Chapter 10
Distances in Algebra

10.1 Group Metrics

A *group* $(G, \cdot, e)$ is a set G of elements with a binary operation $\cdot$, called the *group operation*, that together satisfy the four fundamental properties of *closure* ($x \cdot y \in G$ for any $x, y \in G$), *associativity* ($x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for any $x, y, z \in G$), the *identity property* ($x \cdot e = e \cdot x = x$ for any $x \in G$), and the *inverse property* (for any $x \in G$, there exists an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$).

In additive notation, a group $(G, +, 0)$ is a set G with a binary operation $+$ such that the following properties hold: $x+y \in G$ for any $x, y \in G$, $x+(y+z) = (x+y)+z$ for any $x, y, z \in G$, $x + 0 = 0 + x = x$ for any $x \in G$, and, for any $x \in G$, there exists an element $-x \in G$ such that $x + (-x) = (-x) + x = 0$.

A group $(G, \cdot, e)$ is called *finite* if the set G is finite. A group $(G, \cdot, e)$ is called *Abelian* if it is *commutative*, i.e., $x \cdot y = y \cdot x$ for any $x, y \in G$.

Most metrics considered in this section are **group norm metrics** on a group $(G, \cdot, e)$, defined by

$$||x \cdot y^{-1}||$$

(or, sometimes, by $||y^{-1} \cdot x||$), where $||.||$ is a *group norm*, i.e., a function $||.|| : G \to \mathbb{R}$ such that, for any $x, y \in G$, we have the following properties:

1. $||x|| \geq 0$, with $||x|| = 0$ if and only if $x = e$;
2. $||x|| = ||x^{-1}||$;
3. $||x \cdot y|| \leq ||x|| + ||y||$ (*triangle inequality*).

In additive notation, a group norm metric on a group $(G, +, 0)$ is defined by $||x + (-y)|| = ||x - y||$, or, sometimes, by $||(-y) + x||$.

The simplest example of a group norm metric is the **bi-invariant ultrametric** (sometimes called the *Hamming metric*) $||x \cdot y^{-1}||_H$, where $||x||_H = 1$ for $x \neq e$, and $||e||_H = 0$.

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- **Bi-invariant metric**

A metric (in general, a semimetric) d on a group $(G, \cdot, e)$ is called **bi-invariant** if

$$d(x, y) = d(x \cdot z, y \cdot z) = d(z \cdot x, z \cdot y)$$

for any $x, y, z \in G$ (cf. **translation invariant metric** in Chap. 5). Any **group norm metric** on an Abelian group is bi-invariant.

A metric (in general, a semimetric) d on a group $(G, \cdot, e)$ is called a **right-invariant metric** if $d(x, y) = d(x \cdot z, y \cdot z)$ for any $x, y, z \in G$, i.e., the operation of right multiplication by an element z is a **motion** of the metric space (G, d). Any group norm metric defined by $||x \cdot y^{-1}||$, is right-invariant.

A metric (in general, a semimetric) d on a group $(G, \cdot, e)$ is called a **left-invariant metric** if $d(x, y) = d(z \cdot x, z \cdot y)$ holds for any $x, y, z \in G$, i.e., the operation of left multiplication by an element z is a motion of the metric space (G, d). Any group norm metric defined by $||y^{-1} \cdot x||$, is left-invariant.

Any right-invariant or left-invariant (in particular, bi-invariant) metric d on G is a group norm metric, since one can define a group norm on G by $||x|| = d(x, 0)$.

- ***G*-invariant metric**

Given a metric space (X, d) and an action $g(x)$ of a group G on it, the metric d is called ***G*-invariant** (under this action) if for all $x, y \in X, g \in G$ it holds

$$d(g(x), g(y)) = d(x, y).$$

For every G-invariant metric d_X on X and every point $x \in X$, the function

$$d_G(g_1, g_2) = d_X(g_1(x), g_2(x))$$

is a **left-invariant metric** on G. This metric is called **orbit metric** in [BBI01], since it is the restriction of d on the orbit Gx, which can be identified with G.

- **Positively homogeneous distance**

A distance d on an Abelian group $(G, +, 0)$ is called **positively homogeneous** if

$$d(mx, my) = md(x, y)$$

for all $x, y \in G$ and all $m \in \mathbb{N}$, where mx is the sum of m terms all equal to x.

- **Translation discrete metric**

A **group norm metric** (in general, a group norm semimetric) on a group $(G, \cdot, e)$ is called **translation discrete** if the *translation distances* (or *translation numbers*)

$$\tau_G(x) = \lim_{n \to \infty} \frac{||x^n||}{n}$$

of the *nontorsion elements* x (i.e., such that $x^n \neq e$ for any $n \in \mathbb{N}$) of the group with respect to that metric are bounded away from zero.

If the numbers $\tau_G(x)$ are just nonzero, such a group norm metric is called a **translation proper metric**.

- **Word metric**

Let $(G, \cdot, e)$ be a finitely-generated group with a set A of generators (i.e., A is finite, and every element of G can be expressed as a product of finitely many elements A and their inverses). The *word length* $w_W^A(x)$ of an element $x \in G \backslash \{e\}$ is defined by

$$w_W^A(x) = \inf\{r : x = a_1^{\epsilon_1} \dots a_r^{\epsilon_r}, a_i \in A, \epsilon_i \in \{\pm 1\}\} \text{ and } w_W^A(e) = 0.$$

The **word metric** d_W^A *associated with* A is a **group norm metric** on G defined by

$$w_W^A(x \cdot y^{-1}).$$

As the word length w_W^A is a *group norm* on G, d_W^A is **right-invariant**. Sometimes it is defined as $w_W^A(y^{-1} \cdot x)$, and then it is **left-invariant**. In fact, d_W^A is the maximal metric on G that is right-invariant, and such that the distance from any element of A or A^{-1} to the identity element e is equal to one.

If A and B are two finite sets of generators of the group $(G, \cdot, e)$, then the identity mapping between the metric spaces (G, d_W^A) and (G, d_W^B) is a **quasi-isometry**, i.e., the word metric is unique up to quasi-isometry.

The word metric is the **path metric** of the *Cayley graph* Γ of $(G, \cdot, e)$, constructed with respect to A. Namely, Γ is a graph with the vertex-set G in which two vertices x and $y \in G$ are connected by an edge if and only if $y = a^\epsilon x$, $\epsilon = \pm 1$, $a \in A$.

- **Weighted word metric**

Let $(G, \cdot, e)$ be a finitely-generated group with a set A of generators. Given a bounded *weight function* $w : A \to (0, \infty)$, the *weighted word length* $w_{WW}^A(x)$ of an element $x \in G \backslash \{e\}$ is defined by $w_{WW}^A(e) = 0$ and

$$w_{WW}^A(x) = \inf\left\{\sum_{i=1}^{t} w(a_i), t \in \mathbb{N} : x = a_1^{\epsilon_1} \dots a_t^{\epsilon_t}, a_i \in A, \epsilon_i \in \{\pm 1\}\right\}.$$

The **weighted word metric** d_{WW}^A *associated with* A is a **group norm metric** on G defined by

$$w_{WW}^A(x \cdot y^{-1}).$$

As the weighted word length w_{WW}^A is a *group norm* on G, d_{WW}^A is **right-invariant**. Sometimes it is defined as $w_{WW}^A(y^{-1} \cdot x)$, and then it is **left-invariant**.

The metric d_{WW}^{A} is the supremum of semimetrics d on G with the property that $d(e,a) \leq w(a)$ for any $a \in A$.

The metric d_{WW}^{A} is a **coarse-path metric**, and every right-invariant coarse path metric is a weighted word metric up to **coarse isometry**.

The metric d_{WW}^{A} is the **path metric** of the *weighted Cayley graph* Γ_W of $(G,\cdot,e)$ constructed with respect to A. Namely, Γ_W is a weighted graph with the vertex-set G in which two vertices x and $y \in G$ are connected by an edge with the weight $w(a)$ if and only if $y = a^{\epsilon}x$, $\epsilon = \pm 1$, $a \in A$.

- **Interval norm metric**

  An **interval norm metric** is a **group norm metric** on a finite group $(G,\cdot,e)$ defined by

$$||x \cdot y^{-1}||_{int},$$

where $||.||_{int}$ is an *interval norm* on G, i.e., a *group norm* such that the values of $||.||_{int}$ form a set of consecutive integers starting with 0.

To each interval norm $||.||_{int}$ corresponds an ordered *partition* $\{B_0,\dots,B_m\}$ of G with $B_i = \{x \in G : ||x||_{int} = i\}$; cf. **Sharma–Kaushik distance** in Chap. 16. The *Hamming* and *Lee* norms are special cases of interval norm. A *generalized Lee norm* is an interval norm for which each class has a form $B_i = \{a, a^{-1}\}$.

- ***C*-metric**

  A ***C*-metric** d is a metric on a group $(G,\cdot,e)$ satisfying the following conditions:

1. The values of d form a set of consecutive integers starting with 0;
2. The cardinality of the sphere $B(x,r) = \{y \in G : d(x,y) = r\}$ is independent of the particular choice of $x \in G$.

The **word metric**, the **Hamming metric**, and the **Lee metric** are C-metrics. Any **interval norm metric** is a C-metric.

- **Order norm metric**

  Let $(G,\cdot,e)$ be a finite Abelian group. Let $ord(x)$ be the *order* of an element $x \in G$, i.e., the smallest positive integer n such that $x^n = e$. Then the function $||.||_{ord} : G \to \mathbb{R}$ defined by $||x||_{ord} = \ln ord(x)$, is a *group norm* on G, called the *order norm*.

  The **order norm metric** is a **group norm metric** on G, defined by

$$||x \cdot y^{-1}||_{ord}.$$

- **Tărnăuceanu metric**

  Let $o(a)$ denote the order of the element a of a group. Let C be the class of finite groups G in which $o(ab) < o(a) + o(b)$ for every $a, b \in G$. Tărnăuceanu, 2015, noted that the function $d : G \times G \to \mathbb{N}$ defined by

$$d(x, y) = o(xy^{-1}) - 1$$

for all $x, y \in G$ is a metric on G if and only if $G \in C$.

He found that C contains all Abelian p-groups, Q_8, and A_4, but not nonabelian finite simple groups, alternating groups $A(n)$ with $n \geq 5$, and, for $n \geq 4$, $Sym(n)$, quaternion groups Q_{2^n}, dihedral groups D_{2n}. C is closed under subgroups, but not under direct products or extensions. The centralizers of nontrivial elements of such groups contain only elements of prime power order.

- **Monomorphism norm metric**

Let $(G, +, 0)$ be a group. Let $(H, \cdot, e)$ be a group with a *group norm* $||.||_H$. Let $f : G \to H$ be a *monomorphism* of groups G and H, i.e., an injective function such that $f(x+y) = f(x) \cdot f(y)$ for any $x, y \in G$. Then the function $||.||^f_G : G \to \mathbb{R}$ defined by $||x||^f_G = ||f(x)||_H$, is a *group norm* on G, called the *monomorphism norm*.

The **monomorphism norm metric** is a **group norm metric** on G defined by

$$||x - y||^f_G.$$

- **Product norm metric**

Let $(G, +, 0)$ be a group with a *group norm* $||.||_G$. Let $(H, \cdot, e)$ be a group with a group norm $||.||_H$. Let $G \times H = \{\alpha = (x, y) : x \in G, y \in H\}$ be the Cartesian product of G and H, and $(x, y) \cdot (z, t) = (x + z, y \cdot t)$.

Then the function $||.||_{G\times H} : G\times H \to \mathbb{R}$ defined by $||\alpha||_{G\times H} = ||(x, y)||_{G\times H} = ||x||_G + ||y||_H$, is a group norm on $G \times H$, called the *product norm*.

The **product norm metric** is a **group norm metric** on $G \times H$ defined by

$$||\alpha \cdot \beta^{-1}||_{G\times F}.$$

On the Cartesian product $G \times H$ of two finite groups with the *interval norms* $||.||^{int}_G$ and $||.||^{int}_H$, an interval norm $||.||^{int}_{G\times H}$ can be defined. In fact, $||\alpha||^{int}_{G\times H} = ||(x, y)||^{int}_{G\times H} = ||x||_G + (m + 1)||y||_H$, where $m = \max_{a\in G} ||a||^{int}_G$.

- **Quotient norm metric**

Let $(G, \cdot, e)$ be a group with a *group norm* $||.||_G$. Let $(N, \cdot, e)$ be a *normal subgroup* of $(G, \cdot, e)$, i.e., $xN = Nx$ for any $x \in G$. Let $(G/N, \cdot, eN)$ be the *quotient group* of G, i.e., $G/N = \{xN : x \in G\}$ with $xN = \{x \cdot a : a \in N\}$, and $xN \cdot yN = xyN$. Then the function $||.||_{G/N} : G/N \to \mathbb{R}$ defined by $||xN||_{G/N} = \min_{a\in N} ||xa||_X$, is a group norm on G/N, called the *quotient norm*.

A **quotient norm metric** is a **group norm metric** on G/N defined by

$$||xN \cdot (yN)^{-1}||_{G/N} = ||xy^{-1}N||_{G/N}.$$

If $G = \mathbb{Z}$ with the norm being the absolute value, and $N = m\mathbb{Z}$, $m \in \mathbb{N}$, then the quotient norm on $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$ coincides with the *Lee norm*.

If a metric d on a group $(G, \cdot, e)$ is **right-invariant**, then for any normal subgroup $(N, \cdot, e)$ of $(G, \cdot, e)$ the metric d induces a right-invariant metric (in fact, the **Hausdorff metric**) d^* on G/N by

$$d^*(xN, yN) = \max\{\max_{b \in yN} \min_{a \in xN} d(a, b), \max_{a \in xN} \min_{b \in yN} d(a, b)\}.$$

- **Commutation distance**

Let $(G, \cdot, e)$ be a finite nonabelian group. Let $Z(G) = \{c \in G : x \cdot c = c \cdot x \text{ for any } x \in G\}$ be the *center* of G.

The *commutation graph* of G is defined as a graph with the vertex-set G in which distinct elements $x, y \in G$ are connected by an edge whenever they *commute*, i.e., $x \cdot y = y \cdot x$. (Darafsheh, 2009, consider noncommuting graph on $G \setminus Z(G)$.)

Any two noncommuting elements $x, y \in G$ are connected in this graph by the path x, c, y, where c is any element of $Z(G)$ (for example, e). A path $x = x^1, x^2, \dots, x^k = y$ in the commutation graph is called an $(x - y)$ N-path if $x^i \notin Z(G)$ for any $i \in \{1, \dots, k\}$. In this case the elements $x, y \in G \backslash Z(G)$ are called *N-connected*.

The **commutation distance** (see [DeHu98]) d is an extended distance on G defined by the following conditions:

1. $d(x, x) = 0$;
2. $d(x, y) = 1$ if $x \neq y$, and $x \cdot y = y \cdot x$;
3. $d(x, y)$ is the minimum length of an $(x - y)$ N-path for any N-connected elements x and $y \in G \backslash Z(G)$;
4. $d(x, y) = \infty$ if $x, y \in G \backslash Z(G)$ are not connected by any N-path.

Given a group G and a G-conjugacy class X in it, Bates–Bundy–Perkins–Rowley in 2003, 2004, 2007, 2008 considered *commuting graph* (X, E) whose vertex set is X and distinct vertices $x, y \in X$ are joined by an edge $e \in E$ whenever they commute.

- **Modular distance**

Let $(\mathbb{Z}_m, +, 0)$, $m \geq 2$, be a finite *cyclic group*. Let $r \in \mathbb{N}$, $r \geq 2$. The *modular r-weight* $w_r(x)$ of an element $x \in \mathbb{Z}_m = \{0, 1, \dots, m\}$ is defined as $w_r(x) = \min\{w_r(x), w_r(m - x)\}$, where $w_r(x)$ is the *arithmetic r-weight* of the integer x.

The value $w_r(x)$ can be obtained as the number of nonzero coefficients in the *generalized nonadjacent form* $x = e_n r^n + \dots e_1 r + e_0$ with $e_i \in \mathbb{Z}$, $|e_i| < r$, $|e_i + e_{i+1}| < r$, and $|e_i| < |e_{i+1}|$ if $e_i e_{i+1} < 0$. Cf. **arithmetic *r*-norm metric** in Chap. 12.

The **modular distance** is a distance on $\mathbb{Z}_m$, defined by

$$w_r(x - y).$$

The modular distance is a metric for $w_r(m) = 1$, $w_r(m) = 2$, and for several special cases with $w_r(m) = 3$ or 4. In particular, it is a metric for $m = r^n$ or $m = r^n - 1$; if $r = 2$, it is a metric also for $m = 2^n + 1$ (see, for example, [Ernv85]).

The most popular metric on $\mathbb{Z}_m$ is the **Lee metric** defined by $||x-y||_{Lee}$, where $||x||_{Lee} = \min\{x, m - x\}$ is the *Lee norm* of an element $x \in \mathbb{Z}_m$.

- ***G*-norm metric**

Consider a finite field $\mathbb{F}_{p^n}$ for a prime p and a natural number n. Given a compact convex centrally-symmetric body G in $\mathbb{R}^n$, define the *G-norm* of an element $x \in \mathbb{F}_{p^n}$ by $||x||_G = \inf\{\mu \geq 0 : x \in p\mathbb{Z}^n + \mu G\}$.

The ***G*-norm metric** is a **group norm metric** on $\mathbb{F}_{p^n}$ defined by

$$||x \cdot y^{-1}||_G.$$

- **Permutation norm metric**

Given a finite metric space (X, d), the **permutation norm metric** is a **group norm metric** on the group $(Sym_X, \cdot, id)$ of all permutations of X (*id* is the *identity mapping*) defined by

$$||f \cdot g^{-1}||_{Sym},$$

where the *group norm* $||.||_{Sym}$ on Sym_X is given by $||f||_{Sym} = \max_{x \in X} d(x, f(x))$.

- **Metric of motions**

Let (X, d) be a metric space, and let $p \in X$ be a fixed element of X.

The **metric of motions** (see [Buse55]) is a metric on the group $(\Omega, \cdot, id)$ of all **motions** of (X, d) (*id* is the *identity mapping*) defined by

$$\sup_{x \in X} d(f(x), g(x)) \cdot e^{-d(p,x)}$$

for any $f, g \in \Omega$ (cf. **Busemann metric of sets** in Chap. 3). If the space (X, d) is bounded, a similar metric on Ω can be defined as

$$\sup_{x \in X} d(f(x), g(x)).$$

Given a semimetric space (X, d), the **semimetric of motions** on $(\Omega, \cdot, id)$ is

$$d(f(p), g(p)).$$

- **General linear group semimetric**

Let $\mathbb{F}$ be a locally compact nondiscrete *topological field*. Let $(\mathbb{F}^n, ||.||_{\mathbb{F}^n})$, $n \geq 2$, be a *normed vector space* over $\mathbb{F}$. Let $||.||$ be the *operator norm* associated with the normed vector space $(\mathbb{F}^n, ||.||_{\mathbb{F}^n})$. Let $GL(n, \mathbb{F})$ be the *general linear group* over $\mathbb{F}$. Then the function $|.|_{op} : GL(n, \mathbb{F}) \to \mathbb{R}$ defined by $|g|_{op} = \sup\{|\ln ||g|| |, |\ln ||g^{-1}|| |\}$, is a seminorm on $GL(n, \mathbb{F})$.

The **general linear group semimetric** on the group $GL(n, \mathbb{F})$ is defined by

$$|g \cdot h^{-1}|_{op}.$$

It is a **right-invariant** semimetric which is unique, up to **coarse isometry**, since any two norms on $\mathbb{F}^n$ are **bi-Lipschitz equivalent**.

- **Generalized torus semimetric**

  Let $(T, \cdot, e)$ be a *generalized torus*, i.e., a *topological group* which is isomorphic to a direct product of n multiplicative groups $\mathbb{F}_i^*$ of locally compact nondiscrete *topological fields* $\mathbb{F}_i$. Then there is a proper continuous homomorphism $v : T \to \mathbb{R}^n$, namely, $v(x_1, \dots, x_n) = (v_1(x_1), \dots, v_n(x_n))$, where $v_i : \mathbb{F}_i^* \to \mathbb{R}$ are proper continuous homomorphisms from the $\mathbb{F}_i^*$ to the additive group $\mathbb{R}$, given by the logarithm of the *valuation*. Every other proper continuous homomorphism $v' : T \to \mathbb{R}^n$ is of the form $v' = \alpha \cdot v$ with $\alpha \in GL(n, \mathbb{R})$. If $||.||$ is a norm on $\mathbb{R}^n$, one obtains the corresponding seminorm $||x||_T = ||v(x)||$ on T.

  The **generalized torus semimetric** is defined on the group $(T, \cdot, e)$ by

$$||xy^{-1}||_T = ||v(xy^{-1})|| = ||v(x) - v(y)||.$$

- **Stable norm metric**

  Given a Riemannian manifold (M, g), the **stable norm metric** is a **group norm metric** on its *real homology group* $H_k(M, \mathbb{R})$ defined by the following *stable norm* $||h||_s$: the infimum of the Riemannian k-volumes of real cycles representing h.

  The Riemannian manifold $(\mathbb{R}^n, g)$ is within finite **Gromov–Hausdorff distance** (cf. Chap. 1) from an n-dimensional normed vector space $(\mathbb{R}^n, ||.||_s)$.

  If (M, g) is a compact connected oriented Riemannian manifold, then the manifold $H_1(M, \mathbb{R})/H_1(M, \mathbb{R})$ with metric induced by $||.||_s$ is called the *Albanese torus* (or *Jacobi torus*) of (M, g). This **Albanese metric** is a **flat metric** (Chap. 8).

- **Heisenberg metric**

  Let $(H, \cdot, e)$ be the (real) *Heisenberg group* $\mathcal{H}^n$, i.e., a group on the set $H = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law $h \cdot h' = (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2\sum_{i=1}^n (x'_i y_i - x_i y'_i)$, and the identity $e = (0, 0, 0)$. Let $|.|_{Heis}$ be the *Heisenberg gauge* (Cygan, 1978) on $\mathcal{H}^n$ defined by $|h|_{Heis} = |(x, y, t)|_{Heis} = ((\sum_{i=1}^n (x_i^2 + y_i^2))^2 + t^2)^{1/4}$.

  The **Heisenberg metric** (or **Korányi metric**, **Cygan metric**, **gauge metric**) d_{Heis} is a **group norm metric** on $\mathcal{H}^n$ defined by

$$|x^{-1} \cdot y|_{Heis}.$$

  One can identify the Heisenberg group $\mathcal{H}^{n-1} = \mathbb{C}^{n-1} \times \mathbb{R}$ with $\partial \mathbb{H}^n_{\mathbb{C}} \setminus \{\infty\}$, where $\mathbb{H}^n_{\mathbb{C}}$ is the Hermitian (i.e., complex) hyperbolic n-space, and ∞ is any point of its boundary $\partial \mathbb{H}^n_{\mathbb{C}}$. So, the usual hyperbolic metric of $\mathbb{H}^{n+1}_{\mathbb{C}}$ induces a metric on $\mathcal{H}^n$. The **Hamenstädt distance** on $\partial\mathbb{H}^n_{\mathbb{C}} \setminus \{\infty\}$ (Hersonsky–Paulin, 2004) is $\frac{1}{\sqrt{2}} d_{Heis}$.

Sometimes, the term *Cygan metric* is reserved for the extension of the metric d_{Heis} on whole $\mathbb{H}^n_{\mathbb{C}}$ and (Apanasov, 2004) for its generalization (via the *Carnot group* $\mathbb{F}^{n-1} \times Im\mathbb{F}$) on $\mathbb{F}$-hyperbolic spaces $\mathbb{H}^n_{\mathbb{F}}$ over numbers $\mathbb{F}$ that can be complex numbers, or quaternions or, for $n = 2$, octonions. Also, the generalization of d_{Heis} on Carnot groups *of Heisenberg type* is called the *Cygan metric*.

The second natural metric on $\mathcal{H}^n$ is the **Carnot–Carathéodory metric** (or **CC metric**, **sub-Riemannian metric**; cf. Chap. 7) d_C defined as the **length metric** (Chap. 6) using *horizontal vector fields* on $\mathcal{H}^n$. This metric is the **internal metric** (Chap. 4) corresponding to d_{Heis}.

The metric d_{Heis} is **bi-Lipschitz equivalent** with d_C but not with any Riemannian distance and, in particular, not with any Euclidean metric. For both metrics, the Heisenberg group $\mathcal{H}^n$ is a **fractal** since its **Hausdorff dimension**, $2n + 2$, is strictly greater than its **topological dimension**, $2n + 1$.

- **Metric between intervals**

  Let G be the set of all intervals $[a, b]$ of $\mathbb{R}$. The set G forms semigroups $(G, +)$ and $(G, \cdot)$ under addition $I + J = \{x + y : x \in I, y \in J\}$ and under multiplication $I \cdot J = \{x \cdot y : x \in I, y \in J\}$, respectively.

  The **metric between intervals** is a metric on G, defined by

$$\max\{|I|, |J|\}$$

  for all $I, J \in G$, where, for $K = [a, b]$, one has $|K| = |a - b|$.

- **Metric between games**

  Consider *positional games*, i.e., two-player nonrandom games of perfect information with real-valued outcomes. Play is alternating with a nonterminated game having move options for both players. Real-world examples include Chess, Go and Tic-Tac-Toe. Formally, let $F_{\mathbb{R}}$ be the universe of games defined inductively as follows:

  1. Every real number $r \in \mathbb{R}$ belongs to $F_{\mathbb{R}}$ and is called an *atomic game*.
  2. If $A, B \subset F_{\mathbb{R}}$ with $1 \le |A|, |B| < \infty$, then $\{A|B\} \in F_{\mathbb{R}}$ (*nonatomic game*).

  Write any game $G = \{A|B\}$ as $\{G^L|G^R\}$, where $G^L = A$ and $G^R = B$ are the set of left and right moves of G, respectively.

  $F_{\mathbb{R}}$ becomes a commutative semigroup under the following addition operation:

  1. If p and q are atomic games, then $p + q$ is the usual addition in $\mathbb{R}$.
  2. $p + \{g_{l_1}, \dots |g_{r_1}, \dots\} = \{g_{l_1} + p, \dots |g_{r_1} + p, \dots\}$.
  3. If G and H are both nonatomic, then $\{G^L|G^R\} + \{H^L|H^R\} = \{I^L|I^R\}$, where $I^L = \{g_l + H, G + h_l : g_l \in G^L, h_l \in H^L\}$ and $I^R = \{g_r + H, G + h_r : g_r \in G^R, h_r \in H^R\}$.

For any game $G \in F_{\mathbb{R}}$, define the optimal outcomes $\overline{L}(G)$ and $\overline{R}(G)$ (if both players play optimally with Left and Right starting, respectively) as follows:

$\overline{L}(p) = \overline{R}(p) = p$ and $\overline{L}(G) = \max\{\overline{R}(g_l) : g_l \in G^L\}$, $\overline{R}(G) = \max\{\overline{L}(g_r) : g_r \in G^R\}$.

The **metric between games** G and H defined by Ettinger, 2000, is the following **extended metric** on $F_{\mathbb{R}}$:

$$\sup_X |\overline{L}(G+X) - \overline{L}(H+X)| = \sup_X |\overline{R}(G+X) - \overline{R}(H+X)|.$$

- **Helly semimetric**

  Consider a game $(\mathcal{A}, \mathcal{B}, H)$ between players A and B with *strategy sets* $\mathcal{A}$ and $\mathcal{B}$, respectively. Here $H = H(\cdot, \cdot)$ is the *payoff function*, i.e., if player A plays $a \in \mathcal{A}$ and player B plays $b \in \mathcal{B}$, then A pays H(a,b) to B. A player's *strategy set* is the set of available to him *pure strategies*, i.e., complete algorithms for playing the game, indicating the move for every possible situation throughout it.

  The **Helly semimetric** between strategies $a_1 \in \mathcal{A}$ and $a_2 \in \mathcal{A}$ of A is defined by

$$\sup_{b \in \mathcal{B}} |H(a_1, b) - H(a_2, b)|.$$

- **Factorial ring semimetric**

  Let $(A, +, \cdot)$ be a *factorial ring*, i.e., an *integral domain* (nonzero commutative ring with no nonzero zero divisors), in which every nonzero nonunit element can be written as a product of (nonunit) irreducible elements, and such factorization is unique up to permutation.

  The **factorial ring semimetric** is a semimetric on the set $A \backslash \{0\}$, defined by

$$\ln \frac{lcm(x, y)}{gcd(x, y)},$$

  where $lcm(x, y)$ is the *least common multiple*, and $gcd(x, y)$ is the *greatest common divisor* of elements $x, y \in A \backslash \{0\}$.

- **Frankild–Sather–Wagstaff metric**

  Let $\mathcal{G}(R)$ be the set of isomorphism classes, up to a shift, of semidualizing complexes over a local Noetherian commutative ring R. An *R-complex* is a particular sequence of R-module homomorphisms; see [FrSa07]) for exact definitions.

  The **Frankild–Sather–Wagstaff metric** ([FrSa07]) is a metric on $\mathcal{G}(R)$ defined, for any classes $[K], [L] \in \mathcal{G}(R)$, as the infimum of the *lengths* of chains of pairwise comparable elements starting with $[K]$ and ending with $[L]$.

10.2 Metrics on Binary Relations

A *binary relation* R on a set X is a subset of $X \times X$; it is the arc-set of the directed graph (X, R) with the vertex-set X.

A binary relation R which is *symmetric* ($(x,y) \in R$ implies $(y,x) \in R$), *reflexive* (all $(x,x) \in R$), and *transitive* ($(x,y),(y,z) \in R$ imply $(x,z) \in R$) is called an *equivalence relation* or a *partition* (of X into equivalence classes). Any *q-ary sequence* $x = (x_1, \dots, x_n)$, $q \geq 2$ (i.e., with $0 \leq x_i \leq q-1$ for $1 \leq i \leq n$), corresponds to the partition $\{B_0, \dots, B_{q-1}\}$ of $V_n = \{1, \dots, n\}$, where $B_j = \{1 \leq i \leq n : x_i = j\}$ are the equivalence classes.

A binary relation R which is *antisymmetric* ($(x,y),(y,x) \in R$ imply $x = y$), reflexive, and transitive is called a *partial order*, and the pair (X, R) is called a *poset* (partially ordered set). A partial order R on X is denoted also by $\preceq$ with $x \preceq y$ if and only if $(x,y) \in R$. The order $\preceq$ is called *linear* if any elements $x, y \in X$ are *compatible*, i.e., $x \preceq y$ or $y \preceq x$.

A poset $(L, \preceq)$ is called a *lattice* if every two elements $x, y \in L$ have the *join* $x \vee y$ and the *meet* $x \wedge y$. All partitions of X form a lattice $\mathbb{P}_X$ by refinement; it is a sublattice of the lattice (by set-inclusion) of all binary relations.

- **Kemeny distance**

  The **Kemeny distance** between binary relations R_1 and R_2 on a set X is the **Hamming metric** $|R_1 \triangle R_2|$. It is twice the minimal number of inversions of pairs of adjacent elements of X which is necessary to obtain R_2 from R_1.

  If R_1, R_2 are *partitions*, then the Kemeny distance coincides with the **Mirkin–Tcherny distance**, and $1 - \frac{|R_1 \triangle R_2|}{n(n-1)}$ is the *Rand index*.

  If binary relations R_1, R_2 are *linear orders* (or *permutations*) on the set X, then the Kemeny distance coincides with the **Kendall τ distance** (Chap. 11).
- **Drápal–Kepka distance**

  The **Drápal–Kepka distance** between distinct *quasigroups* (differing from groups in that they need not be associative) $(X, +)$ and $(X, \cdot)$ is the **Hamming metric** $|\{(x,y) : x + y \neq x \cdot y\}|$ between their *Cayley tables*.

  For finite nonisomorphic groups, this distance is (Ivanyos, Le Gall and Yoshida, 2012) at least $2(\frac{|X|}{3})^2$ with equality (Drápal, 2003) for some 3-groups.
- **Editing metrics between partitions**

  Let X be a finite set, $|X| = n$, and let A, B be nonempty subsets of X. Let $\mathcal{P}_X$ be the set of partitions of X, and $P, Q \in \mathcal{P}_X$. Let $P_1, \dots, P_q$ be *blocks* in the partition P, i.e., the pairwise disjoint sets such that $X = P_1 \cup \dots \cup P_q$, $q \geq 1$. Let $P \vee Q$ and $P \wedge Q$ be the *join* and *meet* of P and Q in the *lattice* $\mathbb{P}_X$ *of partitions* of X.

  Consider the following *editing operations* on partitions (clusterings):

  – An *augmentation* transforms a partition P of $A \backslash \{B\}$ into a partition of A by either including the objects of B in a block, or including B as a new block;
  – An *removal* transforms a partition P of A into a partition of $A \backslash \{B\}$ by deleting the objects in B from each block that contains them;

– A *division* transforms one partition P into another by the simultaneous removal of B from P_i (where $B \subset P_i$, $B \neq P_i$), and augmentation of B as a new block;
– A *merging* transforms one partition P into another by the simultaneous removal of B from P_i (where $B = P_i$), and augmentation of B to P_j (where $j \neq i$);
– A *transfer* transforms one partition P into another by the simultaneous removal of B from P_i (where $B \subset P_i$), and augmentation of B to P_j (where $j \neq i$).

Define (see, say, [Day81]), using above operations, the following metrics on $\mathcal{P}_X$:

1. The minimum number of augmentations and removals of single objects needed to transform P into Q;
2. The minimum number of divisions, mergings, and transfers of single objects needed to transform P into Q;
3. The minimum number of divisions, mergings, and transfers needed to transform P into Q;
4. The minimum number of divisions and mergings needed to transform P into Q; in fact, it is equal to $|P| + |Q| - 2|P \vee Q|$;
5. $\sigma(P) + \sigma(Q) - 2\sigma(P \wedge Q)$, where $\sigma(P) = \sum_{P_i \in P} |P_i|(|P_i| - 1)$;
6. $e(P) + e(Q) - 2e(P \wedge Q)$, where $e(P) = \log_2 n + \sum_{P_i \in P} \frac{|P_i|}{n} \log_2 \frac{|P_i|}{n}$;
7. $2n - \sum_{P_i \in P} \max_{Q_j \in Q} |P_i \cap Q_j| - \sum_{Q_j \in Q} \max_{P_i \in P} |P_i \cap Q_j|$ (van Dongen, 2000).

The **maximum matching distance** (or *partition-distance* as defined in Gusfield, 2002) is (Réignier, 1965) the minimum number of elements that must be moved between the blocks of partition P in order to transform it into Q.

- **Rossi–Hamming metric**
  Given a partition $P = (P_1, \dots, P_q)$ of a finite set X, its *size* is defined as $s(P) = \frac{1}{2} \sum_{1 \leq i \leq q} |P_i|(|P_i| - 1)$. We call the **Rossi–Hamming metric** the metric between partitions P and Q, defined in Rossi, 2014, as

$$d_{RH}(P, Q) = s(P) + s(Q) - 2s(P \wedge Q).$$

One has $d_{RH}(P, Q) \leq s(P \vee Q) - s(P \wedge Q)$, where the right-hand side is the *size-based distance* (Rossi, 2011). The inequality is strict only for some noncomparable P, Q.

10.3 Metrics on Semilattices

Consider a poset $(L, \preceq)$. The *meet* (or *infimum*) $x \wedge y$ (if it exists) of two elements x and y is the unique element satisfying $x \wedge y \preceq x, y$, and $z \preceq x \wedge y$ if $z \preceq x, y$. The *join* (or *supremum*) $x \vee y$ (if it exists) is the unique element such that $x, y \preceq x \vee y$, and $x \vee y \preceq z$ if $x, y \preceq z$. A poset $(L, \preceq)$ is called a *lattice* if every its elements x, y have the join $x \vee y$ and the meet $x \wedge y$. A poset is a *meet* (or *lower*) *semilattice* if only the meet-operation is defined. A poset is a *join* (or *upper*) *semilattice* if only the join-operation is defined.

A lattice $\mathbb{L} = (L, \preceq, \vee, \wedge)$ is called a *semimodular lattice* if the *modularity relation xMy* is symmetric: *xMy* implies *yMx* for any $x, y \in L$. Here two elements x and y are said to constitute a *modular pair*, in symbols *xMy*, if $x \wedge (y \vee z) = (x \wedge y) \vee z$ for any $z \preceq x$. A lattice $\mathbb{L}$ in which every pair of elements is modular, is called a *modular lattice*.

Given a lattice $\mathbb{L}$, a function $v : L \to \mathbb{R}_{\geq 0}$, satisfying $v(x \vee y) + v(x \wedge y) \leq v(x) + v(y)$ for all $x, y \in L$, is called a *subvaluation* on $\mathbb{L}$. A subvaluation v is *isotone* if $v(x) \leq v(y)$ whenever $x \preceq y$, and it is *positive* if $v(x) < v(y)$ whenever $x \preceq y$, $x \neq y$. A subvaluation v is called a *valuation* if it is isotone and $v(x \vee y) + v(x \wedge y) = v(x) + v(y)$ for all $x, y \in L$.

- **Lattice valuation metric**

  Let $\mathbb{L} = (L, \preceq, \vee, \wedge)$ be a lattice, and let v be an isotone subvaluation on $\mathbb{L}$. The *lattice subvaluation semimetric* d_v on L is defined by

$$2v(x \vee y) - v(x) - v(y).$$

  (It can be defined also on some semilattices.) If v is a positive subvaluation on $\mathbb{L}$, one obtains a metric, called the **lattice subvaluation metric**. If v is a valuation, d_v is called the *valuation semimetric* and can be written as

$$v(x \vee y) - v(x \wedge y) = v(x) + v(y) - 2v(x \wedge y).$$

  If v is a positive valuation on $\mathbb{L}$, one obtains a metric, called the **lattice valuation metric**, and the lattice is called a **metric lattice**.

  An example is the **Hamming distance** $d_H(A, B) = |A \cup B| - |A \cap B|$ on the lattice $(P(X), \cup, \cap)$ of all subsets of the set X. Cf. also the **Shannon distance** (Chap. 14), which can be seen as a distance on partitions.

  If $L = \mathbb{N}$ (the set of positive integers), $x \vee y = lcm(x, y)$ (least common multiple), $x \wedge y = gcd(x, y)$ (greatest common divisor), and the positive valuation $v(x) = \ln x$, then $d_v(x, y) = \ln \frac{lcm(x,y)}{gcd(x,y)}$.

  This metric can be generalized on any *factorial ring* equipped with a positive valuation v such that $v(x) \geq 0$ with equality only for the multiplicative unit of the ring, and $v(xy) = v(x) + v(y)$. Cf. **factorial ring semimetric**.

- **Finite subgroup metric**

  Let $(G, \cdot, e)$ be a group. Let $\mathbb{L} = (L, \subset, \cap)$ be the meet semilattice of all finite subgroups of the group $(G, \cdot, e)$ with the meet $X \cap Y$ and the valuation $v(X) = \ln |X|$.

  The **finite subgroup metric** is a **valuation metric** on L defined by

$$v(X) + v(Y) - 2v(X \wedge Y) = \ln \frac{|X||Y|}{(|X \cap Y|)^2}.$$

- **Join semilattice distances**

  Let $\mathbb{L} = (L, \preceq, \vee)$ be a join semilattice, finite or infinite, such that every maximal chain in every interval $[x, y]$ is finite. For $x \preceq y$, the *height* $h(x, y)$ *of* y *above* x is the least cardinality of a finite maximal (by inclusion) chain of $[x, y]$ minus 1. Call the join semilattice $\mathbb{L}$ *semimodular* if for all $x, y \in L$, whenever there exists an element z covered by both x and y, the join $x \vee y$ covers both x and y, or, in other words, whenever elements x, y have a common lower bound z, it holds $h(x, x \vee y) \leq h(z, y)$. Any *tree* (i.e., all intervals $[x, z]$ are finite, each pair x, y of uncomparable elements have a least common upper bound $x \vee y$ but they never have a common lower bound) is semimodular. Consider the following distances on L:

  $d_{\text{path}}(x, y)$ is the path metric of the *Hasse diagram* of $(L, \preceq)$, i.e., a graph with vertex-set L and an edge between two elements if they are comparable.

  $d_{a.\text{path}}(x, y)$ is the smallest number of the form $h(x, z) + h(y, z)$, where z is a common upper bound of x and y, i.e., it is the **ancestral path distance**; cf. **pedigree-based distances** in Chap. 23. This and next distance reflect the way how Roman civil law and medieval canon law, respectively, measured degree of kinship.

  $d_{\max}(x, y)$ is defined by $\max(h(x, x \vee y), h(y, x \vee y))$.

  It holds $d_{a.\text{path}}(x, y) \geq d_{\text{path}}(x, y) \geq d_{\max}(x, y)$. Foldes, 2013, proved that $d_{\max}(x, y)$ is a metric if $\mathbb{L}$ is semimodular and that $d_{a.\text{path}}(x, y)$ is a metric if and only if $\mathbb{L}$ is semimodular, in which case $d_{a.\text{path}}(x, y) = d_{\text{path}}(x, y)$.

- **Gallery distance of flags**

  Let $\mathbb{L}$ be a lattice. A *chain* C in $\mathbb{L}$ is a subset of L which is *linearly ordered*, i.e., any two elements of C are compatible. A *flag* is a chain in $\mathbb{L}$ which is maximal with respect to inclusion. If $\mathbb{L}$ is a semimodular lattice, containing a finite flag, then $\mathbb{L}$ has a unique minimal and a unique maximal element, and any two flags C, D in $\mathbb{L}$ have the same cardinality, $n + 1$. Then n is the *height* of the lattice $\mathbb{L}$.

  Two flags C, D are called *adjacent* if either they are equal or D contains exactly one element not in C. A *gallery* from C to D of length m is a sequence of flags $C = C_0, C_1, \dots, C_m = D$ such that C_{i-1} and C_i are adjacent for $i = 1, \dots, m$.

  A **gallery distance of flags** (see [Abel91]) is a distance on the set of all flags of a semimodular lattice $\mathbb{L}$ with finite height defined as the minimum of lengths of galleries from C to D. It can be written as

  $$|C \vee D| - |C| = |C \vee D| - |D|,$$

  where $C \vee D = \{c \vee d : c \in C, d \in D\}$ is the subsemilattice generated by C and D. This distance is the **gallery metric** of the *chamber system* consisting of flags.

- **Scalar and vectorial metrics**

  Let $\mathbb{L} = (L, \leq, \max, \min)$ be a lattice with the join $\max\{x, y\}$, and the meet $\min\{x, y\}$ on a set $L \subset [0, \infty)$ which has a fixed number a as the greatest element and is closed under *negation*, i.e., for any $x \in L$, one has $\overline{x} = a - x \in L$.

The **scalar metric** d on L is defined, for $x \neq y$, by

$$d(x, y) = \max\{\min\{x, \overline{y}\}, \min\{\overline{x}, y\}\}.$$

The **scalar metric** d^* on $L^* = L \cup \{*\}$, $* \notin L$, is defined, for $x \neq y$, by

$$d^*(x, y) = \begin{cases} d(x, y), & \text{if} \quad x, y \in L, \\ \max\{x, \overline{x}\}, & \text{if} \quad y = *, x \neq *, \\ \max\{y, \overline{y}\}, & \text{if} \quad x = *, y \neq *. \end{cases}$$

Given a norm $||.||$ on $\mathbb{R}^n$, $n \geq 2$, the **vectorial metric** on L^n is defined by

$$||(d(x_1, y_1), \ldots, d(x_n, y_n))||,$$

and the **vectorial metric** on $(L^*)^n$ is defined by

$$||(d^*(x_1, y_1), \ldots, d^*(x_n, y_n))||.$$

The vectorial metric on $L_2^n = \{0, 1\}^n$ with l_1-*norm* on $\mathbb{R}^n$ is the **Fréchet–Nikodym–Aronszyan distance**. The vectorial metric on $L_m^n = \{0, \frac{1}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}^n$ with l_1-*norm* on $\mathbb{R}^n$ is the **Sgarro m-valued metric**. The vectorial metric on $[0, 1]^n$ with l_1-*norm* on $\mathbb{R}^n$ is the **Sgarro fuzzy metric**.

If L is L_m or $[0, 1]$, and $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r})$, $y = (y_1, \ldots, y_n, *, \ldots, *)$, where $*$ stands in r places, then the vectorial metric between x and y is the **Sgarro metric** (see, for example, [CSY01]).

- **Metrics on Riesz space**

A *Riesz space* (or *vector lattice*) is a partially ordered vector space $(V_{Ri}, \preceq)$ in which the following conditions hold:

1. The vector space structure and the partial order structure are compatible: $x \preceq y$ implies $x + z \preceq y + z$, and $x \succ 0$, $\lambda \in \mathbb{R}$, $\lambda > 0$ implies $\lambda x \succ 0$;
2. For any two elements $x, y \in V_{Ri}$ there exists the join $x \vee y \in V_{Ri}$ (in particular, the join and the meet of any finite set of elements from V_{Ri} exist).

The **Riesz norm metric** is a **norm metric** on V_{Ri} defined by

$$||x - y||_{Ri},$$

where $||.||_{Ri}$ is a *Riesz norm*, i.e., a *norm* on V_{Ri} such that, for any $x, y \in V_{Ri}$, the inequality $|x| \leq |y|$, where $|x| = (-x) \vee (x)$, implies $||x||_{Ri} \leq ||y||_{Ri}$.

The space $(V_{Ri}, ||.||_{Ri})$ is called a *normed Riesz space*. In the case of completeness it is called a *Banach lattice*. All Riesz norms on a Banach lattice are equivalent.

An element $e \in V_{Ri}^+ = \{x \in V_{Ri} : x \succ 0\}$ is called a *strong unit* of V_{Ri} if for each $x \in V_{Ri}$ there exists $\lambda \in \mathbb{R}$ such that $|x| \preceq \lambda e$. If a Riesz space V_{Ri} has a strong unit e, then $||x|| = \inf\{\lambda \in \mathbb{R} : |x| \preceq \lambda e\}$ is a Riesz norm, and one obtains on V_{Ri} a Riesz norm metric

$$\inf\{\lambda \in \mathbb{R} : |x-y| \preceq \lambda e\}.$$

A *weak unit* of V_{Ri} is an element e of V_{Ri}^{+} such that $e \wedge |x| = 0$ implies $x = 0$. A Riesz space V_{Ri} is called *Archimedean* if, for any two $x, y \in V_{Ri}^{+}$, there exists a natural number n, such that $nx \preceq y$. The **uniform metric** on an Archimedean Riesz space with a weak unit e is defined by

$$\inf\{\lambda \in \mathbb{R} : |x-y| \wedge e \preceq \lambda e\}.$$

- **Machida metric**

For a fixed integer $k \geq 2$ and the set $V_k = \{0, 1, \dots, k-1\}$, let $O_k^{(n)}$ be the set of all n-ary functions from $(V_k)^n$ into V_k and $O_k = \cup_{n=1}^{\infty} O_k^{(n)}$. Let Pr_k be the set of all *projections* pr_i^n over V_k, where $pr_i^n(x_1, \dots, x_i, \dots, x_n) = x_i$ for any $x_1, \dots, x_n \in V_k$.

A *clone over* V_k is a subset C of O_k containing Pr_k and closed under (functional) composition. The set L_k of all clones over V_k is a lattice. The *Post lattice* L_2 defined over Boolean functions, is countable but any L_k with $k \geq 3$ is not. For $n \geq 1$ and a clone $C \in L_k$, let $C^{(n)}$ denote *n-slice* $C \cap O_k^{(n)}$.

For any two clones $C_1, C_2 \in L_k$, Machida, 1998, defined the distance to be 0 if $C_1 = C_2$ and $(\min\{n : C_1^{(n)} \neq C_2^{(n)}\})^{-1}$, otherwise. The lattice L_k of clones with this distance is a compact ultrametric space. Cf. **Baire metric** in Chap. 11.