Chapter 1
General Definitions

1.1 Basic Definitions

- **Distance**

  A **distance space** (X, d) is a set X (*carrier*) equipped with a **distance** d.

  A function $d : X \times X \to \mathbb{R}$ is called a **distance** (or **dissimilarity**) on X if, for all $x, y \in X$, it holds:

  1. $d(x, y) \geq 0$ (*nonnegativity*);
  2. $d(x, y) = d(y, x)$ (*symmetry*);
  3. $d(x, x) = 0$ (*reflexivity*).

  In Topology, a distance with $d(x, y) = 0$ implying $x = y$ is called a **symmetric**.

  For any distance d, the function D_1 defined for $x \neq y$ by $D_1(x, y) = d(x, y) + c$, where $c = \max_{x,y,z \in X}(d(x, y) - d(x, z) - d(y, z))$, and $D(x, x) = 0$, is a **metric**. Also, $D_2(x, y) = d(x, y)^c$ is a metric for sufficiently small $c \geq 0$.

  The function $D_3(x, y) = \inf \sum_i d(z_i, z_{i+1})$, where the infimum is taken over all sequences $x = z_0, \dots, z_{n+1} = y$, is the **path semimetric** of the complete weighted graph on X, where, for any $x, y \in X$, the weight of edge xy is $d(x, y)$.

- **Similarity**

  Let X be a set. A function $s : X \times X \to \mathbb{R}$ is called a **similarity** on X if s is nonnegative, symmetric and the inequality

  $$s(x, y) \leq s(x, x)$$

  holds for all $x, y \in X$, with equality if and only if $x = y$.

  The main transforms used to obtain a distance (dissimilarity) d from a similarity s bounded by 1 from above are: $d = 1 - s$, $d = \frac{1-s}{s}$, $d = \sqrt{1-s}$, $d = \sqrt{2(1-s^2)}$, $d = \arccos s$, $d = -\ln s$ (cf. Chap. 4).

M.M. Deza, E. Deza, *Encyclopedia of Distances*,
DOI 10.1007/978-3-662-52844-0_1

- **Semimetric**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **semimetric** on X if d is nonnegative, symmetric, *reflexive* ($d(x, x) = 0$ for $x \in X$) and it holds

$$d(x, y) \leq d(x, z) + d(z, y)$$

  for all $x, y, z \in X$ (**triangle inequality** or, sometimes, *triangular inequality*).

  In Topology, it is called a **pseudo-metric** (or, rarely, **semidistance**, *gauge*), while the term *semimetric* is sometimes used for a **symmetric** (a distance $d(x, y)$ with $d(x, y) = 0$ only if $x = y$); cf. **symmetrizable space** in Chap. 2.

  For a semimetric d, the triangle inequality is equivalent, for each fixed $n \geq 4$ and all $x, y, z_1, \ldots, z_{n-2} \in X$, to the following *n-gon inequality*

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{n-2}, y).$$

  Equivalent *rectangle inequality* is $|d(x, y) - d(z_1, z_2)| \leq d(x, z_1) + +d(y, z_2)$.

  For a semimetric d on X, define an equivalence relation, called **metric identification**, by $x \sim y$ if $d(x, y) = 0$; equivalent points are equidistant from all other points. Let $[x]$ denote the equivalence class containing x; then $D([x], [y]) = d(x, y)$ is a **metric** on the set $\{[x] : x \in X\}$ of equivalence classes.

- **Metric**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if, for all $x, y, z \in X$, it holds:

  1. $d(x, y) \geq 0$ (*nonnegativity*);
  2. $d(x, y) = 0$ if and only if $x = y$ (*identity of indiscernibles*);
  3. $d(x, y) = d(y, x)$ (*symmetry*);
  4. $d(x, y) \leq d(x, z) + d(z, y)$ (**triangle inequality**).

  In fact, the above condition 1. follows from above 2., 3. and 4.

  If 2. is dropped, then d is called (Bukatin, 2002) **relaxed semimetric**. If 2. is weakened to "$d(x, x) = d(x, y) = d(y, y)$ implies $x = y$", then d is called **relaxed metric**. A **partial metric** is a **partial semimetric**, which is a relaxed metric.

  If above 2. is weakened to "$d(x, y) = 0$ implies $x = y$", then d is called (Amini-Harandi, 2012) **metric-like function**. Any **partial metric** is metric-like.

- **Metric space**

  A **metric space** (X, d) is a set X equipped with a metric d.

  It is called a **metric frame** (or *metric scheme*, *integral*) if d is integer-valued.

  A **pointed metric space** (or *rooted metric space*) (X, d, x_0) is a metric space (X, d) with a selected base point $x_0 \in X$.

- **Extended metric**

  An **extended metric** is a generalization of the notion of metric: the value ∞ is allowed for a metric d.

- **Quasi-distance**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **quasi-distance** on X if d is nonnegative, and $d(x, x) = 0$ holds for all $x \in X$. It is also called a **premetric** or **prametric** in Topology and a *divergence* in Probability.

If a quasi-distance d satisfies the **strong triangle inequality** $d(x,y) \leq d(x,z) + d(y,z)$, then (Lindenbaum, 1926) it is symmetric and so, a semimetric. A **quasi-semimetric** d is a semimetric if and only if (Weiss, 2012) it satisfies the **full triangle inequality** $|d(x,z) - d(z,y)| \leq d(x,z) \leq d(x,z) + d(z,y)$.

The distance/metric notions are usually named as weakenings or modifications of the fundamental notion of **metric**, using various prefixes and modifiers. But, perhaps, extended (i.e., the value ∞ is allowed) semimetric and quasi-semimetric should be (as suggested in Lawvere, 2002) used as the basic terms, since, together with their **short mappings**, they are best behaved of the **metric space categories**.

- **Quasi-semimetric**

A function $d : X \times X \to \mathbb{R}$ is called a **quasi-semimetric** (or **hemimetric**, *ostensible metric*) on X if $d(x,x) = 0$, $d(x,y) \geq 0$ and the **oriented triangle inequality**

$$d(x,y) \leq d(x,z) + d(z,y)$$

holds for all $x,y,z \in X$. The set X can be partially ordered by the *specialization order*: $x \preceq y$ if and only if $d(x,y) = 0$.

A **weak quasi-metric** is a quasi-semimetric d on X with *weak symmetry*, i.e., for all $x,y \in X$ the equality $d(x,y) = 0$ implies $d(y,x) = 0$.

An **Albert quasi-metric** is a quasi-semimetric d on X with *weak definiteness*, i.e., for all $x,y \in X$ the equality $d(x,y) = d(y,x) = 0$ implies $x = y$.

Both, weak and Albert, quasi-metric, is a usual **quasi-metric**.

Any *pre-order* $(X, \prec)$ (satisfying for all $x,y,z \in X$, $x \prec x$ and if $x \prec y$ and $y \prec z$ then $x \prec z$) can be viewed as a **pre-order extended quasi-semimetric** (X,d) by defining $d(x,y) = 0$ if $x \prec y$ and $d(x,y) = \infty$, otherwise.

A **weightable quasi-semimetric** is a quasi-semimetric d on X with *relaxed symmetry*, i.e., for all $x,y,z \in X$

$$d(x,y) + d(y,z) + d(z,x) = d(x,z) + d(z,y) + d(y,x),$$

holds or, equivalently, there exists a weight function $w(x) \in \mathbb{R}$ on X with $d(x,y) - d(y,x) = w(y) - w(x)$ for all $x,y \in X$ (i.e., $d(x,y) + \frac{1}{2}(w(x) - w(y))$ is a semimetric). If d is a weightable quasi-semimetric, then $d(x,y) + w(x)$ is a **partial semimetric** (moreover, a **partial metric** if d is an Albert quasi-metric).

- **Partial metric**

Let X be a set. A nonnegative symmetric function $p : X \times X \to \mathbb{R}$ is called a **partial metric** ([Matt92]) if, for all $x,y,z \in X$, it holds:

1. $p(x,x) \leq p(x,y)$, i.e., every **self-distance** (or *extent*) $p(x,x)$ is *small*;
2. $x = y$ if $p(x,x) = p(x,y) = p(y,y) = 0$ (T_0 *separation axiom*);
3. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (**sharp triangle inequality**).

The 1-st above condition means that p is a **forward resemblance**, cf. Chap. 3.

If the 2-nd above condition is dropped, the function p is called a **partial semimetric**. The nonnegative function p is a partial semimetric if and only if $p(x,y) - p(x,x)$ is a **weightable quasi-semimetric** with $w(x) = p(x,x)$.

If the 1-st above condition is also dropped, the function p is called (Heckmann, 1999) a **weak partial semimetric**. The nonnegative function p is a weak partial semimetric if and only if $2p(x,y) - p(x,x) - p(y,y)$ is a semimetric.

Sometimes, the term *partial metric* is used when a metric $d(x,y)$ is defined only on a subset of the set of all pairs x, y of points.

- **Protometric**

A function $p : X \times X \to \mathbb{R}$ is called a **protometric** if, for all (equivalently, for all different) $x, y, z \in X$, the **sharp triangle inequality** holds:

$$p(x,y) \le p(x,z) + p(z,y) - p(z,z).$$

For finite X, the matrix $((p(x,y)))$ is (Burkard et al., 1996) *weak Monge array*.

A **strong protometric** is a protometric p with $p(x,x) = 0$ for all $x \in X$. Such a protometric is exactly a quasi-semimetric, but with the condition $p(x,y) \ge 0$ (for any $x, y \in X$) being relaxed to $p(x,y) + p(y,x) \ge 0$.

A **partial semimetric** is a **symmetric protometric** (i.e., $p(x,y) = p(y,x)$ with $p(x,y) \ge p(x,x) \ge 0$ for all $x, y \in X$.) An example of a nonpositive symmetric protometric is given by $p(x,y) = -(x.y)_{x_0} = \frac{1}{2}(d(x,y) - d(x,x_0) - d(y,y_0))$, where (X,d) is a metric space with a fixed base point $x_0 \in X$; see **Gromov product similarity** $(x.y)_{x_0}$ and, in Chap. 4, **Farris transform metric** $C - (x.y)_{x_0}$.

A 0-**protometric** is a protometric p for which all sharp triangle inequalities (equivalently, all inequalities $p(x,y) + p(y,x) \ge p(x,x) + p(y,y)$ implied by them) hold as equalities. For any $u \in X$, denote by A'_u, A''_u the 0-protometrics p with $p(x,y) = 1_{x=u}, 1_{y=u}$, respectively. The protometrics on X form a flat convex cone in which the 0-protometrics form the largest linear space. For finite X, a basis of this space is given by all but one A'_u, A''_u (since $\sum_u A'_u = \sum_u A''_u$) and, for the flat subcone of all symmetric 0-protometrics on X, by all $A'_u + A''_u$.

A **weighted protometric** on X is a protometric with a point-weight function $w : X \to \mathbb{R}$. The mappings $p(x,y) = \frac{1}{2}(d(x,y) + w(x) + w(y))$ and $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, $w(x) = p(x,x)$ establish a bijection between the weighted strong protometrics (d,w) and the protometrics p on X, as well as between the weighted semimetrics and the symmetric protometrics. For example, a weighted semimetric (d,w) with $w(x) = -d(x,x_0)$ corresponds to a protometric $-(x.y)_{x_0}$. For finite $|X|$, the above mappings amount to the representation

$$2p = d + \sum_{u \in X} p(u,u)(A'_u + A''_u).$$

- **Quasi-metric**

A function $d : X \times X \to \mathbb{R}$ is called a **quasi-metric** (or *asymmetric metric*, *directed metric*) on X if $d(x,y) \ge 0$ holds for all $x, y \in X$ with equality if and only if $x = y$, and for all $x, y, z \in X$ the **oriented triangle inequality**

$$d(x, y) \leq d(x, z) + d(z, y)$$

holds. A *quasi-metric space* (X, d) is a set X equipped with a quasi-metric d.

For any quasi-metric d, the functions $\max\{d(x, y), d(y, x)\}$ (called sometimes *bi-distance*), $\min\{d(x, y), d(y, x)\}$, $\frac{1}{2}(d^p(x, y) + d^p(y, x))^{\frac{1}{p}}$ with given $p \geq 1$ are **metric generating**; cf. Chap. 4.

A **non-Archimedean quasi-metric** d is a quasi-distance on X which, for all $x, y, z \in X$, satisfies the following strengthened oriented triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

- **Directed-metric**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called (Jegede, 2005) a **directed-metric** on X if, for all $x, y, z \in X$, it holds $d(x, y) = -d(y, x)$ and

  $$|d(x, y)| \leq |d(x, z)| + |d(z, y)|.$$

  Cf. **displacement** in Chap. 24 and **rigid motion of metric space**.
- **Coarse-path metric**

  Let X be a set. A metric d on X is called a **coarse-path metric** if, for a fixed $C \geq 0$ and for every pair of points $x, y \in X$, there exists a sequence $x = x_0, x_1, \ldots, x_t = y$ for which $d(x_{i-1}, x_i) \leq C$ for $i = 1, \ldots, t$, and it holds

  $$d(x, y) \geq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{t-1}, x_t) - C.$$

- **Near-metric**

  Let X be a set. A distance d on X is called a **near-metric** (or *C-near-metric*) if $d(x, y) > 0$ for $x \neq y$ and the *C-relaxed triangle inequality*

  $$d(x, y) \leq C(d(x, z) + d(z, y))$$

  holds for all $x, y, z \in X$ and some constant $C \geq 1$.

  If $d(x, y) > 0$ for $x \neq y$ and the *C-asymmetric triangle inequality* $d(x, y) \leq d(x, z) + Cd(z, y)$ holds, d is a $\frac{C+1}{2}$-near-metric.

  A **C-inframetric** is a C-near-metric, while a C-near-metric is a $2C$-inframetric.

  Some recent papers use the term *quasi-triangle inequality* for the above inequality and so, *quasi-metric* for the notion of near-metric.

  The **power transform** (Chap. 4) $(d(x, y))^\alpha$ of any near-metric is a near-metric for any $\alpha > 0$. Also, any near-metric d admits a **bi-Lipschitz mapping** on $(D(x, y))^\alpha$ for some semimetric D on the same set and a positive number α.

A near-metric d on X is called a **Hölder near-metric** if the inequality

$$|d(x,y) - d(x,z)| \le \beta d^{\alpha}(y,z)(d(x,y) + d(x,z))^{1-\alpha}$$

holds for some $\beta > 0$, $0 < \alpha \le 1$ and all $x, y, z \in X$. Cf. **Hölder mapping**.

A distance d on set X is said (Greenhoe, 2015) to satisfy (C, p) **power triangle inequality** if, for given positive C, p and any $x, y, z \in X$, it holds

$$d(x,y) \le 2C|\frac{1}{2}d^p(x,z) + \frac{1}{2}d^p(z,y)|^{\frac{1}{p}}.$$

- ***f*-quasi-metric**

  Let $f(t,t') : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function with $\lim_{(t,t') \to 0}, f(t,t') = f(0,0) = 0$.

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called (Arutyunov et al., 2016) a ***f*-quasi-metric** on X if $d(x,y) \ge 0$ holds for all $x, y \in X$ with equality if and only if $x = y$, and for all $x, y, z \in X$ holds the ***f*-triangle inequality**

$$d(x,y) \le f(d(x,z), d(z,y)).$$

  The *f-quasi-metric space* (X, d) with symmetric d and $f(t,t') = \max(t,t')$ is exactly the **Fréchet *V*-space** (1906); cf. the **partially ordered distance** in Sect. 3.4.

  The case $f(t,t') = t + t'$ of a f-quasi-metric corresponds to a **quasi-metric**. Given $q, q' \ge 1$, the f-quasi-metric with $f(t,t') = qt + q't'$ is called (q,q')*-quasi-metric*.

  The inequality $d(x,y) \le F(d(x,z), d(y,z))$ implies $d(x,y) \le f(d(x,z), d(z,y))$ for the function $f(t,t') = F(t, F(0,t'))$.
- **Weak ultrametric**

  A **weak ultrametric** (or ***C*-inframetric**, *C-pseudo-distance*) d is a distance on X such that $d(x,y) > 0$ for $x \ne y$ and the *C-inframetric inequality*

$$d(x,y) \le C \max\{d(x,z), d(z,y)\}$$

  holds for all $x, y, z \in X$ and some constant $C \ge 1$.

  The term **pseudo-distance** is also used, in some applications, for any of a **pseudo-metric**, a **quasi-distance**, a **near-metric**, a distance which can be infinite, a distance with an error, etc. Another unsettled term is **weak metric**: it is used for both a **near-metric** and a **quasi-semimetric**.
- **Ultrametric**

  An **ultrametric** (or *non-Archimedean metric*) is (Krasner, 1944) a metric d on X which satisfies, for all $x, y, z \in X$, the following strengthened version of the triangle inequality (Hausdorff, 1934), called the **ultrametric inequality**:

$$d(x,y) \le \max\{d(x,z), d(z,y)\}$$

An ultrametric space is also called an *isosceles space* since at least two of $d(x,y)$, $d(z,y)$, $d(x,z)$ are equal. An ultrametric on a set V has at most $|V|$ values.

A metric d is an ultrametric if and only if its **power transform** (see Chap. 4) d^{α} is a metric for any real positive number α. Any ultrametric satisfies the **four-point inequality**. A metric d is an ultrametric if and only if it is a **Farris transform metric** (Chap. 4) of a **four-point inequality metric**.

- **Robinsonian distance**

A distance d on X is called a **Robinsonian distance** (or *monotone distance*) if there exists a total order $\preceq$ on X *compatible* with it, i.e., for $x,y,w,z \in X$,

$$x \preceq y \preceq w \preceq z \text{ implies } d(y,w) \leq d(x,z),$$

or, equivalently, for $x,y,z \in X$, it holds

$$x \preceq y \preceq z \text{ implies } d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

Any **ultrametric** is a Robinsonian distance.

- **Four-point inequality metric**

A metric d on X is a **four-point inequality metric** (or **additive metric**) if it satisfies the following strengthened version of the triangle inequality called the **four-point inequality** (Buneman, 1974): for all $x,y,z,u \in X$

$$d(x,y) + d(z,u) \leq \max\{d(x,z) + d(y,u), d(x,u) + d(y,z)\}$$

holds. Equivalently, among the three sums $d(x,y) + d(z,u)$, $d(x,z) + d(y,u)$, $d(x,u) + d(y,z)$ the two largest sums are equal.

A metric satisfies the four-point inequality if and only if it is a **tree-like metric**.

Any metric, satisfying the four-point inequality, is a **Ptolemaic metric** and an L_1*-metric*. Cf. L_p**-metric** in Chap. 5.

A **bush metric** is a metric for which all four-point inequalities are equalities, i.e., $d(x,y) + d(u,z) = d(x,u) + d(y,z)$ holds for any $u,x,y,z \in X$.

- **Relaxed four-point inequality metric**

A metric d on X satisfies the **relaxed four-point inequality** if, for all $x,y,z,u \in X$, among the three sums

$$d(x,y) + d(z,u), d(x,z) + d(y,u), d(x,u) + d(y,z)$$

at least two (not necessarily the two largest) are equal. A metric satisfies this inequality if and only if it is a **relaxed tree-like metric**.

- **Ptolemaic metric**

A **Ptolemaic metric** d is a metric on X which satisfies the **Ptolemaic inequality**

$$d(x,y)d(u,z) \leq d(x,u)d(y,z) + d(x,z)d(y,u)$$

for all $x, y, u, z \in X$. A classical result, attributed to Ptolemy, says that this inequality holds in the Euclidean plane, with equality if and only if the points x, y, u, z lie on a circle in that order.

A *Ptolemaic space* is a *normed vector space* $(V, ||.||)$ such that its norm metric $||x - y||$ is a Ptolemaic metric. A normed vector space is a Ptolemaic space if and only if it is an **inner product space** (Chap. 5); so, a **Minkowskian metric** (Chap. 6) is Euclidean if and only if it is Ptolemaic.

For any metric d, the metric $\sqrt{d}$ is Ptolemaic ([FoSc06]).

- **δ-hyperbolic metric**

  Given a number $\delta \geq 0$, a metric d on a set X is called **δ-hyperbolic** if it satisfies the following *Gromov δ-hyperbolic inequality* (another weakening of the **four-point inequality**): for all $x, y, z, u \in X$, it holds that

  $$d(x, y) + d(z, u) \leq 2\delta + \max\{d(x, z) + d(y, u), d(x, u) + d(y, z)\}.$$

  A metric space (X, d) is δ-hyperbolic if and only if for all $x_0, x, y, z \in X$ it holds

  $$(x.y)_{x_0} \geq \min\{(x.z)_{x_0}, (y.z)_{x_0}\} - \delta,$$

  where $(x.y)_{x_0} = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y))$ is the **Gromov product** of the points x and y of X with respect to the base point $x_0 \in X$.

  A metric space (X, d) is 0-hyperbolic exactly when d satisfies the **four-point inequality**. Every bounded metric space of diameter D is D-hyperbolic. The n-dimensional *hyperbolic space* is $\ln 3$-hyperbolic.

  Every δ-hyperbolic metric space is isometrically embeddable into a **geodesic metric space** (Bonk and Schramm, 2000).

- **Gromov product similarity**

  Given a metric space (X, d) with a fixed point $x_0 \in X$, the **Gromov product similarity** (or *Gromov product*, *covariance*, *overlap function*) $(.)_{x_0}$ is a similarity on X defined by

  $$(x.y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)).$$

  The triangle inequality for d implies $(x.y)_{x_0} \geq (x.z)_{x_0} + (y.z)_{x_0} - (z.z)_{x_0}$ (**covariance triangle inequality**), i.e., **sharp triangle inequality** for **protometric** $-(x.y)_{x_0}$.

  If (X, d) is a tree, then $(x.y)_{x_0} = d(x_0, [x, y])$. If (X, d) is a **measure semimetric space**, i.e., $d(x, y) = \mu(x \triangle y)$ for a Borel measure μ on X, then $(x.y)_\emptyset = \mu(x \cap y)$. If d is a **distance of negative type**, i.e., $d(x, y) = d_E^2(x, y)$ for a subset X of a Euclidean space $\mathbb{E}^n$, then $(x.y)_0$ is the usual *inner product* on $\mathbb{E}^n$.

  Cf. **Farris transform metric** $d_{x_0}(x, y) = C - (x.y)_{x_0}$ in Chap. 4.

- **Cross-difference**

  Given a metric space (X, d) and quadruple (x, y, z, w) of its points, the **cross-difference** is the real number cd defined by

  $$cd(x, y, z, w) = d(x, y) + d(z, w) - d(x, z) - d(y, w).$$

In terms of the **Gromov product similarity**, for all $x, y, z, w, p \in X$, it holds

$$\frac{1}{2}cd(x, y, z, w) = -(x.y)_p - (z.w)_p + (x.z)_p + (y.w)_p;$$

in particular, it becomes $(x.y)_p$ if $y = w = p$.

If $x \neq z$ and $y \neq w$, the **cross-ratio** is the positive number defined by

$$cr((x, y, z, w), d) = \frac{d(x, y)d(z, w)}{d(x, z)d(y, w)}.$$

- $2k$**-gonal distance**

  A $2k$**-gonal distance** d is a distance on X which satisfies, for all distinct elements $x_1, \dots, x_n \in X$, the $2k$**-gonal inequality**

  $$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^n b_i = 0$ and $\sum_{i=1}^n |b_i| = 2k$.
- **Distance of negative type**

  A **distance of negative type** d is a distance on X which is $2k$**-gonal** for any $k \geq 1$, i.e., satisfies the **negative type inequality**

  $$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^n b_i = 0$, and for all distinct elements $x_1, \dots, x_n \in X$.

  A distance can be of negative type without being a semimetric. Cayley proved that a metric d is an L_2**-metric** if and only if d^2 is a distance of negative type.
- $(2k + 1)$**-gonal distance**

  A $(2k+1)$**-gonal distance** d is a distance on X which satisfies, for all distinct elements $x_1, \dots, x_n \in X$, the $(2k + 1)$**-gonal inequality**

  $$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^n b_i = 1$ and $\sum_{i=1}^n |b_i| = 2k + 1$.

  The $(2k + 1)$-gonal inequality with $k = 1$ is the usual triangle inequality. The $(2k + 1)$-gonal inequality implies the $2k$**-gonal inequality**.
- **Hypermetric**

  A **hypermetric** d is a distance on X which is $(2k + 1)$**-gonal** for any $k \geq 1$, i.e., satisfies the **hypermetric inequality** (Deza, 1960)

  $$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^n b_i = 1$, and for all distinct elements $x_1, \dots, x_n \in X$.

Any hypermetric is a semimetric, a **distance of negative type** and, moreover, it can be isometrically embedded into some n-sphere $\mathbb{S}^n$ with squared Euclidean distance. Any L_1-*metric* (cf. L_p**-metric** in Chap. 5) is a hypermetric.

- ***P*-metric**

  A ***P*-metric** d is a metric on X with values in $[0, 1]$ which satisfies the **correlation triangle inequality**

$$d(x, y) \leq d(x, z) + d(z, y) - d(x, z)d(z, y).$$

  The equivalent inequality $1-d(x, y) \geq (1-d(x, z))(1-d(z, y))$ expresses that the probability, say, to reach x from y via z is either equal to $(1 - d(x, z))(1 - d(z, y))$ (independence of reaching z from x and y from z), or greater than it (positive correlation). A metric is a P-metric if and only if it is a **Schoenberg transform metric** (Chap. 4).

1.2 Main Distance-Related Notions

- **Metric ball**

  Given a metric space (X, d), the **metric ball** (or *closed metric ball*) with center $x_0 \in X$ and radius $r > 0$ is defined by $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$, and the **open metric ball** with center $x_0 \in X$ and radius $r > 0$ is defined by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. The closed ball is a subset of the closure of the open ball; it is a proper subset for, say, the **discrete metric** on X.

  The **metric sphere** with center $x_0 \in X$ and radius $r > 0$ is defined by $S(x_0, r) = \{x \in X : d(x_0, x) = r\}$.

  For the **norm metric** on an n-dimensional *normed vector space* $(V, ||.||)$, the metric ball $\overline{B}^n = \{x \in V : ||x|| \leq 1\}$ is called the *unit ball*, and the set $S^{n-1} = \{x \in V : ||x|| = 1\}$ is called the *unit sphere*. In a two-dimensional vector space, a metric ball (closed or open) is called a **metric disk** (closed or open, respectively).

- **Metric hull**

  Given a metric space (X, d), let M be a **bounded** subset of X.

  The **metric hull** $H(M)$ of M is the intersection of all metric balls containing M.

  The set of *surface points* $S(M)$ of M is the set of all $x \in H(M)$ such that x lies on the sphere of one of the metric balls containing M.

- **Distance-invariant metric space**

  A metric space (X, d) is **distance-invariant** if all **metric balls** $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$ of the same radius have the same number of elements.

  Then the **growth rate of a metric space** (X, d) is the function $f(n) = |\overline{B}(x, n)|$.

(X, d) is a *metric space of polynomial growth* if there are some positive constants k, C such that $f(n) \leq Cn^k$ for all $n \geq 0$. Cf. **graph of polynomial growth**, including the group case, in Chap. 15.

For a **metrically discrete metric space** (X, d) (i.e., with $a = \inf_{x,y \in X, x \neq y} d(x, y) > 0$), its *growth rate* was defined also (Gordon–Linial–Rabinovich, 1998) by

$$\max_{x \in X, r \geq 2} \frac{\log |\overline{B}(x, ar)|}{\log r}.$$

- **Ahlfors q-regular metric space**

  A metric space (X, d) endowed with a Borel measure μ is called **Ahlfors q-regular** if there exists a constant $C \geq 1$ such that for every ball in (X, d) with radius $r < diam(X, d)$ it holds

$$C^{-1} r^q \leq \mu(\overline{B}(x_0, r)) \leq Cr^q.$$

  If such an (X, d) is locally compact, then the **Hausdorff q-measure** can be taken as μ and q is the **Hausdorff dimension**. For two disjoint **continua** (nonempty **connected compact metric subspaces**) C_1, C_2 of such space (X, d), let Γ be the set of rectifiable curves connecting C_1 to C_2. The *q-modulus* between C_1 and C_2 is $M_q(C_1, C_2) = \inf\{\int_X \rho^q : \inf_{\gamma \in \Gamma} \int_\gamma \rho \geq 1\}$, where $\rho : X \to \mathbb{R}_{>0}$ is any density function on X; cf. the **modulus metric** in Chap. 6.

  The *relative distance* between C_1 and C_2 is $\delta(C_1, C_2) = \frac{\inf\{d(p_1,p_2): p_1 \in C_1, p_2 \in C_2\}}{\min\{diam(C_1), diam(C_2)\}}$. (X, d) is a **q-Loewner space** if there are increasing functions $f, g : [0, \infty) \to [0, \infty)$ such that for all C_1, C_2 it holds $f(\delta(C_1, C_2)) \leq M_q(C_1, C_2) \leq g(\delta(C_1, C_2))$.
- **Connected metric space**

  A metric space (X, d) is called **connected** if it cannot be partitioned into two nonempty **open** sets. Cf. **connected space** in Chap. 2.

  The maximal connected subspaces of a metric space are called its *connected components*. A **totally disconnected metric space** is a space in which all connected subsets are $\emptyset$ and one-point sets.

  A **path-connected metric space** is a connected metric space such that any two its points can be joined by an **arc** (cf. **metric curve**).
- **Cantor connected metric space**

  A metric space (X, d) is called **Cantor** (or *pre-*) *connected* if, for any two its points x, y and any $\epsilon > 0$, there exists an *ϵ-chain* joining them, i.e., a sequence of points $x = z_0, z_1, \dots, z_{n-1}, z_n = y$ such that $d(z_k, z_{k+1}) \leq \epsilon$ for every $0 \leq k \leq n$. A metric space (X, d) is Cantor connected if and only if it cannot be partitioned into two *remote parts* A and B, i.e., such that $\inf\{d(x, y) : x \in A, y \in B\} > 0$.

  The maximal Cantor connected subspaces of a metric space are called its *Cantor connected components*. A **totally Cantor disconnected metric** is the metric of a metric space in which all Cantor connected components are one-point sets.

- **Indivisible metric space**

  A metric space (X, d) is called **indivisible** if it cannot be partitioned into two parts, neither of which contains an isometric copy of (X, d). Any indivisible metric space with $|X| \geq 2$ is infinite, bounded and **totally Cantor disconnected** (Delhomme–Laflamme–Pouzet–Sauer, 2007).

  A metric space (X, d) is called an **oscillation stable metric space** (Nguyen Van Thé, 2006) if, given any $\epsilon > 0$ and any partition of X into finitely many pieces, the **ϵ-neighborhood** of one of the pieces includes an isometric copy of (X, d).
- **Closed subset of metric space**

  Given a subset M of a metric space (X, d), a point $x \in X$ is called a *limit* (or *accumulation*) *point of* M if any **open metric ball** $B(x, r) = \{y \in X : d(x, y) < r\}$ contains a point $x' \in M$ with $x' \neq x$. The *boundary* $\vartheta(M)$ of M is the set of all its limit points. The *closure of* M, denoted by $cl(M)$, is $M \cup \vartheta(M)$, and M is called **closed subset**, if $M = cl(M)$, and **dense subset**, if $X = cl(M)$.

  Every point of M which is not its limit point, is called an *isolated point*. The *interior* $int(M)$ of M is the set of all its isolated points, and the *exterior* $ext(M)$ of M is $int(X \backslash M)$. A subset M is called *nowhere dense* if $int(cl(M)) = \emptyset$.

  A subset M is called *topologically discrete* (cf. **metrically discrete metric space**) if $int(M) = M$ and *dense-in-itself* if $int(M) = \emptyset$. A dense-in-itself subset is called *perfect* (cf. **perfect metric space**) if it is closed. The subsets *Irr* (irrational numbers) and $\mathbb{Q}$ (rational numbers) of $\mathbb{R}$ are dense, dense-in-itself but not perfect. The set $\mathbb{Q} \cap [0, 1]$ is dense-in-itself but not dense in $\mathbb{R}$.
- **Open subset of metric space**

  A subset M of a metric space (X, d) is called *open* if, given any point $x \in M$, the **open metric ball** $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in M for some number $r > 0$. The family of open subsets of a metric space forms a natural topology on it. A **closed subset** is the complement of an open subset.

  An open subset is called *clopen*, if it is closed, and a *domain* if it is **connected**.

  A *door space* is a metric (in general, topological) space in which every subset is either open or closed.
- **Metric topology**

  A **metric topology** is a *topology* induced by a metric; cf. **equivalent metrics**. More exactly, the **metric topology** on a metric space (X, d) is the set of all *open sets* of X, i.e., arbitrary unions of (finitely or infinitely many) open metric balls $B(x, r) = \{y \in X : d(x, y) < r\}, x \in X, r \in \mathbb{R}, r > 0$.

  A topological space which can arise in this way from a metric space is called a **metrizable space** (Chap. 2). **Metrization theorems** are theorems which give sufficient conditions for a topological space to be metrizable.

  On the other hand, the adjective *metric* in several important mathematical terms indicates connection to a measure, rather than distance, for example, *metric Number Theory*, *metric Theory of Functions*, *metric transitivity*.
- **Equivalent metrics**

  Two metrics d_1 and d_2 on a set X are called **equivalent** if they define the same *topology* on X, i.e., if, for every point $x_0 \in X$, every open metric ball with center at x_0 defined with respect to d_1, contains an open metric ball with the same center but defined with respect to d_2, and conversely.

Two metrics d_1 and d_2 are equivalent if and only if, for every $\epsilon > 0$ and every $x \in X$, there exists $\delta > 0$ such that $d_1(x, y) \leq \delta$ implies $d_2(x, y) \leq \epsilon$ and, conversely, $d_2(x, y) \leq \delta$ implies $d_1(x, y) \leq \epsilon$.

All metrics on a finite set are equivalent; they generate the *discrete topology*.

- **Metric betweenness**

  The **metric betweenness** of a metric space (X, d) is (Menger, 1928) the set of all ordered triples (x, y, z) such that x, y, z are (not necessarily distinct) points of X for which the **triangle equality** $d(x, y) + d(y, z) = d(x, z)$ holds.

- **Monometric**

  A ternary relation R on a set X is called a *betweenness relation* if $(x, y, z) \in R$ if and only if $(z, y, x) \in R$ and $(x, y, z), (x, z, y) \in R$ if and only if $y = z$.

  Given a such relation R, a **monometric** is (Perez-Fernández et al., 2016) a function $d : X \times X \to \mathbb{R}_{\geq 0}$ with $d(x, y) = 0$ if and only if $x = y$ and (x, y, z) implying $d(x, y) \leq d(x, z)$. Clearly, any metric is a monometric.

  Cf. a **distance-rationalizable voting rule** in Sect. 11.2.

- **Closed metric interval**

  Given two different points $x, y \in X$ of a metric space (X, d), the **closed metric interval** between them (or *line induced by*) them is the set of the points z, for which the **triangle equality** (or **metric betweenness** (x, z, y)) holds:

$$I(x, y) = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}.$$

  Cf. **inner product space** (Chap. 5) and **cutpoint additive metric** (Chap. 15).

  Let $Ext(x, y) = \{z : y \in I(x, z) \setminus \{x, z\}\}$. A *CC-line* $CC(x, y)$ is $I(x, y) \cup Ext(x, y) \cup Ext(y, x)$. Chen–Chvátal, 2008, conjectured that every metric space on $n, n \geq 2$, points, either has at least n distinct CC-lines or consists of a unique CC-line.

- **Underlying graph of a metric space**

  The **underlying graph** (or *neighborhood graph*) of a metric space (X, d) is a graph with the vertex-set X and xy being an edge if $I(x, y) = \{x, y\}$, i.e., there is no third point $z \in X$, for which $d(x, y) = d(x, z) + d(z, y)$.

- **Distance monotone metric space**

  A metric space (X, d) is called **distance monotone** if for any its **closed metric interval** $I(x, y)$ and $u \in X \setminus I(x, y)$, there exists $z \in I(x, xy)$ with $d(u, z) > d(x, y)$.

- **Metric triangle**

  Three distinct points $x, y, z \in X$ of a metric space (X, d) form a **metric triangle** if the **closed metric intervals** $I(x, y), I(y, z)$ and $I(z, x)$ intersect only in the common endpoints.

- **Metric space having collinearity**

  A metric space (X, d) has **collinearity** if for any $\epsilon > 0$ each of its infinite subsets contains distinct ϵ*-collinear* (i.e., with $d(x, y) + d(y, z) - d(x, z) \leq \epsilon$) points x, y, z.

- **Modular metric space**

  A metric space (X, d) is called **modular** if, for any three different points $x, y, z \in X$, there exists a point $u \in I(x, y) \cap I(y, z) \cap I(z, x)$. This should not be confused with **modular distance** in Chap. 10 and **modulus metric** in Chap. 6.
- **Median metric space**

  A metric space (X, d) is called a **median metric space** if, for any three points $x, y, z \in X$, there exists a unique point $u \in I(x, y) \cap I(y, z) \cap I(z, x)$, or, equivalently,

$$d(x, u) + d(y, u) + d(z, u) = \frac{1}{2}((x, y) + d(y, z) + d(z, x)).$$

  The point u is called *median for* $\{x, y, z\}$, since it minimises the sum of distances to them. Any median metric space is an L_1*-metric*; cf. L_p**-metric** in Chap. 5 and **median graph** in Chap. 15.

  A metric space (X, d) is called an **antimedian metric space** if, for any three points $x, y, z \in X$, there exists a unique point $u \in X$ maximizing $d(x, u) + d(y, u) + d(z, u)$.
- **Metric quadrangle**

  Four different points $x, y, z, u \in X$ of a metric space (X, d) form a **metric quadrangle** if $x, z \in I(y, u)$ and $y, u \in I(x, z)$; then $d(x, y) = d(z, u)$ and $d(x, u) = d(y, z)$.

  A metric space (X, d) is called *weakly spherical* if any three different points $x, y, z \in X$ with $y \in I(x, z)$, form a metric quadrangle with some point $u \in X$.
- **Metric curve**

  A **metric curve** (or, simply, *curve*) γ in a metric space (X, d) is a continuous mapping $\gamma : I \to X$ from an interval I of $\mathbb{R}$ into X. A curve is called an **arc** (or **path**, *simple curve*) if it is injective. A curve $\gamma : [a, b] \to X$ is called a *Jordan curve* (or *simple closed curve*) if it does not cross itself, and $\gamma(a) = \gamma(b)$.

  The **length of a curve** $\gamma : [a, b] \to X$ is the number $l(\gamma)$ defined by

$$l(\gamma) = \sup\{\sum_{1 \le i \le n} d(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_n = b\}.$$

  A *rectifiable curve* is a curve with a finite length. A metric space (X, d), where every two points can be joined by a rectifiable curve, is called a **quasi-convex metric space** (or, specifically, C**-quasi-convex metric space**) if there exists a constant $C \ge 1$ such that every pair $x, y \in X$ can be joined by a rectifiable curve of length at most $Cd(x, y)$. If $C = 1$, then this length is equal to $d(x, y)$, i.e., (X, d) is a **geodesic metric space** (Chap. 6).

  In a quasi-convex metric space (X, d), the infimum of the lengths of all rectifiable curves, connecting $x, y \in X$ is called the **internal metric**.

  The metric d on X is called the **intrinsic metric** (and then (X, d) is called a **length space**) if it coincides with the internal metric of (X, d).

  If, moreover, any pair x, y of points can be joined by a curve of length $d(x, y)$, the metric d is called **strictly intrinsic**, and the length space (X, d) is a geodesic metric space. Hopf–Rinow, 1931, showed that any complete locally compact length space is geodesic and **proper**. The **punctured plane** $(\mathbb{R}^2 \setminus \{0\}, ||x - y||_2)$ is **locally compact** and **path-connected** but not geodesic: the distance between $(-1, 0)$ and $(1, 0)$ is 2 but there is no geodesic realizing this distance.

The **metric derivative** of a metric curve $\gamma : [a, b] \to X$ at a limit point t is

$$\lim_{s \to 0} \frac{d(\gamma(t + s), \gamma(t))}{|s|},$$

if it exists. It is the rate of change, with respect to t, of the length of the curve at almost every point, i.e., a generalization of the notion of *speed* to metric spaces.

- **Geodesic**

Given a metric space (X, d), a **geodesic** is a locally shortest **metric curve**, i.e., it is a locally isometric embedding of $\mathbb{R}$ into X; cf. Chap. 6.

A subset S of X is called a **geodesic segment** (or **metric segment**, *shortest* path, *minimizing geodesic*) between two distinct points x and y in X, if there exists a *segment* (closed interval) [a,b] on the real line $\mathbb{R}$ and an isometric embedding $\gamma : [a, b] \to X$, such that $\gamma[a, b] = S$, $\gamma(a) = x$ and $\gamma(b) = y$.

A **metric straight line** is a geodesic which is minimal between any two of its points; it is an isometric embedding of the whole of $\mathbb{R}$ into X. A **metric ray** and **metric great circle** are isometric embeddings of, respectively, the half-line $\mathbb{R}_{\geq 0}$ and a circle $S^1(0, r)$ into X.

A **geodesic metric space** (Chap. 6) is a metric space in which any two points are joined by a geodesic segment. If, moreover, the geodesic is unique, the space is called *totally geodesic* (or *uniquely geodesic*).

A geodesic metric space (X, d) is called *geodesically complete* if every geodesic is a subarc of a metric straight line. If (X, d) is **complete**, then it is geodesically complete. The **punctured plane** $(\mathbb{R}^2 \setminus \{0\}, ||x - y||_2)$ is not geodesically complete: any geodesic going to 0 is not a subarc of a metric straight line.

- **Length spectrum**

Given a metric space (X, d), a *closed geodesic* is a map $\gamma : \mathbb{S}^1 \to X$ which is locally minimizing around every point of $\mathbb{S}^1$.

If (X, d) is a compact **length space**, its **length spectrum** is the collection of lengths of closed geodesics. Each length is counted with *multiplicity* equal to the number of distinct *free homotopy* classes that contain a closed geodesic of such length. The **minimal length spectrum** is the set of lengths of closed geodesics which are the shortest in their free homotopy class. Cf. the **distance list**.

- **Systole of metric space**

Given a compact metric space (X, d), its **systole** $sys(X, d)$ is the length of the shortest noncontractible loop in X; such a loop is a closed geodesic. So, $sys(X, d) = 0$ exactly if (X, d) is **simply connected**. Cf. **connected space** in Chap. 2.

If (X, d) is a graph with path metric, then its systole is referred to as the *girth*. If (X, d) is a closed surface, then its *systolic ratio* is the ratio $SR = \frac{sys^2(X,d)}{area(X,d)}$.

Some tight upper bounds of SR for every metric on a surface are: $\frac{2}{\sqrt{3}} = \gamma_2$ (*Hermite constant* in 2D) for 2-torus (Loewner, 1949), $\frac{\pi}{2}$ for the real projective plane (Pu, 1952) and $\frac{\pi}{\sqrt{8}}$ for the Klein bottle (Bavard, 1986). Tight asymptotic bounds for a surface S of large genus g are $\frac{4}{9} \cdot \frac{\log^2 g}{\pi g} \leq SR(S) \leq \frac{\log^2 g}{\pi g}$ (Katz et al., 2007).

- **Shankar–Sormani radii**

  Given a **geodesic metric space** (X, d), Shankar and Sormani, 2009, defined its **unique injectivity radius** $Uirad(X)$ as the supremum over all $r \geq 0$ such that any two points at distance at most r are joined by a unique geodesic, and its **minimal radius** $Mrad(X)$ as $\inf_{p \in X} d(p, MinCut(p))$.

  Here the *minimal cut locus of p* $MinCut(p)$ is the set of points $q \in X$ for which there is a geodesic γ running from p to q such that γ extends past q but is not minimizing from p to any point past q. If (X, d) is a Riemannian space, then the distance function from p is a smooth function except at p itself and the cut locus. Cf. **medial axis and skeleton** in Chap. 21.

  It holds $Uirad(X) \leq Mrad(X)$ with equality if (X, d) is a Riemannian space in which case it is the **injectivity radius**. It holds $Uirad(X) = \infty$ for a flat disk but $Mrad(X) < \infty$ if (X, d) is compact and at least one geodesic is extendible.

- **Geodesic convexity**

  Given a **geodesic metric space** (X, d) and a subset $M \subset X$, the set M is called **geodesically convex** (or *convex*) if, for any two points of M, there exists a geodesic segment connecting them which lies entirely in M; the space is **strongly convex** if such a segment is unique and no other geodesic connecting those points lies entirely in M. The space is called **locally convex** if such a segment exists for any two sufficiently close points in M.

  For a given point $x \in M$, the **radius of convexity** is $r_x = \sup\{r \geq 0 : B(x, r) \subset M\}$, where the **metric ball** $B(x, r)$ is convex. The point x is called the *center of mass* of points $y_1, \dots, y_k \in M$ if it minimizes the function $\sum_i d(x, y_i)^2$ (cf. **Fréchet mean**); such point is unique if $d(y_i, y_j) < r_x$ for all $1 \leq i < j \leq k$.

  The **injectivity radius** of the set M is the supremum over all $r \geq 0$ such that any two points in M at distance $\leq r$ are joined by unique geodesic segment which lies in M. The **Hawaiian Earring** is a compact complete metric space consisting of a set of circles of radius $\frac{1}{i}$ for each $i \in \mathbb{N}$ all joined at a common point; its injectivity radius is 0. It is **path-connected** but not **simply connected**.

  The set $M \subset X$ is called a **totally convex metric subspace** of (X, d) if, for any two points of M, any geodesic segment connecting them lies entirely in M.

- **Busemann convexity**

  A **geodesic metric space** (X, d) is called **Busemann convex** (or *Busemann space*, *nonpositively curved in the sense of Busemann*) if, for any three points $x, y, z \in X$ and *midpoints* $m(x, z)$ and $m(y, z)$ (i.e., $d(x, m(x, z)) = d(m(x, z), z) = \frac{1}{2}d(x, z)$ and $d(y, m(y, z)) = d(m(y, z), z) = \frac{1}{2}d(y, z)$), there holds

$$d(m(x, z), m(y, z)) \leq \frac{1}{2}d(x, y).$$

The *flat Euclidean strip* $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1\}$ is **Gromov hyperbolic metric space** (Chap. 6) but not Busemann convex one. In a complete Busemann convex metric space any two points are joined by a unique geodesic segment.

A locally geodesic metric space (X, d) is called **Busemann locally convex** if the above inequality holds locally. Any **locally CAT(0)** metric space is Busemann locally convex.

- **Menger convexity**

A metric space (X, d) is called **Menger convex** if, for any different points $x, y \in X$, there exists a third point $z \in X$ for which $d(x, y) = d(x, z) + d(z, y)$, i.e., $|I(x, y)| > 2$ holds for the **closed metric interval** $I(x, y) = \{z \in X : (x, y) = d(x, z) + d(z, y)\}$. It is called **strictly Menger convex** if such a z is unique for all $x, y \in X$.

Geodesic convexity implies Menger convexity. The converse holds for **complete** metric spaces.

A subset $M \subset X$ is called (Menger, 1928) a *d-convex set* (or *interval-convex set*) if $I(x, y) \subset M$ for any different points $x, y \in M$. A function $f : M \to \mathbb{R}$ defined on a d-convex set $M \subset X$ is a d**-convex function** if for any $z \in I(x, y) \subset M$

$$f(z) \leq \frac{d(y, z)}{d(x, y)} f(x) + \frac{d(x, z)}{d(x, y)} f(y).$$

A subset $M \subset X$ is a *gated set* if for every $x \in X$ there exists a unique $x' \in M$, the *gate*, such that $d(x, y) = d(x, x') + d(x', y)$ for $y \in M$. Any such set is d-convex.

- **Midpoint convexity**

A metric space (X, d) is called **midpoint convex** (or **having midpoints**, *admitting a midpoint map*) if, for any different points $x, y \in X$, there exists a third point $m(x, y) \in X$ for which $d(x, m(x, y)) = d(m(x, y), y) = \frac{1}{2} d(x, y)$. Such a point $m(x, y)$ is called a *midpoint* and the map $m : X \times X \to X$ is called a *midpoint map* (cf. **midset**); this map is unique if $m(x, y)$ is unique for all $x, y \in X$.

For example, the geometric mean $\sqrt{xy}$ is the midpoint map for the metric space $(\mathbb{R}_{>0}, d(x, y) = |\log x - \log y|)$.

A **complete** metric space is **geodesic** if and only if it is midpoint convex.

A metric space (X, d) is said to have **approximate midpoints** if, for any points $x, y \in X$ and any $\epsilon > 0$, there exists an *ϵ-midpoint*, i.e., a point $z \in X$ such that $d(x, z) \leq \frac{1}{2} d(x, y) + \epsilon \geq d(z, y)$.

- **Ball convexity**

A **midpoint convex** metric space (X, d) is called **ball convex** if

$$d(m(x, y), z) \leq \max\{d(x, z), d(y, z)\}$$

for all $x, y, z \in X$ and any midpoint map $m(x, y)$.

Ball convexity implies that all metric balls are **totally convex** and, in the case of a **geodesic** metric space, vice versa. Ball convexity implies also the uniqueness of a midpoint map (geodesics in the case of **complete** metric space).

The metric space $(\mathbb{R}^2, d(x,y) = \sum_{i=1}^{2}\sqrt{|x_i - y_i|})$ is not ball convex.

- **Distance convexity**

  A **midpoint convex** metric space (X, d) is called **distance convex** if

$$d(m(x,y), z) \leq \frac{1}{2}(d(x,z) + d(y,z)).$$

  A **geodesic metric space** is distance convex if and only if the restriction of the distance function $d(x, \cdot)$, $x \in X$, to every geodesic segment is a convex function.

  Distance convexity implies **ball convexity** and, in the case of **Busemann convex** metric space, vice versa.

- **Metric convexity**

  A metric space (X, d) is called **metrically convex** if, for any different points $x, y \in X$ and any $\lambda \in (0, 1)$, there exists a third point $z = z(x, y, \lambda) \in X$ for which $d(x, y) = d(x, z) + d(z, y)$ and $d(x, z) = \lambda d(x, y)$.

  The space is called **strictly metrically convex** if such a point $z(x, y, \lambda)$ is unique for all $x, y \in X$ and any $\lambda \in (0, 1)$.

  A metric space (X, d) is called **strongly metrically convex** if, for any different points $x, y \in X$ and any $\lambda_1, \lambda_2 \in (0, 1)$, there exists a third point $z = z(x, y, \lambda) \in X$ for which $d(z(x, y, \lambda_1), z(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$.

  Metric convexity implies **Menger convexity**, and every Menger convex **complete** metric space is strongly metrically convex.

  A metric space (X, d) is called **nearly convex** (Mandelkern, 1983) if, for any different points $x, y \in X$ and any $\lambda, \mu > 0$ such that $d(x, y) < \lambda + \mu$, there exists a third point $z \in X$ for which $d(x, z) < \lambda$ and $d(z, y) < \mu$, i.e., $z \in B(x, \lambda) \cap B(y, \mu)$. Metric convexity implies near convexity.

- **Takahashi convexity**

  A metric space (X, d) is called **Takahashi convex** if, for any different points $x, y \in X$ and any $\lambda \in (0, 1)$, there exists a third point $z = z(x, y, \lambda) \in X$ such that $d(z(x, y, \lambda), u) \leq \lambda d(x, u) + (1 - \lambda) d(y, u)$ for all $u \in X$. Any convex subset of a normed space is a Takahashi convex metric space with $z(x, y, \lambda) = \lambda x + (1 - \lambda) y$.

  A set $M \subset X$ is *Takahashi convex* if $z(x, y, \lambda) \in M$ for all $x, y \in X$ and any $\lambda \in [0, 1]$. In a Takahashi convex metric space, all metric balls, open metric balls, and arbitrary intersections of Takahashi convex subsets are all Takahashi convex.

- **Hyperconvexity**

  A metric space (X, d) is called **hyperconvex** (Aronszajn–Panitchpakdi, 1956) if it is **metrically convex** and its metric balls have the *infinite Helly property*, i.e., any family of mutually intersecting closed balls in X has nonempty intersection. A metric space (X, d) is hyperconvex if and only if it is an **injective metric space**.

  The spaces l_∞^n, l_∞^∞ and l_1^2 are hyperconvex but l_2^∞ is not.

- **Distance matrix**

  Given a finite metric space $(X = \{x_1, \cdots, x_n\}, d)$, its **distance matrix** is the symmetric $n \times n$ matrix $((d_{ij}))$, where $d_{ij} = d(x_i, x_j)$ for any $1 \leq i, j \leq n$.

The probability that a symmetric $n \times n$ matrix, whose diagonal elements are zeros and all other elements are uniformly random real numbers, is a distance matrix is (Mascioni, 2005) $\frac{1}{2}, \frac{17}{120}$ for $n = 3, 4$, respectively.

- **Magnitude of a finite metric space**

  Let $(X = \{x_1, \dots, x_n\}, d)$ be a finite metric space, such that there exists a vector $w = \{w_1, \dots, w_n\}$ with $((e^{-d(x_i,x_j)}))w = (1, \dots, 1)^T$.

  Then the **magnitude** of (X, d) is (Leinster–Meckes, 2016) the sum $\sum_{i=1}^{n} w_i$. In fact, the definition of *Euler characteristic of a category* was generalized to *enriched categories*, renamed *magnitude*, then re-specialized to metric spaces.

- **Distance product of matrices**

  Given $n \times n$ matrices $A = ((a_{ij}))$ and $B = ((b_{ij}))$, their **distance** (or *min-plus*) **product** is the $n \times n$ matrix $C = ((c_{ij}))$ with $c_{ij} = \min_{k=1}^{n}(a_{ik} + b_{kj})$.

  It is the usual matrix multiplication in the *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \min, +)$ (Chap. 18). If A is the matrix of weights of an edge-weighted complete graph K_n, then its *direct power* A^n is the (shortest path) **distance matrix** of this graph.

- **Distance list**

  Given a metric space (X, d), its **distance set** and **distance list** are the set and the *multiset* (i.e., multiplicities are counted) and of all pairwise distances.

  Two subsets $A, B \subset X$ are said to be *homometric sets* if they have the same distance list. Cf. **homometric structures** in Chap. 24.

  A finite metric space is called *tie-breaking* if all pairwise distances are distinct.

- **Degree of distance near-equality**

  Given a finite metric space (X, d) with $|X| = n \geq 3$, let $f = \min |\frac{d(x,y)}{d(a,b)} - 1|$ (**degree of distance near-equality**) and $f' = \min |\frac{d(x,y)}{d(x,b)} - 1|$, where the minimum is over different 2-subsets $\{x, y\}, \{a, b\}$ of X and, respectively, over different $x, y, b \in X$. [OpPi14] proved $f \leq \frac{9 \log n}{n^2}$ and $f' \leq \frac{3}{n}$, while $f \geq \frac{\log n}{20n^2}$ and $f' \geq \frac{1}{2n}$ for some (X, d).

- **Semimetric cone**

  The **semimetric cone** MET_n is the polyhedral cone in $\mathbb{R}^{\binom{n}{2}}$ of all **distance matrices** of semimetrics on the set $V_n = \{1, \dots, n\}$. Vershik, 2004, considers MET_∞, i.e., the weakly closed convex cone of infinite distance matrices of semimetrics on $\mathbb{N}$.

  The cone of n-point **weightable quasi-semimetrics** is a projection along an extreme ray of the semimetric cone Met_{n+1} (Grishukhin–Deza–Deza, 2011).

  The **metric fan** is a canonical decomposition MF_n of MET_n into subcones whose faces belong to the fan, and the intersection of any two of them is their common boundary. Two semimetrics $d, d' \in MET_n$ lie in the same cone of the metric fan if the subdivisions $\delta_d, \delta_{d'}$ of the polyhedron $\delta(n, 2) = conv\{e_i + e_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n$ are equal. Here a subpolytope P of $\delta(n, 2)$ is a cell of the subdivision δ_d if there exists $y \in \mathbb{R}^n$ satisfying $y_i + y_j = d_{ij}$ if $e_i + e_j$ is a vertex of P, and $y_i + y_j > d_{ij}$, otherwise. The complex of bounded faces of the polyhedron dual to δ_d is the **tight span** of the semimetric d.

- **Cayley–Menger matrix**

  Given a finite metric space $(X = \{x_1, \cdots, x_n\}, d)$, its **Cayley–Menger matrix** is the symmetric $(n+1) \times (n+1)$ matrix

$$CM(X, d) = \begin{pmatrix} 0 & e \\ e^T & D \end{pmatrix},$$

  where $D = ((d^2(x_i, x_j)))$ and e is the n-vector all components of which are 1.

  The determinant of $CM(X, d)$ is called the *Cayley–Menger determinant*. If (X, d) is a metric subspace of the Euclidean space $\mathbb{E}^{n-1}$, then $CM(X, d)$ is $(-1)^n 2^{n-1}((n-1)!)^2$ times the squared $(n-1)$-*dimensional volume* of the convex hull of X in $\mathbb{R}^{n-1}$.

- **Gram matrix**

  Given elements $v_1, \dots, v_k$ of a Euclidean space, their **Gram matrix** is the symmetric $k \times k$ matrix VV^T, where $V = ((v_{ij}))$, of pairwise *inner products* of $v_1, \dots, v_k$:

$$G(v_1, \dots, v_k) = ((\langle v_i, v_j \rangle)).$$

  It holds $G(v_1, \dots, v_k) = \frac{1}{2}((d_E^2(v_0, v_i) + d_E^2(v_0, v_j) - d_E^2(v_i, v_j)))$, i.e., the inner product $\langle \cdot, \cdot \rangle$ is the **Gromov product similarity** of the **squared Euclidean distance** d_E^2. A $k \times k$ matrix $((d_E^2(v_i, v_j)))$ is called *Euclidean distance matrix* (or *EDM*). It defines a **distance of negative type** on $\{1, \dots, k\}$; all such matrices form the (nonpolyhedral) closed convex cone of all such distances.

  The determinant of a Gram matrix is called the *Gram determinant*; it is equal to the square of the *k-dimensional volume* of the *parallelotope* constructed on $v_1, \dots v_k$.

  A symmetric $k \times k$ real matrix M is said to be *positive-semidefinite* (PSD) if $xMx^T \geq 0$ for any nonzero $x \in \mathbb{R}^k$ and *positive-definite* (PD) if $xMx^T > 0$. A matrix is PSD if and only if it is a Gram matrix; it is PD if and only the vectors $v_1, \dots, v_k$ are linearly independent. In Statistics, the *covariance matrices* and *correlation matrices* are exactly PSD and PD ones, respectively.

- **Midset**

  Given a metric space (X, d) and distinct $y, z \in X$, the **midset** (or *bisector*) of points y and z is the set $M = \{x \in X : d(x, y) = d(x, z)\}$ of *midpoints* x.

  A metric space is said to have the *n-point midset property* if, for every pair of its points, the midset has exactly n points. The one-point midset property means uniqueness of the *midpoint map*. Cf. **midpoint convexity**.

- **Distance *k*-sector**

  Given a metric space (X, d) and disjoint subsets $Y, Z \subset X$, the *bisector* of Y and Z is the set $M = \{x \in X : \inf_{y \in Y} d(x, y) = \inf_{z \in Z} d(x, z)\}$.

  The **distance *k*-sector** of Y and Z is the sequence $M_1, \dots, M_{k-1}$ of subsets of X such that M_i, for any $1 \leq i \leq k-1$, is the bisector of sets M_{i-1} and M_{i+1}, where $Y = M_0$ and $Z = M_k$. Asano–Matousek–Tokuyama, 2006, considered the distance k-sector on the Euclidean plane $(\mathbb{R}^2, l_2)$; for compact sets Y and Z, the sets $M_1, \dots, M_{k-1}$ are curves partitioning the plane into k parts.

- **Metric basis**

Given a metric space (X, d) and a subset $M \subset X$, for any point $x \in X$, its *metric M-representation* is the set $\{(m, d(x, m)) : m \in M\}$ of its *metric M-coordinates* $(m, d(x, m))$. The set M is called (Blumenthal, 1953) a **metric basis** (or *resolving set*, *locating set*, *set of uniqueness*, *set of landmarks*) if distinct points $x \in X$ have distinct M-representations. A vertex-subset M of a connected graph is (Okamoto et al., 2009) a *local metric basis* if adjacent vertices have distinct M-representations.

The *resolving number* of a finite (X, d) is (Chartrand–Poisson–Zhang, 2000) minimum k such that any k-subset of X is a metric basis.

The vertices of a non degenerate simplex form a metric basis of $\mathbb{E}^n$, but l_1- and l_∞-metrics on $\mathbb{R}^n$, $n > 1$, have no finite metric basis.

The **distance similarity** is (Saenpholphat–Zhang, 2003) an equivalence relation on X defined by $x \sim y$ if $d(z, x) = d(z, y)$ for any $z \in X \setminus \{x, y\}$. Any metric basis contains all or all but one elements from each equivalence class.

1.3 Metric Numerical Invariants

- **Resolving dimension**

Given a metric space (X, d), its **resolving dimension** (or **location number** (Slater, 1975), *metric dimension* (Harary–Melter, 1976)) is the minimum cardinality of its **metric basis**. The **upper resolving dimension** of (X, d) is the maximum cardinality of its metric basis not containing another metric basis as a proper subset. *Adjacency dimension* of (X, d) is the metric dimension of $(X, \min(2, d))$.

A **metric independence number** of (X, d) is (Currie–Oellermann, 2001) the maximum cardinality I of a collection of pairs of points of X, such that for any two, (say, (x, y) and (x', y')) of them there is no point $z \in X$ with $d(z, x) \neq d(z, y)$ and $d(z, x') \neq d(z, y')$. A function $f : X \to [0, 1]$ is a *resolving function* of (X, d) if $\sum_{z \in X : d(x,z) \neq d(y,z)} f(z) \geq 1$ for any distinct $x, y \in X$. The *fractional resolving dimension* of (X, d) is $F = \min \sum_{x \in X} g(x)$, where the minimum is taken over resolving functions f such that any function f' with f', f is not resolving.

The *partition dimension* of (X, d) is (Chartrand–Salevi–Zhang, 1998) the minimum cardinality P of its *resolving partition*, i.e., a partition $X = \cup_{1 \leq i \leq k} S_i$ such that no two points have, for $1 \leq i \leq k$, the same minimal distances to the set S_i.

Related *locating a robber* game on a graph $G = (V, E)$ was considered in 2012 by Seager and by Carraher et al.: *cop win* on G if every sequence $r = r_1, \dots, r_n$ of robber's steps ($r_i \in V$ and $d_{\text{path}}(r_i, r_{i+1}) \leq 1$) is uniquely identified by a sequence $d(r_1, c_1), \dots, d(r_n, c_n)$ of cop's distance queries for some $c_1, \dots, c_n \in V$.

- **Metric dimension**

  For a metric space (X, d) and a number $\epsilon > 0$, let C_ϵ be the minimal size of an **ϵ-net** of (X, d), i.e., a subset $M \subset X$ with $\cup_{x \in M} B(x, \epsilon) = X$. The number

$$dim(X, d) = \lim_{\epsilon \to 0} \frac{\ln C_\epsilon}{-\ln \epsilon}$$

(if it exists) is called the **metric dimension** (or **Minkowski–Bouligand dimension**, **box-counting dimension**) of X. If the limit above does not exist, then the following notions of dimension are considered:

1. $\underline{dim}(X, d) = \underline{\lim}_{\epsilon \to 0} \frac{\ln C_\epsilon}{-\ln \epsilon}$ called the **lower Minkowski dimension** (or *lower dimension*, *lower box dimension*, *Pontryagin–Snirelman dimension*);
2. $\overline{dim}(X, d) = \overline{\lim}_{\epsilon \to 0} \frac{\ln C_\epsilon}{-\ln \epsilon}$ called the **Kolmogorov–Tikhomirov dimension** (or *upper dimension*, *entropy dimension*, *upper box dimension*).

See below examples of other, less prominent, notions of *metric dimension*.

1. The (equilateral) *metric dimension* of a metric space is the maximum cardinality of its *equidistant* subset, i.e., such that any two of its distinct points are at the same distance. For a normed space, this dimension is equal to the maximum number of translates of its unit ball that touch pairwise.
2. For any $c > 1$, the (normed space) *metric dimension* $dim_c(X)$ of a finite metric space (X, d) is the least dimension of a real *normed space* $(V, ||.||)$ such that there is an embedding $f : X \to V$ with $\frac{1}{c} d(x, y) \le ||f(x) - f(y)|| \le d(x, y)$.
3. The (Euclidean) *metric dimension* of a finite metric space (X, d) is the least dimension n of a Euclidean space $\mathbb{E}^n$ such that $(X, f(d))$ is its metric subspace, where the minimum is taken over all continuous monotone increasing functions $f(t)$ of $t \ge 0$.
4. The *dimensionality* of a metric space is $\frac{\mu^2}{2\sigma^2}$, where μ and σ^2 are the mean and variance of its histogram of distance values; this notion is used in Information Retrieval for proximity searching.

   The term *dimensionality* is also used for the minimal dimension, if it is finite, of Euclidean space in which a given metric space embeds isometrically.

- **Hausdorff dimension**

  Given a metric space (X, d) and $p, q > 0$, let $H_p^q = \inf \sum_{i=1}^{\infty} (diam(A_i))^p$, where the infimum is taken over all countable coverings $\{A_i\}$ with diameter of A_i less than q. The **Hausdorff q-measure** of X is the **metric outer measure** defined by

$$H^p = \lim_{q \to 0} H_p^q.$$

The **Hausdorff dimension** (or **fractal dimension**) of (X, d) is defined by

$$dim_{Haus}(X, d) = \inf\{p \ge 0 : H^p(X) = 0\}.$$

Any countable metric space has $dim_{Haus} = 0$, $dim_{Haus}(\mathbb{E}^n) = n$, and any $X \subset \mathbb{E}^n$ with $Int\ X \neq \emptyset$ has $dim_{Haus} = \overline{dim}$. For any **totally bounded** (X, d), it holds

$$dim_{top} \leq dim_{Haus} \leq \underline{dim} \leq dim \leq \overline{dim}.$$

- **Rough dimension**

  Given a metric space (X, d), its *rough n-volume* $Vol_n X$ is $\overline{\lim}_{\epsilon \to 0} \epsilon^n \beta_X(\epsilon)$, where $\epsilon > 0$ and $\beta_X(\epsilon) = \max |Y|$ for $Y \subseteq X$ with $d(a, b) \geq \epsilon$ if $a \in Y, b \in Y \setminus \{a\}$; $\beta_X(\epsilon) = \infty$ is permitted. The **rough dimension** is defined ([BBI01]) by

$$dim_{rough}(X, d) = \sup\{n : Vol_n X = \infty\} \text{ or, equivalently, } = \inf\{n : Vol_n X = 0\}.$$

  The space (X, d) can be not locally compact. It holds $dim_{Haus} \leq dim_{rough}$.

- **Packing dimension**

  Given a metric space (X, d) and $p, q > 0$, let $P_p^q = \sup \sum_{i=1}^{\infty} (diam(B_i))^p$, where the supremum is taken over all countable packings (by disjoint balls) $\{B_i\}$ with the diameter of B_i less than q.

  The **packing** q**-pre-measure** is $P_0^p = \lim_{q \to 0} P_p^q$. The **packing** q**-measure** is the **metric outer measure** which is the infimum of packing q-pre-measures of countable coverings of X. The **packing dimension** of (X, d) is defined by

$$dim_{pack}(X, d) = \inf\{p \geq 0 : P^p(X) = 0\}.$$

- **Topological dimension**

  For any compact metric space (X, d) its **topological dimension** (or **Lebesgue covering dimension**) is defined by

$$\dim_{top}(X, d) = \inf_{d'}\{\dim_{Haus}(X, d')\},$$

  where d' is any metric on X **equivalent** to d. So, it holds $\dim_{top} \leq \dim_{Haus}$. A **fractal** (Chap. 18) is a metric space for which this inequality is strict.

  This dimension does not exceed also the **Assouad–Nagata dimension** of (X, d).

  In general, the **topological dimension** of a topological space X is the smallest integer n such that, for any finite open covering of X, there exists a finite open refinement of it with no point of X belonging to more than $n + 1$ elements.

  The *geometric dimension* is (Kleiner, 1999; [BBI01]) $\sup dim_{top}(Y, d)$ over compact $Y \subset X$.

- **Doubling dimension**

  The **doubling dimension** ($dim_{doubl}(X, d)$) of a metric space (X, d) is the smallest integer n (or ∞ if such an n does not exist) such that every metric ball (or, say, a set of finite diameter) can be covered by a family of at most 2^n metric balls (respectively, sets) of half the diameter.

If (X, d) has finite doubling dimension, then d is called a **doubling metric** and the smallest integer m such that every metric ball can be covered by a family of at most m metric balls of half the diameter is called *doubling constant*.

- **Assouad–Nagata dimension**

The **Assouad–Nagata dimension** $dim_{AN}(X, d)$ of a metric space (X, d) is the smallest integer n (or ∞ if such an n does not exist) for which there exists a constant $C > 0$ such that, for all $s > 0$, there exists a covering of X by its subsets of diameter $\leq Cs$ with every subset of X of diameter $\leq s$ meeting $\leq n + 1$ elements of covering. It holds (LeDonne–Rajala, 2014) $dim_{AN} \leq dim_{doubl}$; but $dim_{AN} = 1$, while $dim_{doubl} = \infty$, holds (Lang–Schlichenmaier, 2014) for some **real trees** (X, d).

Replacing "for all $s > 0$" in the above definition by "for $s > 0$ sufficiently large" or by "for $s > 0$ sufficiently small", gives the *microscopic mi-*$dim_{AN}(X, d)$ and *macroscopic ma-*$dim_{AN}(X, d)$ Assouad–Nagata dimensions, respectively. Then (Brodskiy et al., 2006) *mi-*$dim_{AN}(X, d) = dim_{AN}(X, \min\{d, 1\})$ and

ma-$dim_{AN}(X, d) = dim_{AN}(X, \max\{d, 1\})$ (here $\max\{d(x, y), 1\}$ means 0 for $x = y$).

The Assouad–Nagata dimension is preserved (Lang–Schlichenmaier, 2004) under **quasi-symmetric mapping** but, in general, not under **quasi-isometry**.

- **Vol'berg–Konyagin dimension**

The **Vol'berg–Konyagin dimension** of a metric space (X, d) is the smallest constant $C > 1$ (or ∞ if such a C does not exist) for which X carries a *doubling measure*, i.e., a Borel measure μ such that, for all $x \in X$ and $r > 0$, it holds that

$$\mu(\overline{B}(x, 2r)) \leq C\mu(\overline{B}(x, r)).$$

A metric space (X, d) carries a doubling measure if and only if d is a **doubling metric**, and any complete doubling metric carries a doubling measure.

The **Karger–Ruhl constant** of a metric space (X, d) is the smallest $c > 1$ (or ∞ if such a c does not exist) such that for all $x \in X$ and $r > 0$ it holds

$$|\overline{B}(x, 2r)| \leq c|\overline{B}(x, r)|.$$

If c is finite, then the **doubling dimension** of (X, d) is at most $4c$.

- **Hyperbolic dimension**

A metric space (X, d) is called an (R, N)*-large-scale doubling* if there exists a number $R > 0$ and integer $N > 0$ such that every ball of radius $r \geq R$ in (X, d) can be covered by N balls of radius $\frac{r}{2}$.

The **hyperbolic dimension** $hypdim(X, d)$ of a metric space (X, d) (Buyalo–Schroeder, 2004) is the smallest integer n such that for every $r > 0$ there are $R > 0$, an integer $N > 0$ and a covering of X with the following properties:

1. Every ball of radius r meets at most $n + 1$ elements of the covering;
2. The covering is an (R, N)-large-scale doubling, and any finite union of its elements is an (R', N)-large-scale doubling for some $R' > 0$.

The hyperbolic dimension is 0 if (X, d) is a large-scale doubling, and it is n if (X, d) is n-dimensional hyperbolic space.

Also, $hypdim(X, d) \leq asdim(X, d)$ since the **asymptotic dimension** $asdim(X, d)$ corresponds to the case $N = 1$ in the definition of $hypdim(X, d)$.

The hyperbolic dimension is preserved under a **quasi-isometry**.

- **Asymptotic dimension**

  The **asymptotic dimension** $asdim(X, d)$ of a metric space (X, d) (Gromov, 1993) is the smallest integer n such that, for every $r > 0$, there exists a constant $D = D(r)$ and a covering of X by its subsets of diameter at most D such that every ball of radius r meets at most $n + 1$ elements of the covering.

  The asymptotic dimension is preserved under a **quasi-isometry**.

- **Width dimension**

  Let (X, d) be a compact metric space. For a given number $\epsilon > 0$, the **width dimension** $Widim_{\epsilon}(X, d)$ of (X, d) is (Gromov, 1999) the minimum integer n such that there exists an n-dimensional polyhedron P and a continuous map $f : X \to P$ (called an *ϵ-embedding*) with $diam(f^{-1}(y)) \leq \epsilon$ for all $y \in P$.

  The width dimension is a *macroscopic dimension at the scale* $\geq \epsilon$ of (X, d), because its limit for $\epsilon \to 0$ is the **topological dimension** of (X, d).

- **Godsil–McKay dimension**

  We say that a metric space (X, d) has **Godsil–McKay dimension** $n \geq 0$ if there exists an element $x_0 \in X$ and two positive constants c and C such that the inequality $ck^n \leq |\{x \in X : d(x, x_0) \leq k\}| \leq Ck^n$ holds for every integer $k \geq 0$.

  This notion was introduced in [GoMc80] for the **path metric** of a countable locally finite graph. They proved that, if the group $\mathbb{Z}^n$ acts faithfully and with a finite number of orbits on the vertices of the graph, then this dimension is n.

- **Metric outer measure**

  A *σ-algebra over X* is any nonempty collection Σ of subsets of X, including X itself, that is closed under complementation and countable unions of its members.

  Given a σ-algebra Σ over X, a *measure* on (X, Σ) is a function $\mu : \Sigma \to [0, \infty]$ with the following properties:

  1. $\mu(\emptyset) = 0$;
  2. For any sequence $\{A_i\}$ of pairwise disjoint subsets of X, $\mu(\sum_i A_i) = \sum_i \mu(A_i)$ (*countable σ-additivity*).

  The triple (X, Σ, μ) is called a *measure space*. If $M \subset A \in \Sigma$ and $\mu(A) = 0$ implies $M \in \Sigma$, then (X, Σ, μ) is called a *complete measure space*. A measure μ with $\mu(X) = 1$ is called a *probability measure*.

  If X is a *topological space* (see Chap. 2), then the σ-algebra over X, consisting of all *open* and *closed sets* of X, is called the *Borel σ-algebra* of X, (X, Σ) is called a *Borel space*, and a measure on Σ is called a *Borel measure*. So, any metric space (X, d) admits a Borel measure coming from its **metric topology**, where the *open set* is an arbitrary union of open **metric d-balls**.

  An *outer measure* on X is a function $\nu : P(X) \to [0, \infty]$ (where $P(X)$ is the set of all subsets of X) with the following properties:

  1. $\nu(\emptyset) = 0$;
  2. For any subsets $A, B \subset X$, $A \subset B$ implies $\nu(A) \leq \nu(B)$ (*monotonicity*);

3. For any sequence $\{A_i\}$ of subsets of X, $\nu(\sum_i A_i) \leq \sum_i \nu(A_i)$ (*countable subadditivity*).

A subset $M \subset X$ is called ν-*measurable* if $\nu(A) = \nu(A \cup M) + \nu(A \setminus M)$ for any $A \subset X$. The set Σ' of all ν-measurable sets forms a σ-algebra over X, and (X, Σ', ν) is a complete measure space.

A **metric outer measure** is an outer measure ν defined on the subsets of a given metric space (X, d) such that $\nu(A \cup B) = \nu(A) + \nu(B)$ for every pair of nonempty subsets $A, B \subset X$ with positive **set-set distance** $\inf_{a \in A, b \in B} d(a, b)$. An example is **Hausdorff q-measure**; cf. **Hausdorff dimension**.

- **Length of metric space**

The **Fremlin length** of a metric space (X, d) is its **Hausdorff 1-measure** $H^1(X)$.

The **Hejcman length** $lng(M)$ of a subset $M \subset X$ of a metric space (X, d) is $\sup\{lng(M') : M' \subset M, |M'| < \infty\}$. Here $lng(\emptyset) = 0$ and, for a finite subset $M' \subset X$, $lng(M') = \min \sum_{i=1}^{n} d(x_{i-1}, x_i)$ over all sequences $x_0, \dots, x_n$ such that $\{x_i : i = 0, 1, \dots, n\} = M'$.

The **Schechtman length** of a finite metric space (X, d) is $\inf \sqrt{\sum_{i=1}^{n} a_i^2}$ over all sequences $a_1, \dots, a_n$ of positive numbers such that there exists a sequence $X_0, \dots, X_n$ of partitions of X with following properties:

1. $X_0 = \{X\}$ and $X_n = \{\{x\} : x \in X\}$;
2. X_i refines X_{i-1} for $i = 1, \dots, n$;
3. For $i = 1, \dots, n$ and $B, C \subset A \in X_{i-1}$ with $B, C \in X_i$, there exists a one-to-one map f from B onto C such that $d(x, f(x)) \leq a_i$ for all $x \in B$.

- **Volume of finite metric space**

Given a metric space (X, d) with $|X| = k < \infty$, its **volume** (Feige, 2000) is the maximal $(k-1)$-dimensional volume of the simplex with vertices $\{f(x) : x \in X\}$ over all **metric mappings** $f : (X, d) \to (\mathbb{R}^{k-1}, l_2)$. The volume coincides with the metric for $k = 2$. It is monotonically increasing and continuous in the metric d.

- **Rank of metric space**

The **Minkowski rank of a metric space** (X, d) is the maximal dimension of a normed vector space $(V, ||.||)$ such that there is an isometry $(V, ||.||) \to (X, d)$.

The **Euclidean rank of a metric space** (X, d) is the maximal dimension of a *flat* in it, that is of a Euclidean space $\mathbb{E}^n$ such that there is an isometric embedding $\mathbb{E}^n \to (X, d)$.

The **quasi-Euclidean rank of a metric space** (X, d) is the maximal dimension of a *quasi-flat* in it, i.e., of an Euclidean space $\mathbb{E}^n$ admitting a **quasi-isometry** $\mathbb{E}^n \to (X, d)$. Every **Gromov hyperbolic metric space** has this rank 1.

- **Roundness of metric space**

The **roundness of a metric space** (X, d) is the supremum of all q such that

$$d(x_1, x_2)^q + d(y_1, y_2)^q \leq d(x_1, y_1)^q + d(x_1, y_2)^q + d(x_2, y_1)^q + d(x_2, y_2)^q$$

for any four points $x_1, x_2, y_1, y_2 \in X$.

Every metric space has roundness ≥ 1; it is ≤ 2 if the space has **approximate midpoints**. The roundness of L_p**-space** is p if $1 \leq p \leq 2$.

The *generalized roundness of a metric space* (X, d) is (Enflo, 1969) the supremum of all q such that, for any $2k \geq 4$ points $x_i, y_i \in X$ with $1 \leq i \leq k$,

$$\sum_{1 \leq i < j \leq k} d^q(x_i, x_j) + d^q(y_i, y_j) \leq \sum_{1 \leq i, j \leq k} d^q(x_i, y_j).$$

Lennard–Tonge–Weston, 1997, have shown that the generalized roundness is the supremum of q such that d is of q**-negative type**, i.e., d^q is of **negative type**.

Every **CAT(0) space** (Chap. 6) has roundness 2, but some of them have generalized roundness 0 (Lafont–Prassidis, 2006).

- **Type of metric space**

  The **Enflo type** of a metric space (X, d) is p if there exists a constant $1 \leq C < \infty$ such that, for every $n \in \mathbb{N}$ and every function $f : \{-1, 1\}^n \to X$, $\sum_{\epsilon \in \{-1,1\}^n} d^p(f(\epsilon), f(-\epsilon))$ is at most

  $C^p \sum_{j=1}^{n} \sum_{\epsilon \in \{-1,1\}^n} d^p(f(\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_j, \epsilon_{j+1}, \ldots, \epsilon_n), f(\epsilon_1, \ldots, \epsilon_{j-1}, -\epsilon_j, \epsilon_{j+1}, \ldots, \epsilon_n))$.

  A Banach space $(V, ||.||)$ of Enflo type p has *Rademacher type* p, i.e., for every $x_1, \ldots, x_n \in V$, it holds

$$\sum_{\epsilon \in \{-1,1\}^n} ||\sum_{j=1}^{n} \epsilon_j x_j||^p \leq C^p \sum_{j=1}^{n} ||x_j||^p.$$

  Given a metric space (X, d), a *symmetric Markov chain on* X is a Markov chain $\{Z_l\}_{l=0}^{\infty}$ on a state space $\{x_1, \ldots, x_m\} \subset X$ with a symmetrical transition $m \times m$ matrix $((a_{ij}))$, such that $P(Z_{l+1} = x_j : Z_l = x_i) = a_{ij}$ and $P(Z_0 = x_i) = \frac{1}{m}$ for all integers $1 \leq i, j \leq m$ and $l \geq 0$. A metric space (X, d) has **Markov type** p (Ball, 1992) if $\sup_T M_p(X, T) < \infty$ where $M_p(X, T)$ is the smallest constant $C > 0$ such that the inequality

$$\mathbb{E}d^p(Z_T, Z_0) \leq TC^p \mathbb{E}d^p(Z_1, Z_0)$$

  holds for every symmetric Markov chain $\{Z_l\}_{l=0}^{\infty}$ on X holds, in terms of expected value (mean) $\mathbb{E}[X] = \sum_x xp(x)$ of the discrete random variable X.

  A metric space of Markov type p has Enflo type p.

- **Strength of metric space**

  Given a finite metric space (X, d) with s different nonzero values of $d_{ij} = d(i, j)$, its **strength** is the largest number t such that, for any integers $p, q \geq 0$ with $p + q \leq t$, there is a polynomial $f_{pq}(s)$ of degree at most $\min\{p, q\}$ such that $((d_{ij}^{2p}))((d_{ij}^{2q})) = ((f_{pq}(d_{ij}^2)))$.

- **Rendez-vous number**

  Given a metric space (X, d), its **rendez-vous number** (or *Gross number, magic number*) is a positive real number $r(X, d)$ (if it exists) defined by the property that for each integer n and all (not necessarily distinct) $x_1, \dots, x_n \in X$ there exists a point $x \in X$ such that

$$r(X, d) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, x).$$

If the number $r(X, d)$ exists, then it is said that (X, d) has the **average distance property**. Every compact connected metric space has this property. The *unit ball* $\{x \in V : ||x|| \le 1\}$ of a **Banach space** $(V, ||.||)$ has the rendez-vous number 1.

- **Wiener-like distance indices**

  Given a finite subset M of a metric space (X, d) and a parameter q, the **Wiener polynomial** of M (as defined by Hosoya, 1988, for the graphic metric d_{path}) is

$$W(M; q) = \frac{1}{2} \sum_{x,y \in M:\, x \neq y} q^{d(x,y)}.$$

It is a *generating function* for the **distance distribution** (Chap. 16) of M, i.e., the coefficient of q^i in $W(M; q)$ is the number $|\{\{x, y\} \in M \times M : d(x, y) = i\}|$.

In the main case when M is the vertex-set V of a connected graph $G = (V, E)$ and d is the **path metric** of G, the number $W(M; 1) = \frac{1}{2} \sum_{x,y \in M} d(x, y)$ is called the **Wiener index** of G. This notion is originated (Wiener, 1947) and applied, together with its many analogs, in Chemistry; cf. **chemical distance** in Chap. 24.

The *hyper-Wiener index* is $\sum_{x,y \in M}(d(x, y) + d(x, y)^2)$. The *reverse-Wiener index* is $\frac{1}{2} \sum_{x,y \in M}(D - d(x, y))$, where D is the diameter of M. The *complementary reciprocal Wiener index* is $\frac{1}{2} \sum_{x,y \in M}(1 + D - d(x, y))^{-1}$. The *Harary index* is $\sum_{x,y \in M}(d(x, y))^{-1}$. The *Szeged index* and the *vertex PI index* are $\sum_{e \in E} n_x(e) n_y(e)$ and $\sum_{e \in E}(n_x(e) + n_y(e))$, where $e = (xy)$ and $n_x(e) = |\{z \in V : d(x, z) < d(y, z)\}|$.

Two studied *edge-Wiener indices* of G are the Wiener index of its *line graph* and $\sum_{(xy),(x'y') \in E} \max\{d(x, x'), d(x, y'), d(y, x'), d(y, y')\}$.

The *Gutman–Schultz index*, **degree distance** (Dobrynin–Kochetova, 1994), *reciprocal degree distance* and *terminal Wiener index* are:

$$\sum_{x,y \in M} r_x r_y d(x, y),\ \sum_{x,y \in M} d(x, y)(r_x + r_y),\ \sum_{x,y \in M} \frac{1}{d(x, y)}(r_x + r_y),\ \sum_{x,y \in \{z \in M : r_z = 1\}} d(x, y),$$

where r_z is the degree of the vertex $z \in M$. The **eccentric distance sum** (Gupta et al., 2002) is $\sum_{y \in M}(\max\{d(x, y) : x \in M\} d_y)$, where d_y is $\sum_{x \in M} d(x, y)$. The *Balaban index* is $\frac{|E|}{c+1} \sum_{(yz) \in E}(\sqrt{d_y d_z})^{-1}$, where c is the number of primitive cycles. The *multiplicative Wiener index* is (Das–Gutman, 2016) $\prod_{x,y \in M, x \neq y} d(x, y)$.

Given a partition $P = \{V_1, \dots, V_k\}$ of the vertex-set V, set $f_P(x) = i$ for $x \in V_i$. The *colored distance* (Dankelman et al., 2001) and the *partition distance* (Klavžar, 2016) of G are $\sum_{f_P(x) \neq f_P(y)} d(x, y)$ and $\sum_{f_P(x) = f_P(y)} d(x, y)$, respectively.

Above indices are called (corresponding) *Kirchhoff indices* if d the **resistance metric** (Chap. 15) of G.

The *average distance* of M is the number $\frac{1}{|M|(|M|-1)}\sum_{x,y\in M} d(x,y)$. In general, for a quasi-metric space (X,d), the numbers $\sum_{x,y\in M} d(x,y)$ and $\frac{1}{|M|(|M|-1)}\sum_{x,y\in M, x\neq y}\frac{1}{d(x,y)}$ are called, respectively, the *transmission* and *global efficiency* of M.

- **Distance polynomial**

Given an ordered finite subset M of a metric space (X,d), let D be the **distance matrix** of M. The **distance polynomial** of M is the *characteristic polynomial* of D, i.e., the determinant $det(D-\lambda I)$.

Usually, D is the distance matrix of the **path metric** of a graph. Sometimes, the distance polynomial is defined as $det(\lambda I - D)$ or $(-1)^n det(D-\lambda I)$.

The roots of the distance polynomial constitute the **distance spectrum** (or *D-spectrum* of *D-eigenvalues*) of M. Let $\rho_{\max}$ and $\rho_{\min}$ be the largest and the smallest roots; then $\rho_{\max}$ and $\rho_{\max}-\rho_{\min}$ are called (distance spectral) *radius* and *spread* of M. The **distance degree** of $x\in M$ is $\sum_{y\in M} d(x,y)$. The **distance energy** of M is the sum of the absolute values of its D-eigenvalues. It is $2\rho_{\max}$ if (as, for example, for the path metric of a tree) exactly one D-eigenvalue is positive.

- ***s*-energy**

Given a finite subset M of a metric space (X,d) and a number $s>0$, the *s*-**energy** and 0-**energy** of M are, respectively, the numbers

$$\sum_{x,y\in M, x\neq y}\frac{1}{d^s(x,y)} \text{ and } \sum_{x,y\in M, x\neq y}\log\frac{1}{d(x,y)} = -\log\prod_{x,y\in M, x\neq y} d(x,y).$$

The (unnormalized) *s*-**moment** of M is the number $\sum_{x,y\in M} d^s(x,y)$.

The *discrete Riesz s-energy* is the s-energy for Euclidean distance d. In general, let μ be a finite Borel probability measure on (X,d). Then $U_s^{\mu}(x)=\int\frac{\mu(dy)}{d(x,y)^s}$ is the (abstract) *s-potential* at a point $x\in X$. The *Newton gravitational potential* is the case $(X,d)=(\mathbb{R}^3,|x-y|)$, $s=1$, for the mass distribution μ.

The *s-energy* of μ is $E_s^{\mu}=\int U_s^{\mu}(x)\mu(dx)=\int\int\frac{\mu(dx)\mu(dy)}{d(x,y)^s}$, and the *s-capacity* of (X,d) is $(\inf_{\mu} E_s^{\mu})^{-1}$. Cf. the **metric capacity**.

- **Fréchet mean**

Given a metric space (X,d) and a number $s>0$, the *Fréchet function* is $F_s(x)=\mathbb{E}[d^s(x,y)]$. For a finite subset M of X, this expected value is the mean $F_s(x)=\sum_{y\in M} w(y)d^s(x,y)$, where $w(y)$ is a weight function on M.

The points, minimizing $F_1(x)$ and $F_2(x)$, are called the **Fréchet median** (or *weighted geometric median*) and **Fréchet mean** (or *Karcher mean*), respectively.

If $(X,d)=(\mathbb{R}^n,||x-y||_2)$ and the weights are equal, these points are called the *geometric median* (or *Fermat–Weber point*, 1-*median*) and the *centroid* (or *geometric center*, *barycenter*), respectively.

The *k-median* and *k-mean* of M are the k-sets C minimizing, respectively, the sums $\sum_{y\in M}\min_{c\in C} d(y,c)=\sum_{y\in M} d(y,C)$ and $\sum_{y\in M} d^2(y,C)$.

Let (X, d) be the metric space $(\mathbb{R}_{>0}, |f(x) - f(y)|)$, where $f : \mathbb{R}_{>0} \to \mathbb{R}$ is a given injective and continuous function. Then the Fréchet mean of $M \subset \mathbb{R}_{>0}$ is the *f-mean* (or *Kolmogorov mean*, *quasi-arithmetic mean*) $f^{-1}(\frac{\sum_{x \in M} f(x)}{|M|})$. It is the arithmetic, geometric, harmonic, and power mean if $f = x, \log(x), \frac{1}{x}$, and $f = x^p$ (for a given $p \neq 0$), respectively. The cases $p \to +\infty, p \to -\infty$ correspond to maximum and minimum, while $p = 2, = 1, \to 0, \to -1$ correspond to the quadratic, arithmetic, geometric and harmonic mean.

Given a *completely monotonic* (i.e., $(-1)^k f^{(k)} \geq 0$ for any k) function $f \in \mathbb{C}^{\infty}$, the f-**potential energy** of a finite subset M of (X, d) is $\sum_{x,y \in M, x \neq y} f(d^2(x, y))$. The set M is called (Cohn–Kumar, 2007) *universally optimal* if it minimizes, among sets $M' \subset X$ with $|M'| = |M|$, the f-potential energy for any such f. Among universally optimal subsets of $(\mathbb{S}^{n-1}, ||x - y||_2)$, there are the vertex-sets of a polygon, simplex, cross-polytope, icosahedron, 600-cell, E_8 root system.

- **Distance-weighted mean**

  In Statistics, the **distance-weighted mean** between given data points $x_1, \dots, x_n$ is defined (Dodonov–Dodonova, 2011) by

$$\frac{\sum_{1 \leq i \leq n} w_i x_i}{\sum_{1 \leq i \leq n} w_i} \quad \text{with} \quad w_i = \frac{n-1}{\sum_{1 \leq j \leq n} |x_i - x_j|}.$$

  The case $w_i = 1$ for all i corresponds to the arithmetic mean.

- **Inverse distance weighting**

  In Numerical Analysis, *multivariate* (or *spatial*) interpolation is interpolation on functions of more than one variable. **Inverse distance weighting** is a method (Shepard, 1968) for multivariate interpolation. Let $x_1, \dots, x_n$ be interpolating points (i.e., samples $u_i = u(x_i)$ are known), x be an interpolated (unknown) point and $d(x, x_i)$ be a given distance. A general form of interpolated value $u(x)$ is

$$u(x) = \frac{\sum_{1 \leq i \leq n} w_i(x) u_i}{\sum_{1 \leq i \leq n} w_i(x)}, \text{ with } w_i(x) = \frac{1}{(d(x, x_i))^p},$$

  where $p > 0$ (usually $p = 2$) is a fixed *power parameter*.

- **Transfinite diameter**

  The *n-th diameter* $D_n(M)$ and the *n-th Chebyshev constant* $C_n(M)$ of a set $M \subseteq X$ in a metric space (X, d) are defined (Fekete, 1923, for the complex plane $\mathbb{C}$) as

$$D_n(M) = \sup_{x_1, \dots, x_n \in M} \prod_{i \neq j} d(x_i, x_j)^{\frac{1}{n(n-1)}} \text{ and } C_n(M) = \inf_{x \in X} \sup_{x_1, \dots, x_n \in M} \prod_{j=1}^{n} d(x, x_j)^{\frac{1}{n}}.$$

  The number $\log D_n(M)$ (the supremum of the average distance) is called the *n-extent* of M. The numbers $D_n(M), C_n(M)$ come from the geometric mean averaging; they also come as the limit case $s \to 0$ of the *s-moment* $\sum_{i \neq j} d(x_i, x_j)^s$ averaging.

The **transfinite diameter** (or *∞-th diameter*) and the *∞-th Chebyshev constant* $C_\infty(M)$ of M are defined as

$$D_\infty(M) = \lim_{n\to\infty} D_n(M) \text{ and } C_\infty(M) = \lim_{n\to\infty} C_n(M);$$

these limits existing since $\{D_n(M)\}$ and $\{C_n(M)\}$ are nonincreasing sequences of nonnegative real numbers. Define $D_\infty(\emptyset) = 0$.

The transfinite diameter of a compact subset of $\mathbb{C}$ is its **conformal radius** at infinity (cf. Chap. 6); for a segment in $\mathbb{C}$, it is $\frac{1}{4}$ of its length.

- **Metric diameter**

The **metric diameter** (or **diameter**, *width*) *diam*(M) of a set $M \subseteq X$ in a metric space (X, d) is defined by

$$\sup_{x,y\in M} d(x, y).$$

The *diameter graph* of M has, as vertices, all points $x \in M$ with $d(x, y) =$ *diam*(M) for some $y \in M$; it has, as edges, all pairs of its vertices at distance *diam*(M) in (X, d). (X, d) is called a **diametrical metric space** if any $x \in X$ has the *antipode*, i.e., a unique $x' \in X$ such that the **closed metric interval** $I(x, x')$ is X.

The *furthest neighbor digraph* of M is a directed graph on M, where xy is an arc (called a *furthest neighbor pair*) whenever y is at maximal distance from x.

In a metric space endowed with a measure, one says that the *isodiametric inequality* holds if the metric balls maximize the measure among all sets with given diameter. It holds for the volume in Euclidean space but not, for example, for the **Heisenberg metric** on the *Heisenberg group* (Chap. 10).

The k**-ameter** (Grove–Markvorsen, 1992) is $\sup_{K\subseteq X:\,|K|=k} \frac{1}{2} \sum_{x,y\in K} d(x, y)$, and the k**-diameter** (Chung–Delorme–Sole, 1999) is $\sup_{K\subseteq X:\,|K|=k} \inf_{x,y\in K:\,x\neq y} d(x, y)$.

Given a property $P \subseteq X \times X$ of a pair (K, K') of subsets of a finite metric space (X, d), the **conditional diameter** (called *P-diameter* in Balbuena et al., 1996) is $\max_{(K,K')\in P} \min_{(x,y)\in K\times K'} d(x, y)$. It is *diam*$(X, d)$ if $P = \{(K, K') \in X \times X : |K| = |K'| = 1\}$. When (X, d) models an interconnection network, the P-diameter corresponds to the maximum delay of the messages interchanged between any pair of clusters of nodes, K and K', satisfying a given property P of interest.

- **Metric spread**

A subset M of a metric space (X, d) is called **Delone set** (or *separated* ϵ**-net**, (A, a)*-Delone set*) if it is **bounded** (with a finite **diameter** $A = \sup_{x,y\in M} d(x, y)$) and **metrically discrete** (with a *separation* $a = \inf_{x,y\in M, x\neq y} d(x, y) > 0$).

The **metric spread** (or *distance ratio*, *normalized diameter*) of M is the ratio $\frac{A}{a}$.

The **aspect ratio** (or *axial ratio*) of a shape is the ratio of its longer and shorter dimensions, say, the length and diameter of a rod, major and minor axes of a torus or width and height of a rectangle (image, display, pixel, etc.).

For a mesh M with separation a and **covering radius** (or *mesh norm*) $c = \sup_{y \in X} \inf_{x \in M} d(x, y)$, the *mesh ratio* is $\frac{c}{a}$.

In Physics, the *aspect ratio* is the ratio of height-to-length scale characteristics. Cf. the *wing's aspect ratio* among **aircraft distances** in Chap. 29.

Dynamic range DNR is the ratio between the largest and smallest possible values of a quantity, such as in sound or light signals; cf. **SNR distance** in Chap. 21.

- **Eccentricity**

Given a bounded metric space (X, d), the **eccentricity** (or *Koenig number*) of a point $x \in X$ is the number $e(x) = \max_{y \in X} d(x, y)$.

The numbers $D = \max_{x \in X} e(x)$ and $r = \min_{x \in X} e(x)$ are called the **diameter** and the **radius** of (X, d), respectively. The point $z \in X$ is called *central* if $e(z) = r$, *peripheral* if $e(z) = D$, and *pseudo-peripheral* if for each point x with $d(z, x) = e(z)$ it holds that $e(z) = e(x)$. For finite $|X|$, the *average eccentricity* is $\frac{1}{|X|} \sum_{x \in X} e(x)$, and the *contour* of (X, d) is the set of points $x \in X$ such that no *neighbor* (closest point) of x has an eccentricity greater than x.

The **eccentric digraph** (Buckley, 2001) of (X, d) has, as vertices, all points $x \in X$ and, as arcs, all ordered pairs (x, y) of points with $d(x, y) = e(y)$. The **eccentric graph** (Akyiama–Ando–Avis, 1976) of (X, d) has, as vertices, all points $x \in X$ and, as edges, all pairs (x, y) of points at distance $\min\{e(x), e(y)\}$.

The **super-eccentric graph** (Iqbalunnisa–Janairaman–Srinivasan, 1989) of (X, d) has, as vertices, all points $x \in X$ and, as edges, all pairs (x, y) of points at distance no less than the radius of (X, d). The **radial graph** (Kathiresan–Marimuthu, 2009) of (X, d) has, as vertices, all points $x \in X$ and, as edges, all pairs (x, y) of points at distance equal to the radius of (X, d).

The sets $\{x \in X : e(x) \leq e(z) \text{ for any } z \in X\}$, $\{x \in X : e(x) \geq e(z) \text{ for any } z \in X\}$ and $\{x \in X : \sum_{y \in X} d(x, y) \leq \sum_{y \in X} d(z, y) \text{ for any } z \in X\}$ are called, respectively, the **metric center** (or *eccentricity center*, *center*), **metric antimedian** (or *periphery*) and the **metric median** (or *distance center*) of (X, d).

- **Radii of metric space**

Given a bounded metric space (X, d) and a set $M \subseteq X$ of **diameter** D, its **metric radius** (or **radius**) Mr, **covering radius** (or **directed Hausdorff distance** from X to M) Cr and **remoteness** (or **Chebyshev radius**) Re are the numbers $\inf_{x \in M} \sup_{y \in M} d(x, y)$, $\sup_{x \in X} \inf_{y \in M} d(x, y)$ and $\inf_{x \in X} \sup_{y \in M} d(x, y)$, respectively. It holds that $\frac{D}{2} \leq Re \leq Mr \leq D$ with $Mr = \frac{D}{2}$ in any **injective metric space**. Somemimes, $\frac{D}{2}$ is called the *radius*.

For $m > 0$, a **minimax distance design of size** m is an m-subset of X having smallest covering radius. This radius is called the *m-point mesh norm* of (X, d).

The **packing radius** Pr of M is the number $\sup\{r : \inf_{x,y \in M, x \neq y} d(x, y) > 2r\}$. For $m > 0$, a **maximum distance design of size** m is an m-subset of X having largest packing radius. This radius is the *m-point best packing distance* on (X, d).

- **ϵ-net**

Given a metric space (X, d), a subset $M \subset X$, and a number $\epsilon > 0$, the **ϵ-neighborhood** of M is the set $M^\epsilon = \cup_{x \in M} B(x, \epsilon)$.

The set M is called an **ϵ-net** (or *ϵ-covering*, *ϵ-approximation*) of (X, d) if $M^\epsilon = X$, i.e., the **covering radius** of M is at most ϵ.

Let C_ϵ denote the *ϵ-covering number*, i.e., the smallest size of an ϵ-net in (X, d). The number $\lg_2 C_\epsilon$ is called (Kolmogorov–Tikhomirov, 1959) the **metric entropy** (or *ϵ-entropy*) of (X, d). It holds $P_\epsilon \leq C_\epsilon \leq P_{\frac{\epsilon}{2}}$, where P_ϵ denote the *ϵ-packing number* of (X, d), i.e., $\sup\{|M| : M \subset X, \overline{B}(x, \epsilon) \cap \overline{B}(y, \epsilon) = \emptyset$ for any $x, y \in M, x \neq y\}$. The number $\lg_2 P_\epsilon$ is called the **metric capacity** (or *ϵ-capacity*) of (X, d).

- **Steiner ratio**

Given a metric space (X, d) and a finite subset $V \subset X$, let $G = (V, E)$ be the complete weighted graph on V with edge-weights $d(x, y)$ for all $x, y \in V$.

Given a tree T, its *weight* is the sum $d(T)$ of its edge-weights. A *spanning tree* of V is a subset of $|V| - 1$ edges forming a tree on V. Let $MSpT_V$ be a *minimum spanning tree* of V, i.e., a spanning tree with the minimal weight $d(MSpT_V)$.

A *Steiner tree* of V is a tree on Y, $V \subset Y \subset X$, connecting vertices from V; elements of $Y \setminus V$ are called *Steiner points*. Let $StMT_V$ be a *minimum Steiner tree* of V, i.e., a Steiner tree with the minimal weight $d(StMT_V) = \inf_{Y \subset X : V \subset Y} d(MSpT_Y)$. This weight is called the **Steiner diversity** of V; cf. **diversity** in Chap. 3. It is the **Steiner distance of set** V (Chap. 15) if (X, d) is graphic metric space.

The **Steiner ratio** $St(X, d)$ of the metric space (X, d) is defined by

$$\inf_{V \subset X} \frac{d(StMT_V)}{d(MSpT_V)}.$$

Cf. **arc routing problems** in Chap. 15.

- **Chromatic numbers of metric space**

Given a metric space (X, d) and a set D of positive real numbers, the **D-chromatic number** of (X, d) is the standard *chromatic number* of its **D-distance graph**, i.e., the graph (X, E) with the vertex-set X and the edge-set $E = \{xy : d(x, y) \in D\}$ (Chap. 15). Usually, (X, d) is an **l_p-space** and $D = \{1\}$ (**Benda–Perles chromatic number**) or $D = [1 - \epsilon, 1 + \epsilon]$.

For a metric space (X, d), the **polychromatic number** is the minimum number of colors needed to color all the points $x \in X$ so that, for each color class C_i, there is a distance d_i such that no two points of C_i are at distance d_i.

For a metric space (X, d), the **packing chromatic number** is the minimum number of colors needed to color all the points $x \in X$ so that, for each color class C_i, no two distinct points of C_i are at distance at most i.

For any integer $t > 0$, the **t-distance chromatic number** of a metric space (X, d) is the minimum number of colors needed to color all the points $x \in X$ so that any two points whose distance is $\leq t$ have distinct colors. Cf. **k-distance chromatic number** in Chap. 15.

For any integer $t > 0$, the **t-th Babai number** of a metric space (X, d) is the minimum number of colors needed to color all the points in X so that, for any set D of positive distances with $|D| \leq t$, any two points $x, y \in X$ with $d(x, y) \in D$ have distinct colors.

- **Congruence order of metric space**

  A metric space (X, d) has **congruence order** n if every finite metric space which is not **isometrically embeddable** in (X, d) has a subspace with at most n points which is not isometrically embeddable in (X, d). For example, the congruence order of l_2^n is $n + 3$ (Menger, 1928); it is 4 for the **path metric** of a tree.

1.4 Main Mappings of Metric Spaces

- **Distance function**

  In Topology, the term *distance function* is often used for **distance**. But, in general, a **distance function** (or *ray function*) is a continuous function on a metric space (X, d) (usually, on a Euclidean space $\mathbb{E}^n$) $f : X \to \mathbb{R}_{\geq 0}$ which is *homogeneous*, i.e., $f(tx) = tf(x)$ for all $t \geq 0$ and all $x \in X$.

  Such function f is called *positive* if $f(x) > 0$ for all $x \neq 0$, *symmetric* if $f(x) = f(-x)$, *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for any $0 < t < 1$ and $x \neq y$, and *strictly convex* if this inequality is strict.

  If $X = \mathbb{E}^n$, the set $S_f = \{x \in \mathbb{R}^n : f(x) < 1\}$ is *star body*, i.e., $x \in S_f$ implies $[0, x] \subset S_f$. Any star body S corresponds to a unique distance function $g(x) = \inf_{tx \in S, t>0} \frac{1}{t}$, and $S = S_g$. The star body is bounded if f is positive, symmetric about the origin if f is symmetric, convex if f is convex, and *strictly convex* (i.e., the boundary ∂B does not contain a segment) if f is strictly convex.

  For a quadratic distance function of the form $f_A = xAx^T$, where A is a real matrix and $x \in \mathbb{R}^n$, the matrix A is *positive-definite* (i.e., the **Gram matrix** $VV^T = ((\langle v_i, v_j \rangle))$ of n linearly independent vectors $v_i = (v_{i1}, \ldots, v_{in})$) if and only if f_A is symmetric and strictly convex function. The *homogeneous minimum* of f_A is

$$\min(f_A) = \inf_{x \in \mathbb{Z}^n \setminus \{0\}} f_A(x) = \inf_{x \in L \setminus \{0\}} \sum_{1 \leq i \leq n} x_i^2,$$

  where $L = \{\sum x_i v_i : x_i \in \mathbb{Z}\}$ is a *lattice*, i.e., a discrete subgroup of $\mathbb{R}^n$ spanning it. The *Hermite constant* γ_n, a central notion in Geometry of Numbers, is the supremum, over all positive-definite $(n \times n)$-matrices, of $\min(f_A) \det(A)^{\frac{1}{n}}$. It is known only for $2 \leq n \leq 8$ and $n = 24$; cf. **systole of metric space**.

- **Convex distance function**

  Given a compact convex region $B \subset \mathbb{R}^n$ containing the origin O in its interior, the **convex distance function** (or **Minkowski distance function**, *Minkowski* *seminorm*, *gauge*) is the function $||P||_B$ whose value at a point $P \in \mathbb{R}^n$ is the *distance ratio* $\frac{OP}{OQ}$, where $Q \in B$ is the furthest from O point on the ray OP.

Then $d_B(x, y) = ||x - y||_B$ is the quasi-metric on $\mathbb{R}^n$ defined, for $x \neq y$, by

$$\inf\{\alpha > 0 : y - x \in \alpha B\},$$

and $B = \{x \in \mathbb{R}^n : d_B(0, x) \leq 1\}$ with equality only for $x \in \partial B$.

The function $||P||_B$ is called a **polyhedral distance function** if B is a n-polytope, *simplicial distance function* if it is a n-simplex, and so on.

If B is centrally-symmetric with respect to the origin, then d_B is a **Minkowskian metric** (Chap. 6) whose unit ball is B. This is the l_1-metric if B is the n-cross-polytope and the l_∞-metric if B is the n-cube.

- **Funk distance**

  Let B be an nonempty open convex subset of $\mathbb{R}^n$. For any $x, y \in B$, denote by $R(x, y)$ the ray from x through y. The **Funk distance** (Funk, 1929) on B is the quasi-semimetric defined, for any $x, y \in B$, as 0 if the boundary $\partial(B)$ and $R(x, y)$ are disjoint, and, otherwise, i.e., if $R(x, y) \cap \partial B = \{z\}$, by

$$\ln \frac{||x - z||_2}{||y - z||_2}.$$

  The **Hilbert projective metric** in Chap. 6 is a symmetrization of this distance.

- **Metric projection**

  Given a metric space (X, d) and a subset $M \subset X$, an element $u_0 \in M$ is called an **element of best approximation** (or **nearest point**) to a given element $x \in X$ if $d(x, u_0) = \inf_{u \in M} d(x, u)$, i.e., if $d(x, u_0)$ is the **point-set distance** $d(x, M)$.

  A **metric projection** (or *operator of best approximation*, *nearest point map*) is a multivalued mapping associating to each element $x \in X$ the set of elements of best approximation from the set M (cf. **distance map**).

  A **Chebyshev set** in a metric space (X, d) is a subset $M \subset X$ containing a unique element of best approximation for every $x \in X$.

  A subset $M \subset X$ is called a **semi-Chebyshev set** if the number of such elements is at most one, and a **proximinal set** if this number is at least one.

  While the **Chebyshev radius** (or **remoteness**; cf. **radii of metric space**) of the set M is $\inf_{x \in X} \sup_{y \in M} d(x, y)$, a **Chebyshev center** of M is an element $x_0 \in X$ realizing this infimum. Sometimes (say, for a finite graphic metric space), $\frac{1}{|M|} \inf_{x \in X} \sum_{y \in M} d(x, y)$ and $\frac{1}{|M|} \sup_{x \in X} \sum_{y \in M} d(x, y)$ are called *proximity* and *remoteness* of M.

- **Distance map**

  Given a metric space (X, d) and a subset $M \subset X$, the **distance map** is a function $f_M : X \to \mathbb{R}_{\geq 0}$, where $f_M(x) = \inf_{u \in M} d(x, u)$ is the **point-set distance** $d(x, M)$ (cf. **metric projection**).

If the boundary $B(M)$ of the set M is defined, then the **signed distance function** g_M is defined by $g_M(x) = -\inf_{u \in B(M)} d(x,u)$ for $x \in M$, and $g_M(x) = \inf_{u \in B(M)} d(x,u)$, otherwise. If M is a (closed orientable) n**-manifold** (Chap. 2), then g_M is the solution of the *eikonal equation* $|\nabla g| = 1$ for its *gradient* ∇.

If $X = \mathbb{R}^n$ and, for every $x \in X$, there is unique element $u(x)$ with $d(x,M) = d(x,u(x))$ (i.e., M is a **Chebyshev set**), then $||x - u(x)||$ is called a **vector distance function**.

Distance maps are used in Robot Motion (M being the set of obstacle points) and, especially, in Image Processing (M being the set of all or only boundary pixels of the image). For $X = \mathbb{R}^2$, the graph $\{(x, f_M(x)) : x \in X\}$ of $d(x,M)$ is called the *Voronoi surface* of M.

- **Isometry**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called an **isometric embedding** of X into Y if it is injective and the equality $d_Y(f(x), f(y)) = d_X(x,y)$ holds for all $x, y \in X$.

An **isometry** (or *congruence mapping*) is a bijective isometric embedding. Two metric spaces are called **isometric** (or *isometrically isomorphic*) if there exists an isometry between them.

A property of metric spaces which is invariant with respect to isometries (completeness, boundedness, etc.) is called a **metric property** (or *metric invariant*).

A **path isometry** (or *arcwise isometry*) is a mapping from X into Y (not necessarily bijective) preserving lengths of curves.

- **Rigid motion of metric space**

A **rigid motion** (or, simply, **motion**) of a metric space (X, d) is an **isometry** of (X, d) onto itself.

For a motion f, the **displacement function** $d_f(x)$ is $d(x, f(x))$. The motion f is called *semisimple* if $\inf_{x \in X} d_f(x) = d(x_0, f(x_0))$ for some $x_0 \in X$, and *parabolic*, otherwise. A semisimple motion is called *elliptic* if $\inf_{x \in X} d_f(x) = 0$, and *axial* (or *hyperbolic*), otherwise. A motion is called a *Clifford translation* if the displacement function $d_f(x)$ is a constant for all $x \in X$.

- **Symmetric metric space**

A metric space (X, d) is called **symmetric** if, for any point $p \in X$, there exists a *symmetry* relative to that point, i.e., a **motion** f_p of this metric space such that $f_p(f_p(x)) = x$ for all $x \in X$, and p is an isolated fixed point of f_p.

- **Homogeneous metric space**

A metric space is called **homogeneous** (or *point-homogeneous*) if, for any two points of it, there exists a motion mapping one of the points to the other.

In general, a *homogeneous space* is a set together with a given transitive group of *symmetries*. Moss, 1992, defined similar *distance-homogeneous distanced graph*.

A metric space is called **ultrahomogeneous space** (or *highly transitive*) if any isometry between two of its finite subspaces extends to the whole space.

A metric space (X, d) is called (Grünbaum–Kelly) a **metrically homogeneous metric space** if $\{d(x,z) : z \in X\} = \{d(y,z) : z \in X\}$ for any $x, y \in X$.

- **Flat space**

  A **flat space** is any metric space with **local isometry** to some $\mathbb{E}^n$, i.e., each point has a neighborhood isometric to an open set in $\mathbb{E}^n$. A space is *locally Euclidean* if every point has a neighborhood homeomorphic to an open subset in $\mathbb{E}^n$.

- **Dilation of metric space**

  Given a metric space (X, d), its **dilation** (or r**-dilation**) is a mapping $f : X \to X$ with $d(f(x), f(y)) = rd(x, y)$ for some $r > 0$ and any $x \in X$.

- **Wobbling of metric space**

  Given a metric space (X, d), its **wobbling** (or r**-wobbling**) is a mapping $f : X \to X$ with $d(x, f(x)) < r$ for some $r > 0$ and any $x \in X$.

- **Paradoxical metric space**

  Given a metric space (X, d) and an equivalence relation on the subsets of X, the space (X, d) is called **paradoxical** if X can be decomposed into two disjoint sets M_1, M_2 so that M_1, M_2 and X are pairwise equivalent.

  Deuber, Simonovitz and Sós, 1995, introduced this idea for *wobbling equivalent* subsets $M_1, M_2 \subset X$, i.e., there is a bijective r**-wobbling** $f : M_1 \to M_2$. For example, $(\mathbb{R}^2, l_2)$ is paradoxical for wobbling but not for isometry equivalence.

- **Metric cone**

  A **pointed metric space** (X, d, x_0) is called a **metric cone**, if it is isometric to $(\lambda X, d, x_0)$ for all $\lambda > 0$. A **metric cone structure** on (X, d, x_0) is a (pointwise) continuous family f_t ($t \in \mathbb{R}_{>0}$) of **dilations** of X, leaving the point x_0 invariant, such that $d(f_t(x), f_t(y)) = td(x, y)$ for all x, y and $f_t \circ f_s = f_{ts}$. A Banach space has such a structure for the dilations $f_t(x) = tx$ ($t \in \mathbb{R}_{>0}$). The *Euclidean cone over a metric space* (cf. **cone over metric space** in Chap. 9) is another example.

  The **tangent metric cone** over a metric space (X, d) at a point x_0 is (for all dilations $tX = (X, td)$) the closure of $\cup_{t>0} tX$, i.e., of $\lim_{t\to\infty} tX$ taken in the pointed Gromov–Hausdorff topology (cf. **Gromov–Hausdorff metric**).

  The **asymptotic metric cone** over (X, d) is its tangent metric cone "at infinity", i.e., $\cap_{t>0} tX = \lim_{t\to 0} tX$. Cf. **boundary of metric space** in Chap. 6.

  The term *metric cone* was also used by Bronshtein, 1998, for a convex cone C equipped with a complete metric compatible with its operations of addition (continuous on $C \times C$) and multiplication (continuous on $C \times \mathbb{R}_{\geq 0}$). by all $\lambda \geq 0$.

- **Metric fibration**

  Given a **complete** metric space (X, d), two subsets M_1 and M_2 of X are called *equidistant* if for each $x \in M_1$ there exists $y \in M_2$ with $d(x, y)$ being equal to the **Hausdorff metric** between the sets M_1 and M_2. A **metric fibration** of (X, d) is a partition $\mathcal{F}$ of X into isometric mutually equidistant closed sets.

  The quotient metric space $X/\mathcal{F}$ inherits a natural metric for which the **distance map** is a **submetry**.

- **Homeomorphic metric spaces**

  Two metric spaces (X, d_X) and (Y, d_Y) are called **homeomorphic** (or *topologically isomorphic*) if there exists a *homeomorphism* from X to Y, i.e., a bijective function $f : X \to Y$ such that f and f^{-1} are *continuous* (the preimage of every open set in Y is open in X).

Two metric spaces (X, d_X) and (Y, d_Y) are called *uniformly isomorphic* if there exists a bijective function $f : X \to Y$ such that f and f^{-1} are *uniformly continuous*. A function g is *uniformly continuous* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in X$, the inequality $d_X(x, y) < \delta$ implies that $d_Y(g(x), f(y)) < \epsilon$; a continuous function is uniformly continuous if X is compact.

- **Möbius mapping**

Given distinct points x, y, z, w of a metric space (X, d), their **cross-ratio** is

$$cr((x, y, z, w), d) = \frac{d(x, y)d(z, w)}{d(x, z)d(y, w)} > 0.$$

Given metric spaces (X, d_X) and (Y, d_Y), a **homeomorphism** $f : X \to Y$ is called a **Möbius mapping** if, for every distinct points $x, y, z, w \in X$, it holds

$$cr((x, y, z, w), d_X) = cr((f(x), f(y), f(z), f(w)), d_Y).$$

A homeomorphism $f : X \to Y$ is called a **quasi-Möbius mapping** (Väisälä, 1984) if there exists a homeomorphism $\tau : [0, \infty) \to [0, \infty)$ such that, for every quadruple x, y, z, w of distinct points of X, it holds

$$cr((f(x), f(y), f(z), f(w)), d_Y) \le \tau(cr((x, y, z, w), d_X)).$$

A metric space (X, d) is called *metrically dense* (or *μ-dense* for given $\mu > 1$, Aseev–Trotsenko, 1987) if for any $x, y \in X$, there exists a sequence $\{z_i, i \in \mathbb{Z}\}$ with $z_i \to x$ as $i \to -\infty$, $z_i \to y$ as $i \to \infty$, and $\log cr((x, z_i, z_{i+1}, y), d) \le \log \mu$ for all $i \in \mathbb{Z}$. The space (X, d) is μ-dense if and only if (Tukia-Väisälä, 1980), for any $x, y \in X$, there exists $z \in X$ with $\frac{d(x,y)}{6\mu} \le d(x, z) \le \frac{d(x,y)}{4}$.

- **Quasi-symmetric mapping**

Given metric spaces (X, d_X) and (Y, d_Y), a **homeomorphism** $f : X \to Y$ is called a **quasi-symmetric mapping** (Tukia–Väisälä, 1980) if there is a homeomorphism $\tau : [0, \infty) \to [0, \infty)$ such that, for every triple (x, y, z) of distinct points of X,

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \le \tau \frac{d_X(x, y)}{d_X(x, z)}.$$

Quasi-symmetric mappings are **quasi-Möbius**, and quasi-Möbius mappings between bounded metric spaces are quasi-symmetric. In the case $f : \mathbb{R}^n \to \mathbb{R}^n$, quasi-symmetric mappings are exactly the same as **quasi-conformal mappings**.

- **Conformal metric mapping**

Given metric spaces (X, d_X) and (Y, d_Y) which are domains in $\mathbb{R}^n$, a **homeomorphism** $f : X \to Y$ is called a **conformal metric mapping** if, for any nonisolated point $x \in X$, the limit $\lim_{y \to x} \frac{d_Y(f(x), f(y))}{d(x,y)}$ exists, is finite and positive.

A homeomorphism $f : X \to Y$ is called a **quasi-conformal mapping** (or, specifically, *C-quasi-conformal mapping*) if there exists a constant C such that

$$\lim_{r\to 0} \sup \frac{\max\{d_Y(f(x), f(y)) : d_X(x,y) \le r\}}{\min\{d_Y(f(x), f(y)) : d_X(x,y) \ge r\}} \le C$$

for each $x \in X$. The smallest such constant C is called the **conformal dilation**.

The **conformal dimension** of a metric space (X, d) (Pansu, 1989) is the infimum of the **Hausdorff dimension** over all quasi-conformal mappings of (X, d) into some metric space. For the middle-third Cantor set on $[0, 1]$, it is 0 but, for any of its quasi-conformal images, it is positive.

- **Hölder mapping**

Let $c, \alpha \ge 0$ be constants. Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called the **Hölder mapping** (or *α-Hölder mapping* if the constant α should be mentioned) if for all $x, y \in X$

$$d_Y(f(x), f(y)) \le c(d_X(x,y))^{\alpha}.$$

A 1-Hölder mapping is a **Lipschitz mapping**; 0-Hölder mapping means that the metric d_Y is bounded.

- **Lipschitz mapping**

Let c be a positive constant. Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **Lipschitz** (or **Lipschitz continuous**, *c-Lipschitz* if the constant c should be mentioned) **mapping** if for all $x, y \in X$ it holds

$$d_Y(f(x), f(y)) \le c d_X(x,y).$$

A c-Lipschitz mapping is called a **metric mapping** if $c = 1$, and is called a **contraction** if $c < 1$.

- **Bi-Lipschitz mapping**

Given metric spaces (X, d_X), (Y, d_Y) and a constant $c > 1$, a function $f : X \to Y$ is called a **bi-Lipschitz mapping** (or *c-bi-Lipschitz mapping*, **c-embedding**) if there exists a number $r > 0$ such that for any $x, y \in X$ it holds

$$r d_X(x,y) \le d_Y(f(x), f(y)) \le c r d_X(x,y).$$

Every bi-Lipschitz mapping is a **quasi-symmetric mapping**.

The smallest c for which f is a c-bi-Lipschitz mapping is called the **distortion** of f. Bourgain, 1985, proved that every k-point metric space c-embeds into a Euclidean space with distortion $O(\ln k)$. Gromov's *distortion for curves* is the maximum ratio of arc length to chord length.

Two metrics d_1 and d_2 on X are called **bi-Lipschitz equivalent metrics** if there are positive constants c and C such that $cd_1(x,y) \le d_2(x,y) \le Cd_1(x,y)$ for all $x, y \in X$, i.e., the identity mapping is a bi-Lipschitz mapping from (X, d_1) into (X, d_2). Bi-Lipschitz equivalent metrics are **equivalent**, i.e., generate the same topology but, for example, equivalent L_1-metric and L_2-metric (cf. L_p**-metric** in Chap. 5) on $\mathbb{R}$ are not bi-Lipschitz equivalent.

A bi-Lipschitz mapping $f : X \to Y$ is a c**-isomorphism** $f : X \to f(X)$.

- c**-isomorphism of metric spaces**

  Given two metric spaces (X, d_X) and (Y, d_Y), the *Lipschitz norm* $||.||_{Lip}$ on the set of all injective mappings $f : X \to Y$ is defined by

$$||f||_{Lip} = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

  Two metric spaces X and Y are called c**-isomorphic** if there exists an injective mapping $f : X \to Y$ such that $||f||_{Lip}||f^{-1}||_{Lip} \leq c$.

- **Metric Ramsey number**

  For a given class $\mathcal{M}$ of metric spaces (usually, l_p*-spaces*), an integer $n \geq 1$, and a real number $c \geq 1$, the **metric Ramsey number** (or *c-metric Ramsey number*) $R_{\mathcal{M}}(c, n)$ is the largest integer m such that every n-point metric space has a subspace of cardinality m that c-embeds into a member of $\mathcal{M}$ (see [BLMN05]).

  The *Ramsey number* R_n is the minimal number of vertices of a complete graph such that any edge-coloring with n colors produces a monochromatic triangle. The following metric analog of R_n was considered in [Masc04]: the least number of points a finite metric space must contain in order to contain an equilateral triangle, i.e., to have **equilateral metric dimension** greater than two.

- **Uniform metric mapping**

  Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **uniform metric mapping** if there are two nondecreasing functions g_1 and g_2 from $\mathbb{R}_{\geq 0}$ to itself with $\lim_{r \to \infty} g_i(r) = \infty$ for $i = 1, 2$, such that the inequality

$$g_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq g_2(d_X(x, y))$$

  holds for all $x, y \in X$. A **bi-Lipschitz mapping** is a uniform metric mapping with linear functions g_1, g_2.

- **Metric compression**

  Given metric spaces (X, d_X) (unbounded) and (Y, d_Y), a function $f : X \to Y$ is a *large scale Lipschitz mapping* if, for some $c > 0, D \geq 0$ and all $x, y \in X$,

$$d_Y(f(x), f(y)) \leq c d_X(x, y) + D.$$

  The *compression* of such a mapping f is $\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))$.

  The **metric compression** of (X, d_X) in (Y, d_Y) is defined by

$$R(X, Y) = \sup_f \{\underline{\lim}_{r \to \infty} \frac{\log \max\{\rho_f(r), 1\}}{\log r}\},$$

  where the supremum is over all large scale Lipschitz mappings f.

In the main interesting case—when (Y, d_Y) is a Hilbert space and (X, d_X) is a (finitely generated discrete) group with **word metric**—$R(X, Y) = 0$ if there is no (Guentner–Kaminker, 2004) **uniform metric mapping** $(X, d_X) \to (Y, d_Y)$, and $R(X, Y) = 1$ for free groups, even if there is no **quasi-isometry**. Arzhantzeva–Guba–Sapir, 2006, found groups with $\frac{1}{2} \le R(X, Y) \le \frac{3}{4}$.

- **Quasi-isometry**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **quasi-isometry** (or (C, c)**-quasi-isometry**) if it holds

$$C^{-1} d_X(x, y) - c \le d_Y(f(x), f(y)) \le C d_X(x, y) + c,$$

for some $C \ge 1, c \ge 0$, and $Y = \cup_{x \in X} B_{d_Y}(f(x), c)$, i.e., for every point $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) < \frac{c}{2}$. Quasi-isometry is an equivalence relation on metric spaces; it is a bi-Lipschitz equivalence up to small distances. Quasi-isometry means that metric spaces contain bi-Lipschitz equivalent **Delone sets**.

A quasi-isometry with $C = 1$ is called a **coarse isometry** (or *rough isometry*, *almost isometry*). Cf. **quasi-Euclidean rank of a metric space**.

- **Coarse embedding**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **coarse embedding** if there exist nondecreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ with $\rho_1(d_X(x, x')) \le d_Y(f(x), f(x')) \le \rho_2(d_X(x, x'))$ if $x, x' \in X$ and $\lim_{t \to \infty} \rho_1(t) = +\infty$.

Metrics d_1, d_2 on X are called **coarsely equivalent metrics** if there exist nondecreasing functions $f, g : [0, \infty) \to [0, \infty)$ such that $d_1 \le f(d_2), d_2 \le g(d_1)$.

- **Metrically regular mapping**

Let (X, d_X) and (Y, d_Y) be metric spaces, and let F be a set-valued mapping from X to Y, having *inverse* F^{-1}, i.e., with $x \in F^{-1}(y)$ if and only if $y \in F(x)$.

The mapping F is said to be **metrically regular at** $\bar{x}$ **for** $\bar{y}$ (Dontchev–Lewis–Rockafeller, 2002) if there exists $c > 0$ such that it holds

$$d_X(x, F^{-1}(y)) \le c d_Y(y, F(x))$$

for all (x, y) close to $(\bar{x}, \bar{y})$. Here $d(z, A) = \inf_{a \in A} d(z, a)$ and $d(z, \emptyset) = +\infty$.

- **Contraction**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **contraction** if the inequality

$$d_Y(f(x), f(y)) \le c d_X(x, y)$$

holds for all $x, y \in X$ and some real number c, $0 \le c < 1$.

Every contraction is a **contractive mapping**, and it is uniformly continuous. *Banach fixed point theorem* (or *contraction principle*): every contraction from a **complete** metric space into itself has a unique fixed point.

- **Contractive mapping**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **contractive** (or *strictly short*, *distance-decreasing*) **mapping** if

$$d_Y(f(x), f(y)) < d_X(x, y)$$

holds for all different $x, y \in X$. A function $f : X \to Y$ is called a **noncontractive mapping** (or *dominating mapping*) if for all $x, y \in X$ it holds

$$d_Y(f(x), f(y)) \geq d_X(x, y).$$

Every noncontractive bijection from a **totally bounded** metric space onto itself is an **isometry**.

- **Short mapping**

Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called a **short** (or *1-Lipschitz*, *nonexpansive*, *distance-noninreasing*, *metric*) **mapping** (or *semicontraction*) if for all $x, y \in X$ it holds

$$d_Y(f(x), f(y)) \leq d_X(x, y).$$

A **submetry** is a short mapping such that the image of any metric ball is a metric ball of the same radius.

The set of short mappings $f : X \to Y$ for bounded metric spaces (X, d_X) and (Y, d_Y) is a metric space under the **uniform metric** $\sup\{d_Y(f(x), g(x)) : x \in X\}$.

Two subsets A and B of a metric space (X, d) are called (Gowers, 2000) **similar** if there exist short mappings $f : A \to X$, $g : B \to X$ and a small $\epsilon > 0$ such that every point of A is within ϵ of some point of B, every point of B is within ϵ of some point of A, and $|d(x, g(f(x))) - d(y, f(g(y)))| \leq \epsilon$ for any $x \in A, y \in B$.

- **Category of metric spaces**

A *category* Ψ consists of a class $Ob(\Psi)$ of *objects* and a class $Mor(\Psi)$ of *morphisms* (or *arrows*) satisfying the following conditions:

1. To each ordered pair of objects A, B is associated a set $\Psi(A, B)$ of morphisms, and each morphism belongs to only one set $\Psi(A, B)$;
2. The composition $f \cdot g$ of two morphisms $f : A \to B$, $g : C \to D$ is defined if $B = C$ in which case it belongs to $\Psi(A, D)$, and it is associative;
3. Each set $\Psi(A, A)$ contains, as an *identity*, a morphism id_A such that $f \cdot id_A = f$ and $id_A \cdot g = g$ for any morphisms $f : X \to A$ and $g : A \to Y$.

The **category of metric spaces**, denoted by *Met* (see [Isbe64]), is a category which has metric spaces as objects and **short mappings** as morphisms. A unique **injective envelope** exists in this category for every one of its objects; it can be identified with its **tight span**. In *Met*, the *monomorphisms* are injective short mappings, and *isomorphisms* are **isometries**. *Met* is a subcategory of the category which has metric spaces as objects and **Lipschitz mappings** as morphisms.

Cf. **metric 1-space** on the objects of a category in Chap. 3.

- **Injective metric space**

A metric space (X, d) is called **injective** if, for every isometric embedding $f : X \to X'$ of (X, d) into another metric space (X', d'), there exists a **short mapping** f' from X' into X with $f' \cdot f = id_X$, i.e., X is a *retract* of X'.

Equivalently, X is an *absolute retract*, i.e., a retract of every metric space into which it embeds isometrically. A metric space (X, d) is injective if and only if it is **hyperconvex**. Examples of such metric spaces are l_1^2-space, l_∞^n-space, any **real tree** and the **tight span** of a metric space.

- **Injective envelope**

The **injective envelope** (introduced first in [Isbe64] as *injective hull*) is a generalization of **Cauchy completion**. Given a metric space (X, d), it can be embedded isometrically into an **injective metric space** $(\hat{X}, \hat{d})$; given any such isometric embedding $f : X \to \hat{X}$, there exists a unique smallest injective subspace $(\overline{X}, \overline{d})$ of $(\hat{X}, \hat{d})$ containing $f(X)$ which is called the **injective envelope** of X. It is isometrically identified with the **tight span** of (X, d).

A metric space coincides with its injective envelope if and only if it is injective.

- **Tight extension**

An extension (X', d') of a metric space (X, d) is called a **tight extension** if, for every semimetric d'' on X' satisfying the conditions $d''(x_1, x_2) = d(x_1, x_2)$ for all $x_1, x_2 \in X$, and $d''(y_1, y_2) \leq d'(y_1, y_2)$ for any $y_1, y_2 \in X'$, one has $d''(y_1, y_2) = d'(y_1, y_2)$ for all $y_1, y_2 \in X'$.

The **tight span** is the *universal tight extension* of X, i.e., it contains, up to isometries, every tight extension of X, and it has no proper tight extension itself.

- **Tight span**

Given a metric space (X, d) of finite diameter, consider the set $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$. The **tight span** $T(X, d)$ of (X, d) is defined as the set $T(X, d) = \{f \in \mathbb{R}^X : f(x) = \sup_{y \in X}(d(x, y) - f(y))$ for all $x \in X\}$, endowed with the metric induced on $T(X, d)$ by the *sup norm* $||f|| = \sup_{x \in X} |f(x)|$.

The set X can be identified with the set $\{h_x \in T(X, d) : h_x(y) = d(y, x)\}$ or, equivalently, with the set $T^0(X, d) = \{f \in T(X, d) : 0 \in f(X)\}$. The **injective envelope** $(\overline{X}, \overline{d})$ of X is isometrically identified with the tight span $T(X, d)$ by

$$\overline{X} \to T(X, d), \ \overline{x} \to h_{\overline{x}} \in T(X, d) : h_{\overline{x}}(y) = \overline{d}(f(y), \overline{x}).$$

The tight span $T(X, d)$ of a finite metric space is the metric space $(T(X), D(f, g) = \max |f(x) - g(x)|)$, where $T(X)$ is the set of functions $f : X \to \mathbb{R}$ such that for any $x, y \in X$, $f(x) + f(y) \geq d(x, y)$ and, for each $x \in X$, there exists $y \in X$ with $f(x) + f(y) = d(x, y)$. The mapping of any x into the function $f_x(y) = d(x, y)$ gives an isometric embedding of (X, d) into $T(X, d)$. For example, if $X = \{x_1, x_2\}$, then $T(X, d)$ is the interval of length $d(x_1, x_2)$.

The tight span of a metric space (X, d) of finite diameter can be considered as a polytopal complex of bounded faces of the polyhedron

$$\{y \in \mathbb{R}^n_{\geq 0} : y_i + y_j \geq d(x_i, x_j) \text{ for } 1 \leq i < j \leq n\}$$

if, for example, $X = \{x_1, \dots, x_n\}$. The dimension of this complex is called (Dress, 1984) the **combinatorial dimension** of (X, d).

- **Real tree**

  A metric space (X, d) is called (Tits, 1977) a **real tree** (or $\mathbb{R}$**-tree**) if, for all $x, y \in X$, there exists a unique **arc** from x to y, and this arc is a **geodesic segment**. So, an $\mathbb{R}$-tree is a (uniquely) arcwise connected metric space in which each arc is isometric to a subarc of $\mathbb{R}$. $\mathbb{R}$-tree is not related to a **metric tree** in Chap. 17.

  A metric space (X, d) is a real tree if and only if it is **path-connected** and Gromov 0**-hyperbolic** (i.e., satisfies the **four-point inequality**). The plane $\mathbb{R}^2$ with the **Paris metric** or **lift metric** (Chap. 19) are examples of an $\mathbb{R}$-tree.

  Real trees are exactly **tree-like** metric spaces which are **geodesic**; they are **injective** metric spaces among tree-like spaces. Tree-like metric spaces are by definition metric subspaces of real trees.

  If (X, d) is a finite metric space, then the **tight span** $T(X, d)$ is a real tree and can be viewed as an edge-weighted graph-theoretical tree.

  A metric space is a complete real tree if and only if it is **hyperconvex** and any two points are joined by a **metric segment**.

  A geodesic metric space (X, d) is called (Druţu–Sapir, 2005) *tree-graded with respect to* a collection $\mathcal{P}$ of connected proper subsets with $|P \cap P'| \leq 1$ for any distinct $P, P' \in \mathcal{P}$, if every its simple loop composed of three geodesics is contained in one $P \in \mathcal{P}$. $\mathbb{R}$-trees are tree-graded with respect to the empty set.

1.5 General Distances

- **Discrete metric**

  Given a set X, the **discrete metric** (or **trivial metric**, **sorting distance**, **drastic distance**, **Dirac distance**, *overlap*) is a metric on X, defined by $d(x, y) = 1$ for all distinct $x, y \in X$ and $d(x, x) = 0$. Cf. the much more general notion of a (metrically or topologically) **discrete metric space**.

- **Indiscrete semimetric**

  Given a set X, the **indiscrete semimetric** d is a semimetric on X defined by $d(x, y) = 0$ for all $x, y \in X$.

- **Equidistant metric**

  Given a set X and a positive real number t, the **equidistant metric** d is a metric on X defined by $d(x, y) = t$ for all distinct $x, y \in X$ (and $d(x, x) = 0$).

- $(1, 2) - B$**-metric**

  Given a set X, the $(1, 2) - B$**-metric** d is a metric on X such that, for any $x \in X$, the number of points $y \in X$ with $d(x, y) = 1$ is at most B, and all other distances are equal to 2. The $(1,2)-B$**-metric** is the **truncated metric** of a graph with maximal vertex degree B.

- **Permutation metric**

  Given a finite set X, a metric d on it is called a **permutation metric** (or *linear arrangement metric*) if there exists a bijection $\omega : X \to \{1, \dots, |X|\}$ such that

$$d(x, y) = |\omega(x) - \omega(y)|$$

holds for all $x, y \in X$. Even–Naor–Rao–Schieber, 2000, defined a more general **spreading metric**, i.e., any metric d on $\{1, \dots, n\}$ such that $\sum_{y \in M} d(x, y) \geq \frac{|M|(|M|+2)}{4}$ for any $1 \leq x \leq n$ and $M \subseteq \{1, \dots, n\} \setminus \{x\}$ with $|M| \geq 2$.

- **Induced metric**

  Given a metric space (X, d) and a subset $X' \subset X$, an **induced metric** (or **submetric**) is the restriction d' of d to X'. A metric space (X', d') is called a **metric subspace** of (X, d), and (X, d) is called a **metric extension** of (X', d').

- **Katětov mapping**

  Given a metric space (X, d), the mapping $f : X \to \mathbb{R}$ is a **Katětov mapping** if

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for any $x, y \in X$, i.e., setting $d(x, z) = f(x)$ defines a one-point **metric extension** $(X \cup \{z\}, d)$ of (X, d).

The set $E(X)$ of Katětov mappings on X is a complete metric space with metric $D(f, g) = \sup_{x \in X} |f(x) - g(x)|$; (X, d) embeds isometrically in it via the *Kuratowski mapping* $x \to d(x, .)$, with unique extension of each isometry of X to one of $E(X)$.

- **Dominating metric**

  Given metrics d and d_1 on a set X, d_1 **dominates** d if $d_1(x, y) \geq d(x, y)$ for all $x, y \in X$. Cf. **noncontractive mapping** (or *dominating mapping*).

- **Barbilian semimetric**

  Given sets X and P, the function $f : P \times X \to \mathbb{R}_{>0}$ is called an *influence* (of P over X) if for any $x, y \in X$ the ratio $g_{xy}(p) = \frac{f(p,x)}{f(p,y)}$ has a maximum when $p \in P$.

  The **Barbilian semimetric** is defined on the set X by

$$\ln \frac{\max_{p \in P} g_{xy}(p)}{\min_{p \in P} g_{xy}(p)}$$

for any $x, y \in X$. Barbilian, 1959, proved that the above function is well defined (moreover, $\min_{p \in P} g_{xy}(p) = \frac{1}{\max_{p \in P} g_{yx}(p)}$) and is a semimetric. Also, it is a metric if the influence f is *effective*, i.e., there is no pair $x, y \in X$ such that $g_{xy}(p)$ is constant for all $p \in P$. Cf. a special case **Barbilian metric** in Chap. 6.

- **Metric transform**

  A **metric transform** is a distance obtained as a function of a given metric (cf. Chap. 4).

- **Complete metric**

  Given a metric space (X, d), a sequence $\{x_n\}$, $x_n \in X$, is said to have *convergence to* $x^* \in X$ if $\lim_{n\to\infty} d(x_n, x^*) = 0$, i.e., for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x^*) < \epsilon$ for any $n > n_0$. Any sequence converges to at most one limit in X; it is not so, in general, if d is a semimetric.

  A sequence $\{x_n\}_n$, $x_n \in X$, is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for any $m, n > n_0$.

  A metric space (X, d) is called a **complete metric space** if every *Cauchy sequence* in it converges. In this case the metric d is called a **complete metric**. An example of an incomplete metric space is $(\mathbb{N}, d(m, n) = \frac{|m-n|}{mn})$.

- **Cauchy completion**

  Given a metric space (X, d), its **Cauchy completion** is a metric space (X^*, d^*) on the set X^* of all equivalence classes of *Cauchy sequences*, where the sequence $\{x_n\}_n$ is called *equivalent to* $\{y_n\}_n$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. The metric d^* is defined by

$$d^*(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n),$$

  for any $x^*, y^* \in X^*$, where $\{z_n\}_n$ is any element in the equivalence class z^*.

  The Cauchy completion (X^*, d^*) is a unique, up to isometry, **complete** metric space, into which the metric space (X, d) embeds as a dense metric subspace.

  The Cauchy completion of the metric space $(\mathbb{Q}, |x - y|)$ of rational numbers is the *real line* $(\mathbb{R}, |x - y|)$. A **Banach space** is the Cauchy completion of a *normed vector space* $(V, ||.||)$ with the **norm metric** $||x - y||$. A **Hilbert space** corresponds to the case an *inner product norm* $||x|| = \sqrt{\langle x, x\rangle}$.

- **Perfect metric space**

  A complete metric space (X, d) is called **perfect** if every point $x \in X$ is a *limit point*, i.e., $|B(x, r) = \{y \in X : d(x, y) < r\}| > 1$ holds for any $r > 0$.

  A topological space is a **Cantor space** (i.e., *homeomorphic* to the *Cantor set* with the natural metric $|x - y|$) if and only if it is nonempty, perfect, **totally disconnected**, compact and metrizable. The totally disconnected countable metric space $(\mathbb{Q}, |x-y|)$ of rational numbers also consists only of limit points but it is not complete and not **locally compact**.

  Every proper metric ball of radius r in a metric space has diameter at most $2r$. Given a number $0 < c \le 1$, a metric space is called a **c-uniformly perfect metric space** if this diameter is at least $2cr$. Cf. the **radii of metric space**.

- **Metrically discrete metric space**

  A metric space (X, d) is called **metrically** (or *uniformly*) **discrete** if there exists a number $r > 0$ such that $B(x, r) = \{y \in X : d(x, y) < r\} = \{x\}$ for every $x \in X$.

  (X, d) is a **topologically discrete metric space** (or a *discrete metric space*) if the underlying topological space is **discrete**, i.e., each point $x \in X$ is an *isolated point*: there exists a number $r(x) > 0$ such that $B(x, r(x)) = \{x\}$. For $X = \{\frac{1}{n}$:

$n = 1, 2, 3, \ldots\}$, the metric space $(X, |x - y|)$ is topologically but not metrically discrete. Cf. **translation discrete metric** in Chap. 10.

Alternatively, a metric space (X, d) is called *discrete* if any of the following holds:

1. (Burdyuk–Burdyuk 1991) it has a proper *isolated subset*, i.e., $M \subset X$ with $\inf\{d(x, y) : x \in M, y \notin M\} > 0$ (any such space admits a unique decomposition into *continuous*, i.e., nondiscrete, components);
2. (Lebedeva–Sergienko–Soltan, 1984) for any distinct points $x, y \in X$, there exists a point z of the **closed metric interval** $I(x, y)$ with $I(x, z) = \{x, z\}$;
3. a stronger property holds: for any two distinct points $x, y \in X$, every sequence of points $z_1, z_2, \ldots$ with $z_k \in I(x, y)$ but $z_{k+1} \in I(x, z_k) \setminus \{z_k\}$ for $k = 1, 2, \ldots$ is a finite sequence.

- **Locally finite metric space**

  Let (X, d) be a **metrically discrete metric space**. Then it is called **locally finite** if for every $x \in X$ and every $r \geq 0$, the ball $|B(x, r)|$ is finite.

  If, moreover, $|B(x, r)| \leq C(r)$ for some number $C(r)$ depending only on r, then (X, d) is said to have *bounded geometry*.
- **Bounded metric space**

  A metric (moreover, a distance) d on a set X is called **bounded** if there exists a constant $C > 0$ such that $d(x, y) \leq C$ for any $x, y \in X$.

  For example, given a metric d on X, the metric D on X, defined by $D(x, y) = \frac{d(x,y)}{1+d(x,y)}$, is bounded with $C = 1$.

  A metric space (X, d) with a bounded metric d is called a **bounded metric space**.
- **Totally bounded metric space**

  A metric space (X, d) is called **totally bounded** if, for every $\epsilon > 0$, there exists a finite **ϵ-net**, i.e., a finite subset $M \subset X$ with the **point-set distance** $d(x, M) < \epsilon$ for any $x \in X$ (cf. **totally bounded space** in Chap. 2).

  Every totally bounded metric space is **bounded** and **separable**. A metric space is totally bounded if and only if its **Cauchy completion** is **compact**.
- **Separable metric space**

  A metric space (X, d) is called **separable** if it contains a countable **dense subset** M, i.e., a subset with which all its elements can be approached: X is the *closure* $cl(M)$ (M together with all its limit points).

  A metric space is separable if and only if it is **second-countable** (cf. Chap. 2).
- **Compact metric space**

  A **compact metric space** (or **metric compactum**) is a metric space in which every sequence has a *Cauchy subsequence*, and those subsequences are convergent. A metric space is compact if and only if it is **totally bounded** and **complete**.

  Every bounded and closed subset of a Euclidean space is compact. Every finite metric space is compact. Every compact metric space is **second-countable**.

  A **continuum** is a nonempty **connected** metric compactum.

- **Proper metric space**

  A metric space is called **proper** (or *finitely compact*, *having the Heine–Borel property*) if every its closed metric ball is compact. Any such space is **complete**.
- **UC metric space**

  A metric space is called a **UC metric space** (or *Atsuji space*) if any continuous function from it into an arbitrary metric space is *uniformly continuous*.

  Every such space is **complete**. Every **metric compactum** is a UC metric space.
- **Metric measure space**

  A **metric measure space** (or *mm-space*, *metric triple*) is a triple (X, d, μ), where (X, d) is a *Polish* (i.e., complete separable; cf. Chap. 2) *metric space* and (X, Σ, μ) is a *probability measure space* ($\mu(X) = 1$) with Σ being a *Borel σ-algebra* of all open and closed sets of the **metric topology** (Chap. 2) induced by the metric d on X. Cf. **metric outer measure**.
- **Norm metric**

  Given a *normed vector space* $(V, ||.||)$, the **norm metric** on V is defined by

$$||x - y||.$$

  The metric space $(V, ||x - y||)$ is called a **Banach space** if it is **complete**. Examples of norm metrics are l_p**- and** L_p**-metrics**, in particular, the **Euclidean metric**.

  Any metric space (X, d) admits an isometric embedding into a Banach space B such that its convex hull is dense in B (cf. **Monge–Kantorovich metric** in Chap. 14); (X, d) is a **linearly rigid metric space** if such an embedding is unique up to isometry. A metric space isometrically embeds into the unit sphere of a Banach space if and only if its diameter is at most 2.
- **Path metric**

  Given a connected graph $G = (V, E)$, its **path metric** (or *graphic metric*) d_{path} is a metric on V defined as the length (i.e., the number of edges) of a shortest path connecting two given vertices x and y from V (cf. Chap. 15).
- **Editing metric**

  Given a finite set X and a finite set $\mathcal{O}$ of (unary) *editing operations* on X, the **editing metric** on X is the **path metric** of the graph with the vertex-set X and xy being an edge if y can be obtained from x by one of the operations from $\mathcal{O}$.
- **Gallery metric**

  A *chamber system* is a set X (its elements are called *chambers*) equipped with n equivalence relations $\sim_i$, $1 \leq i \leq n$. A *gallery* is a sequence of chambers $x_1, \dots, x_m$ such that $x_i \sim_j x_{i+1}$ for every i and some j depending on i.

  The **gallery metric** is an **extended metric** on X which is the length of the shortest gallery connecting x and $y \in X$ (and is equal to ∞ if there is no connecting gallery). The gallery metric is the (extended) **path metric** of the graph with the vertex-set X and xy being an edge if $x \sim_i y$ for some $1 \leq i \leq n$.

- **Metric on incidence structure**

An *incidence structure* (P, L, I) consists of 3 sets: points P, lines L and flags $I \subset P \times L$, where a point $p \in P$ is said to be *incident* with a line $l \in L$ if $(p, l) \in I$.

If, moreover, for any pair of distinct points, there is at most one line incident with both of them, then the collinearity graph is a graph whose vertices are the points with two vertices being adjacent if they determine a line.

The **metric on incidence structure** is the **path metric** of this graph.

- **Riemannian metric**

Given a connected n-dimensional smooth *manifold* M^n (cf. Chaps. 2 and 7), its **Riemannian metric** is a collection of positive-definite symmetric bilinear forms $((g_{ij}))$ on the tangent spaces of M^n which varies smoothly from point to point.

The length of a curve γ on M^n is expressed as $\int_\gamma \sqrt{\sum_{i,j} g_{ij} dx_i dx_j}$, and the **intrinsic metric** on M^n, also called the **Riemannian distance**, is the infimum of lengths of curves connecting any two given points $x, y \in M^n$. Cf. Chap. 7.

- **Linearly additive metric**

A **linearly additive** (or *additive on lines*) **metric** is a continuous metric d on $\mathbb{R}^n$ which, for any points x, y, z lying in that order on a common line, satisfies

$$d(x, z) = d(x, y) + d(y, z).$$

Hilbert's 4-th problem asked in 1900 to classify such metrics; it is solved only for dimension $n = 2$ ([Amba76]). Cf. **projective metric** in Chap. 6.

Every **norm metric** on $\mathbb{R}^n$ is linearly additive. Every linearly additive metric on $\mathbb{R}^2$ is a **hypermetric**.

- **Hamming metric**

The **Hamming metric** d_H (called sometimes *Dalal distance* in Semantics) is a metric on $\mathbb{R}^n$ defined (Hamming, 1950) by

$$|\{i : 1 \le i \le n, x_i \ne y_i\}|.$$

On binary vectors $x, y \in \{0, 1\}^n$ the Hamming metric and the l_1*-metric* (cf. **L_p-metric** in Chap. 5) coincide; they are equal to $|I(x) \Delta I(y)| = |I(x) \setminus I(y)| + |I(y) \setminus I(x)|$, where $I(z) = \{1 \le t \le n : z_i = 1\}$.

In fact, $\max\{|I(x) \setminus I(y)|, |I(y) \setminus I(x)|\}$ is also a metric.

- **Lee metric**

Given $m, n \in \mathbb{N}$, $m \ge 2$, the **Lee metric** d_{Lee} is a metric on $\mathbb{Z}_m^n = \{0, 1, \dots, m-1\}^n$ defined (Lee, 1958) by

$$\sum_{1 \le i \le n} \min\{|x_i - y_i|, m - |x_i - y_i|\}.$$

The metric space (Z_m^n, d_{Lee}) is a discrete analog of the *elliptic space*.

The Lee metric coincides with the Hamming metric d_H if $m = 2$ or $m = 3$. The metric spaces (Z_4^n, d_{Lee}) and $Z_2^{2n}, d_H)$ are isometric. Lee and Hamming metrics are applied for phase and orthogonal modulation, respectively.

Cf. **absolute summation distance** and **generalized Lee metric** in Chap. 16.

- **Enomoto–Katona metric**

Given a finite set X and an integer k, $2k \leq |X|$, the **Enomoto–Katona metric** (2001) is the distance between unordered pairs (X_1, X_2) and (Y_1, Y_2) of disjoint k-subsets of X defined by

$$\min\{|X_1 \setminus Y_1| + |X_2 \setminus Y_2|, |X_1 \setminus Y_2| + |X_2 \setminus Y_1|\}.$$

Cf. **Earth Mover's distance**, **transportation distance** in Chaps. 21 and 14.

- **Symmetric difference metric**

Given a *measure space* $(\Omega, \mathcal{A}, \mu)$, the **symmetric difference** (or *measure*) *semimetric* on the set $\mathcal{A}_\mu = \{A \in \mathcal{A} : \mu(A) < \infty\}$ is defined by

$$od_\triangle(A, B) = \mu(A \triangle B),$$

where $A \triangle B = (A \cup B) \backslash (A \cap B)$ is the *symmetric difference* of A and $B \in \mathcal{A}_\mu$.

The value $d_\triangle(A, B) = 0$ if and only if $\mu(A \triangle B) = 0$, i.e., A and B are equal *almost everywhere*. Identifying two sets $A, B \in \mathcal{A}_\mu$ if $\mu(A \triangle B) = 0$, we obtain the **symmetric difference metric** (or **Fréchet–Nikodym–Aronszyan distance**, **measure metric**).

If μ is the *cardinality measure*, i.e., $\mu(A) = |A|$, then $d_\triangle(A, B) = |A \triangle B| = |A \setminus B| + |B \setminus A|$. In this case $|A \triangle B| = 0$ if and only if $A = B$.

The metrics $d_{\max}(A, B) = \max(|A \setminus B|, |B \setminus A|)$ and $1 - \frac{|A \cap B|}{\max(|A|, |B|)}$ (its normalised version) are special cases of **Zelinka distance** and **Bunke–Shearer metric** in Chap. 15. For each $p \geq 1$, the **p-difference metric** (Noradam–Nyblom, 2014) is $d_p(A, B) = (|A \setminus B|^p + |B \setminus A|^p)^{\frac{1}{p}}$; so, $d_1 = d_\triangle$ and $\lim_{p \to \infty} d_p = d_{\max}$.

The **Johnson distance** between k-sets A and B is $\frac{|A \triangle B|}{2} = k - |A \cap B|$.

The *symmetric difference metric between ordered q-partitions* $A = (A_1, \ldots, A_q)$ and $B = (B_1, \ldots, B_q)$ is $\sum_{i=1}^{q} |A_i \Delta B_i|$. Cf. **metrics between partitions** in Chap. 10.

- **Steinhaus distance**

Given a *measure space* $(\Omega, \mathcal{A}, \mu)$, the **Steinhaus distance** d_{St} is a semimetric on the set $\mathcal{A}_\mu = \{A \in \mathcal{A} : \mu(A) < \infty\}$ defined as 0 if $\mu(A) = \mu(B) = 0$, and by

$$\frac{\mu(A \triangle B)}{\mu(A \cup B)} = 1 - \frac{\mu(A \cap B)}{\mu(A \cup B)}$$

if $\mu(A \cup B) > 0$. It becomes a metric on the set of equivalence classes of elements from $\mathcal{A}_\mu$; here $A, B \in \mathcal{A}_\mu$ are called *equivalent* if $\mu(A \triangle B) = 0$.

The **biotope** (or **Tanimoto**) **distance** $\frac{|A \triangle B|}{|A \cup B|}$ is the special case of Steinhaus distance obtained for the *cardinality measure* $\mu(A) = |A|$ for finite sets.

Cf. also the **generalized biotope transform metric** in Chap. 4.

- **Fréchet metric**

  Let (X, d) be a metric space. Consider a set $\mathcal{F}$ of all continuous mappings $f : A \to X, g : B \to X, \ldots$, where $A, B, \ldots$ are subsets of $\mathbb{R}^n$, homeomorphic to $[0, 1]^n$ for a fixed dimension $n \in \mathbb{N}$.

  The *Fréchet semimetric* d_F is a semimetric on $\mathcal{F}$ defined by

$$\inf_{\sigma} \sup_{x \in A} d(f(x), g(\sigma(x))),$$

  where the infimum is taken over all orientation preserving homeomorphisms $\sigma : A \to B$. It becomes the **Fréchet metric** on the set of equivalence classes $f^* = \{g : d_F(g,f) = 0\}$. Cf. the **Fréchet surface metric** in Chap. 8.

- **Hausdorff metric**

  Given a metric space (X, d), the **Hausdorff metric** (or *two-sided Hausdorff distance*) is a metric on the family $\mathcal{F}$ of nonempty compact subsets of X defined by

$$d_{Haus} = \max\{d_{dHaus}(A, B), d_{dHaus}(B, A)\},$$

  where $d_{dHaus}(A, B) = \max_{x \in A} \min_{y \in B} d(x, y)$ is the **directed Hausdorff distance** (or *one-sided Hausdorff distance*) from A to B. The metric space $(\mathcal{F}, d_{Haus})$ is called **hyperspace of metric space** (X, d); cf. **hyperspace** in Chap. 2.

  In other words, $d_{Haus}(A, B)$ is the minimal number ϵ (called also the **Blaschke distance**) such that a closed **ϵ-neighborhood** of A contains B and a closed ϵ-neighborhood of B contains A. Then $d_{Haus}(A, B)$ is equal to

$$\sup_{x \in X} |d(x, A) - d(x, B)|,$$

  where $d(x, A) = \min_{y \in A} d(x, y)$ is the **point-set distance**.

  If the above definition is extended for noncompact closed subsets A and B of X, then $d_{Haus}(A, B)$ can be infinite, i.e., it becomes an **extended metric**.

  For not necessarily closed subsets A and B of X, the **Hausdorff semimetric** between them is defined as the Hausdorff metric between their closures. If X is finite, d_{Haus} is a metric on the class of all subsets of X.

- **L_p-Hausdorff distance**

  Given a finite metric space (X, d), the **L_p-Hausdorff distance** ([Badd92]) between two subsets A and B of X is defined by

$$\left(\sum_{x \in X} |d(x, A) - d(x, B)|^p\right)^{\frac{1}{p}},$$

where $d(x, A)$ is the **point-set distance**. The usual **Hausdorff metric** corresponds to the case $p = \infty$.

- **Generalized G-Hausdorff metric**

  Given a group $(G, \cdot, e)$ acting on a metric space (X, d), the **generalized G-Hausdorff metric** between two closed bounded subsets A and B of X is

$$\min_{g_1, g_2 \in G} d_{Haus}(g_1(A), g_2(B)),$$

  where d_{Haus} is the **Hausdorff metric**. If $d(g(x), g(y)) = d(x, y)$ for any $g \in G$ (i.e., if the metric d is *left-invariant* with respect of G), then above metric is equal to $\min_{g \in G} d_{Haus}(A, g(B))$.

- **Gromov–Hausdorff metric**

  The **Gromov–Hausdorff metric** is a metric on the set of all *isometry classes* of compact metric spaces defined by

$$\inf d_{Haus}(f(X), g(Y))$$

  for any two classes X^* and Y^* with the representatives X and Y, respectively, where d_{Haus} is the **Hausdorff metric**, and the minimum is taken over all metric spaces M and all isometric embeddings $f : X \to M$, $g : Y \to M$. The corresponding metric space is called the *Gromov–Hausdorff space*.

  The **Hausdorff–Lipschitz distance** between isometry classes of compact metric spaces X and Y is defined by

$$\inf\{d_{GH}(X, X_1) + d_L(X_1, Y_1) + d_{GH}(Y, Y_1)\},$$

  where d_{GH} is the Gromov–Hausdorff metric, d_L is the **Lipschitz metric**, and the minimum is taken over all (isometry classes of compact) metric spaces X_1, Y_1.

- **Kadets distance**

  The *gap* (or *opening*) between two closed subspaces X and Y of a Banach space $(V, ||.||)$ is defined by

$$gap(X, Y) = \max\{\delta(X, Y), \delta(Y, X)\},$$

  where $\delta(X, Y) = \sup\{\inf_{y \in Y} ||x - y|| : x \in X, ||x|| = 1\}$ (cf. **gap distance** in Chap. 12 and **gap metric** in Chap. 18).

  The **Kadets distance** between two Banach spaces V and W is a semimetric defined (Kadets, 1975) by

$$\inf_{Z, f, g} gap(\overline{B}_{f(V)}, \overline{B}_{g(W)}),$$

  where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $f : V \to Z$ and $g : W \to Z$; here $\overline{B}_{f(V)}$ and $\overline{B}_{g(W)}$ are the closed unit balls of Banach spaces $f(V)$ and $g(W)$, respectively.

The nonlinear analog of the Kadets distance is the following **Gromov–Hausdorff distance between Banach spaces** U and W:

$$\inf_{Z,f,g} d_{Haus}(f(\overline{B}_V), g(\overline{B}_W)),$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $f : V \to Z$ and $g : W \to Z$; here d_{Haus} is the **Hausdorff metric**.

The **Kadets path distance** between Banach spaces V and W is defined (Ostrovskii, 2000) as the infimum of the lengths (with respect to the Kadets distance) of all curves joining V and W (and is equal to ∞ if there is no such curve).

- **Banach–Mazur distance**

The **Banach–Mazur distance** d_{BM} between two Banach spaces V and W is

$$\ln \inf_T ||T|| \cdot ||T^{-1}||,$$

where the infimum is taken over all isomorphisms $T : V \to W$.

It can also be written as $\ln d(V, W)$, where the number $d(V, W)$ is the smallest positive $d \geq 1$ such that $\overline{B}_W^n \subset T(\overline{B}_V^n) \subset d\overline{B}_W^n$ for some linear invertible transformation $T : V \to W$. Here $\overline{B}_V^n = \{x \in V : ||x||_V \leq 1\}$ and $\overline{B}_W^n = \{x \in W; ||x||_W \leq 1\}$ are the *unit balls* of the normed spaces $(V, ||.||_V)$ and $(W, ||.||_W)$, respectively.

One has $d_{BM}(V, W) = 0$ if and only if V and W are *isometric*, and d_{BM} becomes a metric on the set X^n of all equivalence classes of n-dimensional normed spaces, where $V \sim W$ if they are isometric. The pair (X^n, d_{BM}) is a compact metric space which is called the **Banach–Mazur compactum**.

The **modified Banach–Mazur distance** (Glushkin, 1963, and Khrabrov, 2001) is

$$\inf\{||T||_{X\to Y} : |detT| = 1\} \cdot \inf\{||T||_{Y\to X} : |detT| = 1\}.$$

The **weak Banach–Mazur distance** (Tomczak–Jaegermann, 1984) is

$$\max\{\overline{\gamma}_Y(id_X), \overline{\gamma}_X(id_Y)\},$$

where id is the identity map and, for an operator $U : X \to Y$, $\overline{\gamma}_Z(U)$ denotes $\inf \sum ||W_k|| \cdot ||V_k||$. Here the infimum is taken over all representations $U = \sum W_k V_k$ for $W_k : X \to Z$ and $V_k : Z \to Y$. This distance never exceeds the corresponding Banach–Mazur distance.

- **Lipschitz distance**

Given $\alpha \geq 0$ and two metric spaces (X, d_X), (Y, d_Y), the *α-Hölder norm* $||.||_{Hol}$ on the set of all injective functions $f : X \to Y$ is defined by

$$||f||_{Hol} = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)^\alpha}.$$

The *Lipschitz norm* $||.||_{Lip}$ is the case $\alpha = 1$ of $||.||_{Hol}$.

The **Lipschitz distance** between metric spaces (X, d_X) and (Y, d_Y) is defined by

$$\ln \inf_f ||f||_{Lip} \cdot ||f^{-1}||_{Lip},$$

where the infimum is taken over all bijective functions $f : X \to Y$. Equivalently, it is the infimum of numbers $\ln a$ such that there exists a bijective **bi-Lipschitz mapping** between (X, d_X) and (Y, d_Y) with constants $\exp(-a)$, $\exp(a)$.

It becomes a metric (**Lipschitz metric**) on the set of all isometry classes of compact metric spaces. Cf. **Hausdorff–Lipschitz distance**.

This distance is an analog to the **Banach–Mazur distance** and, in the case of finite-dimensional real Banach spaces, coincides with it.

It also coincides with the **Hilbert projective metric** on *nonnegative* projective spaces, obtained by starting with $\mathbb{R}^n_{>0}$ and identifying any point x with cx, $c > 0$.

- **Lipschitz distance between measures**

Given a compact metric space (X, d), the *Lipschitz seminorm* $||.||_{Lip}$ on the set of all functions $f : X \to \mathbb{R}$ is defined by $||f||_{Lip} = \sup_{x,y \in X, x \neq y} \frac{|f(x)-f(y)|}{d(x,y)}$.

The **Lipschitz distance between measures** μ and ν on X is defined by

$$\sup_{||f||_{Lip} \leq 1} \int f d(\mu - \nu).$$

It is the **transportation distance** (Chap. 14) if μ, ν are probability measures. Let a such measure $m_x(.)$ be attached to any $x \in X$; for distinct x, y the *coarse Ricci curvature along* (xy) is defined (Ollivier, 2009) as $\kappa(x, y) = 1 - \frac{W_1(m_x, m_y)}{d(x,y)}$. *Ollivier's curvature* generalizes the *Ricci curvature* in Riemannian space (cf. Chap. 7).

- **Barycentric metric space**

Given a metric space (X, d), let $(B(X), ||\mu - \nu||_{TV})$ be the metric space, where $B(X)$ is the set of all regular Borel probability measures on X with bounded support, and $||\mu - \nu||_{TV}$ is the **variational distance** $\int_X |p(\mu) - p(\nu)| d\lambda$ (cf. Chap. 14). Here $p(\mu)$ and $p(\nu)$ are the density functions of measures μ and ν, respectively, with respect to the σ-finite measure $\frac{\mu+\nu}{2}$.

A metric space (X, d) is **barycentric** if there exists a constant $\beta > 0$ and a surjection $f : B(X) \to X$ such that for any measures $\mu, \nu \in B(X)$ it holds the inequality

$$d(f(\mu), f(\nu)) \leq \beta diam(supp(\mu + \nu))||\mu - \nu||_{TV}.$$

Any Banach space $(X, d = ||x - y||)$ is a barycentric metric space with the smallest β being 1 and the map $f(\mu)$ being the usual *center of mass* $\int_X x d\mu(x)$.

Any **Hadamard** (i.e., a complete **CAT**(0) **space**, cf. Chap. 6, is barycentric with the smallest β being 1 and the map $f(\mu)$ being the unique minimizer of the function $g(y) = \int_X d^2(x, y) d\mu(x)$ on X.

- **Point-set distance**

Given a metric space (X, d), the **point-set distance** $d(x, A)$ between a point $x \in X$ and a subset A of X is defined as

$$\inf_{y \in A} d(x, y).$$

For any $x, y \in X$ and for any nonempty subset A of X, we have the following version of the triangle inequality: $d(x, A) \leq d(x, y) + d(y, A)$ (cf. **distance map**).

For a given point-measure $\mu(x)$ on X and a *penalty function* p, an **optimal quantizer** is a set $B \subset X$ such that $\int p(d(x, B)) d\mu(x)$ is as small as possible.

- **Set-set distance**

Given a metric space (X, d), the **set-set distance** between two subsets A and B of X is defined by

$$d_{ss}(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

This distance can be 0 even for disjoint sets, for example, for the intervals $(1, 2)$, $(2, 3)$ on $\mathbb{R}$. The sets A and B are *positively separated* if $d_{ss}(A, B) > 0$. A constructive *appartness space* is a generalization of this relation on subsets of X.

The **spanning distance** between A and B is $\sup_{x \in A, y \in B} d(x, y)$.

In Data Analysis, (cf. Chap. 17) the set-set and spanning distances between clusters are called the **single** and **complete linkage**, respectively.

- **Matching distance**

Given a metric space (X, d), the **matching distance** (or *multiset-multiset distance*) between two multisets A and B in X is defined by

$$\inf_{\phi} \max_{x \in A} d(x, \phi(x)),$$

where ϕ runs over all bijections between A and B, as multisets.

The matching distance is not related to the **perfect matching distance** in Chap. 15 and to the **nonlinear elastic matching distance** in Chap. 21. But the **bottleneck distance** in Chap. 21 is a special case of it.

- **Metrics between multisets**

A *multiset* (or *bag*) drawn from a set S is a mapping $m : S \to \mathbb{Z}_{\geq 0}$, where $m(x)$ represents the "multiplicity" of $x \in S$. The *dimensionality*, *cardinality* and *height* of multiset m is $|S|$, $|m| = \sum_{x \in S} m(x)$ and $\max_{x \in S} m(x)$, respectively.

Multisets are good models for multi-attribute objects such as, say, all symbols in a string, all words in a document, etc.

A multiset m is finite if S and all $m(x)$ are finite; the *complement* of a finite multiset m is the multiset $\overline{m} : S \to \mathbb{Z}_{\geq 0}$, where $\overline{m}(x) = \max_{y \in S} m(y) - m(x)$. Given two multisets m_1 and m_2, denote by $m_1 \cup m_2$, $m_1 \cap m_2$, $m_1 \backslash m_2$ and $m_1 \Delta m_2$ the multisets on S defined , for any $x \in S$, by $m_1 \cup m_2(x) = \max\{m_1(x), m_2(x)\}$, $m_1 \cap m_2(x) = \min\{m_1(x), m_2(x)\}$, $m_1 \backslash m_2(x) = \max\{0, m_1(x) - m_2(x)\}$ and $m_1 \Delta m_2(x) = |m_1(x) - m_2(x)|$, respectively. Also, $m_1 \subseteq m_2$ denotes that $m_1(x) \leq m_2(x)$ for all $x \in S$.

The *measure* $\mu(m)$ of a multiset m is a linear combination $\mu(m) = \sum_{x \in S} \lambda(x) m(x)$ with $\lambda(x) \geq 0$. In particular, $|m|$ is the *counting measure*.

For any measure $\mu(m) \in \mathbb{R}_{\geq 0}$, Miyamoto, 1990, and Petrovsky, 2003, proposed several **semimetrics between multisets** m_1 and m_2 including $d_1(m_1, m_2) = \mu(m_1 \Delta m_2)$ and $d_2(m_1, m_2) = \frac{\mu(m_1 \Delta m_2)}{\mu(m_1 \cup m_2)}$ (with $d_2(\emptyset, \emptyset) = 0$ by definition). Cf. **symmetric difference metric** and **Steinhaus distance**.

Among examples of other metrics between multisets are **matching distance**, **metric space of roots** in Chap. 12, **μ-metric** in Chap. 15 and, in Chap. 11, **bag distance** $\max\{|m_1 \backslash m_2|, |m_2 \backslash m_1|\}$ and **q-gram similarity**.

See also **Vitanyi multiset metric** in Chap. 3.

- **Metrics between fuzzy sets**

A *fuzzy subset* of a set S is a mapping $\mu : S \to [0, 1]$, where $\mu(x)$ represents the "degree of membership" of $x \in S$. It is an ordinary (*crisp*) if all $\mu(x)$ are 0 or 1. Fuzzy sets are good models for *gray scale images* (cf. **gray scale images distances** in Chap. 21), random objects and objects with nonsharp boundaries.

Bhutani–Rosenfeld, 2003, introduced the following two metrics between two fuzzy subsets μ and ν of a finite set S. The **diff-dissimilarity** is a metric (a fuzzy generalization of **Hamming metric**), defined by

$$d(\mu, \nu) = \sum_{x \in S} |\mu(x) - \nu(x)|.$$

The **perm-dissimilarity** is a semimetric defined by

$$\min\{d(\mu, p(\nu))\},$$

where the minimum is taken over all permutations p of S.

The **Chaudhuri–Rosenfeld metric** (1996) between two fuzzy sets μ and ν *with crisp points* (i.e., the sets $\{x \in S : \mu(x) = 1\}$ and $\{x \in S : \nu(x) = 1\}$ are nonempty) is an **extended metric**, defined the **Hausdorff metric** d_{Haus} by

$$\int_0^1 2td_{Haus}(\{x \in S : \mu(x) \geq t\}, \{x \in S : \nu(x) \geq t\})dt.$$

A *fuzzy number* is a fuzzy subset μ of the real line $\mathbb{R}$, such that the *level set* (or *t-cut*) $A_\mu(t) = \{x \in \mathbb{R} : \mu(x) \geq t\}$ is convex for every $t \in [0, 1]$. The *sendograph* of a fuzzy set μ is the set $send(\mu) = \{(x, t) \in S \times [0, 1] : \mu(x) > 0,\ \mu(x) \geq t\}$. The **sendograph metric** (Kloeden, 1980) between two fuzzy numbers μ, ν with crisp points and compact sendographs is the **Hausdorff metric**

$$\max\{\sup_{a=(x,t)\in send(\mu)} d(a, send(\nu)), \sup_{b=(x',t')\in send(\nu)} d(b, send(\mu))\},$$

where $d(a, b) = d((x, t), (x', t'))$ is a **box metric** (Chap. 4) $\max\{|x - x'|, |t - t'|\}$.

The **Klement–Puri–Ralesku metric** (1988) between fuzzy numbers μ, ν is

$$\int_0^1 d_{Haus}(A_\mu(t), A_\nu(t))dt,$$

where $d_{Haus}(A_\mu(t), A_\nu(t))$ is the **Hausdorff metric**

$$\max\{\sup_{x\in A_\mu(t)} \inf_{y\in A_\nu(t)} |x - y|, \sup_{x\in A_\nu(t)} \inf_{x\in A_\mu(t)} |x - y|\}.$$

Several other Hausdorff-like metrics on some families of fuzzy sets were proposed by Boxer in 1997, Fan in 1998 and Brass in 2002; Brass also argued the nonexistence of a "good" such metric.

If q is a quasi-metric on $[0, 1]$ and S is a finite set, then $Q(\mu, \nu) = \sup_{x\in S} q(\mu(x), \nu(x))$ is a quasi-metric on fuzzy subsets of S.

Cf. **fuzzy Hamming distance** in Chap. 11 and, in Chap. 23, **fuzzy set distance** and **fuzzy polynucleotide metric**. Cf. **fuzzy metric spaces** in Chap. 3 for fuzzy-valued generalizations of metrics and, for example, [Bloc99] for a survey.

- **Metrics between intuitionistic fuzzy sets**

An *intuitionistic fuzzy subset* of a set S is (Atanassov, 1999) an ordered pair of mappings $\mu, \nu :\to [0, 1]$ with $0 \leq \mu(x) + \nu(x) \leq 1$ for all $x \in S$, representing the "degree of membership" and the "degree of nonmembership" of $x \in S$, respectively. It is an ordinary *fuzzy subset* if $\mu(x) + \nu(x) = 1$ for all $x \in S$.

Given two intuitionistic fuzzy subsets $(\mu(x), \nu(x))$ and $(\mu'(x), \nu'(x))$ of a finite set $S = \{x_1, \dots, x_n\}$, their **Atanassov distances** (1999) are:

$$\frac{1}{2}\sum_{i=1}^{n}(|\mu(x_i) - \mu'(x_i)| + |\nu(x_i) - \nu'(x_i)|) \text{ (Hamming distance)}$$

and, in general, for any given numbers $p \geq 1$ and $0 \leq q \leq 1$, the distance

$$(\sum_{i=1}^{n}(1-q)(\mu(x_i)-\mu'(x_i))^p + q(\nu(x_i)-\nu'(x_i)^p)^{\frac{1}{p}}.$$

Their **Grzegorzewski distances** (2004) are:

$$\sum_{i=1}^{n}\max\{|\mu(x_i)-\mu'(x_i)|, |\nu(x_i)-\nu'(x_i)|\} \text{ (Hamming distance);}$$

$$\sqrt{\sum_{i=1}^{n}\max\{(\mu(x_i)-\mu'(x_i))^2, (\nu(x_i)-\nu'(x_i))^2\}} \text{ (Euclidean distance).}$$

The normalized versions (dividing the above sums by n) were also proposed.

Szmidt–Kacprzyk, 1997, proposed a modification of the above, adding $\pi(x) - \pi'(x)$, where $\pi(x)$ is the third mapping $1-\mu(x)-\nu(x)$.

An *interval-valued fuzzy subset* of a set S is a mapping $\mu :\rightarrow [I]$, where $[I]$ is the set of closed intervals $[a^-, a^+] \subseteq [0,1]$. Let $\mu(x) = [\mu^-(x), \mu^+(x)]$, where $0 \leq \mu^-(x) \leq \mu^+(x) \leq 1$ and an interval-valued fuzzy subset is an ordered pair of mappings μ^- and μ^+. This notion is close to the above intuitionistic one; so, above distance can easily be adapted. For example, $\sum_{i=1}^{n}\max\{|\mu^-(x_i) - \mu'^-(x_i)|, |\mu^+(x_i) - \mu'^+(x_i)|\}$ is a Hamming distance between interval-valued fuzzy subsets (μ^-, μ^+) and (μ'^-, μ'^+).

- **Polynomial metric space**

  Let (X,d) be a metric space with a finite diameter D and a finite normalized measure μ_X. Let the Hilbert space $L_2(X,d)$ of complex-valued functions decompose into a countable (when X is infinite) or a finite (with $D+1$ members when X is finite) direct sum of mutually orthogonal subspaces $L_2(X,d) = V_0 \oplus V_1 \oplus \dots$.

  Then (X,d) is a **polynomial metric space** if there exists an ordering of the spaces $V_0, V_1, \dots$ such that, for $i = 0, 1, \dots$, there exist *zonal spherical functions*, i.e., real polynomials $Q_i(t)$ of degree i such that

$$Q_i(t(d(x,y))) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)}$$

  for all $x, y \in X$, where r_i is the dimension of V_i, $\{v_{ii}(x) : 1 \leq j \leq r_i\}$ is an orthonormal basis of V_i, and $t(d)$ is a continuous decreasing real function such that $t(0) = 1$ and $t(D) = -1$. The zonal spherical functions constitute an orthogonal system of polynomials with respect to some weight $w(t)$.

  The finite polynomial metric spaces are also called *(P and Q)-polynomial association schemes*; cf. **distance-regular graph** in Chap. 15. The infinite polynomial metric spaces are the *compact connected two-point homogeneous* *spaces*. Wang, 1952, classified them as the Euclidean unit spheres, the real, complex, quaternionic projective spaces or the Cayley projective line and plane.

- **Universal metric space**

  A metric space (U, d) is called **universal** for a collection $\mathcal{M}$ of metric spaces if any metric space (M, d_M) from $\mathcal{M}$ is *isometrically embeddable* in (U, d), i.e., there exists a mapping $f : M \to U$ which satisfies $d_M(x, y) = d(f(x), f(y))$ for any $x, y \in M$. Some examples follow.

  Every separable metric space (X, d) isometrically embeds (Fréchet, 1909) in (a nonseparable) **Banach space** l_∞^∞. In fact, $d(x, y) = \sup_i |d(x, a_i) - d(y, a_i)|$, where $(a_1, \dots, a_i, \dots)$ is a dense countable subset of X.

  Every metric space isometrically embeds (Kuratowski, 1935) in the **Banach space** $L^\infty(X)$ of bounded functions $f : X \to \mathbb{R}$ with the norm $\sup_{x\in X} |f(x)|$.

  The **Urysohn space** is a **homogeneous** complete separable space which is the universal metric space for all separable metric spaces. The **Hilbert cube** (Chap. 4) is the universal space for the class of metric spaces with a countable base.

  The **graphic** metric space of the **random graph** (Rado, 1964; the vertex-set consists of all prime numbers $p \equiv 1 \pmod 4$ with pq being an edge if p is a quadratic residue modulo q) is the universal metric space for any finite or countable metric space with distances 0, 1 and 2 only. It is a discrete analog of the Urysohn space.

  There exists a metric d on $\mathbb{R}$, inducing the usual (interval) topology, such that $(\mathbb{R}, d)$ is a universal metric space for all finite metric spaces (Holsztynski, 1978). The Banach space l_∞^n is a universal metric space for all metric spaces (X, d) with $|X| \le n + 2$ (Wolfe, 1967). The Euclidean space $\mathbb{E}^n$ is a universal metric space for all ultrametric spaces (X, d) with $|X| \le n + 1$; the space of all finite functions $f(t) : \mathbb{R}_{\ge 0} \to \mathbb{R}$ equipped with the metric $d(f, g) = \sup\{t : f(t) \ne g(t)\}$ is a universal metric space for all ultrametric spaces (Lemin–Lemin, 1996).

  The universality can be defined also for mappings, other than isometric embeddings, of metric spaces, say, a bi-Lipschitz embedding, etc. For example, any compact metric space is a continuous image of the **Cantor set** with the natural metric $|x - y|$ inherited from $\mathbb{R}$, and any complete separable metric space is a continuous image of the space of irrational numbers.

- **Constructive metric space**

  A **constructive metric space** is a pair (X, d), where X is a set of constructive objects (say, words over an alphabet), and d is an algorithm converting any pair of elements of X into a constructive real number $d(x, y)$ such that d is a metric on X.

- **Computable metric space**

  Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements from a given *Polish* (i.e., complete separable) *metric space* (X, d) such that the set $\{x_n : n \in \mathbb{N}\}$ is dense in (X, d). Let $\mathcal{N}(m, n, k)$ be the *Cantor tuple function* of a triple $(n, m, k) \in \mathbb{N}^3$, and let $\{q_k\}_{k\in\mathbb{N}}$ be a fixed total standard numbering of the set $\mathbb{Q}$ of rational numbers.

The triple $(X, d, \{x_n\}_{n \in \mathbb{N}})$ is called an *effective* (or *semicomputable*) *metric space* ([Hemm02]) if the set $\{\mathcal{N}(n, m, k) : d(x_m, x_n) < q_k\}$ is *recursively enumerable*, i.e., there is an algorithm that enumerates the members of this set. If, moreover, the set $\{\mathcal{N}(n, m, k) : d(s_m, s_m) > q_k\}$ is recursively enumerable, then this triple is called (Lacombe, 1951) **computable metric space**, (or **recursive metric space**). In other words, the map $d \circ (q \times q) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence of real numbers, i.e., is recursive over $\mathbb{R}$.