II. Equations in the Field of Complex Quantities

We shall now remove the restriction to real quantities and shall operate in the field of complex quantities. Of course, we shall endeavour again only to emphasise those things which are susceptible of geometric representation to an extent greater than one finds elsewhere. Let us begin at once with the most important theorem of algebra.

A. The Fundamental Theorem of Algebra

This is, as you know, the theorem *that every algebraic equation of degree n in the field of complex numbers has, in general, n roots, or, more accurately, that every polynomial $f(z)$, of degree n, can be separated into n linear factors*.

All proofs of this theorem basically make use of the *geometric interpretation* of **[110]** the complex quantity $x + iy$ in the x-y-plane. I shall give you the *train of thought of Gauß' first proof* (*1799*), which can be presented quite intuitively. To be sure, the original exposition of Gauß was somewhat different from mine.

Given the polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n,$$

we may write

$$f(x + iy) = u(x, y) + i \cdot v(x, y),$$

where u, v are real polynomials in the two real variables x, y. *The leading thought of Gauß' proof lies now in considering the two curves*

$$u(x, x) = 0 \quad \text{and} \quad v(x, y) = 0$$

in the x-y-plane, and in showing that they must have one point, at least, in common. For this point one would then have $f(x + iy) = 0$, that is, *the existence of a first* "*r o o t*" *of the equation $f = 0$ would be proved*. For this purpose, it turns out to be sufficient, to investigate the *behaviour of both curves at infinity,* i.e., at a distance from the origin which is arbitrarily great.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49442-4_8

If r, the absolute value of z, becomes very large, we may neglect the lower powers of z in $f(z)$, in comparison with z^n. If we introduce polar coordinates r, φ into the x-y-plane, i.e., if we set

$$z = r(\cos\varphi + i\sin\varphi),$$

we have, by De Moivre's formula

$$z^n = r^n(\cos n\varphi + i\sin n\varphi).$$

This expression is approached asymptotically by $f(z)$, as z increases in absolute value. It follows at once that u and v approach, respectively; asymptotically the functions

$$r^n\cos n\varphi, \quad r^n\sin n\varphi.$$

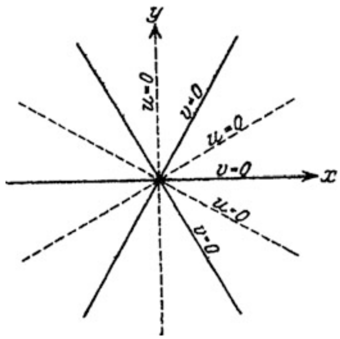

Figure 34

Consequently the ultimate course of the curves $u = 0$, $v = 0$, at infinity, respectively, will be given approximately by the equations

$$\cos n\varphi = 0, \quad \sin n\varphi = 0.$$

Now the curve $\sin n\varphi = 0$ consists of those n straight lines which go through the origin and make with the x-axis the angles $0, \pi/n, 2\pi/n, \ldots, (n-1)\pi/n$, whereas $\cos n\varphi = 0$ consists of the n rays through the origin which bisect these angles (Fig. 34 is drawn for $n = 3$). In the central part of the figure, the true curves $u = 0$,
[111] $v = 0$ can, of course, be essentially different from these straight lines; but they must approach the straight lines asymptotically towards outside as the lines recede from the origin. We *can indicate their course schematically by retaining the straight tines outside of a large circle and replacing them by anything we please, inside the circle* (see Fig. 35). But no matter what the behaviour of the curves may be inside the circle, it is certain that, if one makes the circle about the origin sufficiently large, the branches u, v, outside the circle extending to infinity, must alternate, from which

it is intuitively clear that these branches must cross one another inside the circle. In fact, we can give a rigorous[75] proof of this assertion – and this is the substance of Gauß' proof – if we use the continuity properties of the curves. The preceding argument, however, gives the essentials of the train of thought. If *one* such root has **[112]**
been found, we can divide out a linear factor, and we can then repeat the reasoning for the polynomial factor of degree $(n-1)$ arising. Continuing in this way, *we may finally break up $f(z)$ into n linear factors, i.e., we may prove the existence of n zeros.*

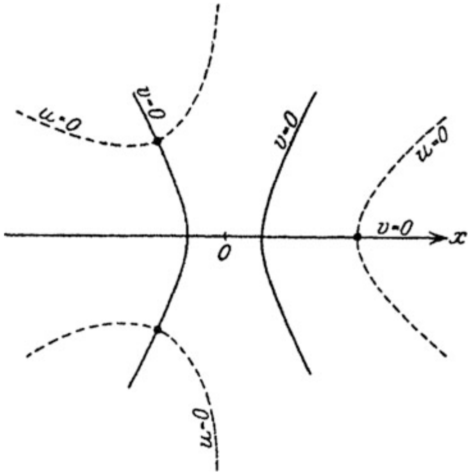

Figure 35

This method of reasoning will be much clearer if you *carry through the construction for special cases.* A simple example would be

$$f(z) = z^3 - 1 = 0.$$

In this case we obviously obtain

$$u = r^3 \cos 3\varphi - 1, \quad v = r^3 \sin 3\varphi,$$

so that $v = 0$ consists simply of three straight lines, while $u = 0$ has three hyperbola-like branches. Figure 36 shows the three intersections of the two curves,

[75] It should be said here that Gauß did not go entirely without geometric considerations. The arithmetization of the proof which he contemplated in his dissertation was first given by Alexander Ostrowski (Göttinger Nachrichten, 1920. or vol. VIII of the materials for a scientific biography of Gauß, 1920). Regarding the history of the Fundamental theorem I want to mention that its first proof was given by D'Alembert. To be sure, there was a gap in his proof, to which Gauß called attention. D'Alembert, namely, failed to distinguish between the upper limit of a function and its maximum, and he made use of the assumption, which in general is false, that a function of a complex variable actually attains its upper limit when this limit exists.

which give the three roots of our equation. I recommend strongly that you work through other and more complicated examples.

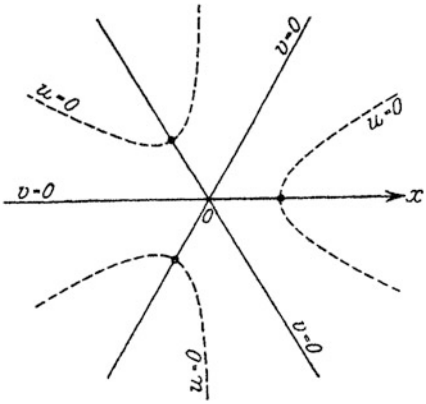

Figure 36

These brief remarks about the fundamental theorem will suffice here, since I am not giving a course of lectures on algebra. Let me close by pointing out that the *significance of the admission of complex numbers into algebra* lies in the fact that it permits a general statement of the fundamental theorem. With the restriction to real quantities one can only say that the equation of degree n has n roots, or fewer, or perhaps none at all.

B. Equations with a Complex Parameter

The remainder of the time which I have set aside for algebra I shall devote to the *discussion, by intuitive methods, of all the roots* (*including the complex ones*) *of complex equations,* as was done earlier for the real roots of real equations. We shall limit ourselves, however, to equations *with one complex parameter* and we shall assume, furthermore, that this occurs *only linearly*. The study of a *simple conformal representation* will then give us all that is required.

Let $z = x + iy$ *be the unknown,* and $w = u + iv$ *the parameter*. Then the type of the equation to be considered has the form

$$\varphi(z) - w \cdot \psi(z) = 0, \tag{1}$$

where φ, ψ, are polynomials in z. Let n be the highest power of z that occurs. According to the fundamental theorem, this equation has for each definite value of w exactly n roots z which, in general, are different. Conversely, however, it follows from (1) that

$$w = \frac{\varphi(z)}{\psi(z)}, \tag{2}$$

i.e., *w is a single-valued rational function of z*, and it is said to be of *degree n*. **[113]**
If we should use, as geometric equivalent of equation (1), simply the conformal representation which this function sets up between the z-plane and the w-plane, the many-valuedness of z as function of w would be disturbing the facility of inspection. We may help ourselves here, as is always the case in function theory, by *thinking of the w-plane as consisting of n sheets, one over another, which are connected in an appropriate manner, by means of branch cuts, into an n-sheeted Riemann surface.* Such surfaces are familiar to you all from the theory of algebraic functions. *Then our function establishes, between the points of the n-sheeted Riemann's surface in the w-plane and the points of the simple z-plane, a one-to-one relation which is, in general, conformal.*

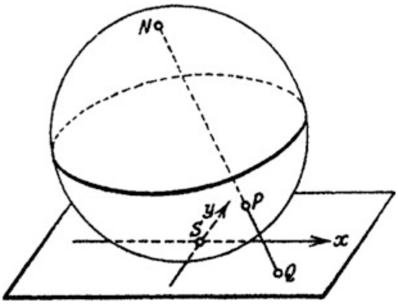

Figure 37

Before we begin a detailed study of this representation, it will be helpful if we set up certain conventions, which will do away with the exceptional role played by infinite values of w and z, a role not justified by the nature of the case, and which will enable us to state theorems in general form. Inasmuch as these conventions are not so widely employed as they should be, you will permit me to say a word or two more about them than I otherwise should. We cannot be satisfied here when one speaks merely symbolically of an *infinitely distant point of the complex plane,* since such a conception gives no adequate concrete perception, so that one must have recourse to special considerations or stipulations, in order to find out what corresponds, for an infinitely distant point, to a definite property of a finite point. *But we can secure all that is desired, if we replace the Gaussian plane, as support of the complex numbers, once for all, by the Riemannian sphere.* For this purpose, we think simply of a sphere of diameter one, tangent to the x-y-plane, its south pole S being at the origin, and from its north pole N we *project the plane stereographically* upon the sphere (see Fig. 37). To every point $Q = (x, y)$ of the plane there corresponds uniquely the second intersection P of the ray NQ with the sphere; and, conversely, to every point P of the sphere, with the exception of N itself, there corresponds a unique point Q with definite coordinate (x, y). *Hence we can consider P as representing the number $x + iy$.* Now if P approaches the north pole **[114]**
N, in any manner, Q moves to infinity; conversely, if Q recedes to infinity in any manner, the corresponding point P approaches the single definite point N. *It seems natural, then, to look upon this point N*, which does not correspond to any finite

complex number, *as the unique representative of all infinitely large $x + iy$, i.e., as the concrete picture of the infinitely distant point of the plane, which is otherwise introduced only symbolically, and to affix to it outright the mark* ∞. In this way we bring about, in the geometric picture, *complete equality between all finite points and the infinitely distant point*.

In order to return now to the geometric interpretation of the algebraic relation (1), we shall *replace the* w*-plane also by a* w*-sphere*. Then our function will be represented by a *mapping of the* z*-sphere upon the* w*-sphere,* and, just as in the case of the mapping of the two planes, this is also *conformal,* since the stereographic mapping of the plane upon the sphere is, according to a well-known theorem, conformal. To a single point on the w-sphere, there will then correspond, in general, n different points on the z-sphere. In order to get a *one-to-one* relation we imagine, again, n sheets on the w-sphere, lying one above another, and connected, in appropriate manner, by means of branch cuts, so as to form *an* n*-sheeted Riemann surface over the* w*-sphere*. This picture presents no greater difficulty than that of the Riemann surface over the plane. *Thus, finally, the algebraic equation* (1) *is interpreted as a one-to-one relation, conformal in general, between the Riemann surface over the* w*-sphere and the simple surface of the* z*-sphere*. This interpretation obviously takes into account, also, infinite values of z and w which may correspond to each other or to finite values.

In order to make the greatest possible use of this geometric device, we must take *a corresponding step in algebra,* one which shall do away with the exceptional role which infinity plays in the formulas, and this step is the *introduction of homogeneous variables*. We set, namely,

$$z = \frac{z_1}{z_2}$$

and consider z_1, z_2 as *two independent complex variables, both of which remain finite, and which cannot both vanish simultaneously*. Each definite value of z will then be given by infinitely many systems of values (cz_1, cz_2), where c is an arbitrary constant factor. We shall look upon all such systems of values (cz_1, cz_2) which differ only by such a factor, as the same "*position*" *in the field of the two homogeneous*
[115] *variables*. Conversely, for every such position there will be a definite value of z, with one exception: to the position (z_1 arbitrary, $z_2 = 0$) there will correspond no finite z; but if one approaches it from other points, the corresponding z becomes infinite itself. *This one point is thus to be looked upon as the arithmetic equivalent of the one infinitely distant point of the* z*-plane or, as the case may be, of the* z*-sphere, and as carrying the mark* $z = \infty$.

In the same way, of course, we put also $w = w_1/w_2$. We shall now set up the "*homogeneous*" *equation between the* "*homogeneous*" *variables* z_1, z_2 *and* w_1, w_2, which corresponds to equation (2). Multiplying by z_2^n in order to clear of fractions, we may write the equation in the form

$$(3) \qquad \frac{w_1}{w_2} = \frac{z_2^n \varphi\left(\dfrac{z_1}{z_2}\right)}{z_2^n \psi\left(\dfrac{z_1}{z_2}\right)} = \frac{\overline{\varphi}(z_1, z_2)}{\overline{\psi}(z_1, z_2)}.$$

In this equation, $\overline{\varphi}(z_1, z_2)$ and $\overline{\psi}(z_1, z_2)$ *are rational integer functions of* z_1 *and* z_2, since $\varphi(z)$ and $\psi(z)$ contain at most the n-th power of $z = z_1/z_2$. Moreover they are *homogeneous polynomials (forms) of dimension* n. For each term z^i of $\overline{\varphi}(z)$ or $\overline{\psi}(z)$ is transformed into the term $z_2^n(z_1/z_2)^i = z_2^{n-i} z_1^i$, of dimension n, by clearing of fractions.

We come now to the detailed study of the functional dependence which our equation (1) *or, as the case may be,* (3) *establishes between* z *and* w. We shall apply consistently our two new aids, mapping upon the complex sphere and homogeneous variables. We shall have solved this problem when we achieve a complete perception of the conformal relation between the z-sphere and the Riemann surface over the w-sphere.

First of all we must inquire as to the *nature and the position of the branch points of the Riemann surface*. I remind you here that a μ-fold branch point is one in which $\mu + 1$ sheets are connected. Since w is a single-valued function of z, we know the branch points when we know the points of the z-sphere which correspond to them, which I am in the habit of calling the *critical or noteworthy points of the* z*-sphere*. To each of these there corresponds a certain *multiplicity* equal to that of the corresponding branch point. I shall now give, without detailed proof, the theorems which make possible the determination of these points. I assume that the rather simple function theoretic facts, which enter into consideration here, are in general familiar to you, though they may not be in the homogeneous form which I prefer to use. I shall illustrate in concrete intuitive form the abstract considerations which I **[116]**
shall present to you, in this connection, by a series of examples.

A little calculation is necessary in order to obtain the *analogue, in homogeneous coordinates,* of the differential coefficient dw/dz. Differentiating equation (3) and omitting the bars over φ and ψ, we obtain

$$\frac{w_2 dw_1 - w_1 dw_2}{w_2^2} = \frac{\psi\, d\varphi - \varphi\, d\psi}{\psi^2}. \tag{3'}$$

We have also

$$d\varphi = \varphi_1\, dz_1 + \varphi_2\, dz_2,$$
$$d\psi = \psi_1\, dz_1 + \psi_2\, dz_2,$$

where

$$\varphi_1 = \frac{\partial\varphi(z_1, z_2)}{\partial z_1}, \quad \varphi_2 = \frac{\partial\varphi(z_1, z_2)}{\partial z_2},$$
$$\psi_1 = \frac{\partial\psi(z_1, z_2)}{\partial z_1}, \quad \psi_2 = \frac{\partial\psi(z_1, z_2)}{\partial z_2}.$$

On the other hand, from Euler's theorem for homogeneous functions of degree n, we have

$$\varphi_1 \cdot z_1 + \varphi_2 \cdot z_2 = n \cdot \varphi$$
$$\psi_1 \cdot z_1 + \psi_2 \cdot z_2 = n \cdot \psi;$$

consequently the numerator on the right side of (3′) may be written in the form

$$\psi\, d\varphi - \varphi\, d\psi = \begin{vmatrix} d\varphi, d\psi \\ \varphi, \psi \end{vmatrix} = \frac{1}{n} \begin{vmatrix} \varphi_1 dz_1 + \varphi_2 dz_2, & \psi_1 dz_1 + \psi_2 dz_2 \\ \varphi_1 z_1 + \varphi_2 z_2, & \psi_1 z_1 + \psi_2 z_2 \end{vmatrix}.$$

This expression, by the multiplication theorem for determinants, becomes

$$= \frac{1}{n} \begin{vmatrix} \varphi_1, \varphi_2 \\ \psi_1, \psi_2 \end{vmatrix} \cdot \begin{vmatrix} dz_1, dz_2 \\ z_1, \ z_2 \end{vmatrix}.$$

Thus (3′) goes over into the equation

$$\frac{w_2 dw_1 - w_1 dw_2}{w_2^2} = \frac{z_2 dz_1 - z_1 dz_2}{n \cdot \psi^2} (\varphi_1 \psi_2 - \psi_1 \varphi_2).$$

This constitutes the *basal formula of the homogeneous theory of our equation,* and the *functional determinant* $\varphi_1\psi_2 - \varphi_2\psi_1$, *of the forms* φ, ψ appears as a crucial expression for all that follows. Except for it and for the factor $z_2^2/(n\psi^2)$, one has on the right the differential of $z = z_1/z_2$, on the left that of $w = w_1/w_2$. Since for finite z and w the critical points are given by $dw/dz = 0$, as is well known, the following theorem appears *plausible*, but I shall here omit the proof: *Each μ-fold*
[117] *zero of the functional determinant is a critical point of multiplicity μ, i.e., there corresponds to it a μ-fold branch point of the Riemann surface over the w-sphere*. The chief advantage of this rule, as compared with those which are otherwise given, lies in the fact that it contains in one statement both finite and infinite values of z and w. It enables us also to make a precise statement concerning the *number of remarkable points*. The four derivatives, namely, are forms of dimension $n-1$, and the functional determinant is therefore a form of dimension $2n-2$. Such a polynomial always has exactly $2n-2$ zeros, if one takes into account their multiplicity. *Thus, $\alpha_1, \alpha_2, \ldots, \alpha_v$ being the remarkable points of the z-sphere (i.e., if $\varphi_1\psi_2 - \varphi_2\psi_1 = 0$ for $z_1 : z_2 = \alpha_1, \alpha_2, \ldots, \alpha_v$) and if $\mu_1, \mu_2, \ldots, \mu_r$ are their respective multiplicities, then their sum is*

$$\mu_1 + \mu_2 + \cdots + \mu_v = 2n - 2.$$

By virtue of the conformal mapping, to these points there correspond the v *branch points*

$$a_1,\ a_2, \ldots, a_v$$

on the Riemann surface over the w-sphere, which must necessarily lie separated on the surface, and about which $\mu_1 + 1, \mu_2 + 1, \ldots, \mu_v + 1$ sheets, respectively, must be cyclically connected. It should be noted, however, that different ones of these branch points may lie over the same position on the w-sphere, since $w = \varphi(z)\psi(z)$ for $z = \alpha_1, \alpha_2, \ldots, \alpha_v$, may give the same value for w more than once. Over such a point, there would be two or more separate series of sheets, each series being in itself connected. Every such position on the w-sphere is called a *branch p o s i t i o n;* we shall denote them, in order, by $A, B, C, \ldots$ It should be noted that their number can be smaller than v.

The statements thus far made furnish only a hazy picture of the *Riemann surface*. We shall now *build it up so that it can assume a gestalt better to be grasped. For this purpose, let us draw on the* w*-sphere through the branch positions* $A, B, C, \ldots$ *an arbitrary closed curve* $\mathfrak{C}$ *not percolating itself and of the simplest possible form* (see Fig. 38), *and distinguish the two spherical caps thus formed as the upper cap and the lower cap*. In all of the examples which we shall discuss later the points $A, B, C, \ldots$ will all be real and we shall then naturally select *as the curve* $\mathfrak{C}$ *the meridian great circle of real numbers,* so that each of our two partial regions will be a hemisphere.

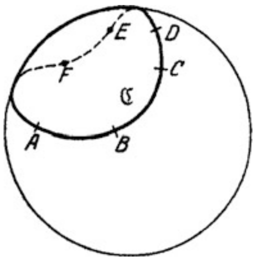

Figure 38

Returning to the general case we see that each pair of sheets of the Riemann **[118]** surface which are connected, percolate along a *branch cut,* which joins two branch points. As you know, the Riemann surface remains unchanged in essence if we move these cuts, leaving the end points fixed, that is, if we think of the same sheets as being connected along other curves, provided these join the same branch points. It is in just this variability that the great generality and also the great difficulty of the idea of the Riemann surface lies. In order to give the surface a definite form, which shall be susceptible to concrete perception, *we move all the branch cuts so that all of them lie over the curve* $\mathfrak{C}$, *which passes through all the branch points*. It may be that several branch cuts lie over the same part of $\mathfrak{C}$, and none at all over other parts.

Now let us cut this entire complex of sheets, i.e., each individual sheet, along the curve $\mathfrak{C}$. Since we had already moved all the branch cuts into a position over $\mathfrak{C}$, the incision just made passes along all of them, so that *our Riemann surface separates into* $2n$ "*half-sheets*" *entirely free from branches,* n *of them over each of the two spherical caps*. If we think of the half-sheets corresponding to the upper cap as being shaded, and those corresponding to the lower as not shaded, we can distinguish briefly, n *shaded and* n *unshaded half-sheets*. We can now describe *the original Riemann surface as follows*. On it *each shaded half-sheet meets only unshaded half-sheets*, *those with which it is connected along segments of the curve* $\mathfrak{C}$ *lying over* $AB, BC, \ldots$; *and, similarly, each unshaded half-sheet is connected along such segments of* $\mathfrak{C}$ *only to shaded half-sheets*. *However, more than two half-sheets may meet only at a branch point; and in fact around any* μ*-fold branch point,* $\mu + 1$ *shaded half-sheets would alternate with* $\mu + 1$ *unshades ones*.

Since the mapping by means of our function $w(z)$ of the z-sphere upon the Riemann surface over the w-sphere is a one-to-one correspondence, we can immediately transfer to the z-sphere the above conditions of connectivity. Because of continuity, the $2n$ half-sheets of the Riemann surface must correspond to $2n$ connected z regions, which we may call the shaded and the unshaded half-regions. These will be separated from one another by the n images of each of the segments $AB, BC, \ldots$ of the curve $\mathfrak{C}$ which the n-valued function $z(w)$ represents upon the z-sphere. *Each shaded half-region meets only shaded half-regions along these image-curves, and each unshaded half-region meets only shaded ones. It is only in a μ–fold critical point that more than two half-regions can meet. At such a point $\mu + 1$ shaded and $\mu + 1$ unshaded half-regions come together.*

[119] This division of the z-sphere into partial regions will help us to follow in detail the course of the function $z(w)$ for a few simple characteristic examples. I shall begin with a preferably simple one.

1. The "Pure" Equation

We shall call the well known equation

$$z^n = w \tag{1}$$

a pure equation. Its solution is given formally by introducing the radical $z = \sqrt[n]{w}$. This gives us no information, however, regarding the functional relation between z and w. We shall proceed according to the general plan by introducing the homogeneous variables

$$\frac{w_1}{w_2} = \frac{z_1^n}{z_2^n},$$

and we shall consider the functional determinant of the numerator and denominator of the right side

$$\begin{vmatrix} nz_1^{n-1}, 0 \\ 0, nz_2^{n-1} \end{vmatrix} = n^2 z_1^{n-1} \cdot z_2^{n-1}.$$

This expression obviously has the $(n-1)$ *fold zeros* $z_1 = 0$ and $z_2 = 0$, or (in non-homogeneous form) $z = 0$ and $z = \infty$. These are the only critical points and they are of total multiplicity $2n - 2$. By our general theorem, therefore, *the only branch points of the Riemann surface over the w-sphere* are at the positions $w = 0$ and $w = \infty$. By the equation $w = z^n$ these correspond to the two points $z = 0$ and $z = \infty$. Each of these two points has the *multiplicity* $n - 1$, so that n leaves are cyclically connected at each of them. Let us now mark on the w-sphere *the meridian of real numbers as the curve* $\mathfrak{C}$ and let us cut all the sheets of the Riemann surface along this meridian, after having appropriately displaced all of the branch cuts. Of the $2n$ hemispheres into which the surface separates we think of those over the *rear* half of the w-sphere, that is, those which correspond to w-values with *positive imaginary parts,* as shaded. Upon the meridian itself, we shall distinguish between the *half-meridian of positive real numbers* (drawn full in Fig. 39) and that of the *negative real numbers* (*dotted*).

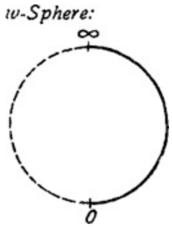

Figure 39

Now we must examine the *mappings of this meridian* $\mathfrak{C}$ *curve upon the z-sphere,* where they bring about the characteristic division into half-regions. Upon the positive half-meridian $w = r$, where r ranges through positive real values from 0 to ∞; **[120]** for these values we have by a well known formula of complex numbers,

$$z = \sqrt[n]{w} = \left|\sqrt[n]{r}\right| \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right), \quad \text{where } k = 0, 1, \ldots, n-1.$$

For the different values of k, this expression gives those n *half-meridians of the* z*-sphere which make with the half-meridian of positive real numbers the angles* $0, 2\pi/n, 4\pi/n, \ldots, 2(n-1)\pi/n$. Thus these curves correspond to the full drawn half of $\mathfrak{C}$. On the negative half-meridian of the w-sphere we must set $w = -r = r \cdot e^{i\pi}$, where again $0 \leqq r \leqq \infty$. This gives

$$z = \sqrt[n]{w} = \left|\sqrt[n]{r}\right| \left(\cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}\right),$$
$$\text{where } k = 0, 1, \ldots, n-1.$$

Corresponding to this we have, on the z-sphere, those n *half-meridians which have the "longitude"* $\pi/n, 3\pi/n, \ldots, 2(n-1)\pi/n$, *which thus bisect the angles between the others. Accordingly, the* z*-sphere is divided into* $2n$ *congruent bi-angular pieces reaching from the north pole to the south pole,* similar to the natural divisions of an orange. This division is exactly in accord with the general theory. In particular, it is only at the remarkable points, the two poles, that more than two half-regions meet. At each of these points $2n$ half-regions meet, corresponding to the multiplicity $n-1$.

As for the *shading of the regions,* we need to fix it for *one* region only. The remainder are then alternately shaded and unshaded. Now note that when we look at the shaded half of the w-sphere (the rear) from the point $w = 0$, the full drawn part of the boundary lies to the left, the dotted part to the right. Since we are concerned with a conformal mapping, in which angles are not reversed, *each shaded region of the z-sphere, looked at from the corresponding point* $z = 0$, *must have the* *same property as to position, that is, it must have a full drawn boundary to the left, and a dotted one to the right.* With this we control completely the division of the z-sphere into regions. Moreover, one notices a characteristic difference in the distribution of the regions upon two z-hemispheres, according as n is even or odd, as can be clearly seen in Figs. 40 and 41 on p. [121] for the first cases $n = 3$, $n = 4$. Let me emphasize how necessary it was to go over to the complex *sphere* in order to get a full understanding of the situation. In the complex z*-plane,* one would have a division into angular sectors by straight lines radiating from $z = 0$, and it would not be at all so obvious that $z = \infty$ and $w = \infty$ have equal significance with $z = 0$ and $w = 0$, as critical point and branch point, respectively.

[121] This furnished us with the essentials for exact knowledge of the functional relation between z and w. We need now study only the *conformal mapping of each of the* $2n$ *spherical sectors upon one or the other of the two* w*-hemispheres*. But I shall not go into the details here. This case, as one of the simplest and most obvious illustrations, will be familiar ground to anyone who has had to do with conformal representation. We shall see later (see p. [141]) how to deduce from this methods for the numerical calculation of z.

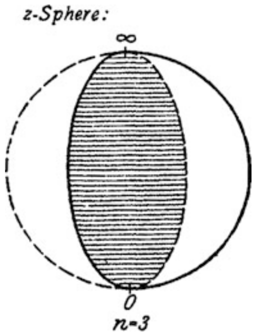

Figure 40

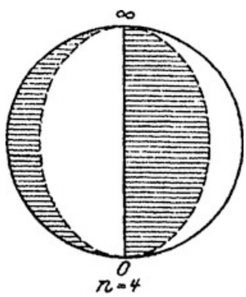

Figure 41

Let us, however, settle here the important question as to the mutual relation *among the various congruent regions of the* z*-sphere*. Speaking more exactly, $w = z^n$ takes on the same value at each point in every of the n shaded regions.

Can the corresponding values of z be expressed in terms of one another? We notice, in fact, that for $z' = z \cdot \varepsilon$ (where ε is any one of the n-th roots of unity) $z'^n = z^n$, that is $w = z^n$ *takes the same value at all the n points*

$$z' = \varepsilon^{\nu} \cdot z = e^{\frac{2\nu i\pi}{n}} \cdot z \quad (\nu = 0, 1, 2, \ldots, n-1). \tag{2}$$

These n values of z' must therefore be distributed so that just one of them lies in each of the n shaded regions of the z-sphere, if z is taken in *one* of the shaded regions and each of them must traverse one of these regions as z traverses its region. The same thing is true of the unshaded regions. Each of the substitutions (2) is represented geometrically by *a rotation of the z-sphere through an angle $\nu \cdot 2\pi/n$ about the vertical axis* 0, ∞ since, as is well known, multiplication in the complex plane by $e^{\frac{2\nu i\pi}{n}}$ denotes a rotation through that angle about the origin. Thus *corresponding points of our spherical regions, as well as the regions themselves, go over into one another by means of these n rotations about the vertical axis*.

If, then, we had determined at the start only one shaded partial region of the sphere, this remark would have furnished *all the similar partial regions*. In this we have made use only of the property of the substitutions (2) *that they transform equation* (1) *into itself* (i.e., $z^n = w$ *into* $z'^n = w$) *and that their number is equal to the degree*. In the examples that follow, we shall always be able to give such linear substitutions at the outset, and by means of them to simplify considerably the determination of the division into subregions. **[122]**

Irreducibility; "Impossibility" of Trisecting the Angle

By using the present example I should like to clarify *an important general notion,* namely, the *notion of irreducibility for equations which contain a parameter w rationally*. We have already discussed irreducibility of equations with rational *numerical* coefficients in connection with the construction of the regular heptagon (p. [55] et seq.). *An equation* $f(z, w) = 0$ (e.g., our equation $z^n - w = 0$), *where $f(z, w)$ is a polynomial in z, whose coefficients are rational functions of w, is called reducible with respect to the parameter w, when f can be split into the product of two polynomials of the same sort, in each of which z really appears*

$$f(z, w) = f_1(z, w) \cdot f_2(z, w);$$

otherwise the equation is called irreducible with respect to w. The entire generalisation, in comparison with the earlier conception, lies in the fact that the "*domain of rationality*" in which we operate and in which the coefficients of the admissible polynomials are to lie, consists of the *totality of rational functions of the parameter w instead of the totality of rational numbers,* in other words, that we pass from a number-theoretic to a function-theoretic conception.

If we illustrate us this, for each equation $f(z, w) = 0$, by means of its Riemann surface, we can set up *a simple criterion for reducibility in this new sense*. If the equation, namely, is reducible, every system of the values z, w, which satisfies it, satisfies either $f_1(z, w) = 0$ or $f_2(z, w) = 0$; now the solutions of $f_1 = 0$ and $f_2 = 0$ are represented by means of their Riemann surfaces, which have nothing to do with each other, and, in particular, cannot be connected. Thus, *the Riemann surface which belongs to a reducible equation* $f(z, w) = 0$ *must break down into at least two separates pieces.*

According to this, we can now assert that the equation $z^n - w = 0$ *is certainly irreducible in the function theoretic sense*. For, on its Riemann surface, which we know exactly, all the n sheets are cyclically connected at each of its branch points. Moreover, the entire surface is mapped upon the unpartitioned z-sphere. Hence such a breaking down cannot occur.

In connection with this, we can answer one of the popular problems of mathematics which we touched earlier (p. [56]), *namely, that of the possibility of dividing an arbitrary angle* φ *into* n *equal parts, in particular, for* $n = 3$, *the possibility of trisecting an angle*. The problem is to give an exact construction *with ruler and compass* for dividing into three equal parts *any angle* φ *whatever*. (It is easy, of course, to give a construction for a series of *special values of* φ). I shall give you the train of thought for the proof of the impossibility of trisecting an angle in the sense just mentioned, and I shall ask you to recall, in this connection, the proof of the impossibility of constructing the regular heptagon with ruler and compass (see p. [56] et seq.). Just as at that time, we shall reduce the problem to that of the solution of an *irreducible cubic equation,* and we shall then show that this equation cannot be solved by a series of square roots; except that, now, the equation will contain *a parameter* (the angle φ), whereas, before, the coefficients were integers. Accordingly, *function-theoretic irreducibility must replace number-theoretic* irreducibility.

[123]

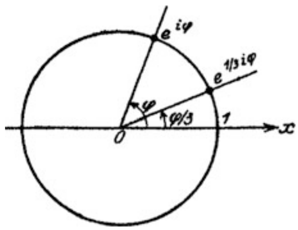

Figure 42

In order to set up the equation of the problem let us think of the angle φ as laid off from the positive real half-axis in the w-plane (see Fig. 42). Then its free arm will cut the unit circle in the point

$$w = e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

Our problem consists in finding, independently of special values of the parameter φ, a construction, involving a finite number of applications of the ruler and compass, which shall give the point of intersection with the unit circle of the arm of the angle $\varphi/3$, i.e., the point

$$z = e^{\frac{i\varphi}{3}} = \cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}.$$

This value of z satisfies the equation:

$$z^3 = \cos\varphi + i\sin\varphi, \tag{3}$$

and the analytic equivalent of our geometric problem consists in *solving this equation* (see p. [56]) *by means of a finite number of square roots, one over another, of rational functions of* $\sin\varphi$ *and* $\cos\varphi$, since these quantities are the coordinates of the point w with which we start the construction.

We must show, first, *that the equation* (3) *is irreducible in the function-theoretic sense*. To be sure, this equation does not have just the form we assumed while explaining the notion, since, instead of the a *complex* parameter w that enters rationally, we have now two functions cos and sin of a *real* parameter φ, both of **[124]** which appear rationally. As a natural extension here of our notion, we shall *call the polynomial* $z^3 - (\cos\varphi + i\sin\varphi)$ *reducible if it can be split into polynomials whose coefficients are likewise rational functions of* $\cos\varphi$ *and* $\sin\varphi$; and we can, as before, assign a criterion for this. If we let φ assume all real values in (3), $w = e^{i\varphi} = \cos\varphi + i\sin\varphi$ will describe the unit circle of the w-plane, to which the equation of the w-sphere corresponds by stereographic projection. The curve which lies over this, on the Riemann surface of the equation $z^3 = w$, and which describes, in one stroke, all three sheets, is mapped by equation (3) uniquely upon the unit circle of the z-sphere. Hence it can be regarded, in a sense as its "*one dimensional Riemann image*". In the same way, we can obviously assign such a Riemann image to every equation of the form $f(z, \cos\varphi, \sin\varphi) = 0$ by taking as many copies of the unit circle with arc length φ as the equation has roots, and stapling them together according to the connectivity of the roots. It follows, just as before, *that the equation* (3) *can be reducible only when its one-dimensional Riemann image breaks down into separate parts,* and this is obviously not the case. *This proves the function-theoretic irreducibility of our equation* (3).

Now, however, the former proof of the theorem, that a cubic equation with rational numerical coefficients is reducible if it can be solved by a series of square roots, can be applied literally to the present case of the function-theoretically irreducible equation (3) (see p. [57] et seq.). We need only to replace "rational numbers" there by "rational functions of $\cos\varphi$ and $\sin\varphi$". *This proves our assertion that the trisection of an arbitrary angle cannot be accomplished by a finite number of applications of ruler and compass*. Hence the endeavours of angle-trisection zealots must always be fruitless!

I pass on now to the treatment of a somewhat more complicated example.

2. The Dihedral Equation

The equation

$$w = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right). \tag{1}$$

is called the dihedral equation, for reasons that will appear later. Clearing of fractions, we see that its degree is $2n$. Introducing homogeneous variables we get

$$\frac{w_1}{w_2} = \frac{z_1^{2n} + z_2^{2n}}{2z_1^n \cdot z_2^n},$$

[125] in which, in fact, forms of dimension $2n$ appear in numerator and denominator. The functional determinant of these forms is

$$\begin{vmatrix} 2nz_1^{2n-1}, & 2nz_2^{2n-1} \\ 2nz_1^{2n-1}z_2^n, & 2nz_1^n z_2^{n-1} \end{vmatrix} = 4n^2 z_1^{n-1} z_2^{n-1}\left(z_1^{2n} - z_2^{2n}\right).$$

It has an $(n-1)$-fold zero at $z_1 = 0$ and at $z_2 = 0$; the other $2n$ zeros are given by

$$z_1^{2n} - z_2^{2n} = 0 \quad \text{or} \quad \left(\frac{z_1}{z_2}\right) = \pm 1.$$

If in addition to the n-th root of unity

$$\varepsilon = e^{\frac{2i\pi}{n}}$$

which we have already used, we introduce also the primitive n-th root of -1:

$$\varepsilon' = e^{\frac{i\pi}{n}}$$

the last $2n$ zeros are given by the equations

$$\frac{z_1}{z_2} = \varepsilon^\nu \quad \text{and} \quad \frac{z_1}{z_2} = \varepsilon' \cdot \varepsilon^\nu, \quad (\nu = 0, 1, \ldots, n-1).$$

Since the values of $z = z_l/z_2$ corresponding to them all have the absolute value one, they all lie therefore on the equator of the z-sphere (corresponding to the unit circle of the z-plane), at equal angular distances of π/n. We have therefore *as critical points on the z-sphere*:

(a) *the south pole $z = 0$ and the north pole $z = \infty$, each of multiplicity $n = 1$*;
(b) *the $2n$ equatorial points $z = \varepsilon^\nu$, $\varepsilon' \cdot \varepsilon^\nu$, each of multiplicity one*.

The sum of all the multiplicities is $2 \cdot (n-1) + 2n \cdot 1 = 4n - 2$, as is demanded by the general theorem on p. [117] for the degree $2n$. By virtue of equation (1) there will correspond to the remarkable points $z = 0$, $z = \infty$ of the z-sphere, the position $w = \infty$ on the w-sphere. Moreover, to all the points $z = \varepsilon^\nu$, corresponds the position $w = +1$; and, to all the points $z = \varepsilon' \cdot \varepsilon^\nu$ the position $w = -1$. There are, accordingly, *only three branch points* $\infty, +1, -1$ *on the* w*-sphere*. These will lie as follows:

$w = \infty$ *two branch points of multiplicity* $n - 1$;
$w = +1$ *n branch points of multiplicity* 1;
$w = -1$ *n branch points of multiplicity* 1.

The $2n$ *sheets of the Riemann surface group themselves therefore over the point* $w = \infty$ *in two separate series, each of n cyclically connected sheets; over* $w = +1$ *and* $w = -1$ *in n series, each of two sheets.* The disposition of the sheets will **[126]** become clear when we study the *corresponding subdivision of the z-sphere* into half-regions.

To this end it will be well, as we remarked above, *to know the linear substitutions which transform equation* (1) *into itself.* As in the case of the pure equation, it is unchanged by the n substitutions

(2a) $$z' = \varepsilon^\nu \cdot z (\nu = 0, 1, \ldots, n-1), \quad \text{where} \quad \varepsilon = e^{\frac{2i\pi}{n}},$$

since for these $z'^n = z^n$. Likewise, however, it is unchanged by the n additional substitutions

(2b) $$z' = \frac{\varepsilon^\nu}{z} (\nu = 0, 1, \ldots, n-1),$$

since these only change z^n into $1/z^n$.

We have therefore $2n$ linear substitutions of equation (1) into itself, exactly *as many as its degree indicates*. Thus, if we know for a given value w_0 of w *one* root z_0 of the equation, we know immediately $2n$ roots $\varepsilon^\nu \cdot z_0$ and ε^ν / z_0 ($\nu = 0, 1, 2, \ldots, n-1$), in general all different, for which w has the same value w_0, i.e., we know *all* the roots of the equation when we have obtained the n-th root of unity ε.

Let us now proceed to examine the subdivision of the z*-sphere corresponding to cuts along the real meridian of the Riemann surface over the* w*-sphere*. In this, as in the previous example, we distinguish on the real meridian of the w-sphere the three segments made by the branch points: that from $+1$ to ∞ (drawn full), that from ∞ to -1 (short dotted), and that from -1 to $+1$ (long dotted) (see Fig. 43). *To each of these three segments there correspond on the* z*-sphere* $2n$ *different curvilinear segments which can be derived from any one of them by means of the* $2n$ *linear substitutions* (2). It will always suffice, therefore, to find one of them. Moreover all these segments must connect the critical points $z = 0, \infty, \varepsilon^\nu, \varepsilon' \cdot \varepsilon^\nu$, which **[127]** we therefore mark on the z-sphere. Just as in the previous case, their form is of a somewhat different type according as n is even or odd. It will suffice if we exhibit a definite case, say for $n = 6$. Fig. 43 shows the front half of the z-sphere in orthogonal projection. One sees, on the equator, from left to right with distances of $60°$, $\varepsilon^3 = -1$, ε^4, ε^5, $\varepsilon^6 = 1$; and of those lying midway between the others, $\varepsilon' \cdot \varepsilon^\nu$, are visible $\varepsilon' \cdot \varepsilon^3$, $\varepsilon' \cdot \varepsilon^4 = -i$, and $\varepsilon' \cdot \varepsilon^5$.

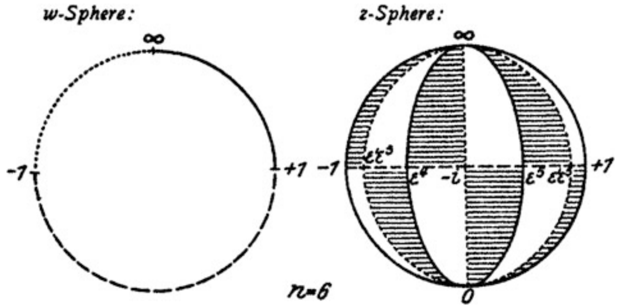

Figure 43

Now we shall see that the quadrant $+1 < z < \infty$ *of the meridian of real* z *corresponds to the part of the real* w*-meridian* $+1 < w < \infty$ (*full drawn*). In fact, if we put $z = r$ and let r range through real values from 1 to ∞, then $w = \frac{1}{2}(z^n + 1/z^n) = \frac{1}{2}(r^n + 1/r^n)$ will vary also through real values that are always increasing, from 1 to ∞. We obtain n other full drawn curves on the z-sphere, from this one, by means of the n linear substitutions (2a). But, as we saw in the previous example, these substitutions mean rotations of the sphere about the vertical axis $(0, \infty)$ through the angles $2\pi/n, 4\pi/n, \ldots, 2(n-1)\pi/n$. *We get in this way the* n *quarter-meridians from the north pole* ∞ *to the points* ε^ν *on the equator*. We get an additional full drawn curve if we apply the substitution $z' = 1/z$, which transforms the meridian quadrant from $+1$ to ∞ into the lower real meridian quadrant from $+1$ to 0. If we subject this quadrant to the n rotations (2a), the composition of which with $z' = 1/z$ gives the n substitutions (2b), we obtain, in addition, *the* n *meridian quadrants which join the south pole with the equatorial points* ε^ν. We have now in fact, as researched, the $2n$ full drawn curves which correspond to the full drawn w-meridian quadrant. In particular, for $n = 6$, they make up the three entire meridians into which the real meridian is transformed by rotations of 0°, 60°, 120°.

We can now also understand that the totality of the values $z = \varepsilon' \cdot r$, where r again ranges through real values from $+1$ to ∞, corresponds to the dotted part of the real w-meridian; for the equation (1) yields then:

$$w = \frac{1}{2}\left(\varepsilon'^n r^n + \frac{1}{\varepsilon'^n r^n}\right) = -\frac{1}{2}\left(r^n + \frac{1}{r^n}\right),$$

and this expression actually always actually decreases from -1 to $-\infty$, But $z = \varepsilon' \cdot r$ represents *the meridian quadrant from* ∞ *to the equatorial point* ε^ν. If we now
[128] apply to it the substitutions (2a), (2b), we find, as before, *that to the dotted part of the real* w*-meridian there correspond all the meridian quadrants joining the poles to the equatorial points* $\varepsilon' \cdot \varepsilon^\nu$, *which thus bisect the angles between the meridian quadrants which we obtained before*. In particular, for $n = 6$, they make up the three entire meridians into which the real meridian is transformed by rotations of 30°, 90°, 150°.

There remain to be found *the* $2n$ *curvilinear segments which correspond to the long-dotted half-meridian* $-1 < w < +1$. I shall prove that they are *the segments of the equator of the* z*-sphere determined by the points* ε^{ν} *and* $\varepsilon' \cdot e^{\nu}$. In fact, the equator represents the points of absolute value one and is given therefore by $z = e^{i\varphi}$ where φ is real and ranges from 0 to 2π, Hence we have

$$w = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \frac{1}{2}\left(e^{ni\varphi} + e^{-ni\varphi}\right) = \cos n\varphi.$$

This expression is always real, and its absolute value is not greater than 1. In fact, it assumes once every value between $+1$ and -1 as φ varies from one multiple of π/n to the next one, i.e., when z traverses one of the segments of which we are speaking.

The curves determined in this manner divide the z*-sphere into* $2 \cdot 2n$ *triangular half-regions which are bounded by one curve of each of the three sorts, and each half-region corresponds to a half-sheet of the Riemann surface. Several regions can meet only at the critical points,* and then in accordance with the table of multiplicities (p. [116]), namely, $2n$ *at the north pole, and at the south pole, and* $2 \cdot 2$ *at each of the points* ε^{ν} *and* $\varepsilon' \cdot \varepsilon^{\nu}$. In order to determine which of these regions are to be shaded, we notice that when w traverses, in order, the full-drawn, the long-dotted, and the short dotted parts of the real w-meridian, the rear half of the w-sphere lies at its left. Since the mapping is conformal with preservation of angles, we should shade those half-regions whose boundaries follow one another in this same sense, and we should leave the others unshaded.

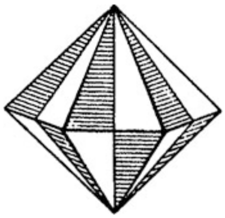

Figure 44

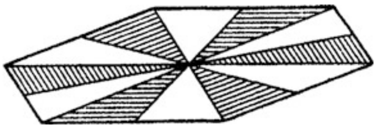

Figure 45

We have now obtained a complete geometric picture of the mutual dependence between z and w which is set up by our equation. We might follow it out in greater detail by examining more closely the conformal mapping of the single triangular regions upon the w-hemisphere, but we shall forego this. *I shall describe only,* and *briefly, the case* $n = 6$, *to which I have already given special attention*. The z-sphere is then divided into twelve shaded and twelve unshaded triangles of which six of each sort are visible in Fig. 44. Six of each sort meet at each pole, and two
[129] of each sort at each of twelve equidistant points of the equator. Each triangle is mapped conformally upon a w half-sheet of the same sort. Of the half-sheets of the Riemann surface, six of each sort are connected at the branch point ∞, and two of each sort at each of the branch points ± 1, corresponding to the grouping of the half-regions on the z-sphere.

We may obtain a convenient picture of the division of the z-sphere, and one which is especially valuable because of its analogy with pictures soon to come, as follows. If we join the n equidistant points on the equator (e.g., all the ε^{ν}) with one another in order by straight lines, and also join each of them to the two poles, one obtains a *double pyramid, with* $2n$ *faces, inscribed in the sphere* (in Fig. 44, six faces). If we now project, from the centre, the subdivision of the z-sphere upon this double pyramid, every pyramid face is divided into a shaded and an unshaded half by the altitude of that face dropped from the pole. If we represent the division of the z-sphere, and consequently our function, by means of this double pyramid, the latter will render a service quite analogous to that which we shall get in the coming examples from the *regular polyhedra*. We obtain a *complete analogy if we think of the double pyramid as collapsed into its base,* and consider the *double regular* n*-gon* (hexagon) which results whose two faces (upper and lower) are divided each into $2n$ triangles by the straight lines which join the centre with the vertices and the middle points of the sides (see Fig. 45). I *have been in the habit of calling this figure a dihedron and of classing it with the five regular polyhedra which have been studied since Plato's time*. It fulfils, in fact, all the conditions by means of which a regular polyhedron is usually defined, since its faces (the two faces of the n-gon) are congruent regular polygons, and since it has congruent edges (the sides of the n-gon) and congruent vertices (the vertices of the n-gon). The only difference is that it does not bound a proper solid body but encloses the volume zero. Thus the theorem of Plato, that there are only five regular solids, is correct only when one includes in the definition the requirement of a *proper* solid, which is always tacitly assumed in the proof

[130] *If we start with the dihedron, we obtain our subdivision of the z-sphere by projecting upon that sphere not only its vertices but also the centres of its edges and its faces,* the projecting rays for the latter being perpendicular to the plane of the dihedron. *Thus the dihedron can also be looked upon as representing the functional relation which our equation sets up between w and z.* Hence the brief name which we have already used, dihedral equation, is appropriate.

In addition, we shall now consider those equations which, as already indicated, are closely related to the platonic regular solids.

3. The Tetrahedral, the Octahedral, and the Icosahedral Equations

We shall see that the last two could, with equal right, be called the cubic and the dodecahedral equations, so that all five regular bodies will have been covered. We shall follow here a route that is the reverse of the one we followed in the preceding example. *Starting from the regular body, we shall first deduce a division of the sphere into regions, and we shall then set up the appropriate algebraic equation, for which that figure* is the proper *geometric visualisation*. I shall have to confine myself frequently to suggestions, however, and I therefore refer you at once to my book: *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*[76], in which you will find a systematic presentation of the entire extensive theory with its numerous relations to allied fields.

Moreover, I shall give a parallel treatment of all three cases and I shall begin by *deducing the subdivision of the sphere for the tetrahedron*.

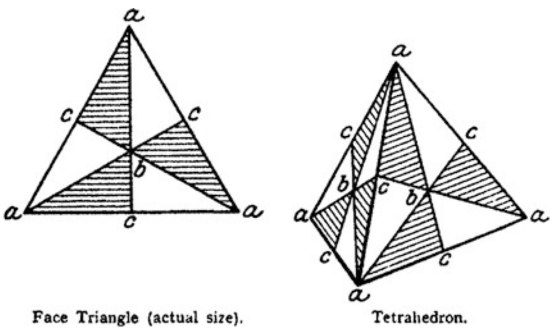

Figure 46

1. *The tetrahedron* (see Fig. 46). We divide each of the four equilateral face-triangles of the tetrahedron, by means of the three altitudes into *six partial triangles*. These are congruent in two groups of three each, while any two non-congruent ones are symmetric. We obtain thus a *division of the entire surface of the tetrahedron into twenty-four triangles, which fall into two groups, each containing twelve congruent triangles, while any triangle of one group is symmetric to every triangle of the* **[131]** *other group*. We shall shade the triangles of one group. Among the vertices of these twenty-four triangles we can distinguish *three sorts,* such that each triangle has one vertex of each sort:

a) *the four vertices of the initial tetrahedron, at each of which three shaded and three unshaded triangles meet;*

[76] Leipzig 1884; referred to hereafter as "*Ikosaeder*". Translation into English by G. C. Morrice: *Lectures on the Icosahedron by Klein*. Revised Edition, 1911, Kegan Paul & Co.

b) *the four centres of gravity of the faces,* which determine again another regular tetrahedron (the co-tetrahedron); *at each of these, three triangles of each kind meet;*

c) *the six middle points of the edges,* which determine a *regular octahedron; at each of these, two triangles of each kind meet.*

If from the centre of gravity of the tetrahedron we project this subdivision into triangles upon the circumscribed sphere, the latter will be *subdivided into* $2 \cdot 12$ *triangles, which are bounded by arcs of great circles and are mutually congruent or symmetric.* About each vertex of the sort a), b), c), there will be respectively 6, 6, 4 equal angles, and since the sum of the angles about a point on a sphere is 2π, *each of the spherical triangles will have an angle* $\pi/3$ *at a vertex of the sort a or b and an angle* $\pi/2$ *at a vertex of the sort c.*

It is a *characteristic property of this division of the sphere that it, as well as the tetrahedron itself, is transformed into itself by a number of rotations of the sphere about its centre.* This will be clear to you in detail if you examine a model of the tetrahedron with its divisions, like the one in our collection. For the lecture, it will suffice if I indicate the *number of possible rotations* (whereby the position of rest is included as the *identical rotation*. If we select a definite vertex of the original tetrahedron, we can, by means of a rotation, transform it into every vertex of the tetrahedron (including itself), which gives four possibilities. If we keep this vertex fixed in any one of these four positions, we can still achieve the tetrahedron to cover itself, in three different ways, namely by rotating the line connecting the centre with that vertex as axis through an angle of 0°, 120° or 240°. This gives *altogether* $4 \cdot 3 = 12$ *rotations* which transform the tetrahedron, or the corresponding triangular division of the circumscribed sphere, into itself. By means of these rotations we can transform a pre-assigned shaded (or unshaded) triangle into every other shaded (or unshaded) triangle, and the particular rotation is determined when that latter triangle is given. These *twelve rotations* form obviously what one calls *a group* G_{12} of twelve operations, i.e., if one performs two of them in succession, the result is one of the twelve rotations.

[132] If we think of this sphere as the z-sphere, each of these twelve rotations will be represented by a *linear transformations* of z, and *the twelve linear transformations which arise in this manner will transform into itself the equation which corresponds to the tetrahedron.* For purposes of comparison, I remark that *one can interpret the* $2n$ *linear substitutions of the dihedral equation as the totality of the rotations of the dihedron into itself.*

2. We shall now treat the *octahedron* similarly (see Fig. 47) and we may be somewhat briefer. We divide each of the eight faces, just as before, into *six partial triangles* and obtain a *division of the entire surface of the octahedron into twenty-four congruent shaded triangles, and twenty-four unshaded triangles which are congruent among themselves but symmetric to the other twenty-four.* We can again distinguish *three sorts of vertices:*

a) *the six vertices of the octahedron, at each of which four triangles of each kind meet;*

b) *the eight centres of gravity of the faces,* which form the *vertices of a cube;* at each of these, three triangles of each kind meet;

c) *the twelve midpoints of the edges, at each of which are situated two triangles of each kind.*

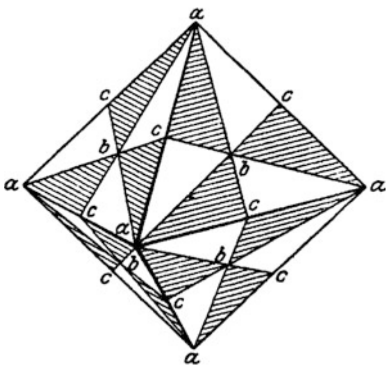

Figure 47

If we pass now to the *circumscribed sphere,* by means of central projection, we obtain a division into $2 \cdot 24$ spherical triangles, which are either congruent or symmetric, and each of which has an angle $\pi/4$ at the vertex a, $\pi/3$ at the vertex b, and $\pi/2$ at the vertex c. Since the vertices b form a cube, it is easy to see that *one would have obtained the same division on the sphere if one had started with a cube and had projected its vertices, and the centres of its faces and edges, upon the sphere*. In other words, we do not need to give special attention to the cube.

Just as in the previous case, it is easy to see *that the octahedron, as well as this division of the sphere, is transformed into itself by twenty-four rotations which form a group* G_{24}; *again each rotation is determined in that it transforms a pre-assigned shaded triangle into another definite shaded triangle*.

3. We come finally to the *icosahedron* (see Fig. 48). Here, also, we start with the same subdivision of each of the twenty-four triangular *faces and obtain altogether sixty shaded and sixty unshaded partial triangles*. The *three sorts of vertices* are:

a) *the twelve vertices of the icosahedron, at each of which are situated five* **[133]**
triangles of each kind;

b) *the twenty centres of gravity of the faces,* which are the *vertices of a regular dodecahedron; at each of them are situated three triangles of each kind;*

c) *the thirty midpoints of the edges, at each of which two triangles of each sort meet*.

When this is carried over to the sphere each spherical triangle has at the vertices a, b, c the angles $\pi/5$, $\pi/3$, $\pi/2$, respectively. From the property of the vertices b one can conclude, as before, *that the same division of the sphere would have resulted if one had considered the dodecahedron*.

Finally, the icosahedron, as well as the corresponding division of the sphere, is transformed into itself by a group G_{60} *of sixty rotations of the sphere about its* *centre*. These rotations, as well as those for the octahedron, will become clear to you upon examination of a model.

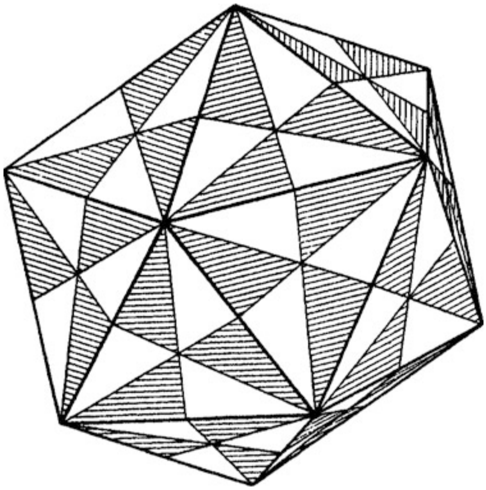

Figure 48

Let me make a list of *the angles of the spherical triangles,* which have appeared in the three cases, which we have considered, to which I shall add the dihedron also ($n \geq 3$); they are

Dihedron: $\pi/2, \pi/n, \pi/2$;
Tetrahedron: $\pi/3, \pi/3, \pi/2$;
Octahedron: $\pi/4, \pi/3, \pi/2$;
Icosahedron: $\pi/5, \pi/3, \pi/2$

As a variation of a joke of Kummer's I might suggest that the natural scientist would at once conclude from this, that there were additional subdivisions of the sphere, having analogous properties, and with angles such as $\pi/6$, $\pi/3$, $\pi/2$; $\pi/7$, $\pi/3$, $\pi/2$. The mathematician, to be sure, does not risk making such inferences by analogy, and his cautiousness justifies itself here, for *the series of possible spherical*
[134] *subdivisions of this sort ends,* in fact, *with our list*. Of course this is connected with the fact *that there are no more regular polyhedra*. We can see its ultimate reason in *a property of whole numbers,* which does not admit a reduction to simpler reasons. It appears, namely, that the angles of each of our triangles must be aliquot parts of π, say π/m, π/n, π/r, such that the denominators satisfy the inequality

$$1/m + 1/n + 1/r > 1.$$

This inequality has the property of existing only for the integer solutions given above. Moreover, we can understand it readily, since it only expresses the fact that the sum of the angles of a spherical triangle exceeds π.

I should like to mention that, as some of you doubtless know, an *appropriate generalisation* of the theory does carry one beyond these apparently too narrow bounds: *The theory of automorphic functions* involves subdividing the sphere into *infinitely many triangles* whose angle sum is less than or equal to π.

4. Continuation: Setting up the Normal Equation

We come now to the second part of our problem, *to set up that equation of the form*

$$\text{(1)} \qquad \varphi(z) - w\psi(z) = 0, \quad \text{or} \quad w = \frac{\varphi(z)}{\psi(z)},$$

which belongs to a definite one of our three spherical subdivisions, that is, which maps the two hemispheres of the w-sphere upon the $2 \cdot 12$, or the $2 \cdot 24$, or the $2 \cdot 60$ partial triangles of the z-sphere. To each value of w there must correspond then, in general, 12, 24, 60 values, respectively, of z, each one in a partial triangle of the right kind. Hence the desired equation must have the *degree* 12, 24, 60 in the three cases respectively, for which we shall write N in general. Now each partial region touches three critical points; hence there must be, in every case, *three branch points* on the w-sphere. We assign these, as is customary, to $w = 0, 1, \infty$; and we choose again the *meridian of real numbers* as the *section curve* $\mathfrak{C}$ through these three points, whose three segments shall correspond to the boundaries of the z triangles.

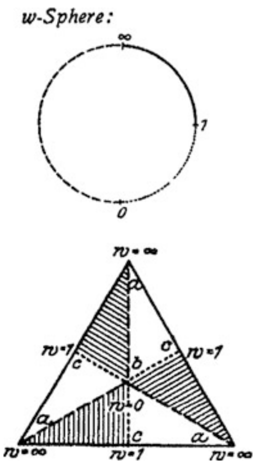

Figure 49

We shall assume moreover (see Fig. 49) that in each of the three cases *the centres of gravity of the faces* (vertices b in the former notation) *correspond, to the point* $w = 0$, *the midpoint of the edges* (vertices c) *to the point* $w = 1$, *and the vertices*

of the polyhedron (vertices a) *to the point* $w = \infty$. The sides of the triangles will then correspond to the three segments of the w-meridian in the manner indicated by
[135] the mapping, and the shaded triangles will correspond to the rear w-hemisphere, the unshaded to the front w-hemisphere. By virtue of these correspondences, the equation (1) is to effect a unique mapping of the z-sphere upon an N-sheeted Riemann surface over the w-sphere with branch points at $0, 1, \infty$.

We might deduce, a priori, a proof for the existence of this equation by means of general function-theoretic theorems. However, I prefer not to presuppose the knowledge which this method would require, but to *construct* the various equations *empirically*. This method will give us perhaps a more vivid perception of the individual cases.

Let us think of equation (1) written in homogeneous variables

$$\frac{w_1}{w_2} = \frac{\Phi_N(z_1, z_2)}{\Psi_N(z_1, z_2)},$$

where Φ_N, Ψ_N are homogeneous polynomials of dimension N in $z_1, z_2 (N = 12, 24, \text{ or } 60)$. In this form of the equation, the points $w_1 = 0, w_2 = 0$ (i.e., $w = 0, \infty$) on the w-sphere seem to be favoured more than the third branch point $w = 1$ (in homogeneous form, $w_1 - w_2 = 0$). Since, however, the three branch points are, for our purpose, of equal importance, it is expedient to consider also the following form of the equation:

$$\frac{w_1 - w_2}{w_2} = \frac{X_N(z_1, z_2)}{\Psi_N(z_1, z_2)},$$

where $X_N = \Phi_N - \Psi_N$ denotes also a form of dimension N. Both forms are embraced in the continued proportion

$$(2) \qquad w_1 : (w_1 - w_2) : w_2 = \Phi_N(z_1, z_2) : X_N(z_1, z_2) : \Psi_N(z_1, z_2).$$

This furnishes us with a completely homogeneous form of equation (1) which gives the same consideration to all the three branch points.

Our problem now is *to set up the forms* Φ_N, X_N, Ψ_N. For this purpose, we shall bring them into relation to our subdivision of the z-sphere. From equation (2) we see that the form $\Phi_N(z_1, z_2) = 0$ for $w_1 = 0$, i.e., *that* $w = 0$ *corresponds to the* N *zeros of* Φ_N *on the z-sphere*. On the other hand, one sees directly from equation (2) that *the midpoints of the faces of the polyhedron* (vertices b in the subdivision), of which there are $N/3$ in every case, must, according to our assumptions, correspond to the branch point $w = 0$. But every one of these centres must be *a triple root*
[136] of our equation, since in each of them there meet three shaded and three unshaded triangles of the z-sphere. *Thus these points, each with multiplicity three, supply all the positions which correspond to* $w = 0$, *and consequently all the zeros of* Φ_N. Hence Φ_N has only triple zeros and must, therefore, be *the third power* of a form $\varphi_N(z_1, z_2)$ of degree $N/3$:

$$\Phi_N = \left[\varphi_{N/3}(z_1, z_2)\right]^3.$$

In the same way, it follows that the zeros of $X_N = 0$ correspond to the position $w = 1$ (i.e., $w_1 - w_2 = 0$), and that these are identical with the $N/2$ midpoints, each counted twice, of the edges of the polyhedron (vertices c of our subdivision). Consequently X_N must be the square of a form of dimension $N/2$:

$$X_N = \left[\chi_{N/2}(z_1, z_2)\right]^2.$$

Finally the zeros of Ψ_N are to correspond to the point $w = \infty$, so that they must be identical with the vertices of the polyhedron (vertices a of the subdivision); but at these vertices 3, 4, or 5 triangles meet, in the several cases, so that we get

$$\Psi_N = [\psi_{N/v}(z_1, z_2)]^v, \quad \text{where } v = 3, 4 \text{ or } 5.$$

Our equation (2) *must then necessarily have the form*

(3) $$w_1 : (w_1 - w_2) : w_2 = \varphi(z_1, z_2)^3 : \chi(z_1, z_2)^2 : \psi(z_1, z_2)^\nu,$$

where the degrees and powers of φ, χ, ψ, and the values of the degree N of the equation are exhibited in the following table:

Tetrahedron: $\varphi_4^3, \chi_6^2, \psi_4^3; N = 12.$
Octahedron: $\varphi_8^3, \chi_{12}^2, \psi_6^4; N = 24.$
Icosahedron: $\varphi_{20}^3, \chi_{30}^2, \psi_{12}^5; N = 60.$

I shall now show briefly *that the dihedral equation which we discussed, fits also into the scheme* (8). We need only to recall that in that case we chose $-1, +1, \infty$ as the branch points on the w-sphere instead of $0, +1, \infty$ which we selected later. We shall, then, obtain actual analogy with (8) only if we throw the dihedral equation into the form

$$(w_1 + w_2) : (w_1 - w_2) : w_2 = \Phi : X : \Psi.$$

Now from the dihedral equation (p. [115]) which we used:

$$\frac{w_1}{w_2} = \frac{z_1^{2n} + z_2^{2n}}{2 \cdot z_1^n z_2^n},$$

we get by simple reduction

$$\begin{aligned}(w_1 + w_2) &: (w_1 - w_2) : w_2 \\ &= \left(z_1^{2n} + z_2^{2n} + 2z_1^n z_2^n\right) : \left(z_1^{2n} + z_2^{2n} - 2z_1^n z_2^n\right) : \left(2z_1^n z_2^n\right) \\ &= \left(z_1^n + z_2^n\right)^2 : \left(z_1^n - z_2^n\right)^2 : 2\,(z_1 z_2)^n.\end{aligned}$$

Thus we can, in fact, add to the above table: **[137]**

Dihedron: $\varphi_n^2, \chi_n^2, \psi_2^n; N = 2n.$

The critical points together with their multiplicities which can at once be read off from this form of the equation are in full agreement with those which we found above (see p. [125]).

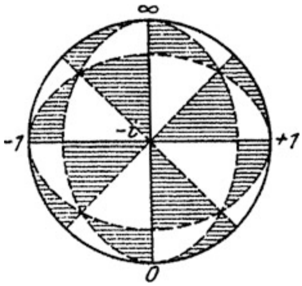

Figure 50

We come now to the *actual setting up of the forms* φ, χ, ψ *in the three new cases. I shall give details here only for the octahedron,* for which the relations turn out to be the simplest. But even here I shall, at times, give only suggestions or results, in order to remain within the confines of a brief survey. For those who desire more, there is easily accessible the detailed exposition in my book on the icosahedron. For the sake of simplicity we think of the octahedron as so inscribed in the z-sphere *that the six vertices fall on* (see Fig. 50):

$$z = 0, \infty, +1, +i, -1, -i.$$

It will then be a simple matter to give the *twenty-four linear substitutions of* z which represent the rotations of the octahedron, i.e., which permute these six points. We begin with the four rotations in which the vertices 0 and ∞ remain fixed

(4a)
$$z' = i^k \cdot z,$$
$$(k = 0, 1, 2, 3).$$

Then we can interchange the points $0, \infty$ by means of the substitution $z' = 1/z$ (i.e., a rotation through 180° about the horizontal axis $(+1, -1)$ which transforms every point of the octahedron into another one. If we now apply the four rotations (4a), we get four new substitutions:

(4b)
$$z' = \frac{i^k}{z}$$
$$(k = 0, 1, 2, 3).$$

In the same way, we now throw in succession the four remaining vertices $z = 1, i, -1, -i$ to ∞ by means of the substitutions

$$z' = \frac{z+1}{z-1}, \quad \frac{z+i}{z-i}, \quad \frac{z-1}{z+1}, \quad \frac{z-i}{z+i},$$

which obviously permute the six vertices of the octahedron, and again apply, each time, the four rotations (4a). Thus we get $4 \cdot 4 = 16$ additional substitutions for the octahedron

$$\text{(4c)} \qquad \begin{cases} z' = i^k \dfrac{z+1}{z-1}, & z' = i^k \dfrac{z-1}{z+1}, \\ z' = i^k \dfrac{z+i}{z-i}, & z' = i^k \dfrac{z-i}{z+i}. \end{cases} \quad (k = 0, 1, 2, 3)$$

We have therefore found all the desired twenty-four substitutions, and we can easily **[138]** confirm, by calculation, that *they really permute the six vertices of the octahedron and that they form a group* G_{24}, i.e., that the successive application of any two of them gives again one of the substitutions in (4).

I shall now construct *the form* ψ_6 which vanishes in each of the vertices of the octahedron. The point $z = 0$ gives the factor z_1, the point $z = \infty$ the factor z_2; the form $z_1^4 - z_2^4$ has a simple zero at each of the points ± 1, $\pm\, i$, so that we obtain finally

$$\text{(5a)} \qquad \psi_6 = z_1 \cdot z_2 \left(z_1^4 - z_2^4\right).$$

It is more difficult to construct the forms φ_8 and χ_{12} which have as zeros the midpoints of the faces and the midpoints of the edges. Without deducing them, I may state that they are[77]

$$\text{(5b)} \qquad \begin{cases} \varphi_8 \;= z_1^8 + 14 z_1^4 z_2^4 + z_2^8 \\ \chi_{12} = z_1^{12} - 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 + z_2^{12}. \end{cases}$$

It goes without saying that there is an undetermined constant multiplier in each of these three forms. If φ_8, ψ_6, χ_{12} stand for the normal forms (5), we must insert, in the octahedral equation (3), two undetermined constants c_1, c_2, and we must write

$$w_1 : (w_1 - w_2) : w_2 = \varphi_8^3 : c_1 \chi_{12}^2 : c_2 \psi_6^4.$$

The constants c are now to be so determined that these two equations give actually only *one* equation between z and w. This is possible when and only when

$$\varphi_8^3 - c_2 \psi_6^4 = c_1 \chi_{12}^2$$

is an identity in z_1 and z_2. Now this relation can be satisfied by definite constants c_1 and c_2. A brief calculation shows that the identity

$$\varphi_8^3 - 108 \psi_6^4 = \chi_{12}^2$$

[77] See *Ikosaeder,* p. [54].

must hold, so that the octahedral equation (3) becomes:

$$(6) \qquad w_1 : (w_1 - w_2) : w_2 = \varphi_8^3 : \chi_{12}^2 : 108\psi_6^4.$$

This equation surely maps *the points* $0, 1, \infty$ *respectively upon the midpoints of the faces, the midpoints of the edges, and the vertices of the octahedron, with, the proper multiplicity,* because the forms φ, χ, ψ, were so constructed. Furthermore, the twenty-four octahedron substitutions (4) transform it into itself, for they transform the zeros of each of the forms φ, χ, ψ into themselves and at the same time change each of the forms by a multiplicative factor. And calculation shows that these factors cancel when the quotients are formed.

[139] It only remains to show that equation (6) really maps *each shaded or unshaded triangle of the z-sphere conformally upon the rear or front w-hemisphere*. We know that the points $0, 1, \infty$ of the real w-meridian correspond to the three vertices of each of the triangles; but the equation has, moreover, twenty-four roots z for each value of w. Since these must distribute themselves among the twenty-four triangles, w can take a given value but once, at most, within a triangle. If we could only show that *w remains real on the three sides of a triangle,* we could then easily show that there is a one-to-one mapping of each side upon a segment of the real w-meridian, and also a *similar mapping of the entire interior of the triangle upon the corresponding hemisphere, one, which is conformal without reversal of angles*. You will be able to make these deductions yourselves by making use of the continuity and the analytic character of the function $w(z)$. I shall indicate the only noteworthy step of the proof, that of showing the reality of w upon the sides of the triangle.

It is more convenient to prove this by showing *that w is real upon all the great circles that arise in the octahedral subdivision*. These are, first, the three mutually perpendicular circles which pass each through four of the six vertices of the octahedron (*principal circles;* full drawn in Fig. 50, p. [137]) and, second, the six circles, corresponding to the altitudes of the faces, which bisect the angles of the principal circles (*auxiliary circles;* long dotted in Fig. 50). By means of the octahedron substitutions, one can transform every principal circle into any other and every auxiliary circle into any other. Hence it will suffice to show *that the function w is real at every point on o n e principal and o n e auxiliary circle,* since it must take the same values on the other circles. Now the meridian of real numbers z is one of the principal circles. By (6), the values on this circle are

$$w = \frac{w_1}{w_2} = \frac{\varphi_8^3}{108\psi_6^4},$$

which are, of course, real, since φ and ψ are real polynomials in z_1 and z_2. Of the auxiliary circles let us prefer the one through 0 and ∞ which makes an angle of 45° with the real meridian and on which z takes the values $z = e^{\frac{i\pi}{4}} \cdot r$, where r ranges through real values from $-\infty$ to $+\infty$. On this circle $z^4 = e^{i\pi} \cdot r^4 = -r^4$ is real. Since by (5) only the fourth powers of z_1 and z_2 occur in φ_8 and in the fourth power of ψ_6, the last formula shows that w is real.

This concludes the proof: *Equation* (6), *in fact, maps the w-hemisphere, or the Riemann surface over it, conformally upon that triangular subdivision of the z-sphere which corresponds to the octahedron, and conversely we have in this case, as completely as in the earlier examples, a geometric control of the dependence which this equation sets up between z and w*. **[140]**

The treatment of the tetrahedron and of the icosahedron proceeds according to the same plan. I shall give only the *results*. As before, these results are those obtained when the subdivision of the z-sphere has the simplest possible position. The *tetrahedral equation*[78] is

$$\begin{aligned} w_1 : (w_1 - w_2) : w_2 = & \left\{z_1^4 - 2\sqrt{-3}z_1^2 z_2^2 + z_2^4\right\}^3 \\ & : -12\sqrt{-3}\left\{z_1 z_2\left(z_1^4 - z_2^4\right)\right\}^2 \\ & : \left\{z_1^4 + 2\sqrt{-3}z_1^2 z_2^2 + z_2^4\right\}^3 \end{aligned}$$

and the *icosahedral equation*[79] is

$$\begin{aligned} & w_1 : (w_1 - w_2) : w_2 \\ & = \left\{-\left(z_1^{20} + z_2^{20}\right) + 228\left(z_1^{15} z_2^5 - z_1^5 z_2^{15}\right) - 494 z_1^{10} z_2^{10}\right\}^3 \\ & \quad : -\left\{\left(z_1^{30} + z_2^{30}\right) + 522\left(z_1^{25} z_2^5 - z_1^5 z_2^{25}\right) - 10005\left(z_1^{20} z_2^{10} + z_1^{10} z_2^{20}\right)\right\}^2 \\ & \quad : 1728\left\{z_1 z_2\left(z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}\right)\right\}^5 , \end{aligned}$$

i.e., *these equations map the w-hemispheres conformally upon the shaded and the unshaded triangles of that subdivision of the z-sphere, which belongs to the tetrahedron and to the icosahedron respectively*.

5. Concerning the Solution of the Normal Equations

Let us now consider somewhat the *common properties of the equations* which we have been discussing up to now as examples of a general theory developed in advance and which we shall call the *normal equations*. Here, too, I will restrict myself to the simplest cases. For in-depth studies, I am referring to my *Ikosaeder* book. Note, first of all, that *the extremely simple nature of all our normal equations* is due to the fact that *they have exactly the same number of linear substitutions into themselves as is indicated by the degree,* i.e., *that all their roots are linear functions of a single one; and, further, that we have, in the divisions of the sphere, a very obvious geometric picture of all the relations that come up for consideration*. Just

[78] See *Ikosaeder,* p. [51], [60].
[79] Loc. cit., p. [56], [60].

how simple many things appear which are ordinarily quite complicated with equations of such high degree will be evident if I raise a certain question in connection with the icosahedral equation.

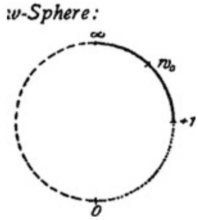

Figure 51

Let a *real value* w_0 be given, say on the segment $(1, \infty)$ of the real w-meridian (see Fig. 51). *Let us inquire about the sixty roots z of the icosahedral equation*
[141] *when* $w = w_0$. Our theory of the mapping tells us at once that *one of them must lie on a side of each of the sixty triangles on the z-sphere, which arise in the case of the icosahedron* (drawn full in Fig. 49, p. [135]). This supplies what one calls, in the theory of equations, the *separation of the roots,* usually a laborious task, which must precede the numerical calculation of the roots. *The task is, namely, that of assigning separated intervals in each of which but one root lies*. But we can also tell at once *how many of the roots are real*. If we take into account, namely, that the form of the icosahedral equation given above implies such a placing[80] of the icosahedron in the z-sphere *that the real meridian contains four vertices of each of the three sorts a, b, c*, then it follows (see Fig. 48, p. [133], and Fig. 49, p. [135]) that four full-drawn triangle sides lie on the real meridian, *so that there are just four real roots*. The same is true if w lies in one of the other two segments of the real w-meridian, *so that for every real w different from* $0, 1, \infty$ *the icosahedral equation has four real and fifty-six imaginary roots; for* $w = 0, 1, \infty$ *there are also four different real roots, but they are multiple roots*.

I shall now say *something about the actual numerical calculation of the roots of our normal equations*. We have here again the great advantage that we *need to calculate but one root,* because the others follow by linear substitutions. Let me remind you, however, *that the numerical calculation of a root is actually a problem of analysis, not of algebra,* since it requires necessarily the application of infinite processes when the root to which one is approximating is irrational, as is the case in general.

I shall go into details only for the *simplest example of all,* the *pure equation*

$$w = z^n.$$

Here I come again *into immediate touch with school mathematics*. For this equation, i.e., the calculation of $\sqrt[n]{w}$, at least for the small values of n and for real values of

[80] See *Ikosaeder,* p. [55].

$w = r$, is treated there also. The method of calculating square and cube root, as you learned it in school, depends, in essence, upon the following procedure. One determines the position which the radicand $w = r$ has in the series of the squares or cubes, respectively, of the natural numbers 1, 2, 3, ... Then, using the decimal notation, one makes the same trial with the tenths of the interval concerned, then with the hundredths, and so on. In this way one can, of course, approximate with any desired degree of closeness.

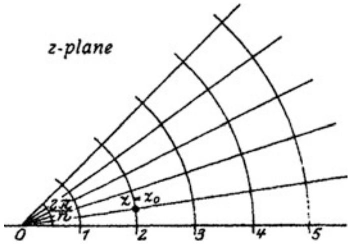

Figure 52

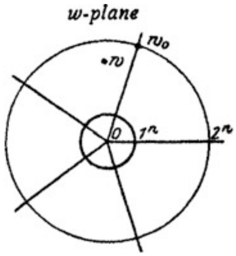

Figure 53

I should like to apply a *more rational procedure,* one in which we can admit not only arbitrary integer values of n but also *arbitrary complex values of* w. Since we need to determine *only one* solution of the equation, we shall seek, in particular, that value $z = \sqrt[n]{w_0}$ which lies *within the angle* $2\pi/n$ *laid off on the axis of real numbers*. Generalising the elementary method mentioned above, we begin by dividing this angular space into ν equal parts ($\nu = 5$ in Fig. 52), and by drawing circles intersecting the dividing rays by circles which have the origin as common centre and whose radii are measured by the numbers $r = 1, 2, 3, \ldots$ In this way, after choosing ν, we find all the points **[142]**

$$z = r \cdot e^{\frac{2i\pi}{n} \cdot \frac{k}{\nu}} \left(\begin{array}{l} k = 0, 1, 2, ..., \nu - 1 \\ r = 1, 2, 3, ..., \end{array} \right)$$

marked within the angular space, and we can at once mark in the w-plane the corresponding w-values

$$w = z^n = r^n \cdot e^{2i\pi \frac{k}{\nu}}.$$

These will be the corners of a corresponding network (see Fig. 53) covering the *entire* w-plane and consisting of circles with radii $1^n, 2^n, 3^n, \ldots$ together with rays inclined to the real axis at angles of $0, 2\pi/\nu, 4\pi/\nu, \ldots, (v-1)2\pi/\nu$. Let the given value of w lie either within or on the contour of one of the meshes of this lattice, and suppose that w_0 is the lattice corner nearest to it. We know a value z_0 of $\sqrt[n]{w_0}$ is a corner of the original lattice in the z-plane; hence the value we are seeking will be

$$Z = \sqrt[n]{w} = \sqrt[n]{w_0 + (w - w_0)} = \sqrt[n]{w_0}\sqrt[n]{1 + \frac{w - w_0}{w_0}} = z_0\left(1 + \frac{w - w_0}{w_0}\right)^{\frac{1}{n}}.$$

We expand the right side by the *binomial theorem,* which we may consider known, inasmuch as we are now, in reality, in the domain of analysis

$$Z = z_0\left\{1 + \frac{1}{n}\cdot\frac{w - w_0}{w_0} + \frac{1-n}{2n^2}\left(\frac{w - w_0}{w_0}\right)^2 + \cdots\right\}.$$

[143] We can decide at once as to the convergence of this series if we look upon it as the *Taylor's expansion of the analytic function* $\sqrt[n]{w}$ and apply the theorem that it converges within the circle which has w_0 as centre and which passes through the nearest singular point. Since $\sqrt[n]{w}$ has only 0 and ∞ as singular points, *our expansion will converge if, and only if, w lies within that circle about w_0 which passes through the origin,* and we can always bring this about by starting, in the z-plane, with a similar lattice which may have smaller meshes, if necessary. *But in order that the convergence should be really good, i.e., in order that the series should be adapted to numerical calculation, $(w - w_0)/w_0$ must – additionally – be sufficiently small.* This can always be effected by a further reduction of the lattice. This is really a very usable method for the actual calculation of numerical roots.

Now is it worthy of remark that the numerical solution of the remaining normal equations of the regular solids is not essentially more difficult, but I shall omit the proof. If we apply, namely, the same method to our normal equations, starting from the mapping upon the w-sphere of two neighbouring triangles, there will appear, in place of the binomial series, certain other series that are well known in analysis and are well adapted to practical use, called *the hypergeometric series*. In the year 1877 I set up[81] this series numerically.

6. Uniformisation of the Normal Irrationalities by Means of Transcendental Functions

I shall now discuss *another method of solving our normal equations* which is characterized by the *systematic employment of transcendental functions*. Instead of proceeding, in each individual case, with series expansions in the neighbourhood

[81] [*Weitere Untersuchungen über das Ikosaeder, Mathematische Annalen*, vol. 12, p. 515. See also F. Klein, *Gesammelte Mathematische Abhandlungen,* vol. 2, p. 331 et seq.]

of a known solution, we try to represent, once for all, *the whole set of number pairs* (w, z) *which satisfy the equation, as single-valued analytic functions of an auxiliary variable: or, as we say, to uniformise the irrationalities defined by the equation.* If we can succeed by using only functions which can easily be tabulated, or of which one already has, perhaps, numerical tables, one can obtain the *numerical solution of the equation without farther calculation.* I am the more willing to discuss this connection with transcendental functions because it sometimes plays a part in *school* **[144]** *teaching,* where it still often has a hazy, almost mysterious, aspect. The reason for this is that one is still clinging to traditional imperfect conceptions, although the modern theory of functions of a complex variable has provided perfect clarity.

I shall apply these general suggestions first to the *pure equation. Even in schools, one always uses logarithms in calculating the positive solution of* $z^n = r$, *for real positive values of* r. We write the equation in the form $z = e^{\log r/n}$, where $\log r$ stands for the positive principal value. The logarithmic tables supply first $\log r$, and then, conversely, z is the number that corresponds to $\frac{\log r}{n}$. Moreover, we ordinarily use 10 as base instead of e. This solution can be extended immediately to complex values. We satisfy the equation

$$z^n = w,$$

by putting x equal to the general complex logarithm, $\log w$, after which we obtain w and z actually as single-valued analytic functions of x:

$$w = e^x, z = e^{\frac{x}{n}}$$

In view of the many-valuedness of $x = \log w$, which we shall study later in detail, one obtains here for the same w precisely n values of z. *We call x the uniformising variable.*

Since the tables contain only the *real logarithms of real numbers,* one is apparently unable to use immediately the given solution numerically. But by the aid of a simple property of logarithms, we can *reduce the calculation to the use of trigonometric tables which are accessible to everybody.* If we put

$$w = u + iv = \left|\sqrt{u^2+v^2}\right| \left(\frac{u}{\left|\sqrt{u^2+v^2}\right|} + i \frac{v}{\left|\sqrt{u^2+v^2}\right|} \right),$$

then the first factor, as a positive real number, has a real logarithm, the second, as a number of absolute value 1, *a pure imaginary logarithm* $i\varphi$ (i.e., the second factor is equal to $e^{i\varphi}$), and we obtain φ from the equation

$$\text{(a)} \qquad \frac{u}{\sqrt{u^2+v^2}} = \cos\varphi, \frac{v}{\sqrt{u^2+v^2}} = \sin\varphi.$$

This gives $x = \log w = \log\left|\sqrt{u^2+v^2}\right| + i\varphi$, and the root of the equation is therefore

$$z = e^{\frac{x}{n}} = e^{\frac{1}{n}\log\left|\sqrt{u^2+v^2}\right|} \cdot e^{\frac{1}{n}i\varphi},$$

i.e., we have

(b) $$z = \sqrt[n]{u + iv} = e^{\frac{1}{n}\log\left|\sqrt{u^2+v^2}\right|}\left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right).$$

[145] Since φ is determined only to within multiples of $2n$, this formula supplies all the n roots. With the aid of ordinary logarithmic and trigonometric tables, we can now get first φ from (a) und then z from (b). We have obtained this "*trigonometric solution*" *from the logarithms of complex numbers in an entirely natural way*. However, if we assume that these are not known and try to develop this trigonometric solution, as is done in the schools, it must appear as something entirely foreign and unintelligible.

The "Casus Irreducibilis" and the Trigonometric Solution of the Cubic Equation

It is in particular one topic of school mathematics where it becomes necessary to find roots of numbers that are not real. Thus, in school teaching, such roots must be found in the so called *Cardan's solution of the cubic equation* about which I should like to interpolate here a few remarks. If this equation is given in the reduced form

(1) $$x^3 + px - q = 0,$$

then the formula of Cardan states that its three roots x_l, x_2, x_3 are contained in the expression

(2) $$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Since every cube root is three valued, this expression has, all told, nine values, in general all different; among these, x_1, x_2, x_3 are determined by the condition *that the product of the two cube roots employed each time is* $-\frac{p}{3}$. If we replace the coefficients p, q in the well-known manner by their expressions as symmetric functions of x_1, x_2, x_3, and if we note that the coefficient of x^2 vanishes, that is, $x_1 + x_2 + x_3 = 0$, we get

$$\frac{q^2}{4} + \frac{p^3}{27} = -\frac{(x_1 - x_2)^2\,(x_2 - x_3)^2\,(x_3 - x_1)^2}{108},$$

that is, the radicand of the square root is, to within a negative factor, the discriminant of the equation. This shows at once that *it is negative when all three roots are real, but positive when one root is real and the other two conjugate imaginary*. It is precisely in the apparently simplest case of the cubic equation, namely when all the roots are real, that the formula of Cardan requires the extraction of the square root of a negative number, and hence of the cube root of an imaginary number.

This passage through the complex must have seemed something quite impossible to the mediaeval algebraists at a time when one was still far removed from a theory of complex numbers, 250 years before Gauß gave his interpretation of them in the plane! One talked of the "*Casus irreducibilis*" of the cubic equation and said that the Cardan formula failed here to give a reasonable usable solution. When it was discovered later that it was possible, precisely in this case, to establish a simple relation between the cubic equation and the trisection of an angle, and to get in this way a real "trigonometric solution" in place of the defective Cardan formula, it was believed that something new had been discovered which had no connection with the old formula. Unfortunately this is the position taken occasionally even today in elementary teaching! **[146]**

In opposition to this view, I should like to insist here emphatically *that this trigonometric solution is nothing else than the application, in calculating the roots of complex radicands, of the process* which we have just discussed. It is obtained therefore in a perfectly natural way in this case, where the cube root has a complex radicand, if we transform the Cardan formula, for numerical calculation, in the same convenient way that one pursues in school for the case of the real radicand. In fact, let us suppose

$$\frac{q^2}{4} + \frac{p^3}{27} < 0,$$

where p must be negative if q is real. If we then write the first cube root in (2) in the form

$$\sqrt[3]{\frac{q}{2} + i\left|\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}\right|}.$$

We note that its absolute value (as positive cube root of the value $\sqrt{-p^3/27}$ of the radicand) is equal to $|\sqrt{-p/3}|$; but since the product of this by the second cube root is equal to $-p/3$, that second cube root must be the conjugate complex of this, and the sum of the two, i.e., the solution of the cubic equation, is simply twice the real part, that is,

$$x_1, x_2, x_3 = 2\Re\left(\sqrt[3]{\frac{q}{2} + i\left|\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}\right|}\right).$$

Now let us apply the general procedure of p. [144]. We write the radicand of the cube root, after separating out its absolute value, in the form

$$\left|\sqrt{-\frac{p^3}{27}}\right|\left\{\frac{\frac{q}{2}}{\left|\sqrt{-\frac{p^3}{27}}\right|} + i\,\frac{\left|\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}\right|}{\left|\sqrt{-\frac{p^3}{27}}\right|}\right\}$$

[147] and determine an angle φ from the equations

$$\cos\varphi = \frac{\frac{q}{2}}{\left|\sqrt{-\frac{p^3}{27}}\right|}, \quad \sin\varphi = \frac{\left|\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}}\right|}{\left|\sqrt{-\frac{p^3}{27}}\right|}.$$

Then, since the positive cube root of $\left|\sqrt{-p^3/27}\right|$ is $\left|\sqrt{-p/3}\right|$, our cube root takes the form

$$\left|\sqrt{-\frac{p}{3}}\right| \cdot \left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right),$$

and hence, remembering that φ is determinate only to within multiples of 2π, we obtain

$$x_k = 2\left|\sqrt{-\frac{p}{3}}\right| \cdot \cos\frac{\varphi + 2k\pi}{3} \; (k = 0, 1, 2).$$

But this is the usual form of the trigonometric solution.

I should like to take this opportunity to make a remark about the expression "*casus irreducibilis*". "Irreducible" is used here in a sense entirely different from the one in use today and which we shall often use in this lecture course. In the sense here used it implies that the solution of the cubic equation cannot be reduced to the cube roots of real numbers. This is not in the least the modern meaning of the word. You see how the unfortunate use of words, together with the general fear of complex numbers, has created at least the possibility for a good deal of misunderstanding in just this field. I hope that my words may serve as a preventive, at least among you.

Let us now inquire briefly *about uniformisation by means of transcendental functions in the case of the remaining normal irrationalities*. In the dihedral equation

$$z^n + \frac{1}{z^n} = 2w$$

we put simply

$$w = \cos\varphi.$$

De Moivre's formula shows that the equation is then satisfied by

$$z = \cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}.$$

[148] Since all values of $\varphi + 2k\pi$ and of $2k\pi - \varphi$ give the same value w, this formula gives, in fact, for every w, $2n$ values of z, which we can write

$$z = \cos\frac{\varphi + 2k\pi}{n} \pm i\sin\frac{\varphi + 2k\pi}{n} \; (k = 0, 1, 2, ..., n-1)$$

In the case of the equations of the octahedron, tetrahedron, and icosahedron these "elementary" transcendental functions do not suffice. However, *we can obtain the* *corresponding solution by means of elliptic modular functions*. Although one may not consider this solution as belonging to elementary mathematics, I should, nevertheless, like to give, at least, the formulas[82] which relate to the *icosahedron*. They are, namely, *closely related to the solution of the general equation of degree five by means of elliptic functions,* to which allusion is always made in textbooks and about which I give later some explanation. The icosahedral equation had the form (see pp. [136], [140])

$$w = \frac{\varphi_{20}(z)^3}{\psi_{12}(z)^5}.$$

Now we identify w with the absolute invariant J from the theory of elliptic functions and think of J as a function of the period quotient $\omega = \omega_1/\omega_2$ (in Jacobi's notation iK'/K), i.e., we set

$$w = J\,(\omega) = \frac{g_2^3\,(\omega_1, \omega_2)}{\Delta\,(\omega_1, \omega_2)},$$

where g_2 and Δ are certain transcendental forms of dimension -4 and -12, respectively, in ω_1 and ω_2, which play an important role. If we introduce the usual abbreviation of Jacobi

$$q = e^{i\pi\omega} = e^{-\pi\frac{K'}{K}}$$

the roots z of the icosahedral equation will be given by the following quotients of ϑ functions

$$z = -q^{\frac{3}{5}}\frac{\vartheta_1\left(2\pi\omega, q^5\right)}{\vartheta_1\left(\pi\omega, q^5\right)}.$$

If we take into account that a) as a function of w, coming from the first equation, is infinitely many-valued, then this formula yields in fact all sixty roots of the icosahedral equation for a given w.

7. Solvability in Terms of Radical Signs

There is one question in the theory of the normal equations which I have not yet touched, namely, whether or not our normal equations yield algebraically anything that is essentially new; and whether or not they can be resolved into one another or, in particular, into a sequence of pure equations. In other words, is it possible to **[149]** build up the solution z of these equations in terms of w by means of a finite number of radical signs, one above another?

So far as the equations of the dihedron, tetrahedron, and octahedron are concerned, it is easy to show, by means of algebraic theory, that they can be reduced, in

[82] See „Über die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades", *Mathematische Annalen,* vol. 14 (1878/79), p. 111 et seq., or Klein, *Gesammelte Abhandlungen,* vol. 3, p. 13 et seq., also *Ikosaeder,* p. 131.]

fact, to pure equations. It will be sufficient if I give the details here for the dihedral equation only:

$$z^n + \frac{1}{z^n} = 2w.$$

If we set:

$$z^n = \zeta,$$

the equation goes over into

$$\zeta^2 - 2w\zeta + 1 = 0.$$

It follows from this that

$$\zeta = w \pm \sqrt{w^2 - 1},$$

and consequently

$$z = \sqrt[n]{w \pm \sqrt{w^2 - 1}},$$

which is the desired solution by means of radical signs.

On the other hand, however, the icosahedral equation does not admit such a solution by means of radical signs, so that this equation defines an essentially new algebraic function. I am going to give you a particularly intuitive proof of this, which I have recently published (*Mathematische Annalen,* Vol. 61 [1905])[83] , and which follows from consideration of the familiar function-theoretic construction of the icosahedral function $z(w)$. For this purpose I shall need the following lemma, due to *Abel*, a proof of which you will find in every textbook of algebra: *If the solution of an algebraic equation can be expressed as a sequence of radical signs, then every radical of the sequence can be expressed as a rational function of the n roots of the given equation.*

Let us now apply this lemma to the icosahedral equation. If we assume its root z can be expressed as a sequence of radical signs of rational functions of the coefficients, i.e., of rational functions of w, then every radical in the sequence is a rational function of the sixty roots:

$$R(z_1, z_2, \ldots, z_{60}).$$

(We shall show that this leads to a contradiction.) In the first place, we can replace this expression by a rational function $R(z)$ of z alone since all the roots can be derived from any one of them by a linear substitution. Let us now convert this $R(z)$
[150] into a function of w by writing for z the sixty-valued icosahedral function $z(w)$, and consider the result. Since every circuit in the w-plane, which returns z to its initial value, must of necessity return $R(z)$ also to its initial value, it follows that $+R[z(w)]$ can have branch points only at the points $w = 0, 1, \infty$ (where $z(w)$ has branch points), and the number of sheets of the Riemann surface for $R[z(w)]$, which are cyclically connected at each of these points must be a divisor of the corresponding number belonging to $z(w)$. We know that this number is $3, 2, 5$ at the three points, respectively. Hence every rational function $R(z)$ of an icosahedral root, and

[83] „Beweis für die Nichtauflösbarkeit der Ikosaedergleichung durch Wurzelzeichen", pp. 369–371 [see also: Felix Klein: Gesammelte mathematische Abhandlungen, vol. II, p. 385].

consequently every radical which appears in the assumed solution, considered as function of w, can have branch points, if at all, only at $w = 0, w = 1, w = \infty$. If branching occurs, then there must be three sheets connected at $w = 0$, two at $w = 1$, and five at $w = \infty$, since $3, 2, 5$ have no divisor other than 1.

We shall now see that this result leads to a contradiction. To this end let us examine the innermost radical, which appears in our hypothetically assumed expression for $z(w)$. Its radicand must be a rational function $\mathrm{P}(w)$. We can assume that the index of the radical is a prime number p, since we could otherwise build it up out of radicals with prime indices. Moreover $\mathrm{P}(w)$ cannot be the *p-th* power of a rational function $\varrho(w)$ of w. for if it were, our radical would be superfluous, and we could direct our attention to the next really essential radical.

Let us now see what kind of branching the function $\sqrt[p]{\mathrm{P}(w)}$ can have. For this purpose it will be convenient to write it in the homogeneous form

$$\mathrm{P}(w) = \frac{g\,(w_1, w_2)}{h\,(w_1, w_2)},$$

where g and h are forms of the same dimension in the variables w_1, w_2 ($w = w_1/w_2$). According to the fundamental theorem of algebra we can separate g and h into linear factors and write

$$\mathrm{P}(w) = \frac{l^{\alpha} \cdot m^{\beta} \cdot n^{\gamma} \ldots}{l'^{\alpha'} \cdot m'^{\beta'} \cdot n'^{\gamma'} \ldots}.$$

where

$$\alpha + \beta + \gamma + \cdots = \alpha' + \beta' + \gamma' + \cdots$$

since the numerator and the denominator are of the same degree. Not all the exponents $\alpha, \beta, \ldots, \alpha', \beta' \ldots$ can be divisible by p, since P would then be a perfect p-th power. On the other hand, $\alpha + \beta + \cdots - \alpha' - \beta' - \ldots$ is equal to zero, and is therefore divisible by p. Consequently at least *two* of these numbers are not divisible by p, and not just *one*. *It follows that the zeros of both the corresponding linear factors must* **[151]** *be branch points of* $\sqrt[p]{\mathrm{P}(w)}$, *at each of which p sheets are cyclically connected. But herein lies the contradiction of the previous theorem*, which, of course, must be equally valid for $\sqrt[p]{\mathrm{P}(w)}$. For we enumerated at that time all possible branch points, and we found among them no two at which the same number of sheets were connected. Our assumption is therefore *not tenable, and the icosahedral equation cannot be solved by radical signs.*

This proof depends essentially upon the fact that the numbers $3, 2, 5$ which are characteristic for the icosahedron have no common divisor. When such a common divisor appears, as in the case of the numbers $1, 2, 4$ of the octahedron, it is at once possible to have rational functions $R[z(w)]$ which exhibit the same kind of branching at two points, e.g., one in which two sheets are connected at 1 and at ∞, and these can then be really represented as roots of a rational function $\mathrm{P}(w)$. *It is in this way that the solution by means of radical signs comes about in the case of the octahedron and tetrahedron* (*with the numbers* $3, 2, 3$)*, and of the dihedron* $(2, 2, n)$.

I should like to show you here how slightly the language used in wide mathematical circles keeps pace with knowledge. The word "root" is used today nearly everywhere in two senses: once for the solution of any algebraic equation, and, secondly, in particular, for the solution of a pure equation. The latter use, of course, dates from a time when only pure equations were studied. Today it is, if not actually harmful, at least rather inconvenient. Think only of the formulation that the "*roots*" of an equation cannot be expressed by means of "radical signs". But there is another form of expression which has lingered on from the beginnings of algebra and which is a more serious source of misunderstanding, namely, that algebraic equations are said to be "not algebraically solvable", if they cannot be solved in terms of radical signs, i.e. if they cannot be reduced to pure equations. This use is in immediate contradiction with the modern meaning of the word "algebraic". Today we say *that an equation can be solved algebraically when we can reduce it to a chain of simplest algebraic equations in which one controls the dependence of the solutions upon the parameters, the relation of the different roots to one another, etc. as completely as one does in the case of the pure equation. It is not at all necessary that these equations should be pure equations.* In this sense we may say that the icosahedral equation can be solved algebraically, for our discussion shows that we
[152] can construct its theory in a manner that meets all the demands mentioned above. The fact that this equation cannot be solved by radical signs rather lends it special interest by suggesting it *as an appropriate normal equation to which one might try to reduce,* (i.e., completely solve) *still other equations which are in the old sense algebraically unsolvable*.

The last remark leads us to the last section of this chapter, in which we shall try to get a general view of such reductions.

8. Reduction of General Equations to Normal Equations

It turns out, namely, that the following reductions are possible:

The most general equation of the third degree to the dihedral equation for $n = 3$;

The general equation of the fourth degree to the tetrahedral or to the octahedral equation;

The general equation of the fifth degree to the icosahedral equation.

This result is the *most recent triumph of the theory of the regular bodies,* which have always played such an important role since the beginning of mathematical history, and which have a decisive influence in the most widely separated fields of modern mathematics.

In order to show you the meaning of my general assertion I shall go somewhat more into details *for the equation of degree three,* without, however, fully proving the formulas. We again take the cubic equation in the reduced form

$$x^3 + px - q = 0. \tag{1}$$

Denoting solutions by x_1, x_2, x_3, we try to set up a rational function z of them which undergoes the six linear substitutions of the dihedron for $n = 3$ when we interchange the x_i in all six possible ways. The values that z should take on are

$$z, \varepsilon z, \varepsilon^2 z, \frac{1}{z}, \frac{\varepsilon}{z}, \frac{\varepsilon^2}{z} \left(\text{where } \varepsilon = e^{\frac{2i\pi}{3}}\right).$$

It is easily seen that

$$z = \frac{x_1 + \varepsilon x_2 + \varepsilon^2 x_3}{x_1 + \varepsilon^2 x_2 + \varepsilon x_3} \tag{2}$$

satisfies these conditions. The dihedral function $z^3 + 1/z^3$ of this quantity must remain unaltered by all the interchanges of the x_k, since the six linear substitutions of z leave it unchanged. Hence, by a well-known theorem of algebra, it must be a rational function of the coefficients of (1). A calculation shows that

$$z^3 + \frac{1}{z^3} = -27\frac{q^2}{p^3} - 2. \tag{3}$$

Conversely, if we solve this dihedral equation, and if z is one of its roots, we can **[153]** express the three values x_1, x_2, x_3 rationally in terms of z, p, and q by means of (3) and the well-known relations

$$x_1 + x_2 + x_3 = 0, x_1x_2 + x_2x_3 + x_3x_1 = p, x_1x_2x_3 = q.$$

Doing this, we find

$$\begin{cases} x_1 = -\dfrac{3q}{p} \cdot \dfrac{z\,(1+z)}{1+z^3}, \\[2ex] x_2 = -\dfrac{3q}{p} \cdot \dfrac{\varepsilon z\,(1+\varepsilon z)}{1+z^3}, \\[2ex] x_3 = -\dfrac{3q}{p} \cdot \dfrac{\varepsilon^2 z\left(1+\varepsilon^2 z\right)}{1+z^3}. \end{cases} \tag{4}$$

Thus, as soon as the dihedral equation (3) has been solved, the formulas (4) give at once the solution of the cubic (1).

In the same way we may reduce the general equations of the fourth and fifth degrees. The equations would be, of course, somewhat longer, but not more difficult in principle. The only new thing would be that the parameter w of the normal equation, which was expressed above rationally in the coefficients of the equation

$$\left(2w = -27\frac{q^2}{p^2} - 2\right)$$

would now contain *square roots*. You will find this theory for the equation of degree five given fully in the second part of my lectures on the icosahedron. Not only are the formulas calculated, but also the essential reasons for the appearance of the equations are explained.

Finally, let me say a word about the relation of this development to the usual presentation of the theory of equations of the third, fourth, and fifth degree. In the first place, we can obtain the usual solutions of the cubic and biquadratic from our formulas by appropriate reductions, if we use the solutions of the equations of the dihedron, octahedron, and tetrahedron in terms of radical signs. In the case of equations of degree five, most of the textbooks confine themselves unfortunately to the establishment of the negative result that the equation cannot be solved by radical signs, to which is then added the vague hint that the solution is possible by elliptic functions, to be exact one should say elliptic *modular* functions. I take exception to this procedure because it exhibits a one-sided contrast and hinders rather than promotes a real understanding of the situation. In view of the preceding survey,
[154] distinguishing an algebraic and an analytic part, we may say:

1. *The general equation of the fifth degree cannot be reduced, indeed, to pure equations, but it is possible to reduce it to the icosahedral equation as the simplest normal equation*. This is the real problem of its algebraic solution.

2. *The icosahedral equation, on the other hand, can be solved by elliptic modular functions*. For purposes of numerical calculation, this is the full analogue of the solution of pure equations by means of logarithms.

This supplies the complete solution of the problem of the equation of fifth degree. Remember that when the usual road does not lead to success, one should not be content with this determination of impossibility, but should bestir oneself to find a new and more promising route. Mathematical thought, as such, does never end. If someone says to you that mathematical reasoning cannot be carried beyond a certain point, you may be sure that the really interesting problem begins precisely there.

In conclusion, it might be remarked that these theories do not stop with equations of degree five. On the contrary, one can set up analogous developments for equations of the sixth and higher degrees if one will only make use of the higher-dimensional analogues of the regular bodies. If you are interested in this, you might read my article[84] *Über die Auflösung der allgemeinen Gleichung fünften und sechsten Grades**. In connection with this article the problem was successfully attacked by Paul Gordan[85] and Arthur Byron Coble[86]. The investigation is somewhat simplified in the latter memoir[87].

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[84] *Journal für reine und angewandte Mathematik*, vol. 129 (1905), p. 151; and *Mathematische Annalen,* vol. 61 (1905), p. 50.

* *Concerning the solution of the general equation of fifth and of sixth degree.*

[85] *Mathematische Annalen,* vol. 61 (1905), p. 50; and vol.68 (1910), p. 1.

[86] *Mathematische Annalen,* vol. 70 (1911), p. 337.

[87] See also F. Klein, *Gesammelte Mathematische Abhandlungen*, vol. 2, p. 502–503.