I. Logarithmic and Exponential Functions

Let me recall briefly the familiar curriculum of the school, and the continuation of it to the point at which the so called *algebraic analysis* begins.

1. Systematic Account of Algebraic Analysis

One starts with powers of the form $a = b^c$, where the exponent c is a positive integer, and extends the notion step by step *for negative integer values of c, then for fractional values of c, and finally, if circumstances warrant it, to irrational values of c*. In this process the concept of *root* becomes subordinated to the general concept of power. Without going into the details of involution, I will only recall the *rule for multiplication*

$$b^c \cdot b^{c'} = b^{c+c'},$$

which reduces the multiplication of two numbers to the addition of exponents. The possibility of this reduction, which, as you know, is fundamental for logarithmic calculation, lies in the fact that the fundamental laws for multiplication and addition are so largely identical, that both operations, namely, are commutative as well associative. The operation inverse to that of raising to a power yields the logarithm. The quantity c is called the *logarithm* of a to the base b:

$$c = \underset{(b)}{\log} a.$$

At this point a number of essential difficulties already appear which are usually passed over without any attempt at explanation. For this reason I shall try to be **[156]** especially clear at this point. For the sake of convenience we shall write x and y instead of a and c, inasmuch as we wish to study the mutual dependence of these two variables. Our fundamental equations then become

$$x = b^y, \quad y = \underset{(b)}{\log} x.$$

Let us first of all notice that b is always assumed to be positive. If b were negative, x would be alternately positive and negative for integer values of y, and would even

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49442-4_9

include imaginary values for fractional values of y, so that the *totality of number pairs (x, y) would not give a continuous curve*. But even with $b > 0$ one cannot get along *without making stipulations that appear to be quite arbitrary*. For if y is rational, say $y = m/n$, where m and n are integers prime to each other, $x = b^{m/n}$ is, as you know, defined to be $\sqrt[n]{b^m}$ and it has accordingly n values, of which, for even values of n, we should have *two* to deal with even if we confined ourselves to *real* numbers. It is customary to stipulate that x shall always be the positive root, the so-called *principal* root.

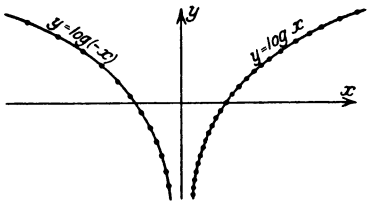

Figure 54

If you will permit me to use, somewhat prematurely, the familiar graph of the logarithm curve $y = \log x$ (Fig. 54), you will see that neither the above stipulation nor its suitableness is by any means self-evident. If y traverses the dense set of rational values, the corresponding points whose abscissas are the positive principal values $x = b^y$ constitute a dense set on our curve. If, now, when the denominator n of y is even, we should mark the points which correspond to negative values of x, we have a set of points which would be, one might say, only *half so dense*, but nevertheless still "*everywhere dense*" on the curve, which is the reflection in the y axis of our curve [$y = \log(-x)$]. If we now admit all real, including irrational, values of y, it is certainly not immediately clear why the principal values which we have been marking on the right now constitute a continuous curve and whether or not the set of negative values which we have marked on the left do similarly permit such a completion. We shall see later that this can be made clear only with the profounder resources of function theory, an aid which is not at the command of school teaching. For this reason, one does desist in the schools to strive for a
[157] conceptual understanding. One adopts rather an authoritative convention, which is quite convincing to the pupils, namely that one must take $b > 0$ and must select the positive principal values of x, that everything else is prohibited. Then the theorem follows, of course, that the logarithm is a single-valued function defined only for a positive argument.

Once the theory is carried to this point, the logarithmic tables are put into the hands of the pupil and he must learn to use them in practical calculation. There may still be some schools – in my school days this was the rule – where little or nothing is said as to how these tables are made. That was despicable utilitarianism which is scornful of every higher principle of teaching, and which we must surely and severely condemn. Today, however, the calculation of logarithms is probably discussed in the majority of cases, and in many schools indeed the theory of natural logarithms and the expansion into series is taught for this purpose.

As for the first of these, the base of the system of natural logarithms is, as you know, the number

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2{,}7182818\ldots.$$

This definition of e is usually, in imitation of the French models, placed at the very beginning in the great text books of analysis, and entirely unmotivated, whereby the really valuable element is missed, the one which mediates the understanding, namely, *an explanation why precisely this remarkable limit is used as base and why the resulting logarithms are called natural*. Likewise the expansion into series is often introduced with equal abruptness. There is a formal assumption of the expansion

$$\log(1+x) = a_0 + a_1x + a_2x^2 + \cdots,$$

the coefficients a_0, a_1, ..., are calculated by means of the known properties of logarithms, and perhaps the convergence is shown for $-1 < x \leq 1$. But again there is no explanation *as to why one would ever even suspect the possibility of a series expansion in the case of a function of such arbitrary composition as is the logarithm according to the school definition*.

2. The Historical Development of the Theory

If we wish to find all the *conceptual connections*, whose absence we have noted, and to ascertain the deeper reasons why those apparently arbitrary conventions must lead to a reasonable result, in short, *if we wish really to press forward to a full understanding of the theory of logarithms, it will be best to follow the historical development in its broad outlines*. You will see that it by no means corresponds to **[158]** the school practice mentioned above, but rather that this practice is, so to speak, a projection of that development from a most unfavourable standpoint.

We shall mention first a German mathematician of the sixteenth century, the Swabian, Michael Stifel, whose *Arithmetica Integra* appeared in Nürnberg in 1544. This was the time of the first beginnings of our present algebra, a year before the appearance, also in Nürnberg, of the book by Cardanus, which we have mentioned. I can show you this book, as well as most of those which I shall mention later, thanks to our unusually complete university library. You will find that it uses, for the first time, operations with powers where the exponents are any rational numbers, and, in particular, emphasizes the rule for multiplication. Indeed, Stifel gives, in a sense, the very first logarithmic table (on p. [250]) which, to be sure, is quite rudimentary. It contains only the integers from –3 to 6 as exponents of 2, along with the corresponding powers $\frac{1}{8}$ to 64. Stifel appears to have appreciated the significance of the development of which we have here the beginning. He declares, namely, that one might devote an entire book to these remarkable number relations.

Neper and Bürgi

But in order to make logarithms really available for practical calculation Stifel lacked still an important device, namely, decimal fractions; and it was only when these became common property, after 1600, that the possibility arose of constructing real logarithmic tables. The first tables were due to the Scotsman Napier (or Neper), who lived 1550–1617. They appeared in 1614, in Edinburgh, under the title *Mirifici logarithmorum canonis descriptio*, and the enthusiasm which they aroused is evidenced by the verses with which different authors in its preface sang the virtues of logarithms. However, Napier's method for calculating logarithms was not published until 1619, after his death, as *Mirifici logarithmorum canonis constructio*[88].

The Swiss, Jobst Bürgi (1552–1632), had calculated a table independently of Napier, which did not appear, however, until 1620, in Prag, under the title *Arithmetische und geometrische Progresstabuln*. We, in Göttingen, should have a peculiar interest in Bürgi, as one of our countrymen, since he lived for a long time in Kassel. In general, Kassel, particularly the old observatory there, has been of importance for the development of arithmetic, astronomy, and of optics prior to the discovery of infinitesimal calculus, just as Hannover became important later as the home of Leibniz. Thus our immediate neighbourhood was historically significant for our science long before this university was founded.

[159] It is very instructive to follow the train of thought of Napier and Bürgi. Both start from values of $x = b^y$ for integer values of y and seek an arrangement whereby the numbers x shall be as close together as possible. Their object was to find for every number x, as nearly as possible, a logarithm y. This is achieved today, in school, by considering fractional values of y, as we saw before. But Napier and Bürgi, with the intuition of genius, avoided the difficulties which thus present themselves by grasping the thing by the smooth handle. *They had, namely, the simple and happy thought of choosing the base b close to one, when, in fact, the successive integer powers of b are close to one another*. Bürgi takes

$$b = 1{,}0001,$$

while Napier selects a value *less* than one, but still closer to it:

$$b = 1 - 0{,}0000001 = 0{,}9999999.$$

The reason for this departure by Napier from the method of today is that he had in mind *from the first the application to trigonometric calculation*, where one has to do primarily with logarithms of proper fractions (sine and cosine) and these are negative for $b > 1$ but positive for $b < 1$. But with both investigators the chief thing *was that they made use only of integer powers of this b and so avoided, completely, the many-valuedness which embarrassed us above*.

[88] Lugduni 1620. There is a later edition in phototype. (Paris 1895)

Let us now calculate, in the *system of Bürgi*, the powers for two neighbouring exponents, y and $y+1$:

$$x = (1{,}0001)^y, \ x + \Delta x = (1{,}0001)^{y+1}.$$

By subtraction, then, we have

$$\Delta x = (1{,}0001)^y (1{,}0001 - 1) = \frac{x}{10^4}$$

or, writing Δy for the differences, 1, of the values of the exponent:

(1a) $$\frac{\Delta y}{\Delta x} = \frac{10^4}{x}.$$

We have thus obtained a difference equation for the Bürgi logarithms, one which Bürgi himself used directly in the calculation of his tables. After he had determined the x corresponding to a y he obtained the following x belonging to $y+1$ by the addition of $x/10^4$. In the same way it follows that the logarithms of Napier satisfy the difference equation

(1b) $$\frac{\Delta y}{\Delta x} = -\frac{10^7}{x}.$$

In order to see the close relationship between the two systems, we need only write for y on the one hand $y/10^4$, on the other hand $y/10^7$, i.e., we need only **[160]** displace the decimal point in the logarithm. If we denote the new numbers so obtained simply by y, *we shall have in each case a series of numbers, which satisfy the difference equation*

(2) $$\frac{\Delta y}{\Delta x} = \frac{1}{x}$$

and in which the values of y proceed by steps of 0,0001 in the one case and of $-0{,}0000001$ in the other.

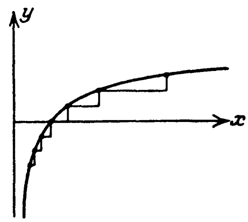

Figure 55

If, for the sake of convenience, we now make use of the graph of the continuous exponential curve (we ought really to obtain it as the result of our discussion) we shall have a tangible representation of the points which correspond to the number series of Napier and of Bürgi. These points will be the corners of a stairway inscribed in one of the two exponential curves

$$x = (1{,}0001)^{10\,000y}, \text{ and } x = (0{,}9999999)^{10\,000\,000y} \tag{3}$$

respectively, where the risers have the constant value $\Delta y = 0{,}0001$ and $\Delta y = 0{,}0000001$ in the two systems, respectively (see Fig. 55).[89]

We can get another geometric interpretation in which we do not need to presuppose the exponential curve, which will rather point out the natural way to obtain that curve, if w replace the difference equation (2) by a summation equation (that is, so to speak, if we "integrate" it):

$$\eta = \sum \frac{\Delta\xi}{\xi}. \tag{4}$$

During this summation ξ increases discontinuously, from unity on, by such steps that the corresponding $\Delta\eta = \Delta\xi/\xi$ is always constant and equal to 10^{-4} and -10^{-7} respectively, so that $\Delta\xi = \xi/10^4$ and $-\xi/10^7$, in the two cases. With the last step ξ attains the value x. One can easily give a geometric expression to this procedure. For this purpose *let us draw the hyperbola* $\eta = 1/\xi$ *in an* ξ-η-*plane* (*see Fig.* 56)
[161] *and, beginning at* $\xi = 1$, *construct successively on the* ξ-*axis all the points that are given by the law of progression* $\Delta\xi = \xi/10^4$ (confining ourselves to the Bürgi formulation). *The rectangle of altitude* $1/\xi$ *erected upon each of the intervals so obtained will have the constant area* $\Delta\xi \cdot 1/\xi = 1/10^4$. *The Bürgi logarithm will then be, according to* (4), *the* 10^4-*fold sum of all these rectangles inscribed in the hyperbola and lying between* 1 and x. A similar result is obtained for the logarithm of Napier.

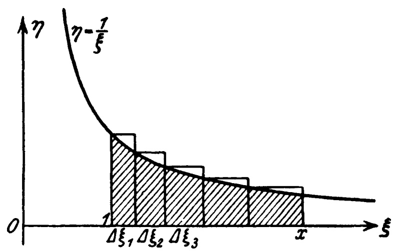

Figure 56

[89] [According to Klein, the determination of logarithms by Bürgi and by Neper are based on the same procedure. As a matter of fact, this is not the case. Only in the first stage of his considerations, Neper's approach coincides with that of Bürgi. In order to calculate the logarithms in a tolerably period of time with the accuracy aspired by Napier, new and much deeper thoughts were necessary than those by Bürgi. See: I) Conrad Müller: John Napier, Laird of Merchiston und die Entdeckungsgeschichte seiner Logarithmen, *Naturwissenschaften* 1914, Heft 28 and 2) Lord Moulton: The invention of logarithms, its genesis and growth (in the "Napier Memoria Volume", London 1915).]

Proceeding from this last interpretation, one is led immediately to the natural logarithm if, instead of the sum of the rectangles, one takes the area under the hyperbola itself between the ordinates $\xi = 1$ *and* $\xi = x$ (shaded in the figure). This finds expression in the well-known formula

$$\log \text{nat}\, x = \int_1^x \frac{d\xi}{\xi}.$$

This was, in fact, the *historical way*, and the decisive step was taken *about 1650*, when analytic geometry had become the common possession of mathematicians and when infinitesimal calculus originated with efforts to achieve the quadrature of known curves.

If we desire to use this definition of the natural logarithm as our starting point, we must, of course, convince ourselves that it possesses the fundamental property of replacing the multiplication of numbers by the addition of logarithms; or, in modern terms, we must show that the function

$$f(x) = \int_1^x \frac{d\xi}{\xi}$$

defined thus by means of the area under the hyperbola, has the simple *addition theorem*

$$f(x_1) + f(x_2) = f(x_1 \cdot x_2).$$

In fact, if we vary x_1 and x_2, then, according to the definition of an integral, the increments of the two sides $dx_1/x_1 + dx_2/x_2$ and $d(x_1 \cdot x_2)/(x_1 \cdot x_2)$ are equal. **[162]**
Consequently $f(x_1) + f(x_2)$ and $f(x_1 \cdot x_2)$ can differ only by a constant, and this turns out to be zero when we put $x_1 = 1$ (since $f(1) = 0$).

If we wish to determine, eventually, the *"base" of the logarithms obtained in this way*, we need only notice that one can realise the transition from the series of rectangles to the area under the hyperbola by progressing on the x-axis not by $\Delta\xi = \xi/10^4$ but by $\Delta\xi = \xi/n$ and allowing n to become infinite. This is the same thing as replacing the Bürgi sequence $x = (1{,}0001)^{10\,000}$ by $x = (1+1/n)^{ny}$, where $n\,y$ passes through all integer values. According to the general definition of a power, this amounts to saying that x is the y-th power of $(1 + 1/n)^n$. Accordingly it seems plausible to say that – after passing to the limit – the base becomes $\lim_{n\to\infty} (1 + 1/n)^n$, the very limit which is ordinarily assumed at the start as the definition of e. It is interesting to note, moreover, that Bürgi's base $(1{,}0001)^{10\,000} = 2{,}718146$ coincides with e to three decimal places.[90]

[90] Extensive analyses of the basis of the logarithms with Bürgi and with Napier can be found in: *Otto Mautz,* Zur Basisbestimmung der Napierschen und Bürgischen Logarithmen, Jahresbericht des Gymnasiums, Basel 1919, and *Otto Mautz,* Zur Stellung des Dezimalkommas in der Bürgischen Logarithmentafel, Basel 1921.]

The 17th Century: The Area of the Hyperbola

Let us now examine the *historical development of the theory of the logarithm after Napier and Bürgi*. First of all I have to mention:

1. Mercator, whom we have already met in these pages (see p. [88]), was one of the first to make use of the definition of the logarithm by means of the area of the hyperbola. In his book *Logarithmotechnica* of 1668, as well as in memoirs in the *Philosophical Transactions* of the London Royal Society in 1667 and 1668, he shows, by means of the same argument which I have just given you in modern terms, that $f(x) = \int_1^x d\xi/\xi$ differs from the common logarithm with the base 10, which was already the base used in calculations, only by a constant factor, the so called modulus of the system of logarithms. Moreover he had already introduced[91] the name "natural logarithm" or "hyperbolic logarithm". But the greatest achievement of Mercator was the setting up of the power series for the logarithm, which he
[163] obtained (essentially, at least) from the integral representation by dividing out and integrating term by term. I mentioned this to you (p. [88]) as an epoch-making advance in mathematics.

2. In that same connection, I told you also that Newton had taken up these ideas of Mercator's and had enriched them with two important results, namely, *the general binomial theorem and the method for the reversion of series*. This last was established in a work of Newton's youth *De analyst per aequationes numero terminorum infinitas* which appeared late in print but which from 1669 on was distributed in manuscript form[92]. In this[93] Newton derives the exponential series

$$x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

for the first time by reverting Mercator's series for $y = \log x$. This yields, as the number whose natural logarithm is $y = 1$:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots,$$

and it is now easy, with the aid of the functional equation for the logarithm, to derive that, for every real rational y, x is one of the values of e^y, and in fact the positive value, in the sense of the customary definition of power. We shall go into this more in detail later on. The function $y = \log \operatorname{nat} x$ thus turns out to be precisely what one would call the logarithm of x to the base e, according to the ordinary definition, in which e is defined by means of the series and not as $\lim\limits_{n\to\infty} (1 + 1/n)^n$.

3. Brook Taylor could follow a more convenient path in deriving the exponential series, after he had devised the general series-expansion, which bears his name,

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[91] Philosophical Transactions of the Royal Society of London, vol. 3 (1668), p. 761.
[92] Isaac Newton: *Opuscula*, Tome I, op. I, Lausanne 1744. First published in 1711.
[93] Loc. cit., p. [20].

which appeared in his work *Methodus Incrementorum*[94] and of which we shall have much to say later on. He could then use the relation

$$\frac{d \log x}{dx} = \frac{1}{x},$$

which is implied in the integral definition of the logarithm, infer from it the inverse relation

$$\frac{de^y}{dy} = e^y$$

and so write down at once the exponential series as a special case of his general series.

We have already seen (p. [89]) how this *productive period* was followed by the period of *criticism*, I should almost like to say the period of moral despair, in which **[164]** every effort was directed toward placing the new results upon a sound basis and in separating out what was maybe false.

Euler and Lagrange: Algebraic Analysis

Let us now see what attitude was taken toward the exponential function and the logarithm in the books of Euler and Lagrange, which tended in this new direction.

We shall begin with Euler's *Introductio in analysin infinitorum*[95]. Let me, first of all, praise the extraordinary and admirable analytic skill, which Euler shows in all his developments, noting, however, at the same time, that he shows no trace of the rigour which is demanded today.

At the head of his developments Euler places the binomial theorem

$$(1+k)^l = 1 + \frac{l}{1}k + \frac{l\,(l-1)}{1\cdot 2}k^2 + \frac{l\,(l-1)\,(l-2)}{1\cdot 2\cdot 3}k^3 + \cdots,$$

in which the exponent l is assumed to be an integer. Non-integer exponents are not considered in the *Introductio*. This development is specialized for the expression

$$\left(1+\frac{1}{n}\right)^{ny},$$

in which $n\,y$ is integer. He then allows n to become infinite, applies *this limit process to each term of the series*, thinks of e as defined by $\lim_{n\to\infty}(1+1/n)^n$, and so obtains the exponential series

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots.$$

[94] London, 1715.
[95] Lausanne, 1748, Caput VII, p. 85 et seq. Translation by Maser, Berlin 1885, p. 70. [See also vol. VIII (1923) of Euler's *Opera Omnia*, edited by Ferdinand Rudio, Adolf Krazer, and Paul Stäckel.]

To be sure, Euler is not in the least concerned here as to whether or not the individual steps in this process are rigorous, in the modern sense; in particular, whether the sum of the limits of the separate terms of the series is really the limit of the sum of the terms, or not. Now this derivation of the exponential has been, as you know, a model for numerous textbooks on infinitesimal calculus, although, as time went on, the different steps have been more and more elaborated and their legitimacy put to the test of rigour. You will see how influential Euler's work has been for the entire course of these things if you recall that the use of the letter e for that important number is due to him. "Ponamus autem brevitatis gratia pro numero hoc 2.71828... constanter litteram e", as he writes on p. 90.

I might add that Euler immediately follows this with an entirely analogous
[165] derivation of the series for the sine and cosine. For this purpose he starts with the expansion of $\sin\varphi$ in powers of $\sin(q/n)$ and lets n converge towards ∞. This is nothing else than a limit process applied to the binomial theorem, as is evident if one obtains the power series in question from De Moivre's formula:

$$\cos\varphi + i\sin\varphi = \left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right)^n = \left(\cos\frac{\varphi}{n}\right)^n \cdot \left(1 + i\,\mathrm{tg}\,\frac{\varphi}{n}\right)^n.$$

Let us now consider Lagrange's *Théorie des fonctions analytiques*[96]. Again it is to be noted that questions of convergence are treated, at most, only incidentally. I have already stated (p 83) that Lagrange *considers only those functions that are given by power series*, and defines their derivatives formally by means of the derived power series. Consequently the Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

is for him simply the result of a formal reordering of the series for $f(x+h)$ proceeding originally according to powers of $x+h$. Of course, if one wishes then to apply this series to a given function, one ought really to show in advance that this function is *analytic*, i.e., that it can be developed into a power series.

Lagrange begins with the investigation of the *function* $f(x) = x^n$, *for rational* n, and determines $f'(x)$ as the coefficient of h in the expansion of $(x+h)^n$, the first two terms of which he thinks of as really calculated. Then, by the same law, he obtains at once $f''(x), f'''(x), \ldots$, and the *binomial expansion of* $(x+h)^n$ appears as a *special case of Taylor*'s *series* for $f(x+h)$. Moreover, let me note expressly that Lagrange does not give special consideration to the case of irrational exponents, but rather looks upon it as obviously settled when he has considered all rational values. It is interesting to contemplate this fact, since it is upon the rigorous justification of precisely this sort of transition that the greatest importance is laid today.

Lagrange uses these results in a similar treatment of the function $f(x) = (1+b)^x$. By recording the binomial series for $(1+b)^{x+h}$ he finds, namely, $f'(x)$ as the

[96] Paris, 1797, Reprinted in Lagrange, *Œuvres*, vol. 4. Paris 1881. Compare especially chapter 3, p. 34 et seq.

coefficient of h, then determines $f''(x)$, $f'''(x)$, ... according to the same law, and forms, finally, the Taylor series for $f(x+h) = (1+b)^{x+h}$. He is then in possession, for $h = 0$, of the *desired exponential series*. **[166]**

The 19th Century: Functions of Complex Variables

I should like now to finish this brief historical sketch, in which I have, of course, mentioned only names of the very first rank, by indicating *what essentially new turns came with the nineteenth century*.

1. At the head of this list I should place *the precise conceptual developments concerning the convergence of infinite series and other infinite processes*. Gauß takes precedence here with his *Abhandlung über die hypergeometrische Reihe** in 1812 (*Disquisitiones generales circa seriem infinitam* $1+[(a\cdot b)/(1\cdot c)]x+\cdots$)[97]. After him comes Niels Henrik Abel with his memoir on the binomial series in 1826 (*Untersuchungen über die Reihe* $1+(m/1)x+\cdots$[98]), while Augustin-Louis Cauchy, in the early twenties in his *Cours d'Analyse*[99] undertook, for the first time, a general discussion of the convergence of series. The result of these investigations, for the series which we have under consideration, is that *all the earlier developments are sometimes correct, although the rigorous proofs are very complicated.* For the detailed consideration of such proofs, in modern form, I refer you again to Burkhardt's *Algebraische Analysis* or to Weber-Wellstein.

2. Although we shall have occasion to talk about it in detail later, I must mention *already here the final foundation by Cauchy of the infinitesimal calculus. By means of it the theory of the logarithm, which we discussed above as taking its start at the hands of Bürgi and Napier in the seventeenth century, was established with full mathematical exactness.*

3. Finally, we must mention the rise of that theory which is indispensable to a complete understanding of the logarithmic and exponential functions, namely, the theory of functions of a complex argument, often called, briefly, "*function theory*". Gauß was the first, again, to have a complete view of the foundations of this theory, even though he published little or nothing concerning it. In a letter to Friedrich Wilhelm Bessel, dated December 18, 1811, but published much later[100], he sketches and explains with admirable clearness the significance of the integral $\int_1^z d\zeta/\zeta$ in the complex plane, in so far as it is an infinitely many-valued function. The fame of **[167]**

* Memoir on the hypergeometric series.

[97] Commentationes societatis regiae Göttingiensis recentiores, vol. 11 (1813), No. 1, pp. 1–46. Werke, vol. 3, pp. 123–162. German translation by Max Simon, Berlin 1888.

[98] *Journal für reine und angewandte Mathematik*, vol. 1 (1826), pp. 311–339 = Ostwalds Klassiker No. 71.

[99] *Première Partie, Analyse Algébrique*. Paris 1821. = *Œuvres*, 2nd series, vol. 3, Paris, 1897. German translation by Carl Itzigsohn. Berlin 1885.

[100] *Briefwechsel zwischen Gauß und Bessel*, edited by Arthur Auwers. Berlin 1880; or Gauß, *Werke*, vol. 8 (1900), p. 90.

having also created independently the complex function theory and of having made it known to the mathematical world belongs, however, again to *Cauchy*.

The result of these developments implemented at the beginning of the nineteenth century, insofar as it concerns our special subject, might be briefly stated as follows: *The introduction of the logarithm by means of the quadrature of the hyperbola is the equal in rigour of any other method, whereas it surpasses all others, as we have seen, in simplicity and clarity*.

3. Remarks about Teaching in Schools

It is remarkable that *this modern development has passed over the schools without having, for the most part, the slightest effect on the teaching*, an evil to which I have often alluded. The teacher manages to get along still with the cumbersome *algebraic analysis*, in spite of its difficulties and imperfections, and avoids *any smooth infinitesimal calculus*, although the eighteenth century shyness toward it has long lost all point. The reason for this probably lies in the fact that *mathematical teaching in the schools and the advance of research lost all touch with each other after the beginning of the nineteenth century*. And this is the more strange since the specific training of future teachers of mathematics dates from the early decades of that century. I called attention in the preface to this *discontinuity*, which was of long standing, and which impeded every reform of the school tradition: In the schools, namely, one cared little whether and how the approaches taught might be extended within higher education and one was therefore satisfied often with definitions which were perhaps sufficient for the present, but which failed to meet more far-reaching demands. In a word, Euler remained the standard for the secondary schools. And conversely, the university frequently takes little trouble to make connection with what has been taught in the schools, but builds up its own system, sometimes dismissing this or that with brief consideration and with the sometimes inappropriate remark: "You had this at school".

In view of this, it is interesting to note that those university teachers who give lectures to wider circles, e.g. to students of natural science and engineers, have, of their own accord, *adopted a method of introducing the logarithm, which is quite similar to the one which I am recommending*. Let me mention here, in particular, Georg Scheffers' *Lehrbuch der Mathematik für Studierende der Naturwissenschaften und*
[168] *Technik*[*,101]. You will find there in chapters six and seven a very detailed theory of the logarithm and the exponential function, which coincides entirely with our plan and which is followed in chapter eight by a similar theory of the trigonometric functions. I urge you to make the acquaintance of this book. It is very appropriate for teachers, for whom it is designed, in that the material is presented fully, in readable form, and adapted to the comprehension even of the less gifted. Note, too, the great *pedagogic skill* of Scheffers when he (to cite one example) continually draws atten-

[*] *Textbook of Mathematics for Students of Natural Science and Technology.*
[101] Leipzig, 1905;[Seventh edition 1932.]

tion to the small number of formulas in the theory of logarithms that one needs to know by heart, provided the subject is once *understood;* for one can then easily look them up when they are needed. In this way he encourages the reader to persevere in face of the great mass of new material. I call your attention also to the fact that although Scheffers takes it for granted that the subject has been studied in school, he nevertheless develops it here in detail, on the assumption that most of what was learned in school has been forgotten. In spite of this, it does not occur to Scheffers to make proposals for a reform of instruction in the schools, as I am doing.

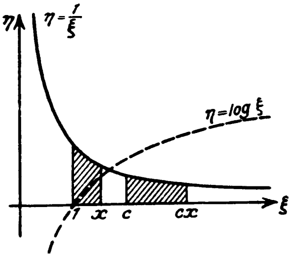

Figure 57

I should like to outline briefly once more my plan for *introducing the logarithm into the schools in this simple and natural way*. The *first principle is that the proper source from which to bring in new functions is the quadrature of known curves*. This corresponds, as I have shown, not only to the *historical situation* but also to the *procedure in the higher fields of mathematics*, e.g., in elliptic functions. Following this principle one would start with the *hyperbola* $\eta = 1/\xi$ and define the *logarithm of x as the area under this curve between the ordinates* $\xi = 1$ *and* $\xi = x$ (see Fig. 57). If the final ordinate is allowed to vary, it is easy to see how the area changes with ξ and hence to draw approximately the curve $\eta = \log \xi$.

In order now to obtain the functional equation of the logarithm in the most simple manner we can start with the relation

$$\int_1^x \frac{d\xi}{\xi} = \int_c^{cx} \frac{d\xi}{\xi},$$

which is obtained by applying the transformation $c\xi = \xi'$ to the variable of integration. This means that the area between the ordinates 1 and x is the same as that between the ordinates c and $c\,x$ which are c times as far from the origin. We can make this clear geometrically by observing *that the area remains the same when we slide it along the* ξ*-axis under the curve provided we stretch the width in the same ratio as we shrink the height*. From this the addition theorem follows at once: **[169]**

$$\int_1^{x_1} \frac{d\xi}{\xi} + \int_1^{x_2} \frac{d\xi}{\xi} = \int_1^{x_1} \frac{d\xi}{\xi} + \int_{x_1}^{x_1 \cdot x_2} \frac{d\xi}{\xi} = \int_1^{x_1 \cdot x_2} \frac{d\xi}{\xi}.$$

I wish very much that someone would give this plan a practical test in the schools. Just how it should be carried out in detail must, of course, be decided by the experienced school man. In the Meran school curriculum we did not quite venture to propose this as the standard method.

4. The Standpoint of Function Theory

Let us, finally, see how the modern theory of functions disposes of the logarithm. We shall find that all the difficulties which we met in our earlier discussion will be fully cleared away. From now on we shall use, instead of y and x, the complex variables $w = u + iv$ and $z = x + iy$. Then

1. The logarithm is defined by means of the integral

$$w = \int_1^z \frac{d\zeta}{\zeta}, \tag{1}$$

where the path of integration (Fig. 58) is any curve in the ζ-plane joining $\zeta = 1$ to $\zeta = z$.

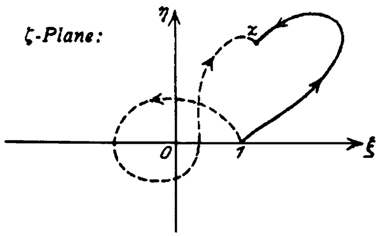

Figure 58

2. The integral has infinitely many values according as the path of integration traverses around the origin 0, 1, 2, ... times, so that log z is an *infinitely-many-valued function*. One definite value, the principal value [log z], is determined if we slit the plane along the negative real axis and agree that the path of integration shall not cross this cut. It still remains arbitrary, of course, whether we shall choose to reach the negative real values from above or from below. According to the decision
[170] on this point the logarithm has $+\pi i$ or $-\pi i$ for its imaginary part. *The general value of the logarithm is obtained from the principal value by the addition of an arbitrary multiple of* $2i\pi$:

$$\log z = [\log z] + 2k\pi i, \quad (k = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

3. It follows from the integral definition of $w = \log z$ *that the inverse function* $z = f(w)$ *satisfies the differential equation*

$$\frac{df}{dw} = f. \tag{3}$$

From this we can at once write down the power series for f

$$z = f(w) = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots .$$

Since this series converges for every finite w, we can infer *that the inverse function is a single-valued function which can be singular only for* $w = \infty$, i.e., that it is an "*integer*" *transcendental function*.

4. The *addition theorem for the logarithm* is derived from the integral definition, just as for real variables. From it we obtain for the inverse function the equation

$$f(w_1) \cdot f(w_2) = f(w_1 + w_2). \tag{4}$$

Similarly, it follows from (2) that

$$f(w + 2k\pi i) = f(w), (k = 0, \pm 1, \pm 2, \ldots) \tag{5}$$

i.e., $f(w)$ is a simply periodic function with the period $2\pi i$.

5. If we put $f(1) = e$, it follows from (3) that for every rational value m/n of w the function $f(w)$ will be one of the n values of $\sqrt[n]{e^m}$, as this expression is usually defined; that is

$$f\left(\frac{m}{n}\right) = \sqrt[n]{e^m} = e^{\frac{m}{n}}.$$

We shall adopt the customary notation, and denote this one value of $f(w)$ by $e^w = e^{m \cdot n}$, so that e^w is a well-defined single-valued function, and indeed, the one given by equation (3).

6. What sort of a function, then, shall we understand, in the *most general sense, by the power* b^w *with a non-zero, but otherwise arbitrary base* b? We must adopt such conventions, of course, that the formal rules for exponents remain valid. In order then to reduce b^w to the function e^w, which we have just defined, let us put b equal to $e^{\log b}$, where $\log b$ has the infinitely many values

$$\log b = [\log b] + 2k\pi i, \ (k = 0, \pm 1, \pm 2, \ldots)$$

It follow then necessarily that

$$b^w = \left(e^{\log b}\right)^w = e^{w \cdot \log b} = e^{w[\log b]} \cdot e^{2k\pi i w}, \ (k = 0, \pm 1, \pm 2, \ldots),$$

and this expression represents, for the values of k different from each other, functions different from each other and definitely unconnected. We have thus the re- **[171]**
markable result *that the values of the general exponential expression* b^w, *as these are obtained by the processes of raising to a power and extracting a root, do not belong at all to one coherent analytic function, but to infinitely many different functions of w, each of which is by all means single-valued.*

The values of these functions are, to be sure, related to each other in various ways. In particular they are all equal when w is an integer; and there are only a finite number of different ones among them (namely, n) when w is a fraction m/n in its lowest terms. These n values are $e^{(m/n)[\log b]} \cdot e^{2k\pi i(m/n)}$ for $k = 0, 1, \ldots, n-1$, that is, the n values of $\sqrt[n]{b^m}$, as we should expect.

7. It is only now that we can appreciate the inappropriateness of the traditional systematics which wants to ascend from involution and evolution to the single-valued exponential function. It finds itself in an outright labyrinth in which it cannot possibly find its way by so called "elementary" means, especially since it restricts itself to real quantities. You will see this clearly if you will consider the situation when b is negative, with the aid of the illuminating results which we have just obtained. In this connection I merely remind you that we are only now in a position to understand the suitableness of the *definition of the principal value* ($b > 0$ and $b^{m/n} > 0$; see p. [156]) which at the time seemed arbitrary. It yields the values of *one only of our infinitely many functions, namely those of the function*

$$[b^w] = e^{w[\log b]}.$$

On the other hand, if n is even, the negative real values of $b^{m/n}$ will constitute a set which is everywhere dense, but they belong to an entirely *different one of our infinitely many functions*, and cannot possibly combine to form a continuous analytic curve.

I should now like to add a few remarks of a more profound kind concerning the *function-theoretic nature of the logarithm.* Since $w = \log z$ suffers an increment of $2\pi i$ every time z makes a circuit about $z = 0$, the corresponding Riemann surface of infinitely many sheets must have at $z = 0$ a *branch point of infinitely high order* so that each circuit means a passage from one sheet into the next one. If one goes over to the Riemann sphere it is easy to see that $z = \infty$ is *a second branch point of the same order* and that there are no others. We can now make clear what one calls the *uniformising power of the logarithm* of which we have already spoken in
[172] connection with the solution of certain algebraic equations (see p. [143] et sq.). To fix ideas let us consider a rational power, $z^{m/n}$. By reason of the relation

$$z^{\frac{m}{n}} = e^{\frac{m}{n}\log z}$$

this power will be a *single-valued function of $w = \log z$. This is expressed by saying that it is uniformised by means of the logarithm.* In order to understand this, let us think of the Riemann surface of $z^{m/n}$ as well as that of the logarithm, both spread over the z-plane. This will have n sheets and its branch points will also be at $z = 0$ and $z = \infty$, at each of which all the n sheets will be cyclically connected. If we now think of any closed path in the z-plane (see Fig. 59) along which the logarithm returns to its initial value, which implies that its path on the infinitely many-sheeted surface is also closed, it is easy to see that the image of this path will likewise be closed when it is mapped upon the n-sheeted surface of $z^{m/n}$. We infer from this geometric consideration that $z^{m/n}$ will always return to its initial value when $\log z$ does, and hence that it is a single-valued function of $\log z$. I am the more willing to give this brief explanation because we have here the simplest case of the *principle of uniformisation*, which plays such an important part in modern function theory.

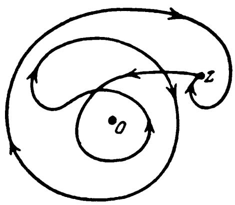

Figure 59

We shall now try to make clearer the *nature of the functional relation* $w = \log z$ *by considering the conformal mapping upon the w-plane of the z-plane and of the Riemann surface* spread upon it. In order not to be obliged to go back too far, let us refrain from including the corresponding spheres within the scope of our deliberations, in spite of the fact that it would be preferable to do so. As before, we divide the z-plane along the axis of real numbers into a *shaded* (upper) and an *unshaded* (lower) half-plane. Each of these must have *infinitely many images* in the w-plane, since $\log z$ is infinitely many-valued, and all these images must lie simply side by side with one another since the inverse function $z = e^w$ is single-valued. This means that the w-plane is divided into parallel stripes of width n separated from one another by parallels to the real z-axis (see Fig. 60). These stripes are to be alternately shaded and left blank (the first one above the real axis is shaded) and they represent, accordingly, alternate conformal maps of the upper and lower z-half-planes while the limiting parallels correspond to the parts of the real z-axis. As to the correspondence in detail, I shall remark only that z *always approaches* 0 *when* w, within a stripe, *tends to the left toward infinity*, *that* z *becomes infinite when* w **[173]** *approaches infinity to the right*, and that the inverse function e^w has an essential singularity at $w = \infty$.

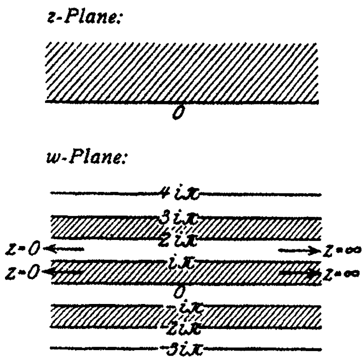

Figure 60

I must not omit here to draw attention to the connection between this representation and the theorem of *Émile Picard*, since that is one of the most interesting theorems of the recent function theory. Let $z(w)$ be an *integer transcendental* function, that is, a function which has an essential singularity only at $w = \infty$ (e.g. e^w). The question is whether there can be values z, and how many of them, which cannot be taken at any finite value of w, but which are approached as a limit when w becomes infinite in an appropriate way. The theorem of Picard states that a function in the neighbourhood of an essential singularity can omit at most two different values; that an integer transcendental function, therefore, can omit, besides $z = \infty$, (which it of necessity omits), at most one other value. e^w is an example of a function which really omits one other value besides ∞, namely $z = 0$. In each of the parallel stripes of our division e^w approaches each of these values but it assumes neither of them for any finite value of w. The function $\sin w$ is an example of a function which omits no value except $z = \infty$.

The Passage to the Limit from the Power to the Exponential Function

I should like to conclude this discussion by bringing up again a point, which we have repeatedly touched by applying to it these geometric aids. I refer to the passage to the limit from the power to the exponential function, which is given by the formula

$$e^w = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nw} .$$

If we put $n \cdot w = v$ this takes the form

$$e^w = \lim_{v\to\infty} \left(1 + \frac{w}{v}\right)^{v} .$$

Let us, before passing to the limit, consider the function

$$f_v(w) = \left(1 + \frac{w}{v}\right)^{v} ,$$

whose function-theoretic behaviour, as a power, is known to us. It has as critical points $w = -v$ and $w = \infty$, where the base becomes 0 and ∞ respectively, and it maps the f_v-half-planes conformally upon sectors of the w-plane which have
[174] $w = -v$ as common vertex and an angular opening of π/v each (see Fig. 61). If v is not an integer this series of sectors can cover the w-plane a finite or even an infinite number of times, corresponding to the then occurring many-valuedness of f_v. If now v becomes infinite, the vertex, $-v$, of the sectors moves off without limit to the left and it is clear that *these sectors lying to the right of $-v$ go over into the parallel stripes of the w-plane, which belong to the limit function e^w*. This explains geometrically that limit definition of e^w. One can verify by calculation that the width of the sectors at the point $w = 0$ goes over into the stripe width π of the parallel division.

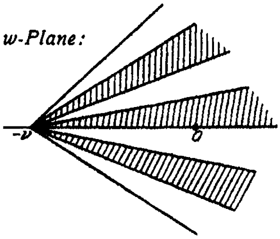

Figure 61

But a doubt arises here. If v becomes infinite continuously, it passes through not only integer but also rational and irrational values, for which the f_v will be *many-valued* and will correspond to *many-sheeted surfaces. How can these go over into the simple plane, which corresponds to the single-valued function* e^w? If, for example, we allow v to approach infinity only through rational values having π for a denominator, converging towards the infinite, each $f_v(w)$ will have an n-sheeted Riemann surface. In order to study the limit process, let us, for a moment, consider the w-sphere. It is covered for each $f_v(w)$ with n sheets which are connected at the branch points $-v$ and ∞. Let the branch cut lie along the minor meridian segment joining these points, as shown in Fig. 62. If now v approaches ∞ the branch points coincide and the branch cut disappears. Thus the bridge is destroyed that supplied the connection between the sheets, there emerge n separate sheets and, corresponding to them, n single-valued functions, of which only one is our e^w. If we now allow v to vary through *all real values*, we shall have, in general, surfaces with infinitely many sheets whose connection is broken in the limit. *The values on one sheet of each of these surfaces converge toward the single-valued function* e^w, which is spread over the simple sphere, while the sequences of values on the other sheets have, in general, no limit whatever. We thus have a complete explanation of the surely quite complicated and wonderful passage to the limit from the many-valued power to the single-valued exponential function.

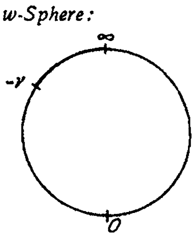

Figure 62

As a general moral of these last considerations we might say *that a complete conceptual understanding of such problems is possible only when they are taken into the field of complex numbers*. Is this, then, not a sufficient reason for teaching complex function theory in the schools? Max Simon, for one, has in fact supported **[175]** similar demands. I hardly believe, however, that the average pupils, even in the *Prima*, can be carried so far, and I think, therefore, *that we should abandon the method of algebraic analysis in the schools which leads toward such considerations, in favour of the simple and natural way, which we have developed above. I am, to be sure, all the more desirous that the teacher shall be in full possession of all the function-theoretic connections that come up here; for the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them.*

After these detailed discussions we can now be briefer in the corresponding consideration of the goniometric functions.