Chapter 9
Distances on Convex Bodies, Cones, and Simplicial Complexes

9.1 Distances on Convex Bodies

A *convex body* in the n-dimensional Euclidean space $\mathbb{E}^n$ is a convex *compact connected* subset of $\mathbb{E}^n$. It is called *solid* (or *proper*) if it has nonempty interior. Let K denote the space of all convex bodies in $\mathbb{E}^n$, and let K_p be the subspace of all proper convex bodies. Given a set $X \subset \mathbb{E}^n$, its *convex hull* $conv(X)$ is the minimal *convex* set containing X.

Any metric space (K, d) on K is called a *metric space of convex bodies*. Such spaces, in particular the metrization by the **Hausdorff metric**, or by the **symmetric difference metric**, play a basic role in Convex Geometry (see, for example, [Grub93]).

For $C, D \in K \backslash \{\emptyset\}$, the *Minkowski addition* and the *Minkowski nonnegative scalar multiplication* are defined by $C + D = \{x + y : x \in C, y \in D\}$, and $\alpha C = \{\alpha x : x \in C\}$, $\alpha \geq 0$, respectively. The Abelian semigroup $(K, +)$ equipped with nonnegative scalar multiplication operators can be considered as a *convex cone*.

The *support function* $h_C : S^{n-1} \to \mathbb{R}$ of $C \in K$ is defined by $h_C(u) = \sup\{\langle u, x\rangle : x \in C\}$ for any $u \in S^{n-1}$, where S^{n-1} is the $(n-1)$-dimensional *unit sphere* in $\mathbb{E}^n$, and $\langle , \rangle$ is the *inner product* in $\mathbb{E}^n$. The *width* $w_C(u)$ is $h_C(u) + h_C(-u) = h_{C-C}(u)$. It is the perpendicular distance between the parallel supporting hyperplanes perpendicular to given direction. The *mean width* is the average of width over all directions in S^{n-1}.

- **Area deviation**

  The **area deviation** (or **template metric**) is a metric on the set K_p in $\mathbb{E}^2$ (i.e., on the set of plane convex disks) defined by

$$A(C \triangle D),$$

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where $A(.)$ is the *area*, and $\triangle$ is the *symmetric difference*. If $C \subset D$, then it is equal to $A(D) - A(C)$.

- **Perimeter deviation**

  The **perimeter deviation** is a metric on K_p in $\mathbb{E}^2$ defined by

$$2p(conv(C \cup D)) - p(C) - p(D),$$

where $p(.)$ is the *perimeter*. In the case $C \subset D$, it is equal to $p(D) - p(C)$.

- **Mean width metric**

  The **mean width metric** is a metric on K_p in $\mathbb{E}^2$ defined by

$$v2W(conv(C \cup D)) - W(C) - W(D),$$

where $W(.)$ is the *mean width*: $W(C) = p(C)/\pi$, and $p(.)$ is the *perimeter*.

- **Florian metric**

  The **Florian metric** is a metric on K defined by

$$\int_{S^{n-1}} |h_C(u) - h_D(u)| d\sigma(u) = ||h_C - h_D||_1.$$

It can be expressed in the form $2S(conv(C \cup D)) - S(C) - S(D)$ for $n = 2$ (cf. **perimeter deviation**); it can be expressed also in the form $nk_n(2W(conv(C \cup D)) - W(C) - W(D))$ for $n \geq 2$ (cf. **mean width metric**).

Here $S(.)$ is the *surface area*, k_n is the *volume* of the *unit ball* $\overline{B}^n$ of $\mathbb{E}^n$, and $W(.)$ is the *mean width*: $W(C) = \frac{1}{nk_n} \int_{S^{n-1}} (h_C(u) + h_C(-u)) d\sigma(u)$.

- **McClure–Vitale metric**

  Given $1 \leq p \leq \infty$, the **McClure–Vitale metric** is a metric on K, defined by

$$\left(\int_{S^{n-1}} |h_C(u) - h_D(u)|^p d\sigma(u) \right)^{\frac{1}{p}} = ||h_C - h_D||_p.$$

- **Pompeiu–Hausdorff–Blaschke metric**

  The **Pompeiu–Hausdorff–Blaschke metric** is a metric on K defined by

$$\max\{\sup_{x \in C} \inf_{y \in D} ||x - y||_2, \sup_{y \in D} \inf_{x \in C} ||x - y||_2\},$$

where $||.||_2$ is the Euclidean norm on $\mathbb{E}^n$.

In terms of support functions and using Minkowski addition, this metric is

$$\sup_{u \in S^{n-1}} |h_C(u) - h_D(u)| = ||h_C - h_D||_\infty = \inf\{\lambda \geq 0 : C \subset D + \lambda \overline{B}^n, D \subset C + \lambda \overline{B}^n\},$$

where $\overline{B}^n$ is the *unit ball* of $\mathbb{E}^n$. This metric can be defined using any norm on $\mathbb{R}^n$ and for the space of bounded closed subsets of any metric space.

- **Pompeiu–Eggleston metric**
  The **Pompeiu–Eggleston metric** is a metric on K defined by

$$\sup_{x\in C}\inf_{y\in D}||x-y||_2+\sup_{y\in D}\inf_{x\in C}||x-y||_2,$$

  where $||.||_2$ is the Euclidean norm on $\mathbb{E}^n$.
  In terms of support functions and using Minkowski addition, this metric is

$$\max\{0,\sup_{u\in S^{n-1}}(h_C(u)-h_D(u))\}+\max\{0,\sup_{u\in S^{n-1}}(h_D(u)-h_C(u))\}=$$
$$=\inf\{\lambda\geq 0: C\subset D+\lambda\overline{B}^n\}+\inf\{\lambda\geq 0: D\subset C+\lambda\overline{B}^n\},$$

  where $\overline{B}^n$ is the *unit ball* of $\mathbb{E}^n$. This metric can be defined using any norm on $\mathbb{R}^n$ and for the space of bounded closed subsets of any metric space.
- **Sobolev distance**
  The **Sobolev distance** is a metric on K defined by

$$||h_C-h_D||_w,$$

  where $||.||_w$ is the *Sobolev* 1*-norm* on the set $G_{S^{n-1}}$ of all real continuous functions on the *unit sphere* S^{n-1} of $\mathbb{E}^n$.
  The *Sobolev* 1*-norm* is defined by $||f||_w=\langle f,f\rangle_w^{1/2}$, where $\langle,\rangle_w$ is an *inner product* on $G_{S^{n-1}}$, given by

$$\langle f,g\rangle_w=\int_{S^{n-1}}(fg+\nabla_s(f,g))dw_0,\quad w_0=\frac{1}{n\cdot k_n}w,$$

  where $\nabla_s(f,g)=\langle grad_s f, grad_s g\rangle$, $\langle,\rangle$ is the *inner product* in $\mathbb{E}^n$, and $grad_s$ is the *gradient* on S^{n-1} (see [ArWe92]).
- **Shephard metric**
  The **Shephard metric** is a metric on K_p defined by

$$\ln(1+2\inf\{\lambda\geq 0: C\subset D+\lambda(D-D), D\subset C+\lambda(C-C)\}).$$

- **Nikodym metric**
  The **Nikodym metric** (or **volume of symmetric difference**, **Dinghas distance**) is a metric on K_p defined by

$$V(C\triangle D)=\int(1_{x\in C}-1_{x\in D})^2dx,$$

  where $V(.)$ is the *volume* (i.e., the Lebesgue n-dimensional measure), and $\triangle$ is the *symmetric difference*. For $n=2$, one obtains the **area deviation**.

Normalized volume of symmetric difference is a variant of **Steinhaus distance** defined by

$$\frac{V(C\triangle D)}{V(C\cup D)}.$$

- **Eggleston distance**

  The **Eggleston distance** (or **symmetric surface area deviation**) is a distance on K_p defined by

$$S(C\cup D)-S(C\cap D),$$

  where $S(.)$ is the *surface area*. It is not a metric.
- **Asplund metric**

  The **Asplund metric** is a metric on the space $K_p/\approx$ of affine-equivalence classes in K_p defined by

$$\ln\inf\{\lambda\geq 1:\exists T:\mathbb{E}^n\to\mathbb{E}^n \text{ affine}, x\in\mathbb{E}^n, C\subset T(D)\subset\lambda C+x\}$$

  for any equivalence classes C^* and D^* with the representatives C and D, respectively.
- **Macbeath metric**

  The **Macbeath metric** is a metric on the space $K_p/\approx$ of affine-equivalence classes in K_p defined by

$$\ln\inf\{|\det T\cdot P|:\exists T,P:\mathbb{E}^n\to\mathbb{E}^n \text{ regular affine}, C\subset T(D), D\subset P(C)\}$$

  for any equivalence classes C^* and D^* with the representatives C and D, respectively.

  Equivalently, it can be written as $\ln\delta(C,D)+\ ln\delta(D,C)$, where $\delta(C,D)=\inf_T\{\frac{V(T(D))}{V(C)}; C\subset T(D)\}$, and T is a regular affine mapping of $\mathbb{E}^n$ onto itself.
- **Banach–Mazur metric**

  The **Banach–Mazur metric** is a metric on the space $K_{po}/\sim$ of the equivalence classes of proper 0-symmetric convex bodies with respect to linear transformations defined by

$$\ln\inf\{\lambda\geq 1:\exists T:\mathbb{E}^n\to\mathbb{E}^n \text{ linear}, C\subset T(D)\subset\lambda C\}$$

  for any equivalence classes C^* and D^* with the representatives C and D, respectively.

  It is a special case of the **Banach–Mazur distance** (Chap. 1).
- **Separation distance**

  The **separation distance** between two disjoint convex bodies C and D in $\mathbb{E}^n$ (in general, between any two disjoint subsets) $\mathbb{E}^n$) is (Buckley, 1985) their

Euclidean **set-set distance** $\inf\{||x-y||_2 : x \in C, y \in D\}$, while $\sup\{||x-y||_2 : x \in C, y \in D\}$ is their **spanning distance**.

- **Penetration depth distance**

  The **penetration depth distance** between two interpenetrating convex bodies C and D in $\mathbb{E}^n$ (in general, between any two interpenetrating subsets of $\mathbb{E}^n$) is (Cameron–Culley, 1986) defined as the minimum *translation distance* that one body undergoes to make the interiors of C and D disjoint:

  $$\min\{||t||_2 : interior(C+t) \cap D = \emptyset\}.$$

  Keerthi–Sridharan, 1991, considered $||t||_1$- and $||t||_\infty$-analogs of this distance.

  Cf. **penetration distance** in Chap. 23 and **penetration depth** in Chap. 24.

- **Growth distances**

  Let $C, D \in K_p$ be two compact convex proper bodies. Fix their *seed points* $p_C \in int\, C$ and $p_D \in int\, D$; usually, they are the centroids of C and D. The *growth function* $g(C, D)$ is the minimal number $\lambda > 0$, such that

  $$(\{p_C\} + \lambda(C \backslash \{p_C\})) \cap (\{p_D\} + \lambda(D \backslash \{p_D\})) \neq \emptyset.$$

  It is the amount objects must be grown if $g(C, D) > 1$ (i.e., $C \cap D = \emptyset$), or contracted if $g(C, D) > 1$ (i.e., $int\, C \cap int\, D \neq \emptyset$) from their internal seed points until their surfaces just touch. The **growth separation distance** $d_S(C, D)$ and the **growth penetration distance** $d_P(C, D)$ ([OnGi96]) are defined as

  $$d_S(C, D) = \max\{0, r_{CD}(g(C, D) - 1)\} \text{ and } d_P(C, D) = \max\{0, r_{CD}(1 - g(C, D))\},$$

  where r_{CD} is the scaling coefficient (usually, the sum of radii of circumscribing spheres for the sets $C \backslash \{p_C\}$ and $D \backslash \{p_D\}$).

  The *one-sided growth distance* between disjoint C and D (Leven–Sharir, 1987) is

  $$-1 + \min \lambda > 0 : (\{p_C\} + \lambda\{(C \backslash \{p_C\})) \cap D \neq \emptyset\}.$$

- **Minkowski difference**

  The **Minkowski difference** on the set of all compact subsets, in particular, on the set of all *sculptured objects* (or *free form objects*), of $\mathbb{R}^3$ is defined by

  $$A - B = \{x - y : x \in A, y \in B\}.$$

  If we consider object B to be free to move with fixed orientation, the Minkowski difference is a set containing all the translations that bring B to intersect with A. The closest point from the Minkowski difference boundary, $\partial(A - B)$, to the origin gives the **separation distance** between A and B.

If both objects intersect, the origin is inside of their Minkowski difference, and the obtained distance can be interpreted as a **penetration depth distance**.

- **Demyanov distance**

Given $C \in K_p$ and $u \in S^{n-1}$, denote, if $|\{c \in C : \langle u, c\rangle = h_C(u)\}| = 1$, this unique point by $y(u, C)$ (*exposed point of C in direction u*).

The *Demyanov difference* $A \ominus B$ of two subsets $A, B \in K_p$ is the closure of

$$conv(\cup_{T(A)\cap T(B)}\{y(u,A) - y(u,B)\}),$$

where $T(C) = \{u \in S^{n-1} : |\{c \in C : \langle u, c\rangle = h_C(u)\}| = 1\}$.

The **Demyanov distance** between two subsets $A, B \in K_p$ is defined by

$$||A \ominus B|| = \max_{c \in A \ominus B} ||c||_2.$$

It is shown in [BaFa07] that $||A \ominus B|| = \sup_\alpha ||St_\alpha(A) - St_\alpha(M)||_2$, where $St_\alpha(C)$ is a *generalized Steiner point* and the supremum is over all "sufficiently smooth" probabilistic measures α.

- **Maximum polygon distance**

The **maximum polygon distance** is a distance between two convex polygons $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_m)$ defined by

$$\max_{i,j} ||p_i - q_j||_2, \; i \in \{1, \dots, n\}, \; j \in \{1, \dots, m\}.$$

- **Grenander distance**

Let $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_m)$ be two disjoint convex polygons, and let $L(p_i, q_j), L(p_l, q_m)$ be two intersecting *critical support lines* for P and Q. Then the **Grenander distance** between P and Q is defined by

$$||p_i - q_j||_2 + ||p_l - q_m||_2 - \Sigma(p_i, p_l) - \Sigma(g_j, q_m),$$

where $||.||_2$ is the Euclidean norm, and $\Sigma(p_i, p_l)$ is the sum of the edges lengths of the polynomial chain $p_i, \dots, p_l$.

Here $P = (p_1, \dots, p_n)$ is a convex polygon with the vertices in standard form, i.e., the vertices are specified according to Cartesian coordinates in a clockwise order, and no three consecutive vertices are collinear. A line L is a *line of support* of P if the interior of P lies completely to one side of L.

Given two disjoint polygons P and Q, the line $L(p_i, q_j)$ is a *critical support line* if it is a line of support for P at p_i, a line of support for Q at q_j, and P and Q lie on opposite sides of $L(p_i, q_j)$. In general, a chord $[a, b]$ of a convex body C is called its **affine diameter** if there is a pair of different hyperplanes each containing one of the endpoints a, b and supporting C.

9.2 Distances on Cones

A *convex cone* C in a real vector space V is a subset C of V such that $C + C \subset C$, $\lambda C \subset C$ for any $\lambda \geq 0$. A cone C induces a *partial order* on V by

$$x \preceq y \text{ if and only if } y - x \in C.$$

The order $\preceq$ respects the vector structure of V, i.e., if $x \preceq y$ and $z \preceq u$, then $x + z \preceq y + u$, and if $x \preceq y$, then $\lambda x \preceq \lambda y$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$. Elements $x, y \in V$ are called *comparable* and denoted by $x \sim y$ if there exist positive real numbers α and β such that $\alpha y \preceq x \preceq \beta y$. Comparability is an equivalence relation; its equivalence classes (which belong to C or to $-C$) are called *parts* (or *components*, *constituents*).

Given a convex cone C, a subset $S = \{x \in C : T(x) = 1\}$, where $T : V \to \mathbb{R}$ is a positive linear functional, is called a *cross-section* of C. A convex cone C is called *almost Archimedean* if the closure of its restriction to any 2D subspace is also a cone.

A convex cone C is called *pointed* if $C \cup (-C) = \{0\}$ and *solid* if $int\ C \neq \emptyset$.

- **Koszul–Vinberg metric**

  Given an open pointed convex cone C, let C^* be its dual cone.

  The **Koszul–Vinberg metric** on C (Vinberg, 1963, and Koszul, 1965) is an affine invariant Riemannian metric defined as the Hessian $g = d^2\psi_C$, where $\psi_C(x) = -\log \int_{C^*} e^{-\langle \epsilon, x \rangle} d\epsilon$ for any $x \in C$.

  The Hessian of the *entropy* (Legendre transform of $\psi_C(x)$) defines a metric on C^*, which ([Barb14]) is equivalent to the **Fisher–Rao metric** (Sect. 7.2). [Barb14] also observed that *Fisher–Souriau metric* ([Sour70]) generalises Fisher–Rao metric for Lie group thermodynamics and interpreted it as a geometric *heat capacity*.
- **Invariant distances on symmetric cones**

  An open convex cone C in an Euclidean space V is said to be *homogeneous* if its group of linear automorphisms $G = \{g \in GL(V) : g(C) = C\}$ act transitively on C. If, moreover, $\overline{C}$ is pointed and C is self-dual with respect to the given inner product $\langle , \rangle$, then it is called a *symmetric cone*. Any symmetric cone is a Cartesian product of such cones of only 5 types: the cones $Sym(n, \mathbb{R})^+$, $Her(n, \mathbb{C})^+$ (cf. Chap. 12), $Her(n, \mathbb{H})^+$ of positive-definite Hermitian matrices with real, complex or quaternion entries, the *Lorentz cone* (or *forward light cone*) $\{(t, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : t^2 > x_1^2 + \dots + x_n^2\}$ and 27-dimensional exceptional cone of 3×3 positive-definite matrices over the octonions $\mathbb{O}$. An $n \times n$ quaternion matrix A can be seen as a $2n \times 2n$ complex matrix A'; so, $A \in Her(n, \mathbb{H})^+$ means $A' \in Her(2n, \mathbb{C})^+$.

  Let V be an *Euclidean Jordan algebra*, i.e., a finite-dimensional *Jordan algebra* (commutative algebra satisfying $x^2(xy) = x(x^2y)$ and having a multiplicative identity e) equipped with an *associative* ($\langle xy, z \rangle = \langle y, xz \rangle$) inner product $\langle , \rangle$. Then the set of square elements of V is a symmetric cone, and every symmetric cone arises in this way. Denote $P(x)y = 2x(xy) - x^2y$ for any $x, y \in C$.

For example, for $C = PD_n(\mathbb{R})$, the group G is $GL(n, \mathbb{R})$, the inner product is $\langle X, Y\rangle = \mathrm{Tr}(XY)$, the Jordan product is $\frac{1}{2}(XY + YX)$, and $P(X)Y = XYX$, where the multiplication on the right-hand side is the usual matrix multiplication.

If r is the rank of V, then for any $x \in V$ there is a complete set of orthogonal primitive idempotents $c_1, \dots, c_r \neq 0$ (i.e., $c_i^2 = c_i$, c_i indecomposable, $c_i c_j = 0$ if $i \neq j$, $\sum_{i=1}^{r} c_i = e$) and real numbers $\lambda_1, \dots, \lambda_r$, called *eigenvalues* of x, such that $x = \sum_{i=1}^{r} \lambda_i c_i$. Let $x, y \in C$ and $\lambda_1, \dots, \lambda_r$ be the eigenvalues of $P(x^{-\frac{1}{2}})y$. Lim, 2001, defined following three G-invariant distances on any symmetric cone C:

$$d_R = (\sum_{1 \le i \le r} \ln^2 \lambda_i)^{\frac{1}{2}},\ d_F = \max_{1 \le i \le r} \ln |\lambda_i|,\ d_H = \ln(\max_{1 \le i \le r} \lambda_i (\min_{1 \le i \le r} \lambda_i)^{-1}).$$

For above distances, the geometric mean $P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}}y))^{\frac{1}{2}}$ is the midpoint of x and y. The distances $d_R(x, y)$, $d_F(x, y)$ are the intrinsic metrics of G-invariant Riemannian and Finsler metrics on C. The Riemannian geodesic curve $\alpha(t) = P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}}y))^t$ is one of infinitely many shortest Finsler curves passing through x and y. The space $(C, d_R(x, y))$ is an **Hadamard space** (Chap. 6), while $(C, d_F(x, y))$ is not. The distance $d_F(x, y)$ is the **Thompson's part metric** on C, and the distance $d_H(x, y)$ is the **Hilbert projective semimetric** on C which is a complete metric on the unit sphere on C.

- **Thompson's part metric**

  Given a convex cone C in a real Banach space V, the **Thompson's part metric** on a *part* $K \subset C \backslash \{0\}$ is defined (Thompson, 1963) by

$$\log \max\{m(x, y), m(y, x)\}$$

  for any $x, y \in K$, where $m(x, y) = \inf\{\lambda \in \mathbb{R} : y \preceq \lambda x\}$.

  If C is *almost Archimedean*, then K equipped with this metric is a **complete** metric space. If C is finite-dimensional, then one obtains a **chord space** (Chap. 6). The *positive cone* $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) : x_i \ge 0 \text{ for } 1 \le i \le n\}$ equipped with this metric is isometric to a *normed space* which can be seen as being flat. The same holds for the **Hilbert projective semimetric** on $\mathbb{R}^n_+$.

  If C is a closed solid cone in $\mathbb{R}^n$, then *int* C can be seen as an n-dimensional manifold M^n. If for any tangent vector $v \in T_p(M^n)$, $p \in M^n$, we define a norm $||v||_p^T = \inf\{\alpha > 0 : -\alpha p \preceq v \preceq \alpha p\}$, then the length of any piecewise differentiable curve $\gamma : [0, 1] \to M^n$ is $l(\gamma) = \int_0^1 ||\gamma'(t)||^T_{\gamma(t)} dt$, and the distance between x and y is $\inf_\gamma l(\gamma)$, where the infimum is taken over all such curves γ with $\gamma(0) = x$, $\gamma(1) = y$.

- **Hilbert projective semimetric**

  Given a pointed closed convex cone C in a real Banach space V, the **Hilbert projective semimetric** on $C \backslash \{0\}$ is defined (Bushell, 1973), for $x, y \in C \backslash \{0\}$, by

$$h(x, y) = \log(m(x, y)m(y, x)),$$

where $m(x, y) = \inf\{\lambda \in \mathbb{R} : y \preceq \lambda x\}$; it holds $\frac{1}{m(y,x)} = \sup\{\lambda \in \mathbb{R} : \lambda y \preceq x\}$. This semimetric is finite on the interior of C and $h(\lambda x, \lambda' y) = h(x, y)$ for $\lambda, \lambda' > 0$. So, $h(x, y)$ is a metric on the *projectivization* of C, i.e., the space of rays of this cone.

If C is finite-dimensional, and S is a *cross-section* of C (in particular, $S = \{x \in C : ||x|| = 1\}$, where $||.||$ is a norm on V), then, for any distinct points $x, y \in S$, it holds $h(x, y) = |\ln(x, y, z, t)|$, where z, t are the points of the intersection of the line $l_{x,y}$ with the boundary of S, and (x, y, z, t) is the **cross-ratio** of x, y, z, t. Cf. the **Hilbert projective metric** in Chap. 6.

If C is finite-dimensional and *almost Archimedean*, then each part of C is a **chord space** (Chap. 6) under the Hilbert projective semimetric. On the *Lorentz cone* $L = \{x = (t, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : t^2 > x_1^2 + \cdots + x_n^2\}$, this semimetric is isometric to the n-dimensional *hyperbolic space*. On the hyperbolic subspace $H = \{x \in L : \det(x) = 1\}$, it holds $h(x, y) = 2d(x, y)$, where $d(x, y)$ is the **Thompson's part metric** which is (on H) the usual hyperbolic distance $\operatorname{arccosh}\langle x, y\rangle$.

If C is a closed solid cone in $\mathbb{R}^n$, then *int* C can be seen as an **n-manifold** M^n (Chap. 2). If for any tangent vector $v \in T_p(M^n)$, $p \in M^n$, we define a seminorm $||v||_p^H = m(p, v) - m(v, p)$, then the length of any piecewise differentiable curve $\gamma : [0, 1] \to M^n$ is $l(\gamma) = \int_0^1 ||\gamma'(t)||_{\gamma(t)}^H dt$, and $h(x, y) = \inf_\gamma l(\gamma)$, where the infimum is taken over all such curves γ with $\gamma(0) = x$ and $\gamma(1) = y$.

- **Bushell metric**

  Given a convex cone C in a real Banach space V, the **Bushell metric** on the set $S = \{x \in C : \sum_{i=1}^n |x_i| = 1\}$ (in general, on any *cross-section* of C) is defined by

$$\frac{1 - m(x, y) \cdot m(y, x)}{1 + m(x, y) \cdot m(y, x)}$$

  for any $x, y \in S$, where $m(x, y) = \inf\{\lambda \in \mathbb{R} : y \preceq \lambda x\}$. In fact, it is equal to $\tanh(\frac{1}{2}h(x, y))$, where h is the **Hilbert projective semimetric**.

- **k-oriented distance**

  A *simplicial cone* C in $\mathbb{R}^n$ is defined as the intersection of n (open or closed) half-spaces, each of whose supporting planes contain the origin 0. For any set M of n points on the *unit sphere*, there is a unique simplicial cone C that contains these points. The *axes* of the cone C can be constructed as the set of the n rays, where each ray originates at the origin, and contains one of the points from M.

  Given a *partition* $\{C_1, \dots, C_k\}$ of $\mathbb{R}^n$ into a set of simplicial cones $C_1, \dots, C_k$, the **k-oriented distance** is a metric on $\mathbb{R}^n$ defined by

$$d_k(x - y)$$

  for all $x, y \in \mathbb{R}^n$, where, for any $x \in C_i$, the value $d_k(x)$ is the length of the shortest path from the origin 0 to x traveling only in directions parallel to the axes of C_i.

- **Cones over metric space**

  A **cone over a metric space** (X, d) is the quotient space $Con(X, d)=(X \times [0, 1])/(X \times \{0\})$ obtained from the product $X \times \mathbb{R}_{\geq 0}$ by collapsing the *fiber* (subspace $X \times \{0\}$) to a point (the apex of the cone). Cf. **metric cone** in Chap. 1.

  The *Euclidean cone over the metric space* (X, d) is the cone $Con(X, d)$ with a metric d defined, for any $(x, t), (y, s) \in Con(X, d)$, by

$$\sqrt{t^2 + s^2 - 2ts\cos(\min\{d(x, y), \pi\})}.$$

  If (X, d) is a compact metric space with diameter < 2, the **Krakus metric** is a metric on $Con(X, d)$ defined, for any $(x, t), (y, s) \in Con(X, d)$, by

$$\min\{s, t\}d(x, y) + |t - s|.$$

  The cone $Con(X, d)$ with the Krakus metric admits a unique *midpoint* for each pair of its points if (X, d) has this property.

  If M^n is a manifold with a pseudo-Riemannian metric g, one can consider a metric dr^2+r^2g (in general, a metric $\frac{1}{k}dr^2+r^2g, k \neq 0$) on $Con(M^n) = M^n \times \mathbb{R}_{>0}$. For example, $Con(M^n) = \mathbb{R}^n \setminus \{0\}$ if (M^n, g) is the unit sphere in $\mathbb{R}^n$.

  A *spherical cone* (or *suspension*) $\Sigma(X)$ over a metric space (X, d) is the quotient of the product $X \times [0, a]$ obtained by identifying all points in the fibers $X \times \{0\}$ and $X \times \{a\}$. If (X, d) is a **length space** (Chap. 6) with $diam(X) \leq \pi$, and $a = \pi$, the **suspension metric** on $\Sigma(X)$ is defined, for any $(x, t), (y, s) \in \Sigma(X)$, by

$$\arccos(\cos t \cos s + \sin t \sin s \cos d(x, y)).$$

9.3 Distances on Simplicial Complexes

An r-dimensional *simplex* (or *geometrical simplex*, *hypertetrahedron*) is the *convex hull* of $r + 1$ points of $\mathbb{E}^n$ which do not lie in any $(r - 1)$-plane. The boundary of an r-simplex has $r + 1$ 0-*faces* (polytope vertices), $\frac{r(r+1)}{2}$ 1-*faces* (polytope edges), and $\binom{r+1}{i+1}$ *i-faces*, where $\binom{r}{i}$ is the binomial coefficient. The *content* (i.e., the *hypervolume*) of a simplex can be computed using the *Cayley–Menger determinant*. The regular simplex of dimension r is denoted by α_r. *Simplicial depth of a point* $p \in \mathbb{E}^n$ *relative to a set* $P \subset \mathbb{E}^n$ is the number of simplices S, generated by $(n + 1)$-subsets of P and containing p.

Roughly, a *geometrical simplicial complex* is a space with a *triangulation*, i.e., a decomposition of it into closed simplices such that any two simplices either do not intersect or intersect only along a common face.

An *abstract simplicial complex* S is a set, whose elements are called *vertices*, in which a family of finite nonempty subsets, called *simplices*, is distinguished, such that every nonempty subset of a simplex s is a simplex, called a *face* of s, and every one-element subset is a simplex. A simplex is called i-dimensional if it consists of $i + 1$ vertices. The *dimension* of S is the maximal dimension of its simplices. For every simplicial complex S there exists a triangulation of a polyhedron whose simplicial complex is S. This geometric simplicial complex, denoted by GS, is called the *geometric realization* of S.

- **Vietoris–Rips complex**

  Given a metric space (X, d) and distance δ, their **Vietoris–Rips complex** is an abstract simplicial complex, the simplexes of which are the finite subsets M of (X, d) having diameter at most δ; the dimension of a simplex defined by M is $|M| - 1$.
- **Simplicial metric**

  Given an abstract simplicial complex S, the points of geometric simplicial complex GS, realizing S, can be identified with the functions $\alpha : S \to [0, 1]$ for which the set $\{x \in S : \alpha(x) \neq 0\}$ is a simplex in S, and $\sum_{x \in S} \alpha(x) = 1$. The number $\alpha(x)$ is called the *x-th barycentric coordinate* of α.

  The **simplicial metric** on GS (Lefschetz, 1939) is the Euclidean metric on it:

$$\sqrt{\sum_{x \in S} (\alpha(x) - \beta(x))^2}.$$

  Tukey, 1939, found another metric on GS, topologically equivalent to a simplicial one. His **polyhedral metric** is the **intrinsic metric**, defined as the infimum of the lengths of the polygonal lines joining the points α and β such that each link is within one of the simplices. An example of a polyhedral metric is the intrinsic metric on the surface of a convex polyhedron in $\mathbb{E}^3$.
- **Polyhedral space**

  A Euclidean **polyhedral space** is a simplicial complex with a **polyhedral metric**. Every simplex is a **flat space** (a metric space locally isometric to some $\mathbb{E}^n$; cf. Chap. 1), and the metrics of any two simplices coincide on their intersection. The metric is the maximal metric not exceeding the metrics of simplices.

  If such a space is an n-**manifold** (Chap. 2), a point in it is a **metric singularity** if it has no neighborhood isometric to an open subset of $\mathbb{E}^n$.

  A polyhedral metric on a simplicial complex in a space of constant (positive or negative) curvature results in *spherical* and *hyperbolic polyhedral spaces*.

  The *dimension* of a polyhedral space is the maximal dimension of simplices used to glue it. **Metric graphs** (Chap. 15) are just one-dimensional polyhedral spaces.

  The surface of a convex polyhedron is a 2D polyhedral space. A polyhedral metric d on a triangulated surface is a **circle-packing metric** (Thurston, 1985) if there exists a vertex-weighting $w(x) > 0$ with $d(x, y) = w(x) + w(y)$ for any edge xy.

- **Manifold edge-distance**

  A (boundaryless) *combinatorial n-manifold* is an abstract n-dimensional simplicial complex M^n in which the *link* of each r-simplex is an $(n-r-1)$-sphere. The category of such spaces is equivalent to the category of piecewise-linear (PL) manifolds.

  The *link* of a simplex S is $Cl(Star_S) - Star_S$, where $Star_S$ is the set of all simplices in M^n having a face S, and $Cl(Star_S)$ is the smallest simplicial subcomplex of M^n containing $Star_S$.

  The **edge-distance** between vertices $u, v \in M^n$ is the minimum number of edges needed to connect them.

- **Manifold triangulation metric**

  Let M^n be a compact PL (piecewise-linear) n-dimensional manifold. A *triangulation* of M^n is a simplicial complex such that its corresponding polyhedron is PL-homeomorphic to M^n. Let T_{M^n} be the set of all *combinatorial types* of triangulations, where two triangulations are equivalent if they are simplicially isomorphic.

  Every such triangulation can be seen as a metric on the smooth manifold M if one assigns the unit length for any of its 1-dimensional simplices; so, T_{M^n} can be seen as a discrete analog of the space of Riemannian structures, i.e., isometry classes of Riemannian metrics on M^n.

  A **manifold triangulation metric** between two triangulations x and y is (Nabutovsky and Ben-Av, 1993) an **editing metric** on T_{M^n}, i.e., the minimal number of elementary moves, from a given finite list of operations, needed to obtain y from x.

  For example, the *bistellar move* consists of replacing a subcomplex of a given triangulation, which is simplicially isomorphic to a subcomplex of the boundary of the standard $(n+1)$-simplex, by the complementary subcomplex of the boundary of an $(n+1)$-simplex, containing all remaining n-simplices and their faces. Every triangulation can be obtained from any other one by a finite sequence of bistellar moves.

- **Polyhedral chain metric**

  An r-dimensional *polyhedral chain A* in $\mathbb{E}^n$ is a linear expression $\sum_{i=1}^{m} d_i t_i^r$, where, for any i, the value t_i^r is an r-dimensional simplex of $\mathbb{E}^n$. The *boundary* ∂A of a chain AD is the linear combination of boundaries of the simplices in the chain. The boundary of an r-dimensional chain is an $(r-1)$-dimensional chain.

  A **polyhedral chain metric** is a **norm metric** $||A - B||$ on the set $C_r(\mathbb{E}^n)$ of all r-dimensional polyhedral chains. As a norm $||.||$ on $C_r(\mathbb{E}^n)$ one can take:

  1. The *mass* of a polyhedral chain, i.e., $|A| = \sum_{i=1}^{m} |d_i||t_i^r|$, where $|t^r|$ is the volume of the cell t_i^r;
  2. The *flat norm* of a polyhedral chain, i.e., $|A|^\flat = \inf_D\{|A - \partial D| + |D|\}$, where the infimum is taken over all $(r+1)$-dimensional polyhedral chains;

3. The *sharp norm* of a polyhedral chain, i.e.,

$$|A|^{\sharp} = \inf\left(\frac{\sum_{i=1}^{m} |d_i||t_i^r||v_i|}{r+1} + |\sum_{i=1}^{m} d_i T_{v_i} t_i^r|^{\flat}\right),$$

where the infimum is taken over all *shifts* v (here $T_v t^r$ is the cell obtained by shifting t^r by a vector v of length $|v|$). A flat chain of finite mass is a sharp chain. If $r = 0$, than $|A|^{\flat} = |A|^{\sharp}$.

The metric space of *polyhedral co-chains* (i.e., linear functions of polyhedral chains) can be defined similarly. As a norm of a polyhedral co-chain X one can take:

1. The *co-mass* of a polyhedral co-chain, i.e., $|X| = \sup_{|A|=1} |X(A)|$, where $X(A)$ is the value of the co-chain X on a chain A;
2. The *flat co-norm* of a polyhedral co-chain, i.e., $|X|^{\flat} = \sup_{|A|^{\flat}=1} |X(A)|$;
3. The *sharp co-norm* of a polyhedral co-chain, i.e., $|X|^{\sharp} = \sup_{|A|^{\sharp}=1} |X(A)|$.