# Linear Vector Optimization and European Option Pricing Under Proportional Transaction Costs

**Alet Roux and Tomasz Zastawniak**

**Abstract** A method for pricing and superhedging European options under proportional transaction costs based on linear vector optimisation and geometric duality developed by Löhne and Rudloff (Int. J. Theor. Appl. Finance 17(2): 1450012–1–1450012–33, 2014) is compared to a special case of the algorithms for American type derivatives due to Roux and Zastawniak (Acta Applicandae Mathematicae, published online 2015). An equivalence between these two approaches is established by means of a general result linking the support function of the upper image of a linear vector optimisation problem with the lower image of the dual linear optimisation problem.

**Keywords** Option pricing · Superhedging · Transaction costs · Linear vector optimisation

## 1 Introduction

We compare two existing methods for the computational pricing and superhedging of European options in the presence of proportional transaction costs, and investigate the relationships between them, highlighting their similarities, differences and relative strengths.and dual constructions stated in Sect. 3.3, goes back to [20, 21], where it was developed for the much more general class of American type derivative securities, of which European options are a special case. The other method, which relies on linear vector optimisation and geometric duality, was proposed by [17] and named the SHP-algorithm by them; see Sect. 3.4.

As a by-product, we prove a general result establishing one-to-one correspondence between the support function of the upper image of a linear vector optimisation

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problem on the one hand, and the lower image of the dual linear vector optimisation problem on the other hand; see Proposition 2.1. This result provides a link between the two methods for pricing and superhedging European options, and it is also interesting in its own right.

We work within the general model of a currency exchange market of [9], with proportional transaction costs included in the form of exchange rate bid ask spreads. This model has been extensively studied, for example, by [10, 11, 22].

All three algorithms, the primal construction, the dual construction and the SHP-algorithm lend themselves well to computer implementation. For the primal and dual constructions this has been done by [21] with the aid of the *Maple* package *Convex* developed by [3]. To implement the SHP-algorithm [17] used Benson's linear vector optimisation technique; see [2, 4]. We illustrate the results by a numerical example computed by means of the primal and dual constructions and compare this with a similar example presented by [17], who employed the SHP-algorithm.

We conclude by suggesting a possible extension of the SHP-algorithm to hedge and price the seller's (short) position in an American option, and pointing out an inherent difficulty in hedging and pricing the buyer's (long) position in an American option due to the essential non-convexity of the problem.

## 2 A General Duality Result

In this section we present a simple observation that links support functions with duality in linear vector optimization. The related work of [18] provides further insight on the connection between support functions and duality. This result will prove useful in comparing the various pricing and hedging algorithms in the following sections.

For a cone $C \subseteq \mathbb{R}^q$ we define a partial ordering $\leq_C$ on $\mathbb{R}^q$ by

$$y \leq_C z \iff z - y \in C$$

and denote by $C^+$ the dual (or positive polar) cone of $C$, i.e.

$$C^+ = \left\{x \in \mathbb{R}^q : x^T y \geq 0 \forall y \in C\right\}.$$

In what follows we assume that $C$ is a polyhedral cone with non-empty interior, and there exists some $c \in \operatorname{int} C$ with $c_q = 1$. Suppose that matrices $P \in \mathbb{R}^{q\times d}$ and $B \in \mathbb{R}^{m\times d}$ and a vector $b \in \mathbb{R}^m$ are given, and consider the linear vector optimization problem

$$\text{minimize } Px \text{ with respect to } \leq_C \text{ over } x \in S, \tag{P}$$

with feasible set

$$S = \{x \in \mathbb{R}^d : Bx \geq b\}.$$

The *upper image* of problem (P) is the set

$$\mathcal{P} = P[S] + C.$$

The dual problem to (P) is

$$\text{maximize } D^*(u, w) \text{ with respect to } \leq_K \text{ over } (u, w) \in T, \tag{D$^*$}$$

where the linear operator $D^* : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^q$ is defined as

$$D^*(u, w) = (w_1, \ldots, w_{q-1}, b^T u)^T \text{ for } (u, w) \in \mathbb{R}^m \times \mathbb{R}^q,$$

with $K = \text{cone}\{e^q\}$ for $e^q = (0, \ldots, 0, 1) \in \mathbb{R}^q$, and with

$$T = \{(u, w) \in \mathbb{R}^m \times \mathbb{R}^q : u \geq 0, B^T u = P^T w, c^T w = 1, w \in C^+\}.$$

The *lower image* of problem (D$^*$) is the set

$$\mathcal{D}^* = D^*[T] - K.$$

We now state and prove a general result that links the lower image $\mathcal{D}^*$ of (D$^*$) with the support function of $-\mathcal{P}$, where $\mathcal{P}$ is the upper image of (P). The support function $Z : \mathbb{R}^q \to \mathbb{R}$ of $-\mathcal{P}$ is defined as (see e.g. [19] p. 28)

$$Z(x) = \sup \left\{x^T z : z \in -\mathcal{P}\right\} \text{ for all } x \in \mathbb{R}^q.$$

Note that $Z(x)$ is the negative of a scalarization of $\mathcal{P}$ with respect to the weighting vector $x$ (see e.g. [15] Sect. 4.1.1). Thus the following result can be regarded as a reformulation of strong geometric duality (see [15] Theorems 4.40, 4.41) by means of the family of scalarizations of $\mathcal{P}$.

**Proposition 2.1** *If $C$ contains no lines, i.e. if $C \cap (-C) = \{0\}$, then*

$$\mathcal{D}^* = \left\{ w \in \mathbb{R}^q : -w_q \geq Z\left( w_1, \ldots w_{q-1}, 1 - \sum_{i=1}^{q-1} c_i w_i \right) \right\}, \tag{2.1}$$

$$Z(w) = \begin{cases} -\sup\left\{ y \in \mathbb{R} : \frac{1}{c^T w}\left(w_1, \ldots, w_{q-1}, y\right) \in \mathcal{D}^* \right\} & \text{if } c^T w > 0, \\ 0 & \text{if } w = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

*Proof* If $C$ contains no lines, then Theorems 4.40 and 4.41 of [15] (see also [4] Remark 3.7) give

$$\mathcal{D}^* = \left\{ w \in \mathbb{R}^q : \varphi(y, w) \geq 0 \forall y \in \mathcal{P} \right\},$$

where the bi-affine coupling function $\varphi : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ is defined as

$$\varphi(y, w) = \sum_{i=1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) - w_q \text{ for } (y, w) \in \mathbb{R}^q \times \mathbb{R}^q.$$

The function $\varphi$ was first introduced for the special case $c = (1, \ldots, 1)^T$ by [7] and for general $c$ by [17].

Observe that $\varphi(y, w) \geq 0$ for all $y \in \mathcal{P}$ if and only if

$$-w_q \geq \sum_{i-1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) \quad \text{for all } y \in -\mathcal{P},$$

that is, if and only if

$$\begin{aligned} -w_q &\geq \sup \left\{ \sum_{i-1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) : y \in -\mathcal{P} \right\} \\ &= Z\left(w_1, \ldots w_{q-1}, 1 - \sum_{i=1}^{q-1} c_i w_i\right). \end{aligned}$$

This proves (2.1).

Now take any $w \in \mathbb{R}^d$ such that $c^T w > 0$. Then $-y \geq Z(w)$ is equivalent to $-\frac{y}{c^T w} \geq Z\left(\frac{w}{c^T w}\right)$ since the support function is positively homogeneous. By (2.1), the last inequality is in turn equivalent to $\frac{1}{c^T w}\left(w_1, \ldots, w_{q-1}, y\right) \in \mathcal{D}^*$. This shows that

$$\begin{aligned} Z(w) &= -\sup\{y \in \mathbb{R} : -y \geq Z(w)\} \\ &= -\sup\left\{y \in \mathbb{R} : \frac{1}{c^T w}\left(w_1, \ldots, w_{q-1}, y\right) \in \mathcal{D}^*\right\} \end{aligned}$$

when $c^T w > 0$. If $w = 0$, then $Z(w) = 0$ by the definition of the support function. Finally, take any $w \neq 0$ such that $c^T w \leq 0$. Since $c \in \operatorname{int} C$, there is an $\varepsilon > 0$ such that $c - \varepsilon w \in C$. It follows that $(c - \varepsilon w)^T w = c^T w - \varepsilon w^T w < 0$ because $w^T w > 0$. As $\mathcal{P} = \mathcal{P} + C$, for any fixed $x \in \mathcal{P}$ and for each $\lambda > 0$ we have $x + \lambda(c - \varepsilon w) \in \mathcal{P}$. Hence, by the definition of the support function,

$$Z(w) \geq -(x + \lambda(c - \varepsilon w))^T w = -x^T w - \lambda(c - \varepsilon w)^T w$$

for each $\lambda > 0$. Since $(c - \varepsilon w)^T w < 0$, this means that $Z(w) = \infty$, completing the proof of (2.2). □

*Remark 2.2* According to Proposition 2.1,

$$\mathcal{D}^* = \left\{(w_1, \ldots, w_{q-1}, y) \in \mathbb{R}^q : (w, y) \in -\operatorname{epi} Z, c^T w = 1\right\}, \tag{2.3}$$

so $\mathcal{D}^*$ can be identified with the section of the cone $-\operatorname{epi} Z$ by the hyperplane $\{(w, y) \in \mathbb{R}^q \times \mathbb{R} : c^T w = 1\}$ in $\mathbb{R}^{q+1}$. The convex set $\mathcal{D}^*$ (which depends on $c$) captures the same information as the support function $Z$. This is remarkable given that $Z$ is independent of the arbitrary choice of $c$. Also note the similarity between (2.3) and the representation by [6, p. 828] of the dual image in a more general setting.

This section concludes with a simple example.

*Example 2.3* Suppose that

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \quad C = \operatorname{cone}\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$

and fix $c = (0, 1)^T \in \operatorname{int} C$. For this data we have

$$\mathcal{P} = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq \tfrac{1}{3} z_1 + 4, z_2 \geq z_1, z_2 \geq -\tfrac{1}{3} z_1 + 4\},$$
$$\mathcal{D}^* = \{(w_1, y) \in \mathbb{R}^2 : -1 \leq w_1 \leq \tfrac{1}{3}, y \leq 4, y - 6w_1 \leq 6\}$$

(full details in [16] Example 6.4]. The sets $\mathcal{P}$ and $\mathcal{D}^*$ are represented graphically in Fig. 1.

The support function $Z$ is finite on its effective domain, which consists of vectors $w \in \mathbb{R}^2$ such that $x^T w \leq 0$ for each $x \in -\mathcal{P}$, so

$$\operatorname{dom} Z = \{w \in \mathbb{R}^2 : Z(w) < \infty\} = \{(w_1, w_2) \in \mathbb{R}^2 : w_2 \geq -w_1, w_2 \geq 3w_1\}.$$

For each $w \in \operatorname{dom} Z$ the linear function $x \mapsto x^T w$ takes a maximum at one of the extreme points $(0, -4)$, $(-6, -6)$ of the convex set $-\mathcal{P}$, hence

$$Z(w) = \sup\{x^T w : x \in -\mathcal{P}\} = \max\{-4w_2, -6w_1 - 6w_2\}.$$

This means that

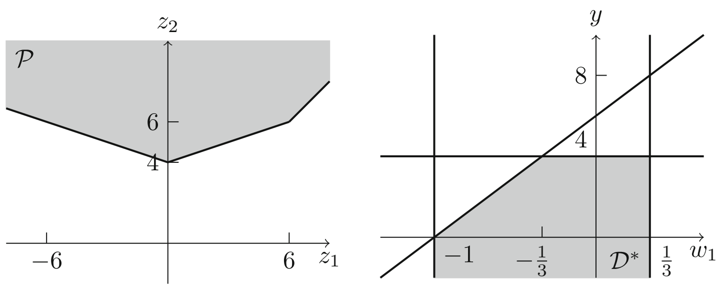



**Fig. 1** Upper and lower images in Example 2.3

$$\begin{aligned}\{(w_1, y) \in \mathbb{R}^2 : (w, y) \in -\operatorname{epi} Z, c^T w = 1\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \le -Z(w_1, w_2), (w_1, w_2) \in \operatorname{dom} Z, w_2 = 1\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \le -Z(w_1, 1), -1 \le w_1 \le \tfrac{1}{3}\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \le 4, y \le 6w_1 + 6, -1 \le w_1 \le \tfrac{1}{3}\} = \mathcal{D}^*.\end{aligned}$$

This identifies $\mathcal{D}^*$ with the section of $-\operatorname{epi} Z$ by the hyperplane

$$\{(w, y) \in \mathbb{R}^2 \times \mathbb{R} : c^T w = 1\} = \{(w_1, w_2, y) \in \mathbb{R}^3 : w_2 = 1\}.$$

## 3 Pricing and Hedging European Options Under Proportional Transaction costs

### *3.1 Currency Model*

The model is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t=0}^T)$. We assume that $\Omega$ is finite, that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, that $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For each $t$ denote by $\Omega_t$ the collection of atoms of $\mathcal{F}_t$, called the time $t$ *nodes* of the associated stock price tree model. Note that $\Omega_0 = \{\Omega\}$ and $\Omega_T = \{\{w\} : \omega \in \Omega\}$. For every $t < T$ a node $\nu \in \Omega_{t+1}$ is said to be a *successor* of a node $\mu \in \Omega_t$ if $\nu \subseteq \mu$. We denote for all $\mu \in \Omega_t$

$$\operatorname{succ} \mu = \{\nu \in \Omega_{t+1} : \nu \text{ a successor of } \mu\}.$$

For each $t$ let $\mathcal{L}_t = \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}_t)$ be the collection of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables. We identify elements of $\mathcal{L}_t$ with functions on $\Omega_t$ whenever convenient.

We consider the discrete-time currency model introduced by [9] and studied by others. The model contains $d$ assets or currencies. At each trading date $t = 0, 1, \dots, T$ one unit of each asset $k = 1, \dots, d$ can be obtained by exchanging $\pi_t^{jk} > 0$ units of asset $j = 1, \dots, d$. We assume that the exchange rates $\pi_t^{jk}$ are $\mathcal{F}_t$-measurable and $\pi_t^{jj} = 1$ for all $t$ and $j, k$.

We say that a portfolio $x \in \mathcal{L}_t$ can be *exchanged* into a portfolio $y \in \mathcal{L}_t$ at time $t$ whenever there are $\mathcal{F}_t$-measurable random variables $\beta^{jk} \geq 0$, $j, k = 1, \dots, d$ such that for all $k = 1, \dots, d$

$$y^k = x^k + \sum_{j=1}^{d} \beta^{jk} - \sum_{j=1}^{d} \beta^{kj} \pi_t^{kj},$$

where $\beta^{jk}$ represents the number of units of asset $k$ received as a result of exchanging some units of asset $j$.

The *solvency cone* $\mathcal{K}_t \subseteq \mathcal{L}_t$ is the set of portfolios that are *solvent* at time $t$, i.e. those portfolios at time $t$ that can be exchanged into portfolios with non-negative holdings in all $d$ assets. It is straightforward to show that $\mathcal{K}_t$ is the convex cone generated by the canonical basis $e^1, \dots, e^d$ of $\mathbb{R}^d$ and the vectors $\pi_t^{jk} e^j - e^k$ for $j, k = 1, \dots, d$, and so $\mathcal{K}_t$ is a polyhedral cone. Note that $\mathcal{K}_t$ contains all the non-negative elements of $\mathcal{L}_t$.

A *self-financing strategy* $y = (y_t)_{t=0}^T$ is a predictable $\mathbb{R}^d$-valued process (i.e. $y_0 \in \mathcal{L}_0$ and $y_t \in \mathcal{L}_{t-1}$ for $t = 1, \dots, T$) such that

$$y_t - y_{t+1} \in \mathcal{K}_t \quad \text{for all } t = 0, \dots, T-1$$

Here $y_0 \in \mathcal{L}_0$ is the initial endowment, and $y_t \in \mathcal{L}_{t-1}$ for each $t = 1, \dots, T$ is the portfolio held from time $t-1$ to time $t$. Let $\Phi$ be the set of self-financing strategies.

A self-financing strategy $y = (y_t) \in \Phi$ is called an *arbitrage opportunity* if $y_0 = 0$ and there is a portfolio $x \in \mathcal{L}_T \setminus \{0\}$ with non-negative holdings in all $d$ assets such that $y_T - x \in \mathcal{K}_T$. This notion of arbitrage was considered by [22], and its absence is formally different but equivalent to the weak no-arbitrage condition introduced by [11].

**Theorem 3.1** ([11, 22]) *The model admits no arbitrage opportunity if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an $\mathbb{R}^d$-valued $\mathbb{Q}$-martingale $S = (S_t)$ such that*

$$S_t \in \mathcal{K}_t^+ \setminus \{0\} \text{ for all } t, \tag{3.1}$$

*where $\mathcal{K}_t^+$ is the dual cone of $\mathcal{K}_t$.*

*Remark 3.2* A pair $(\mathbb{Q}, S)$ satisfying the conditions in Theorem 3.1 is called a *consistent pricing pair*. In place of such a pair $(\mathbb{Q}, S)$ one can equivalently use the so-called *consistent price process* $S_t \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t)$; see [22].

## *3.2 European Options*

A *European option* with expiry time $T > 0$ and payoff $\xi \in \mathcal{L}_T$ is a contract that gives its holder (i.e. the option buyer) the right to receive a portfolio $\xi$ of currencies at time $T$. On the other hand, the writer (seller) of the option is obliged to deliver this portfolio to the buyer.

To hedge against this liability the writer can follow a self-financing strategy $y \in \Phi$ such that $y_T - \xi \in \mathcal{K}_T$. The initial endowment $y_0$ of such a strategy $y$ is called a *superhedging portfolio*, and the strategy $y$ itself is called a *superhedging strategy* for the European option $\xi$.

The *ask price* (*seller's price*, *superhedging price*) $\pi_i^a(\xi)$ of the European option in currency $i = 1, \ldots, d$ can be understood as the lowest value $x$ such that the portfolio consisting of $x$ units of currency $i$ and no other currency is a superhedging portfolio for $\xi$. In other words,

$$\pi_i^a(\xi) = \min \left\{ x \in \mathbb{R} : xe^i \text{ is a superhedging portfolio for } \xi \right\}.$$

On the other hand, to hedge his position the option buyer would like to follow a self-financing strategy $y \in \Phi$ such that $y_T + \xi \in \mathcal{K}_T$. Here $-y_0$ is a portfolio of currencies which the option buyer could borrow at time 0 and would be able to settle later by following the strategy $y$ and using the payoff $\xi$ to be received on exercising the option at time $T$. We call $-y_0$ a *subhedging portfolio* and $-y$ a *subhedging strategy* for the European option $\xi$.

The *bid price* (*buyer's price*, *subhedging price*) $\pi_i^b(\xi)$ of the European option in currency $i = 1, \ldots, d$ can be understood as the highest value $x$ such that the portfolio consisting of $x$ units of currency $i$ and no other currency is a subhedging portfolio for $\xi$,

$$\pi_i^b(\xi) = \max \left\{ x \in \mathbb{R} : xe^i \text{ is a subhedging portfolio for } \xi \right\}.$$

It is the highest amount in currency $i$ that an option holder could raise by using the option as collateral.

Observe that $-y$ is a subhedging strategy for a European option $\xi$ if and only if $y$ is a superhedging strategy for $-\xi$. It follows immediately that

$$\pi_i^b(\xi) = -\pi_i^a(-\xi).$$

Because of these relationships it is sufficient to develop algorithms for hedging and pricing the seller's (short) position in a European option.

### *3.3 Primal and Dual Constructions*

The constructions presented here for European options are a special case of those developed by [21] to hedge and price the much wider class of American type options under proportional transaction costs. Construction 4.2 in [21], which produces the set of superhedging portfolios, takes a particularly simple form in this special case:

- For each $\omega \in \Omega_T$ put

$$\mathcal{Z}_T^\omega = \xi^\omega + \mathcal{K}_T^\omega.$$

- If $\mathcal{Z}_{t+1}$ has already been constructed for some $t = 0, 1, \ldots, T-1$, then for each $\omega \in \Omega_t$ put

$$\mathcal{W}_t^\omega = \bigcap_{\omega' \in \operatorname{succ} \omega} \mathcal{Z}_{t+1}^{\omega'},$$
$$\mathcal{Z}_t^\omega = \mathcal{W}_t^\omega + \mathcal{K}_t^\omega$$

(To link this with Construction 4.2 in [21] observe that the formula for $\mathcal{W}_t$ can be written concisely as $\mathcal{W}_t = \mathcal{Z}_{t+1} \cap \mathcal{L}_t$.)

For each $t$ the set $\mathcal{Z}_t$ consists of all portfolios that allow the seller to hedge the option by following a self-financing strategy between times $t$ and $T$. In particular, $\mathcal{Z}_0$ is the set of superhedging portfolios. The ask price of the option can be expressed in terms of $\mathcal{Z}_0$ as

$$\pi_i^a(\xi) = \min\left\{x \in \mathbb{R} : xe^i \in \mathcal{Z}_0\right\}. \quad (3.2)$$

The above construction involves two standard operations on polyhedral convex sets, namely the intersection of finitely many such sets and the algebraic sum of such a set and a polyhedral convex cone. Both operations can be implemented using standard geometric methods in existing software libraries, for example, *Parma Polyhedra Library* [1] and *PolyLib* [8, 13, 14, 23, among others]. As soon as the set $\mathcal{Z}_0$ of superhedging portfolios has been computed in this manner, it becomes a routine task to evaluate the option price $\pi_i^a(\xi)$ using (3.2). Roux and Zastawniak [21] provided a numerical implementation of this procedure for hedging and pricing European options (and much more generally, American type options) in currency markets with transaction costs by using the *Maple* package *Convex* [3].

Moreover, once the $\mathcal{Z}_t$ have been constructed, it is straightforward to compute a superhedging strategy starting from any superhedging portfolio $y_0 \in \mathcal{Z}_0$. Namely, if $y_t \in \mathcal{Z}_t$ has already been computed for some $t = 0, 1, \ldots, T-1$, we can take $y_{t+1} \in (y_t - \mathcal{K}_t) \cap \mathcal{W}_t$. The intersection is non-empty since $\mathcal{Z}_t = \mathcal{W}_t + \mathcal{K}_t$, so it is always possible to find such $y_{t+1}$, though it may be non-unique. The self-financing condition $y_t - y_{t+1} \in \mathcal{K}_t$ is clearly satisfied. Moreover, since $\mathcal{W}_t = \mathcal{Z}_{t+1} \cap \mathcal{L}_t$, it follows that $y_{t+1}$ is $\mathcal{F}_t$-measurable, so $y$ constructed in this manner will be a predictable process. It also follows that $y_{t+1} \in \mathcal{Z}_{t+1}$, which makes it possible to iterate the procedure.

It is also possible to follow the construction using convex dual objects to the $\mathcal{Z}_t$. We introduce the support functions

$$Z_t(x) = \sup\left\{x^T z : z \in -\mathcal{Z}_t\right\}, \quad W_t(x) = \sup\left\{x^T z : z \in -\mathcal{W}_t\right\}$$

and the linear function

$$U(x) = -x^T \xi$$

defined for all $x \in \mathbb{R}^d$. If we need to make the dependence on $\omega \in \Omega$ explicit in these functions, we shall write $Z_t^\omega$, $W_t^\omega$, $U^\omega$. The above construction (we call it the *primal construction*) can now be written in the following equivalent form (called the *dual construction*); see Lemma 5.5 in [21]:

- For each $\omega \in \Omega_T$

$$Z_T^\omega = \begin{cases} U^\omega & \text{on } \mathcal{K}_T^{+\omega}, \\ \infty & \text{otherwise.} \end{cases}$$

  This is the linear function $U^\omega$ restricted to the domain $\mathcal{K}_T^{+\omega}$.
- Suppose that $Z_{t+1}$ has been constructed for some $t = 0, 1, \dots, T-1$. Then, for each node $\omega \in \Omega_t$ let $W_t^\omega$ be the convex hull of the family of convex functions $Z_{t+1}^{\omega'}$ indexed by $\omega' \in \operatorname{succ}\omega$, and let $Z_t^\omega$ be the restriction of $W_t^\omega$ to the domain $\mathcal{K}_t^{+\omega}$:

$$W_t^\omega = \operatorname{conv}\left\{Z_{t+1}^{\omega'} : \omega' \in \operatorname{succ}\omega\right\},$$
$$Z_t^\omega = \begin{cases} W_t^\omega & \text{on } \mathcal{K}_t^{+\omega}, \\ \infty & \text{otherwise.} \end{cases}$$

Once $Z_0$ has been computed, the ask price of the option can be obtained as (see Theorem 4.4 in [21])

$$\pi_i^a(\xi) = -\min\left\{Z_0(x) : x \in \mathbb{R}^d, x_i = 1\right\}.$$

This dual construction also lends itself well to computer implementation. Taking the convex hull of finitely many polyhedral convex functions and restricting the domain of such a function to a given polyhedral convex cone are operations equivalent to some standard operations on polyhedral convex sets, which are widely available in computer packages such as the *Convex* library in *Maple* used by [21].

Observe that the dual construction, which follows from Lemma 5.5 in [21] specialised to the case of European options, is equivalent to the construction in Corollary 6.3 of [17]. The only difference is that the dual construction is expressed in terms of the support functions $Z_t$ and $W_t$, whereas [17] use $\widetilde{V}_t(x) = -Z_t(x)$ and $V_t(x) = -W_t(x)$ defined for all $x$'s on the hyperplane in $\mathbb{R}^d$ given by the condition $x^i = 1$. Both are a straightforward extension to $d$ assets of the construction stated in Algorithm 4.1 of [20] in the case of 2 assets.

### 3.4 SHP-Algorithm

Löhne and Rudloff [17] consider the same problem of pricing and hedging European options (though not options of American type). In particular, the same sets as in the primal construction above are denoted by [17] as

$$SHP_t(\xi) = \mathcal{Z}_t.$$

These authors propose a different construction of the $\mathcal{Z}_t$ based on linear vector optimisation methods and geometric duality.

From this perspective, $S = \mathcal{W}_t$ can be viewed as the feasible set of a linear vector optimisation problem (P). If the solvency cone $\mathcal{K}_t$ contains no lines, which means that there are non-zero transaction costs between any two currencies, then the matrix $P$ in (P) is just the $d \times d$ unit matrix, and the ordering cone is $C = \mathcal{K}_t$. The upper image of the linear vector optimisation problem (P) is

$$\mathcal{P} = P[S] + C = \mathcal{W}_t + \mathcal{K}_t = \mathcal{Z}_t.$$

Because $C$ contains no lines, Benson's algorithm, see [2] or [4], can be applied to compute a solution to the dual problem (D*) and hence the corresponding lower image $\mathcal{D}^*$. The Benson algorithm yields simultaneously a solution to (P) and gives the upper image $\mathcal{P} = \mathcal{Z}_t$. We know from Proposition 2.1 that if $C$ contains no lines, then $\mathcal{D}^*$ can be identified with a section of the epigraph of the support function $Z$ of $-\mathcal{P}$. Since $\mathcal{P} = \mathcal{Z}_t$, it follows that $Z = Z_t$ is the function from the dual construction in Sect. 3.3.

A complication arises when the solvency cone $\mathcal{K}_t$ contains some lines, which means that there are currencies which can be exchanged into one another without incurring any transaction costs. This is dealt with by taking $P$ to be the matrix representing the so-called liquidation map, a linear map which amounts to liquidating all but one of the assets that can be exchanged into one another without transaction costs; see (4.1) in [17] for the precise definition of $P$. In this case $C = P[\mathcal{K}_t]$ contains no lines because there are no longer any assets that can be exchanged into one another without transaction costs. Then the upper image of the linear vector optimisation problem (P) is

$$\mathcal{P} = P[S] + C = P[\mathcal{W}_t + \mathcal{K}_t] = P[\mathcal{Z}_t].$$

Since $C$ contains no lines, Benson's algorithm can also be applied in this case to compute a solution to the dual problem (D*) and hence the corresponding lower image $\mathcal{D}^*$. The Benson algorithm yields simultaneously a solution to (P) and gives the upper image $\mathcal{P} = P[\mathcal{Z}_t]$. This then gives $\mathcal{Z}_t = \{x \in \mathcal{L}_t : Px \in \mathcal{P}\}$ as the inverse image of $\mathcal{P}$ under $P$. Once again by Proposition 2.1, since $C$ contains no lines, it follows that $\mathcal{D}^*$ can be identified with a section of the epigraph of the support function $Z$ of $-\mathcal{P} = -P[\mathcal{Z}_t]$. This is related to $Z_t$, the support function of $-\mathcal{Z}_t$, by $Z(x) = Z_t(P^T x)$.

## 4 Example

In this section we present an example to illustrate the numerical procedures discussed in Sect. 3.3. Consider a model involving three assets, with time horizon $\tau = 1$ and with $T = 4$ time steps. Two of the assets are risky with correlated returns, and follow the two-asset recombinant [12] model with Cholesky decomposition. That is, there are $(t + 1)^2$ possibilities for the stock prices $S_t = (S^1, S^2)$ at each time step $t = 0, \ldots, T$, indexed by pairs $(j_1, j_2)$ where $1 \le j_1, j_2 \le t + 1$, and each non-terminal node with stock price $S_t(j_1, j_2)$ has four successors, associated with the stock prices $S_{t+1}(j_1, j_2)$, $S_{t+1}(j_1 + 1, j_2)$, $S_{t+1}(j_1, j_2 + 1)$ and $S_{t+1}(j_1 + 1, j_2 + 1)$. With $\Delta = \frac{\tau}{T}$ defined for convenience, the stock prices are given by

$$
\begin{aligned}
S_t^1(j_1, j_2) &= S_0^1 e^{\left(r - \frac{1}{2}\sigma_1^2\right)t\Delta + (2j_1 - t - 2)\sigma_1\sqrt{\Delta}}, \\
S_t^2(j_1, j_2) &= S_0^2 e^{\left(r - \frac{1}{2}\sigma_2^2\right)t\Delta + \left((2j_1 - t - 2)\rho + (2j_2 - t - 2)\sqrt{1-\rho^2}\right)\sigma_2\sqrt{\Delta}}
\end{aligned}
$$

for $t = 0, \ldots, T$ and $j_1, j_2 = 1, \ldots, t + 1$, where $S_0^1 = 45$ and $S_0^2 = 50$ are the initial stock prices, $\sigma_1 = 15$ and $\sigma_2 = 20\,\%$ are the volatilities of the returns and $\rho = 20\,\%$ is the correlation between the log returns on the two stocks. The third asset is a risk-free bond with nominal interest rate $r = 5\,\%$ and value process

$$
B_t = (1 + r\Delta)^{-(T-t)} \text{ for } t = 0, \ldots, T.
$$

Proportional transaction costs are introduced by allowing the asset prices to have constant (proportional) bid-ask spreads, i.e. the bid and ask prices are

$$
\begin{aligned}
S_t^{1b} &= (1 - k_1)S_t^1, & S_t^{1a} &= (1 + k_1)S_t^1, \\
S_t^{2b} &= (1 - k_2)S_t^2, & S_t^{2a} &= (1 + k_2)S_t^2, \\
B_t^b &= (1 - k_3)B_t, & B_t^a &= (1 + k_3)B_t
\end{aligned}
$$

for $t = 0, \ldots, T$, where $k_1 = 2$, $k_2 = 4$ and $k_3 = 1\,\%$. The matrix of exchange rates at each time step $t$ is then

$$
\begin{pmatrix} \pi_t^{11} & \pi_t^{12} & \pi_t^{13} \\ \pi_t^{21} & \pi_t^{22} & \pi_t^{23} \\ \pi_t^{31} & \pi_t^{32} & \pi_t^{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{S_t^{2a}}{S_t^{1b}} & \frac{B_t^a}{S_t^{1b}} \\ \frac{S_t^{1a}}{S_t^{2b}} & 1 & \frac{B_t^a}{S_t^{2b}} \\ \frac{S_t^{1a}}{B_t^b} & \frac{S_t^{2a}}{B_t^b} & 1 \end{pmatrix},
$$

and the solvency cone is

$$
\mathcal{K}_t = \operatorname{cone}\left\{ \begin{pmatrix} S_t^{2a} \\ -S_t^{1b} \\ 0 \end{pmatrix}, \begin{pmatrix} B_t^a \\ 0 \\ -S_t^{1b} \end{pmatrix}, \begin{pmatrix} -S_t^{2b} \\ S_t^{1a} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_t^a \\ -S_t^{1b} \end{pmatrix}, \begin{pmatrix} -B_t^b \\ 0 \\ S_t^{1a} \end{pmatrix}, \begin{pmatrix} 0 \\ -B_t^b \\ S_t^{2a} \end{pmatrix} \right\}.
$$

This model was also considered by [17, Section 5.2]; note that the assets have been reordered in the present paper.

Consider an exchange option with physical delivery and payoff

$$\xi = (\mathbf{1}_{\{S_T^{1a} \geq S_T^{2a}\}}, -\mathbf{1}_{\{S_T^{1a} \geq S_T^{2a}\}}, 0)$$

that matures at time step $T$. [17, Example 5.3] reported

$$SHP_0 = \operatorname{conv}\left\{\begin{pmatrix} 0.584 \\ -0.260 \\ -7.760 \end{pmatrix}, \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}, \begin{pmatrix} 0.347 \\ -0.446 \\ 13.341 \end{pmatrix}\right\} + \mathcal{K}_0,$$

and gave the ask price of the exchange option in terms of the bond as

$$\pi_3^a(\xi) = 7.418.$$

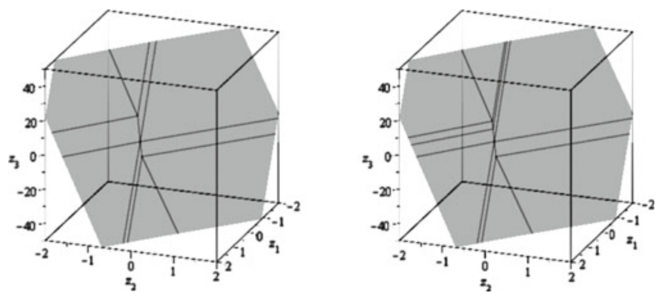
The boundary of $SHP_0$ is depicted in Fig. 2. Application of the primal construction in Sect. 3.3 produces

$$\mathcal{Z}_0 = \operatorname{conv}\left\{\begin{pmatrix} 0.584 \\ -0.260 \\ -7.760 \end{pmatrix}, \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}, \begin{pmatrix} 0.399 \\ -0.406 \\ 8.714 \end{pmatrix}, \begin{pmatrix} 0.424 \\ -0.388 \\ 6.564 \end{pmatrix}\right\} + \mathcal{K}_0,$$

from which the ask price of the exchange option in terms of each of three assets can be computed as

$$\pi_1^a(\xi) = 0.152, \qquad \pi_2^a(\xi) = 0.146, \qquad \pi_3^a(\xi) = 7.418.$$

There is substantial agreement between $SHP_0$ and $\mathcal{Z}_0$, which can be confirmed visually (see Fig. 2), and in view of the agreement on the ask price $\pi_3^a(\xi)$, we ascribe the differences in the specifications of $SHP_0$ and $\mathcal{Z}_0$ to the error level chosen in




$SHP_0$ $\mathcal{Z}_0$

**Fig. 2** Boundary of the set of superhedging endowments

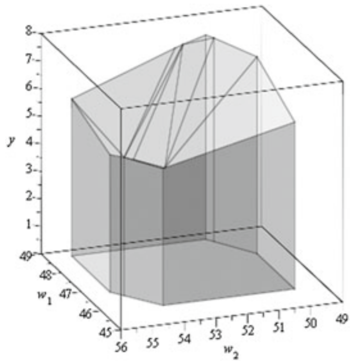



**Fig. 3** Lower image $\mathcal{D}_0^*$ associated with $Z_0$

Benson's algorithm. Finally, application of the dual construction in Sect. 3.3 produces the support function $Z_0$ of $-\mathcal{Z}_0$. The set

$$\mathcal{D}_0^* = \{(w_1, w_2, y) : y \leq -Z_0(w_1, w_2, 1)\}$$

is the lower image of the dual problem (D*) with the choice $c = (0, 0, 1)^T$. It has 12 vertices

$$\begin{pmatrix} 48.726 \\ 51.930 \\ 7.081 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 51.681 \\ 7.178 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 54.050 \\ 4.981 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 55.201 \\ 5.702 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 49.946 \\ 6.048 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 50.955 \\ 7.418 \end{pmatrix},$$
$$\begin{pmatrix} 48.573 \\ 50.796 \\ 7.395 \end{pmatrix}, \begin{pmatrix} 47.761 \\ 49.946 \\ 7.141 \end{pmatrix}, \begin{pmatrix} 46.565 \\ 54.907 \\ 5.012 \end{pmatrix}, \begin{pmatrix} 46.815 \\ 55.201 \\ 4.982 \end{pmatrix}, \begin{pmatrix} 46.405 \\ 54.718 \\ 5.018 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 54.108 \\ 4.962 \end{pmatrix},$$

and is depicted in Fig. 3. The maximum of $\mathcal{D}_0^*$ in the $y$-direction is

$$\pi_3^a(\xi) = 7.418.$$

We conclude this numerical example by demonstrating the procedure of finding a superhedging strategy $y = (y_t)_{t=0}^T$ starting from the initial endowment

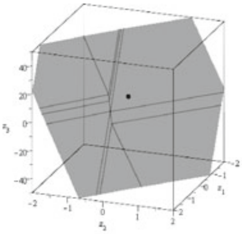
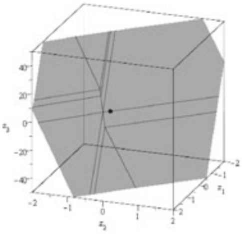
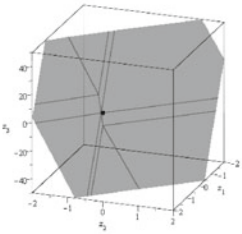
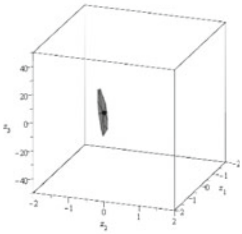
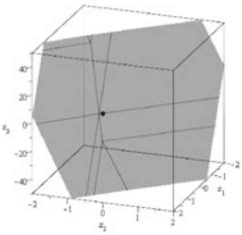
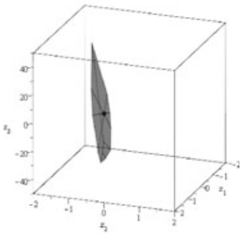
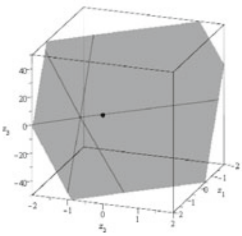
$$y_0 = (0, 0, \pi_3^a(\xi))^T \in \mathcal{Z}_0$$

along the price path in Table 1. At each time step $t$ the portfolio $y_t$ (indicated by a dot on the graph of the boundary of $\mathcal{Z}_t$ in Table 1) is rebalanced into a portfolio

$$y_{t+1} \in (y_t - \mathcal{K}_t) \cap \mathcal{W}_t \subseteq \mathcal{Z}_{t+1}.$$

As can be seen in Table 1, for this particular path the set $(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$ is a singleton at time steps $t = 0$ and $t = 1$, which means that there is only one choice for $y_{t+1}$. At

**Table 1** Superhedging strategy along a path

| $t$ | $(j_1, j_2)$ | $y_t$ | $\mathcal{Z}_t$ | $(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$ |
|---|---|---|---|---|
| 0 | (1,1) | $\begin{pmatrix} 0.000 \\ 0.000 \\ 7.418 \end{pmatrix}$ |  | $\left\{ \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix} \right\}$ |
| 1 | (2,1) | $\begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}$ |  | $\left\{ \begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix} \right\}$ |
| 2 | (2,1) | $\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$ |  |  |
| 3 | (3,2) | $\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$ |  |  |
| 4 | (3,2) | $\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$ |  | N/A |

time steps $t = 2$ and $t = 3$ this set is a convex polytope, and the choice of $y_{t+1}$ is no longer unique, which means that other considerations (e.g. a preference for holding one asset over another, or a preference not to trade) may be used to select $y_{t+1}$ in $(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$. In this demonstration we adopted a minimum-trading rule, that is, whenever possible we selected $y_{t+1} = y_t$. At the final time step $t = 4$ we have

$$y_4 - \xi = \begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix} - \begin{pmatrix} 1.000 \\ -1.000 \\ 0.000 \end{pmatrix} = \begin{pmatrix} -0.359 \\ 0.509 \\ 0.000 \end{pmatrix} \in \mathcal{K}_4.$$

## 5 Representation of Superhedging Price

In this section we briefly present and compare the result of [17, 21] concerning the representation of the superhedging price of a European option in terms of risk-neutral expectations of the payoff $\xi$:

$$\pi_i^a(\xi) = \sup_{(\mathbb{Q},S)\in\mathcal{P}^i} \mathbb{E}_{\mathbb{Q}}((\xi^T S_T)), \tag{5.1}$$

where $\mathcal{P}^i$ is the set of pairs $(\mathbb{Q}, S)$ consisting of a probability measure $\mathbb{Q}$ and an $\mathbb{R}^d$-valued martingale $S$ under $\mathbb{Q}$ satisfying the conditions of Theorem 3.1 and such that $S_t^i = 1$ for each $t = 0, \ldots, T$.

In Theorem 6.1 of [17] this result was proved under the so-called robust no-arbitrage condition of [22] and subject to the simplifying assumption that the solvency cone $\mathcal{K}_t$ contains no lines for any $t$ (that is, the transaction costs are non-zero for any $t$). Their proof is based on the scalarisation procedure of [5] for the dual representation of the set $SHP_0$ of superhedging portfolios.

By comparison, the result in [21] is free of these restrictions: it works under the assumption that there is no arbitrage opportunity as defined in Sect. 3.1, which is weaker than the robust no-arbitrage condition, and without the need to assume that the solvency cone $\mathcal{K}_t$ contain no lines. It is also a much more general result that applies to American type derivatives, which reduces to (5.1) for European options. The proof is based on the dual construction from Sect. 3.3, which can in fact be used to produce a pair $(\mathbb{Q}, S)$ that realises the supremum in (5.1) (though in general such a pair does not lie in $\mathcal{P}^i$ as $\mathbb{Q}$ may be a degenerate measure, absolutely continuous with respect to but not necessarily equivalent to $\mathbb{P}$).

## 6 Conclusions

We have established a close link, indeed an equivalence between the three approaches: the above primal and dual constructions and the SHP-algorithm of [17]. The primal construction involves primal objects only. The dual construction deals exclusively with dual objects (support functions). Meanwhile, the SHP-algorithm switches back and forth between primal and dual objects (in this case the lower images of the dual problem ($D^*$). By Proposition 2.1, these two types of dual objects are in one-to-one correspondence, which means that the apparent differences between the algorithms are merely superficial.

Moreover, all three approaches lend themselves well to numerical implementation: the primal and dual constructions utilise available software libraries for handling convex sets, whereas the SHP-algorithm makes an innovative use of Benson's procedure. In both approaches the procedure limiting computational efficiency is vertex enumeration. An advantage offered by Benson's algorithm is the ability to control the accuracy versus efficiency by choosing an error level. On the other hand, the *Maple* package *Convex* used by [21] employs exact arithmetic with rational numbers, hence there is no rounding beyond the conversion (as accurate as one needs it to be) of input data from real to rational numbers. While accurate rational arithmetic carries obvious computational overheads, the primal and dual algorithms are efficient enough so this does not become a problem in realistic multi-step and multi-asset examples that have been investigated, where the computation times were of the order of a couple of minutes on a standard PC machine.

One major difference as compared with the SHP-algorithm approach is that the primal and dual constructions have been developed in [21] for the much wider class of American type options, and can handle early exercise problems. In this context, European options are a particularly straightforward special case. It remains an open question whether or not the SHP-algorithm of [17] could be extended to American options, at least in the case of hedging and pricing the seller's position. It would be exciting to see this happen.

On the other hand, there are limits to what can be expected of the SHP-algorithm. American options present a particular obstacle that this approach is unlikely to be able to overcome. Namely, the case of hedging and pricing the buyer's (rather than the seller's) position in an American option leads to a non-convex optimisation problem, which is unlikely to yield to the power of linear vector optimisation methods and geometric duality. For the same reason, the dual construction collapses as there are no convex dual objects to work with in the first place. Nonetheless, the primal construction can still be adapted to handle this case; see Example 7.1 in [21] for details.

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