Compressing Bounded Degree Graphs

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Abstract. Recently, Aravind et al. (IPEC 2014) showed that for any finite set of connected graphs $\mathcal{H}$, the problem $\mathcal{H}$-Free Edge Deletion admits a polynomial kernelization on bounded degree input graphs. We generalize this theorem by no longer requiring the graphs in $\mathcal{H}$ to be connected. Furthermore, we complement this result by showing that also $\mathcal{H}$-Free Edge Editing admits a polynomial kernelization on bounded degree input graphs.

We show that there exists a finite set $\mathcal{H}$ of connected graphs such that $\mathcal{H}$-Free Edge Completion is incompressible even on input graphs of maximum degree 5, unless the polynomial hierarchy collapses to the third level. Under the same assumption, we show that C_{11}-free Edge Deletion—as well as $\mathcal{H}$-Free Edge Editing—is incompressible on 2-degenerate graphs.

1 Introduction

Graph modification problems have been a fundamental part of computational graph theory throughout its history [11, A1. GraphTheory]. In these classical problems you are to apply at most k modifications to an input graph G to make it adhere to a specific set of properties, where both the modifying operations and the target properties are problem specific. Unfortunately, even when considering vertex deletion to hereditary graph classes, the modification problems often regarded as the most tractable, almost all of them are NP-complete [17]. A similar dichotomy is yet to appear for edge modification problems and hence the classical complexity landscape seems far more involved. However, various results display the NP-hardness of the edge variants as well [3,7,19]. Due to this inherent intractability we need to find other ways of coping. A well-established tool for tackling hard problems, in practice as well as in theory, is preprocessing of data. In theoretical computer science, preprocessing is best described within the framework of parameterized complexity as kernelization. For our purposes a problem admits a kernel of size $f(k)$ if given a graph G and an integer k as

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input, one can in polynomial time output an equivalent instance (G', k') such that both the size of G' and the value k' is bounded by $f(k)$. If f is a polynomial we say that the problem admits a polynomial kernelization.

In this paper we will restrict our attention to hereditary graph classes characterized by finite sets of forbidden induced subgraphs. Hence, for every graph class studied there is a set of finite graphs $\mathcal{H}$ such that a graph G is in the graph class if and only if no graph in $\mathcal{H}$ is an induced subgraph of G. In this situation Cai's theorem [4] shows that all $\mathcal{H}$-free modification problems are *fixed parameter tractable*, that is, they are all solvable in time $f(k) \cdot \text{poly}(n)$. And furthermore, every vertex deletion problem admits a classic $O(k^d)$ polynomial kernel, based on the sunflower lemma [1,10]. However, for edge modification problems the landscape is much less understood. In particular, P_4-free edge deletion admits a polynomial kernel, C_4-free edge deletion does not and for S_4 and the claw $(K_{1,3})$, nobody knows.

The edge modification problems characterized by a finite set of forbidden induced subgraphs $\mathcal{H}$ are often referred to as $\mathcal{H}$-Free Edge Completion, $\mathcal{H}$-Free Edge Deletion and $\mathcal{H}$-Free Edge Editing, where one is to add, remove or both add and remove k edges to make the graph $\mathcal{H}$-free. In dealing with the inherent intractability of graph modification problems Natanzon, Shamir, and Sharan [18] suggested to study $\mathcal{H}$-Free Edge Deletion on bounded degree input graphs. Recently, following this direction of research, Aravind, Sandeep and Sivadasan [2] were able to show that as long as every graph $H \in \mathcal{H}$ is connected, the problem $\mathcal{H}$-Free Edge Deletion admits a polynomial kernel of size

$$O\left(\Delta^c \cdot k^d\right),$$

where c is depending only on $\mathcal{H}$ and d on $\mathcal{H}$ and Δ. In particular, this yields a polynomial kernel for every fixed maximum degree Δ.

The first result of the paper is several, simultaneously applicable improvements upon the above mentioned result. First, we are able to remove the condition requiring all graphs of $\mathcal{H}$ to be connected. As many interesting graph classes (threshold graphs, split graphs e.g.) are described by disconnected forbidden subgraphs, this is a major extension. Second, we complement it by proving that the same kernels can be obtained when considering $\mathcal{H}$-Free Edge Editing. And third, we improve the kernel dependency on Δ. The novelty of our approach lies within a better understanding of how forbidden subgraphs are introduced when edges are modified in the input graph. Due to this, we can localize the crucial part of the instance even when both forbidden subgraphs and modifications are spread throughout the graph.

We continue by providing several hardness results. First, we prove that somewhat surprisingly the positive result does not extend to the completion variant. Due to page restrictions, we have deferred some of the proofs from the kernelization section to the full version. The statements to which these proofs belong have been marked with a ♠.

Table 1. Overview of polynomial kernelization complexity for graph modification on bounded degree and degenerate input graphs. The table shows that there is no distinction between disconnected graphs, and that the completion variant is notoriously incompressible—bounded degree does not help compressing completion problems.

	Deletion	Completion	Editing	Vertex deletion
bounded degree	Yes ([2], Theorem 4)	No (Theorem 1)	Yes (Theorem 4)	Yes
2-degenerate	No (Theorem 3)	No (Theorem 1)	No (Theorem 2)	Yes

Theorem 1 (♠). *There exists a finite set $\mathcal{H}$ such that $\mathcal{H}$-*Free Edge Completion *does not admit a polynomial kernel, even on input graphs of maximum degree* 5*, unless* NP ⊆ coNP/poly*.*

Furthermore, we prove that for both $\mathcal{H}$-Free Edge Editing and $\mathcal{H}$-Free Edge Deletion there is no hope for polynomial kernels, even when restricted to 2-degenerate graphs. It can easily be observed that the same proofs can be applied to generalize the results to K_9-minor free graphs.

Theorem 2 (♠). *There is a finite set of connected graphs $\mathcal{H}$ such that $\mathcal{H}$-*Free Edge Editing *does not admit a polynomial kernel, even on 2-degenerate graphs, unless* NP ⊆ coNP/poly*.*

Theorem 3 (♠). *There is a finite set of connected graphs $\mathcal{H}$ such that $\mathcal{H}$-*Free Edge Deletion *does not admit a polynomial kernel, even on 2-degenerate graphs, unless* NP ⊆ coNP/poly*.*

We now have complete information on the kernelization complexity of edge and vertex modification problems when the target graph class is characterized by a finite set of forbidden induced subgraphs, on bounded degree and 2-degenerate input graphs. Recall that the yes answer for the vertex deletion version on general graphs is obtained by a simple reduction from the $\mathcal{H}$-Free Vertex Deletion problem to the d-Hitting Set problem, which, using the sunflower lemma [8], can be shown to admit a polynomial kernel [1].

Related work. One should note that many modification problems remains NP-complete for bounded degree graphs. Komusiewicz and Uhlmann showed [15] that even for simple cases like $\mathcal{H} = \{P_3\}$, the path on three vertices, $\mathcal{H}$-Free Edge Deletion—also known as Cluster Deletion—is NP-complete, even on graphs of maximum degree 6. Later, it was also shown that P_4-free Edge Deletion and Editing (Cograph Editing) and $\{C_4, P_4\}$-free Edge Deletion and Editing (Trivially Perfect Editing) [6] had similar properties; NP-complete, even on graphs of maximum degree 4.

Gramm et al. [12], and Guo [14] showed kernels for several graph modification problems to graph classes characterized by a finite set of forbidden induced subgraphs. Several positive results followed, which led Fellows, Langston, Rosamond, and Shaw to ask whether all $\mathcal{H}$-free modification problems admit polynomial kernels [9].

This was refuted by Kratsch and Wahlströ [16] who showed that for $\mathcal{H} = \{H\}$ where H is a certain graph on seven vertices, $\mathcal{H}$-Free Edge Deletion, as well as $\mathcal{H}$-Free Edge Editing, does not admit a polynomial kernel unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.[1] Without stating it explicitly, but revealed by a more careful analysis of the inner workings of their proofs, Kratsch and Wahlström actually showed something even stronger; namely that the result holds when restricted to 6-degenerate graphs, both for the deletion and for the editing version.

This line of research was followed up by Guillemot, Havet, Paul, and Perez [13] showing large classes of simple graphs for which $\mathcal{H}$-Free Edge Deletion is incompressible, which was further developed by Cai and Cai [5]; Combining these results, we now know that when H is a path or a cycle, $\mathcal{H}$-Free Edge Deletion, Editing *and* Completion is compressible if and only if H has at most three edges, that is, only for the simplest graphs.

Notation

We consider only simple finite undirected graphs. Let $G = (V, E)$ be a graph on n vertices with $v \in V$. When $X \subseteq V(G)$, we write $G - X$ to denote the graph $(V \setminus X, E)$. Similarly, when $F \subseteq [V]^2$, we write $G - F$ to denote the graph $(V, E \setminus F)$ and $G \triangle F$ to mean $(V, E \triangle F)$ where $\triangle$ is the *symmetric difference operator*, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

We say that a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap [V(H)]^2$. Furthermore, we say that H is an *induced subgraph* of G if H is a subgraph of G and $E(H) = E(G) \cap [V(H)]^2$. For a set $X \subseteq V(G)$ we denote the induced subgraph of G with X as its vertices by $G[X]$. We lift the notion of neighborhoods to subgraphs by letting $N(H) = N_G(V(H))$ and $N[H] = N_G[V(H)]$. In addition, if H is a subgraph of G and $F \subseteq E(G)$ we denote by $H \triangle F$ the graph $H \triangle (F \cap [V(H)]^2)$. The *diameter* of a connected graph G, denoted $\mathrm{diam}(G)$, is defined as the number of edges in a longest shortest path of G, $\mathrm{diam}(G) = \max_{u,v \in V(G)} \mathrm{dist}_G(u, v)$. If G is disconnected, we define $\mathrm{diam}(G)$ to be $\max_C \mathrm{diam}(C)$, over all connected components C of G. For a graph G, a vertex $v \in V(G)$ and a set of vertices $X \subseteq V(G)$ we define the *distance from v to X*, denoted $\mathrm{dist}(v, X)$ as $\min_{u \in X} \mathrm{dist}(v, u)$. When provided with a non-negative integer r in addition, we define the *ball around X of radius r*, denoted $B(X, r)$, as the set $\{v \in V(G) \text{ such that } \mathrm{dist}(v, X) \leq r\}$.

Obstructions. An *obstruction set* $\mathcal{H}$ is a finite set of graphs. Given an obstruction set $\mathcal{H}$, a graph G and an induced subgraph H of G we say that H is an *obstruction* in G if H is isomorphic to some element of $\mathcal{H}$. If there is no obstruction H in G we say that G is $\mathcal{H}$-free. The size of the largest graph in $\mathcal{H}$ we denote by

[1] $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ implies that PH is contained in Σ_3^p. It is widely believed that PH does not collapse, and hence it is also believed that $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$. We will throughout this section assume that $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$.

$n_{\mathcal{H}} = \max\{|V(H)| \text{ for } H \in \mathcal{H}\}$. In addition, we lift the notation of diameter to account for a finite set of graphs $\mathcal{H}$, denoted $\text{diam}(\mathcal{H})$, being the maximum of $\text{diam}\, G$ for $G \in \mathcal{H}$.

Given a graph G and an integer k the problem $\mathcal{H}$-Free Edge Deletion asks whether there is a set $F \subseteq E(G)$ with $|F| \leq k$ such that $G - F$ is $\mathcal{H}$-free. And similarly, $\mathcal{H}$-Free Edge Editing asks whether there is a set $F \subseteq E(G)$ with $|F| \leq k$ such that $G \triangle F$ is $\mathcal{H}$-free. We say that a set of edges F is an *$\mathcal{H}$-solution* if $G \triangle F$ is $\mathcal{H}$-free. When $\mathcal{H}$ is clear from context, we will refer to F simply as a solution. When the problem at hand is the deletion problem, we furthermore assume $F \subseteq E(G)$, and when the problem at hand is the completion problem, we assume $F \cap E(G) = \emptyset$, as is expected.

Definition 1 (H-packing). *Given a graph G and an obstruction H we say that $\mathcal{X} \subseteq 2^{V(G)}$ forms an H-*packing *in G if*

(i) $G[X]$ and H are isomorphic for every $X \in \mathcal{X}$, and
(ii) X and Y are disjoint for every $X, Y \in \mathcal{X}$.

Observation 1. Given a graph G and an obstruction H we can obtain a maximal H-packing $\mathcal{X}$ in $O(n^{|V(H)|+1})$ time.

The problem we are dealing with in this article is the following, where we may replace *editing* with *deletion* or *completion*, by simply putting restrictions on where we chose F from.

$\mathcal{H}$-Free Edge Editing
Input: A graph G and an integer k
Parameter: k
Question: Is there a set F of at most k edges s.t. $G \triangle F$ is $\mathcal{H}$-free?

2 Graph Modification on Bounded Degree Graphs

In this section we prove that for any finite set of obstructions $\mathcal{H}$, the problem of deleting or editing at most k edges to make an input graph of bounded degree $\mathcal{H}$-free admits polynomial kernels. More precisely, both $\mathcal{H}$-Free Edge Editing and $\mathcal{H}$-Free Edge Deletion admits polynomial kernels on bounded degree graphs.

The argument consists of two parts. First, we identify a set of critical vertices in the input graph G, called the obstruction core Z. Based on this set we can decompose any set of modifications F in G. The decomposition leads to the construction of a set of vertices in the graph, called the extended obstruction core Z^+. The first crucial property of Z^+ is that if F modifies $G[Z^+]$ into an $\mathcal{H}$-free graph, then F also modifies G into an $\mathcal{H}$-free graph. In other words, the obstructions you want to eliminate in your graph, should be eliminated within the extended obstruction core. The second crucial property is that the extended obstruction core can be proved to live within a ball around the obstruction core, were the radius depends on how well the solution decomposes. This ball will in the end constitute the kernel.

In the second part of the argument we prove that every minimal solution decomposes well. Hence we can bound the size of the ball containing the extended obstruction core and obtain a kernel.

We point out that we have considered the editing variant of the problem where you are allowed to surpass the original maximum degree in the graph by adding edges. However, it is the case that there is always a solution that at most doubles the maximum degree of the graph since if more edges are added one might as well remove all edges incident to the vertex. The validity of this is proved in Lemma 9. It can furthermore be argued that this version of the problem is the most general one. This is due to the fact that adding every supergraph of the star with $\Delta(G) + 1$ leaves to the obstruction set ensures that any solution respects the current maximum degree.

2.1 Cores and Layers

In this section we introduce the concepts of obstruction cores and extended obstruction cores. They are heavily based on the notion of shattered obstructions; the set of obstructions you get from $\mathcal{H}$ if you take every connected component as an obstruction. It follows immediately that every shattered obstruction is connected.

Definition 2. (Shattered obstructions). *Given a set of obstructions $\mathcal{H}$ we define the* shattered obstructions*, denoted $\mathcal{H}^{\blacktriangledown}$ as the set of all connected components of graphs of $\mathcal{H}$.*

Based on shattered obstructions we now define an obstruction core and explain how such a set of not too large size can be obtained.

Definition 3. (Obstruction core). *Let (G, k) be an instance of $\mathcal{H}$-*Free Edge Editing *($\mathcal{H}$-*Free Edge Deletion*). We then say that a set $Z \subseteq V(G)$ is an* obstruction core in *G if for every shattered obstruction H in G it holds that either:*

(i) $V(H) \subseteq Z$ or
(ii) there is an H-packing in $G[Z]$ of size at least $(\Delta(G) + 1) \cdot n_{\mathcal{H}} + 2k + 1$.

Observe that for every H satisfying *(ii)* it holds that even if you discard an arbitrary obstruction in G and its entire neighborhood, together with all vertices touched by a set of at most k edges, it still holds that H occurs in $G[Z]$. This is very useful if you want to replace some part of an obstruction.

Observation 2. Given an instance (G, k) of $\mathcal{H}$-Free Edge Editing ($\mathcal{H}$-Free Edge Deletion) we can in $O(|\mathcal{H}^{\blacktriangledown}|n^{n_{\mathcal{H}}+1})$ time obtain an obstruction core Z in G of size at most $|\mathcal{H}^{\blacktriangledown}|((\Delta(G) + 1) \cdot n_{\mathcal{H}} + 2k + 1)$.

Proof. Let Z be the empty set initially. Then for every shattered obstruction H we find a maximal H-packing $\mathcal{X} = X_1, \ldots, X_t$ and add the following set $\bigcup_{i=1}^{p} X_i$ to Z, where $p = \min(t, (\Delta(G) + 1) \cdot n_{\mathcal{H}} + 2k + 1)$. The time complexity follows by Observation 1.

The next definitions are the ones of layer decompositions and core extensions, arguably the most central definitions of the kernelization algorithm. They are both with respect to a fixed obstruction core Z and set of edges F. The solution is decomposed into several layers such that the first layer consists of the edges of F that are contained in Z. The second layer consists of the edges of F that are contained in scattered obstructions created when the modifications in Z was done, and so forth. The extended core is a set of vertices encapsulating all scattered obstructions either in $G[Z]$ or created in G when doing the modifications of the layers. It should be observed by the reader that the consider solution F is not constructed, but analyzed implicitly with the intention to locate a part of the input graph that encapsulates all the crucial information of the instance.

Layer decompositions and core extensions. Let (G, k) be an instance of $\mathcal{H}$-FREE EDGE EDITING ($\mathcal{H}$-FREE EDGE DELETION), $F \subseteq [V(G)]^2$ and Z an obstruction core. We construct the *layer decomposition* $F_1, \ldots, F_\ell$ of F as follows: Let $G_1 = G$, $R_1 = F$ and $Z_1 = Z$. Then, inductively we construct the set $X = R_i \cap [Z_i]^2$. If X is empty we stop the process, otherwise we let $F_i = X$, $G_{i+1} = G_i \triangle F_i$ and $R_{i+1} = R_i \setminus F_i$. Furthermore, we let

$$W_{i+1} = \{v \in H : H \text{ is a shattered obstruction in } G_{i+1} \text{ with } [V(H)]^2 \cap F_i \neq \emptyset\},$$

and based on this we let $Z_{i+1} = Z_i \cup W_{i+1}$.

With the construction above in mind, we will refer to G_i as the *ith intermediate graph*, R_i as the *ith remainder*, Z_i as the *ith core extension* and ℓ as the *solution depth* (all with respect to G, Z and F). Furthermore, we will refer to $G^+ = G_{\ell+1}$ as the *resulting graph* and $Z^+ = Z_{\ell+1}$ as the *extended core*.

The next lemma says that if there is an obstruction in some intermediate graph such that every connected component of the obstruction is either inside the corresponding core extension or not modified at all so far by the layers, then there is an isomorphic obstruction contained entirely within the core extension. The intuition is that any untouched connected component has a large packing in Z and hence it can be replaced by an isomorphic subgraph inside Z that both avoids the modifications and the neighborhood of the rest of the obstruction.

Lemma 3. *Let (G, k) be an instance of $\mathcal{H}$-FREE EDGE EDITING ($\mathcal{H}$-FREE EDGE DELETION), Z an obstruction core of G, and $F \subseteq [V(G)]^2$ with $|F| \leq k$ and $F_1, \ldots, F_\ell$ a layer decomposition of F. For an integer $j \in [1, \ell + 1]$ let G_j be the intermediate graph and Z_j the core extension with respect to G, Z and F. Let H be an obstruction in G_j with connected components $H_1, \ldots, H_t$ such that every H_i satisfies either: (i) $V(H_i) \subseteq Z_j$ or (ii) $H_i = G[V(H_i)]$. Then there is an obstruction H' in G_j isomorphic to H with $V(H') \subseteq Z_j$ and $V(H') \setminus V(H) \subseteq Z$.*

Proof. For convenience we denote neighborhoods in G_j by N_j. Let H' be the disjoint union of every H_i such that $V(H_i) \subseteq Z_j$ and $\mathcal{L}$ the list containing every H_i not added to H'. Let H_i be an element of $\mathcal{L}$. We will now prove that there is an H'_i in $G_j[Z_j \setminus N_j[H']]$ such that H_i and H'_i are isomorphic. Let $\mathcal{X}_i$ be the maximal H_i-packing obtained when constructing Z. Since $V(H_i)$ is not contained in Z_j (and hence Z) and H_i's edges are as in G it holds that $|\mathcal{X}_i| \geq (\Delta(G)+1) \cdot n_{\mathcal{H}} + 2k + 1$ by the definition of obstruction cores. This yields that $(\Delta(G) + 1) \cdot n_{\mathcal{H}} + 2k + 1$ of the elements of the packing was added to Z. Furthermore, we observe that $|V(H')| \leq n_{\mathcal{H}}$ and hence that $|N_G(H')| \leq \Delta \cdot n_{\mathcal{H}}$. It follows immediately that $|N_j(H')| \leq \Delta \cdot n_{\mathcal{H}} + k$ and hence that $|N_j[H']| \leq (\Delta+1) \cdot n_{\mathcal{H}} + k$. By the previous arguments it follows that there is an H_i-packing in $G_j[Z \setminus N_j[H']]$ of size at least $k+1$. And hence, by the pigeon hole principle there is an H_i' isomorphic to H_i in $G_j[Z_j \setminus N_j[H']]$ such that $[V(H_i')]^2$ and F' are disjoint.

To complete the proof we do the following for every H_i in $\mathcal{L}$. We find an H_i' as described above, add H_i' to H' and remove H_i from $\mathcal{L}$. Since $H_1, \ldots, H_t$ are the connected components of H it follows that H and H' are isomorphic. Furthermore, $V(H')$ is clearly contained in Z_j and $V(H') \setminus V(H)$ in Z.

This possibility of moving obstructions to the inside of core extensions immediately yields several very useful lemmata.

Lemma 4. *Let (G, k) be an instance of* $\mathcal{H}$-Free Edge Editing *(*$\mathcal{H}$-Free Edge Deletion*), Z an obstruction core of G, and $F \subseteq [V(G)]^2$. Construct the layer decomposition $F_1, \ldots, F_\ell$ of F with respect to Z, let $F' = \cup_{i=1}^{\ell} F_i$ and let Z^+ be the extended core with respect to Z and F. It then holds that: $(G \triangle F')[Z^+]$ is $\mathcal{H}$-free if and only if $G \triangle F'$ is $\mathcal{H}$-free.*

Proof. Recall that $G^+ = G \triangle F'$. It is trivial that if there is an obstruction H in $G^+[Z^+]$ then H is also an obstruction in G^+. For the other direction, let H be an obstruction in G^+ and $H_1, \ldots, H_t$ the connected components of H. Observe that by the definition of Z^+ it holds that every H_i satisfies either *(i)* or *(ii)* of Lemma 3 with $j = \ell + 1$. It follows that there is an obstruction H' in G^+ with $V(H') \subseteq Z^+$. Hence H' is an obstruction in $G^+[Z^+]$, which completes the argument.

Lemma 5. *Let (G, k) be an instance of* $\mathcal{H}$-Free Edge Editing *(*$\mathcal{H}$-Free Edge Deletion*), Z an obstruction core of G, F a minimal solution and $F_1, \ldots, F_\ell$ the layer decomposition of F with respect to Z. It then holds that $F_1, \ldots, F_\ell$ forms a partition of F.*

Proof. Let F_i and F_j be two layers with $i < j$. It follows immediately from the definition of layer decomposition that $F_j \subseteq R_j \subseteq R_i \setminus F_i$ and hence F_i and F_j are disjoint. For convenience we let $F' = \cup_{i \in [1,\ell]} F_i$. We now prove that $F' = F$. It follows from the definition of layer decomposition that $F' \subseteq F$. Assume for a contradiction that $F' \subsetneq F$. Consider the final graph $G^+ = G \triangle F'$. If G^+ is $\mathcal{H}$-free it follows that F is not a minimal solution, yielding a contradiction.

Hence, G^+ is not $\mathcal{H}$-free. It follows immediately from Lemma 4 that $G^+[Z^+]$ is also not $\mathcal{H}$-free. Furthermore, we know by the definition of layer decompositions that $G^+[Z^+] = (G \triangle F)[Z^+]$. And hence $G \triangle F$ is not $\mathcal{H}$-free, contradicting that F is a solution.

We finish the section by stating two important properties of the core; The first one gives the true power of an extended core, namely that if a set of edges is a solution for the graph induced on its extended core it also is a solution for the entire graph. The second lemma gives us a partial tool for encapsulating an extended core without knowing the solution beforehand. The next section is dedicated to turning this partial tool into a true hammer.

Lemma 6. *Let (G, k) be an instance of $\mathcal{H}$-Free Edge Editing ($\mathcal{H}$-Free Edge Deletion), Z an obstruction core of G, $F \subseteq [V(G)]^2$ and Z^+ the extended core with respect to Z and F. If $F \subseteq [Z^+]^2$ then $(G \triangle F)[Z^+]$ is $\mathcal{H}$-free if and only if $G \triangle F$ is $\mathcal{H}$-free.*

Proof. Since $F \subseteq [Z^+]^2$ it holds that $G^+ = G \triangle F$. It trivially holds that if G^+ is $\mathcal{H}$-free, then so is $G^+[Z^+]$. Let H be an obstruction in G^+ with connected components $H_1, \ldots, H_t$. Observe that if H_i contains an edge of F then $V(H_i) \subseteq Z^+$ due to the definition of Z^+ and the assumption that $F \subseteq [Z^+]$. Apply Lemma 3 with $j = \ell + 1$ to obtain an obstruction H' in Z^+.

Lemma 7. *Let (G, k) be an instance of $\mathcal{H}$-Free Edge Editing ($\mathcal{H}$-Free Edge Deletion), Z an obstruction core of G, $F \subseteq [V(G)]^2$ and Z^+ the extended core with respect to Z and F. It then holds that: $Z^+ \subseteq B(Z, \ell \cdot \operatorname{diam}(\mathcal{H}))$.*

Proof. Let $Z_1, \ldots, Z_{\ell+1}$ be the extended cores. Instead of proving the lemma directly we prove the following, stronger claim:

($\star$) For every Z_i it holds that $Z_i \subseteq B(Z, (i-1) \cdot \operatorname{diam}(\mathcal{H}))$.

Since $Z^+ = Z_{\ell+1}$, it is clear that ($\star$) implies the lemma. The proof of ($\star$) is by induction. First, we observe that ($\star$) holds for $i = 1$ by the definition of balls, since $Z = Z_1$. Assume for the induction step that ($\star$) holds for i. Let v be a vertex in Z_{i+1}. If v is also in Z_i we are done by assumption. Hence, we assume v to be a vertex in $Z_{i+1} \setminus Z_i$. Or in other words, v is in W_{i+1}. By definition there is a scattered obstruction H in G_{i+1} and an edge uw in F_i such that both u, v and w are in H. Observe that the distance between u and v is at most $\operatorname{diam}(H)$ and recall that u is in $Z_i \subseteq B(Z, (i-1) \cdot \operatorname{diam}(\mathcal{H}))$. It follows immediately that v is in $B(Z, i \cdot \operatorname{diam}(\mathcal{H}))$ and hence the proof is complete.

2.2 Solutions are Shallow

In this section we prove that the depth of any solution is bounded logarithmically by the size of the solution. This, combined with Lemma 7 gives that linearly in k many balls of logarithmic radius is sufficient to encapsulate an extended core. To motivate that we obtain a polynomial kernel, observe that a ball of logarithmic radius in a bounded degree graph is of polynomial size.

First, we prove that when considering any layer we can always find a set of vertices of the same size, the removal of which would result in an $\mathcal{H}$-free graph. Next we prove that as long as the graph is not very small, removing a set of vertices from the graph has the same effect as modifying the graph such that the set becomes a set of isolates.

Lemma 8. *Let* (G, k) *be an instance of* $\mathcal{H}$-FREE EDGE EDITING*(*$\mathcal{H}$-FREE EDGE DELETION*),* Z *an obstruction core of* G*,* F *a minimal solution of the instance and* $F_1, \ldots, F_\ell$ *the layer partition of* F *with respect to* Z*. For every* $i \in [1, \ell]$ *there exist a set* Y *with* $Y \leq |F_i|$ *such that* $G_i - Y$ *is* $\mathcal{H}$*-free.*

Proof. We construct Y as follows: For every edge uv in F_i we add to Z the endpoint furthest away from Z. If it is a tie, we choose an arbitrary endpoint. Assume for a contradiction that $G_i - Y$ is not $\mathcal{H}$-free. Let H be an obstruction in $G_i - Y$ and $H_1, \ldots, H_t$ the connected components of H.

First, we consider the case when $i = 1$. We then apply a modification of the proof of Lemma 3. The idea is as follows: Let H' be the disjoint union of the components of H contained in Z and H_x a component not in Z. Then there is a H_x-packing of size $k+1$ in Z avoiding the closed neighborhood of H'. We observe that Y intersects with at most k of the elements of the packing and hence we can find a subgraph H'_x in $G[Z]$ not intersecting with Y such that H_x and H'_x are isomorphic. Add H'_x to H' and continue with the next component not contained in Z. It follows immediately that H' is also an obstruction in G_2. By definition $G_2[Z] = G^+[Z]$ and hence H' is an obstruction in G^+. This contradicts F being a solution.

If $i \geq 2$ it holds that Y and Z are disjoint. This is true since if both endpoints of an edge are included in Z, the edge would be in F_1 and not F_i. It holds by the definition of Y that $[V(H)]^2 \cap F_i$ is empty. Furthermore, by the definition of layer decompositions it holds that if some H_x intersects with some F_j with $j < i$ then $V(H_x) \subseteq Z_{j+1} \subseteq Z_i$. Hence we can apply Lemma 3 to obtain an obstruction H' in G_i with $V(H') \subseteq Z_i$. Since $V(H) \subseteq V(G) \setminus Y$ and $V(H') \setminus V(H) \subseteq Z$ it follows that H' is an obstruction in $G_i \setminus Y$. It follows immediately that H' is also an obstruction in G_{i+1}. By definition $G_{i+1}[Z_i] = G^+[Z_i]$ and hence H' is an obstruction in G^+. This contradicts F being a solution and completes the proof.

Lemma 9. *Let* (G, k) *be an instance of* $\mathcal{H}$-FREE EDGE EDITING *(*$\mathcal{H}$-FREE EDGE DELETION*),* X *a set of vertices in* G *and* E_X *the set of edges incident to vertices in* X*. It then holds that either*

(i) $|V(G)| < |X| + k + 2(\Delta(G) + 1)n_{\mathcal{H}}$ *or*
(ii) the instances $(G - X, k')$ *and* $(G - E_X, k')$ *are equivalent for every* k'*.*

Proof. We assume that *(i)* does not apply and prove that this implies *(ii)*. It is trivial that if $(G - E_X, k')$ is a yes-instance then $(G - X, k')$ is also a yes-instance. For the other direction, assume for a contradiction that $(G - X, k')$ is a yes-instance and that $(G - E_X, k')$ is a no-instance. Let F be a solution of $(G - X, k')$. For convenience we define $G_V = (G - X) \triangle F$ and $G_E = (G - E_X) \triangle F$. Let H an obstruction in G_E and B the set of vertices $V(H) \setminus X$. Observe that $G_V[B] = G_E[B]$ and that $|N_{G_E}(V(H))| \leq \Delta(G) \cdot n_{\mathcal{H}} + k$. It follows immediately that

$$\begin{aligned}
&|V(G_E) \setminus (X \cup N_{G_E}[V(H)])| \\
&\geq |V(G_E)| - |X| - |N_{G_E}[V(H)]| \\
&\geq |X| + k + 2(\Delta(G)+1)n_{\mathcal{H}} - |X| - n_{\mathcal{H}} - \Delta(G) \cdot LH - k \\
&= 2(\Delta(G)+1)n_{\mathcal{H}} - n_{\mathcal{H}} - \Delta(G) \cdot n_{\mathcal{H}} \\
&= (\Delta(G)+1)n_{\mathcal{H}}.
\end{aligned}$$

Hence we can obtain an independent set I of size $X \cap V(H)$ that is contained entirely outside of both X and $N_{G_E}[V(H)]$. Let $H' = G_V[I \cup B]$ and observe that H' is isomorphic to H, contradicting G_V being $\mathcal{H}$-free.

With the two previous lemmata in mind we present the main intuition of the shallowness of a solution. Basically, if for any level of a decomposed solution you do a factor $\Delta(G)$ more modifications in the future than you do in this particular level you could instead remove a set of edges related to this layer and stop any further propagation. This ensures that in any optimal solution the size of the union of the remaining layers are bounded by a layer and the maximum degree of the graph.

Lemma 10. *Given an instance (G,k) of $\mathcal{H}$-Free Edge Editing($\mathcal{H}$-Free Edge Deletion), an obstruction core Z, an optimal solution F and its layer decomposition $F_1, \ldots, F_\ell$ it holds that either*

(i) $|V(G)| \leq k + 2(\Delta(G)+1) \cdot n_{\mathcal{H}}$ *or*
(ii) $\Delta(G) \cdot |F_i| \geq |R_{i+1}|$ *for every* $i \in [1,\ell]$.

Proof. We assume that *(i)* does not apply and hence that $|V(G)| > k + (\Delta(G)+2) \cdot n_{\mathcal{H}}$. Assume for a contradiction that there is an $i \in [1,\ell]$ such that *(ii)* does not hold. Specifically, i is so that $\Delta(G) \cdot |F_i| < |R_{i+1}|$. By Lemma 8 there is a set of vertices Y with $|Y| \leq |F_i|$ such that $G_i - Y$ is $\mathcal{H}$-free. It follows by Lemma 9 with $k' = 0$ that $G_i - E_X$ is also $\mathcal{H}$-free. Let $F' = (\cup_{j \in [1,i-1]} F_j) \cup E_X$ and observe that $G \triangle F'$ is $\mathcal{H}$-free. By the following calculations;

$$|F'| \leq |\cup_{j \in [1,i-1]} F_j| + |E_X| < |\cup_{j \in [1,i-1]} F_j| + |R_{i+1}| = |F|,$$

we conclude that $|F'| < |F|$. This contradicts the optimality of $|F|$ and hence our proof is complete.

Lemma 11. *Given a instance (G,k) of $\mathcal{H}$-Free Edge Editing($\mathcal{H}$-Free Edge Deletion), an optimal solution F and its layer decomposition $F_1, \ldots, F_\ell$ it holds that either*

(i) $|V(G)| \leq k + 2(\Delta(G)+1) \cdot n_{\mathcal{H}}$ *or*
(ii) $\ell \leq 1 + \log_{\frac{\Delta(G)+1}{\Delta(G)}} |F|$.

Proof. Assume that *(i)* does not hold and hence that $|V(G)| > k + 2(\Delta(G) + 1) \cdot n_{\mathcal{H}}$. It follows immediately that *(ii)* in Lemma 10 applies.

$$\begin{aligned}
|F| &= |R_1| = |F_1| + |R_2| \\
&\geq \frac{|R_2|}{\Delta(G)} + |R_2| = \frac{\Delta(G)+1}{\Delta(G)} \cdot |R_2| = \frac{\Delta(G)+1}{\Delta(G)} \cdot (|F_2| + |R_3|) \\
&\geq \cdots \geq \left(\frac{\Delta(G)+1}{\Delta(G)}\right)^{\ell-1} \cdot |R_\ell| \\
&= \left(\frac{\Delta(G)+1}{\Delta(G)}\right)^{\ell-1} \cdot |F_\ell| \\
&\geq \left(\frac{\Delta(G)+1}{\Delta(G)}\right)^{\ell-1}
\end{aligned}$$

This gives that $\ell \leq 1 + \log_{\frac{\Delta(G)+1}{\Delta(G)}} |F|$ and hence the argument is complete.

2.3 Obtaining the Kernels

We now have all the necessary tools for providing the kernels. We reduce the graph to a ball of small radius around any obstruction core Z and by this obtain a kernelized instance. Both the size bounds and the correctness of the reduction rule follows by combining the tools developed during the section.

Rule 1. *Given an instance* (G, k) $\mathcal{H}$-FREE EDGE EDITING *(*$\mathcal{H}$-FREE EDGE DELETION*) such that* $|V(G)| > k + 2(\Delta(G)+1) \cdot n_{\mathcal{H}}$*, we find an obstruction core* Z *in* G *and return* $(G[B(Z, r)], k)$ *where* $r = \text{diam}(\mathcal{H}) \cdot (1 + \log_{\frac{\Delta(G)+1}{\Delta(G)}} k)$.

Lemma 12. *Let* (G, k) *be an instance of* $\mathcal{H}$-FREE EDGE EDITING *(*$\mathcal{H}$-FREE EDGE DELETION*) and* (G', k) *the instance obtained when applying* Rule 1 *to* (G, k)*. Then* (G, k) *is a yes-instance if and only if* (G', k) *is a yes-instance.*

Proof. It follows immediately from G' being an induced subgraph of G that if (G, k) is a yes-instance, then so is (G', k). For the other direction, let (G', k) be a yes-instance and let Z be the obstruction core of G obtained when applying Rule 1. Clearly, Z is also an obstruction core of G'. Let F be an optimal solution of (G', k) and construct the layer decomposition $F'_1, \ldots, F'_{\ell'}$ and the core extensions Z'_i with respect to Z and F in G'. Now we construct the layer decomposition $F_1, \ldots, F_\ell$ and the core extensions Z_i with respect to Z and F in G. By the definition core extensions it holds that $Z'_i \subseteq Z_i$ and hence $\ell \leq \ell'$. By Lemma 6 it holds that $Z^+_G = Z_{\ell+1} \subseteq B_G(Z, \ell \cdot \text{diam}(\mathcal{H})) \subseteq B_G(Z, \ell' \cdot \text{diam}(\mathcal{H}))$. By Lemma 11 applied to F in G' it holds that $\ell \leq 1 + \log_{\frac{\Delta(G)+1}{\Delta(G)}} |F| \leq 1 + \log_{\frac{\Delta(G)+1}{\Delta(G)}} k$ and hence $Z^+_G \subseteq V(G')$. It follows immediately that $(G \triangle F)[Z^+_G]$ is $\mathcal{H}$-free. By Lemma 5 it holds that $F \subseteq [Z'_{\ell'+1}]^2$ and hence $F \subseteq [Z_{\ell+1}]^2$. It follows immediately that Lemma 6 applies and hence $G \triangle F$ is $\mathcal{H}$-free. Hence (G, k) is a yes-instance and the proof is complete.

For ease of readability, we denote diam($\mathcal{H}$) simply by D and $\Delta(G)$ by Δ.

Theorem 4 (♠). *Both* $\mathcal{H}$-Free Edge Deletion *and* $\mathcal{H}$-Free Edge Editing *admit kernels with at most* $2n_{\mathcal{H}}|\mathcal{H}^{\blacktriangledown}|\Delta^{D+1}k^{1+D(\Delta \log \Delta)}$ *vertices. For fixed* $\mathcal{H}$ *and* Δ *this is a kernel with* $k^{O(1)}$ *vertices.*

3 Conclusion

We showed that for any finite set $\mathcal{H}$ of forbidden induced subgraphs, both $\mathcal{H}$-Free Edge Editing and $\mathcal{H}$-Free Edge Deletion admit polynomial kernelizations on bounded degree input graphs. This extendes and generalizes the result of Aravind et al. [2], who showed that $\mathcal{H}$-Free Edge Deletion admits kernel when $\mathcal{H}$ is connected on bounded degree input. We not only extend their kernel, but also improve on the size of their kernel.

We showed two lower bounds: (1) for a finite set $\mathcal{H}$ of connected graphs, $\mathcal{H}$-Free Edge Completion does not admit a polynomial kernel on bounded degree input graphs, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. (2) Under the same assumption, C_{11}-Free Edge Deletion does not have a polynomial kernel on 2-degenerate graphs, nor does $\mathcal{H}$-Free Edge Editing.

Since there is a finite set $\mathcal{H}$ of connected graphs such $\mathcal{H}$-Free Edge Completion does not admit a polynomial kernel, we encourage a further study of these problems. We leave it as an open problem whether there is a dichotomy for when $\mathcal{H}$-Free Edge Completion admits a polynomial kernel, restricted to bounded degree graphs and connected, forbidden induced subgraphs.

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