IV. Complex Numbers

1. Ordinary Complex Numbers

Let me give, as a preliminary, *some historical facts*. Imaginary numbers are said to have been used first, incidentally, to be sure, by *Cardan* in 1545, in his solution of the cubic equation. As for the further development, we can make the same statement as in the case of negative numbers, *that imaginary numbers made their own way into arithmetic calculation without the approval, and even against the desires of individual mathematicians, as they occurred ever again by themselves during calculations, and obtained wider circulation only gradually and to the extent to which they showed themselves useful*. Meanwhile the mathematicians were not altogether happy about it. Imaginary numbers long retained a somewhat *mystic* colouring, just as they have today for every pupil who hears for the first time about that remarkable $i = \sqrt{-1}$. As evidence, I mention a very significant utterance by *Leibniz* in the year 1702, "Imaginary numbers are a fine and wonderful refuge of the divine spirit, almost an amphibian between being and non-being". In the eighteenth century, the notion involved was indeed by no means cleared up, although *Euler*, above all, *recognized their fundamental significance for the theory of functions*. In 1748 Euler set up that remarkable relation:

$$e^{ix} = \cos x + i \ \sin x$$

by means of which one recognizes the fundamental relationship among the kinds of functions which appear in elementary analysis. The *nineteenth century finally brought the clear understanding of the nature of complex numbers*. In the first place, we must emphasize here the *geometric interpretation* to which various investigators were led about the turn of the century, almost simultaneously. It will suffice if I mention the man who certainly went deepest into the essence of the thing and **[62]**
who exercised the most lasting influence upon the public, namely Gauß. As his diary, mentioned above, proves incontrovertibly, he was, in 1797, already in full possession of that interpretation, although, to be sure, it was published very much later. The second achievement of the nineteenth century is the creation of a *purely formal foundation* for complex numbers, which reduces them to dependence upon real numbers. This originated with English mathematicians of the thirties, the details of which I shall omit here, but which you will find in Hankel's book, mentioned above.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49442-4_5

Let me now explain these *two prevailing foundation methods*. We shall take first the *purely formal standpoint*, from which the consistency of the rules of operation among themselves, rather than the meaning of the objects, guarantees the correctness of the concepts. According to this view, complex numbers are introduced in the following manner, which precludes every trace of the mysterious.

1. The complex number $x + iy$ is the *combination of two real numbers* x, y, that is, a *number-pair*, concerning which one adopts the conventions which follow.

2. Two complex numbers $x + iy$, $x' + iy'$ are called *equal* when $x = x'$, $y = y'$.

3. Addition and subtraction are defined by the relation

$$(x + iy) \pm (x' + iy') = (x \pm x') + i\,(y \pm y').$$

All the *rules of addition* follow from this, as is easily verified. The *monotonic law* alone loses its validity in its original form, since complex numbers, by their nature, do not have the same simple order in which natural or real numbers appear by virtue of their magnitude. For the sake of brevity I shall not discuss the modified form which this gives to the monotonic law.

4. We stipulate that in *multiplication* one operates as with ordinary letters, except that one always puts $i^2 = -1$; in particular, that

$$(x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y).$$

It is easy to see that, with this, *all the laws of multiplication hold, with the exception of the monotonic law, which does not enter into consideration.*

5. *Division* is defined as the *inverse of multiplication;* in particular, we may easily verify that

$$\frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}.$$

[63] This number always exists except for $x = y = 0$, i.e., *division by zero has the same exceptional place here as in the domain of real numbers.*

It follows from this that operations with complex numbers cannot lead to contradictions, since they depend exclusively upon real numbers and known operations with them. We shall assume here that these are devoid of contradiction.

Besides this purely formal treatment, we should of course like to have a geometric, or otherwise visual, interpretation of complex numbers and of operations with them, in which we might see an *intuitive foundation of consistency*. This is supplied by that Gaussian interpretation, which, as you all know and as we have already mentioned, *looks upon the totality of points* (x, y) *of the plane in an* x-y-*coordinate system as representing the totality of complex numbers* $z = x + iy$. The sum of two numbers z, a follows by means of the familiar *parallelogram construction* with the two corresponding points and the origin 0, while the product $z \cdot a$ is obtained, adding the unit point 1 ($x = 1$, $y = 0$), by constructing on the segment $0z$ a triangle similar to a01 (Fig. 14). In brief, *addition* $z' = z + a$ *is represented by a translation of the plane into itself, multiplication* $z' = za$ *by a similarity transformation*, i.e., *by a turning and a stretching, the origin remaining fixed.* From the order of the points in the plane, considered as representatives of complex numbers, one sees at once what takes the place here of the monotonic laws for real numbers. These suggestions will suffice, I hope, to recall the subject clearly to your memory.

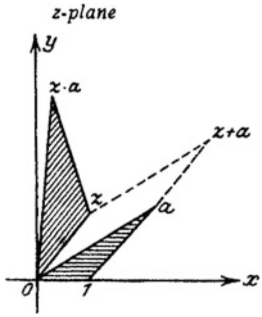

Figure 14

I must call to your attention the place in Gauß in which this foundation of complex numbers, by means of their geometric interpretation, is set out with full emphasis, since it was this which first exhibited the general importance of complex numbers. In a paper published in 1831, Gauß exposed the theory especially of *integer complex numbers* $a + ib$, where a, b are real integers, in which he developed for the new numbers the theorems of ordinary *number theory* concerning prime factors, quadratic and biquadratic residues, etc. We mentioned such generalizations of number theory, in connection with our discussion of Fermat's theorem. In his own abstract[50] of this paper Gauß expresses himself concerning what he calls the "true **[64]** metaphysics of imaginary numbers". For him, the right to operate with complex numbers is justified by that intuitive geometric interpretation which one can give to them and to the operations with them. Thus he takes *by no means the formal standpoint*. Moreover, these long, beautifully written expositions of Gauß are extremely well worth reading. I mention here, also, that Gauß proposes the clearer word "complex", instead of "imaginary", a name that has, in fact, been adopted.

2. Higher Complex Numbers, Especially Quaternions

It has occurred to everyone who has worked seriously with complex numbers to ask if we cannot set up other, higher, complex numbers, with more new units than the one i and if we cannot operate with them reasonably. Positive results in this direction were obtained about 1840 by *Hermann Graßmann*, in Stettin, and *W*illiam *R. Hamilton*, in Dublin, independently of each other. We shall examine the invention of Hamilton, the *calculus of quaternions*, somewhat carefully later on. For the present let us look at the general problem.

[50] For this "Selbstanzeige" see Gauß Werke, vol. II.

We can look upon the ordinary complex number $x + iy$ as a *linear combination*

$$x \cdot 1 + y \cdot i$$

formed from two different "*units*" 1 and i, by means of the *real parameters* x and y. Similarly, let us now imagine an arbitrary number, n, of units $e_1, e_2, \ldots, e_n$ all different from one another, and let us call the totality of combinations of the form $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ a *higher complex number system* formed from them with n arbitrary real numbers $x_1, x_2, \ldots, x_n$. If there are given two such numbers, say x, defined above, and

$$y = y_1e_1 + y_2e_2 + \ldots + y_ne_n.$$

It is almost obvious that we should call them *equal when, and only when, the coefficients of the individual units, the so called "components" of the number, are equal in pairs*

$$x_1 = y_1, \quad x_2 = y_2, \ldots, x_n = y_n.$$

The definition of addition and subtraction, which reduces these operations simply to the *addition and subtraction of the components*,

$$x \pm y = (x_1 \pm y_1)\, e_1 + (x_2 \pm y_2)\, e_2 + \cdots + (x_n \pm y_n)\, e_n$$

is equally obvious.

The matter is more difficult and more interesting in the case of *multiplication*.
[65] To start with, we shall proceed according to the general rule for multiplying letters, i.e., multiply each i-th term of x by every k-th term of y $(i, k = 1, 2, \ldots, n)$. This gives:

$$x \cdot y = \sum_{(i,k=1,\ldots,n)} x_i y_k e_i e_k.$$

In order that this expression should be a number in our system, one must have a rule which represents the *products $e_i \cdot e_k$ as complex numbers of the system*, i.e., as linear combinations of the units. Thus one must have n^2 equations of the form:

$$e_i e_k = \sum_{(l=1,\ldots,n)} c_{ikl} \cdot e_l \quad (i, k = 1, \ldots, n).$$

Then we may say that the number

$$x \cdot y = \sum_{(l=1,\ldots,n)} \left\{ \sum_{(i,k=1,\ldots,n)} x_i y_k c_{ikl} \right\} e_l$$

will always belong to our complex number system. *The convention of determining this rule for multiplication, i.e., the table of the coefficients c_{ikl}, provides the characteristic feature of each particular complex number system.*

If one now defines *division as the operation inverse to multiplication*, it turns out that, under this general arrangement, division *is not always uniquely possible*, even when the divisor does not vanish. For, the determination of y from $x \cdot y = z$ requires the solution of the n linear equations $\sum_{i,k} x_i y_k c_{ikl} = z_l$ for the n unknowns $y_1, \ldots, y_n$, and these would have either no solution, or infinitely many solutions, if their determinant happened to vanish. Moreover, all the z_l may be zero even when not all the x_i or not all the y_k vanish, i.e., *the product of two numbers can vanish without either factor being zero*. It is only by a skilful special choice of the numbers c_{ikl} that one can bring about accord here with the behaviour of ordinary numbers. To be sure, a closer investigation *shows, when* $n > 2$, *that, to attain this, we must sacrifice one of the other rules of operation*. We choose as the rule that fails to be satisfied, one which appears less important under the circumstances.

Let us now follow up these general explanations by a more detailed discussion of *quaternions* as the example which, by reason of its applications in physics and mathematics, constitutes the *most important higher complex number system*. As the name indicates, these are *four-term numbers* ($n = 4$); as a subclass, they include the *three-term vectors*, which are generally known today, and which are sometimes discussed in the schools.

As the first of the four units with which we shall construct quaternions, we shall select the *real unit* 1, (as in the case of ordinary complex numbers). We ordinarily denote the other three units, as did Hamilton, by i, j, k, so that the general form of **[66]**
the quaternion is

$$p = d + ia + jb + kc,$$

where a, b, c, d are *real parameters*, the *coefficients of the quaternion*. We call the first component *d*, the one which is multiplied by 1, and which corresponds to the real part of the common complex number, the "*scalar part*" *of the quaternion*, the aggregate $ai + bj + ck$ of the other three terms its "*vector part*".

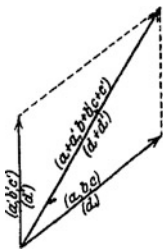

Figure 15

The addition of quaternions follows from the preceding general remarks. I shall give an obvious *geometric interpretation*, which goes back to that interpretation of vectors which is familiar to you. We imagine the *segment*, corresponding to the vector part of p, and having the projections a, b, c on the coordinate axes, as loaded with a *weight* equal to the scalar part. Then the addition of p and $p' = d' + ia' + jb' + kc'$ is accomplished by constructing the resultant of the two segments, according to the well known parallelogram law of vector addition (see Fig. 15), and then loading it with the sum of the weights, for this would then in fact represent the quaternion:

$$p + p' = (d + d') + i(a + a') + j(b + b') + k(c + c'). \tag{1}$$

We come to specific properties of quaternions only when we turn to *multiplication*. As we saw in the general case, these properties must be implicit in the *conventions adopted as to the products of the units*. To begin with, I shall indicate the quaternions to which *Hamilton* equated the sixteen products of two units each. As its symbol indicates, we shall operate with the first unit 1 as with the real number 1, so that:

$$1^2 = 1,\ \ i \cdot 1 = 1 \cdot i = i,\ \ j \cdot 1 = 1 \cdot j = j,\ \ k \cdot 1 = 1 \cdot k = k. \tag{2a}$$

As something essentially new, however, we agree that, for the squares of the other units:

$$i^2 = j^2 = k^2 = -1, \tag{2b}$$

and for their binary products:

$$jk = +i,\ \ ki = +j,\ \ ij = +k \tag{2c}$$

whereas one convenes for the inverted position of the factors:

$$kj = -i,\ \ ik = -j,\ \ ji = -k. \tag{2d}$$

One is struck here by the fact that the *commutative law for multiplication is not obeyed*. This is the inconvenience in quaternions which one must accept in order to rescue the uniqueness of division, as well as the theorem that a product should
[67] vanish only when one of the factors vanishes. *We shall show at once that not only this theorem but also all the other laws of addition and multiplication remain valid, with this one exception, in other words, that these simple agreements are very expedient.*

We construct, first, the product of two general quaternions

$$p = d + ia + jb + kc \ \text{ and } \ q = w + ix + jy + kz.$$

Let us start from the equation

$$q' = p \cdot q = (d + ia + jb + kc) \cdot (w + ix + jy + kz);$$

and let us multiply out term by term. In carrying out this multiplication, we must note the order in the case of the units i, j, k. We must follow the commutative law for products composed of the components a, b, c, d, and for products of components and one unit, we must replace the products of units in accordance with our multiplication table, and we must then collect the terms having the same unit. We then have

$$\left.\begin{aligned} q' = pq = w' + ix' + jy' + kz' = &(dw - ax - by - cz) \\ &+ i\left(aw + dx \underline{+ bz - cy}\right) \\ &+ j\left(bw + dy \underline{+ cx - az}\right) \\ &+ k\left(cw + dz \underline{+ ay - bx}\right). \end{aligned}\right\} \tag{3}$$

The components of the product quaternion are thus definite simple, *bilinear combinations* of the components of the two factors. If we invert the order of the factors, the six underscored terms change their signs, so that $q \cdot p$, *in general, is different from* $p \cdot q$, and the difference is more than a change of sign as was the case with the individual units. Although the commutative law fails for multiplication, the *distributive and associative laws hold without change*. For, if we construct on the one hand $p(q + q_1)$, on the other $pq + pq_1$ by multiplying out formally without replacing the products of the units, we must, of necessity, get identical results, and no change can be brought about by then using the multiplication table. Further, the associative law must hold in general, if it holds for the multiplication of the units. But this follows at once from the multiplication table, as the following example shows:

$$(ij)k = i(jk).$$

In fact, we have:

$$(ij)k = k \cdot k = -1,$$

and

$$i(jk) = i \cdot i = -1.$$

We shall now take up *division*. It will suffice to show *that for every quaternion* $p = d + ia + jb + kc$ *there is a definite second one*, q, *such that:*

$$p \cdot q = 1.$$

We shall denote q appropriately by $1/p$. Division in general can be reduced easily to this special case, as we shall show later. In order to determine q, let us put, in equation (3), **[68]**

$$q' = 1 = 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k,$$

and obtain, by equating components, the following four equations for four unknown components x, y, z, w of q:

$$\begin{aligned} dw - ax - by - cz &= 1 \\ aw + dx - cy + bz &= 0 \\ bw + cx + dy - az &= 0 \\ cw - bx + ay + dz &= 0. \end{aligned}$$

The solvability of such a system of equations depends, as is well known, upon its determinant, which, in the case before us, is a skew symmetric determinant, in which all the elements of the principal diagonal are the same, and all the pairs of elements which are symmetrically placed with respect to that diagonal are equal and opposite in sign. According to the theory of determinants, such determinants are easily calculated; and we find

$$\begin{vmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{vmatrix} = \left(a^2 + b^2 + c^2 + d^2\right)^2,$$

By direct calculation this result can be easily verified. The real *elegance of Hamilton's conventions* depends upon this result, that the determinant is a power of the sum of squares of the four components of *p;* for it follows *that the determinant is always different from zero except when* $a = b = c = d = 0$. With this one self-evident exception ($p = 0$), the equations are *uniquely solvable* and the *reciprocal quaternion q is uniquely determined.*

The quantity

$$T = \sqrt{a^2 + b^2 + c^2 + d^2}$$

plays an important role in the theory, and is called the *tensor of* p. It is easy to show that these unique solutions are

$$x = -\frac{a}{T^2}, \quad y = -\frac{b}{T^2}, \quad z = -\frac{c}{T^2}, \quad w = \frac{d}{T^2}$$

so that we have as the final result

$$\frac{1}{p} = \frac{1}{d + ia + jb + kc} = \frac{d - ia - jb - kc}{a^2 + b^2 + c^2 + d^2}.$$

[69] If we introduce the *conjugate value* of p, as in ordinary complex numbers:

$$\overline{p} = d - ia - jb - kc,$$

we can write the last formula in the form

$$\frac{1}{p} = \frac{\overline{p}}{T^2}$$

or

$$p \cdot \overline{p} = T^2 = a^2 + b^2 + c^2 + d^2.$$

These formulas which are immediate generalizations of certain properties of ordinary complex numbers. Since p is also the number conjugate to $\overline{p}$, it follows also that:

$$\overline{p} \cdot p = T^2,$$

so that the commutative law holds in this special case.

The general problem of division can now be solved. For, from the equation

$$p \cdot q = q',$$

it follows, by multiplication by $1/p$, that

$$q = \frac{1}{p} \cdot q' = \frac{\overline{p}}{T^2} \cdot q',$$

whereas the equation

$$q \cdot p = q',$$

which one gets by changing the order of the factors, has the solution

$$q = q' \cdot \frac{1}{p} = q' \cdot \frac{\overline{p}}{T^2}.$$

This solution is different, in general, from the other.

Remarks on Vector Calculus

Now we must inquire whether there is a *geometric interpretation of quaternions* in which these operations, together with their laws, appear in a natural form. In order to arrive at it, we start with the special case in which *both factors reduce to simple vectors*, i.e., in which the scalar parts w and d, are zero. The formula (3) for multiplication then becomes

$$\begin{aligned} q' = p \cdot q &= (ia + jb + kc)(ix + jy + kz) \\ &= -(ax + by + cz) + i(bz - cy) + j(cx - az) + k(ay - bx), \end{aligned}$$

i.e., *when each of two quaternions reduces to a vector, their product consists of a scalar and a vector part.* We can easily bring these two parts into relation with the different kinds of *vector multiplication,* which are in use with us in Germany. The notions of vector calculus, which is far more wide spread than quaternion calculus, go back to, although the word *vector* is of English origin. The two kinds of vector product with which one usually operates are designated now, mostly, by *inner* **[70]** (*scalar*) *product* $ax + by + cz$ (i.e., the scalar part of the above quaternion product, except for the sign), and *outer* (*vector*) *product* $i(bz - cy) + j(cx - az) + k(ay - bx)$, (i.e., the vector part of the quaternion product). We shall give a geometric interpretation of each part separately.

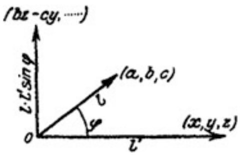

Figure 16

Let us lay off both vectors (a, b, c) and (x, y, z), as segments, from the origin O (Fig. 16). They terminate in the points (a, b, c) and (x, y, z) respectively, and have the lengths $l = \sqrt{a^2 + b^2 + c^2}$ and $l' = \sqrt{x^2 + y^2 + z^2}$. If φ is the angle between these two segments, then, according to well-known formulas of analytic geometry, which I do not need to develop here, the *inner product* is:

$$ax + by + cz = l \cdot l' \cdot \cos\varphi;$$

and the *outer product*, on the other hand, is itself a *vector*, which, as is easily seen, is *perpendicular to the plane of* l *and* l' and has the *length* $l \cdot l' \cdot \sin\varphi$.

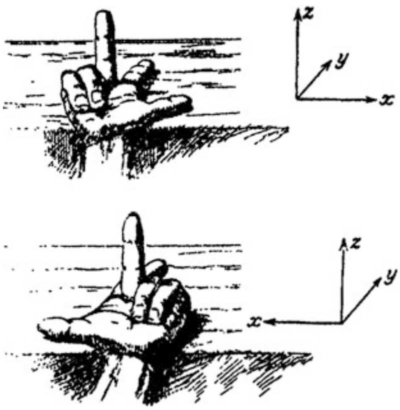

Figure 17

It is essential now to decide as to the *direction of the product vector*, i.e., toward which side of the plane determined by l and l' one is to lay off this vector. This direction is *different according to the coordinate system* which one chooses. As you know, one can choose *two rectangular coordinate systems which are not congruent*, i.e., *which cannot be made to coincide with one another*, by holding, say, the y- and the z-axis fixed and reversing the direction of the x-axis. These systems are then *symmetric to each other*, like *the right and the left hand* (Fig. 17). The distinction between them can be borne in mind by the following rule: *In the one system, the* x-, y-, *and* z*-axis lie like the outstretched thumb, fore finger and middle finger, respectively, of the right hand; in the other, like the same fingers of the left hand.* These two systems are used confusedly in the literature; different habits obtain in different countries, in different fields, and, finally, with different writers, or even with the same writer. Let us now examine the simplest case, where $p = i$,
[71] $q = j$, these being the unit lengths laid off on the x- and y-axis. Then, since $i \cdot j = k$, the outer vector product is the unit length laid off on the z-axis. (See Fig. 18.) Now one can transform i and j continuously into two arbitrary vectors p and q so that k transforms continuously into the vector component of $p \cdot q$ without vanishing during the transition. Consequently the *first factor, the second factor, and the vector product must always lie, with respect to each other, like the* x-, y-, *and*

z-axis of the system of coordinates, i.e., right-handed (as in Fig. 18) *or left-handed* (as in Fig. 16), *according to the choice of coordinate system.* (In Germany, now, the choice indicated in Fig. 18 is customary.)

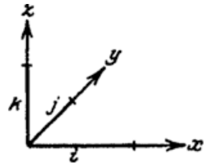

Figure 18

I should like to add a few words concerning the much disputed *question of notation in vector analysis*. There are, namely, a great many different symbols used for each of the vector operations, and it has been impossible, thus far, to bring about a generally accepted notation. At the meeting of natural scientists at Kassel (1903) a commission was set up for this purpose. Its members, however, were not able even to come to a complete understanding among themselves. Since their intentions were good, however, each member was willing to meet the others part way, so that the only result was that about three new notations came into existence! My experience in such things inclines me to the belief that real agreement could be brought about only if important material interests stood behind it. It was only after such pressure that, in 1881, the uniform system of measures according to volts, amperes, and ohms was generally adopted in electro-technics and afterward settled by public legislation, due to the fact that industry was in urgent need of such uniformity as a basis for all of its calculations. But there are no such strong material interests behind vector calculus, as yet, and hence one must agree, for better or worse, to let every mathematician cling to the notation which he finds the most convenient, or – if he is dogmatically inclined – the only correct one.

3. Quaternion Multiplication – Rotation-Dilation[51]

Before we proceed to the consideration of the geometric interpretation of multiplication of general quaternions, let us consider the following question. Let us consider the product $q' = p \cdot q$ of two quaternions p and q, and let us replace p and q by their conjugates $\overline{p}$ and $\overline{q}$, that is, let us change the signs of a, b, c, x, y, z. Then the scalar part of the product, as given in (3), p. [67], remains unchanged, and only **[72]**
those factors of i, j, k which are *not* underscored will change sign. On the other hand, if we also reverse the order of the factors $\overline{p}$ and $\overline{q}$, the factors of i, j, k which are underscored will change sign. Hence the product $\overline{q}' = \overline{q} \cdot \overline{p}$ is precisely the

[51] [Translator's note: In German, there is the more handy term „Drehstreckung".]

conjugate of the original product $q' = p \cdot q$; and we have

$$q' = p \cdot q, \quad \overline{q}' = \overline{q} \cdot \overline{p},$$

where $\overline{q}'$ is the conjugate of q'. If we multiply these two equations together, we obtain

$$q' \cdot \overline{q}' = p \cdot q \cdot \overline{q} \cdot \overline{p}.$$

In this equation the order of the factors is essential, since the commutative law does not hold. We may apply the associative law, however, and we may write

$$q' \cdot \overline{q}' = p \cdot (q \cdot \overline{q}) \cdot \overline{p}.$$

Since we have, by p. [66],

$$q \cdot \overline{q} = x^2 + y^2 + z^2 + w^2,$$

we may write

$$w'^2 + x'^2 + y'^2 + z'^2 = p\left(w^2 + x^2 + y^2 + z^2\right)\overline{p}.$$

The middle factor on the right is a scalar, and the commutative law does hold for multiplication of a scalar by a quaternion, since $M \cdot p = Md + i(Ma) + j(Mb) + k(Mc) = pM$. Hence we have

$$w'^2 + x'^2 + y'^2 + z'^2 = p\overline{p}\left(w^2 + x^2 + y^2 + z^2\right),$$

and, since $p \cdot \overline{p}$ is the square of the tensor of p, we find[52]

$$\text{(I)} \qquad w'^2 + x'^2 + y'^2 + z'^2 = \left(d^2 + a^2 + b^2 + c^2\right)\left(w^2 + x^2 + y^2 + z^2\right),$$

that is, *the tensor of the product of quaternions is equal to the product of the tensors of the factors*. This formula can be obtained also by direct calculation, by taking the values of w', x', y', z' from the formula for a product given on p. [67].

We shall now represent a quaternion q as the segment joining the origin of a four-dimensional space to the point (x, y, z, w) in it, in a manner exactly analogous to the representation of a vector in three-dimensional space. It is no longer necessary to apologize for making use of four-dimensional space, as one always had to do when I was a student. All of you are fully aware that no metaphysical meaning is intended, and that higher dimensional space is nothing more than a convenient mathematical expression which permits us to use terminology analogous to that of actual spatial perception. If we regard p as a constant, that is, if we regard a, b, c, d as constants, the quaternion equation

$$q' = p \cdot q$$

[52] The essence of this formula can already be found in Lagrange's works.

represents a certain *linear transformation* of the points (x, y, z, w) of the four-dimensional space into the points (x', y', z', w'), since the equation assigns to every four-dimensional vector q another vector q' linearly. The explicit equations for this transformation, i.e., the expressions for x', y', z', w' as linear functions of x, y, z, w, may be obtained by comparison of the coefficients of the product formula (3), p. [67]. The tensor equation (I) shows that the distance of any point from the origin, $\sqrt{x^2 + y^2 + z^2 + w^2}$, is multiplied by the same constant factor $T = \sqrt{a^2 + b^2 + c^2 + d^2}$, for all points of the space. Finally, by p. [68], the determinant of the linear transformation is surely positive. **[73]**

It is shown in analytic geometry of three-dimensional space that if a linear transformation of the coordinates x, y, z is *orthogonal* (that is, if it carries the expression $x^2 + y^2 + z^2$ into itself), and if the determinant of the transformation is positive, the transformation represents a *rotation about the origin*. Conversely, *any rotation* can be obtained in this manner. If the linear transformation carries $x^2 + y^2 + z^2$ into the similar expression in x', y', z' multiplied by a constant factor T^2, however, and if the determinant again is positive, the transformation represents a *rotation about the origin combined with an expansion in the ratio T about the origin*, or, briefly, *a rotation-dilation*.

The facts just mentioned for three-dimensional space may be extended to four-dimensional space. We shall say that our transformation of four-dimensional space represents in precisely the same sense a *rotation-dilation about the origin*. It is easy to see, however, that in this case we do *not* obtain the most general rotation-dilation about the origin. For our transformation contains only four arbitrary constants, namely, the components a, b, c, d of p, whereas, as we shall show immediately, the most general rotation-dilation about the origin in the four-dimensional space R_4 contains seven arbitrary constants. Indeed, in order that the general linear transformation should be a rotation-dilation, we must have

$$x'^2 + y'^2 + z'^2 + w'^2 = T^2(x^2 + y^2 + z^2 + w^2).$$

By comparing the coefficients this yields 10 conditions, since, on the left side, one replaces x', y', z', w' by linear integer functions of x, y, z, w, one obtains a quadratic form in four variables, which contains $(4 \cdot 5)/2 = 10$ terms. Since T is still arbitrary, these reduce to nine equations among the sixteen coefficients of the transformation. Hence there remain $16 - 9 = 7$ arbitrary constants.

It is remarkable that in spite of this *the most general rotation-dilation can be obtained by quaternion multiplication.* Let $\pi = \delta + i\alpha + j\beta + k\gamma$ be another **[74]** constant quaternion. Then we may show, just as before, that the transformation $q' = q \cdot \pi$, which differs from the preceding one only in that the order is reversed, represents a rotation-dilation of R_4. Hence the combined transformation

$$\text{(II)} \qquad q' = p \cdot q \cdot \pi = (d + ia + jb + kc) \cdot q \cdot (\delta + i\alpha + j\beta + k\gamma)$$

also represents such a rotation-dilation. This transformation contains only seven (not eight) arbitrary constants, for the transformation remains unchanged if we multiply a, b, c, d by any real number and divide $\alpha, \beta, \gamma, \delta$ by the same number. It is therefore plausible that this combined transformation *represents the general rotation-dilation of four-dimensional space*. This beautiful result is actually true, as was shown by Arthur Cayley. I shall restrict myself to the mention of the historical fact, in order not to be drawn into too great detail of this interpretation. The formula is given in Cayley's paper *on the homographic transformation of a surface of the second order into itself*[53], in 1854, and also in certain other papers of his[54].

This formula of Cayley's has the great advantage that it enables us to grasp at once the combination of two rotation-dilations in a very easy way. Thus, if a second rotation-dilation be given by the equation

$$q'' = w'' + ix'' + jy'' + kz'' = p' \cdot q' \cdot \pi',$$

where p' and π' are new given quaternions, we find, by (II) for the value q',

$$q'' = p' \cdot (p \cdot q \cdot \pi) \cdot \pi',$$

whence, by the associative law,

$$q'' = (p' \cdot p) \cdot q \cdot (\pi \cdot \pi')$$

or

$$q'' = r \cdot q \cdot \varrho,$$

where $r = p' \cdot p$ and $\varrho = \pi \cdot \pi'$ are definite new quaternions. We have therefore obtained an expression for the rotation-dilation that carries q into q'' in precisely the old form, and we see that the multipliers, which precede and follow q in the quaternion product arise, respectively, from the products of the corresponding two multipliers of q in the separate transformations which were combined, the order of the factors being necessarily as shown in the formula.

Interpretation in Three-dimensional Space

This four-dimensional representation may seem unsatisfactory, and there may be a desire for something more tangible, which can be represented in ordinary *three-dimensional* space intuition. We shall therefore show that we can obtain similar formulas for the *same three-dimensional operations* by *simple specialization* of
[75] the formulas just given. Indeed the importance of quaternion multiplication for ordinary physics and mechanics is based upon these very formulas. I have said "ordinary", because I do not desire at this point to anticipate generalizations of

[53] *Journal für reine und angewandte Mathematik*, 1855. Reprinted in Cayley's Collected Papers, vol. 2, p. 133. Cambridge 1889.
[54] See, for example, *Recherches ultérieures sur les déterminants gauches*, loc. cit., p. 214.

these sciences where the preceding formulas should apply without any modification. These generalizations are more immediate, however, than you may suppose. The new developments of electrodynamics which are associated with the *principle of relativity*, are essentially nothing else than the logical use of rotations-dilations in a four-dimensional space. These ideas have been presented and enlarged upon recently by Hermann Minkowski[55].

Let us remain, however, in three-dimensional space. In such a space, a rotation-dilation carries a point (x, y, z) into a point (x', y', z') in such a way that

$$x'^2 + y'^2 + z'^2 = M^2(x^2 + y^2 + z^2),$$

where M denotes the ratio of linear dilation of every length. Since the general linear transformation of (x, y, z) into (x', y', z') contains $3 \cdot 3 = 9$ coefficients, and since the left-hand side of the preceding equation, after the insertion of the values of x', y', z', becomes a quadratic form in x, y, z with $\frac{3\cdot 4}{2} = 6$ terms, the comparison of coefficients in the preceding equation leads to six equations, which reduce to five if the value of M is supposed arbitrary. Therefore the nine original coefficients of the linear transformation, which are subject to these five conditions, are reduced to four *arbitrary parameters*. (Compare p. [73].) If such a substitution has a *positive determinant*, it represents, as was stated on p. [73], a rotation of space about the origin, together with an dilation in the ratio $1 : M$. If the determinant is *negative*, however, the substitution represents a rotation-dilation, combined with a *reflection of the space*, such as, for example, the reflection defined by the equations $x = -x'$, $y = -y'$, $z = -z'$. Moreover, it can be shown easily that the determinant of the transformation must have one of the two values $\pm M^3$.

In order to represent these relationships by means of quaternions, let us first reduce the variable quaternions q and q' to their vectorial parts:

$$q' = ix' + jy' + kz', q = ix + jy + kz,$$

which we shall think of as the three-dimensional vectors joining the origin to the positions of the point before and after the transformation, respectively. We claim now that *the general rotation-dilation of the three-dimensional space is given by the formula* (II) *if* p *and* π *have conjugate values, that is, if we write* $q' = p \cdot q \cdot \overline{p}$; *or, in expanded form*,

$$(1) \qquad \begin{cases} ix' + jy' + kz' \\ \quad = (d + ia + jb + kc)(ix + jy + kz)(d - ia - jb - kc). \end{cases}$$

In order to prove this, we must show first that the scalar part of the product on the right vanishes; that is, that q' is indeed a *vector*. To do this, we first multiply p by **[76]**

[55] Since this was written, an extensive literature on the special theory of relativity mentioned above has appeared. Let me mention here my address *Über die geometrischen Grundlagen der Lorentz-gruppe*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 19 (1910), p. 299, reprinted in Klein's *Gesammelte mathematische Abhandlungen*, vol. 1, p. 533.

q according to the rule for quaternion multiplication, and we find

$$q' = [-ax - by - cz + i(dz + bz - cy) \\ + j(dy + cx - az) + k(dz + ay - bx)] \cdot [d - ia - jb - kc].$$

After another quaternion multiplication, we actually find the scalar part of q' to be zero, whereas we find for the components of the vector part the expressions

(2)
$$\begin{cases} x' = \left(d^2 + a^2 - b^2 - c^2\right) x + 2\,(ab - cd)\, y + 2\,(ac + bd)\, z \\ y' = 2\,(ba + cd)\, x + \left(d^2 + b^2 - c^2 - a^2\right) y + 2\,(bc - ad)\, z \\ z' = 2\,(ca - bd)\, x + 2\,(cb + ad)\, y + \left(d^2 + c^2 - a^2 - b^2\right) z \end{cases}$$

That these formulas actually represent a rotation-dilation becomes evident if we write the tensor equation for (1), which, by (I), is

$$x'^2 + y'^2 + z'^2 = \left(d^2 + a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right)\left(d^2 + a^2 + b^2 + c^2\right),$$

or

$$x'^2 + y'^2 + z'^2 = T^4 \cdot \left(x^2 + y^2 + z^2\right),$$

where $T = \sqrt{d^2 + a^2 + b^2 + c^2}$ denotes the tensor of p. Hence, our transformation is precisely a rotation-dilation (see p. [75]), provided the determinant is positive; otherwise it is such a transformation combined with a reflection. In any case, the ratio of dilation is $M = T^2$. As remarked above, the determinant must have one of the two values $\pm M^3 = \pm T^6$. If we consider the transformation for all possible values of the parameters a, b, c, d which correspond to the same tensor value T, which must obviously be different from zero, we see that the determinant must *always* have the value $+T^6$ if it has that value for any *single* system of values of a, b, c, d; for the determinant is a continuous function of a, b, c, d, and therefore it cannot suddenly change in value from $+T^6$ to $-T^6$ without taking on intermediate values. One set of values for which the determinant is positive is $a = b = c = 0, d = T$, since, by (2), the value of the determinant for these values of a, b, c, d, is

$$\begin{vmatrix} d^2, & 0, & 0 \\ 0, & d^2, & 0 \\ 0, & 0, & d^2 \end{vmatrix} = d^6 = +T^6.$$

It follows that the sign is *always* positive, and hence (1) *always represents a genuine rotation together with a dilation*. It is easy to write down a transformation which combines a reflection with a rotation-dilation, for we need only to combine the preceding transformation with the reflection $x' = -x$, $y' = -y$, $z' = -z$, which is equivalent to writing the quaternion equation $\overline{q}' = p \cdot q \cdot \overline{p}$.

If we want to understand that, conversely, every rotation-dilation is contained in the form (1), or in the equivalent form (2), we have to observe, in the first place, that this formula in fact contains the *four arbitrary parameters*, which, according to the counting made on p. [75], are necessary for the general case. That we can **[77]** actually obtain any desired value of the linear dilation-ratio $M = T^2$, any desired *position of the axis of rotation*, and any desired *angle of rotation*, by a suitable choice of these parameters, can be seen by means of the following formulas. Let ξ, η, ζ denote the direction cosines of the axis of rotation, and let ω denote the angle of rotation (amplitude of rotation). We have, of course, the well known relation

$$\xi^2 + \eta^2 + \zeta^2 = 1. \tag{3}$$

I shall now prove that a, b, c, d are given by the equations

$$\begin{cases} d = T \cdot \cos\frac{\omega}{2} \\ a = T \cdot \xi \cdot \sin\frac{\omega}{2}, \quad b = T \cdot \eta \cdot \sin\frac{\omega}{2}, \quad c = T \cdot \zeta \cdot \sin\frac{\omega}{2}, \end{cases} \tag{4}$$

which, by (3), obviously satisfy the condition

$$d^2 + a^2 + b^2 + c^2 = T^2.$$

When these relations have been proved, we can evidently obtain the correct values of a, b, c, d for any given values of $T, \xi, \eta, \zeta, \omega$.

To *prove* the relations (4), let us remark first that if a, b, c, d are given, the quantities ω, ξ, η, ζ are directly determined, and in such a way that (3) is satisfied. For, squaring and adding the equations (4), since T is the tensor of the quaternion $p = d + ia + jb + kc$, we have

$$1 = \cos^2\frac{\omega}{2} + \sin^2\frac{\omega}{2}\left(\xi^2 + \eta^2 + \zeta^2\right),$$

whence we see that (3) holds. It follows that ξ, η, ζ, are fully determined by the relations

$$a : b : c = \xi : \eta : \zeta, \tag{4'}$$

which appear directly from (4). These equations express the fact that the point (a, b, c) lies on the axis of revolution of the transformation. This fact is easy to verify, for if we put $x = a$, $y = b$, $z = c$ in (2), we find

$$\begin{aligned} x' &= \left(d^2 + a^2 + b^2 + c^2\right) a = T^2 \cdot a, \\ y' &= \left(d^2 + a^2 + b^2 + c^2\right) b = T^2 \cdot b, \\ z' &= \left(d^2 + a^2 + b^2 + c^2\right) c = T^2 \cdot c, \end{aligned}$$

that is, the point (a, b, c) remains on the same ray through the origin, which identifies it as a point on the axis of revolution. It remains only to prove that the angle ω defined by (4) is *actually the amplitude of rotation*. This demonstration requires extended discussion which I can avoid now by remarking that the *transformation*
[78] *formulas* (2) *for* $T = 1$ *transform – due to* (4) *– precisely into the formulas given by Euler established for the rotation of the coordinate system, which has* ξ, η, ζ *as axes and of the angle* ω. This is to be found more in detail, for example, in Klein-Sommerfeld, *Theorie des Kreisels*, volume 1[56], where explicit mention of the theory of quaternions is given, or in Richard Baltzer, *Theorie und Anwendung der Determinanten*[57].

Finally, if we substitute the values given by (4) in the equation (1), we obtain the very brief and convenient equation in quaternion form for the *rotation through an angle* ω *about an axis whose direction cosines are* ξ, η, ζ, *combined with a dilation of ratio* T^2:

(5)
$$\begin{cases} ix' + jy' + kz' = T^2 \left\{\cos\frac{\omega}{2} + \sin\frac{\omega}{2}(i\xi + j\eta + k\zeta)\right\} \cdot \{ix + jy + kz\} \\ \qquad \cdot \left\{\cos\frac{\omega}{2} - \sin\frac{\omega}{2}(i\xi + j\eta + k\zeta)\right\}. \end{cases}$$

This formula expresses in a form that is easy to remember all of Euler's formulas for rotation in one single equation: the multipliers which precede and follow the vector $ix + jy + kz$, are, respectively, the two conjugate quaternions whose tensor is unity (so-called *versor*, that is, "rotator", in contradistinction to *tensor*, "stretcher"), and then the whole result is to be multiplied by a scalar factor which is the dilation-ratio.

We shall proceed now to show that when we specialize these formulas still further to two-dimensions, they become the well-known formulas for the representation of a rotation-dilation of the x-y-plane by means of the multiplication of two complex numbers. (See p. [62].) For this purpose, let us choose the axis of rotation as the z-axis ($\xi = \eta = 0$, $\zeta = 1$). Then the formula (5), for $z = z' = 0$, may be written in the form

$$ix' + jy' = T^2\left(\cos\frac{\omega}{2} + k\sin\frac{\omega}{2}\right)(ix + jy)\left(\cos\frac{\omega}{2} - k\sin\frac{\omega}{2}\right),$$

or, upon multiplication with due regard to the rules for products of the units,

$$\begin{aligned} ix' + jy' &= T^2\left\{\cos\frac{\omega}{2}(ix + jy) + \sin\frac{\omega}{2}(jx - iy)\right\}\left\{\cos\frac{\omega}{2} - k\sin\frac{\omega}{2}\right\} \\ &= T^2\left\{\cos^2\frac{\omega}{2}(ix + jy) + 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}(jx - iy) - \sin^2\frac{\omega}{2}(ix + jy)\right\} \\ &= T^2\{(ix + jy)\cos\omega + (jx - iy)\sin\omega\} \\ &= T^2(\cos\omega + k\sin\omega)(ix + jy). \end{aligned}$$

[56] Leipzig 1897; 2nd printing, 1914.
[57] Fifth edition, Leipzig 1881.

If we now multiply both sides by the right-hand factor $(-i)$, we obtain

$$x' + ky' = T^2(\cos\omega + k\sin\omega)(x + ky),$$

which is precisely the rule for multiplying two ordinary complex numbers, and **[79]** which can be interpreted as a rotation through an angle ω, together with a dilation in the ratio T^2, except that we have used the letter k in place of the usual letter i to denote the imaginary unit $\sqrt{-1}$.

Let us now return to three-dimensional space, and let us modify the formula (1) so that it shall represent a pure rotation without a dilation. To do so, we must replace x', y', z' by $x' \cdot T^2$, $y' \cdot T^2$, $z' \cdot T^2$, that is, we must replace q' by $q' \cdot T^2$. If we notice that $p^{-1} = 1/p = p/T^2$, we may write the formula for a pure rotation in the form

$$ix' + jy' + kz' = p \cdot (ix + jy + kz) \cdot p^{-1}. \tag{6}$$

There is no loss of generality if we assume that p is a quaternion whose tensor is unity, that is,

$$p = \cos\frac{\omega}{2} + \sin\frac{\omega}{2}(i\xi + j\eta + k\zeta), \quad \text{where } \xi^2 + \eta^2 + \zeta^2 = 1,$$

whence we see that (6) results from (5) if T is set equal to unity. The formula was first stated in this form by Cayley in 1845[58]. We may express the composition of two rotations in a particularly simple form, precisely as we did above for four-dimensional space. Given a second rotation

$$ix'' + jy'' + kz'' = p'(ix' + jy' + kz')p'^{-1},$$

where

$$p' = \cos\frac{\omega'}{2} + \sin\frac{\omega'}{2}(i\xi' + j\eta' + k\zeta')$$

the direction cosines of the axis of rotation being ξ', η', ζ', and the angle of rotation being ω', we may write

$$ix'' + jy'' + kz'' = p' \cdot p \cdot (ix + jy + kz) \cdot p^{-1} \cdot p'^{-1}$$

as the equation for the resultant rotation. Hence the direction cosines of the axis or rotation, ξ'', η'', ζ'', and the angle of rotation, ω'', for the resultant rotation, are given by the equation

$$p'' = \cos\frac{\omega''}{2} + \sin\frac{\omega''}{2}(i\xi'' + j\eta'' + k\zeta'') = p' \cdot p.$$

[58] *On certain results relating to quaternions*, Collected Mathematical Papers, vol. 1 (1889), p. 123. – According to Cayley's own statement (vol. 1, p. 586), however, Hamilton had discovered the same formula independently.

We have therefore found a brief and simple expression for the composition of two rotations about the origin, whereas the ordinary formulas for expressing the resultant rotation appear rather complicated. Since any quaternion may be expressed as the product of a real number (its tensor) and the versor of a rotation, we have also found a *simple geometric interpretation of quaternion multiplication as the composition of the rotations. The fact that quaternion multiplication is not commutative*
[80] *then corresponds to the well-known fact that the order of two rotations about a point cannot be interchanged, in general, without changing the result.*

If you desire to know more about the historical development of the interpretations and applications of quaternions and on the theory of rotations of a coordinate system, which we have discussed, I would recommend to you an extremely valuable report on dynamics written by Cayley himself: *Report on the progress of the solution of certain special problems of dynamics*[59].

I shall close with certain *general remarks on the value and the dissemination of quaternions*. For such a purpose, one should distinguish between the *general quaternion calculus* and *quaternion multiplication* properly. The latter, at least, is certainly of very great usefulness, as appears sufficiently from the preceding discussion. The general quaternion calculus, on the other hand, as Hamilton conceived it, embraced addition, multiplication, and division of quaternions, carried to an arbitrary number of steps. Thus Hamilton studied the algebra of quaternions; and, since he investigated also infinite processes, he can even develop a quaternion theory of functions. Since the commutative law does not hold, such a theory takes on a totally different aspect from the theory of ordinary complex variables. It is just to say, however, *that these general and far-reaching ideas of Hamilton did not stand the test of time*, for there have not arisen any vital relationships and interdependencies with other branches of mathematics and its applications. For this reason, the general theory has aroused little general interest.

It is in mathematics, however, as it is in other human affairs: there are those whose views are calmly objective; but there are always some who form impassionate personal convictions. Thus the theory of quaternions has enthusiastic supporters and bitter opponents. The supporters, who are to be found chiefly in England and in America, adopted in 1907 the modern means to found an "Association for the Promotion of the Study of Quaternions". This organization was established as a thoroughly international institution by the Japanese mathematician Shinkichi Kimura, who had studied in America. Sir Robert Ball was for some time its president. They foresaw great possible developments of mathematics to be secured through intensive study of quaternions. On the other hand, there are those who refuse to listen to anything about quaternions, and who go so far as to refuse to consider the very useful idea of quaternion multiplication. According to the view of such persons, all computation with quaternions amounts to nothing but computa-

[59] Report of the British Association for the Advancement of Science, 1862; reprinted in Cayley's Collected Mathematical Papers, Cambridge, vol. 4 (1891), pp. 552ff.

tion with the four components; the units and the multiplication table appear to them to be superfluous luxuries. Between these two extremes, there is a mediating tendency who holds that we should always distinguish carefully between scalars and vectors. **[81]**

4. Complex Numbers in School Teaching

I shall now leave the theory of quaternions and close this chapter with some remarks about the role which these concepts play in the curriculum of the schools. No one would ever think of teaching quaternions in a secondary school, but *the common complex numbers* $x + iy$ *always come up as teaching subjects*. Perhaps it will be more interesting if, instead of telling you at length how it is done and how it ought to be done, I exhibit to you, by means of three books from different periods, *how its teaching has developed historically*.

I put before you, first, a book by Abraham G. Kästner who had a leading position in Göttingen in the second half of the eighteenth century. In those days one still studied, at the university, those elementary mathematical things which later, in the thirties of the nineteenth century, went over to the schools. Accordingly, Kästner also gave lecture courses on elementary mathematics, which were heard by large numbers of non-mathematical students. His textbook, which formed the basis of these lecture courses, was called *Mathematische Anfangsgründe**. The portion which interests us here is the second division of the third, part: *Anfangsgründe der Analysis endlicher Größen*[**,60]. The treatment of imaginary quantities begins there on p. 20 in something like the following words: "Whoever demands the extraction of an even root of a 'denied' quantity (one said 'denied', then, instead of 'negative'), demands an impossibility, for there is no 'denied' quantity which would be such a power". This is, in fact, quite correct. But on p. 34 one finds: "Such roots *are called* impossible or imaginary", and, without much investigation as to justification, one proceeds quietly to operate with them as with ordinary numbers, notwithstanding their existence has just been disputed – as though, so to speak, the meaningless became suddenly usable through receiving a name. You recognize here a reflex of Leibniz's point of view, according to which, imaginary numbers were really something quite foolish but they led, nevertheless, in some incomprehensible way, to useful results.

Kästner was, moreover, a stimulating writer; he achieved quite a place in the literature as a coiner of epigrams. To cite only one of many examples, he expatiates, in the introduction of this textbook mentioned above, on the *origin of the word algebra*, which, indeed, as the article "al" indicates, comes from the Arabic.

* *Elements of Mathematics*.
** *Elements of Analysis of Finite Quantities*.
[60] Third edition. Göttingen 1794.

According to Kästner, an algebraist is a man who "makes" fractions "whole", who, that is, treats rational functions and reduces them to a common denominator, etc. It is said to have referred, originally, to the practice of a surgeon in mending broken
[82] bones. Kästner then cites Don Quixote, who went to an algebraist to get his broken ribs set. Of course, I shall leave undecided, whether Cervantes really adopted this form of expression or whether this is only a lampoon.

The second work which I put before you is more recent, by a couple of years, and comes from the Berlin professor Martin Ohm: *Versuch eines vollständig konsequenten Systems der Mathematik*[***,61]; a book with a purpose similar to that of Kästner and at one time widely used. But Ohm is much nearer the modern point of view, in that he speaks clearly of the *principle of the extension of the number system*. He says, for example, that, just like negative numbers, so $\sqrt{-1}$ must be added to the real numbers as a *new thing*. But even his book lacks a geometric interpretation, since it appeared before the epoch-making publication by Gauß (1831).

Finally, I lay before you, out of the long list of modern school books, one that is widely used: *Bardeys Aufgabensammlung*[62]. The *principle of extension* comes to the fore here, and, in due course, the *geometric interpretation* is explained. This may be taken as the general position of school teaching today, even if, at isolated places, the development has remained at the earlier level. The point of view adopted in this book seems to me to yield the treatment best adapted to the schools. Without tiring the pupil with a systematic development, and without, of course, going into logically abstract explanations, one should explain *complex numbers as an extension of the familiar number concept*, and should avoid any touch of mystery. Above all, one should accustom the pupil, at once, to the *intuitive geometric interpretation in the complex plane!*

With this, we come to the end of the first main part of the course, which was dedicated to arithmetic. Before going over to similar discussions of algebra and analysis, I should like to insert a somewhat extended historical appendix in order to throw new light upon the general conduct of teaching at present, and upon those features of it which we would improve.

*** *An Attempt to Construct a Consistent System of Mathematics*.

61 Nine volumes. Berlin 1828. Vol. I: *Arithmetik und Algebra*, p. 276.

62 [See also the *Reformausgabe* of Bardeys *Aufgabensammlung*, revised by Walther Lietzmann and Paul Zühlke. Oberstufe. Verlag Teubner. Leipzig.] – See also H. Fine *The Number-System in Algebra*. Heath; H. Fine, College Algebra. Ginn.