# I. Calculating with Natural Numbers

We begin with the foundation of all arithmetic, calculation with positive integers. Here, as always in the course of these lectures, we first raise the question as to how these things are handled in the schools; then we shall proceed to the question as to what they imply when viewed from a higher standpoint.

## 1. Introduction of Numbers in the Schools

I shall confine myself to brief suggestions. These will enable you to recall how you yourselves learned your numbers. In such an exposition it is, of course, not my purpose to induct you into the practice of teaching, as should be done in the Seminars of the secondary schools. I shall merely exhibit the material upon which we shall base our critique.

The problem of teaching children the properties of integers and how to reckon with them, and of leading them on to complete mastery, is very difficult and requires the labour of several years, from the first school year until the first or second year of the Gymnasium. The *manner of teaching* as it is carried on in this field in Germany can perhaps best be designated by the words *intuitive* and *genetic*, i.e., the entire structure is gradually erected on the basis of familiar, concrete things, in marked contrast to the customary *logical* and *systematic* method in higher education.

One usually divides up this topic of teaching roughly as follows: The entire *first year* is occupied with *reckoning in the number domain from 1 to 20*, the first half being devoted to the range 1 to 10. The integers appear at first as *number pictures of points* or as *quantities* of all sorts of objects familiar to the children. Addition **[7]**
and multiplication are then presented by intuitional methods, and are fixed in mind.

In the second stage, the *number domain from 1 to 100* are taught and the *Arabic numerals, together with the notion of positional value* and the *decimal system*, are introduced intensely. Let us note, incidentally, that the name “Arabic numerals”, like so many others in science, is historically wrong. This form of writing was invented by the Hindus, not by the Arabs. Another principal aim of the second stage is knowledge of the *multiplication table*. One must know what $5 \times 7$ or $3 \times 8$ is in “one’s sleep”, so to speak. Consequently the pupil must learn the multiplication



F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49442-4_2

table by heart to this degree of thoroughness, to be sure only after it has been made clear to him visually with concrete objects. To this end the *abacus* is used to advantage. It consists, as you all know, of 10 wires stretched one above another, upon each of which there are strung ten movable beads. By sliding these beads in the proper way, one can read off the result of multiplication and also its decimal form.

The *third stage*, finally, teaches calculation with numbers of more than one digit, based on the known simple rules whose general validity is evident, or should be evident, to the pupil. To be sure, this evidence does not always enable the pupil to make the rules completely his own; they are often instilled with the authoritative dictum: "It is thus and so, and if you don't know it yet, so much the worse for you!"

I should like to emphasize another point in this teaching, which is usually neglected in higher education. It is that the application of numbers to practical life is strongly emphasized. From the beginning, the pupil is dealing with numbers taken from real situations, with *coins, measures, and weights;* and the question, "*What does it cost*?", which is so important in daily life, forms the pivot of much of the material of teaching. This plan rises soon to the stage of *word problems*, when deliberate thought is necessary in order to determine what calculation is demanded. It leads to the problems in *regeldetri, alligation*, etc. To the words *intuitive* and *genetic*, which we used above to designate the character of this teaching, we can add a third word, *applications*.

We might summarize the *purpose of the number work* by saying: *It aims at*
**[8]** *reliability in the use of the rules of operation, based on a parallel development of the intellectual abilities involved, and without special concern for logical relations*.

Incidentally, I should like to direct your attention to a contrast which often plays a mischievous role in the schools, viz., the contrast between the university-trained teachers and those who have attended normal schools ("Seminar")[12] for the preparation of elementary school teachers. The former displace the latter, as teachers of arithmetic, during or after the *Quinta* (the second year of the Gymnasium), with the result that a regrettable discontinuity often manifests itself. The poor youngsters must suddenly make the acquaintance of new expressions, whereas the old ones are now forbidden. A simple example is the different multiplication signs, the × being preferred by the elementary teacher, the point by the one who has attended the university. Such conflicts can be dispelled, if the more highly trained teacher will give more heed to his Seminar-trained colleague and will try to meet him on common ground. That will become easier for you, if you will realize what high regard one must have for the performance of the elementary school teachers. Imagine what methodical training is necessary to inculcate over and over again a hundred thousand stupid, unprepared children with the principles of reckoning! Try it with your university training; you will not have great success!

[12] This refers to the "Seminare" for the training of primary school teachers, which have nothing to do with the before mentioned "Seminare" at secondary schools. [Translator's note: These served for giving some practical advise for teaching, for those beginning teachers who had just qualified at universities.]

Returning, after this digression, to the subject matter of teaching, we note that after the *Quarta* of the Gymnasium[*], and especially in the *Tertia*, arithmetic begins to take on the *more noble dress of mathematics*, for which the *transition to operations with letters* is characteristic. One designates by $a$, $b$, $c$, or $x$, $y$, $z$ any numbers, at first only positive integers, and applies the rules and operations of arithmetic to the *numbers thus symbolized by letters*, whereby the numbers are devoid of concrete intuitive content. This represents such a *long step in abstraction* that one may well declare that *real mathematics begins with operations with letters*. Naturally this transition must not be accomplished suddenly. The pupils must accustom themselves gradually to such marked abstraction.

It seems unquestionably necessary that, for this teaching, the teacher should know thoroughly the logical laws and foundations of reckoning and of the theory of integers – although he evidently will not teach them directly to the pupil. Let us now study this more in detail. **[9]**

## 2. The Fundamental Laws of Reckoning

Addition and multiplication were familiar operations long before any one inquired as to the fundamental laws governing these operations. It was in *the twenties and thirties of the last century* that particularly *English and French mathematicians* formulated the fundamental properties of the operations, but I will not enter into historical details here. If you wish to know more, I recommend to you, as I shall often do, the great *Enzyklopädie der Mathematischen Wissenschaften mit Einschluß ihrer Anwendungen*[13], and also the French translation: *Encyclopédie des Sciences mathématiques pures et appliquées*[14] which bears in part the character of a revised and enlarged edition. If a school library has only one mathematical work, it ought to be this encyclopedia, for through it the teacher of mathematics would be placed in position to continue his work in any direction that might interest him. For us, at this place, the article of interest is the first one in the first volume[15] Hermann Schubert: "*Grundlagen der Arithmetik*", of which the translation into French is by Jules Tannery and Jules Molk.

Going back to our theme, I shall enumerate the *five fundamental laws* upon which *addition* depends:

[*] The German Gymnasium was a nine-year secondary school. Its grades were named: Sexta, Quinta, Quarta, Unter-Tertia, Ober-Tertia, Unter-Sekunda, Ober-Sekunda, Unter-Prima, Ober-Prima. Preparatory schooling for those intending to enter the Gymnasium, used to be different from the Volksschulen for the lower social classes.

[13] Leipzig (B. G. Teubner) from 1908 on. Volume I has appeared complete, Volumes II–VI are nearing completion.

[14] Paris (Gauthier-Villars) and Leipzig (Teubner) from 1904 on; unfortunately the undertaking had to be abandoned after the death of its editor J. Molk (1914).

[15] *Arithmetik und Algebra*, edited by Wilhelm Franz Meyer (1896–1904); in the French edition, the editor was Jules Molk.

1. *$a + b$ is always again a number*, i.e., *addition is possible* without restrictions (in contrast to subtraction, which is not always possible in the domain of positive integers).

2. *$a + b$ is* one-valued.

3. *The* associative *law holds:*

$$(a + b) + c = a + (b + c),$$

so that one may omit the parentheses entirely.

4. *The* commutative *law holds*:

$$a + b = b + a.$$

5. *The* monotonic *law holds*:

$$\text{If } b > c, \text{ then } a + b > a + c.$$

These properties are all obvious immediately if one recalls the notion of quantity,
**[10]** which was deduced directly from intuition; but they must be peeled out formally in order to support logically the later developments.

For multiplication there are *five exactly analogous laws*:

1. *$a \cdot b$ is always a number.*

2. *$a \cdot b$ is one-valued.*

3. *Associative law*: $a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c$.

4. *Commutative law*: $a \cdot b = b \cdot a$.

5. *Monotonic law*: *If $b > c$, then $a \cdot b > a \cdot c$.*

A *connection of multiplication with addition* is given by the following law.

6. *Distributive law*:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

It is easy to show that all elementary reckoning is based only upon these eleven laws. It will be sufficient to illustrate this fact by a simple example, say the multiplication of 7 and 12. From the distributive law we have:

$$7 \cdot 12 = 7 \cdot (10 + 2) = 70 + 14,$$

and if we separate 14 into $10 + 4$ (carrying the tens), we have, by the associative law of addition,

$$70 + (10 + 4) = (70 + 10) + 4 = 80 + 4 = 84.$$

You will recognize in this procedure the steps of the usual decimal reckoning. It would be well for you to construct for yourselves more complicated examples. We might summarize by saying *that ordinary reckoning with integers consists in repeated use of the eleven fundamental laws together with the memorized results of the addition and multiplication tables.*

But where does one use the monotonic laws? In ordinary formal reckoning, to be sure, they do not appear, but in certain other problems. Let me remind you of the process called *abridged multiplication and division* with decimal numbers[16]. That is a thing of great practical importance which unfortunately is too little known in the schools, as well as among university students, although it is sometimes mentioned in the *Quinta*. As an example, suppose that one wished to compute $567 \cdot 134$, and that the units digit in each number was of questionable accuracy, say as a result of physical measurement. It would be unnecessary work, then, to determine the product *exactly*, since one could not guarantee an exact result. It is, however, important **[11]** to know the *order* of *magnitude* of the product, i.e., to know between which tens or between which hundreds the exact value lies. The monotonic law supplies this estimate at once; for it follows by that law that the desired value lies between $560 \cdot 134$ and $570 \cdot 134$ or between $560 \cdot 130$ and $570 \cdot 140$. I leave to you the carrying out of the details; at least you see that *the monotonic law is continually used in abridged reckoning*.

A systematic exposition of these fundamental laws is, of course, not to be thought of in the real teaching in schools. Only after the pupils have gained a concrete understanding and a secure mastery of reckoning with numbers, and are ready for the transition to operations with letters, the teacher should take the opportunity to state, at least, the associative, commutative, and distributive laws and to illustrate them by means of numerous obvious numerical examples.

## 3. The Logical Foundations of Operations with Integers

While teaching in the schools will naturally not rise to still more difficult questions, *present mathematical investigation* really begins with the question: *How does one justify the above-mentioned fundamental laws, how does one explain the concept of number at all*? I shall try to explain this matter in accordance with the announced purpose of this lecture course to endeavour to get new light upon school topics by looking at them from a higher point of view. I am all the more willing to do this because these modern thoughts crowd in upon you from all sides during your academic years, but not always accompanied by any indication of their psychological significance.

First of all, so far as the concept of number is concerned, it is very difficult to discover its origin. Perhaps one is happiest if one decides to ignore these most difficult questions. For more complete information as to these questions, which are so earnestly discussed by the philosophers, I must refer you to the article, already mentioned, in the French *Enzyclopädie*, and I shall confine myself to a few remarks. A widely accepted belief is that the concept of number is closely connected with the notion of time, with temporal succession. The philosopher Kant and the mathematician Hamilton represent this view. Others think that number has more to do

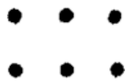
[16] The monotonic laws will be used later, in the theory of irrational numbers.

with space intuition. They base the concept of number upon the *simultaneous perception of different objects which are near each other*. Still others see, in number [12] representations, the expression of a *peculiar faculty of the mind*, which exists independently of, and coordinate with, or even above, the intuition of space and time. I think that this conception would be well characterized by quoting from Faust the lines, which Minkowski, in the preface of his book on *Diophantine Approximation*, applies to numbers:

> "Göttinnen thronen hehr in Einsamkeit,
> Um sie kein Ort, noch weniger eine Zeit."

While this problem involves primarily questions of psychology and epistemology, the justification of our eleven laws, at least the recent researches regarding their compatibility, implies questions of logic. We shall distinguish the following four points of view.

1. According 1to the first of these, best represented perhaps by Kant, the rules of reckoning are immediate necessary results of *Anschauung*, whereby this word is to be understood, in its broadest sense, as "inner perception" or intuition. It is not to be understood by this that mathematics rests throughout upon experimentally controllable facts of rough external experience. To mention a simple example, the commutative law is established by examining the accompanying picture, which consists of two rows of three points each, that is, $2 \cdot 3 = 3 \cdot 2$.



If the objection is raised that in the case of only moderately large numbers, this immediate perception would not suffice, the reply is that we call to our assistance the *theorem of mathematical induction. If a theorem holds for small numbers, and if an assumption of its validity for a number n always insures its validity for* $n + 1$, *then it holds generally for every number*. This theorem, which I consider to be really of an intuitive origin, carries us over the boundary where sense perception fails. This standpoint is more or less that of Henri Poincaré in his well-known philosophical writings.

If we would realize the significance of this question as to the source of the validity of our eleven fundamental rules of reckoning, we should remember that, along with arithmetic, mathematics as a whole rests ultimately upon them. Thus it is not asserting too much to say, that, according to the conception of the rules of reckoning which we have just outlined, *the certainty of the entire structure of mathematics rests upon intuition, where this word is to be understood in its most general sense*.

2. The second point of view is a *modification of the first*. According to it, one tries to separate the eleven fundamental laws into a larger number of shorter steps [13] of which one need take only the simplest directly from intuition, while the remainder are deduced from these by rules of logic without any further use of intuition. Whereas, before, the possibility of logical operation began only *after* the eleven fundamental laws had been set up, it can start earlier here, after the simpler ones

have been selected. *The boundary between intuition and logic is displaced in favour of the latter*. Hermann Graßmann did pioneer work in this direction in his *Lehrbuch der Arithmetik*[17] in 1861. As an example from it, I mention merely that the commutative law can be derived from the associative law by the aid of the principle of mathematical induction. – Because of the precision of his presentation, one might place by the side of this book of Graßmann one by the Italian Giuseppe Peano, *Arithmetices principia nova methodo exposita*[18]. Do not assume, however, because of this title, that the book was written in Latin! It is written in a peculiar symbolic language designed by the author to display each logical step of the proof and emphasize it as such. Peano wishes to have a guarantee in this way, that he is making use only of those principles which he explicitly introduces, while nothing additionally whatever coming from intuition. He wishes to avoid the danger that countless uncontrollable associations of ideas and reminders of perception might creep in if he used our ordinary language. Note, too, that Peano is the *leader of an extensive Italian school* which is trying in a similar way to separate into small groups the premises of each individual branch of mathematics, and, with the aid of such a symbolic language, to investigate their exact logical connections.

3. We come now to a *modem development of these ideas*, which has, moreover, been influenced by Peano. I refer to that treatment of the foundations of arithmetic which is based on the concept of set. You will be able to form a notion of the wide range of the general idea of a set if I tell you that the sequence of all integers, as well as the totality of all points on a line segment, are special examples of sets. Georg Cantor, as is generally known, was the first to make this general idea the object of systematic mathematical speculation.

The *theory of sets*, which he created, is now claiming the profound attention **[14]** of the younger generation of mathematicians. Later I shall endeavour to give you a cursory view of set theory. For the present, it is sufficient to characterize as follows the tendency of the new foundation of arithmetic which has been based upon it: *The properties of integers and of operations with them are to be deduced from the general properties and abstract relations of sets*, in order that the foundation may be as sound and general as possible.

One of the pioneers along this path was Richard Dedekind, who, in his small but important book *Was sind und was sollen die Zahlen?*[19], attempted such a foundation for integers. *Heinrich Weber* essentially follows this point of view in the first part of Weber-Wellstein, volume I (See p. 4). To be sure, the deduction is quite abstract and offers, still, certain grave difficulties, so that Weber, in an Appendix to Volume III[20], gave a more elementary presentation, using only finite sets. In later editions, this

[17] With the addition to the title "*für höhere Lehranstalten*" (Berlin 1861). The corresponding chapters are reprinted in H. Graßmann's *Gesammelte mathematische und, physikalische Werke* (edited by Friedrich Engel), Vol. II, 1, pp. 295–349, Leipzig 1904.

[18] Augustae Taurinorum. Torino 1889 – [There is a more comprehensive presentation in Peano's *Formulaire de Mathématiques* (1892–1899)].

[19] Braunschweig 1888; third edition 1911.

[20] *Angewandte Elementarmathematik*. Revised by Heinrich Weber, Josef Wellstein, Rudolf Heinrich Weber. Leipzig 1907.

appendix is incorporated into Volume I. Those of you who are interested in such questions are especially referred to this presentation.

4. Finally, I shall mention the *purely formal theory of numbers*, which, indeed, goes back to Leibniz and which has recently been brought into the foreground again by Hilbert. His address *Über die Grundlagen der Logik und Arithmetik** at the Heidelberg Congress in 1904 is important for arithmetic[21]. His fundamental conception is as follows: Once one has the eleven fundamental rules of reckoning, one can operate with the letters *a*, *b*, *c*, ..., which actually represent arbitrary integers, without bearing in mind that they have a real meaning as numbers. In other words: let *a*, *b*, *c*, ..., be things devoid of meaning, or things of whose meaning we know nothing; let us agree only that one may combine them according to those eleven rules, but that these combinations need not have any real known meaning. Obviously one can than operate with *a*, *b*, *c*, ..., precisely as one ordinarily does with actual numbers. Only the question arises here *whether these operations could lead once to*
**[15]** *contradictions*. Now ordinarily one says that intuition shows us the existence of numbers for which these eleven laws hold, and that it is consequently impossible for contradictions to lurk in these laws. But in the present case, where we are not thinking of the symbols as having definite meaning, such an appeal to intuition is not permissible. *In fact, there arises the entirely new problem, to p r o v e logically that no operations with our symbols which are based on the eleven fundamental laws can ever lead to a contradiction, i.e., that these eleven laws are consistent, or compatible*. While we were discussing the first point of view, we took the position that the certainty of mathematics rests upon the existence of intuitional things which fit its theorems. The adherents of this formal standpoint, on the other hand, must hold that *the certainty of mathematics rests upon the possibility of showing that the fundamental laws considered formally and without reference to their intuitional content, constitute a logically consistent system.*

I shall close this discussion with the following remarks:

a) Hilbert indicated all of these points of view in his Heidelberg address, but he performed none of them completely. Afterwards he pushed them somewhat farther in a course of lectures, but then abandoned them. We can thus say that *here is constituted a research programme*[22].

b) The tendency to crowd intuition completely off the field and to attain to really *pure* logical investigations seems to me not completely realisable. It seems to me that *one must retain a remainder*, *albeit a minimum, of intuition*. One must always tie a certain intuition, even in the most abstract formulation, with the symbols one uses in operations, in order to be able always to recognise the symbols again, even if one thinks only about the shape of the letters.

c) Let us even assume that the proposed problem has been solved in a way free from objection, that the compatibility of the eleven fundamental laws has been

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* *On the foundations of logic and arithmetic.*

[21] *Verhandlungen des 3. internationalen Mathematikerkongresses in Heidelberg* August 8–13. 1904, p. 174 et seq., Leipzig 1905.

[22] [These investigations have in the meantime been decisively promoted by Hilbert and his disciples.]

proved purely logically. Precisely at this point an opening is offered for a remark which I should like to make with the utmost emphasis. One must namely *become conscious that the p r o p e r arithmetic, the theory of a c t u a l integers, is neither established, nor can ever be established, by considerations of this nature*. It is impossible to show in a purely logical way that the laws whose consistency is established in that manner are actually valid for the numbers with which we are intuitionally familiar; that the undetermined things, which are referred to, and the operations which we apply to them, can be identified with the actual processes of addition and multiplication in their intuitively clear significance. What **[16]** is accomplished is, rather, that the tremendous *task of justification of arithmetic*, unassailable in its complexity, *is split into two parts*, and that the first, the purely logical problem, *the establishing of independent fundamental laws or axioms and the investigation of them as to independence and consistency* has been made available to study. The second, the more epistemological part of the problem, which has to do with the justification of the *application of these laws to actual conditions*, is not even touched, although it must of course be solved also if one will really perform the justification of arithmetic. This second part presents, in itself, an *extremely profound problem*, whose difficulties lie in the general field of epistemology. I can characterize its status most clearly perhaps, by the somewhat paradoxical remark that anyone who accepts only *pure logical* investigations as *pure* mathematics must, to be consistent, ascribe this second part of the problem of the foundation of arithmetic, and hence arithmetic itself, as belonging to *applied* mathematics.

I have felt obliged to go into detail here very carefully, in as much as misunderstandings occur so often at this point, because people simply overlook the existence of the second problem. This is by no means the case with Hilbert himself, and neither disagreements nor agreements based on such an assumption can hold. Johannes Thomae of Jena, coined the neat expression "thoughtless thinkers" for those persons who confine themselves exclusively to these abstract investigations concerning things that are devoid of meaning, and to theorems that tell nothing, and who forget not only that second problem but often also all the rest of mathematics. This facetious term cannot apply, of course, to people who carry on those investigations alongside of many others of a different sort.

In connection with this brief survey of the foundation of arithmetic, I shall bring to your notice a *few general matters*. Many have thought that one could, or that one indeed must, teach all mathematics deductively *throughout*, by starting with a definite number of axioms and deducing everything from these by means of logic. This method, which some seek to maintain upon the authority of Euclid, certainly does not correspond to the historical development of mathematics. In fact, mathematics has grown like a tree, which does not start at its tiniest rootlets and grows merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upward. Just so – if we may drop the figure of speech – mathematics began its development from a certain standpoint corresponding to normal human understanding, and has progressed, from that point, according **[17]** to the demands of science itself and of the then prevailing interests, now in the one direction toward new knowledge, now in the other through the study of fundamental principles. For example, our standpoint today with regard to foundations is different from that of the researchers of a few decades ago; and what we today would state as ultimate principles, will certainly be outstripped after a time, in that the latest truths will be still more meticulously analysed and referred back to something still more general. *We see, then, that as regards the fundamental investigations in mathematics, there is no final ending, and therefore, on the other hand, no first beginning, which could offer an absolute basis for teaching.*

Still another remark concerning the *relation between the logical and the intuitional handling of mathematics, between pure and applied mathematics.* I have already emphasized the fact that, in the schools, applications accompany arithmetic from the beginning, that the pupil learns not only to understand the rules, but to do something with them. And it should always be so in the practice of mathematics! Of course, the purely logical connections, must remain – one might say – *the rigid skeleton in the mathematical organism*, in order to give it its peculiar stability and trustworthiness. But the living thing in mathematics, its most important stimuli, its effectiveness outwardly, depend entirely upon the *applications*, i.e., upon the mutual relations between those purely logical things and all other domains. To banish applications from mathematics would be comparable to seeking the essence of the living animal in the skeleton alone, without considering muscles, nerves and tissues, instincts, in short, the very life of the animal.

In scientific *research* there will be often, to be sure, a *division of labour* between pure and applied science, but provision must be made otherwise for maintaining their connection if conditions are to remain sound. In any case, and this should be especially emphasized here, *for the school such a division of labour, such a far reaching specialisation of the individual teacher, is not possible.* To put the matter crassly, imagine that at some school a teacher is appointed who treats numbers only as meaningless symbols, a second teacher who knows how to bridge the gap from these empty symbols to actual numbers, a third, a fourth, a fifth, finally, who understands the application of these numbers to geometry, to mechanics, and to physics; and that these different teachers are all assigned the same pupils. You see that such an organisation of teaching is impossible. In this way, the things could
**[18]** not be brought to the comprehension of the pupils, neither would the individual teachers be able even to understand each other. The needs of school teaching itself require precisely a certain many sidedness of the individual teacher, a comprehensive orientation in the field of pure and applied mathematics, in the broadest sense, and include thus a *desirable remedy against a too extensive splitting up of science.*

In order to give a practical turn to the last remarks I refer again to our above mentioned *Dresden Proposals*. There we recommend outright that *applied mathematics*, which since 1898 has been a special subject in the examination for prospective teachers, be made *a required part in all normal mathematical training*, so that competence to teach pure and applied mathematics should always be combined. In addition to this, it should be noted that, in the Meran Curriculum[23] of the Breslau

[23] *Reformvorschläge für den mathematischen und naturwissenschaftlichen Unterricht, überreicht der Versammlung der Naturforscher und Arzte zu Meran.* Leipzig, 1905. – See also a reprint in the *Gesamtbericht der Kommission*, p. 93. as well as in Klein-Schimmack, p. 208.

Commission of Instruction, the following three tasks are announced as the *purpose of mathematics teaching in the last school year (Oberprima)*:

1. *A scientific survey of the systematic structure of mathematics*.

2. *A certain degree of skill in the complete handling of problems, numerical and graphical*.

3. *An appreciation of the significance of mathematical thought for a knowledge of nature and for modern culture in general*.

All these formulations I approve with deep conviction.

## 4. Practice in Calculating with Integers

Turning from the last discussions which have been chiefly abstract, let us give our attention to more concrete things by considering the *carrying out of numerical calculation*. As suitable literature for collateral reading, I should mention first of all, the article on *Numerisches Rechnen* by Rudolf Mehmke[24] in the *Enzyklopädie*. I can best give you a general view of the issues that belong here by giving a brief account of this article. It is divided into two parts: A. *Die Lehre vom genauen Rechnen*[*], and B. *Die Lehre vom genäherten Rechnen*[**]. Under A occur all methods for simplifying exact calculation with large integers. *Convenient devices for calculating, tables of products and squares*, and in particular, *calculating machines*, which we shall discuss soon. Under B, on the other hand, one finds a discussion of the methods and devices for all calculating in which only the *order of magnitude of* **[19]** *the result* is important, especially *logarithmic tables and allied devices*, the *slide rule*, which is only an especially well-arranged graphical logarithmic table; finally, also, the numerous important *graphical methods*. In addition to this reference I can recommend the small book by Jacob Lüroth, *Vorlesungen über numerisches Rechnen*[25,***], which, written in pleasant form by a master of the subject, gives a rapid survey of this field.

### *Description of the Calculating Machine "Brunsviga"*

From the many topics that have to do with calculating with integers, I shall present you more in detail only the calculating machine, which you will find in use, in a great variety of ingenious forms, in each larger bank and business house, and which is therefore in fact of the greatest practical significance. We have in our

[24] Enzyklopädie der mathematischen Wissenschaften, Band I, Teil II. See also *Horst von Sanden*, Practical Mathematical Analysis (Translation by Levy), Dutton *St*. Co. – *Horsburgh, E. M.*, Modern Instruments and Methods of Calculation. Bell & Sons.

[*] The *Theory of Exact Calculation*.

[**] The *Theory of Approximate Calculation*.

[25] Leipzig 1900.

[***] *Lectures on Numerical Calculation*.

mathematical collection one of the most widely used types, the “Brunsviga”, manufactured by the firm Brunsviga-Maschinenwerke Grimme, Natalis & Co. A.-G. in Braunschweig. The design originated with the Swedish engineer Odhner, but it has been much changed and improved. I shall describe the machine here in some detail, as a typical example. You will find other kinds described in the books mentioned above[26]. My description of course can give you a real understanding of the machine only if you examine it afterwards personally and if you convince yourself, by actual use, how it is operated. The machine will be at your disposal, for that purpose, after the lecture.

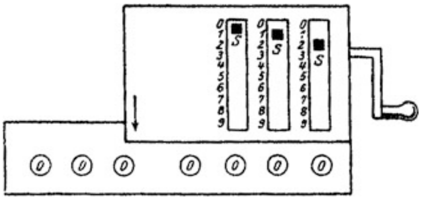


**Figure 1** Before the first turn.

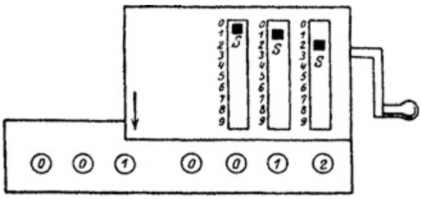


**Figure 2** After the first turn.

So far as the *external appearance* of the Brunsviga is concerned, it presents schematically a picture somewhat as follows (see Fig. 1). There is a fixed frame, the “*drum*”, below which and sliding on it, is a smaller longish case, the “*slide*”. A handle which projects from the dram on the right, is operated by hand. On the drum there is a series of parallel slits, each of which carries the digits $0, 1, 2, \ldots, 9$, read downwards; a peg $S$ projects from each slit and can be set at pleasure at any one of the ten digits. Corresponding to each of these slits there is an opening on the slide under which a digit can appear. Figures 3 and 3a, p. [19] give a view of newer models of the machine.

[26] [Concerning other types of calculating machines, see also Andreas Galle, *Mathematische Instrumente*, Leipzig 1912.]

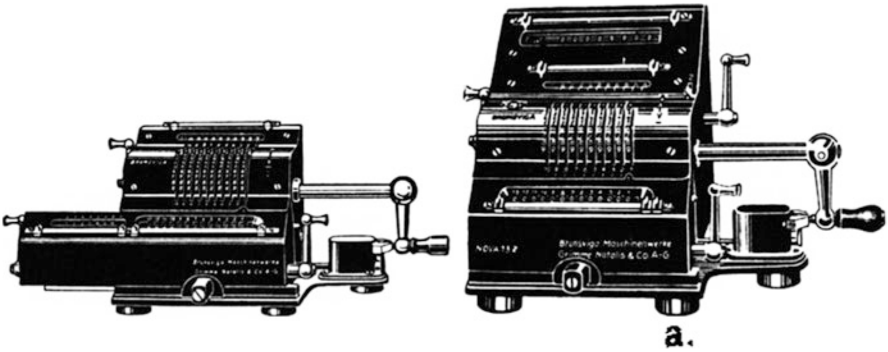


**Figure 3**

I think that the arrangement of the machine will become clearer if I describe to you the process of carrying out a definite calculation, and the way in which the machine brings it about. For this I select *multiplication*.

The procedure is as follows: One *first sets the drum pegs on the multiplicand,* **[20]**
*i.e.*, beginning at the right, one puts the first lever at the one's digit, the second at the ten's digit of the multiplicand, etc. If, for example, the multiplicand is 12, one sets the first lever at 2, the second lever at 1; all the other levers remain at zero (see Fig. 1).

*Now turn the handle once around, clockwise*. The multiplicand appears under the openings of the slide, in our case a 2 in the first opening from the right, a 1 in the second, while zeros remain in all the others. Simultaneously, however, in the first of a series of openings in the slide, at the left, the digit 1 appears to indicate that we have turned the handle once (Fig. 2). *If now one has to do with a multiplier of one digit, one turns the handle as many times as this digit indicates; the multiplier will then be exhibited on the slide to the left, while the product will appear on the slide to the right*.

How does the apparatus bring this result about? In the first place there is attached to the underside of the slide, at the left, a *cogwheel* which carries, equally spaced on its rim, the digits $0, 1, 2, \ldots, 9$. By means of a driver, this cogwheel is rotated through one tenth of its perimeter with every turn of the handle, so that a digit becomes visible through the opening in the slide, which actually indicates the number of revolutions, in other words the multiplier.

Now as to the *obtaining of the product*, it is brought about by similar cogwheels, one under each opening at the right of the slide. But how is it that by one and the same turning of the handle, one of these wheels, in the above case, moves by one unit, the other by two? This is where the peculiarity in construction of the Brunsviga appears. Under each slit of the drum there is a flat wheel-shaped disc (*driver*) attached to the axle of the handle, upon which there are nine teeth which are movable in a radial direction (see Fig. 4). By means of the projecting peg *S*, mentioned above, one can turn a ring *R* which rests upon the periphery of the disc, **[21]** so that, according to the mark upon which one sets $S$ in the slit, $0, 1, 2, \ldots, 9$ of the movable teeth spring outward (in Fig. 4, two teeth). These teeth engage the cogs under the corresponding openings of the slide, so that *with one turn of the handle each driver thrusts forward the corresponding cogwheel by as many units as there are teeth pushed out*, i.e., *by as many teeth as one has set with the corresponding peg S*. Accordingly, in the above illustration, when we start at the zero position, and turn the handle once, the units wheel must jump to 2, the ten's wheel to 1, so that 12 appears. A second turn of the handle moves the units wheel another 2 and the tens wheel another 1, so that 24 appears, and similarly, we get, after 3 or 4 times, $3 \cdot 12 = 36$ or $4 \cdot 12 = 48$, respectively.

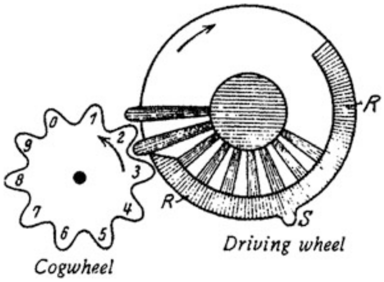


**Figure 4**

But now turn the handle a fifth time: Again, according to the account above, the units wheel should jump again by two units, in other words back to 0, the tens wheel by one, or to 5, and we should have the false result $5 \cdot 12 = 50$. In the actual turning, however, the slide shows 50, to be sure, until just before the completion of the turn; but at the last instant the 5 changes into 6, so that the correct result appears. Something has come into action now that we have not yet described, and
**[22]** which is really the most remarkable point of such machines: the so called *carrying the tens*. Its principle is as follows: *when one of the number bearing cogwheels under the slide* (e.g., *the units wheel*) *goes through zero, it presses an otherwise inoperative tooth of the neighbouring driver* (*for the tens*) *into position, so that it engages the corresponding cogwheel* (*the tens wheel*) *and pushes this forward one place farther than it would have gone otherwise*. You can understand the details of this construction only by examining the apparatus itself. There is the less need for my going into particulars here because it is just the method of carrying the tens that is worked out in the greatest variety of ways in the different makes of machines, but I recommend a careful examination of our machine as an example of a most ingenious model. Our collection contains separately the most important parts of the Brunsviga – which are for the most part invisible in the assembled machine – so that you can, by examining them, get a complete picture of its arrangement.

We can best characterize the operation of the machine, so far as we have made its acquaintance, by the words *adding machine, because*, *with every turn of the handle,* *it adds, once, to the number on the slide at the right, the number which has been set on the drum.*

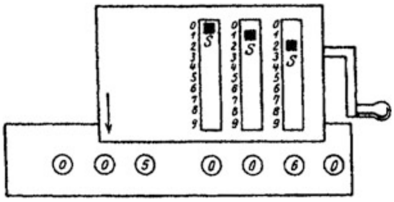


**Figure 5**

Finally, I shall describe in general that arrangement of the machine which permits convenient operation with *multipliers of more than one digit*. If we wish to calculate, say, $15 \cdot 12$ we should have to turn the handle fifteen times, according to the procedure already outlined; moreover, if one wished to have the multiplier indicated by the counter at the left of the slide, it would be necessary to have, there also, a device for carrying the tens. Both of these difficulties are avoided by the following arrangement[27]. We first perform the multiplication by five, so that 5 appears on the slide at the left and 60 at the right (see Fig. 5). Now *we push the slide one place to the right*, so that, as shown in Fig. 5, its units cogwheel is cut out, its tens cogwheel is moved under the units slit of the drum, its hundreds cogwheel under the tens slit, etc., while, at the left, this shift brings it about that the tens cogwheel, instead of the units, is connected with the driver which the handle carries. If we now turn the handle once, 1 appears at the left, in ten's place, so that we read 15; **[23]**
at the right, however, we do not get the addition $\begin{cases} \ \ 60 \\ +12 \end{cases}$ but $\begin{cases} \ \cdot 60 \\ +12\cdot \end{cases}$ or, in other words, $60+120$, since the 2 is "carried over" to the tens wheel, the 1 to the hundreds wheel. Thus we get correctly $15 \cdot 12 = 180$. It is, as you see, *the exact mechanical translation of the customary process of written multiplication*, in which one writes down under one another, the products of the multiplicand by the successive digits of the multiplier, each product moved to the left one place farther than the preceding, and then adds. *In just the same way one proceeds quite generally when the multiplier has three or more digits, that is, after the usual multiplication by the ones, one moves the slide* $1, 2, \ldots$ *places to the right and turns the handle in each place as many times as the digit in the tens, hundreds, ... place of the multiplier indicates.*

Direct examination of the machine will disclose how one can perform other calculations with it; the remark here will suffice that *subtraction and division are effected by turning the handle in the direction opposite to that employed in addition.*

Permit me to summarize by remarking that *the theoretical principle of the machine is quite elementary and represents merely a technical realization of the rules which one always uses in numerical calculation*. That the machine really functions reliably, that all the parts engage one another with unfailing certainty, so that there

[27] In the newer models also this the cogwheel device realises the complete "carrying over".

is no jamming, that the wheels do not turn farther than is necessary, is, of course, the remarkable accomplishment of the man who made the design, and the mechanician who carried it out.

Let us consider for a moment the *general significance of the fact that there really are such calculating machines*, which relieve the mathematician of the purely mechanical work of numerical calculation, and which do this work faster, and, to a higher degree free from error, than he himself could do it, since the errors of human carelessness do not creep into the machine. In the existence of such a machine we see an outright confirmation *that the rules of operation alone, and not the meaning of the numbers themselves, are of importance in calculating;* for it is only these that the machine can follow; it is constructed to do just that; it could not possibly have an intuitive appreciation of the *meaning* of the numbers. We shall not, then, wish to consider it as accidental that such a man as Leibniz, who was both an abstract thinker of first rank and a man of the highest practical gifts, was, at the same time, both the father of purely formal mathematics and the inventor of a calculating machine. His machine is, to this day, one of the most prized possessions of the Kästner Museum in Hannover. Although it is not historically authenticated, still
**[24]** I like to assume that when Leibniz invented the calculating machine, he not only followed a useful purpose, but that he also wished to exhibit, clearly, the purely formal character of mathematical calculation.

With the construction of the calculating machine Leibniz certainly did not wish to minimize the *value of mathematical thinking*, and yet it is just such conclusions, which are now sometimes drawn from the existence of the calculating machine. If the activity of a science can be supplied by a machine, that science cannot amount to much, so it is said; and hence it deserves a subordinate place. The answer to such arguments, however, is that the mathematician, even when he is himself operating with numbers and formulas, is by no means an inferior counterpart of the errorless machine, and by no means the "thoughtless thinker" of Thomae; but rather, he sets for himself his problems with definite, interesting, and valuable ends in view, and carries them to solution in always anew appropriate and original manner. He turns over to the machine only certain operations which recur frequently in the same way, and it is precisely the mathematician – one must not forget this – who invented the machine for his own relief, and who, for his own intelligent ends, designates the tasks which it shall perform.

Let me close this chapter with the wish that the calculating machine, in view of its great importance, may become known in wider circles than is now the case. Above all, every teacher of mathematics should become familiar with it, and it ought to be possible to have it demonstrated once to each pupil of the last grades ("Primaner") of our secondary schools!