Chapter 2
The Theory: Mechanics. An Example: Collision of a Point and a Plane

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2.1 A System Made of a Point and a Plane

The system we consider is made of a point and an obstacle, for instance a plane. The point moves above the plane, can collide it, can slide on it. But it cannot interpenetrate it. Indeed, the two elements of the system, the point and the plane are not deformable. And it is not possible to speak of deformation when each of them is considered by itself. But when we consider the system made of the point and the plane, this system is deformable because the distance of the point to the plane changes, [14]. The development of this idea in the sequel gives a productive and elegant theory.

We assume also that the duration of the collisions of the point with the plane, i.e., the time for the point to adapt to the kinematic incompatibility, is negligible with respect to the time scale of the theory. For instance, with respect to the time flight. Thus we assume, the collisions are instantaneous. As for an example of such a motion, one may think of the motion of a soccer ball over a soccer field. The system is made of the soccer ball and the field. Collisions result from external actions due to the players and from internal actions due to the kinematic constraint which is that the soccer ball call cannot interpenetrate the field.

Remark 2.1 The collision theory has mainly been developed for solids. The traditional approach (see for example, [4, 36–38]) is based on the coefficients of restitution which is appropriated in simple situations but may be in contradiction with the basic principles of mechanics in more complex setups [3, 4, 7, 25, 37, 38].

2.2 The Velocities

The velocity of the point is a smooth function of time t when the flight is smooth. When it is not, for instance when colliding the plane or when hitted by some external percussion. At such a time, the velocity is discontinuous. There is velocity

$$\mathbf{U}^{-}(t) = \lim_{\Delta t \to 0, \Delta t > 0} \mathbf{U}(t - \Delta t), \tag{2.1}$$

before the collision and velocity

$$\mathbf{U}^{+}(t) = \lim_{\Delta t \to 0, \Delta t > 0} \mathbf{U}(t + \Delta t), \tag{2.2}$$

after the collision. The virtual velocities $\mathbf{V}$ are possible velocities of the point or possible velocities of the point we may think of. In terms of mechanics and mathematics, the virtual velocities are elements of the linear space which is induced by the formula giving the actual velocity. In this example, we choose the space of bounded variation 3-D vectors denoted $\mathcal{V}$, [1, 2, 31]. For the sake of simplicity, we assume the plane is fixed in a Galilean reference frame and remains immobile even when collided by the point. This is the case if the plane, the obstacle, is very massive compared to the point. Note that this is the case of the soccer field.

The discontinuity of velocity is denoted

$$[\mathbf{U}(t)] = \mathbf{U}^{+}(t) - \mathbf{U}^{-}(t). \tag{2.3}$$

More generally discontinuity of quantity A is denoted

$$[A(t)] = A^{+}(t) - A^{-}(t), \tag{2.4}$$

with

$$A^{+}(t) = \lim_{\Delta t \to 0, \Delta t > 0} A(t + \Delta t), \ A^{-}(t) = \lim_{\Delta t \to 0, \Delta t > 0} A(t - \Delta t) \tag{2.5}$$

2.3 The Velocity of Deformation

The choice of the velocity of deformation of the system has to be in agreement with observations. It is reasonable to choose as velocity of deformation of the system, the velocity of the point with respect to the plane obstacle which is assumed to be immobile, for instance assuming it is very massive. It is obvious that if the velocity of the point with respect to the plane is null, the shape of the system does not change. To be precise, if the velocity of the point is parallel to the plane, the form of the system changes. In particular, if the point slides on the plane, the shape changes because the distance of a reference point of the plane to the point changes.

2.4 The Principle of Virtual Work

A productive way to derive the equations of motion in mechanics is to use the Principle of Virtual Work, [16–19]. We think it may be founded on experimental observations. Let us consider a system and apply actions. We see that the work we provide to the system between times t_1 and t_2, $t_1 \leq t_2$ is used to modify its velocity and to modify its shape. We assume that this work, the work of the external forces, $\mathcal{W}_{ext}(t_1, t_2)$, is the sum of two works:

- the work to modify the velocity which is the actual work of the acceleration forces, $\mathcal{W}_{acc}(t_1, t_2)$;
- the work to modify the shape or the form which is the opposite of the classical actual work of the internal forces, $-\mathcal{W}_{int}(t_1, t_2)$.

$$\mathcal{W}_{ext}(t_1, t_2) = \mathcal{W}_{acc}(t_1, t_2) - \mathcal{W}_{int}(t_1, t_2)), \tag{2.6}$$

Experiments lead also to assume the works are additive functions of time t and linear functions of the velocities. Thus we have

$$\mathcal{W}_{ext}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{ext}(\mathbf{V}(\boldsymbol{\tau}))d\tau + \sum_t \mathcal{T}_{ext}(\mathbf{V}, t), \tag{2.7}$$

$$\mathcal{W}_{acc}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{acc}(\mathbf{V}(\boldsymbol{\tau}))d\tau + \sum_t \mathcal{T}_{acc}(\mathbf{V}, t), \tag{2.8}$$

$$\mathcal{W}_{int}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{int}(\mathbf{V}(\boldsymbol{\tau}))d\tau + \sum_t \mathcal{T}_{int}(\mathbf{V}, t). \tag{2.9}$$

The works have a density with respect to the Lebesgue measure, the power $\mathcal{P}_{ext}(\mathbf{V}(\boldsymbol{\tau}))$ and a density with respect to the atomic measure, the work $\mathcal{T}_{ext}(\mathbf{V}, t)$. Experimental result (2.6) becomes

$$\mathcal{W}_{ext}(\mathbf{U}, t_1, t_2) = \mathcal{W}_{acc}(\mathbf{U}, t_1, t_2) - \mathcal{W}_{int}(\mathbf{U}, t_1, t_2), \tag{2.10}$$

The principle of virtual works extends this relationship to virtual velocities $\mathbf{V}$ and to any times t_1 and t_2, $t_1 \leq t_2$,

$$\forall \mathbf{V} \in \mathcal{V}, \ \forall t_1, \ \forall t_2, \ t_1 \leq t_2, \tag{2.11}$$
$$\mathcal{W}_{ext}(\mathbf{V}, t_1, t_2) = \mathcal{W}_{acc}(\mathbf{V}, t_1, t_2) - \mathcal{W}_{int}(\mathbf{V}, t_1, t_2), \tag{2.12}$$

or in its classical formulation,

$$\forall \mathbf{V} \in \mathcal{V}, \ \forall t_1, \ \forall t_2, \ t_1 \leq t_2, \tag{2.13}$$
$$\mathcal{W}_{acc}(\mathbf{V}, t_1, t_2) = \mathcal{W}_{int}(\mathbf{V}, t_1, t_2) + \mathcal{W}_{ext}(\mathbf{V}, t_1, t_2). \tag{2.14}$$

Let us investigate the principle of virtual work and define the different works. To be didactic, we assume only one time t between times t_1 and t_2 where there is an atomic part in the works. The work of the acceleration forces is

2.4.1 The Work of the Acceleration Forces

In the smooth evolution, the work of the acceleration force is the sum of the classical power

$$\mathcal{P}_{acc}(\mathbf{V}(\tau)) = m\frac{d\mathbf{U}}{dt}(\tau) \cdot \mathbf{V}(\tau), \tag{2.15}$$

where m is the mass of the point. Let us note that we have

Theorem 2.1 *In a smooth evolution, the variation of the kinetic energy is equal to the actual work of the acceleration*

$$\mathcal{W}_{acc}(\mathbf{U}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{acc}(\mathbf{U}(\tau))d\tau = \frac{m}{2}(\mathbf{U}(t_2))^2 - \frac{m}{2}(\mathbf{U}(t_1))^2. \tag{2.16}$$

In the non smooth evolution, the work $\mathcal{T}_{acc}(\mathbf{V}, t)$ of the non smooth acceleration force, the discontinuity of the velocity $(\mathbf{U}^+(t) - \mathbf{U}^-(t))$ is

$$\mathcal{T}_{acc}(\mathbf{V}, t) = m(\mathbf{U}^+(t) - \mathbf{U}^-(t)) \cdot L(\mathbf{V}), \tag{2.17}$$

where $L(\mathbf{V})$ is a linear function of velocity $\mathbf{V} \in \mathcal{V}$. Because we want to keep Theorem 2.1, we choose

$$L(\mathbf{V}) = \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \tag{2.18}$$

and have

$$\mathcal{T}_{acc}(\mathbf{V}, t) = m(\mathbf{U}^+(t) - \mathbf{U}^-(t)) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \tag{2.19}$$

with theorem

Theorem 2.2 *In a non smooth evolution, the variation of the kinetic energy is equal to the actual work of the acceleration*

$$\mathcal{T}_{acc}(\mathbf{U}, t) = \frac{m}{2}(\mathbf{U}^+(t))^2 - \frac{m}{2}(\mathbf{U}^-(t))^2. \tag{2.20}$$

The work of the acceleration with an atomic part at time t is

$$\mathcal{W}_{acc}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{acc}(\mathbf{V}(\tau))d\tau + \mathcal{T}_{acc}(\mathbf{V}, t) \tag{2.21}$$

$$= \int_{t_1}^{t_2} m \frac{d\mathbf{U}}{dt} \cdot \mathbf{V} d\tau + m(\mathbf{U}^+(t) - \mathbf{U}^-(t)) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}. \qquad (2.22)$$

The choice of the non smooth part of the virtual work of the acceleration forces is based on the wish to have its actual value to be equal to the variation of the kinetic energy.

2.4.1.1 The Theorem of the Kinetic Energy

The actual work of the acceleration forces, $\mathcal{W}_{acc}(\mathbf{U}, t_1, t_2)$ satisfies the theorem of kinetic energy, the théorème de l'énergie cinétique in French and il teorema dell'energia cinetica in Italian.

Theorem 2.3 *In an evolution, the variation of the kinetic energy is equal to the sum of the actual, internal and external works*

$$\frac{m}{2}(\mathbf{U}^+(t))^2 - \frac{m}{2}(\mathbf{U}^-(t))^2 = \mathcal{W}_{int}(\mathbf{U}, t_1, t_2) + \mathcal{W}_{ext}(\mathbf{U}, t_1, t_2). \qquad (2.23)$$

Proof It is a direct application of Theorems 2.1 and 2.2. □

2.4.2 The Work of the External Forces

To define this work let us go back to the soccer field where a player kicks the ball which is flying. The external action of the player modifies instantaneously the ball velocity. The player has applied an instantaneous work. An other interesting observation is to look at a ball of a pin-ball machine hitting a bumper and getting an impulse from an electrical device which modifies instantaneously its velocity. Considering the system to be the ball. The system has received an external work which can be measured (at least we may think of such a measure) through the electrical energy consumption. We conclude that the work of the external forces we consider has jumps: it is a bounded variation function of time. Its time derivative has a density with respect to the Lebesgue measure and a density with respect to the atomic measure. The density with respect to the Lebesgue measure is the classical power we have defined in the previous section. Based on this idea, we choose the work of the external forces to be

$$\mathcal{W}_{ext}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{ext}(\mathbf{V}(\tau)) d\tau + \mathcal{T}_{ext}(\mathbf{V}, t) \qquad (2.24)$$

$$= \int_{t_1}^{t_2} \mathbf{f}^{ext}(\tau) \cdot \mathbf{V}(\tau) d\tau + \mathbf{P}^{ext}(t) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \qquad (2.25)$$

with

$$\mathcal{P}_{ext}(\mathbf{V}(\boldsymbol{\tau})) = \mathbf{f}^{ext}(\boldsymbol{\tau}) \cdot \mathbf{V}(\boldsymbol{\tau})), \tag{2.26}$$

$$\mathcal{T}_{ext}(\mathbf{V}, t) = \mathbf{P}^{ext}(t) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \tag{2.27}$$

where $\mathbf{f}^{ext}$ is the external force, for instance the gravity, and $\mathbf{P}^{ext}(t)$ the external percussion, the kick of the soccer player in the example, applied to the point.

Remark 2.2 The work of the external percussion may be

$$\mathcal{T}_{ext}(\mathbf{V}, t) = \mathbf{P}^{ext+}(t) \cdot \mathbf{V}^+(t) + \mathbf{P}^{ext-}(t) \cdot \mathbf{V}^-(t). \tag{2.28}$$

In fact, our choice is not restrictive, [18–20]. It has the advantage of being simple.

2.4.3 The Work of the Internal Forces

The virtual work of the internal force is a linear function of the virtual velocities of deformation which is null for any rigid system velocities. Because the plane is immobile, the rigid system velocities are $\mathbf{V}^+ = \mathbf{V}^- = 0$. The work of the internal forces has to satisfy the Galilean relativity: the work of the internal force has to be null for any constant translation velocity of the system. Let us note that this property is equivalent to the internal work is null in any rigid body motion. A rigid body motion is such that the distance of the material points remains constant, thus in this situation the position of the points remain constant (remember the plane is immobile). The quantity which measures the evolution has to be null in such a motion. The distance of the material points of the system does not change. Because, the plane is immobile the only rigid body motion has a null velocity. Because the work of the internal forces is a linear function of the velocity, it is null for any rigid body motion. The work of the internal forces is a linear function of the velocity of deformation. Our choice is

$$\mathcal{W}_{int}(\mathbf{V}, t_1, t_2) = \int_{t_1}^{t_2} \mathcal{P}_{int}(\mathbf{V}(\boldsymbol{\tau}))d\tau + \mathcal{T}_{int}(\mathbf{V}, t) \tag{2.29}$$

$$= -\int_{t_1}^{t_2} \mathbf{R}^{int}(\boldsymbol{\tau}) \cdot \mathbf{V}(\boldsymbol{\tau})d\tau - \mathbf{P}^{int}(t) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \tag{2.30}$$

with

$$\mathcal{P}_{int}(\mathbf{V}(\boldsymbol{\tau})) = -\mathbf{R}^{int}(\boldsymbol{\tau}) \cdot \mathbf{V}(\boldsymbol{\tau}), \tag{2.31}$$

$$\mathcal{T}_{int}(\mathbf{V}, t) = -\mathbf{P}^{int}(t) \cdot \frac{\mathbf{V}^+(t) + \mathbf{V}^-(t)}{2}, \tag{2.32}$$

Internal force $\mathbf{R}^{int}$ is a classical force which intervenes in smooth evolution, when the point is sliding on the plane or if there are at a distance interactions between the plane and the point, for instance if the point is tight to the plane by a long elastic string. Internal percussion $\mathbf{P}^{int}$ intervenes when the point collides with the plane.

Remark 2.3 The work of the internal percussion may be

$$\mathcal{T}_{int}(\mathbf{V}, t) = -\mathbf{P}^{int+}(t) \cdot \mathbf{V}^+(t) - \mathbf{P}^{int-}(t) \cdot \mathbf{V}^-(t). \tag{2.33}$$

In fact, this choice does not provide new opportunities, [18–20]. As said in Remark 2.2, it has the advantage of being simple. In case no external percussion is applied, the equation of motion, (2.41) down below, gives

$$\mathbf{P}^{int}(t) = \frac{\mathbf{P}^{int+}(t) + \mathbf{P}^{int-}(t)}{2}. \tag{2.34}$$

It is the choice of the works of the acceleration forces which is the important and leading choice.

2.5 The Equations of Motion

We derive the equations of motion using the assumption that time interval $]t_1, t[$ contains only one time t, $t_1 < t < t_2$, where two of the densities, i.e., the works, of the atomic measure are not null. By choosing virtual velocity with compact support in interval $]t_1, t[$, we get

$$\forall \mathbf{V} \in \mathcal{V}, \quad (2.35)$$

$$\int_{t_1}^{t} m \frac{d\mathbf{U}}{dt}(\tau) \cdot \mathbf{V}(\tau) d\tau = -\int_{t_1}^{t} \mathbf{R}^{int}(\tau) \cdot \mathbf{V}(\tau) d\tau + \int_{t_1}^{t} \mathbf{f}^{ext}(\tau) \cdot \mathbf{V}(\tau) d\tau. \quad (2.36)$$

The fondamental lemma of the variation calculus gives

$$m \frac{d\mathbf{U}}{dt} = -\mathbf{R}^{int} + \mathbf{f}^{ext}, \; a.e. \; in \;]t_1, t[. \tag{2.37}$$

Remark 2.4 a.e. means almost everywhere or almost always in this context.

This relationship is also valid almost everywhere in interval $]t, t_2[$. It results it is valid almost everywhere in whole interval $]t_1, t_2[$.

Then the principle becomes

$$\forall \mathbf{V} \in \mathcal{V}, \tag{2.38}$$

$$m\left[\mathbf{U}(t)\right] \cdot \frac{\mathbf{V}^{+}(t) + \mathbf{V}^{-}(t)}{2} \tag{2.39}$$

$$= -\mathbf{P}^{int}(t) \cdot \frac{\mathbf{V}^{+}(t) + \mathbf{V}^{-}(t)}{2} + \mathbf{P}^{ext}(t) \cdot \frac{\mathbf{V}^{+}(t) + \mathbf{V}^{-}(t)}{2}. \tag{2.40}$$

It gives immediately

$$m\left[\mathbf{U}(t)\right] = -\mathbf{P}^{int}(t) + \mathbf{P}^{ext}(t). \tag{2.41}$$

Previous relationship shows that at time t at least two of its quantities are non null Because at times different from t, its three quantities are null, relationship (2.41) is satisfied at any time

$$\forall t \in \,]t_1, t_2[\,, \ m\left[\mathbf{U}(t)\right] = -\mathbf{P}^{int}(t) + \mathbf{P}^{ext}(t). \tag{2.42}$$

Remark 2.5 For a soccer ball, previous relationship is

1. when $m\left[\mathbf{U}(t)\right]$ and $\mathbf{P}^{int}(t)$ are non null with $\mathbf{P}^{ext}(t) = 0$, there is a collision: soccer ball hits the field;
2. when $m\left[\mathbf{U}(t)\right]$ and $\mathbf{P}^{ext}(t)$ are non null with $\mathbf{P}^{int}(t) = 0$, a player kicks the ball which is flying;
3. when $\mathbf{P}^{int}(t)$ and $\mathbf{P}^{ext}(t)$ are non null with $m\left[\mathbf{U}(t)\right] = 0$, a player kicks the ball vertically downward. Nothing occurs, the percussion applied by the foot of the player is equilibrated by the reaction of the field. Note that because it is impossible that the ball surges from the field, we have $\mathbf{U}^{-}(t) \cdot \mathbf{N} \leq 0$, where $\mathbf{N}$ is the upward normal vector to the field. Because the ball cannot interpenetrate the field, we have $\mathbf{U}^{+}(t) \cdot \mathbf{N} \geq 0$. Because the discontinuity of velocity is null, we get $\mathbf{U}^{-}(t) \cdot \mathbf{N} = \mathbf{U}^{+}(t) \cdot \mathbf{N} = 0$. Thus the ball is either at rest or sliding on the field;
4. when the three quantities of relationship (2.37) are non null there is a concomitant collision with the field and a kick by a player.

The other example of external percussion concomitant to a collision has been mentioned by Jean Jacques Moreau: in an electrical pin ball machine an impulse due to an electrical device is applied to a steel ball whenever it collides some obstacle called bumper.

A more general derivation of the equation of motion from the principle of virtual power is given in the book [19].

2.5.1 Properties of the Equations of Motion

The two Eqs. (2.37) and (2.42) describe:

- the smooth evolution, the free flight and the sliding of the point on the plane with Eq. (2.37) where the Lebesgue measure intervenes;
- the non smooth evolution or the collisions, with Eq. (2.41) where the atomic Dirac measure intervenes. This equation may be read two way: if there is a discontinuity of velocity, there is either an internal or an external percussion and if there is a percussion either external or internal, there is a discontinuity of velocity.

When Eq. (2.37) is no longer valid, quantities of Eq. (2.41) are non null. This is the case either if there is a kinematic constraint: the point hits the plane, or if there is a constraint related to the force, a sthenic constraint. This unexpected and not very common cause of discontinuity of velocity, an example of which is known as the Painlevé's paradox, [24, 26, 33–35], is not a paradox and it is taken into account by our theory, [19, 22].

Remark 2.6 Equations of motion (2.37) and (2.42) may be understood as a unique equality of measures

$$md\mathbf{U} = -\mathbf{Z}^{int} + \mathbf{Z}^{ext}. \tag{2.43}$$

Differential measure $d\mathbf{U}$ is defined by

$$\langle d\mathbf{U}, \boldsymbol{\varphi} \rangle = \mathcal{T}_{acc}(t_1, t_2, \boldsymbol{\varphi}), \tag{2.44}$$

where $\boldsymbol{\varphi}$ is a continuous virtual velocity with compact support. This measure satisfies

$$\langle d\mathbf{U}, \boldsymbol{\varphi} \rangle = -\int_{t_1}^{t_2} \mathbf{U} \frac{d\boldsymbol{\varphi}}{dt} d\tau, \tag{2.45}$$

if $\boldsymbol{\varphi}$ is smooth enough. Measures $\mathbf{Z}^{int}$, the internal forces, and $\mathbf{Z}^{ext}$, the externals forces, are defined by

$$\langle \mathbf{Z}^{int}, \boldsymbol{\varphi} \rangle = \int_{t_1}^{t_2} \mathbf{R}^{int} \boldsymbol{\varphi} d\tau + \mathbf{P}^{int}(t)\boldsymbol{\varphi}(t), \tag{2.46}$$

$$\langle \mathbf{Z}^{ext}, \boldsymbol{\varphi} \rangle = \int_{t_1}^{t_2} \mathbf{f} \boldsymbol{\varphi} d\tau + \mathbf{P}^{ext}(t)\boldsymbol{\varphi}(t), \tag{2.47}$$

[29, 31]. In [31], Jean Jacques Moreau describe these equations with differential inclusions. Relationship (2.43) is an equality in the dual space of the space of bounded variations functions. An existence mathematical result of solutions of (2.43) is given in [9].

2.6 The Laws of Thermodynamics

For the sake of simplicity, we assume the point and plane have the same constant temperature T. Thus any heat which is produced is expelled toward the exterior. The case where the temperatures of the colliding solids evolve is investigated in Chap. 3 and in [6, 15, 19, 28].

2.6.1 The First Law

The energy balance is

$$\mathcal{E}^-(t_2) - \mathcal{E}^+(t_1) + \mathcal{K}^-(t_2) - \mathcal{K}^+(t_1) = \mathcal{T}_{ext}(t_1, t_2, \mathbf{U}) + \mathcal{C}(t_1, t_2), \quad (2.48)$$

where $\mathcal{E}$ is the internal energy of the system and $\mathcal{C}(t_1, t_2)$ is the amount of heat received by the system between times t_1 and t_2

$$\mathcal{C}(t_1, t_2) = \int_{t_1}^{t_2} TQ(\tau)d\tau + TB(t), \quad (2.49)$$

where $TQ(\tau)$ is the Lebesgue density of heat received at temperature T and $TB(t)$ is the heat received instantaneously at temperature T at collision time t. It is possible to measure those heat quantities, (see Chap. 3). If internal energy depends only on temperature, $-TQ(\tau)$ is the heat resulting from friction and $-TB(t)$ is the heat burst resulting from the collision. We have

$$\mathcal{E}^-(t_2) - \mathcal{E}^+(t_1) = \int_{t_1}^{t_2} \frac{d\mathcal{E}}{dt}d\tau + [\mathcal{E}(t)], \quad (2.50)$$

Theorem of kinetic energy 2.3

$$\mathcal{T}_{acc}(t_1, t_2, \mathbf{U}) = \mathcal{T}_{int}(t_1, t_2, \mathbf{U}) + \mathcal{T}_{ext}(t_1, t_2, \mathbf{U}), \quad (2.51)$$

gives with the energy balance

$$\mathcal{E}^-(t_2) - \mathcal{E}^+(t_1) = -\mathcal{T}_{int}(t_1, t_2, \mathbf{U}) + \mathcal{C}(t_1, t_2). \quad (2.52)$$

This relationship satisfied for any times t_1 and t_2, gives

$$\frac{d\mathcal{E}}{dt} = \mathbf{R}^{int} \cdot \mathbf{U} + TQ, \; a.e. \; in \;]t_1, t_2[, \quad (2.53)$$

and

$$[\mathcal{E}(t)] = \mathbf{P}^{int}(t) \cdot \frac{\mathbf{U}^+(t) + \mathbf{U}^-(t)}{2} + TB(t), \ \forall t \in \,]t_1, t_2[\, . \qquad (2.54)$$

The last relationship describes the effects of the internal mechanical heat burst and of a possible external heat burst. Its quantities are null when there is neither collision, i.e., neither internal heat burst, nor external heat burst. If the internal energy depends only on temperature T, heat produced by a collision is expelled toward the external without modifying the temperature in agreement with our assumption. In a smooth evolution, the heat produced by friction is also immediately expelled toward the exterior. The case where the temperatures are no longer constant is investigated in [19].

2.6.2 The Second Law

The second law is

$$\mathcal{S}^-(t_2) - \mathcal{S}^+(t_1) \geq \int_{t_1}^{t_2} Q(\tau)d\tau + B(t), \qquad (2.55)$$

where $\mathcal{S}$ is the entropy of the system and t is the collision time. We have

$$\mathcal{S}^-(t_2) - \mathcal{S}^+(t_1) = \int_{t_1}^{t_2} \frac{d\mathcal{S}}{dt}d\tau + [\mathcal{S}(t)], \qquad (2.56)$$

The law which is satisfied for any times t_1 and t_2 gives

$$\frac{d\mathcal{S}}{dt} \geq Q, \ a.e. \ in \ ]t_1, t_2[\, , \qquad (2.57)$$

and

$$[\mathcal{S}(t)] \geq B(t), \quad \forall t \in \,]t_1, t_2[\, . \qquad (2.58)$$

If the entropy depends only on temperature T and if there is no external burst of heat at times different from time t, the elements of this relationship are null at times different from time t.

2.6.2.1 A Useful Inequality

Let us recall that the free energy is $\Psi = \mathcal{E} - T\mathcal{S}$. Combining relationships (2.53) and (2.57), we get

$$\frac{d\Psi}{dt} \leq \mathbf{R}^{int} \cdot \mathbf{U}, \ a.e. \ in \ ]t_1, t_2[\, , \qquad (2.59)$$

and combining relationships (2.54) and (2.58)

$$[\Psi(t)] \leq \mathbf{P}^{int}(t) \cdot \frac{\mathbf{U}^+(t) + \mathbf{U}^-(t)}{2}, \quad \forall t \in \,]t_1, t_2[\,. \tag{2.60}$$

If the free energy depends only on temperature T and if there is no external burst of heat at times different from time t, the elements of this relationship are null at times different from time t.

2.7 The Constitutive Laws

The two internal forces $\mathbf{R}^{int}$ and $\mathbf{P}^{int}(t)$ result from theoretical choices and experiments. Theoretical choices are controlled by relationships (2.59) and (2.60). They are a guide for the general expression of the internal forces which depend on some parameters. Experiments can be used to quantify them. In any case it is always possible to have an idea of their value. The theoretical results given by the predictive theory is compared to practical results. If we do not get what is expected and useful for the engineering project, adaptation are to be made. The simplest is to change the expression of the internal forces. The more sophisticated is to change some of the assumptions of the predictive theory. For instance to assume the solid, the soccer ball, is no longer represented by a point but it is represented either by a rigid solid or by a deformable solid and use the theories given either in Chaps. 4, 5, 6 or in 7. Let us recall a predictive theory is subjective: it is the engineer who chooses the sophistication of the theory he needs. In this book, we keep the instantaneity assumption for the different collisions but the other assumptions may be changed.

Internal forces are split into non dissipative or elastic forces and dissipative forces

$$\mathbf{R}^{int} = \mathbf{R}^{inte} + \mathbf{R}^{intd}, \; \mathbf{P}^{int} = \mathbf{P}^{inte} + \mathbf{P}^{intd}. \tag{2.61}$$

The definition of the internal forces is based on this splitting and on the structure of relationships (2.59) and (2.60): non dissipative forces are defined with the system free energy and the dissipative forces are defined either with a pseudo-potential of dissipation or with a dissipation function as for the Coulomb's friction law. This method described further down is not very demanding and offers innumerable openings with the advantage of satisfying automatically the mechanical laws (2.59) and (2.60). Of course, it is possible to follow another way but there are risks and verifications are to be performed. An example of such a situation is the restitution coefficient which is perfect for the collision of two spheres but is misleading in case of collisions of two or many solids (see [5, 7, 19]).

In this chapter, we want to give the basic elements of the collision theory and to show its large scope. For this purpose, we focus on simple constitutive laws, in fact in many cases on linear constitutive laws besides the non interpenetration conditions and thresholds we cannot avoid. We are convince that with those simple laws we

may capture the main physical properties. The fine and sophisticated constitutive laws will be easily integrated and adapted to each particular system.

2.7.1 The Free Energy and the Non Dissipative Forces

The free energy depends on the system state variables and defines the non dissipative or elastic forces. In the present situation, we may choose the position of the point with respect to the plane $\mathbf{x}(t)$ as state variable. It intervenes when there are at a distance interactions between the point and the plane, for instance when the ball is connected to a point of the plane by a long elastic wire. In this case the free energy depends on $\mathbf{x}(t) - \mathbf{x}_o$ where $\mathbf{x}_o$ is the elastic wire fixation point we choose as origin, [21]. Because the free energy accounts for the static properties, it has to take into account that the point is above the plane:

$$d(\mathbf{x}(t)) = \mathbf{x}(t) \cdot \mathbf{N} \geq 0, \tag{2.62}$$

where $\mathbf{N}$ is the upward normal vector to the plane. We forget the at a distance interactions and choose simple free energy Ψ

$$\Psi(\mathbf{x}) = I_+(d(\mathbf{x})), \tag{2.63}$$

where I_+ is the indicator function of $\mathbb{R}^+$ (see [12, 13, 32] or the appendix). Non dissipative forces are the generalized derivatives of the free energy

$$\mathbf{R}^{inte} \in \partial I_+(d(\mathbf{x}))\mathbf{N} = \partial \Psi(\mathbf{x}), \tag{2.64}$$

$$\mathbf{P}^{inte} = 0, \tag{2.65}$$

where ∂I_+ is the subdifferential set of function I_+, (see [12, 13, 32] or Sect. A.2.1 and A.4 of the appendix). Non dissipative force $-\mathbf{R}^{inte}$ is the non interpenetration reaction force of the plane: the action of the plane on the point. It is null if the point is not in contact with the plane. It is directed upward when the point slides on the plane. The non dissipative percussion $\mathbf{P}^{inte}$ is null because $[\Psi(t)] = 0$. In a collision, the non interpenetration is not ensured by a non dissipative percussion. It is to be ensured by a dissipative percussion. This means that the non interpenetration reaction works. This reaction cannot be workless or perfect (liaison parfaite in French, vincolo perfetto in Italian). The dissipative character of collisions appears. Some properties we are accustomed to disappear. We mention them in the sequel.

Remark 2.7 Adjective elastic is used with its abstract meaning equivalent to non dissipative.

2.7.2 The Dissipative Forces

Relationships (2.59) and (2.60) show that it is wise to assume that the internal forces depend not only on d, as we already know, but also on $\mathbf{U}$ and $\left(\mathbf{U}^+(t) + \mathbf{U}^-(t)\right)/2$ which describe how the form of the system evolves.

2.7.2.1 The Dissipative Forces Defined with a Pseudo-potential of Dissipation

A pseudo-potential of dissipation $\Phi(X, \chi)$, introduced by Jean Jacques Moreau, is a convex function of X, with value in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, positive and null at the origin, $\Phi(0, \chi) = 0$, [23, 27, 30]. At any time $\chi(t)$ depends on the past but not on X. Internal dissipative forces are given by the subdifferential set with respect to X of $\Phi(X, \chi)$ (see [12, 13, 32] or Sect. A.4 of the appendix).

In the present situation, X is either $\mathbf{U}$ or $\left(\mathbf{U}^+(t) + \mathbf{U}^-(t)\right)/2$. The internal dissipative forces are either $\mathbf{R}^{intd}$ or $\mathbf{P}^{intd}$

$$\mathbf{R}^{intd} \in \partial \tilde{\varphi}(\mathbf{U}, \chi), \ \ \mathbf{P}^{intd} \in \partial \tilde{\Phi}(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}, \chi). \tag{2.66}$$

Assuming there is no at a distance actions, the dissipative forces are null when the point is not in contact with the plane. Thus we let d be a χ quantity and choose pseudo-potentials of dissipation null for $d > 0$

$$\tilde{\varphi}(\mathbf{U}, d) = \begin{cases} 0, \ if \ d > 0, \\ \varphi(\mathbf{U}), \ if \ d = 0, \end{cases} \tag{2.67}$$

for smooth evolution, and

$$\tilde{\Phi}(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}, d, \mathbf{U}^- \cdot \mathbf{N}) = \begin{cases} 0, \ if \ d > 0, \\ \Phi(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}) + I_+(\mathbf{U}^+ \cdot \mathbf{N}), \ if \ d = 0, \end{cases} \tag{2.68}$$

for the collisions; i.e., for the non smooth evolutions. We have added $\mathbf{U}^- \cdot \mathbf{N}$ in quantities χ for the non interpenetration condition. Because when $d = 0$, non interpenetration is insured if normal velocity after collision is directed upward, $\mathbf{U}^+ \cdot \mathbf{N} \geq 0$. This is why indicator function

$$I_+(\mathbf{U}^+ \cdot \mathbf{N}) = I_+(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2} \cdot \mathbf{N} - \frac{\mathbf{U}^-}{2} \cdot \mathbf{N}), \tag{2.69}$$

intervenes. Clearly it is a function of $(\mathbf{U}^+ + \mathbf{U}^-)/2$ and $\chi = (d, \mathbf{U}^- \cdot \mathbf{N})$. For $X = 0$, $X = (\mathbf{U}^+ + \mathbf{U}^-)/2 = 0$, indicator function

$$I_+(0 - \frac{\mathbf{U}^-}{2} \cdot \mathbf{N}), \tag{2.70}$$

is actually null because normal velocity before collision, $\mathbf{U}^- \cdot \mathbf{N} \leq 0$, is non positive.

Remark 2.8 Situation where $\mathbf{U}^- \cdot \mathbf{N} > 0$, is physically impossible because the point cannot surge from the plane Moreover $\tilde{\Phi}$ is no longer a pseudo-potential of dissipation because $\tilde{\Phi}(0, d, \mathbf{U}^- \cdot \mathbf{N}) = +\infty$ for $d = 0$.

Dissipative forces are defined by

$$\mathbf{R}^{intd} \in \partial\tilde{\varphi}(\mathbf{U}, d) = \begin{cases} 0, \ if \ d > 0, \\ \partial\varphi(\mathbf{U}), \ if \ d = 0, \end{cases} \tag{2.71}$$

$$\mathbf{P}^{intd} = \mathbf{P}^{int} \in \partial\tilde{\Phi}(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}, d, \mathbf{U}^- \cdot \mathbf{N}) \tag{2.72}$$

$$= \begin{cases} 0, \ if \ d > 0, \\ \partial\Phi(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}) + \partial I_+(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2} \cdot \mathbf{N} - \frac{\mathbf{U}^-}{2} \cdot \mathbf{N})\mathbf{N}, \ if \ d = 0. \end{cases} \tag{2.73}$$

Remark 2.9 Subdifferential set $\partial\tilde{\Phi}(X, \chi)$ is computed with respect to $X \in \mathbb{R}^3$ with the usual scalar product in $\mathbb{R}^3$.

These forces are actually null when there is no contact. Non interpenetration percussion

$$P^{reac}\mathbf{N} \in \partial I_+(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2} \cdot \mathbf{N} - \frac{\mathbf{U}^-}{2} \cdot \mathbf{N})\mathbf{N} = \partial I_+(\mathbf{U}^+ \cdot \mathbf{N})\mathbf{N}, \tag{2.74}$$

is normal to the plane. It is active only if $\mathbf{U}^+ \cdot \mathbf{N} = 0$, i.e., if the future normal velocity is null meaning that there is a risk of interpenetration. We remark that the non interpenetration reaction appears when the other mechanical effects are not sufficient to prevent interpenetration. This percussion works with work

$$P^{reac}\mathbf{N} \cdot \frac{\mathbf{U}^+ + \mathbf{U}^-}{2} = P^{reac}\frac{\mathbf{U}^- \cdot \mathbf{N}}{2} \geq 0. \tag{2.75}$$

We denote in the sequel

$$\mathbf{P}^{int} = \mathbf{P}^d + P^{reac}\mathbf{N} = \mathbf{P}^d + \mathbf{P}^{reac}. \tag{2.76}$$

2.7.2.2 Internal Forces Defined with a Dissipation Function

A dissipation function is a real function $\Phi(X, \chi)$ where at any time $\chi(t)$ depends on the past and possibly on X. This function is differentiable or has a generalized derivative with respect to X, in a domain which contains at least the origin, $X = 0$. It satisfies

$$\forall Y, \; grad_X \Phi(Y, \chi) \cdot Y \geq 0. \tag{2.77}$$

The dissipative forces are given by the derivative of $\Phi(X, \chi)$ with respect to X.

2.7.2.3   A Property of Dissipative Internal Forces Defined Either with a Pseudo-potential of Dissipation or with a Dissipation Function

An advantage of defining the dissipative forces either with a pseudo-potential of dissipation or with a function of dissipation is that relationships (2.59) and (2.60) are satisfied. Moreover experimental results fit very often with pseudo-potential of dissipation. There are some cases but important cases where experimental results are more easily taken into account by a dissipation function.

Theorem 2.4 *If dissipative internal forces are defined either by a pseudo-potential of dissipation or by a dissipation function, relationships (2.59) et (2.60) are satisfied.*

Proof In case of a pseudo-potential, the proof is classical [18, 27]. In case of a dissipation function, the proof relies on relationship (2.77).

Examples are given in following section. Let us note that a pseudo-potential of dissipation is a dissipation function but the converse is not true.

The presentation of the predictive theory of the motion of a point above a plane is achieved. It remains to identify the constitutive laws with experiments giving hints to choose either a pseudo-potential of dissipation or a function of dissipations.

2.8   Examples of Collisions with Internal Forces Defined with a Pseudo-potential of Dissipation

In these examples, we decouple the collision normal to the plane phenomena from the tangential phenomena.

We define the normal and tangential percussions

$$\mathbf{P}^{int} = P_N \mathbf{N} + \mathbf{P}_T, \; P_N = \mathbf{P}^{int} \cdot \mathbf{N}, \tag{2.78}$$

with the normal and tangential velocities

$$\mathbf{U} = U_N \mathbf{N} + \mathbf{U}_T, \; U_N = \mathbf{U} \cdot \mathbf{N}. \tag{2.79}$$

We split the pseudo-potential into a part giving the normal percussion depending on the normal velocity and a part giving the tangential percussion depending on the tangential velocity

$$\Phi(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}) + I_+(\mathbf{U}^+ \cdot \mathbf{N}) = \Phi_T(\frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2}) + I_+(\mathbf{U}^+ \cdot \mathbf{N}) + \Phi_N(\frac{U_N^+ + U_N^-}{2}).$$

(2.80)

2.8.1   First Example

The normal and tangential evolutions are decoupled. And we choose $\Phi_T = 0$ and $\Phi_N(X) = kX^2$ quadratic functions giving linear internal forces (besides the non interpenetration reaction). Moreover $\Phi_T = 0$, gives a tangential percussion null We have

$$P_N \in \frac{\partial \Phi_N}{\partial X}(\frac{U_N^+ + U_N^-}{2}) + \partial I_+(\frac{U_N^+ + U_N^-}{2} - \frac{U_N^-}{2})$$

(2.81)

$$= k(U_N^+ + U_N^-) + \partial I_+(U_N^+),$$

(2.82)

$$P_N^d = k(U_N^+ + U_N^-), \ P_N^{reacd} \in \partial I_+(U_N^+),$$

(2.83)

$$\mathbf{P}_T = 0.$$

(2.84)

2.8.1.1   A Collision

Let us consider le point falling on the plane. Equations of motion and constitutive laws give

$$m\,[\mathbf{U}_T] = -\mathbf{P}_T = 0,$$

(2.85)

$$m\,[U_N] = -P_N \in -k(U_N^+ + U_N^-) - \partial I_+(U_N^+).$$

(2.86)

First equation proves that the tangential velocity is continuous: it is not modified by the collision. Second equation is easy to solve

- if $m < k$, i.e., if the solid is light or the dissipation is important in collisions, the point bounces with velocity

$$U_N^+ = \frac{m - k}{m + k} U_N^-,$$

(2.87)

and non interpenetration reaction percussion is null

$$P_N^{reacd} = 0 \ ;$$

(2.88)

- if $m > k$, if the solid is heavy or if the dissipation is not important in collisions, the point does not bounce. It remains in contact with the plane and slides

$$U_N^+ = 0,$$

(2.89)

and non interpenetration reaction percussion is

$$P_N^{reacd} = (m - k)U_N^- \leq 0, \tag{2.90}$$

the dissipative percussion being

$$P_N^d = kU_N^-. \tag{2.91}$$

Dissipated energy is

$$\mathbf{P}^{int} \cdot \frac{\mathbf{U}^+ + \mathbf{U}^-}{2} = (P_N^d + P_N^{reacd})\frac{U_N^+ + U_N^-}{2} = \tag{2.92}$$

$$= \frac{k}{2}(\frac{2m}{m+k})^2 \left(U_N^-\right)^2, \text{ if } m < k, \tag{2.93}$$

$$= \frac{k}{2} \left(U_N^-\right)^2 + \frac{m-k}{2} \left(U_N^-\right)^2 = \frac{m}{2} \left(U_N^-\right)^2, \text{ if } m > k. \tag{2.94}$$

We remark again that the non interpenetration reaction percussion is dissipative and it intervenes only if interpenetration is at risk, i.e., if the point does not bounce. The energy dissipated by this reaction is maximal for $k = 0$. It decreases to 0 when k increases up to m. We may note that if k is small, dissipation is mostly due to the reaction percussion and that if k is large the whole dissipation is due to the dissipative percussion. This is in agreement with our vocabulary. Let us also note another property which may be surprising, when dissipative parameter k is infinite, dissipation is null. But when k tends toward infinity, deformation velocity $U_N^+ + U_N^-$ tends to 0. The system tends to become rigid! Thus it is correct that the dissipation tends to 0, the dissipation in a rigid or undeformable system. The pseudo-potential tends to $I_0(U_N^+ + U_N^-)$ where I_0 is the indicator function of the origin. If we assume the same type of constitutive law for the smooth evolution pseudo-potential $I_0(U_N)$ implies rigidity for the vertical displacement. As soon as the point gets into contact with the plane, it is irremediably fixed to it. It remains free to slide on the plane without dissipation with the tangential constitutive law we have chosen. Thus it is good that the dissipated energy tends to 0 when k tends to ∞.

2.8.2   Second Example

We may choose sophisticated pseudo-potentials. For instance

$$\Phi_N \left(\frac{U_N^+ + U_N^-}{2}\right) = k_0 np \left(\frac{U_N^+ + U_N^- + 2U_{\lim}}{2}\right) \tag{2.95}$$

$$+k \left(np \left(\frac{U_N^+ + U_N^- + 2U_{\lim}}{2}\right)\right)^2, \tag{2.96}$$

where we assume $(k_0/m) \geq U_{\lim} \geq 0$ and $np(x)$ denotes the negative part function see (A.2.1)

$$np(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ 0, & \text{if } x \geq 0. \end{cases} \tag{2.97}$$

Remark 2.10 We have

$$\left| \left(np(\frac{U_N^+ + U_N^- + 2U_{\lim}}{2}) \right) \right| = np \left(\frac{U_N^+ + U_N^- + 2U_{\lim}}{2} \right). \tag{2.98}$$

The subdifferential set of $np(x)$ is

$$G(x) = \begin{cases} 0, & \text{if } x > 0, \\ [-1, 0], & \text{siif } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{2.99}$$

Graph $G(x) = -H(-x)$ where graph $H(x)$ is the subdifferential set of the positive part function defined by formula (A.2.1). Constitutive law is

$$P_N \in k_0 G(U_N^+ + U_N^- + 2U_{\lim}) - k \left(np(U_N^+ + U_N^- + 2U_{\lim}) \right) + \partial I_+(U_N^+). \tag{2.100}$$

We let $X = U_N^+ + U_N^- + 2U_{\lim}$

$$mX + k_0 G(X) - knp(X) + \partial I_+(X - U_N^- - 2U_{\lim}) \ni 2m \left(U_N^- + U_{\lim} \right). \tag{2.101}$$

In the case $m - k \geq 0$, i.e., if the point is heavy or the dissipation is not important in collisions we have, (remember we have assumed $U_{\lim} \leq k_0/(2m)$)

$$U_N^+ = 0, \text{ if } 0 \leq -U_N^- \leq 2U_{\lim} \tag{2.102}$$

$$U_N^+ = -U_N^- - 2U_{\lim}, \text{ if } 2U_{\lim} \leq -U_N^- \leq U_{\lim} + \frac{k_0}{2m}, \tag{2.103}$$

$$U_N^+ = \frac{k_0}{m+k} + \frac{m-k}{m+k} U_N^- - \frac{2k}{m+k} U_{\lim}, \tag{2.104}$$

$$\text{if } U_{\lim} + \frac{k_0}{2m} \leq -U_N^- \leq \frac{k_0}{m-k} - \frac{2k}{m-k} U_{\lim}, \tag{2.105}$$

$$U_N^+ = 0, \text{ if } -U_N^- \geq \frac{k_0}{m-k} - \frac{2k}{m-k} U_{\lim}. \tag{2.106}$$

Function $-U_N^- \rightarrow U_N^+$ shown on Fig. 2.1, illustrates what happened when a gangsters' car, pursued by a police one, drives on speed bumps (the car is shown by a point):

- when the car travels at low speed, it takes off on the first speed bump and falls back at low vertical speed. $0 \leq -U_N^- \leq 2U_{\lim}$. The car does not bounce back, $U_N^+ = 0$, the gangsters haven't found out that the police follows them;
- on the next speed bump, the gangsters understand the police is after them. They try to escape and increase their speed. The car takes off and falls back at a vertical speed $-U_N^-$ satisfying

$$2U_{\lim} \leq -U_N^- \leq \frac{k_0}{m-k} - \frac{2k}{m-k}U_{\lim}. \tag{2.107}$$

The car bounces as it falls back, it keeps travelling, potentially bouncing back several times while accelerating more and more;

- on the following speed bump, the gangsters' car is at a breakneck speed, it flies off and falls back at a high vertical speed,

$$-U_N^- \geq \frac{k_0}{m-k} - \frac{2k}{m-k}U_{\lim}. \tag{2.108}$$

As it falls back, the springs and shock absorbers are not sufficient to withstand the impact. They break and thus unable the car to bounce. As such, the car does not bounce, $U_N^+ = 0$ and is not in a state to travel as it is not suspended anymore. The police can now stops the gangsters.

Function U_N^+ versus $-U_N^-$ plotted on Fig. 2.1 shows that the physical behaviours may be non monotonous and sophisticated whereas the constitutive laws are rather simple.
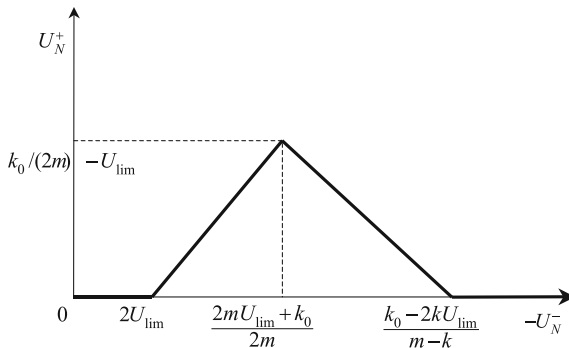
Fig. 2.1 The vertical velocity U_N^+ of a solid after collision versus its velocity $-U_N^-$ before collision. The solid shown by a point, bounces only at medium vertical velocity. At low or high velocity, it does not bounce. This is what happens when a car drives on speed bumps. After taking off on the bump, when falling down with a low vertical velocity, it does not bounce, it bounces when falling down at medium velocity due to its springs and shock absorbers and does not bounce when falling at high velocity, breaking its springs and shock absorbers

2.8.3 Third Example. Interpenetration Is Possible

Let us consider a steel ball falling on a tightened sheet of paper. Depending on its velocity and mass, it can get through the sheet of paper. We choose the following pseudo-potential of dissipation no longer involving the non interpenetration condition (2.68).

$$\tilde{\Phi}(\frac{\mathbf{U}^+ + \mathbf{U}^-}{2}, \frac{\mathbf{U}^-}{2}) = k_0 np \left(\frac{U_N^+ + U_N^-}{2} - \frac{U_N^-}{2}\right) \tag{2.109}$$

$$+k \left(\frac{U_N^+ + U_N^-}{2}\right)^2, \tag{2.110}$$

It satisfies $\tilde{\Phi}(0, \mathbf{U}^-/2) = 0$ because $\mathbf{U}^- \cdot \mathbf{N} < 0$. Internal percussion is

$$P_N = \frac{\partial \tilde{\Phi}}{\partial((\mathbf{U}^+ + \mathbf{U}^-)/2)} \in k \left(U_N^+ + U_N^-\right) + k_0 G \left(U_N^+\right), \tag{2.111}$$

where graph G, subdifferential set of negative part function np, is defined by (2.99). Equation giving normal velocity

$$m(U_N^+ - U_N^-) = -P_N \in -k_0 G \left(U_N^+\right) - k(U_N^+ + U_N^-), \tag{2.112}$$

is solved by letting $X = U_N^+ + U_N^-$. It becomes

$$mX + k_0 G(X - U_N^-) + kX \ni 2mU_N^-. \tag{2.113}$$

Because graph $Y \to mY + k_0 G(Y - U_N^-) + kY$ is monotone, maximal and surjective, Previous equation has one and only one solution which is

1. si $m - k \leq 0$, i.e., either the ball is light or the system is very dissipative, the ball bounces with velocity
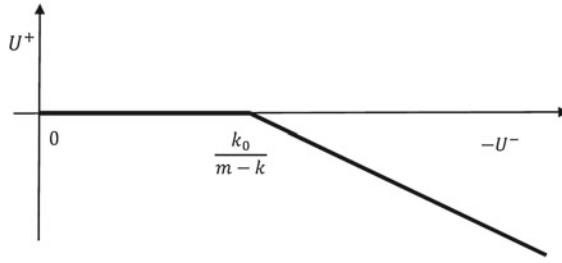
$$U_N^+ = \frac{m - k}{m + k} U_N^- ; \tag{2.114}$$

2. if $m - k \geq 0$, i.e., if the ball is heavy or the system is weakly dissipative,

   a. if falling velocity is low,

$$- U_N^- \leq \frac{k_0}{m - k}, \tag{2.115}$$

   the ball does not bounce. It is stopped by the sheet of paper

$$U_N^+ = 0 ; \tag{2.116}$$

Fig. 2.2 The velocity U^+ of the steel ball versus its velocity $-U^-$ before colliding the sheet of paper. When the falling velocity is low, $-U^- \leq k_0/(m-k)$, the ball is stopped by the sheet of paper. When it is high, $-U^- \geq k_0/(m-k)$, the steel ball transfixes the sheet of paper

b. if falling velocity is large

$$- U_N^- \geq \frac{k_0}{m-k}, \tag{2.117}$$

the ball transfixes the sheet of paper with velocity

$$U_N^+ = \frac{k_0}{m+k} + \frac{m-k}{m+k} U_N^-, \tag{2.118}$$

$$U_N^- < U_N^+ \leq 0. \tag{2.119}$$

The ball is still falling but it is slowed down when getting through the sheet of paper, (Fig. 2.2).

This idea has been used by to predict avalanches of rocks in mountain forests. The trees can be broken by the falling rocks, [11].

Remark 2.11 Quasi-static collision. If we neglect the inertia effects. Collision equation of motion becomes

$$0 = m\,[\mathbf{U}] = -\mathbf{P}^{int} + \mathbf{P}^{ext}. \tag{2.120}$$

When there is no external percussion and the constitutive law is given by pseudo-potential of dissipation

$$I_+(U_N^+) + \Phi_N(\frac{U_N^+ + U_N^-}{2}), \tag{2.121}$$

we have

$$0 \in \partial I_+(U_N^+) + \partial\Phi_N(\frac{U_N^+ + U_N^-}{2}). \tag{2.122}$$

This equation has always solution $U_N^+ + U_N^- = 0$. The point rebounces. No information is given on the tangential motion. We may choose the continuity of the tangential velocity.

2.9 Examples of Dissipative Forces Defined with a Function of Dissipation

The important behaviour described by a function of dissipation is the Coulomb behaviour which intervenes in the smooth motion but also in the collisions. Moreover the two constitutive laws are related, [19].

2.9.1 The Coulomb's Friction Law

The normal and tangential forces are

$$\mathbf{R}^{int} = R_N \mathbf{N} + \mathbf{R}_T, \ \ R_N = \mathbf{R}^{int} \cdot \mathbf{N}. \tag{2.123}$$

The normal and tangential velocities are

$$\mathbf{U} = U_N \mathbf{N} + \mathbf{U}_T, \ \ U_N = \mathbf{U} \cdot \mathbf{N}. \tag{2.124}$$

We choose

$$X = \begin{pmatrix} U_N \\ \mathbf{U}_T \end{pmatrix}, \ \ \chi = -R_N, \tag{2.125}$$

and

$$\Phi(X, \chi) = \begin{pmatrix} 0 \\ -R_N \phi(\mathbf{U}_T) \end{pmatrix}, \tag{2.126}$$

where ϕ is a pseudo-potential of dissipation.

Pseudo-potential $\phi(\mathbf{U}_T) = \mu \|\mathbf{U}_T\|$ where $\mathbf{U}_T$ is the tangential velocity, is often chosen. Function $\Phi(X, \chi)$ is not a pseudo-potential of dissipation because normal reaction $\chi = -R_N$ is not a function of the state quantities, i.e., position or distance d. It can depend on $\mathbf{U}_T$. Coulomb's friction law is defined by

$$\mathbf{R}^{inte} = \begin{pmatrix} R_N^e \\ \mathbf{R}_T^e \end{pmatrix} \in \begin{pmatrix} \partial I_+(d) \\ 0 \end{pmatrix}, \tag{2.127}$$

$$\mathbf{R}^{intd} = \begin{pmatrix} R_N^d \\ \mathbf{R}_T^d \end{pmatrix} \in \begin{pmatrix} 0 \\ -R_N \partial \phi(\mathbf{U}_T) \end{pmatrix}, \tag{2.128}$$

$$\mathbf{R}^{int} = \mathbf{R}^{inte} + \mathbf{R}^{intd} = \begin{pmatrix} R_N \\ \mathbf{R}_T \end{pmatrix} \in \begin{pmatrix} \partial I_+(d) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -R_N \partial \phi(\mathbf{U}_T) \end{pmatrix}. \tag{2.129}$$

Tangential force is given by relationships

$$\mathbf{R}_T = -R_N \mu \frac{\mathbf{U}_T}{\|\mathbf{U}_T\|}, \ \ \text{if } \mathbf{U}_T \neq 0, \tag{2.130}$$

$$\|\mathbf{R}_T\| \leq -R_N \mu, \ \ \text{if } \mathbf{U}_T = 0. \tag{2.131}$$

2.9.2 The Coulomb's Collision Law

For collisions, internal percussion is split in the same way

$$\mathbf{P}^{int} = P_N \mathbf{N} + \mathbf{P}_T, \ \ P_N = \mathbf{P}^{int} \cdot \mathbf{N}. \tag{2.132}$$

Coulomb collision dissipation function is

$$X = \begin{pmatrix} \frac{U_N^+ + U_N^-}{2} \\ \frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2} \end{pmatrix}, \ \chi = \left(-P_N, \frac{U_N^-}{2}, d \right), \tag{2.133}$$

and

$$\Phi(X, \chi) = 0, \ if \ d > 0, \tag{2.134}$$

$$\Phi(X, \chi) = \begin{pmatrix} \Phi_N(\frac{U_N^+ + U_N^-}{2}) + I_+(\frac{U_N^+ + U_N^-}{2} - \frac{U_N^-}{2}) \\ -P_N \phi(\frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2}) \end{pmatrix}, \ if \ d = 0. \tag{2.135}$$

Collision Coulomb constitutive law is

$$\mathbf{P}^{int} = \begin{pmatrix} P_N \\ \mathbf{P}_T \end{pmatrix} = \mathbf{P}^d + \mathbf{P}^{reac}, \tag{2.136}$$

$$\mathbf{P}^{int} = 0, \ if \ d > 0, \tag{2.137}$$

$$\mathbf{P}^{int} \in \begin{pmatrix} \partial \Phi_N(\frac{U_N^+ + U_N^-}{2}) + \partial I_+(\frac{U_N^+ + U_N^-}{2} - \frac{U_N^-}{2}) \\ -P_N \partial \phi(\frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2}) \end{pmatrix}, \ if \ d = 0, \tag{2.138}$$

$$\mathbf{P}^d \in \begin{pmatrix} \partial \Phi_N(\frac{U_N^+ + U_N^-}{2}) \\ -P_N \partial \phi(\frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2}) \end{pmatrix}, \ \mathbf{P}^{reac} \in \begin{pmatrix} \partial I_+(\frac{U_N^+ + U_N^-}{2} - \frac{U_N^-}{2}) \\ 0 \end{pmatrix}, \ if \ d = 0, \tag{2.139}$$

where Φ_N and ϕ are pseudo-potentials of dissipation. Function $\Phi(X, \chi)$ is not a pseudo-potential of dissipation because P_N depends on $U_N^+ + U_N^-$. We see again that collisions are uniquely dissipative phenomenons.

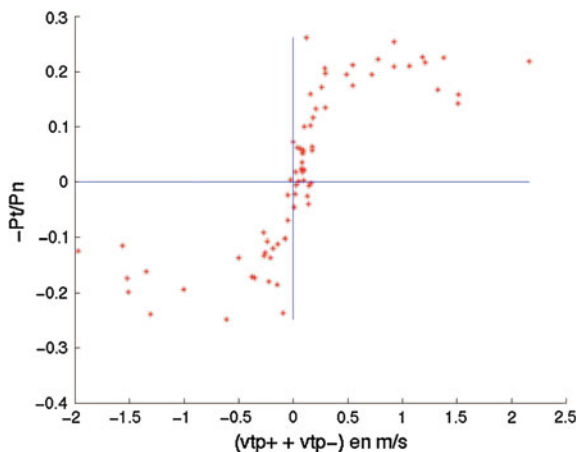### 2.9.3 Experimental Results

Experimental results, [7, 8, 10] show that $\phi(\mathbf{U}_T) = \mu \left\| \frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{2} \right\|$ may be chosen for collisions of small steel squares with a marble table (see Figs. 2.3 and 2.4). Then tangential percussion is
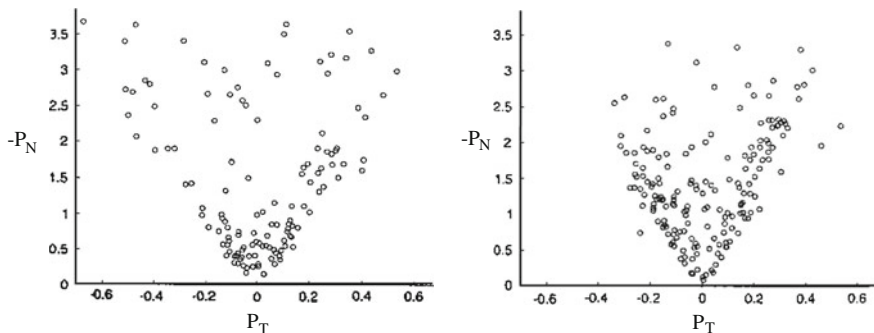
Fig. 2.3 Normal percussion $-P_N$ versus tangential percussion P_T for two small solids colliding a marble table. Coulomb cone with collision coefficient of friction $\mu = 0.22$ is conspicuous. *Left* figure is for small steel triangles and *right* figure for small steel rectangles, [7]

Fig. 2.4 Quotient $-P_T/P_N$ versus $U_T^+ + U_T^-$ in m/s for small steel triangles colliding a marble table. Collision coefficient of friction is 0.22 equal to the smooth motion coefficient of friction Coulomb's law appears clearly even if the experimental results are scattered, [7]. One may compare with Fig. 2.5 where no functional relationship appears


$$\mathbf{P}_T = -P_N\mu\frac{\mathbf{U}_T^+ + \mathbf{U}_T^-}{\|\mathbf{U}_T^+ + \mathbf{U}_T^-\|}, \text{ if } \mathbf{U}_T^+ + \mathbf{U}_T^- \neq 0, \tag{2.140}$$

$$\|\mathbf{P}_T\| \leq -P_N\mu, \text{ if } \mathbf{U}_T^+ + \mathbf{U}_T^- = 0. \tag{2.141}$$

These experimental results show that there are good functional relationships between percussions and velocity of deformation $\mathbf{U}^+ + \mathbf{U}^-$ whereas nothing pertinent appears when investigating the percussions versus either the deformation velocity before collision, $\mathbf{U}^-$ or the deformation velocity after collision, $\mathbf{U}^+$, [7, 17]. This is in agreement with theoretical inequality (2.60) which suggests that to relate $\mathbf{P}^{int}$ and $\mathbf{U}^+ + \mathbf{U}^-$ may be productive. Let us note that there is a priori no hint to look for relationships between these quantities. As already said the important interest of relationship (2.60) is to guide experimental work and theoretical choices.

Fig. 2.5 Tangential internal percussion versus tangential velocity U_T^+ after collision. No functional relationship may be founded. We have the same result for the tangential percussion versus tangential velocity U_T^- before collision
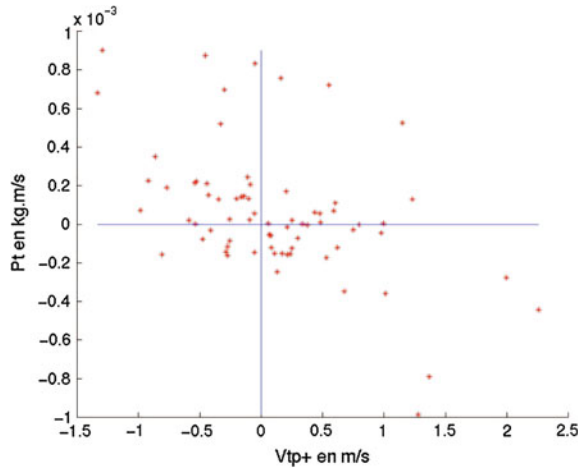
2.9.4 Relationships Between Smooth Friction and Collision Constitutive Laws

The experimental results for collisions between steel small solids and a marble table, in fact a massive polished stone table, give a Coulomb's collision coefficient 0.17 approximatively. The dry coefficient of friction for steel-stone is of this order, around 0.2. Moreover it appears a relationship between collisions with the Coulomb law and smooth sliding on the plane with Coulomb friction law. It may be proved that smooth sliding is the limit of periodic small jumps (Coulomb collisions with rebounds) when the amplitude of the jumps tends to zero, [19]. Thus experimental and theoretical results suggest to have for both smooth friction and non smooth collisions, the Coulomb's law with the same coefficient. Of course this choice is not compulsory but without more experimental information, it is reasonable.

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