the level of the weight of the relevant information by one level, i.e., moderate.

In evidence (3), the relevant information is related to the correlation length of soil properties by an empirical correlation (Phoon and Kulhawy 1999b). The empirical correlation does not directly give the range of correlation length λ of ϕ', but provides the correlation length of other soil properties (e.g., standard penetration test (SPT) N-value, cone tip resistance obtained from cone penetration test (CPT)) of sands. The range of λ of ϕ' is intuitively inferred from the range of correlation length of other soil properties (e.g., SPT N-value) of sands. The empirical correlation and intuitive inference are sequentially used in evidence (3). Therefore, the strength of the evidence is moderate. The relevant information in evidence (3) is collected from an official report of the sand site (Briaud 2000). Thus, the weight of the relevant information is high. The range of λ is qualitatively estimated from the relevant information, so the weight of the evidence is obtained through decreasing the level of the weight of the relevant information by one level, i.e., moderate. However, the assessor believes that the intuitive inference on the correlation length λ of ϕ' from that of other soil properties is not very reliable. Thus, the weight of the evidence is further decreased to the third level, i.e., low.

4.9.3.2 Assembling Evidence and Statistical Analysis

There are only three pieces of evidence available. Evidence (1) gives a possible range of ϕ' with strong strength and moderate weight. The range of ϕ' in evidence (1) provides some information on the mean μ of ϕ'. Evidence (2) gives a possible range of σ with strong strength and moderate weight. Evidence (3) gives a possible range of λ with moderate strength and low weight. For each sub-objective, there is only one piece of evidence available, i.e., evidence (1) for evaluating the probability of μ, evidence (2) for evaluating the probability of σ, and evidence (3) for evaluating the probability of λ.

4.9.4 Results of Subjective Probability Assessment

Based on the evidence on each sub-objective, the percentiles of μ, σ, and λ are elicited from the assessor using the questionnaire shown in Appendix 4.1. Because there is only one piece of evidence for each sub-objective, only the 1 % and 99 % percentiles of μ, σ, and λ are evaluated (i.e., only Q1 to Q3 in the questionnaire (see Appendix 4.1) are answered). As shown in Table 4.4, the 1 % and 99 % percentiles of μ, σ, and λ are $\mu_{0.01} = 27.5°, \mu_{0.99} = 50.0°, \sigma_{0.01} = 3.7°, \sigma_{0.99} = 5.5°, \lambda_{0.01} = 2.0\,\text{m}$, and $\lambda_{0.99} = 6.0\,\text{m}$. Using the simplified method described in Sect. 4.7.4.1, prior distributions of μ, σ, and λ are obtained from their respective percentiles, as discussed in the following three subsections.

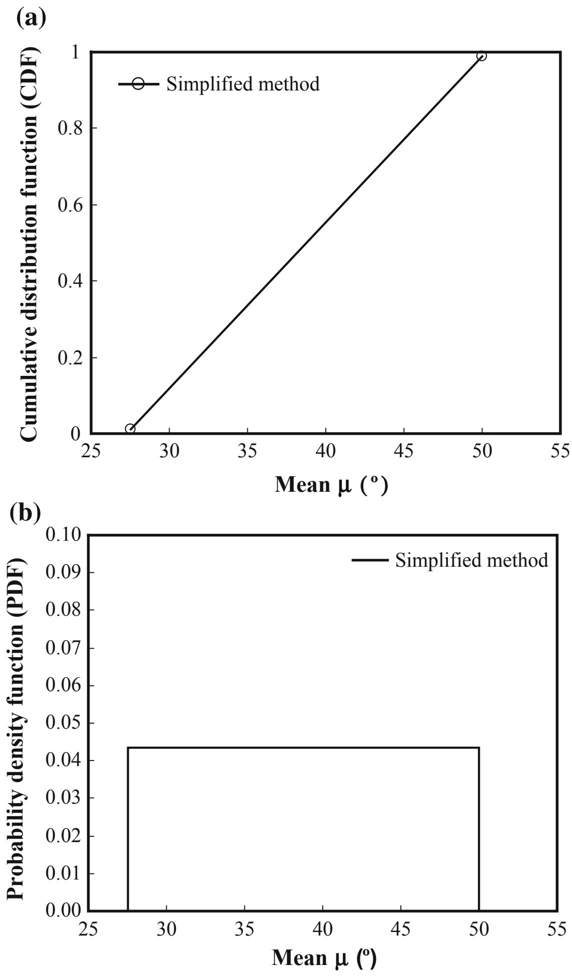
4.9.4.1 Prior Distribution of the Mean μ

Figure 4.3a shows the CDF of μ obtained from the simplified method by a solid line with open circles. The CDF value of μ increases linearly from 0.01 to 0.99 as μ increases from 27.5° to 50.0°. Figure 4.3b shows the PDF of μ obtained from the simplified method by a histogram with only one bin (i.e., a uniform distribution with a range from 27.5° to 50.0°), and the PDF value of μ is about 0.044. The uniform PDF of μ (see Fig. 4.3b) can be taken as the prior distribution of μ in the Bayesian framework developed in Chap. 3.

Table 4.4 Summary of percentiles of the mean, standard deviation, and correlation length for scenario I

Cumulative probability	0.01	0.99
Mean μ (°)	27.5	50
Standard deviation σ (°)	3.7	5.5
Correlation length λ (m)	2.0	6.0

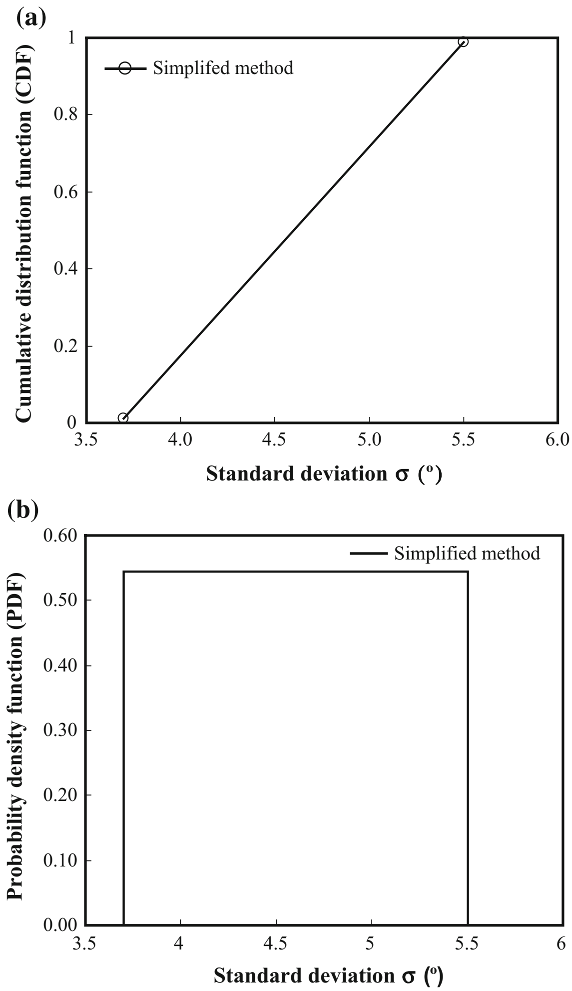
Fig. 4.3 Prior distribution of the mean, μ, in scenario I. **a** Cumulative distribution function (CDF) of the mean, μ. **b** Probability density function of the mean, μ



4.9.4.2 Prior Distribution of the Standard Deviation σ

Figure 4.4a shows the CDF of σ obtained from the simplified method by a solid line with open circles. The CDF value of σ increases linearly from 0.01 to 0.99 as σ increases from 3.7° to 5.5°. Figure 4.4b shows the PDF of σ obtained from the simplified method by a histogram with only one bin (i.e., a uniform distribution with a range from 3.7° to 5.5°), and the PDF value of σ is about 0.54. The uniform PDF of σ (see Fig. 4.4b) can be taken as the prior distribution of σ in the Bayesian framework developed in Chap. 3.

Fig. 4.4 Prior distribution of the standard deviation, σ, in scenario I. **a** Cumulative distribution function (CDF) of the standard deviation, σ. **b** Probability density function (PDF) of the standard deviation, σ



4.9.4.3 Prior Distribution of the Correlation Length λ

Figure 4.5a shows the CDF of λ obtained from the simplified method by a solid line with open circles. The CDF value of λ increases linearly from 0.01 to 0.99 as λ increases from 2.0 to 6.0 m. Figure 4.5b shows the PDF of λ obtained from the simplified method by a histogram with only one bin (i.e., a uniform distribution with a range from 2.0 to 6.0 m), and the PDF value of λ is about 0.25. The uniform PDF of λ (see Fig. 4.5b) can be taken as the prior distribution of λ in the Bayesian framework developed in Chap. 3.

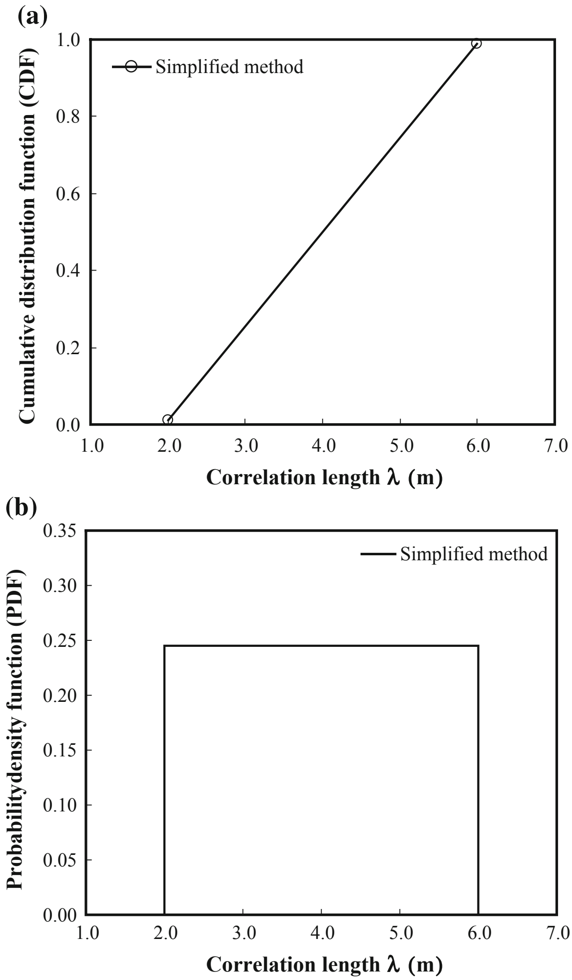


Fig. 4.5 Prior distribution of the correlation length, λ, in scenario I. **a** Cumulative distribution function (CDF) of the correlation length, λ. **b** Probability density function (PDF) of the correlation length, λ

4.9.5 Final Confirmation

All the areas under the respective PDFs (see Figs. 4.3b, 4.4b, and 4.5b) of μ, σ, and λ are summed up to unity. This is consistent with the basic probability axiom that integration on a PDF is unity. The prior distributions of μ, σ, and λ are consistent with the information on them provided by evidence (1), (2), and (3), respectively. All the three pieces of evidence are taken into account properly. Because the information on μ, σ, and λ is very sparse in this scenario, the prior distributions of μ, σ, and λ obtained from the SPAF are uninformative (i.e., uniform distributions). The prior distributions of μ, σ, and λ reflect properly the prior knowledge. The assessment outcomes are then confirmed. In the next section, the proposed SPAF is further illustrated under another scenario that has a reasonable amount of prior knowledge.

4.10 Scenario II: Informative Prior Knowledge

4.10.1 Assessment Objectives

The sand effective friction angle ϕ' at the sand site of NGES at Texas A&M university is still of interest under this scenario. The general assessment objective remains unchanged, i.e., "probabilistic characterization of effective friction angle ϕ' at the sand site." Similar to scenario I, it is then decomposed into three sub-objectives: "evaluating probability of μ," "evaluating probability of σ," and "evaluating probability of λ."

4.10.2 Relevant Information and Prior Uncertain Estimates

In this scenario, geological information and soil classification information are obtained from Briaud (2000). Table 4.5 summarizes the relevant information (i.e., Column 2) on geology and soil classification, available correlations (i.e., Column 3) between the geology and soil classification information and ϕ', μ, σ, and λ; their prior uncertain estimates (i.e., Column 4). As shown in Table 4.5, a total of 11 pieces of evidence are obtained from geological information and soil classification information.

Evidence (1), (2), and (3) are obtained from geological information. The sand site is underlain by fluvial and overbank deposits (Briaud 2000), which can be categorized as alluvium deposits (Heim 1990). Alluvium deposits usually have relatively low in-situ densities (e.g., loose or medium) (Heim 1990). For loose and medium sands, ϕ' varies from 28° to 40° (i.e., evidence (1)) or varies from 30° to 40° (i.e., evidence (2)) (Kulhawy and Mayne 1990). In addition, a textbook (Rollings and Rollings 1996) provides consistent values (i.e., from 30° to 40°) of ϕ' of alluvium deposits, i.e., evidence (3).

The sands underlying the site include silty sand, clean sand, and clayey sand (Briaud 2000). Two peer-reviewed academic papers (i.e., Phoon and Kulhawy 1999a, 1999b) provided six pieces of evidence (i.e., evidence (4)–(9)) on ϕ', μ, σ, and λ of sands. Evidence (4) and (5) give the two possible ranges of ϕ', i.e., from 35° to 41° and from 33° to 43° (Phoon and Kulhawy 1999a), respectively. Evidence (6) and (7) provide two possible values of μ, i.e., 37.6° or 36.7° (Phoon and Kulhawy 1999a), respectively. Evidence (8) and (9) provide the respective ranges of σ (i.e., from 3.7° to 5.5°) and λ (i.e., from 2 to 6 m) (Phoon and Kulhawy 1999a).

Table 4.5 Summary of relevant information and prior uncertain estimates for scenario II

Type	Relevant information	Correlations		Prior uncertain estimates		No. of evidence
Geology	Fluvial and overbank deposits (Briaud 2000)	Alluvium deposit—relatively low in-situ density: loose or medium sand (Heim 1990)	Table 2.8 after Kulhawy and Mayne (1990)	ϕ'	28°–40°	(1)
			Table 2.9 after Kulhawy and Mayne (1990)	ϕ'	30°–40°	(2)
		Alluvium Deposit (Heim 1990)	Rollings and Rollings (1996)	ϕ'	30°–40°	(3)
Geotechnical properties	Classification: silty sand, clean sand, and clayey sand (Briaud 2000)	Sand	Phoon and Kulhawy (1999a)	ϕ'	35°–41°	(4)
					33°–43°	(5)
				μ	37.6°	(6)
					36.7°	(7)
			Phoon and Kulhawy (1999b)	σ	3.7°–5.5°	(8)
			Phoon and Kulhawy (1999a)	λ	2–6 m	(9)
		Dry unit weight: 13.3–21.7 kN/m^3 (Table 2.10 after Kulhawy and Mayne (1990))	Figure 2.7 after Kulhawy and Mayne (1990)	ϕ'	27.0°–37.0°	(10)
		Sand or silty sand	Table 2.11 after Kulhawy and Mayne (1990)	ϕ'	28°–45°	(11)

Note that evidence (8) and (9) in this scenario are the same as evidence (2) and (3) in scenario I (see Table 4.2), respectively. In addition, the dry unit weight of sands usually varies from 13.3 to 21.7 kN/m^3, and it is also related to ϕ' (Kulhawy and Mayne 1990). This leads to evidence (10), i.e., a possible range of ϕ' from 27.0° to 37.0°. Kulhawy and Mayne (1990) also reported that the effective friction angle of sand or silty sand ranges from 28° to 45°, i.e., evidence (11).

4.10.3 *Strength and Weight of the Evidence and Statistical Analysis*

4.10.3.1 Strength and Weight of the Evidence

Table 4.6 summarizes the strength (i.e., Column 7) and weight (i.e., Column 8) of the 11 pieces of evidence and the procedure of evaluating their strength and weight, including source of the information (i.e., Column 2), quantity of data (i.e., Column 3), procedure of estimation (i.e., Column 4), type of analysis (i.e., Column 5), and type of correlation (i.e., Column 6). As mentioned above, evidence (8) and (9) in this scenario are the same as evidence (2) and (3) in scenario I (see Table 4.2), respectively, and the procedures of evaluating their strength and weight have been described in Sect. 4.9.3.1. For further illustration, procedures of evaluating the strength and weight of evidence (3) and (11) in this scenario are described below.

In evidence (3), the type of deposits underlying the sand site is intuitively inferred from the geological information, and the range of ϕ' is then intuitively estimated from the deposit type. Only the intuitive inference is used to estimate the range of ϕ' in evidence (3). Thus, the strength of the evidence is weak. The geological information in evidence (3) is obtained from an official report of the sand site (Briaud 2000). Thus, the weight of the geological information is high. The range of ϕ' in evidence (3) is qualitatively estimated from the geological information, so that the weight of the evidence is obtained through decreasing the level of the weight of the relevant information by one level, i.e., moderate.

In evidence (11), the range of ϕ' is estimated from the soil classification information by an empirical correlation (Kulhawy and Mayne 1990). Thus, the strength of the evidence is strong. The soil classification information in evidence (11) is obtained from an official report of the sand site (Briaud 2000). Thus, the weight of the soil classification information is high. The range of ϕ' in evidence (11) is qualitatively estimated from the geological information, so the weight of the evidence is obtained through decreasing the level of the weight of the relevant information by one level, i.e., moderate.

4.10.3.2 Assembling Evidence and Statistical Analysis

Table 4.7 summarizes the strength and weight of evidence with regard to ϕ', μ, σ, and λ, respectively. Six evidence groups (i.e., evidence groups (I)–(VI)) are obtained with their respective strength and weight.

Table 4.6 Summary of strength and weight of the evidence for scenario II

No. of evidence	Source of information	Quantity	Procedure of estimation	Type of analysis	Type of correlation	Strength	Weight
(1)	An official report of the sand site	N/A	Geological information—type of deposit—geotechnical characteristics—range of ϕ'	Qualitative analysis	Intuitive inference and empirical correlation	Moderate	Moderate
(2)	An official report of the sand site	N/A	Geological information—type of deposit—geotechnical characteristics—range of ϕ'	Qualitative analysis	Intuitive inference and empirical correlation	Moderate	Moderate
(3)	An official report of the sand site	N/A	Geological information—type of deposit—range of ϕ'	Qualitative analysis	Intuitive inference	Weak	Moderate
(4)–(5)	An official report of the sand site	7 data groups and 13 data groups	Soil classification—range of ϕ'	Qualitative analysis	Empirical correlation	Strong	Moderate
(6)–(7)	An official report of the sand site	7 data groups and 13 data groups	Soil classification—values of μ	Qualitative analysis	Empirical correlation	Strong	Moderate
(8)	An official report of the sand site	N/A	Soil classification—range of σ	Qualitative analysis	Empirical correlation	Strong	Moderate
(9)	An official report of the sand site	N/A	Soil classification—range of λ	Qualitative analysis	Empirical correlation and intuitive inference	Moderate	Low
(10)	An official report of the sand site	N/A	Soil classification—dry unit weight—range of ϕ'	Qualitative analysis	Empirical correlation	Moderate	Moderate
(11)	An official report of the sand site	N/A	Soil classification—range of ϕ'	Qualitative analysis	Empirical correlation	Strong	Moderate

Table 4.7 Summary of the evidence for scenario II

Variable	No. of evidence	Prior uncertain estimates	Strength	Weight	No. of evidence group
ϕ'	(3)	30°–40°	Weak	Moderate	(I)
	(1)	28°–40°	Moderate	Moderate	(II)
	(2)	30°–40°	Moderate	Moderate	
	(10)	27°–37°	Moderate	Moderate	
	(4)	35°–41°	Strong	Moderate	(III)
	(5)	33°–43°	Strong	Moderate	
	(11)	28°–45°	Strong	Moderate	
μ	(6)	37.6°	Strong	Moderate	(IV)
	(7)	36.7°	Strong	Moderate	
σ	(8)	3.7°–5.5°	Strong	Moderate	(V)
λ	(9)	2–6 m	Moderate	Low	(VI)

For ϕ', there are in total 7 pieces of relevant evidence that are assembled into evidence groups (I)–(III) by their strength and weight. Evidence group (I) has one piece of evidence with weak strength and moderate weight, i.e., evidence (3). Evidence group (II) consists of 3 pieces of evidence with moderate strength and moderate weight, i.e., evidence (1), (2), and (10). Evidence group (III) is comprised of 3 pieces of evidence with strong strength and moderate weight, i.e., evidence (4), (5), and (11).

Using Eqs. (4.3) and (4.4), the range of ϕ' from 30° to 40° in evidence (3) (i.e., evidence group (I)) leads to $\mu = 35.0°$ and $\sigma = 1.7°$, respectively. In evidence group (II), the 3 possible ranges of ϕ' (i.e., evidence (1), (2), and (10)) provide 3 possible values of μ (i.e., 34.0°, 35.0°, and 32.0°) and σ (i.e., 2.0°, 1.7°, and 1.7°) by Eqs. (4.3) and (4.4), respectively. In evidence group (III), the 3 possible ranges of ϕ' (i.e., evidence (4), (5), and (11)) provide 3 possible values of μ (i.e., 38.0°, 38.0°, and 36.5°) and σ (i.e., 1.0°, 1.7°, and 2.8°) by Eqs. (4.3) and (4.4), respectively. Note that each range of ϕ' in evidence group (I), (II), and (III) leads to a pair of estimates of μ and σ. These estimates of μ and σ should be used with caution during subjective probability assessment since they are obtained from only one piece of evidence.

There are 2 pieces of evidence (i.e., evidence (6) and (7)) about the mean value μ of ϕ' in the evidence list, which suggest that μ is equal to 37.6° or 36.7°, i.e., evidence group (IV). For the standard deviation σ of ϕ', there is a possible range (i.e., from 3.7° to 5.5° in evidence (8)) in the evidence list. In addition, there is a possible range (i.e., from 2.0 to 6.0 m in evidence (9)) of λ in the evidence list.

4.10.3.3 Reassembling the Relevant Evidence for Each sub-objective

Table 4.8 summarizes the relevant evidence (i.e., Column 3) for each assessment sub-objective (i.e., Column 1) together with the strength (i.e., Column 4) and

Table 4.8 Summary of relevant evidence for each sub-objective of scenario II

sub-objective	No. of evidence group	No. of Evidence	Strength of the evidence	Weight of the evidence
Evaluating probability of μ	(I)	(3)	Weak	Moderate
	(II)	(1), (2), (10)	Moderate	Moderate
	(III)	(4), (5), (11)	Strong	Moderate
	(IV)	(6), (7)	Strong	Moderate
Evaluating probability of σ	(I)	(3)	Weak	Moderate
	(II)	(1), (2), (10)	Moderate	Moderate
	(III)	(4), (5), (11)	Strong	Moderate
	(V)	(8)	Strong	Moderate
Evaluating probability of λ	(VI)	(9)	Moderate	Low

weight (i.e., Column 5) of the evidence. There are a total of 9 pieces of evidence on evaluating probability of μ, including evidence (3) with weak strength and moderate weight, evidence (1), (2), and (10) with moderate strength and moderate weight, and evidence (4), (5), (6), (7), and (11) with strong strength and moderate weight. 8 pieces of evidence are obtained for evaluating probability of σ, including evidence (3) with weak strength and moderate weight, evidence (1), (2), and (10) with moderate strength and moderate weight, and evidence (4), (5), (8), and (11) with strong strength and moderate weight. Only one piece of evidence is obtained for evaluating probability of λ, i.e., evidence (9), with moderate strength and low weight.

4.10.4 Results of Subjective Probability Assessment

Based on the relevant evidence on each sub-objective (see Tables 4.7 and 4.8), percentiles of μ, σ, and λ are elicited from the assessor using the equivalent lottery method with the aid of the questionnaire shown in Appendix 4.1. Table 4.9 summarizes the percentiles of μ, σ, and λ obtained from the equivalent lottery method in Rows 2, 3, and 4, respectively. For the mean μ, 1 %, 25 %, 50 %, 75 % and 99 % percentiles are obtained from the equivalent lottery method according to the nine pieces of evidence shown in Table 4.8, and they are $\mu_{0.01} = 28.0°, \mu_{0.25} = 33.0°, \mu_{0.5} = 36.0°, \mu_{0.75} = 38.0°$, and $\mu_{0.99} = 45.0°$. For the standard deviation σ, 1 %, 50 %, and 99 % percentiles are evaluated using the eight pieces of evidence shown in Table 4.8, and they are $\sigma_{0.01} = 1.0°, \sigma_{0.5} = 2.5°$, and $\sigma_{0.99} = 5.5°$. The information on λ is much less than that of μ and σ (see Table 4.8). Therefore, only

Table 4.9 Summary of percentiles of the mean, standard deviation, and correlation length for scenario II

Cumulative probability	0.01	0.25	0.5	0.75	0.99
Mean μ (°)	28.0	33.0	36.0	38.0	45.0
Standard deviation σ (°)	1.0	N/A	2.5	N/A	5.5
Correlation length λ (m)	2.0	N/A	N/A	N/A	6.0

1 % and 99 % percentiles of λ are evaluated, and they are $\mu_{0.01} = 2.0\,\mathrm{m}$ and $\mu_{0.99} = 6.0\,\mathrm{m}$. Using the simplified method and the least squares regression method described in Sect. 4.7.4.1, prior distributions of μ, σ, and λ are obtained from their respective percentiles, as discussed in the following three subsections.

4.10.4.1 Prior Distribution of the Mean μ

Figure 4.6a shows the CDF of μ obtained from the simplified method and the least squares regression method by a solid line with open circles and a dark solid line, respectively. By the simplified method, the CDF of μ increases linearly between the adjacent percentiles (i.e., adjacent open circles). By the least squares regression method, a Gaussian CDF with a mean of 35.7° and standard deviation of 3.7° is obtained, and it is in good agreement with the CDF (i.e., the line with open circles) obtained from the simplified method. Figure 4.6a also includes the 95 % confidence interval of the CDF obtained from the least squares regression method by dashed lines. The five percentiles of μ obtained from the SPAF fall within the 95 % confidence interval of the Gaussian CDF. Therefore, the Gaussian CDF represents the assessment outcomes (i.e., percentiles of μ) from the SPAF reasonably well. Figure 4.6b shows the PDF of μ obtained from the simplified method and the least squares regression method by a histogram with four bins and a dark solid line, respectively. The Gaussian PDF of μ obtained from the least squares regression method compares favorably with the histogram of μ obtained from the simplified method. Both can be used as the prior distribution of μ in the Bayesian framework developed in Chap. 3.

4.10.4.2 Prior Distribution of the Standard Deviation σ

Figure 4.7a shows the CDF of σ obtained from the simplified method by a solid line with open circles. The CDF value of σ increases linearly from 0.01 to 0.5 as σ increases from 1.0° to 2.5° and then increases linearly from 0.5 to 0.99 as σ increases from 2.5° to 5.0°. Figure 4.7b shows the PDF of σ obtained from the simplified method by a histogram with two bins (i.e., from 1.0° to 2.5° and from 2.5° to 5.5°), and the PDF values of σ in the two bins are about 0.33 and 0.16,

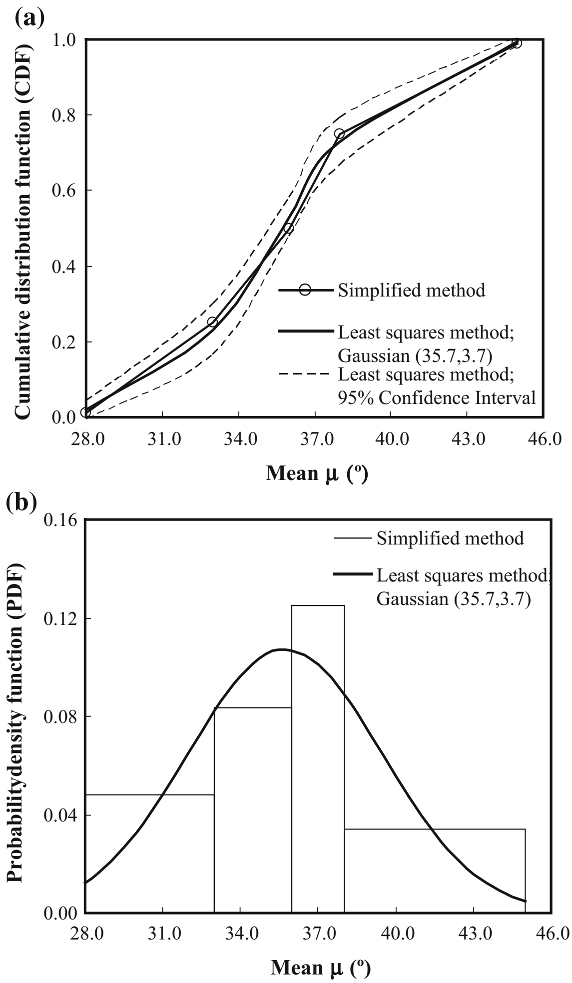


Fig. 4.6 Prior distribution of the mean, μ, in scenario II. **a** Cumulative distribution function (CDF) of the mean, μ. **b** Probability density function of the mean, μ

respectively. The PDF of σ (see Fig. 4.7b) can be taken as the prior distribution of σ in the Bayesian framework developed in Chap. 3.

4.10.4.3 Prior Distribution of the Correlation Length λ

Figure 4.8a shows the CDF of λ obtained from the simplified method by a solid line with open circles. The CDF value of λ increases linearly from 0.01 to 0.99 as λ increases from 2.0 to 6.0 m. Figure 4.8b shows the PDF of λ obtained from the simplified method by a histogram with only one bin (i.e., a uniform distribution with a range from 2.0 to 6.0 m), and the PDF value of λ is about 0.25. The probability distribution of λ obtained in this scenario remains the same as that

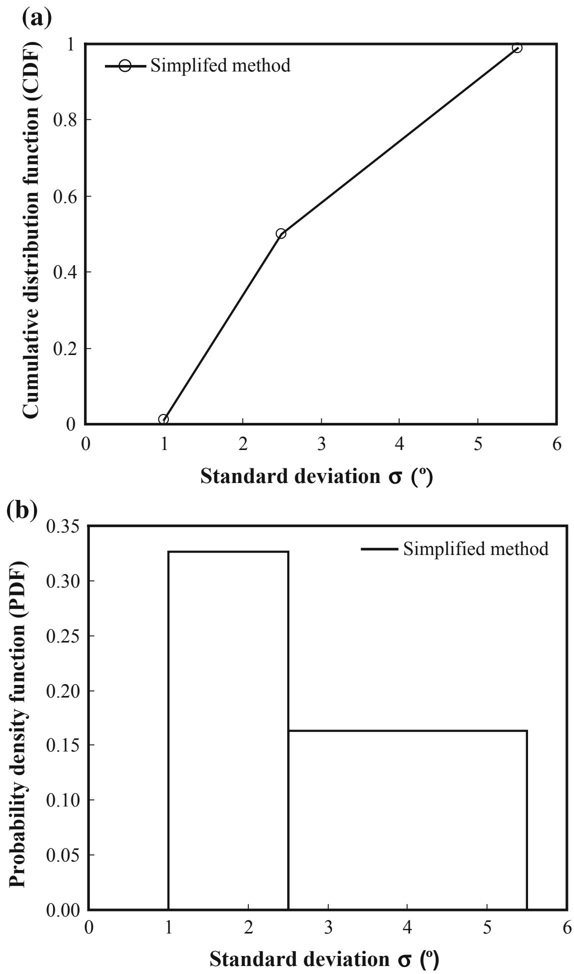


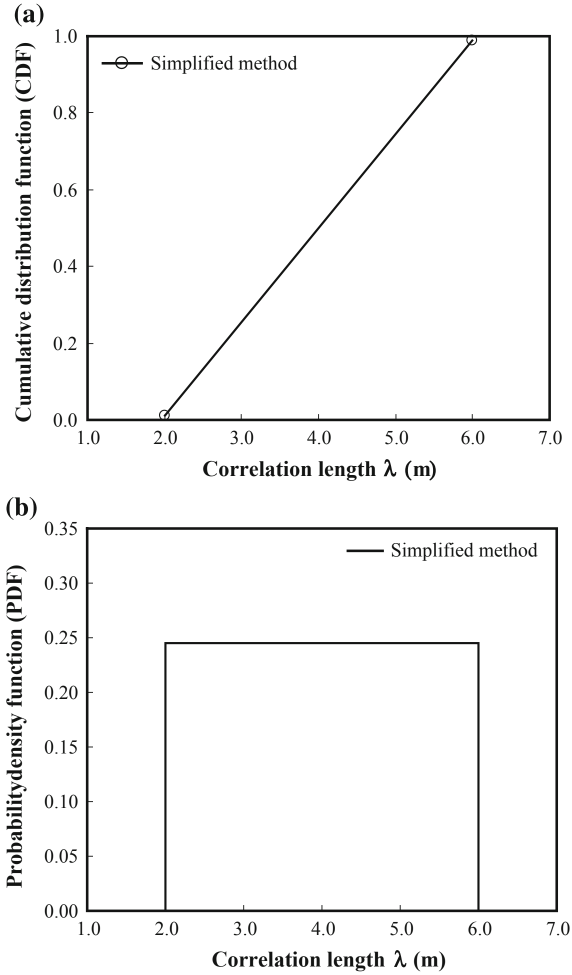
Fig. 4.7 Prior distribution of the standard deviation, σ, in scenario II. **a** Cumulative distribution function (CDF) of the standard deviation, σ. **b** Probability density function (PDF) of the standard deviation, σ

obtained in scenario I. It is not surprising to see this because the same information (i.e., a possible range of λ from 2.0 to 6.0 m) on λ is used in both scenarios. Similar to scenario I, the uniform PDF of λ (see Fig. 4.8b) can be taken as the prior distribution of λ in the Bayesian framework developed in Chap. 3.

4.10.5 Final Confirmation

All the areas under the respective PDFs (see Figs. 4.6b, 4.7b, and 4.8b) of μ, σ, and λ are summed up to unity. This is consistent with the basic probability axiom that the integration on a PDF is unity. The PDF value of μ in the bin from 36.0° to 38.0° is

Fig. 4.8 Prior distribution of the correlation length, λ, in scenario II. **a** Cumulative distribution function (CDF) of the correlation length, λ. **b** Probability density function (PDF) of the correlation length, λ



greater than that of μ in the other bins, as shown in Fig. 4.6b. This is consistent with the information provided by evidence groups (III) and (IV) (see Tables 4.7 and 4.8), both of which have strong strength and moderate weight. Although the information provided by evidence groups (I) and (II) suggests that μ is 35.0° or varies from 32.0° to 35.0° (see Sect. 4.10.3.2), they have relatively weak strength (i.e., weak and moderate) compared with evidence groups (III) and (IV) (see Table 4.8). The PDF of μ reflects properly the information provided by prior knowledge.

The PDF value of σ in the bin from 1.0° to 2.5° is greater than that of σ in the bin from 2.5° to 5.5°. This is consistent with the information provided by evidence groups (I), (II), and (III) (see Sect. 4.10.3.2). Although the evidence group (V) suggests strongly that σ varies from 3.7° to 5.5°, the evidence group (V) only contains one piece of evidence (see Table 4.7) and should not be overemphasized.

The PDF of σ reflects properly the information provided by prior knowledge. Examination of the evidence is performed cautiously. It is found that no evidence is overlooked or overemphasized and the weight and strength of the evidence are properly taken into account when determining the percentiles of μ, σ, and λ.

It is also noted that the prior distributions of μ and σ (Figs. 4.6 and 4.7) obtained in this scenario are more informative than those obtained in scenario I (Figs. 4.3 and 4.4). This is reasonable in the sense that more information on μ and σ is used in this scenario (see Tables 4.7 and 4.8). In addition, the ranges of μ (i.e., from 28.0° to 45.0°), σ (i.e., from 1.0° to 5.5°), and λ (i.e., from 2.0 to 6.0 m) obtained from the SPAF are generally consistent with the actual belief of the assessor. The outcomes obtained from the SPAF are then confirmed. The probability distributions (see Figs. 4.6, 4.7 and 4.8) of μ, σ, and λ reflect the confidence levels of prior knowledge on them, respectively, and quantify properly the information provided by the prior knowledge.

4.11 Summary and Conclusions

This chapter proposed a subjective probability assessment framework (SPAF) based on a stage cognitive model of engineers' cognitive process. The SPAF assists engineers in utilizing prior knowledge in a relatively rational way and expressing quantitatively their engineering judgments in a probabilistic manner. The assessment outcomes (e.g., probability distributions) obtained from the SPAF are then taken as the prior distribution in the Bayesian framework developed in Chap. 3.

The SPAF consists of five steps: specification of assessment objectives (i.e., the soil property and its statistics of interest), collection of relevant information and preliminary estimation, synthesis of the evidence, numerical assignment, and confirmation of assessment outcomes. The steps of the proposed SPAF are corresponding to the stages of cognitive process of engineers. By this means, engineers can formulate their engineering judgments naturally using prior knowledge and express quantitatively the engineering judgments using subjective probability with relative ease. Several suggestions were provided for each step to assist engineers in utilizing prior knowledge in a relatively rational way and reducing the effects of cognitive biases and limitations during subjective probability assessment.

The proposed SPAF is applied to characterize probabilistically the sand effective friction angle at a US National Geotechnical Experimentation Site (NGES) at Texas A&M University, and it is illustrated under two scenarios: one with sparse prior knowledge and the other with a reasonable amount of prior knowledge. It is shown that the SPAF is applicable for both scenarios. When the prior knowledge is sparse, the prior distribution obtained from the proposed approach is relatively uninformative (e.g., uniform distributions). As the information provided by the prior knowledge improves, the proposed approach provides informative prior distribution. The prior distribution obtained from the SPAF quantifies properly the information provided by the prior knowledge.

Appendix 4.1: Questionnaire for Implementing the Equivalent Lottery Method

This appendix provides a questionnaire for implementing the equivalent lottery method. The questionnaire starts with a question (i.e., Q1), that is used to determine a reference prize for the equivalent lottery method, followed by the second and third questions (i.e., Q2 and Q3) for determining 1 % and 99 % percentiles (i.e., $\theta_{i,0.01}$ and $\theta_{i,0.99}$) of the variable θ_i concerned, respectively. Then, the fourth question (i.e., Q4) can be used to estimate the 50 % percentile (i.e., $\theta_{i,0.5}$) of θ_i if sufficient information on θ_i is available. The questionnaire can be continued to determine percentiles of θ_i progressively until engineers believe that there is no sufficient information on θ_i to balance the two lotteries in the equivalent lottery method for a given range of θ_i.

Questionnaire

Q1: What is the prize that you want recently? Please write it down.
Answer: A_1

Q2: What is the minimum possible value of θ_i?
Answer: A_2

Q3: What is the maximum possible value of θ_i?
Answer: A_3

Q4: There are two lotteries as follows.

Lottery 1:

Win A_1 if $A_2 \le \theta_i \le a_s$ occurs.
Win nothing if $a_s < \theta_i \le A_3$ occurs.

Lottery 2:

Win A_1 or nothing equally likely.

Please adjust the value a_s from A_3 to A_2 gradually until you feel indifferent between the two lotteries. Please write down the resulting value of a_s and denote it by A_4.

Note that the questionnaire shall be continued to determine percentiles of θ_i progressively using different ranges of θ_i in the equivalent lottery method if there is sufficient information on θ_i to balance the two lotteries for a given range of θ_i.

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