Chapter 3
Generalizations of Metric Spaces

Some immediate generalizations of the notion of metric, for example, **quasi-metric**, **near-metric**, **extended metric**, were defined in Chap. 1. Here we give some generalizations in the direction of Topology, Probability, Algebra, etc.

3.1 *m*-Tuple Generalizations of Metrics

In the definition of a metric, for every *two points* there is a *unique associated number*. Here we group some generalizations of metrics in which *several points* or *several numbers* are considered instead.

- ***m*-hemimetric**

  Let X be a nonempty set. A function $d : X^{m+1} \to \mathbb{R}_{\geq 0}$ is called a ***m*-hemimetric** (Deza–Rosenberg, 2000) if it have the following properties:

  1. d is *totally symmetric*, i.e., satisfies $d(x_1, \dots, x_{m+1}) = d(x_{\pi(1)}, \dots, x_{\pi(m+1)})$ for all $x_1, \dots, x_{m+1} \in X$ and for any permutation π of $\{1, \dots, m+1\}$;
  2. $d(x_1, \dots, x_{m+1}) = 0$ if $x_1, \dots, x_{m+1}$ are not pairwise distinct;
  3. for all $x_1, \dots, x_{m+2} \in X$, d satisfies the ***m*-simplex inequality**

  $$d(x_1, \dots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+2}).$$

  Cf. unrelated **hemimetric** (i.e., a quasi-semimetric) in Chap. 1.

  If in above 3. $d(x_1, \dots, x_{m+1})$ is replaced by $sd(x_1, \dots, x_{m+1})$ for some $s, 0 < s \leq 1$, then d is called (m, s)**-super-metric** ([DeDu03]). $(m, 1)$- and $(1, s)$-super-metrics are exactly m-hemimetric and $\frac{1}{s}$-*near-semimetric*; cf. **near-metric** in Chap. 1.

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If above 3. is dropped, d is called m-**dissimilarity**. 1-dissimilarity and 1-hemimetric are exactly a distance and a semimetric.

- 2-**metric**

An m-**hemimetric** with $m = 2$ satisfies 2-**simplex** (or *tetrahedron*) **inequality**

$$d(x_1, x_2, x_3) \leq d(x_4, x_2, x_3) + d(x_1, x_4, x_3) + d(x_1, x_2, x_4).$$

A 2-**metric** (Gähler, 1963 and 1966) is a 2-**hemimetric** d in which, for any distinct points x_1, x_2, there is a point x_3 with $d(x_1, x_2, x_3) > 0$. The area of the triangle spanned by x_1, x_2, x_3 on $\mathbb{R}^2$ or $\mathbb{S}^2$ is a 2-metric.

A *D-space* (Dhage, 1992) is an 2-**hemimetric space** (X, d) in which the condition "$d(x_1, x_2, x_3) = 0$ if two of x_1, x_2, x_3 are equal" is replaced by "$d(x_1, x_2, x_3) = 0$ if and only if $x_1 = x_2 = x_3$." Mustafa and Sims, 2003, showed that D-spaces are not suitable for topological constructions. In 2006, they defined instead a function, let us call it $MS-2$-**metric**, $D : X^3 \to \mathbb{R}_{\geq 0}$ which satisfies

1. $D(x_1, x_2, x_3) = 0$ if $x_1 = x_2 = x_3$;
2. $D(x_1, x_1, x_2) > 0$ whenever $x_1 \neq x_2$;
3. $D(x_1, x_2, x_3) \geq D(x_1, x_1, x_2)$ whenever $x_3 \neq x_2$;
4. D is a *totally symmetric* function of its three variables, and
5. $D(x_1, x_2, x_3) \leq D(x_1, x_4, x_4) + D(x_4, x_2, x_3)$ for all $x_1, x_2, x_3, x_4 \in X$.

The perimeter of the triangle spanned by x_1, x_2, x_3 on $\mathbb{R}^2$ is a $MS-2$-metric. If d is a metric, then $\frac{1}{2}(d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3))$ and $\max(d(x_1, x_2), d(x_2, x_3), d(x_1, x_3))$ are $MS-2$-metrics. If D is a $MS-2$-metric, then $D(x_1, x_2, x_2) + D(x_1, x_1, x_2)$ is a metric. If (X, D) is a $MS-2$-metric space, the open *D-ball with center* x_0 *and radius* r is $B_D(x_0, r) = \{x_1 \in X : D(x_0, x_1, x_1) < r\}$.

- **Multidistance**

Given a set X, a function $D : \cup_{m>1} X^m \to \mathbb{R}_{\geq 0}$ is called a **multidistance** (Martin–Major, 2009) if, for all m and all $x_1, \ldots, x_m, y \in X$, it satisfies:

1. $D(x_1, \ldots, x_m) = 0$ if $x_1 = \cdots = x_m$;
2. $D(x_1, \ldots, x_m) = D(x_{\pi(1)}, \ldots, x_{\pi(m)})$ for any permutation π of $\{1, \ldots, m\}$;
3. $D(x_1, \ldots, x_m) \leq \sum_{i=1}^m D(x_i, y)$.

Clearly, the restriction of a multidistance on X^2 is a semimetric.

A multidistance D is called *regular*, if all $D(x_1, \ldots, x_m) \leq D(x_1, \ldots, x_m, y)$ hold, and *stable*, if all $D(x_1, \ldots, x_m) = D(x_1, \ldots, x_m, x_i)$ hold. Given a metric space (X, d), the *Fermat multidistance* is $\min_{x \in X} \sum_{i=1}^m d(x_i, x)$; it is regular, but not stable.

The regular multidistances on X form a convex cone.

- **Multimetric**

In Mao, 2006, a **multimetric space** is the union of some metric spaces $(X_i, d_i), i \in J$. In the case $X_i = X, i \in J$, the **multimetric** is defined as the sequence-valued map $d(x, y) = (d_i), i \in J$, from $X \times X$ to $R_{\geq 0}^{|J|}$.

Cf. **bimetric theory of gravity** in Chap. 24 and (in the item **meter-related terms**) *multimetric crystallography* in Chap. 27.

Also, Jörnsten, 2007, consider *clustering* (Chap. 17) under several distance metrics simultaneously. In Rintanen, 2004, a *linear multimetric* is defined as $d = w_1d_1 + \cdots + w_md_m$, where d_i are metrics and $w_i \in [0, 1]$ are weights.

- **Diversity**

  Given a set X, a function f from its finite subsets to $\mathbb{R}_{\geq 0}$ is called (Bryant–Tupper, 2012) *diversity on X* if $f(A) = 0$ for all $A \subset X$ with $|A| \leq 1$ and

  $$f(A \cup B) + f(B \cup C) \geq f(A \cup C) \text{ for all } A, B, C \subset X \text{ with } B \neq \emptyset.$$

  The **induced diversity metric** $d(x, y)$ is $f(\{x, y\})$.

  For any diversity $f(A)$ with induced metric space (X, d), it holds $f_{diam}(A) \leq f(A) \leq f_S(A) \leq (|A| - 1)f_{diam}(A)$, where the **diameter diversity** $f_{diam}(A)$ is $\max_{x,y \in A} d(x, y) = diam(A)$ and the **Steiner diversity** $f_S(A)$ is the minimum weight of a Steiner tree connecting elements of A.

  l_1-diversity is defined by $f_{m1}(A) = \max |a_i - b_i| : a, b \in A$ for all finite $A \subset \mathbb{R}^m$.

  Any diversity is a **Vitanyi multiset metric**, restricted to subsets. But much of Bryant–Tupper's theory of diversities does not extend on multisets.

- **Vitanyi multiset metric**

  Given two multisets m and m', define $n = mm'$ if n is the multiset consisting of the elements of the multisets m and m', that is, if x occurs once in m and once in m', then it occurs twice in n. A function d on the set of nonempty finite multisets is (Vitanyi, 2011) a **multiset metric** if

  1. $d(m) = 0$ if all elements of m are equal and $d(m) > 0$ otherwise.
  2. $d(X)$ is invariant under all permutations of m.
  3. $d(mm') \leq d(mm'') + d(m''m')$ (*multiset triangle inequality*).

  The usual metric between two elements results if the multiset m has two elements in 1. and 2. and the multisets m, m', m'' have one element each in 3.

  An example is the set of all nonempty finite multisets m of integers with $d(m) = \max\{x : x \in m\} - \min\{x : x \in m\}$. Cohen–Vitanyi, 2012, defined another multiset metric, generalising **normalised web distance** (Chap. 22).

3.2 Indefinite Metrics

- **Indefinite metric**

  An **indefinite metric** (or *G-metric*) on a real (complex) vector space V is a *bilinear* (in the complex case, *sesquilinear*) *form* G on V, i.e., a function $G : V \times V \to \mathbb{R}$ ($\mathbb{C}$), such that, for any $x, y, z \in V$ and for any scalars α, β, we have the following properties: $G(\alpha x + \beta y, z) = \alpha G(x, z) + \beta G(y, z)$, and $G(x, \alpha y + \beta z) = \overline{\alpha} G(x, y) + \overline{\beta} G(x, z)$, where $\overline{\alpha} = \overline{a + bi} = a - bi$ denotes the *complex conjugation*.

If a positive-definite form G is symmetric, then it is an *inner product* on V, and one can use it to canonically introduce a *norm* and the corresponding **norm metric** on V. In the case of a general form G, there is neither a norm, nor a metric canonically related to G, and the term **indefinite metric** only recalls the close relation of such forms with certain metrics in vector spaces (cf. Chaps. 7 and 26).

The pair (V, G) is called a *space with an indefinite metric*. A finite-dimensional space with an indefinite metric is called a *bilinear metric space*. A **Hilbert space** H, endowed with a continuous G-metric, is called a *Hilbert space with an indefinite metric*. The most important example of such space is a *J-space*; cf. **J-metric**.

A subspace L in a space (V, G) with an indefinite metric is called a *positive subspace*, *negative subspace*, or *neutral subspace*, depending on whether $G(x,x) > 0$, $G(x,x) < 0$, or $G(x,x) = 0$ for all $x \in L$.

- **Hermitian G-metric**

A **Hermitian G-metric** is an **indefinite metric** G^H on a complex vector space V such that, for all $x, y \in V$, we have the equality

$$G^H(x,y) = \overline{G^H(y,x)},$$

where $\overline{\alpha} = \overline{a + bi} = a - bi$ denotes the *complex conjugation*.

- **Regular G-metric**

A **regular G-metric** is a continuous **indefinite metric** G on a **Hilbert space** H over $\mathbb{C}$, generated by an invertible *Hermitian operator* T by the formula

$$G(x,y) = \langle T(x), y \rangle,$$

where $\langle, \rangle$ is the *inner product* on H.

A *Hermitian operator* on a Hilbert space H is a *linear operator* T on H defined on a *domain* $D(T)$ of H such that $\langle T(x), y \rangle = \langle x, T(y) \rangle$ for any $x, y \in D(T)$. A bounded Hermitian operator is either defined on the whole of H, or can be so extended by continuity, and then $T = T^*$. On a finite-dimensional space a Hermitian operator can be described by a *Hermitian matrix* $((a_{ij})) = ((\overline{a}_{ji}))$.

- **J-metric**

A **J-metric** is a continuous **indefinite metric** G on a **Hilbert space** H over $\mathbb{C}$ defined by a certain *Hermitian involution* J on H by the formula

$$G(x,y) = \langle J(x), y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the *inner product* on H.

An *involution* is a mapping H onto H whose square is the *identity mapping*. The involution J may be represented as $J = P_+ - P_-$, where P_+ and P_- are orthogonal projections in H, and $P_+ + P_- = H$. The rank *of indefiniteness* of the J-metric is defined as $\min\{\dim P_+, \dim P_-\}$.

The space (H, G) is called a *J-space*. A J-space with finite rank of indefiniteness is called a *Pontryagin space*.

3.3 Topological Generalizations

- **Metametric space**

  A **metametric space** (Väisälä, 2003) is a pair (X, d), where X is a set, and d is a nonnegative symmetric function $d : X \times X \to \mathbb{R}$ such that $d(x, y) = 0$ implies $x = y$ and triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ holds for all $x, y, z \in X$.

  A metametric space is metrizable: the metametric d defines the same topology as the metric d' defined by $d'(x, x) = 0$ and $d'(x, y) = d(x, y)$ if $x \neq y$. A metametric d induces a Hausdorff topology with the usual definition of a *ball* $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. Any **partial metric** (Chap. 1) is a metametric.
- **Resemblance**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called (Batagelj-Bren, 1993) a **resemblance** on X if d is *symmetric* and if, for all $x, y \in X$, either $d(x, x) \leq d(x, y)$ (in which case d is called a **forward resemblance**), or $d(x, x) \geq d(x, y)$ (in which case d is called a **backward resemblance**).

  Every resemblance d induces a *strict partial order* $\prec$ on the set of all unordered pairs of elements of X by defining $\{x, y\} \prec \{u, v\}$ if and only if $d(x, y) < d(u, v)$.
- **w-distance**

  Given a metric space (X, d), a **w-distance** on X (Kada–Suzuki–Takahashi, 1996) is a nonnegative function $p : X \times X \to \mathbb{R}$ which satisfies the following conditions:

  1. $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
  2. for any $x \in X$, the function $p(x, .) : X \to \mathbb{R}$ is *lower semicontinuous*, i.e., if a sequence $\{y_n\}_n$ in X converges to $y \in X$, then $p(x, y) \leq \underline{\lim}_{n\to\infty} p(x, y_n)$;
  3. for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$, for each $x, y, z \in X$.

- **τ-distance space**

  A **τ-distance space** is a pair (X, f), where X is a topological space and f is an Aamri-Moutawakil's *τ-distance* on X, i.e., a nonnegative function $f : X \times X \to \mathbb{R}$ such that, for any $x \in X$ and any neighborhood U of x, there exists $\epsilon > 0$ with $\{y \in X : f(x, y) < \epsilon\} \subset U$.

  Any distance space (X, d) is a τ-distance space for the topology τ_f defined as follows: $A \in \tau_f$ if, for any $x \in X$, there exists $\epsilon > 0$ with $\{y \in X : f(x, y) < \epsilon\} \subset A$. However, there exist nonmetrizable τ-distance spaces. A τ-distance $f(x, y)$ need be neither symmetric, nor vanishing for $x = y$; for example, $e^{|x-y|}$ is a τ-distance on $X = \mathbb{R}$ with usual topology.

- **Proximity space**

  A **proximity space** (Efremovich, 1936) is a set X with a binary relation δ on the *power set* $P(X)$ of all of its subsets which satisfies the following conditions:

  1. $A\delta B$ if and only if $B\delta A$ (*symmetry*);
  2. $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$ (*additivity*);
  3. $A\delta A$ if and only if $A \neq \emptyset$ (*reflexivity*).

  The relation δ defines a **proximity** (or *proximity structure*) on X. If $A\delta B$ fails, the sets A and B are called *remote sets*.

  Every metric space (X, d) is a proximity space: define $A\delta B$ if and only if $d(A, B) = \inf_{x \in A, y \in B} d(x, y) = 0$.

  Every proximity on X induces a (**completely regular**) topology on X by defining the *closure operator* $cl : P(X) \to P(X)$ on the set of all subsets of X as $cl(A) = \{x \in X : \{x\}\delta A\}$.

- **Uniform space**

  A **uniform space** is a topological space (with additional structure) providing a generalization of metric space, based on **set-set distance**.

  A **uniform space** (Weil, 1937) is a set X with an *uniformity* (or *uniform structure*) $\mathcal{U}$, i.e., a nonempty collection of subsets of $X \times X$, called *entourages*, with the following properties:

  1. Every subset of $X \times X$ which contains a set of $\mathcal{U}$ belongs to $\mathcal{U}$;
  2. Every finite intersection of sets of $\mathcal{U}$ belongs to $\mathcal{U}$;
  3. Every set $V \in \mathcal{U}$ contains the *diagonal*, i.e., the set $\{(x, x) : x \in X\} \subset X \times X$;
  4. If V belongs to $\mathcal{U}$, then the set $\{(y, x) : (x, y) \in V\}$ belongs to $\mathcal{U}$;
  5. If V belongs to $\mathcal{U}$, then there exists $V' \in \mathcal{U}$ such that $(x, z) \in V$ whenever $(x, y), (y, z) \in V'$.

  Every metric space (X, d) is a uniform space. An entourage in (X, d) is a subset of $X \times X$ which contains the set $V_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$ for some positive real number ϵ. Other basic example of uniform space are *topological groups*.

  Every uniform space $(X, \mathcal{U})$ generates a topology consisting of all sets $A \subset X$ such that, for any $x \in A$, there is a set $V \in \mathcal{U}$ with $\{y : (x, y) \in V\} \subset A$.

  Every uniformity induces a **proximity** σ where $A\sigma B$ if and only if $A \times B$ has nonempty intersection with any entourage.

  A topological space admits a uniform structure inducing its topology if only if the topology is **completely regular** (Chap. 2) and, also, if only if it is a *gauge space*, i.e., the topology is defined by a $\geq$*-filter* of semimetrics.

- **Nearness space**

  A **nearness space** (Herrich, 1974) is a set X with a *nearness structure*, i.e., a nonempty collection $\mathcal{U}$ of families of subsets of X, called *near families*, with the following properties:

  1. Each family refining a near family is near;
  2. Every family with nonempty intersection is near;

3. V is near if $\{cl(A) : A \in V\}$ is near, where $cl(A)$ is $\{x \in X : \{\{x\}, A\} \in \mathcal{U}\}$;
4. $\emptyset$ is near, while the set of all subsets of X is not;
5. If $\{A \cup B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$ is near family, then so is $\mathcal{F}_1$ or $\mathcal{F}_2$.

   The **uniform spaces** are precisely **paracompact** nearness spaces.

- **Approach space**

  An **approach space** is a topological space providing a generalization of metric space, based on **point-set distance**.

  An **approach space** (Lowen, 1989) is a pair (X, D), where X is a set and D is a **point-set distance**, i.e., a function $X \times P(X) \to [0, \infty]$ (where $P(X)$ is the set of all subsets of X) satisfying, for all $x \in X$ and all $A, B \in P(X)$, the following conditions:

1. $D(x, \{x\}) = 0$;
2. $D(x, \{\emptyset\}) = \infty$;
3. $D(x, A \cup B) = \min\{D(x, A), D(x, B)\}$;
4. $D(x, A) \leq D(x, A^{\epsilon}) + \epsilon$ for any $\epsilon \in [0, \infty]$, where $A^{\epsilon} = \{x : D(x, A) \leq \epsilon\}$ is the "ϵ-ball" with center x.

Every metric space (X, d) (moreover, any extended quasi-semimetric space) is an approach space with $D(x, A)$ being the usual point-set distance $\min_{y \in A} d(x, y)$.

Given a **locally compact separable** metric space (X, d) and the family $\mathcal{F}$ of its nonempty closed subsets, the **Baddeley–Molchanov distance function** gives a tool for another generalization. It is a function $D : X \times \mathcal{F} \to \mathbb{R}$ which is lower semicontinuous with respect to its first argument, measurable with respect to the second, and satisfies the following two conditions: $F = \{x \in X : D(x, F) \leq 0\}$ for $F \in \mathcal{F}$, and $D(x, F_1) \geq D(x, F_2)$ for $x \in X$, whenever $F_1, F_2 \in \mathcal{F}$ and $F_1 \subset F_2$.

The additional conditions $D(x, \{y\}) = D(y, \{x\})$, and $D(x, F) \leq D(x, \{y\}) + D(y, F)$ for all $x, y \in X$ and every $F \in \mathcal{F}$, provide analogs of symmetry and the triangle inequality. The case $D(x, F) = d(x, F)$ corresponds to the usual point-set distance for the metric space (X, d); the case $D(x, F) = d(x, F)$ for $x \in X \backslash F$ and $D(x, F) = -d(x, X \backslash F)$ for $x \in X$ corresponds to the **signed distance function** in Chap. 1.

- **Metric bornology**

  Given a topological space X, a *bornology* of X is any family $\mathcal{A}$ of proper subsets A of X such that the following conditions hold:

1. $\cup_{A \in \mathcal{A}} A = X$;
2. $\mathcal{A}$ is an *ideal*, i.e., contains all subsets and finite unions of its members.

   The family $\mathcal{A}$ is a **metric bornology** ([Beer99]) if, moreover

3. $\mathcal{A}$ contains a countable base;
4. For any $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}$ such that the closure of A coincides with the interior of A'.

The metric bornology is called *trivial* if $\mathcal{A}$ is the set $P(X)$ of all subsets of X; such a metric bornology corresponds to the family of bounded sets of some bounded metric. For any noncompact **metrizable** topological space X, there exists an unbounded metric compatible with this topology. A nontrivial metric bornology on such a space X corresponds to the family of bounded subsets with respect to some such unbounded metric. A noncompact metrizable topological space X admits uncountably many nontrivial metric bornologies.

3.4 Beyond Numbers

- **Metric 1-space**

  A *category* Ψ consists (Eilenberg and MacLane, 1945) of a set $Ob(\Psi)$ of *objects*, a set $Mor(\Psi)$ of *morphisms* (or *arrows*)) and a set-valued map associating a set $\Psi(x, y)$ of arrows to each ordered pair of objects x, y, so that each arrow belongs to only one set $\Psi(x, y)$. An element of $\Psi(x, y)$ is also denoted by $f : x \to y$.

  Moreover, the composition $f \cdot g \in \Psi(x, z)$ of two arrows $f : x \to y, g : y \to z$ is defined, and it is associative. Finally, each set $\Psi(x, x)$ contains an *identity arrow* id_x such that $f \cdot id_x = f$ and $id_x \cdot g = g$ for any arrows $f : y \to x$ and $g : x \to z$. Cf. **category of metric spaces** in Chap. 1.

  Weiss defined in [Weis12] a **metric 1-space** as a category Ψ together with a weight-function $w : \Psi(x, y) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ on arrows, which satisfies

  1. $w(id_x) = 0$ holds for each object $x \in Ob(\Psi)$ (*reflexivity*).
  2. $|w(g) - w(f)| \leq w(g \cdot f) \leq w(g) + w(f)$ holds for any objects x, y, z and arrows $f : x \to y, g : y \to z$ (*full triangle inequality*).

  Any set X produces an *indiscrete category* I_X, in which $Ob(I_X) = X$ and $|I_X(x, y)| = 1$ for all $x, y \in X$. Any metric space (X, d) produces a metric 1-space on I_X by defining $w(f) = d(x, y)$, and it is unique metric 1-space on I_X. But, in general, the function w on arrows can be seen as a multivalued function on $Ob \times Ob$.

  [Weis12] also outlined a **metric** m**-space** as a kind of an m**-hemimetric** on an m*-category* consisting of i-dimensional cells, $0 \leq i \leq m$ (objects, arrows, ...) and a associative-like composition rule for the cells with matching boundaries.
- V**-continuity space**

  Let $(V, \wedge, \vee)$ be a *complete* (having $\wedge S := \wedge_{x \in S} x$ and $\vee S := \vee_{x \in S}$ for all $S \subseteq V$) lattice with bottom element 0. For $a, b \in V$, a is said to be *well above* b, denoted by $b \prec a$, if given any $S \subseteq V$ such that $\wedge S \prec b$, there exists $s \in S$ with $s \prec a$.

  A *value quantale* is a pair $(V, +)$, where V is a complete lattice and $+$ is an associative and commutative operation o such that for all $a, b \in V$ and $S \subseteq V$,

  1. $a + \wedge S = \wedge(a + S)$,
  2. $a + 0 = a$,
  3. $a = \wedge\{b \in Va \prec b\}$,
  4. $0 \prec a \wedge b$ if $0 \prec a, b$.

A V**-continuity space** is (Flagg–Koperman, 1997) a triple (X, d, V), where V is a value quantale, X is a set, and $d : X \times X \to V$ is a function satisfying

$$d(x, x) = 0 \quad \text{and} \quad d(x, z) \leq d(x, y) + d(y, z).$$

Any extended quasi-semimetric space is a V-continuity space, where V is the value quantale $[0, \infty]$, seen as a complete lattice, with ordinary addition.

Weiss, 2013, showed that taken with continuous functions, the categories of all V-continuity spaces and of all topological spaces are equivalent. In particular, every topological space (X, τ) is "metrizable" in the sense that there exists a V-continuity space (X, d, V) such that τ is the topology generated by *open balls* $\{y \in X :\prec \epsilon\}$.

- **Probabilistic metric space**

A notion of **probabilistic metric space** is a generalization of the notion of metric space (see, for example, [ScSk83]) in two ways: distances become probability distributions, and the sum in the triangle inequality becomes a **triangle operation**.

Formally, let A be the set of all *probability distribution functions*, whose support lies in $[0, \infty]$. For any $a \in [0, \infty]$ define *step functions* $\epsilon_a \in A$ by $\epsilon_a(x) = 1$ if $x > a$ or $x = \infty$, and $\epsilon_a(x) = 0$, otherwise. The functions in A are ordered by defining $F \leq G$ to mean $F(x) \leq G(x)$ for all $x \geq 0$; the minimal element is ϵ_0.

A commutative and associative operation τ on A is called a **triangle function** if $\tau(F, \epsilon_0) = F$ for any $F \in A$ and $\tau(E, F) \leq \tau(G, H)$ whenever $E \leq G, F \leq H$. The semigroup (A, τ) generalizes the group $(\mathbb{R}, +)$.

A **probabilistic metric space** is a triple (X, D, τ), where X is a set, D is a function $X \times X \to A$, and τ is a triangle function, such that for any $p, q, r \in X$

1. $D(p, q) = \epsilon_0$ if and only if $p = q$;
2. $D(p, q) = D(q, p)$;
3. $D(p, r) \geq \tau(D(p, q), D(q, r))$.

For any metric space (X, d) and any triangle function τ, such that $\tau(\epsilon_a, \epsilon_b) \geq \epsilon_{a+b}$ for all $a, b \geq 0$, the triple $(X, D = \epsilon_{d(x,y)}, \tau)$ is a probabilistic metric space.

For any $x \geq 0$, the value $D(p, q)$ at x can be interpreted as "the probability that the distance between p and q is less than x"; this was approach of Menger, who proposed in 1942 the original version, *statistical metric space*, of this notion.

A probabilistic metric space is called a *Wald space* if the triangle function is a convolution, i.e., of the form $\tau_x(E, F) = \int_{\mathbb{R}} E(x - t) dF(t)$.

A probabilistic metric space is called a **generalized Menger space** if the triangle function has form $\tau_x(E, F) = \sup_{u+v=x} T(E(u), F(v))$ for a *t-norm* T, i.e., such a commutative and associative operation on $[0, 1]$ that $T(a, 1) = a$, $T(0, 0) = 0$ and $T(c, d) \geq T(a, b)$ whenever $c \geq a, d \geq b$.

- **Fuzzy metric spaces**

A *fuzzy subset* of a set S is a mapping $\mu : S \to [0, 1]$, where $\mu(x)$ represents the "degree of membership" of $x \in S$.

A *continuous t-norm* is a binary commutative and associative continuous operation T on $[0,1]$, such that $T(a,1) = a$ and $T(c,d) \geq T(a,b)$ whenever $c \geq a, d \geq b$.

A **KM fuzzy metric space** (Kramosil–Michalek, 1975) is a pair $(X,(\mu,T))$, where X is a nonempty set and a *fuzzy metric* (μ,T) is a pair comprising a continuous t-norm T and a fuzzy set $\mu : X^2 \times \mathbb{R}_{\geq 0} \to [0,1]$, such that, for $x,y,z \in X$ and $s,t \geq 0$, the following conditions hold:

1. $\mu(x,y,0) = 0$;
2. $\mu(x,y,t) = 1$ if and only if $x = y, t > 0$;
3. $\mu(x,y,t) = \mu(y,x,t)$;
4. $T(\mu(x,y,t),\mu(y,z,s)) \leq \mu(x,z,t+s)$;
5. the function $\mu(x,y,\cdot) : \mathbb{R}_{\geq 0} \to [0,1]$ is left continuous.

A KM fuzzy metric space is called also a **fuzzy Menger space** since by defining $D_t(p,q) = \mu(p,q,t)$ one gets a **generalized Menger space**. The following modification of the above notion, using a stronger form of metric fuzziness, it a generalized Menger space with $D_t(p,q)$ positive and continuous on $\mathbb{R}_{>0}$ for all p,q.

A **GV fuzzy metric space** (George–Veeramani, 1994) is a pair $(X,(\mu,T))$, where X is a nonempty set, and a *fuzzy metric* (μ,T) is a pair comprising a continuous t-norm T and a fuzzy set $\mu : X^2 \times \mathbb{R}_{>0} \to [0,1]$, such that for $x,y,z \in X$ and $s,t > 0$

1. $\mu(x,y,t) > 0$;
2. $\mu(x,y,t) = 1$ if and only if $x = y$;
3. $\mu(x,y,t) = \mu(y,x,t)$;
4. $T(\mu(x,y,t),\mu(y,z,s)) \leq \mu(x,z,t+s)$;
5. the function $\mu(x,y,\cdot) : \mathbb{R}_{>0} \to [0,1]$ is continuous.

An example of a GV fuzzy metric space comes from any metric space (X,d) by defining $T(a,b) = b - ab$ and $\mu(x,y,t) = \frac{t}{t+d(x,y)}$. Conversely, any GV fuzzy metric space (and also any KM fuzzy metric space) generates a metrizable topology. Most GV fuzzy metrics are *strong*, i.e., $T(\mu(x,y,t),\mu(y,z,t)) \leq \mu(x,z,t)$ holds.

A *fuzzy number* is a fuzzy set $\mu : \mathbb{R} \to [0,1]$ which is *normal* ($\{x \in \mathbb{R} : \mu(x) = 1\} \neq \emptyset$), *convex* ($\mu(tx + (1-t)y) \geq \min\{\mu(x),\mu(y)\}$ for every $x,y \in \mathbb{R}$ and $t \in [0,1]$) and *upper semicontinuous* (at each point x_0, the values $\mu(x)$ for x near x_0 are either close to $\mu(x_0)$ or less than $\mu(x_0)$). Denote the set of all fuzzy numbers which are *nonnegative*, i.e., $\mu(x) = 0$ for all $x < 0$, by G. The additive and multiplicative identities of fuzzy numbers are denoted by $\tilde{0}$ and $\tilde{1}$, respectively. The *level set* $[\mu]_t = \{x : \mu(x) \geq t\}$ of a fuzzy number μ is a closed interval.

Given a nonempty set X and a mapping $d : X^2 \to G$, let the mappings $L, R : [0,1]^2 \to [0,1]$ be symmetric and nondecreasing in both arguments and satisfy $L(0,0) = 0$, $R(1,1) = 1$. For all $x,y \in X$ and $t \in (0,1]$, let $[d(x,y)]_t = [\lambda_t(x,y), \rho_t(x,y)]$.

A **KS fuzzy metric space** (Kaleva–Seikkala, 1984) is a quadruple (X, d, L, R) with *fuzzy metric* d, if for all $x, y, z \in X$

1. $d(x, y) = \tilde{0}$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y)(s + t) \geq L(d(x, z)(s), d(z, y)(t))$ whenever $s \leq \lambda_1(x, z)$, $t \leq \lambda_1(z, y)$, and $s + t \leq \lambda_1(x, y)$;
4. $d(x, y)(s + t) \leq R(d(x, z)(s), d(z, y)(t))$ whenever $s \geq \lambda_1(x, z)$, $t \geq \lambda_1(z, y)$, and $s + t \geq \lambda_1(x, y)$.

The following functions are some frequently used choices for L and R:

$$\max\{a + b - 1, 0\}, ab, \min\{a, b\}, \max\{a, b\}, a + b - ab, \min\{a + b, 1\}.$$

Several other notions of **fuzzy metric space** were proposed, including those by Erceg, 1979, Deng, 1982, and Voxman, 1998, Xu–Li, 2001, Tran–Duckstein, 2002, Chakraborty–Chakraborty, 2006. Cf. also **metrics between fuzzy sets**, **fuzzy Hamming distance**, **gray-scale image distances** and **fuzzy polynucleotide metric** in Chaps. 1, 11, 21 and 23, respectively.

- **Interval-valued metric space**

Let $I(\mathbb{R}_{\geq 0})$ denote the set of closed intervals of $\mathbb{R}_{\geq 0}$.

An **interval-valued metric space** (Coppola–Pacelli, 2006) is a pair $((X, \leq), \Delta)$, where $(X, \leq)$ is a partially ordered set and Δ is an interval-valued mapping $\Delta : X \times X \to I(\mathbb{R}_{\geq 0})$, such that for every $x, y, z \in X$

1. $\Delta(x, x) \star [0, 1] = \Delta(x, x)$;
2. $\Delta(x, y) = \Delta(y, x)$;
3. $\Delta(x, y) - \Delta(z, z) \preceq \Delta(x, z) + \Delta(z, y)$;
4. $\Delta(x, y) - \Delta(x, y) \preceq \Delta(x, x) + \Delta(y, y)$;
5. $x \leq x'$ and $y \leq y'$ imply $\Delta(x, y) \subseteq \Delta(x', y')$;
6. $\Delta(x, y) = 0$ if and only if $x = y$ and x, y are *atoms* (minimal elements of $(X, \leq)$).

Here the following *interval arithmetic* rules hold: $[u, v] \preceq [u', v']$ if and only if $u \leq u'$,

$[u, v] + [u', v'] = [u + u', v + v']$, $[u, v] - [u', v'] = [u - u', v - v']$,
$[u, v] \star [u', v'] = [\min\{uu', uv', vu', vv'\}, \max\{uu', uv', vu', vv'\}]$ and
$\frac{[u,v]}{[u',v']} = [\min\{\frac{u}{u'}, \frac{u}{v'}, \frac{v}{u'}, \frac{v}{v'}\}, \max\{\frac{u}{u'}, \frac{u}{v'}, \frac{v}{u'}, \frac{v}{v'}\}]$ when $0 \notin [u', v']$.

The addition and multiplication operations are commutative, associative and *subdistributive*: it holds $X \star (Y + Z) \subset (X \star Y + X \star Z)$.

Cf. **metric between intervals** in Chap. 10.

The usual metric spaces coincide with above spaces in which all $x \in X$ are atoms.

- **Direction distance**

Given a normed real vector space $(V, ||.||)$, for any $x \in V \setminus \{0\}$, denote by $[x]$ the *direction* (ray) $\{\lambda x : \lambda > 0\}$ and by x_0 the point $\frac{x}{||x||}$. An *oriented angle* is an ordered pair $([x], [y])$ of directions. The **direction distance** from x to y is defined (Busch–Ruch, 1992) as the family of distances $||\alpha x_0 - \beta y_0||$ with $\alpha, \beta \in \mathbb{R}_{>0}$.

The **mixing distance** is defined as the restriction of the direction distance to pairs of directions in the cone $\{\lambda v : v \in V, \lambda > 0\}$. In fact, authors introduced these distances on some special normed spaces used in Quantum Mechanics.

- **Generalized metric**

  Let X be a set. Let $(V, +, \leq)$ be an *ordered semigroup* (not necessarily commutative) with a least element θ and with $x \leq y, x_1 \leq y_1$ implying $x + x_1 \leq y + y_1$. Let $(V, +)$ be also endowed with an order-preserving *involution* x^* (i.e., $(x^*)^* = x$), which is operation-reversing, i.e., $(x + y)^* = y^* + x^*$.

  A function $d : X \times X \to G$ is called (Li–Wang–Pouzet, 1987) a **generalized metric** over $(V, +, \leq)$ if the following conditions hold:

  1. $d(x, y) = \theta$ if and only if $x = y$;
  2. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$;
  3. $d^*(x, y) = d(y, x)$.

- **Cone metric**

  Let C be a *proper cone* in a real Banach space W, i.e., C is closed, $C \neq \emptyset$, the interior of C is not equal to $\{\theta\}$ (where θ is the zero vector in W) and

  1. if $x, y \in C$ and $a, b \in \mathbb{R}_{\geq 0}$, then $ax + by \in C$;
  2. if $x \in C$ and $-x \in C$, then $x = 0$.

  Define a partial ordering $(W, \leq)$ on W by letting $x \leq y$ if $y - x \in C$. The following variation of **generalized metric** and **partially ordered distance** was defined in Huang–Zhang, 2007, and, partially, in Rzepecki, 1980. Given a set X, a **cone metric** is a mapping $d : X \times X \to (W, \leq)$ such that

  1. $\theta \leq d(x, y)$ with equality if and only if $x = y$;
  2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
  3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$;

  The pair (X, d) is called a **cone metric space**.

- ***W*-distance on building**

  Let X be a set, and let $(W, \cdot, 1)$ be a group. A *W-distance* on X is a W-valued map $\sigma : X \times X \to W$ having the following properties:

  1. $\sigma(x, y) = 1$ if and only if $x = y$;
  2. $\sigma(y, x) = (\sigma(x, y))^{-1}$.

  A natural W-distance on W is $\sigma(x, y) = x^{-1}y$.

  A *Coxeter group* is a group $(W, \cdot, 1)$ generated by the elements

$$\{w_1, \ldots, w_n : (w_i w_j)^{m_{ij}} = 1, 1 \leq i, j \leq n\}.$$

  Here $M = ((m_{ij}))$ is a *Coxeter matrix*, i.e., an arbitrary symmetric $n \times n$ matrix with $m_{ii} = 1$, and the other values are positive integers or ∞. The *length* $l(x)$ of $x \in W$ is the smallest number of generators $w_1, \ldots, w_n$ needed to represent x.

Let X be a set, let $(W, \cdot, 1)$ be a Coxeter group and let $\sigma(x, y)$ be a W-distance on X. The pair (X, σ) is called (Tits, 1981) a *building* over $(W, \cdot, 1)$ if it holds

1. the relation $\sim_i$ defined by $x \sim_i y$ if $\sigma(x, y) = 1$ or w_i, is an equivalence relation;
2. given $x \in X$ and an equivalence class C of $\sim_i$, there exists a unique $y \in C$ such that $\sigma(x, y)$ is *shortest* (i.e., of smallest length), and $\sigma(x, y') = \sigma(x, y)w_i$ for any $y' \in C, y' \neq y$.

The **gallery distance on building** d is a usual metric on X defined by $l(d(x, y))$. The distance d is the **path metric** in the graph with the vertex-set X and xy being an edge if $\sigma(x, y) = w_i$ for some $1 \leq i \leq n$. The gallery distance on building is a special case of a **gallery metric** (of *chamber system* X).

- **Boolean metric space**

A *Boolean algebra* (or *Boolean lattice*) is a *distributive lattice* $(B, \vee, \wedge)$ admitting a least element 0 and greatest element 1 such that every $x \in B$ has a *complement* $\overline{x}$ with $x \vee \overline{x} = 1$ and $x \wedge \overline{x} = 0$.

Let X be a set, and let $(B, \vee, \wedge)$ be a Boolean algebra. The pair (X, d) is called (Blumenthal, 1953) a **Boolean metric space** over B if the function $d : X \times X \rightarrow B$ has the following properties:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) \leq d(x, z) \vee d(z, y)$ for all $x, y, z \in X$.

- **Space over algebra**

A **space over algebra** is a metric space with a differential-geometric structure, whose points can be provided with coordinates from some *algebra* (usually, an associative algebra with identity).

A *module* over an algebra is a generalization of a vector space over a field, and its definition can be obtained from the definition of a vector space by replacing the field by an associative algebra with identity. An *affine space over an algebra* is a similar generalization of an *affine space* over a field. In affine spaces over algebras one can specify a Hermitian metric, while in the case of commutative algebras even a quadratic metric can be given. To do this one defines in a unital module a *scalar product* $\langle x, y \rangle$, in the first case with the property $\langle x, y \rangle = J(\langle y, x \rangle)$, where J is an *involution* of the algebra, and in the second case with the property $\langle y, x \rangle = \langle x, y \rangle$.

The n-dimensional *projective space over an algebra* is defined as the variety of one-dimensional submodules of an $(n + 1)$-dimensional unital module over this algebra. The introduction of a *scalar product* $\langle x, y \rangle$ in a unital module makes it possible to define a Hermitian metric in a projective space constructed by means of this module or, in the case of a commutative algebra, quadratic elliptic and hyperbolic metrics. The metric invariant of the points of these spaces is the **cross-ratio** $W = \langle x, x \rangle^{-1} \langle x, y \rangle \langle y, y \rangle^{-1} \langle y, x \rangle$. If W is a real number, then $w = \arccos \sqrt{W}$ is called the **distance** between x and y **in the space over algebra**.

- **Partially ordered distance**

  Let X be a set. Let $(G, \leq)$ be a *partially ordered set* with a least element g_0. A **partially ordered distance** is a function $d : X \times X \to G$ such that, for any $x, y \in X$, $d(x, y) = g_0$ if and only if $x = y$.

  A **generalized ultrametric** (Priess-Crampe and Ribenboim, 1993) is a symmetric (i.e., $d(x, y) = d(y, x)$) partially ordered distance, such that $d(z, x) \leq g$ and $d(z, y) \leq g$ imply $d(x, y) \leq g$ for any $x, y, z \in X$ and $g \in G$.

  Suppose that $G' = G \backslash \{g_0\} \neq \emptyset$ and, for any $g_1, g_2 \in G'$, there exists $g_3 \in G'$ such that $g_3 \leq g_1$ and $g_3 \leq g_2$. Consider the following possible properties:

  1. For any $g_1 \in G'$, there exists $g_2 \in G'$ such that, for any $x, y \in X$, from $d(x, y) \leq g_2$ it follows that $d(y, x) \leq g_1$;
  2. For any $g_1 \in G'$, there exist $g_2, g_3 \in G'$ such that, for any $x, y, z \in X$, from $d(x, y) \leq g_2$ and $d(y, z) \leq g_3$ it follows that $d(x, z) \leq g_1$;
  3. For any $g_1 \in G'$, there exists $g_2 \in G'$ such that, for any $x, y, z \in X$, from $d(x, y) \leq g_2$ and $d(y, z) \leq g_2$ it follows that $d(y, x) \leq g_1$;
  4. G' has no first element;
  5. $d(x, y) = d(y, x)$ for any $x, y \in X$;
  6. For any $g_1 \in G'$, there exists $g_2 \in G'$ such that, for any $x, y, z \in X$, from $d(x, y) <^* g_2$ and $d(y, z) <^* g_2$ it follows that $d(x, z) <^* g_1$; here $p <^* q$ means that either $p < q$, or p is not comparable to q;
  7. The order relation $<$ is a total ordering of G.

  In terms of above properties, d is called: the **Appert partially ordered distance** if 1 and 2 hold; the **Golmez partially ordered distance of first type** if 4, 5, and 6 hold; the **Golmez partially ordered distance of second type** if 3, 4, and 5 hold; the **Kurepa–Fréchet distance** if 3, 4, 5, and 7 hold.

  The case $G = \mathbb{R}_{\geq 0}$ of the Kurepa–Fréchet distance corresponds to the **Fréchet V-space**; cf. the **f-quasi-metric** in Sect. 1.1. The general case was considered in Kurepa, 1934, and rediscovered in Fréchet, 1946.

- **Distance from measurement**

  Distance from measurement is an analog of distance on domains in Computer Science; it was developed in [Mart00].

  A *po* (partially ordered set) $(D, \preceq)$ is called *dcpo* (directed-complete po) if every *directed subset* $S \subset D$ (i.e., $S \neq \emptyset$ and any pair $x, y \in S$ is *bounded*: there is $z \in S$ with $x, y \preceq z$) has a *supremum* $\sqcup S$, i.e., the least of such upper bounds z.

  For $x, y \in D$, y is an *approximation* of x if, for all directed subsets $S \subset D$, $x \preceq \sqcup S$ implies $y \preceq s$ for some $s \in S$. A dcpo $(D, \preceq)$ is *continuous* if for all $x \in D$ the set of all approximations of x is directed and x is its supremum. A *domain* is a continuous dcpo $(D, \preceq)$ such that for all $x, y \in D$ there is $z \in D$ with $z \preceq x, y$. A *Scott domain* is a domain with least element, in which any bounded pair has a supremum.

  A subset U of a dcpo $(D, \preceq)$ is *Alexandrov open* if, for any $x \in U$ and $y \in D$, $x \preceq y$ implies $y \in U$; it is *Scott open* if also, for any directed subset $S \subset D$, $\sqcup S \in U$ implies $S \cap U \neq \emptyset$. The set of Scott open sets form the *Scott topology*; it is a **T_0-space** (Chap. 2) with generalized metrization by a **partial metric** (Chap. 1).

A *measurement* is a mapping $\mu : D \to \mathbb{R}_{\geq 0}$ between dcpo $(D, \preceq)$ and dcpo $(\mathbb{R}_{\geq 0}, \preceq)$, where $\mathbb{R}_{\geq 0}$ is ordered as $x \preceq y$ if $y \leq x$, such that

1. $x \preceq y$ implies $\mu(x) \preceq \mu(y)$;
2. $\mu(\sqcup S) = \sqcup(\{\mu(s) : s \in S\})$ for every directed subset $S \subset D$;
3. For all $x \in D$ with $\mu(x) = 0$ and all sequences $(x_n), n \to \infty$, of approximations of x with $\lim_{n\to\infty} \mu(x_n) = \mu(x)$, one has $\sqcup(\cup_{n=1}^{\infty}\{x_n\}) = x$.

Given a measurement μ, the **distance from measurement** is a mapping $d : D \times D \to \mathbb{R}_{\geq 0}$ given by

$$d(x, y) = \inf\{\mu(z) : z \text{ approximates } x, y\} = \inf\{\mu(z) : z \preceq x, y\}.$$

One has $d(x, x) \preceq \mu(x)$. The function $d(x, y)$ is a metric on the set $\{x \in D : \mu(x) = 0\}$ if μ satisfies the following **measurement triangle inequality**: for all bounded pairs $x, y \in D$, there is an element $z \preceq x, y$ such that $\mu(z) \leq \mu(x) + \mu(y)$.

Waszkiewicz, 2001, found topological connections between topologies coming from a distance from measurement and from a **partial metric** defined in Chap. 1.