

Minimum Degree Up to Local Complementation: Bounds, Parameterized Complexity, and Exact Algorithms

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Abstract. The local minimum degree of a graph is the minimum degree that can be reached by means of local complementation. For any n , there exist graphs of order n which have a local minimum degree at least $0.189n$, or at least $0.110n$ when restricted to bipartite graphs. Regarding the upper bound, we show that the local minimum degree is at most $\frac{3}{8}n + o(n)$ for general graphs and $\frac{n}{4} + o(n)$ for bipartite graphs, improving the known $\frac{n}{2}$ upper bound. We also prove that the local minimum degree is smaller than half of the vertex cover number (up to a logarithmic term). The local minimum degree problem is NP-Complete and hard to approximate. We show that this problem, even when restricted to bipartite graphs, is in W[2] and FPT-equivalent to the EVENSET problem, whose W[1]-hardness is a long standing open question. Finally, we show that the local minimum degree is computed by a $\mathcal{O}^*(1.938^n)$ -algorithm, and a $\mathcal{O}^*(1.466^n)$ -algorithm for the bipartite graphs.

1 Introduction

Notations. Given a graph $G = (V, E)$, \sim_G denotes the neighbourhood relation of G i.e., $\forall u, v \in V, u \sim_G v \Leftrightarrow \{u, v\} \in E$. We consider simple ($\forall u \in V, u \not\sim u$), undirected ($u \sim v \Leftrightarrow v \sim u$) graphs. The set $N_G(u) = \{v \mid u \sim_G v\}$ is the neighbourhood of u and its size $\delta_G(u) = |N_G(u)|$ is the degree of u . $\delta(G) = \min_{u \in V} \delta_G(u)$ is the minimum degree of G and $\tau(G)$ is the vertex cover number i.e., the size of the smallest set S such that if $u \sim v$, then $u \in S$ or $v \in S$. For any $D \subseteq V$, $Odd_G(D) = \Delta_{u \in D} N_G(u) = \{v \in V \mid |N_G(v) \cap D| = 1 \pmod 2\}$ is the odd-neighbourhood of D , where Δ denotes the symmetric difference.

Local Complementation. Local complementation of a graph with respect to one of its vertices consists in complementing the neighbourhood of this vertex:

Definition 1. *The local complementation of a graph G with respect to one of its vertices u is the graph $G \star u$ such that $v \sim_{G \star u} w$ iff $(v \sim_G w)$ xor $(u \sim_G v \wedge u \sim_G w)$.*

The local complementation is an involution ($G \star u \star u = G$). Two graphs are LC-equivalent if there exists a sequence of local complementation transforming one into the other: $G \equiv_{LC} H \Leftrightarrow \exists u_0, \dots, u_k, G \star u_0 \dots \star u_k = H$.

Local complementation has been introduced by Kotzig [20]. The study of this quantity is motivated by several applications: Bouchet [4, 5] and de Fraysseix [9] used local complementation to give a characterization of circle graphs, and Oum [22] links the notion of vertex minor of a graph to LC-equivalence. A noticeable property of local complementation proved by Bouchet [2] is that LC-equivalence of graphs can be decided in time polynomial in the order of the graphs.

Cut Rank. Local complementation is related to the cut-rank function¹ [2, 22]: given a graph G and a bipartition $(A, V \setminus A)$ of its vertices, $\text{cutrk}_G(A)$ is the rank of the linear map $L_A : 2^A \rightarrow 2^{V \setminus A} = X \mapsto \text{Odd}_G(X) \cap (V \setminus A)$. L_A is linear with respect to the symmetric difference: $L_A(X \Delta Y) = L_A(X) \Delta L_A(Y)$. The cut-rank can equivalently be defined as the rank of the cut-matrix, a sub-matrix of the adjacency matrix. Notice that for any A , $\text{cutrk}_G(A) = \text{cutrk}_G(V \setminus A)$.

LC-equivalent graphs have the same cutrank ($\text{cutrk}_G(\cdot) = \text{cutrk}_{G \star u}(\cdot)$) [3], however the converse which was conjectured in [2], has been disproved by Fon der Flaass [12]: the counterexample involves two isomorphic Petersen graphs which have the same cut-rank but which are not LC-equivalent.

LU-equivalence. More recently, local complementation has emerged as a key operation in the field of quantum information theory. The graph state formalism consists in representing a quantum state using a graph (see [15] for details). This powerful formalism provides a graphical representation of quantum entanglement: each vertex represent a quantum bit (qubit) and the edges represent intuitively the entanglement between the qubits. Since entanglement is a non local property, the strength of the entanglement can only decrease when *local* operations are applied on the quantum state, and as a consequence the entanglement is invariant by *local reversible* operations. In the field of quantum information theory this intuition is captured by the LU-equivalence of quantum states: two quantum states have the same entanglement if and only if they are LU-equivalent i.e., there is a local unitary operation transforming one state into the other. LU-equivalence of quantum states can be naturally lifted to graphs as follows: two graphs are LU-equivalent if and only if the corresponding quantum states are LU-equivalent. Van den Nest [27] proved that LC-equivalent graphs are LU-equivalent. Moreover Hein et al. [15] proved that LU-equivalent graphs have the same cutrank. Thus LU-equivalence is weaker than LC-equivalence but stronger than the cut-rank equivalence. Using Fon der Flaass's counterexample based on the Petersen graph, one can show that there exist pairs of graphs which are not LU-equivalent but which have the same cutrank [15]. LC- and LU-equivalences were conjectured to coincide [25]. Indeed, LC- and LU-equivalence actually coincide for several families of graphs [26, 28], however a counterexample of order 27 has been discovered using computer assisted methods [19].

¹ It was used by Bouchet [2] and others under the name *connectivity function*, and coined the cut-rank by Oum [22].

Local Minimum Degree. In this paper we will focus on the minimum degree up to local complementation called local minimum degree:

Definition 2. *Given a graph G , the local minimum degree of G is*

$$\delta_{loc}(G) = \min_{H \equiv_{LC} G} \delta(H)$$

The local minimum degree has been used to bound the rate of some quantum codes obtained by graph concatenation [1]. This quantity has also been used to characterise the complexity of preparation of graph states [16] which are used as a resource in measurement-based quantum computation [24] (a model of quantum computation which is very promising in terms of physical implementation), as well as blind quantum computation [6] for instance. The local minimum degree is also used to bound the optimal threshold that can be achieved by graph-based quantum secret sharing [13, 21].

The local minimum degree is related to the cut-rank function and the smallest set of the form $D \cup Odd_G(D)$:

Property 1 [16]. Given a graph $G = (V, E)$,

$$\delta_{loc}(G) + 1 = \min_{\emptyset \subset D \subseteq V} |D \cup Odd_G(D)| = \min\{|A| : A \subseteq V \wedge \text{cutrk}_G(A) < |A|\}$$

The second equation provides a cut-rank characterisation of the local minimum degree which implies that two graphs which have the same cut-rank have the same local minimum degree. As a consequence, since LU-equivalent graphs have the same cut-rank function, they have the same local minimum degree, too. Thus the local minimum degree is invariant for the three closely related, albeit distinct, classes of equivalence based respectively on local complementation, local unitary operations, and cut-rank functions.

Bounds on the Local Minimum Degree. The local minimum degree has been studied for several families of graphs: the local minimum degree of the hypercube is at least logarithmic in the order of the hypercube [16]; the local minimum degree of a Paley graph \mathcal{P}_n of order n is at least \sqrt{n} . There is no known specific upper bound on the local minimum degree of Paley graphs except that not all Paley graphs can have a linear local minimum degree (i.e., $\delta_{loc}(\mathcal{P}_n) = \Theta(n)$), and the existence of an infinite number of Paley graphs with a linear local minimum degree would imply the Bazzi-Mitter conjecture on elliptic curves [17, 18].

There is no known explicit construction which leads to a local minimum degree greater than the square root of the order of the graph, however using probabilistic methods, it has been proven that there exist graphs of order n which have a local minimum degree larger than $0.189n$ [18]. There are even bipartite graphs with a linear local minimum degree: for any n there exists a bipartite graph of order n and local minimum degree at least $0.110n$ [18].

Regarding the upper-bounds, Property 1 implies that the local minimum degree is at most half of the order of the graph, since no set larger than half of the

vertices can have a full cut-rank. In Sect. 2, we improve this upper bound, proving that for any graph of order n , its local minimum degree is at most $\frac{3}{8}n + o(n)$, and $\frac{n}{4} + o(n)$ for bipartite graphs. We also prove that the local minimum degree is smaller than half of the vertex cover number (up to a logarithmic term).

Complexity of the Local Minimum Degree. One motivation for studying the complexity of computing the local minimum degree comes from the problem of producing graphs with a ‘large’ local minimum degree. Indeed, there is no known explicit construction of graphs with a local minimum degree linear in the order of the graph, but a random graph has such a ‘large’ local minimum degree with high probability. So to produce a graph with a large local minimum degree, one can pick a graph at random and then double check that the local minimum degree is actually ‘large’. However, computing the local minimum degree is hard, even for bipartite graphs: the associated decision problem is NP-Complete [18] and hard to approximate [18].

In Sect. 3, we investigate the parameterized complexity of the local minimum degree problem and its restriction to bipartite graphs. We show that both problems are FPT-equivalent to the so-called EVENSET problem, implying their $W[2]$ -membership. However, it does not imply any hardness result since the $W[1]$ -hardness of EvenSet is long standing open question [11].

In Sect. 4, we introduce exponential algorithms for computing the local minimum degree, mainly based on the improved upper bounds. We show that the local minimum degree of any graph of order n can be computed in time $\mathcal{O}^*(1.938^n)$ and more interestingly that the local minimum degree of bipartite graphs can be computed in time $\mathcal{O}^*(1.466^n)$.

2 Upperbounds on the Local Minimum Degree

For improving the known bounds on the local minimum degree, we use as a routine the fact that in any bipartite graph $G = (V_1, V_2, E)$, there exists a non empty subset of V_1 which oddly dominates at most $\frac{|V_2|}{2(1-2^{-|V_1|})}$ vertices, so roughly speaking as long as V_1 is not too small with respect to V_2 there is a non empty subset of V_1 which oddly dominates at most half of the vertices of V_2 . This fact is a direct consequence of the so called Plotkin bound [23] on linear codes:

Lemma 1. *For any bipartite graph $G = (V_1, V_2, E)$, there exists a non empty set $D \subseteq V_1$ s.t.*

$$|\text{Odd}_G(D)| \leq \frac{|V_2|}{2(1 - 2^{-|V_1|})}$$

Proof. $C := \{\text{Odd}_G(D) : D \subseteq V_1\}$ is a linear binary code of length $n = |V_2|$ and rank $k = |V_1|$, where $\text{Odd}_G(D)$ is identified with its indicator vector in V_2 . According to the Plotkin bound [23], the minimum distance d of C is at most $n/(2(1 - 2^{-k}))$, thus there exists a non empty set $D \subseteq V_1$ such that $|\text{Odd}_G(D)| \leq |V_2|/(2(1 - 2^{-|V_1|}))$. \square

The local minimum degree can be bounded by the vertex cover number as follows:

Lemma 2. *Given a graph G of order n and vertex cover number $\tau(G) > 0$,*

$$2\delta_{loc}(G) \leq \tau(G) + \log_2(\tau(G)) + 1$$

Proof. Let $G = (V, E)$ be a graph of order n , and let S be an independent set of size $\alpha = n - \tau(G)$, and $R \subseteq S$ a subset of size k to be fixed later. Let $G' = (R, (V \setminus S) \cup R, E')$ be a bipartite graph s.t. for any $u \in R$, $N_{G'}(u) = \{u\} \cup N_G(u)$. Notice that there are two copies of R in G' , one on each side of the bipartite graph: there is a matching between these two copies of R , the other edges of G' are those of G between R and $V \setminus S$. According to Lemma 1 there exists $D \subseteq R'$ s.t

$$|\text{Odd}_{G'}(D)| \leq \frac{|V| - |S| + |R|}{2(1 - 2^{-|R|})} = \frac{\tau(G) + k}{2(1 - 2^{-k})}$$

The odd-neighbourhood of D in G' is related to the odd-neighbourhood of D in G as follows: $\text{Odd}_{G'}(D) = \Delta_{u \in D} N_{G'}(u) = \Delta_{u \in D} (\{u\} \cup N_G(u)) = D \Delta \text{Odd}_G(D)$. Thus $|\text{Odd}_{G'}(D)| = |D \cup \text{Odd}_G(D)|$. As a consequence, $\delta_{loc}(G) + 1 \leq \frac{\tau(G) + k}{2(1 - 2^{-k})}$.

- If $\lceil \log_2(\tau(G) + 1) \rceil \leq n - \tau(G)$, then we fix $k = \lceil \log_2(\tau(G) + 1) \rceil$:

$$\delta_{loc}(G) + 1 \leq \frac{\tau(G) + \lceil \log_2(\tau(G) + 1) \rceil}{2(1 - 2^{-\lceil \log_2(\tau(G) + 1) \rceil})} < \frac{1}{2}(\tau(G) + \log_2(\tau(G))) + 1 \quad (1)$$

To prove the second inequality of Eq. (1), let $\tau(G) = 2^r + y$ with $y < 2^r$. Notice that $\lceil \log_2(\tau(G) + 1) \rceil = r + 1$, thus

$$\delta_{loc}(G) + 1 \leq \frac{2^r + y + r + 1}{2(1 - 2^{-r-1})}$$

Moreover, standard calculation shows that $\frac{2^r + y + r + 1}{1 - 2^{-r-1}} < 2^r + y + \log_2(2^r + y) + 2$ when $r > 0$. Thus $2\delta_{loc}(G) + 2 < \tau(G) + \log_2(\tau(G)) + 2$. When $r = 0$, $\tau(G) = 1$, thus G is a star (and possibly some isolated vertices), so $2\delta_{loc}(G) \leq 2 = \tau(G) + \log_2(\tau(G)) + 1$.

- If $\lceil \log_2(\tau(G) + 1) \rceil > n - \tau(G)$, then it is enough to prove that $2\delta_{loc}(G) \leq n$ since $\tau(G) + \log_2(\tau(G)) + 1 \geq \tau(G) + \lceil \log_2(\tau(G) + 1) \rceil > n$. For any set S of size $\lfloor \frac{n}{2} \rfloor + 1$, $\text{cutrk}_G(S) < |S|$ since $|V \setminus S| < |S|$, thus according to property 1, $\delta_{loc}(G) < \lfloor \frac{n}{2} \rfloor + 1 \leq n/2$. \square

Remark 1. In Lemma 2, the condition $\tau(G) > 0$ only excludes the empty graph and is used to guarantee that the logarithm is well defined. The bound is tight for star graphs: $\delta_{loc}(S_n) = 1$ and $\tau(S_n) = 1$. This is the only tight case and when $\tau(G) > 1$, the proof can be modified to prove the following statement where the constant factor is removed: if $\tau(G) > 1$, $2\delta_{loc}(G) \leq \tau(G) + \log_2(\tau(G))$.

The vertex cover number-based bound on the local minimal degree leads to an improved general upper bound for bipartite graphs:

Theorem 1. *For any bipartite graph G of order $n > 0$,*

$$\delta_{loc}(G) < \frac{n}{4} + \log_2 n$$

Proof. If $n \leq 2$, the property is satisfied. Otherwise, since G is bipartite $\tau(G) \leq \lfloor \frac{n}{2} \rfloor$, so according to Lemma 2, $\delta_{loc}(G) \leq \frac{1}{2}(\tau(G) + \log_2(\tau(G)) + 1) \leq \frac{n}{4} + \frac{1}{2} \log_2(n/2) + \frac{1}{2} \leq \frac{n}{4} + \frac{1}{2} \log_2 n < \frac{n}{4} + \log_2 n$. \square

Contrary to the bipartite case, the bound involving the vertex cover number does not lead to an improved upper bound for non-bipartite graphs. However, we prove that the local minimum degree of a graph of order n is at most $\frac{3}{8}n + o(n)$ exploiting the structure of the kernels of the linear maps associated with the cuts of the graph:

Theorem 2. *For any graph G of order $n > 0$,*

$$\delta_{loc}(G) < \frac{3}{8}n + \log_2 n$$

Proof. For any integer $0 < k < n/2$, let S be a subset of $\lfloor n/2 \rfloor + k$ vertices. Let $L : S \rightarrow V \setminus S$ be the map $D \mapsto Odd_G(D) \setminus S$ which is linear for the symmetric difference, i.e. $L(D_1 \Delta D_2) = L(D_1) \Delta L(D_2)$. Notice that for any $D \in Ker(L)$, $D \cup Odd(D) \subseteq S$. According to the rank nullity theorem, $dim(Ker(L)) \geq 2k - 1$. Let $R \subseteq S$ be a basis for $Ker(L)$. Let $G' = (R, S \times \{1, 2, 3\}, E')$ be a bipartite graph s.t. for any $D \in R$, $N_{G'}(D) = D \times \{1\} \cup Odd_G(D) \times \{2\} \cup (Odd_G(D) \Delta D) \times \{3\}$: the neighbourhood of D in G' is the disjoint union of D , $Odd_G(D)$ and $D \Delta Odd_G(D)$. Notice that $|R| \geq 2k - 1$ and $|S \times \{1, 2, 3\}| = 3(\lfloor n/2 \rfloor + k)$, so according to Lemma 1, there exists a non empty $R_0 \subseteq R$ such that $|Odd_{G'}(R_0)| \leq \left\lfloor \frac{3}{2} \cdot \frac{\lfloor n/2 \rfloor + k}{1 - 2^{1-2k}} \right\rfloor$.

Let $F := \Delta_{D \in R_0} D$. Since R is a basis and $R_0 \neq \emptyset$, $F \neq \emptyset$. Moreover $Odd_{G'}(R_0) = \Delta_{D \in R_0} N_{G'}(D) = \Delta_{D \in R_0} (D \times \{1\} \cup Odd_G(D) \times \{2\} \cup (Odd_G(D) \Delta D) \times \{3\}) = F \times \{1\} \cup Odd_G(F) \times \{2\} \cup (F \Delta Odd_G(F)) \times \{3\}$. Thus $|Odd_{G'}(R_0)| = |F| + |Odd_G(F)| + |F \Delta Odd_G(F)| = 2|F \cup Odd_G(F)|$. As a consequence,

$$|F \cup Odd_G(F)| \leq \left\lfloor \frac{1}{2} \left[\frac{3}{2} \cdot \frac{\lfloor n/2 \rfloor + k}{1 - 2^{1-2k}} \right] \right\rfloor \tag{2}$$

We choose $k = \lfloor 4 \log_2(n)/3 \rfloor$ to guarantee $|F \cup Odd_G(F)| \leq \frac{3}{8}n + \log_2(n) + O(1)$. More precisely, notice that $|F \cup Odd_G(F)| \leq \frac{3}{8} \cdot \frac{n + 2 \lfloor 4 \log_2(n)/3 \rfloor}{1 - 2 \times 2^{-2 \lfloor 4 \log_2(n)/3 \rfloor}} \leq \frac{3}{8} \cdot \frac{n + 8 \log_2(n)/3}{1 - 8 \cdot n^{-8/3}}$ which is strictly smaller than $\frac{3}{8}n + \log_2 n + 1$ when $n > 60$. For $2 < n \leq 61$, one can double check by direct calculation that the bound in Eq. 2 is actually strictly smaller than $\frac{3}{8}n + \log_2(n) + 1$. Thus for any $n > 2$, $\min_{D \neq \emptyset} |D \cup Odd_G(D)| < \frac{3}{8}n + \log_2 n + 1$, so $\delta_{loc}(G) < \frac{3}{8}n + \log_2 n$. Finally, it is easy to check that $\delta_{loc}(G) < \frac{3}{8}n + \log_2 n$ also holds for $n \leq 2$. \square

Remark 2. Choosing $k = \lfloor \log_2(n)/2 \rfloor$ in the proof of Theorem 2 gives an asymptotically slightly better bound: $\delta_{loc}(G) \leq 3/8n + 3/4 \log_2(n) + O(1)$.

3 Parameterized Complexity

The decision problem associated with the local minimum degree is known to be NP-complete and hard to approximate: there exists no k -approximation algorithm for this problem for any constant k unless $P=NP$ [18]. In this section we consider the parameterized complexity of this problem, and its bipartite version. Please refer to [10] for an introduction to parameterized complexity.

LOCAL MINIMUM DEGREE :
input: A graph G
parameter: An integer k
question: Is $\delta_{loc}(G) \leq k$?

BIPARTITE LOCAL MINIMUM DEGREE :
input: A bipartite graph G
parameter: An integer k
question: Is $\delta_{loc}(G) \leq k$?

We show that both problems are FPT-equivalent to the EVENSET problem [11]:

EVENSET:
input: A bipartite graph $G = (R, B, E)$
parameter: An integer k
question: Is there a non empty $D \subseteq R$, such that $|D| \leq k$ and $Odd_G(D) = \emptyset$ i.e., every vertex in B has an even number of neighbours in D ?

To prove the FPT-equivalence of these three problems, first we prove that EVENSET is harder than LOCAL MINIMUM DEGREE, and then that BIPARTITE LOCAL MINIMUM DEGREE is harder than EVENSET.

Theorem 3. EVENSET is FPT-reducible to LOCAL MINIMUM DEGREE.

Proof. Given an instance (G, k) of LOCAL MINIMUM DEGREE, let (G', k') be an instance of EVENSET where:

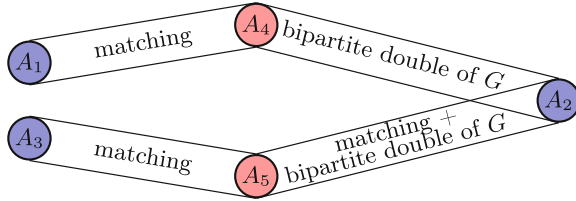
$$G' = (A_1 \cup A_2, \cup A_3, A_4 \cup A_5, E_1 \cup E_2 \cup E_3), k' = 2k+2$$

$$\begin{aligned} \forall i \in [1, 5], A_i &= \{a_{i,u}, \forall u \in V(G)\} \\ E_1 &= \{(a_{1,u}, a_{4,u}), \forall u \in V(G)\}, \\ E_2 &= \{(a_{i,u}, a_{5,u}), \forall i \in \{2, 3\}, \forall u \in V(G)\} \\ E_3 &= \{(a_{2,u}, a_{i,v}), \forall i \in \{4, 5\}, \forall \{u, v\} \in E(G)\} \end{aligned}$$

In other words, G' consists of 5 copies A_i s of $V(G)$, there is a matching between A_1 and A_4 , and between A_3 and A_5 . Moreover, the subgraph induced by $A_2 \cup A_4$ is the bipartite double of G , whereas subgraph induced by $A_2 \cup A_5$ the bipartite double of G augmented with a matching.

– If (G, k) is a positive instance of LOCAL MINIMUM DEGREE with a non empty $D \subseteq V(G)$ such that $|D \cup Odd_G(D)| \leq k+1$. Let $D' = \{a_{1,u} \mid u \in Odd_G(D)\} \cup \{a_{2,u} \mid u \in D\} \cup \{a_{3,u} \mid u \in Odd_G(D)\Delta D\}$, thus D' is composed of the copy of D in A_2 , the copy of $Odd_G(D)$ in A_1 and the copy of $D\Delta Odd_G(D)$ in A_3 . Notice that $Odd_{G'}(D') = \emptyset$, and $D' \neq \emptyset$ since $D \neq \emptyset$. Moreover $|D'| = |Odd_G(D)| + |D| + |D\Delta Odd_G(D)| = 2|D \cup Odd_G(D)| \leq 2k + 2 = k'$. Thus D' makes (G', k') a positive instance of EVENSET.

– If (G', k') is a positive instance of EVENSET with a non empty $D \subseteq A_1 \cup A_2 \cup A_3$ of size at most k' such that $Odd_{G'}(D) = \emptyset$. For $i \in [1, 3]$, let $D_i = \{u \in V(G) \mid a_{i,u} \in D\}$. Notice that $D_1 = Odd_G(D_2)$ and $D_3 = Odd_G(D_2)\Delta D_2$. $D \neq \emptyset$ implies $D_2 \neq \emptyset$, moreover $|D_2 \cup Odd_G(D_2)| = \frac{1}{2}(|D_2| + |Odd_G(D_2)| + |Odd_G(D_2)\Delta D_2|) = \frac{1}{2}|D| \leq \frac{1}{2}k' = k+1$, so D_2 makes (G, k) a positive instance of LOCAL MINIMUM DEGREE. \square



Corollary 1. LOCAL MINIMUM DEGREE is in W[2].

W[2]-membership of LOCAL MINIMUM DEGREE is not surprising in the sense that not only EVENSET but all similar problems of graph domination with parity conditions are known to be in W[2] [8]. We refine this W[2]-membership by proving that both LOCAL MINIMUM DEGREE and BIPARTITE LOCAL MINIMUM DEGREE are FPT-equivalent to EVENSET. They form a peculiar subclass of W[2] for which no hardness results are known: the W[1]-hardness of EVENSET is a long standing open question in parameterized complexity [11]. This contrasts with the subclass of problems FPT-equivalent to the W[1]-hard ODDSET problem which contains problems like WEAK ODD DOMINATION and QUANTUM THRESHOLD [7, 14].

Theorem 4. BIPARTITE LOCAL MINIMUM DEGREE is FPT-reducible to EVENSET.

Proof. If $(G=(R, B, E), k)$ is a positive instance of EVENSET, then it is also a positive instance of BIPARTITE LOCAL MINIMUM DEGREE. But if (G, k) is a positive instance of BIPARTITE LOCAL MINIMUM DEGREE, it may fail to be a positive instance of EVENSET mainly for two reasons:

- (i) A set D such that $|D \cup Odd_G(D)| \leq k+1$ may not be a subset of R
- (ii) For solving EVENSET, one wants to guarantee that $Odd_G(D) = \emptyset$.

Regarding the first point, a gadget with a local minimum degree larger than $k+1$ is attached to each vertex in B to guarantee that no vertex of B can occur in a set D such that $|D \cup Odd(D)| \leq k+1$. Concretely we can use a Paley graph P_q which vertices are $\{0, \dots, q-1\}$ for $q = 1 \pmod 4$ a power of prime, and (i, j) is an edge iff $\exists x, i - j = x^2 \pmod q$. The local minimal degree of a Paley graph is at least square root of its order. However to keep the bipartiteness of the graph we use the bipartite double of a Paley graph rather than a Paley graph. Indeed, it is known that the local minimum degree of a bipartite double graph is as large as the local minimum degree of the original graph ($\delta_{loc}(G^{\oplus 2}) \geq \delta_{loc}(G)$ [17]).

Regarding the second point, each vertex of B is duplicated k times in such a way that for any $D \subseteq R$ if a vertex $v \in B$ is in the odd neighbourhood of D than its k copies are also in the odd-neighbourhood which contradicts the fact that $|D \cup \text{Odd}(D)|$ is at most $k + 1$.

Concretely, let q be a prime number such that $q \geq k^2 + 1$ and $q \equiv 1 \pmod{4}$, let (G', k) be an instance of BIPARTITE LOCAL MINIMUM DEGREE such that $G' = (R \cup P', P, E_G \cup E_{\text{Paley}})$, where $P = \cup_{b \in B, i \in [0, k]} P_{b,i}$, $P' = \cup_{b \in B, i \in [0, k]} P'_{b,i}$, $P_{b,i} = \{p_{b,i,r}, \forall r \in [0, q-1]\}$, $P'_{b,i} = \{p'_{b,i,r}, \forall r \in [0, q-1]\}$, $E_{\text{Paley}} = \cup_{b \in B, i \in [0, k]} E_{\text{Paley}}^{(b,i)}$ and $E_{\text{Paley}}^{(b,i)} = \{(p_{b,i,r}, p'_{b,i,r'}), \forall r, r' \in [0, q-1] \text{ s.t. } \exists \ell \in [0, q-1], \ell^2 = r - r' \pmod{q}\}$.

– If (G, k) is a positive instance of EVENSET with $D \subseteq E$ s.t. $\text{Odd}_G(D) = \emptyset$ then $\text{Odd}_{G'}(D) = \emptyset$ so (G', k) is a positive instance of BIPARTITE LOCAL MINIMUM DEGREE.

– If (G', k) is a positive instance of BIPARTITE LOCAL MINIMUM DEGREE with D s.t. $|D \cup \text{Odd}_{G'}(D)| \leq k+1$. For any $b \in B, i \in [0, k]$, let $D'_{b,i} = D \cap (P_{b,i} \cup P'_{b,i})$, in the subgraph induced by $P_{b,i} \cup P'_{b,i}$ $|D'_{b,i} \cup \text{Odd}_{G'}(D)| \leq k+1$, thus $D'_{b,i} = \emptyset$ since $\delta_{\text{loc}}(\text{Paley}_{k^2+1}) > k$. So $D \subseteq R$. Moreover if there exists $p_{b,i,0} \in \text{Odd}_{G'}(D)$ then $\forall j \in [0, k], p_{b,j,0} \in \text{Odd}_{G'}(D)$, so $|D \cup \text{Odd}_{G'}(D)| > k+1$, so by contradiction $\text{Odd}_{G'}(D) = \emptyset$. Thus (G, k) is a positive of EVENSET. \square

Corollary 2. BIPARTITE LOCAL MINIMUM DEGREE and LOCAL MINIMUM DEGREE are FPT-equivalent to EVENSET.

W[1]-hardness of EVENSET is a long standing open problem, the FPT-equivalence with (BIPARTITE) LOCAL MINIMUM DEGREE might give some more insights and open new perspectives on the parameterized complexity of EVENSET.

4 Exponential Algorithms

In this section we introduce exact exponential algorithms for computing the local minimum degree of a graph.

Property 2. The local minimum degree of a graph of order n can be computed in time $\mathcal{O}^*(1.938^n)$.

Proof. Thanks to Property 1 and Theorem 2, $\delta_{\text{loc}}(G)+1 = \min\{|A| : |A| \leq \frac{3}{8}n + \log_2(n) \wedge \text{cutrk}_G(A) < |A|\}$. The algorithm consists in enumerating all subsets of at most $\frac{3}{8}n + \log_2(n)$ vertices and computing its cut-rank. The cut-rank can be computed in polynomial time, so the complexity of this algorithm is $\mathcal{O}^*(2^{H(\frac{3}{8})n})$ where $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$ is the binary entropy function. \square

Regarding the bipartite case, enumerating all the subsets of size at most $\frac{n}{4} + \log_2(n)$ leads to a $\mathcal{O}^*(1.755^n)$ algorithm. This naive algorithm can be improved:

Theorem 5. The local minimum degree of a bipartite graph of order n can be computed in time $\mathcal{O}^*(1.466^n)$.

Proof. We use the following property of bipartite graphs: given a bipartite graph $G = (V_1, V_2, E)$, $\delta_{loc}(G) + 1 = \min_{\emptyset \subset D \subseteq V_1 \text{ or } \emptyset \subset D \subseteq V_2} |D \cup Odd_G(D)|$. Indeed, for any $D \subseteq V_1 \cup V_2$, both $(D \cap V_1) \cup Odd_G(D \cap V_1)$ and $(D \cap V_2) \cup Odd_G(D \cap V_2)$ are subsets of $D \cup Odd_G(D)$. Let $|V_1| = \alpha n$ and $|V_2| = (1 - \alpha)n$. We assume w.l.o.g. that $\alpha \leq 1/2$. Since V_1 is a vertex cover set, according to Lemma 2, $\delta_{loc}(G) \leq \frac{\alpha}{2}n + \frac{\log_2(\alpha n)}{2}$. Thus to compute the local minimum degree, it is enough to enumerate all sets D of size at most $\frac{\alpha}{2}n + \frac{\log_2(\alpha n)}{2}$ in both V_1 and V_2 and to compute their odd neighbourhood – which can be done in time polynomial in n . There are $\binom{\frac{\alpha n}{2} + \frac{\log_2(\alpha n)}{2}}{\frac{\alpha n}{2} + \frac{\log_2(\alpha n)}{2}} + \binom{\frac{(1-\alpha)n}{2} + \frac{\log_2(\alpha n)}{2}}{\frac{\alpha n}{2} + \frac{\log_2(\alpha n)}{2}} = \mathcal{O}^*(2^{(1-\alpha)nH(\frac{\alpha}{2(1-\alpha)})})$ sets to enumerate. Notice that $\alpha \mapsto (1 - \alpha)H(\frac{\alpha}{2(1-\alpha)})$ is maximal for $\alpha_0 = 0.3885$, and $2^{(1-\alpha_0)H(\frac{\alpha_0}{2(1-\alpha_0)})} = 1.46557$. □

5 Conclusion

After having shown that the local minimum degree is smaller than half of the vertex cover number (up to a logarithmic term), we have improved the best known upper bound on the local minimum degree, proving that it is at most $\frac{3}{8}n + o(n)$ and $\frac{n}{4} + o(n)$ for bipartite graphs. Moreover, we have investigated the parametrized complexity of the problem, showing its W[2]-membership and its FPT-equivalence with the EVENSET problem, even when restricted to bipartite graphs. Finally, we have introduced a $\mathcal{O}^*(1.938^n)$ -algorithm – $\mathcal{O}^*(1.466^n)$ -algorithm for the bipartite graphs – for computing the local minimum degree.

This is noticeable that the bipartite case evolves quite similarly to the general case: same parameterized complexity, and upper bound and algorithm slightly better in the bipartite case. It would be interesting to investigate other families of graphs, in particular those defined by excluded vertex minors, in order to identify a family of graphs which local minimum is large but easy to compute or to approximate.

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