Chapter 5
A Brief Review of Relativistic Gravitational Collapse

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Abstract We review here the basic setup to describe complete gravitational collapse of massive bodies within the general theory of relativity. We derive Einstein's equations describing collapse and solve them in some simple well-known toy models. We study the final outcome of collapse and the quantities that describe the formation of trapped surfaces and of the central singularity.

5.1 Introduction

General relativity became an essential part of the curricula of astrophysicists nearly 50 years after it was first developed by Albert Einstein, when new ultra-dense objects such as pulsars and highly energetic phenomena such as quasars were discovered. By 1963, it was clear that general relativity was necessary to understand those phenomena and that gravitational collapse played a crucial role for many astrophysical scenarios. Nevertheless, the seeds of our modern understanding of stellar collapse have deeper roots. It was Chandrasekhar, back in 1931, who used special relativity to evaluate the pressure necessary to overcome the electron degeneracy in a star and derived the famous upper mass limit for a stable white dwarf. He wrote: *"...the life history of a star of small mass must be essentially different from the life history of a star of large mass. For a star of small mass the natural white dwarf stage is an initial step towards complete extinction. A star of large mass cannot pass into the white dwarf stage, and one is left speculating on other possibilities"* [8]. The "other possibilities" to which Chandrasekhar referred today are called neutron stars and black holes.

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In 1939, Oppenheimer and Volkov performed a similar calculation using neutrons instead of electrons and general relativity instead of special relativity [39]. They concluded that any body with a large enough mass would not be able to sustain its own gravity and undergo complete collapse, although at the time it was not clear what the final state would be. They wrote: *"...actual stellar matter after the exhaustion of thermonuclear sources of energy will, if massive enough, contract indefinitely, although more and more slowly, never reaching equilibrium."* Soon after, Oppenheimer and Snyder and independently Datt developed the first exact solution of Einstein's equations describing a collapsing spherical cloud of non-interacting particles [11, 38]. The end state of such collapse model is a Schwarzschild black hole.

The existence of black holes as astrophysical objects was just a conjecture fifty years ago, while today is almost unanimously accepted by the community of astrophysicists. Nevertheless, the theoretical paradigm upon which the whole theory of black hole formation relies is not much different from the original Oppenheimer–Snyder–Datt (OSD) collapse model. And while our physical knowledge of astrophysical phenomena has progressed enormously in the last fifty years, our theoretical understanding of how black holes form is still very much rooted in simple toy models such as OSD. The reason for this relies mostly in the immense difficulty that one encounters when trying to solve analytically Einstein's equation in more general and physically relevant cases. Also, the fact that we do not know much about the behavior of matter in the strong field regime contributes in making our present theoretical understanding very limited.

On the other hand, from an experimental perspective, new missions and observatories are due to come online in the near future, and there is great hope that they will produce, among other things, an enormous amount of data on gravitational collapse and black hole formation. Astrophysicists of tomorrow will be able to rely on photons, neutrinos, and gravitational waves in order to study and understand what happens at the end of the life cycle of a star. This is usually called multimessenger astronomy. One of the key questions they will have to address is whether white dwarves, neutron stars, and black holes are the only possible objects that are left after a star dies. At present, it is natural to ask whether it is possible that there exists some yet unknown state of matter beyond the neutron degeneracy limit and capable of producing stable ultra-dense remnants. The purpose of this chapter is to pave the way for astrophysicists toward a broad theoretical understanding of the processes that lead to the formation of black holes. We do so by reviewing the paradigm for gravitational collapse in general relativity (GR) and the most fundamental analytical results that were obtained in the field.

The chapter is structured as follows: In Sect. 5.2, we derive the set of differential equations that are used to describe collapse. In Sect. 5.3, we discuss how the collapsing "star" can be matched to a vacuum exterior. Section 5.4 is devoted to the discussion of the conditions for the model to be physically viable. In Sect. 5.5, the apparent horizon and the singularity curve are defined. Section 5.6 presents the simplest solution for the homogenous dust collapse model, while in Sect. 5.7 inhomogeneous dust and homogeneous perfect fluid models are briefly outlined. Section 5.8 is devoted to discussing how the mathematical models can be useful for astrophysics, and finally, in Sect. 5.9, some possible future directions of investigation are discussed.

5.2 Einstein's Equations for the Collapsing Interior

Stars are supported in equilibrium by the balance of the gravitational attraction that pulls inward and the push outward generated by nuclear reactions happening at their center. When a star exhausts the nuclear fuel that was keeping it stable, it implodes under its own weight. At this point, the future evolution of the star depends on its mass. If a star is sufficiently massive, then there is no known force in nature capable of halting collapse. These stars end their lives forming black holes. In order to be able to describe the final stages of collapse, where the gravitational field becomes extremely large, we must use general relativity. Therefore, it is useful to begin our discussion by understanding what is a black hole in general relativity. The simplest and most intuitive definition of a black hole is that of a space-time singularity surrounded by an event horizon. Clearly, there are two elements that are crucial to our definition of a black hole: the singularity and the event horizon. The event horizon acts like a two-dimensional one-way membrane that lets particles and light enter while not letting anything exit. The singularity, strictly speaking, is not a part of the space-time, and it is the boundary that marks the geodesic incompleteness of all paths for particles that enter the horizon. The horizon for a non-rotating Schwarzschild black hole is located at a radius $R_{\rm Sch} = 2GM_{\rm S}/c^2$, where $M_{\rm S}$ is the black hole's mass, G is the Newton's constant, and c is the speed of light. The singularity is ideally "located" at the center of symmetry of the system. Intuitively, we can see that the black hole forms once enough mass is concentrated within a sphere of a small enough radius. Once matter is trapped inside the horizon, it can only fall toward the singularity (if there is no rotation). In principle, the equations of general relativity can be very difficult to solve; therefore, in order to describe the process by which all the matter in the star falls within the horizon radius thus forming a black hole, we need a mathematical framework that is simple enough to solve the equations but that still retains the most important physical features. In the following, we will neglect all the physical processes that happen in the cloud except gravity, we will assume that the cloud is perfectly spherically symmetric and not rotating, and we will assume that the exterior of the cloud is vacuum. Also, we shall consider here only extremely simplified fluid models to describe the state of matter of the collapsing star. Finally, it is custom to make use of natural units, thus setting $G = c = 1$.

5.2.1 Co-moving Coordinates

Our aim is to solve the system of Einstein's equations for a spherical collapsing matter cloud. As it is well known, Einstein's equations possess two distinct parts that both require some assumptions in order to allow us to find physically meaningful solutions. On the left-hand side, we have the geometrical part of the set of equations that is given by the Einstein tensor. This is determined once we know the metric for the space-time. As said, we will consider here only spherically symmetric, non-rotating space-times. A space-time is said to be spherically symmetric if the metric remains invariant under the group of spatial rotations $SO(3)$. This means that we can define the two-dimensional metric induced on the unit two-sphere as

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{5.1}$$

and define a function R for which $4\pi R^2$ represents the area of each two-sphere in the space-time. The full four-dimensional metric then can be written as

$$ds^2 = g_{AB}dx^A dx^B + R(x_A, x_B)^2 d\Omega^2, \tag{5.2}$$

with $A, B = 0, 1$. We can then introduce the coordinates $t = x_0$ and $r = x_1$ that diagonalize the two-dimensional part of the metric g_{AB} and write the most general spherically symmetric line element in the simple form

$$ds^2 = -e^{2\lambda}dt^2 + e^{2\psi}dr^2 + R^2 d\Omega^2 \tag{5.3}$$

where the functions λ, ψ, and R do not depend on the coordinates θ and ϕ. In the following, we will use a dot to express derivatives with respect to t and a prime to express derivatives with respect to r, thus writing $\dot{X} = dX/dt$, $X' = dX/dr$. The above coordinate system for which the metric is diagonal is called co-moving because one can think of the labels t and r as "attached" to each collapsing particle. Then, the functions λ, ψ, and R depend only on r and t. In this reference frame, the fluid is instantaneously at rest and its four-velocity u^μ is $u^t = e^{-\lambda}$, $u^r = u^\theta = u^\phi = 0$.

In order to describe the collapse of a spherically symmetric massive object such as a star, we need to solve Einstein's equations for a space-time described by (5.3) coupled to an energy momentum tensor describing a realistic fluid source. The energy momentum tensor is the right-hand side of Einstein's equations and for a fluid source in the co-moving frame can be written as

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_\theta & 0 \\ 0 & 0 & 0 & p_\theta \end{pmatrix}$$

A perfect fluid is an idealized fluid where no shear stresses, no viscosity, and no heat conduction are present. It can be characterized by its mass density and isotropic pressures alone. Isotropic pressure means that the radial pressure equals the tangential pressure, and this implies $p_r = p_\theta = p$. The energy density is the energy per unit volume of the fluid in the local rest frame with four-velocity u^μ and the energy momentum tensor for a perfect fluid can then be written as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \tag{5.4}$$

Note that in general relativity, the pressure contributes to the gravitational field, and therefore, the total gravitational energy need not be conserved. Nevertheless, the baryon number is conserved. For a gas of non-interacting particles, the so-called dust, the pressure vanishes and we can set $p = 0$. This is the simplest fluid model that can be considered.

5.2.2 Misner–Sharp Mass

The metric (5.3) can be used to describe static sources in equilibrium in the case when λ, ψ, and R do not depend on t. These are static objects with non-vanishing energy momentum. In this case, the area radius R can be used as a radial coordinate setting $R = r$. The simplest interior solution of this kind is given by the constant density Schwarzschild interior and was found by Schwarzschild himself together with the more famous vacuum solution (see, e.g., [47]). For the constant density interior, one sets $\rho =$ const. and solves Einstein's equations that take the form of the famous Tolman–Oppenheimer–Volkov equation [44], to find $p(r)$. Then, the object's boundary r_b is determined by the condition that $p(r_b) = 0$. For a metric describing a static interior case, we can define a function $m(r)$ such that

$$g_{rr} = e^{2\psi(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{5.5}$$

It is easy to see that at the boundary of the static object, the function $m(r)$ must become equal to the Schwarzschild parameter M_S that describes the total mass of the star, and therefore, we can interpret $m(r)$ as describing the amount of matter enclosed within the radius r. We can generalize the above expression in the dynamical case by introducing a function $U(r, t)$ as

$$U = u^\mu \frac{dR(r,t)}{dx^\mu} = e^{-\lambda}\dot{R}, \tag{5.6}$$

then, we get

$$g_{rr} = e^{2\psi(r,t)} = \left(1 + U^2 - \frac{2m(r,t)}{R}\right)^{-1} R'^2. \tag{5.7}$$

which reduces to the static case for $R = r$, so that $R' = 1$ and $\dot{R} = 0$. The Misner–Sharp mass $F(r, t)$ is then defined from $1 - F/R = g_{\mu\nu}\nabla^\mu R \nabla^\nu R$ and it is given by

$$F(r,t) = 2m(r,t) = R(1 - e^{-2\psi}R'^2 + e^{-2\lambda}\dot{R}^2). \tag{5.8}$$

In analogy with what was said before, we can interpret the Misner–Sharp mass as describing the amount of matter enclosed within the radius r at the time t [35].

5.2.3 Einstein's Equations

Einstein's equations couple the space-time geometry given by the metric $g_{\mu\nu}$ appearing in the Einstein's tensor $G_{\mu\nu}$ for the line element (5.3) to the matter content of the collapsing cloud given by the energy momentum tensor (5.4). Einstein's equations take the usual form

$$G_{\mu\nu} = \mathrm{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{R} = T_{\mu\nu}, \tag{5.9}$$

where $\mathrm{R}_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar and where we have absorbed the constant factor $8\pi k$ into the definition of $T_{\mu\nu}$. Then, Einstein's tensor for the collapsing system is given by

$$G^t_t = -\frac{F'}{R^2R'} + \frac{2\dot{R}e^{-2\lambda}}{RR'}\left(\dot{R}' - \dot{R}\lambda' - \dot{\psi}R'\right), \tag{5.10}$$

$$G^r_r = -\frac{\dot{F}}{R^2\dot{R}} - \frac{2R'e^{-2\psi}}{R\dot{R}}\left(\dot{R}' - \dot{R}\lambda' - \dot{\psi}R'\right), \tag{5.11}$$

$$G^t_r = -e^{2\psi-2\lambda}G^r_t = \frac{2e^{-2\lambda}}{R}\left(\dot{R}' - \dot{R}\lambda' - \dot{\psi}R'\right), \tag{5.12}$$

$$G^\theta_\theta = G^\phi_\phi = \frac{e^{-2\psi}}{R}\left((\lambda'' + \lambda'^2 - \lambda'\psi')R + R'' + R'\lambda' - R'\psi'\right) + \\ -\frac{e^{-2\lambda}}{R}\left((\ddot{\psi} + \dot{\psi}^2 - \dot{\lambda}\dot{\psi})R + \ddot{R} + \dot{R}\dot{\psi} - \dot{R}\dot{\lambda}\right). \tag{5.13}$$

These equations need to be supplemented with one more equation coming from the conservation of energy momentum that in general relativity comes as a consequence of the fact that the connection is metric and which can be written as

$$\nabla_\mu T^\mu_\nu = 0. \tag{5.14}$$

Then, in the simple case of pressureless (i.e., dust) collapse, and by making use of the definition of the Misner–Sharp mass given in Eq. (5.8), the first two equations of the above system simplify to

$$\rho = -G^t_t = \frac{F'}{R^2R'}\ , \tag{5.15}$$

$$p = 0 = G^r_r = -\frac{\dot{F}}{R^2\dot{R}}, \tag{5.16}$$

the third and fourth combine to give

$$\dot{R}' = \dot{R}\lambda' + \dot{\psi} R' = 0, \tag{5.17}$$

and the conservation of energy momentum (5.14) becomes

$$\rho\lambda' = 0. \tag{5.18}$$

From Eq. (5.16), we see that for dust, we must have $\dot{F} = 0$ which implies $F = F(r)$. This shows that the amount of matter enclosed within the co-moving radius r does not change with time. In other words, during collapse, there is no inflow or outflow of matter across any co-moving shell r. From Eq. (5.18), since the energy density is nonzero, we see that we must have $\lambda' = 0$, which implies $\lambda = \lambda(t)$. Now, we can define a new co-moving time coordinate $\tilde{t}$ by rescaling in such a way that

$$\frac{d\tilde{t}}{dt} = e^{\lambda}, \tag{5.19}$$

and therefore obtain

$$-e^{2\lambda} dt^2 = -e^{2\lambda} \left(\frac{dt}{d\tilde{t}}\right)^2 d\tilde{t}^2 = -d\tilde{t}^2. \tag{5.20}$$

This means that there is always the gauge freedom to fix the co-moving time t such that $\lambda = 0$, and in the following, we shall take t as such a gauge. Finally, Eq. (5.17) can be written as

$$\frac{\dot{R}'}{R'} = \dot{\psi}, \tag{5.21}$$

from which we get

$$R' = e^{g(r)+\psi}. \tag{5.22}$$

We call $f(r) = e^{2g(r)} - 1$ and the Misner–Sharp mass equation (5.8) can be rewritten in the form of the equation of motion of the system as

$$\dot{R}^2 = \frac{F(r)}{R} + f(r). \tag{5.23}$$

Once Eq. (5.23) is solved to give $R(r, t)$, the whole system of Einstein's equations is solved. The metric becomes

$$ds^2 = -dt^2 + \frac{R'^2}{1+f} dr^2 + R^2 d\Omega^2. \tag{5.24}$$

This is the well-known Lemaitre–Tolman–Bondi (LTB) space-time [6, 33, 43]. We see that the whole problem allows for two free functions of r, namely F and f, to be specified at will. As said, the function F can be thought of as representing the matter profile within the radius r, while from the above line element, the function f can be thought of as an energy profile describing the spatial curvature of the space-time. Then, provided that F and f are sufficiently regular, a unique solution of the equation of motion (5.23) exists for each regular initial condition $R_i = R(r, t_i)$.

5.3 Matching with an Exterior Metric

The metric given in Eq. (5.24) describes the dynamical collapse of a dust sphere. This can be thought of as describing the final stages of the life of a star, provided that gravity prevails on all other forces (thus allowing us to neglect any effect coming from the microphysics of the collapsing fluid) and that the collapsing cloud has a boundary. In the co-moving frame, this boundary can be identified with the surface given by the co-moving radius $r = r_b$. We consider the exterior of the collapsing dust cloud to be static and vacuum. Then, Birkhoff's theorem implies that it must be a portion of the Schwarzschild space-time. The exterior Schwarzschild solution can readily be derived from the metric (5.3) by including the further assumptions of staticity and vanishing of energy momentum tensor. A space-time is said to be stationary if it possess a time-like Killing vector, ∂_t. In such a case, the metric (5.3) becomes invariant under translations in t, and this is reflected in the metric functions λ, ψ, and R that do not depend on t. Further, a space-time is said to be static if the time-like Killing vector is orthogonal to the hypersurfaces of constant t. If we further impose that the energy momentum tensor is that of vacuum, namely $T_{\mu\nu} = 0$, we find that the only static spherically symmetric vacuum solution of Einstein's field equations can be written in the form

$$ds^2 = -\left(1 - \frac{2\mathrm{M_S}}{r_\mathrm{s}}\right) dt_\mathrm{s}^2 + \left(1 - \frac{2\mathrm{M_S}}{r_\mathrm{s}}\right)^{-1} dr_\mathrm{s}^2 + r_\mathrm{s}^2 d\Omega^2. \tag{5.25}$$

which is the well-known Schwarzschild line element expressed in Schwarzschild coordinates $\{t_\mathrm{s}, r_\mathrm{s}, \theta, \phi\}$. As said, the Schwarzschild metric has a singularity at the center $r_\mathrm{s} = 0$. One way to determine the presence of singularities in solutions of Einstein's field equations is by inspecting curvature invariants looking for divergences. The Kretschmann scalar is one of these invariants and it is defined starting from the Riemann tensor as $\mathscr{K} = \mathrm{R}_{\mu\nu\sigma\delta}\mathrm{R}^{\mu\nu\sigma\delta}$. The Kretschmann scalar for Schwarzschild is

$$\mathscr{K} = 12\frac{4{\mathrm{M_S}}^2}{r_\mathrm{s}^6}, \tag{5.26}$$

from which we see that the null surface $r_s = 2M_S$ is not singular; in fact, it is the event horizon [15], and the singularity is located at $r_s = 0$.

The global solution for the collapsing star is obtained by matching the collapsing interior given by the metric (5.24) to the vacuum exterior across a shrinking boundary surface Σ. Mathematically, "matching" means that the induced metric on the boundary hypersurface Σ must be the same on both sides. Also, the rate of change of the unit normal to Σ must be the same on both sides [23]. Let us label the interior metric with $(-)$ and the exterior metric with $(+)$. Then, the two line elements can be written as

$$ds^2_\pm = g^\pm_{\mu\nu} dx^\mu_\pm dx^\nu_\pm. \tag{5.27}$$

The boundary hypersurface is implicitly defined on each side by $\Phi^\pm(x^\mu_\pm(y^a)) = 0$, and the induced metric on Σ can be written as

$$ds^2_\Sigma = \gamma_{ab} dy^a dy^b, \tag{5.28}$$

where $a = 1, 2, 3$. Now define the three basis 4-vectors tangent to Σ as $e^\mu_{(a)} = \partial x^\mu / \partial y^a$. Then, the condition that the induced metric $\gamma^\pm_{ab} = g^\pm_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)}$ agrees on both sides is simply $\gamma^+_{ab} = \gamma^-_{ab}$. Define the unit normal to Σ as

$$n_\mu = \left(g^{\rho\sigma} \frac{\partial \Phi}{\partial x^\rho} \frac{\partial \Phi}{\partial x^\sigma} \right)^{-1/2} \frac{\partial \Phi}{\partial x^\mu}. \tag{5.29}$$

The extrinsic curvature (or second fundamental form) is defined as

$$K_{ab} = g_{\mu\nu} n^\mu \nabla_a e^\nu_{(b)}. \tag{5.30}$$

Then,

$$K^\pm_{ab} = \frac{\partial x^\mu_\pm}{\partial y^a} \frac{\partial x^\nu_\pm}{\partial y^b} \nabla_\mu n_\nu \ , \tag{5.31}$$

and continuity of K across Σ is given by

$$K^+_{ab} = K^-_{ab}. \tag{5.32}$$

In the case of spherical collapse of dust, we have that the boundary hypersurface, in the exterior, with coordinates $\{x^+\} = \{t_s, r_s, \theta, \phi\}$, is given by

$$\Phi^+ = r_s - R_b(t_s) = 0, \tag{5.33}$$

and in the interior, with coordinates $\{x^-\} = \{t, r, \theta, \phi\}$, is given by

$$\Phi^- = r - r_b = 0. \tag{5.34}$$

So that the induced metric on the boundary is

$$ds^2_\Sigma = -dt^2 + R_b(t)^2 d\Omega^2. \tag{5.35}$$

The Schwarzschild time t_s can be written as a function of the co-moving time t from

$$\frac{dt}{dt_s} = \sqrt{\left(1 - \frac{2M_S}{R_b}\right) - \left(1 - \frac{2M_S}{R_b}\right)^{-1}\left(\frac{dR_b}{dt_s}\right)^2}, \tag{5.36}$$

and the matching conditions for the continuity of the metric become

$$R_b(t_s) = R(r_b, t_s(t)), \tag{5.37}$$

$$F(r_b) = 2M_S. \tag{5.38}$$

Finally, continuity of K_{ab} follows identically from the matching conditions. Therefore, we see that the Misner–Sharp mass can be interpreted as the mass enclosed within the co-moving radius r and that on the boundary, it becomes proportional to the Schwarzschild mass M_S. Also, we see that the area function $R(r, t)$ in the interior at the boundary becomes the shrinking area radius in the Schwarzschild portion of the space-time. For more general collapse model, a matching to a suitable exterior space-time can also be defined (see, e.g., [13, 14, 27]).

5.4 Regularity, Scaling, and Energy Conditions

In order for the model to be physically acceptable, we need to choose an initial configuration that satisfies several conditions. The most important ones are regularity, which corresponds to requiring that the initial matter profiles do not present any singularities and are well behaved and the usual energy conditions, which in the dust case can be expressed via positivity of the energy density. Another requirement that is often imposed on the model is the absence of shell crossing singularities. These are caustics like singularities that are due to the overlap of infalling shells. Shell crossing singularities can possibly be removed by a suitable redefinition of the coordinates and generally do not represent a breakdown of the model.

5.4.1 Regularity and Scaling

We now investigate regularity of the matter profiles and the condition for avoidance of singularities at the initial time. In order to study these properties, we first need to express the area radius R, the Misner–Sharp F mass, and the velocity profile f in an appropriate gauge. As mentioned before, the Kretschmann scalar constitutes a valid tool to investigate the occurrence of singularities. In the case of the metric (5.24), this becomes

$$\mathscr{K} = 12\frac{F'^2}{R^4 R'^2} - 32\frac{FF'}{R^5 R'} + 48\frac{F^2}{R^6}. \tag{5.39}$$

We note here that with the present choice of the metric functions, one may be induced to think that the central curve $R = 0$ is always singular, including at the initial time. Nevertheless, this is not a physical singularity, as can be easily verified by evaluating the energy density, which turns out to be finite at the initial time. We notice then that there is a gauge degree of freedom in the scaling of R, namely in the way R is "measured" at the initial time that can be used to remove the above ambiguity. In fact, we can always choose arbitrarily the initial value of R. In the following, we choose this initial scaling condition as

$$R(r, t_i) = r. \tag{5.40}$$

From the choice of the initial data for R given in Eq. (5.40), we see that the gauge freedom allows us to define a scaling function $a(r, t)$ from the area function $R(r, t)$ as

$$R = ra(r, t). \tag{5.41}$$

Now, the scaling factor a is an a-dimensional quantity such that

- at the initial time, we have $a(r, t_i) = 1$,
- at the time of formation of the singularity t_{sing}, we have $a(r, t_{sing}) = 0$,
- collapse is given by $\dot{a} < 0$.

For dust, using Einstein's equation (5.15), the above scaling implies that the initial density must satisfy the following condition:

$$\rho(r, t_i) = \rho_i(r) = \frac{F'}{r^2} > 0. \tag{5.42}$$

Therefore, in order to avoid having ρ diverging at $r = 0$ at the initial time, we must impose a regularity condition on the Misner–Sharp mass. This is given by

$$F(r) = r^3 M(r), \tag{5.43}$$

with $M(r)$ non-diverging and sufficiently regular in the interval $[0, r_b]$. Generally, we assume that the function $M(r)$ can be written as a polynomial expansion in the vicinity of $r = 0$. In general, a physically viable density profile should be non-increasing radially outwards and therefore we must impose that the first non-vanishing term in the polynomial expansion of M is vanishing or negative in $r = 0$. It is reasonable to suppose that $M' \leq 0$ near the center. With the above scaling, the density becomes

$$\rho = \frac{3M + rM'}{a^2(a + ra')}. \tag{5.44}$$

If we add the further requirement that ρ must not present any cusps at $r = 0$, we must impose that $M'(r)$ vanishes in $r = 0$, and therefore, we must require $M'' \leq 0$ near $r = 0$. Note that in the simplest case of homogeneous dust, ρ does not depend on r and so $M(r)$ must be constant M_0. Then,

$$\rho(t) = \frac{3M_0}{a^3}, \tag{5.45}$$

and as a consequence, the scale factor also does not depend on r. With this choice of the scaling factor, it is easy to verify that the central density diverges only at the singularity. Also, we see that the Kretschmann scalar in the homogeneous case reduces to

$$\mathscr{K} = 60\frac{M_0^2}{a^6}, \tag{5.46}$$

and it is regular at the initial time, its value being $\mathscr{K}_i = 60M_0^2$. In general, for inhomogeneous dust, we have

$$\mathscr{K} = 12\frac{(3M + rM')^2}{a^4(a + ra')^2} - 32\frac{M(3M + rM')}{a^5(a + ra')} + 48\frac{M^2}{a^6}. \tag{5.47}$$

Note that by writing $\mathscr{K}$ in terms of M and a, we avoid the problem of divergence along the central line. In the new scaling along $r = 0$, we see that $\mathscr{K}$ diverges only for $a = 0$, thus showing the occurrence of the singularity. The curve $t_{sing}(r)$ for which $a(r, t_{sing}) = 0$ is the singularity curve which describes the time at which the shell r becomes singular. As a consequence of the fact that in the homogeneous case, a depends only on t, we see that for homogeneous dust, the singularity occurs at the same time t_{sing} for every co-moving shell r.

From the equation of motion (5.23), we see that at the initial time, the velocity of the infalling particles is given by

$$\dot{R}_i = -\sqrt{\frac{F}{r} + f}. \tag{5.48}$$

Given the fact that the choice of the free function F corresponds to fixing the initial density profile from the above equation, we see that fixing f corresponds to determining the initial velocity profile for the particles in the cloud. Now, by making use of the scaling above, we can rewrite Eq. (5.48) as

$$\dot{a}_i = -\sqrt{M + \frac{f}{r^2}}, \tag{5.49}$$

from which we see that in order to have a finite initial velocity at all radii, we must set a scaling for f as well. We shall take

$$f(r) = r^2 b(r), \tag{5.50}$$

with $b(r)$ a sufficiently regular function (again which can be given as a polynomial expansion near $r = 0$). To summarize, at the initial time, we have the freedom to specify three functions of r as follows:

- Choose an initial condition for the scaling $R(r, t_i) = R_i(r)$ or equivalently set the value of $a(r, t_i)$.
- Choose a mass function $F(r)$, or equivalently $M(r)$, which implies an initial density $\rho_i = F'/r^2$.
- Choose a velocity function $f(r)$, or equivalently $b(r)$, which implies the initial condition for the velocity $\dot{R}(r, t_i)$.

Then, the system is fully determined and the equation of motion can be written as

$$\dot{a}(r, t) = -\sqrt{\frac{M(r)}{a(r, t)} + b(r)}. \tag{5.51}$$

By solving the above equation for a, we completely solve the system of Einstein's equations. As said, homogeneous dust collapse is given by $\rho = \rho(t) = 3M_0/a^3$. Therefore, from the above, it follows that homogeneous dust collapse can be obtained from the following requirements:

- $a = a(t)$
- $M(r) = M_0 = \text{const.}$
- $b(r) = k = \text{const.}$

In this case, we can give a precise interpretation of the velocity profile if we imagine a dust cloud that extends to infinity. We can think at the constant k as representing the initial velocity of particles at spatial infinity, and we can characterize the geometry of the space-time based on the sign of k in the following way:

- $k = 0$ marginally bound collapse, corresponding to a flat geometry. Shells at radial infinity begin collapse with zero initial velocity.
- $k > 0$ unbound collapse, corresponding to a hyperbolic geometry. Shells at radial infinity have positive initial velocity.
- $k < 0$ bound collapse, corresponding to an elliptic geometry. Shells at radial infinity have negative initial velocity.

Note that if one wishes to have zero initial velocity $\dot{R}_i = 0$ for particles in the collapsing cloud with boundary, then the only possible choice is that of bound collapse with $M_0 = -k$.

5.4.2 Energy Conditions

Einstein's equations are often regarded as made of two different parts. The "golden" half, more elegant, is the left-hand side that contains the Einstein tensor and therefore encodes the information about the geometry of the space-time. The "wooden" half is the right-hand side that contains the energy momentum tensor and in principle should describe the physical properties of matter. As a matter of fact, it is generally practically impossible to fully describe all the properties of the matter fields in the energy momentum tensor, and therefore, one usually resorts to simplifications and averaged properties that are valid for macroscopic fields. Nevertheless, we must keep in mind that the behavior of matter under very strong gravitational fields is not known at present, and therefore, the description of classical macroscopic fluids that is valid in the weak field may not be enough when the curvature becomes very high. To simplify things, one usually imposes that certain inequalities be satisfied by the energy momentum tensor in order for the same to be considered physically viable [18]. The first and most commonly used inequality is the weak energy condition (w.e.c.). To satisfy the w.e.c., the energy momentum tensor must be given in such a way that $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$ for any time-like (and null) vector V^{μ}. This means that the energy density must be nonnegative in any reference frame. The energy momentum tensor for a fluid made of massive particles, with respect to some orthonormal basis, can always be written as $T^{\mu\nu} = diag\{\rho, p_1, p_2, p_3\}$. Then, the weak energy conditions in the co-moving frame can be written as

$$\rho \geq 0 \quad \rho + p_i \geq 0 \quad \text{with} \quad i = 1, 2, 3. \tag{5.52}$$

This is the less demanding of the energy requirements. The weak energy condition allows for violations of the conservation of baryon number as new particles can be created. Still, more stringent conditions can be imposed. If one desires to impose that the total amount of mass in the space-time is conserved, then one must impose the dominant energy condition (d.e.c.) which states that for every time-like vector V^{μ}, the energy momentum tensor must satisfy both $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$ and $T_{\mu\nu}V^{\mu}$ being a null or time-like vector. This means that not only the energy density is nonnegative in any frame, but also the flow of ρ must be locally not space-like. As a consequence, we get that in an orthonormal reference frame, the energy density must be greater than the pressures. Namely,

$$\rho \geq 0\,, \; -\rho \leq p_i \leq \rho \quad \text{with} \quad i = 1, 2, 3. \tag{5.53}$$

Note that if we define the speed of sound waves within the fluid travelling in the direction of p_i as $v_i = dp_i/d\rho$ ($i = 1, 2, 3$), then the d.e.c. does not allow for the speed of sound to be greater that the speed of light. It is a very reasonable assumption that is not implemented by the w.e.c.. A fluid that satisfies the d.e.c. obviously satisfies also the w.e.c.. Finally, let us briefly mention a third energy condition that can be imposed and that is not directly related to the previous two. This is the strong energy condition (s.e.c.), and for a perfect fluid in the co-moving frame, it is equivalent to requiring

$$\rho \geq 0\,,\ \ \rho + p \geq 0\,,\ \ \rho + 3p \geq 0. \tag{5.54}$$

In the following, we shall always require that the fluid satisfies the w.e.c. and whenever possible that it satisfies the d.e.c. as well.

5.4.3 Shell Crossing Singularities

From Eq. (5.47), we see that the Kretschmann scalar diverges when the central singularity forms, namely when $a = 0$, but also when $R' = 0$ if $M' \neq 0$. In this case, we speak of the occurrence of shell crossing singularities. These are true curvature singularities that arise from overlapping radial shells. At the shell crossing singularity, the radial geodesic distance between shells with radial coordinate r and $r + dr$ vanishes. These singularities are equivalent to caustics in wave propagation, and it is reasonable to assume that the space-time can be extended through the singularity by a suitable redefinition of the coordinates. This can also be seen from the fact that shell crossing singularities are gravitationally weak, meaning that geodesics reaching the singularity are not squeezed into a line (as is the case of the central singularity), and thus, observers at the shell crossing are not crushed (see, e.g., [21, 22, 32, 49, 50]). Nevertheless, in any collapse model, a condition that can be required is the absence of shell crossing singularities. In order to avoid shell crossing singularities, we can either impose $R' \neq 0$ or choose $M(r)$ in such a way that $M(r) = 0$ when $R' = 0$ so that $M'/R' < \infty$. During collapse, the mass function M is generally assumed to be positive; therefore, requiring the absence of shell crossing singularities is equivalent to requiring that R' is not vanishing. Note that since $R' = a + ra'$ in a neighborhood of the center, the condition can always be satisfied if $a' \neq 0$.

5.5 Trapped Surfaces and Singularities

In the Schwarzschild space-time, the surface $r_s = 2M_S$, known as the event horizon, is *"...a perfect unidirectional membrane: causal influences can cross it in only one direction"* [15]. The event horizon is the boundary of the region where light rays can not escape to infinity. At the horizon, the time-like Killing vector is null and outgoing null geodesics have zero radial velocity. Nevertheless, the event horizon is not a very useful concept for practical (i.e., astrophysical) purposes. In fact, the event horizon is a global property of the space-time which does not depend on the observer and its determination requires the knowledge of the entire future history of the space-time. What we need in order to be able to make experiments is a local approach to the definition of trapped surfaces that allows us to define when a co-moving observer that is collapsing with the cloud becomes causally disconnected from the outside universe. If matter is present, as in the case of the LTB metric given in Eq. (5.24), the event horizon is not the only possible horizon that can be defined and it is not the most useful concept to investigate the physics that occurs as the black hole forms. We want to know how and when the black hole forms during gravitational collapse. When light will be trapped by the gravitational field? The exterior region will become a black hole solution once the boundary surface $R_b(t)$ passes the Schwarzschild radius. What about the interior? Each collapsing shell will become causally disconnected from the outer universe at some point and will eventually fall into the central singularity. If we want to track the formation of the horizon inside the matter region, first we need to know what we mean by trapped surface in the interior.

Given a $3 + 1$ slicing of the space-time, consider the three-dimensional space-like slice. Then, a "trapped surface" is defined as a smooth closed two-surface in the slice whose future-pointing outgoing null geodesics have negative expansion. This means that all light rays, all null geodesics, emanating from the surface are pointing inward. The "trapped region" in the slice is then defined as the union of all trapped surfaces, and the "apparent horizon" is the outer boundary of the trapped region [7, 20, 40]. One intuitive way to understand the difference between the apparent horizon and the event horizon is to note that the event horizon is the surface at which any light ray directed outward can be initially outgoing and eventually become ingoing, thus falling back inwards at some later time, while the apparent horizon is the surface for which all light rays directed outwards are ingoing, thus directed inward at the time when they are emitted. In vacuum, the two surfaces coincide, and therefore, the apparent horizon and the event horizon in the Schwarzschild space-time are the same. Still, when matter is present, they can be different, as is the case for the LTB metric.

The apparent horizon in general need not be a null surface and it always lies inside the event horizon. Nevertheless, it is the apparent horizon that determines the trapped region in the collapsing cloud. It is a local property of the space-time and is observer dependent, and therefore, it can be experimentally tested, while the event horizon may be undetectable. To understand this, imagine the situation of a thin spherical shell separating a vacuum Minkowski interior from a vacuum Schwarzschild exterior. The shell may be time-like or light-like. Let the shell have total mass M_S and collapse under its own gravity (see Fig. 5.1). As the shell collapses, an event horizon will form at $R = 0$ at the time $t = t_0$. The event horizon curve will expand to larger radii and eventually match the Schwarzschild radius $R = 2M_S$ in the exterior at $t = t_{Sch}$. An observer living at a fixed radial coordinate $R = R_1$ inside the Minkowski region will experience the event horizon passing through him at the time t_1 but will not have any way to detect it. This shows how in principle we can have event horizons where we would not expect and why local experiment cannot detect the presence of an event horizon. For this reason, the apparent horizon is a more useful tool to study the trapped region that develops during the formation of a black hole.

For the spherical dust collapse model, the apparent horizon is the surface for which the surface $R(r, t) = \text{const.}$ is null. This means

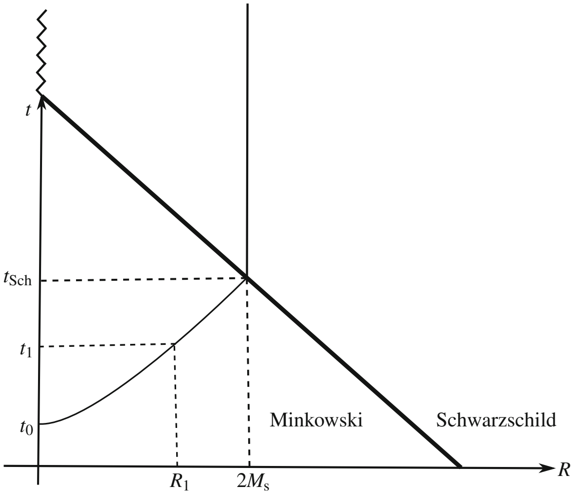

Fig. 5.1 Collapse of a thin spherical null shell (*thick line*) separating a flat vacuum interior from a Schwarzschild exterior. The angular coordinates θ and ϕ are suppressed, so every radius R corresponds to a spherical surface. The *thin line* represents the event horizon, which forms at the center of symmetry of the system at the time t_0 and expands toward bigger radii. As the shell crosses the Schwarzschild radius $2M_S$, the horizon settles to the usual event horizon of a static black hole. Observers living in the interior at a fixed radius R_1 would fall inside the trapped region at the time t_1 but would have no way of detecting the horizon that is passing through them

$$g^{\mu\nu}(\partial_\mu R)(\partial_\nu R) = 1 + f - e^{-2\lambda}\dot{R}^2 = 0. \tag{5.55}$$

From the definition of the Misner–Sharp mass in Eq. (5.8), we get that the condition for the formation of trapped surfaces can be expressed as

$$1 - \frac{F}{R} = 1 - \frac{r^2 M}{a} = 0. \tag{5.56}$$

This condition can be viewed as the implicit definition of the curve $t_{ah}(r)$ for which

$$a(r, t_{ah}(r)) = r^2 M(r). \tag{5.57}$$

The above curve is called the apparent horizon curve and describes the co-moving time t at which the co-moving shell r becomes trapped.

As we have seen, the other important curve to describe the formation of the black hole at the end of collapse is the singularity curve $t_{sing}(r)$. This is the curve that describes the co-moving time t at which the co-moving shell r becomes singular. This curve represents the limit of the space-time manifold, and all geodesics inside the trapped region must terminate at the singularity. From the condition of formation of the singularity, we see that the curve $t_{sing}(r)$ is given implicitly by

$$a(r, t_{sing}(r)) = 0. \tag{5.58}$$

We have already seen that in the case of homogeneous dust, we must have $t_{sing} =$ const., which means that all shells become singular at the same co-moving time.

5.6 Homogeneous Solutions

The equation of motion (5.51) for homogeneous collapse is written as

$$\dot{a} = -\sqrt{\frac{M_0}{a} + k}. \tag{5.59}$$

We can characterize the geometry depending on the sign of the free parameter k by introducing the following change of coordinates

$$r = S_k(\chi) = \begin{cases} \sinh\chi & \text{if } k = +1, \text{ “hyperbolic” region,} \\ \chi & \text{if } k = 0, \text{ “flat” region,} \\ \sin\chi & \text{if } k = -1, \text{ “elliptic” region.} \end{cases}$$

We can then write the Oppenheimer–Snyder metric in a unified form as

$$ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + S_k(\chi)^2 d\Omega^2\right], \tag{5.60}$$

and the solution of the equation of motion is given in parametric form by

$$a(t) = \begin{cases} \frac{M_0}{2k}(\cosh\eta - 1) \quad \text{with} \quad \sinh\eta - \eta = \frac{2k^{3/2}(t-t_s)}{M_0} & \text{if } k > 0, \\ \left(\frac{3M_0(t-t_s)}{2}\right)^{2/3} & \text{if } k = 0, \\ \frac{M_0}{-2k}(1 - \cos\eta) \quad \text{with} \quad \eta - \sin\eta = \frac{2(-k)^{3/2}(t-t_s)}{M_0} & \text{if } k < 0, \end{cases}$$

On the other hand, one can always solve the equation of motion (5.59) to find $t(a)$.

- In the flat region given by $k = 0$, the equation of motion is easily integrated to give

$$t(a) = -\frac{2a^{\frac{3}{2}}}{3\sqrt{M_0}} + t_{sing}.$$

- In the hyperbolic region, given by $k > 0$, we define $X = M_0/k$ and we get

$$t(a) = \frac{a}{\sqrt{k}}\left(\frac{X}{a}\tanh^{-1}\frac{1}{\sqrt{\frac{X}{a}+1}} - \sqrt{\frac{X}{a}+1}\right) + t_{sing}.$$

- In the elliptic region, given by $k < 0$, we define $X = -M_0/k$ and we get

$$t(a) = \frac{a}{\sqrt{-k}}\left(\sqrt{\frac{X}{a}-1} - \frac{X}{a}\tan^{-1}\frac{1}{\sqrt{\frac{X}{a}-1}}\right) + t_{sing}.$$

Finally, for homogeneous dust, the metric (5.3) can also be written as

$$ds^2 = -dt^2 + \frac{a^2}{1+r^2k}dr^2 + r^2a^2d\Omega^2, \qquad (5.61)$$

from which we see that the metric in the interior is just the time reversal of a dust Friedmann–Robertson–Walker cosmological solution.

Marginally bound case is a particularly simple case given by $k = 0$. Then, the initial velocity of collapse is nonzero, as can be seen from $\dot{a}_i = -\sqrt{M_0}$, and the equation of motion is simply $\dot{a} = -\sqrt{M_0/a}$ for which the solution is obtained immediately as (see Fig. 5.2)

$$a(t) = \left(1 - \frac{3}{2}\sqrt{M_0}t\right)^{2/3}. \qquad (5.62)$$

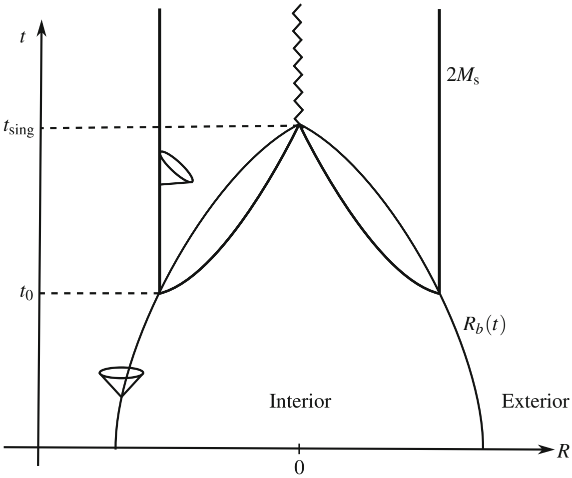

Fig. 5.2 Schematic view of homogeneous dust collapse. At the time $t_i = 0$, no singularities are present. The boundary curve $R_b(t)$ separates the interior from the vacuum exterior. The cloud collapses as t increases, and at the time t_0, the horizon forms at the boundary. In the exterior region, the horizon is the Schwarzschild radius. In the interior, the apparent horizon propagates inward and reaches the center of symmetry at the time of formation of the singularity t_{sing}. For $t > t_{sing}$, the space-time has settled to the usual Schwarzschild solution

5.6.1 Apparent Horizon and Singularity

From the condition for formation of trapped surfaces given in Eq. (5.56), we obtain the curve $t_{ah}(r)$ describing the time at which the shell r crosses the apparent horizon:

$$t_{ah}(r) = \begin{cases} t_{sing}(r) + \frac{F}{f^{\frac{3}{2}}} \tanh^{-1}\sqrt{\frac{f}{1+f}} - \frac{F}{f}\sqrt{1+f}\ , & \text{for } k > 0, \\ t_{sing}(r) - \frac{2}{3}F(r) = \frac{2}{3\sqrt{M_0}} - \frac{2}{3}r^3 M_0\ , & \text{for } k = 0, \\ t_{sing}(r) + \frac{F}{(-f)^{\frac{3}{2}}} \tan^{-1}\sqrt{-\frac{f}{1+f}} - \frac{F}{f}\sqrt{1+f}\ , & \text{for } k > 0. \end{cases}$$

Now, if we look for simplicity at the apparent horizon for marginally bound homogeneous dust model, we see that it forms initially at the boundary of the collapsing cloud at the time $t_0 = t_{ah}(r_b) = 2/3\sqrt{M_0} - 2r_b^3 M_0/3$ and then propagates inward toward the center. The time t_0 is the same time at which the event horizon forms in the exterior spacetime. For $t > t_0$, the apparent horizon curve moves to smaller radii reaching the center at the time $t_{sing} = t_{ah}(0) = 2/3\sqrt{M_0}$, which is the time of formation of the singularity. Inside the trapped region, all geodesics terminate at the singularity; therefore, an observer on the boundary, once this has passed the horizon, falls toward the singularity in a finite time of the order of $\sqrt{R_b^3/GM}$. An observer at infinity sees the boundary approaching the horizon becoming infinitely redshifted and indefinitely slow [36].

The singularity is reached once the density diverges. From the above, we have seen that the shell focusing strong curvature singularity corresponds to $a = 0$. In the homogeneous dust collapse case, this gives

$$t_{sing} = \begin{cases} \frac{1}{\sqrt{-k}}\left(X \tan^{-1}\frac{1}{\sqrt{X-1}} - \sqrt{X-1}\right)\ , & \text{for } k > 0, \\ \frac{2r^{\frac{3}{2}}}{3\sqrt{F}} = \frac{2}{3\sqrt{M_0}}\ , & \text{for } k = 0, \\ \frac{1}{\sqrt{k}}\left(\sqrt{X+1} - X \tanh^{-1}\frac{1}{\sqrt{X+1}}\right)\ , & \text{for } k > 0. \end{cases}$$

The singularity is simultaneous, and all shells fall into the singularity at the same co-moving time.

5.7 Inhomogeneous Dust and Collapse with Pressures

The easiest way to extend the homogeneous dust collapse model is to introduce inhomogeneities in ρ. Inhomogeneous models have been widely considered in cosmology which are obtained by a time reversal of collapse models (see [5, 31] and references therein). This means considering $\rho = \rho(r, t)$, with ρ radially decreasing in order for the matter profile to be physically realistic. This describes a dust cloud that initially has higher density at the center. Following the same procedure as in homogeneous collapse, we can evaluate Einstein's equations as

$$\rho = \frac{F'}{R^2 R'}, \tag{5.63}$$

$$\dot{F} = 0, \tag{5.64}$$

$$\lambda' = 0, \tag{5.65}$$

$$\dot{\psi} = \frac{\dot{R}'}{R'}. \tag{5.66}$$

The main difference with the homogeneous case is that now we will have $M(r)$, $b(r)$, and $a(r,t)$. Then, it is worth asking how the boundary, the trapped surfaces, and singularity are affected by the presence of inhomogeneities. Does a black hole still form at the end of collapse? Yes, the singularity theorems by Hawking and Penrose tell us that once the trapped surfaces form, the formation of the singularity is inevitable. Eventually, all matter falls into the central singularity and we are left with a Schwarzschild black hole [19]. Nevertheless, if we ask whether we get a picture of collapse qualitatively similar to the OS model, then the answer is not always in the affirmative. In fact, the way in which the apparent horizon and the singularity curve develop depends on the form of the density and velocity profiles and some important differences with the OS model may arise. The most striking of these differences is that in the inhomogeneous dust collapse, there is the possibility for the central singularity to be "naked" (i.e., not covered by a horizon) at the instant of formation (see, e.g., [9, 12, 26, 37, 48]).

In general, if $M = M(r)$, we have

$$\rho = \frac{F'}{R^2 R'} = \frac{3M + rM'}{a^2(a + ra')} . \tag{5.67}$$

If we consider $M(r)$ as a polynomial expansion near $r = 0$, we can take

$$M(r) = M_0 + M_1 r + M_2 r^2 + \cdots , \tag{5.68}$$

and the condition for the energy density to be radially decreasing outward is given by $M_1 \leq 0$. If we also wish to impose that the density does not present cusps at the center, we may impose $M_1 = 0$, and then, the condition for ρ to be decreasing becomes $M_2 \leq 0$. This is consistent with the choice of density profiles in astrophysical models that generally present only quadratic terms in r. Similar to the homogeneous case, the value $a = 0$ signals the appearance of the shell focusing singularity. On the other hand, now we need to make sure that shell crossing singularities, given by $R' = a + ra' = 0$, do not occur before the formation of the singularity. For simplicity, let us consider the marginally bound case. Then, $R' = 0$ implies

$$1 - \frac{3}{2}\sqrt{M} - \frac{rtM'}{2\sqrt{M}} = 0, \tag{5.69}$$

which can be used to obtain the time at which the shell r develops a shell crossing singularity as

$$t_{sc}(r) = \frac{2\sqrt{M}}{3M + rM'}. \tag{5.70}$$

For homogeneous dust, then $t_{sc} = t_{sing}$ and we do not have shell crossing singularities during collapse. Similarly, in the inhomogeneous case with $M' < 0$, we have $t_{sc}(r) \geq t_s(r)$, and therefore, no shell crossing singularities occur before the formation of the central singularity. From this, we see that the physical requirement of a radially decreasing density profile is compatible with the condition for avoidance of shell crossing singularities.

The solution for inhomogeneous dust collapse can be obtained form the one for homogeneous dust by replacing M_0 and k with $M(r)$ and $b(r)$. Again, let us consider for simplicity the solution for marginally bound collapse. This is given by

$$a(r,t) = \left(1 - \frac{3}{2}\sqrt{M(r)}t\right)^{2/3}. \tag{5.71}$$

We immediately see that now each shell collapses with a different scale factor and a different velocity. As a consequence, each shell becomes singular at a different time. The apparent horizon curve is also affected, as now it does not necessarily form initially at the boundary. The singularity curve and apparent horizon curve are explicitly given by

$$t_{sing}(r) = \frac{2}{3\sqrt{M(r)}}, \tag{5.72}$$

$$t_{ah}(r) = t_{sing}(r) - \frac{2}{3}r^3 M, \tag{5.73}$$

and near $r = 0$, they have the same behavior up until the third order in r,

$$t_{sing}(r) = \frac{2}{3\sqrt{M_0}} - \frac{M_1 r}{3M_0^{3/2}} + \cdots, \tag{5.74}$$

$$t_{ah}(r) = \frac{2}{3\sqrt{M_0}} - \frac{M_1 r}{3M_0^{3/2}} + \cdots. \tag{5.75}$$

Note that near the center, the apparent horizon curve is increasing and the central line $r = 0$ becomes singular and trapped at the same time. This suggests the possibility for the existence of geodesics that originate at the central singularity and are not trapped inside the horizon as (see Fig. 5.3). Due to the lack of pressures in the model,

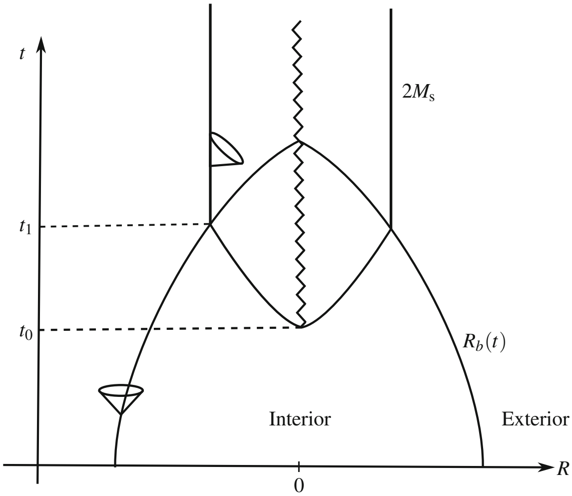

Fig. 5.3 Schematic view of inhomogeneous dust collapse. At the time $t_i = 0$, no singularities are present. The boundary curve $R_b(t)$ separates the interior from the vacuum exterior. The cloud collapses as t increases, and at the time t_0, the horizon forms at the center of the cloud. The singularity forms at the same time. Null geodesic can originate from the singularity and reach distant observers. In the interior, the apparent horizon propagates outward and reaches the boundary at the time $t_1 > t_0$. Once all the matter falls into the singularity, the space-time settles to the usual Schwarzschild solution

the boundary of the star r_b can be chosen arbitrarily. This is a mathematical artifact of the dust solution, and in the case with pressures, the boundary would have to be set at the radius where p vanishes. Therefore, if the boundary is chosen in such a way that t_{ah} is always increasing, it is possible to find null geodesics that originate at $t_s(0)$ and reach observers at infinity. Outgoing radial null geodesics $t_\gamma(r)$ are given by

$$\frac{dt_\gamma}{dr} = R', \tag{5.76}$$

and it can be proven that there are null geodesics coming out of the first instant of the central singularity $t_s(0)$ and reaching the boundary [9, 12, 26, 37, 48]. Naked singularities are found in many solutions of Einstein's equations and can be very different from one another. The question is if they can form from physically realistic processes. How much these models rely on the assumptions? What outcome will come from more realistic models? We shall shortly discuss these issues in the next sections (for a more detailed discussion, see, e.g., [28]).

Analytically, we cannot deal with rotation or departures from spherical symmetry. In fact, when it comes to rotation at present, we do not possess an analytical solution that describes an interior for the Kerr space-time that matches smoothly to the exterior. Nevertheless, some indications on how general is the scenario described in the dust collapse model can be obtained by considering collapse of a fluid source with pressures. For a fluid source, the energy momentum tensor is given by $T_{\mu\nu} = diag\{\rho, p_r, p_\theta, p_\theta\}$ and we can write Einstein's equations and energy momentum conservation

$$\rho = \frac{F'}{R^2 R'}, \tag{5.77}$$

$$p_r = -\frac{\dot{F}}{R^2 \dot{R}}, \tag{5.78}$$

$$\lambda' = 2\frac{p_\theta - p_r}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \tag{5.79}$$

$$\dot{G} = 2\lambda' \frac{\dot{R}}{R} G \ , \quad \text{with} \quad G = R'^2 e^{-2\psi}. \tag{5.80}$$

Then, it is easy to see that the system of equations to be solved becomes much more complicated with respect to the dust case. Together with the Misner–Sharp mass definition, the system has five equations and seven unknown functions. Specifying equations of state for p_r and p_θ then closes the system.

In this case, the Misner–Sharp mass need not be conserved during the evolution, and therefore, there can be an inflow or outflow of matter across each shell r as collapse progresses. As a consequence, the matching with the exterior space-time need not be done with the Schwarzschild solution. Requiring the boundary condition $p_r(r_b, t) = 0$ implies that $F(r_b, t)$ is conserved during collapse and the exterior is Schwarzschild, and on the other hand, the condition that p_r vanishes at the boundary translates in a variable boundary surface $r_b(t)$. It can be shown that matching with the Vaidya solution describing ingoing or outgoing null dust can be done in certain cases and matching with a generalized Vaidya solution is always possible [13, 14, 27].

The simplest model with pressures that can be considered is that of a homogeneous perfect fluid with linear equation of state. Then, requiring that the fluid be perfect implies $p_r = p_\theta = p$, while homogeneity implies $p = p(t)$ and $\rho = \rho(t)$. Finally, a linear equation of state relates p to ρ via

$$p = \gamma\rho, \tag{5.81}$$

with γ being a constant. The presence of the linear equation of state closes the system, and Einstein's equations can be fully integrated in this case. The third Einstein's equation (5.79) becomes again $\lambda' = 0$, and the fourth equation gives again $G = 1 + kr^2$. Then, the equation of motion is again written as

$$\dot{a}^2 = \frac{M}{a} + k, \tag{5.82}$$

where now $M = M(t)$ is to be determined from the equation of state that together with Eqs. (5.77) and (5.78) gives

$$\dot{M} = -\frac{3\gamma M}{a}. \tag{5.83}$$

Integrating the above equation, we get

$$M(t) = \frac{M_0}{a^{3\gamma}}. \tag{5.84}$$

Einstein's equations (5.77) and (5.78) imply that the density is $\rho = 3M/a^3$ and the pressure is $p = -\dot{M}/a^2$. To have a positive pressure, M must decrease in time. If we require a constant co-moving boundary $r = r_b$, then the exterior metric cannot be a portion of the Schwarzschild space-time and some mass must be radiated away in the exterior region. This can be easily seen from the fact that $M(t)$ implies that the total mass within r_b changes with time, and therefore, there must be an outflow of matter from the co-moving boundary. As said, we can always match to a non-vacuum solution describing a radiating null fluid or we can require the matching to be performed at a surface $r_b(t)$.

5.8 Collapse in Astrophysics

We studied here some simple analytical toy models that describe the complete gravitational collapse of a spherical matter cloud made of non-interacting particles in general relativity (GR). If we assume as a first approximation that these models can be used to describe the most relevant features of the collapse of the core of a massive star, we see that a black hole must inevitably form as the final product of collapse. At the time t_0, the horizon forms as the boundary of the star crosses the threshold of the Schwarzschild radius, and at the time $t_{sing} > t_0$, the singularity forms at the center of symmetry of the system. In the end, we are left with a Schwarzschild black hole. The Oppenheimer–Snyder–Datt (OSD) model is very simple and relies on many simplifying assumptions that while on the one hand allow us to solve the equations analytically thus finding a global solution, on the other hand make for a scenario that is not very realistic. The OSD model can be seen as the bridge between mathematical black holes and astrophysical black holes in the sense that it is a simplified mathematical description of a dynamical phenomena that nevertheless captures the most essential features. The main assumptions in the model are spherical symmetry, no rotation, homogeneous density, and no pressures. A real star will have some small but non-vanishing quadrupole moment, it will have angular momentum, it will be composed of several kinds of gases with pressures and different equations of state, and its density will not be homogeneous. Therefore, it is reasonable to ask how general is the picture obtained in the OSD model and how much the collapse of a real star will depart from our mathematical idealization.

What happens to singularity and horizon once we introduce inhomogeneities, pressures, rotation, and asymmetries in the model? Gravitational collapse is a dynamical process, and the structure and evolution of the horizon in a realistic stellar interior is not well understood. We still do not have any analytical model of formation of a Kerr black hole. In fact, we do not even have any analytical solution describing a viable interior metric for the Kerr solution. Nevertheless, the evidence for the existence of black holes is now almost universally accepted, and most people believe that the process that leads to their formation can be roughly described via the OSD model. Still, the question of whether the black hole candidates that we observe in the universe are well described by the Schwarzschild and Kerr metrics is still open. As is the question of whether every collapsing star that is massive enough must inevitably form a black hole as the final end state. In order to study more realistic models, one needs to give up the hope to solve Einstein's equations analytically and resort to numerical simulations. Fully general relativistic simulations have been done in the past years to study (among other things) gravitational core collapse with rotation and magnetic fields, black hole mergers and black hole neutron star mergers, gravitational wave production, recoil from black hole mergers, production of jets, and gamma ray bursts (see, e.g., [24, 25, 42] and references therein).

Numerical simulations have improved dramatically over the last decade. Nevertheless, there are still no fully satisfactory simulations of supernovae explosions that lead to the formation of a black hole. One reason resides in the fact that many elements of classical and quantum physics come into play during the last stages of the life of a star. Describing accurately such scenarios is an enormous task that requires very expansive computations on the most advanced supercomputers. Further to this, numerical simulations must assume that GR holds unchanged at all energy scales and therefore are limited by our lack of knowledge of gravity in the strong field. Stellar evolution and black hole formation still present a lot of open questions, and the possibility exists that black holes are not the only possible final outcome of collapse of very massive stars. For these reasons, despite the increasing amount of observational evidence for the existence of black holes, it is useful to keep an eye open for other, more exotic, possibilities (see, e.g., [10, 16, 34, 45, 46] and references therein).

5.9 Concluding Remarks

Simple analytical models of general relativistic collapse show that a black hole can form as the end state of the life of a massive star. From a mathematical point of view, a black hole is a space-time singularity covered by an event horizon. The curvature singularity at the end of collapse is indicated by the divergence of the scalar $\mathscr{K}$, and approaching the singularity matter reaches infinite density in a finite co-moving time. In some sense, the singularity at the end of collapse is analogous to the infinite density obtained in Newtonian collapse and can be viewed as a limit of the model rather than a physical feature of the system. Singularities are found in many solutions of Einstein's equations. The question is whether they can form from physically realistic processes and how we should interpret their appearance. We may think that GR works well in the strong field regime and nothing can prevent complete collapse from happening. In this case, one has to accept that singularities are there and they may be causally connected to the outside universe. On the other hand, we can believe that GR works well in the strong field, but other effects arise either preventing the formation of singularities or hiding them from view. Or we can think that GR needs modifications in the strong field regime due perhaps to quantum effects. These modifications would then affect the space-time near the formation of classical singularities, thus removing them. The first attitude, although legitimate, is not very common. A singularity in any classical theory such as classical mechanics or electromagnetism is located somewhere in time and space and it does not affect the future predictability of space-time itself. On the other hand in GR, a singularity is not a part of the space-time. The distribution of matter determines the properties of space and time, and the occurrence of singularities translates in geodesic incompleteness and has important consequences for the causal structure of the space-time itself. For this reason, most people believe that singularities must not occur in the real universe. The second attitude can be summarized by the words of Roger Penrose [41]: *"...does there exist a 'cosmic censor' who forbids the appearance of naked singularities, clothing each one in an absolute event horizon?"* This is the famous cosmic censorship conjecture (CCC). At present, there exist counterexamples to the CCC, like the inhomogeneous dust collapse model, but their physical relevance is not entirely clear. On the other hand, it is highly plausible that GR is not enough to describe what happens in the strong field regime. One needs to account for microphysics or for modifications to GR possibly due to quantum effects. This third attitude is a view that was already suggested by Wheeler who saw singularities as possible probes for new physics.

If the occurrence of singularities at the end of collapse signals a breakdown of the fluid model approximation or a breakdown of GR itself, then what could be a better mathematical framework to describe the last stages of the life of a star? Is there any viable model for collapse that does not originate a singularity? The singularity theorems by Hawking and Penrose tell us that if GR is the ultimate ingredient that we need to use to describe collapse and if matter satisfies the usual energy conditions, then a singularity must necessarily form [19]. More precisely, provided that some energy condition is satisfied, the space-time is globally hyperbolic, and a trapped region develops at some point, a singularity must always form. Therefore, in order to develop non-singular model of collapse, one needs to modify GR in some way. Several attempts have been made over the years, and the general scenario that is arising is that singularities may be removed by quantum gravitational effects. Matter in the strong field regime may violate standard energy conditions, and the complete collapse to a black hole may be replaced by a bouncing scenario in which collapsing matter re-expands after reaching a minimal size. The expansion phase may take the form of an explosive event, and it may leave behind an exotic compact remnant (see, e.g., [3, 4, 17] and references therein).

These compact remnants may be less massive, more dense, and smaller than a neutron star and they would not possess an event horizon. Several types of exotic compact objects have been investigated, and their observational properties are of great interest for future astrophysical observations (see, e.g., [1, 2, 29, 30]). Given the small number of astrophysical black hole candidates observed so far and the peculiar features that theoretical compact objects may possess, it is reasonable to suppose that their observation may pose a great challenge for future astrophysics. Nevertheless, if some departure from the black hole paradigm will be observed in the future, this may open a window onto new areas of physics where gravitation and quantum mechanics merge.

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