A Survey of Set Optimization Problems with Set Solutions

Elvira Hernández

Dedicated to the memory of Professor Luis Rodríguez-Marín

Abstract This paper presents a state-of-the-art survey on set-valued optimization problems whose solutions are defined by set criteria. It provides a general framework that allows to give an overview about set-valued optimization problems according to decision concepts based on certain set relations. The first part of this paper (Sects. 1 and 2) motivates and describes the set-valued optimization problem (in short, SVOP). The present survey deals with general problems of set-valued optimization and recall its main properties in order to establish the differences between vector set-valued optimization problems (VOP) and set optimization problems (SOP). In this context, in the second part (Sects. 3–5) we focus on those results existing in the literature related with optimality conditions by using a set approach. We list and quote references devoted to (SOP) from the beginning up to now. In Sect. 5, a particular attention is paid to applications of the set relations considered in other fields as fixed point theory. The last section provides some conclusions and suggestions for further study.

Keywords Set-valued maps · Set optimization · Optimality conditions · Set approach · Solutions of set type

This work was partially supported by Ministerio de Ciencia (Spain), project MTM2009-09493 and by UNED, project 2011V/PUNED/0008.

E. Hernández (✉)
Departamento de Matemática Aplicada, Universidad Nacional de Educación a Distancia, E.T.S.I. Industriales, Juan del Rosal 12, 28040 Madrid, Spain
e-mail: ehernandez@ind.uned.es

A.H. Hamel et al. (eds.), *Set Optimization and Applications - The State of the Art*, Springer Proceedings in Mathematics & Statistics 151,
DOI 10.1007/978-3-662-48670-2_4

1 Introduction

The set-valued maps receive great attention from more and more authors. This is partly due to its wide applications in diverse fields as for example: Control theory, Optimization, Economics or Game theory, to name a few. See, for instance, [4, 51] and references therein.

On the other hand, set-valued optimization problems are very known in Optimization theory and Economics as for example equilibrium theorems for Economies. See, [1–3, 10, 12, 30, 46].

Throughout this paper, we consider preference relations generated by a pre-order (a binary relation which is reflexive and transitive). In the sequel M denotes a nonempty subset of a set X, Y a linear space and $K \subset Y$ a convex cone. If $y,\ y' \in Y$ we denote by $y \leq y'$ if and only if $y' - y \in K$. This relation $\leq$ is obviously a pre-order on Y. Thus, the pair (Y, K) is called a pre-ordered linear space (or partially ordered space) with the ordering $\leq$ induced by K. To consider weakly efficient solutions, we also assume, in addition, that Y is a topological space and K is solid, that is, its topological interior is nonempty, $\operatorname{int} K \neq \emptyset$.

Remark 1.1
1. If K is pointed, $K \cap (-K) = \{0\}$, the preference $\leq$ is also antisymmetric and (Y, K) is an ordered linear space.
2. In spite of the most optimization theory is based on a pre-order on the criteria space, other preferences (non-reflexitivy or non-transitivity) are very important from the practical point of view, for instance, in Economic. Some references and results can be found in [11] where the variational approach developed in this paper allows to obtain new necessary conditions for various types of solutions and to apply it to nonconvex models of welfare economics with finite-dimensional and infinite-dimensional commodity spaces.
3. In this paper the discussion is based on a general framework. Note that it is possible to avoid the topological structure on Y to consider weakly efficient solutions via the algebraic interior of K, that is, $\operatorname{core}(K)$. See also [19].

Given a nonempty set $A \subset Y$, we denote by $\operatorname{Min} A = \{\bar{y} \in A\colon\ y \in A,\ y \leq \bar{y} \text{ imply } \bar{y} \leq y\}$ the set of minimal points of A. In particular, if K is pointed, $\operatorname{Min} A = \{\bar{y} \in A\colon (\bar{y} - K) \cap A = \bar{y}\}$. It is said that $\bar{y} \in A$ is a strongly or ideal minimal point of A, $\bar{y} \in \operatorname{IMin} A$, if $A \subset \bar{y} + K$. By replacing K by $-K$, we can define maximal and ideal maximal point of A.

We denote by (V) the following vector optimization problem:

$$(\mathrm{V}) \quad \begin{cases} \operatorname{Min}\ f(x) \\ \text{subject to } x \in M, \end{cases}$$

where $f\colon M \to Y$. An element $x_0 \in M$ is said to be an efficient solution of (V), $x_0 \in \operatorname{Eff}(f)$, if $f(x_0) \in \operatorname{Min} \bigcup_{x \in M} f(x)$.

The above solution is defined via Edgeworth-Pareto solution. However, for a vector optimization problem there are various solution concepts as for instance proper solutions or strong solutions. For more references about this problem, see [18, 46, 69, 80] and references therein. It is well-known that there exist other solution concepts (different to Edgeworth-Pareto notions) which have been investigated by many authors. We remark that according to [21, 22, 28] it is possible to study a unified vector problem which includes other efficient notions.

The present survey deals with optimization problems where the objective map is more complex than that given in (V). In addition, our presented results address the notions of extended Edgeworth-Pareto optimality.

The general formulation of a set-valued optimization problem is as follows:

$$\text{(SVOP)} \quad \begin{cases} \text{Min } F(x) \\ \text{subject to } x \in M, \end{cases}$$

where $F\colon M \longrightarrow 2^Y$ is a set-valued map with $F(x) \neq \emptyset$ for all $x \in M$.

Unlike the vector optimization problem (V), for the above problem there is not a only one approach of solution associated to it. The solutions of (SVOP) are categorized into

(i) *vector solutions*; when the problem, denoted by (VOP) and called vector set-valued optimization problem, is a vector optimization problem with set-valued maps.
(ii) *set solutions*; when the problem, denoted by (SOP), is a set optimization problem.

Now, we present the above problems to establish the differences between them. For this, we introduce some notations and define theirs solutions.

The general vector set-valued optimization problem is denoted as follows:

$$\text{(VOP)} \quad \begin{cases} \text{Min } F(x) \\ \text{subject to } x \in M. \end{cases}$$

We denote $F(M) = \bigcup_{x \in M} F(x)$ the image set under F on M. To define the solutions of vector type we consider the pre-order $\leq$ defined on Y by the convex cone K. Roughly speaking, the solutions of (VOP) are introduced by means the minimal elements of $F(M)$.

Definition 1.1 We say that $\bar{x} \in M$ is a solution of (VOP), $\bar{x} \in \text{Eff}(F)$, if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in \text{Min } F(M)$. The pair $(\bar{x}, \bar{y})$ is called minimizer of (VOP).

On the contrary, the solutions of set-type are defined via a preference, $\preceq$, on the family of nonempty subsets of Y, $\wp_0(Y)$. We denote a set optimization problem as follows:

$$\text{(SOP)} \quad \begin{cases} \preceq -\text{Min } F(x) \\ \text{subject to } x \in M. \end{cases}$$

The essence of set approach consists in considering the whole set as a solution, not just one point of the image. Following the vector case, the solutions of (SOP) with respect to $\preceq$ are defined by the more preferred sets of $\{F(x)\colon x \in M\}$ as follows:

Definition 1.2 We say that $\bar{x} \in M$ is a $\preceq$-solution of (SOP), $\bar{x} \in \preceq -\mathrm{Eff}\ F$, if $x \in M$ and $F(x) \preceq F(\bar{x})$ imply $F(\bar{x}) \preceq F(x)$.

A natural extension of problem (V) is when $\preceq$ is compatible with the ordering defined by K in the following sense:

Definition 1.3 Let $a,\ b \in Y$ be. We say that $\preceq$ is compatible with $\leq$ if $\{a\} \preceq \{b\}$ is equivalent to $a \leq b$.

Remark 1.2 We point out that we can combine both approach (vector and set) to define new preferences and solutions for a set-valued optimization problem. For instance, $A \subseteq M$ is a set solution of (SVOP) if $A \subseteq \preceq -\mathrm{Eff}\ F$ and $\mathrm{Min} \bigcup_{x \in M} F(x) = \mathrm{Min} \bigcup_{x \in A} F(x)$. Such preferences could be related with Finance according to [44].

Remark 1.3 1. It is clear that it is possible to define different vector solutions of (VOP) from those presented in Definition 1.1 like weak, strong or proper minimizer. See also [9]. Similarly for set solutions of (SOP).
2. Definitions 1.1 and 1.2 are given in a natural way. Both seem to be the most appropriated to generalize the Edgeworth-Pareto notions.
3. A decision maker considers (VOP) or (SOP) depending on his preferences are given on elements of Y or on elements of $\wp_0(Y)$.

In terms of existing literature, we point out that it is usual to call set-valued optimization problem or set-optimization problem to refer to (VOP) or (SOP). In this paper we establish such a difference. On the other hand, about solutions for a set-valued optimization problem, the vector criterion is the most well-known and investigated in the branch of set-valued optimization. Thus, the vast majority of publications on (SVOP) is about optimality conditions for (VOP).

The set approach was introduced by Kuroiwa [52] in 1997 by using set-relations which generalize that given by the ordering cone (Sect. 3). Since the notion of set solution was introduced, there has been rapid growth in the field about it. In this survey, our claim is to show several bibliographic collections reported about the set approach to give a comprehensive listing and to analyse the research covering its first 16 years of history which is not available, as far we know.

This paper is decomposed into six sections. The second one is devoted to establish the main differences between (VOP) and (SOP). In Sect. 3 we introduce the main preferences defined on $\wp_0(Y)$ which have been explored in the literature in terms of (SVOP). In the next section, an extensive listing of set optimization research that covers theoretical developments from the beginning to the year 2013 is given. In Sect. 5, we present several areas different to optimization in which the set-relations have been used implicitly. Finally, in Sect. 6, several remarks and conclusions are presented for new research.

2 Vector Optimization Problem Versus Set Optimization Problem

In this section, firstly we show the main (geometric and analytic) aspects of (VOP) and (SOP) and secondly, the immediate relationships between their solutions.

It is clear that solving a vector set-valued optimization problem is equivalent to solve a rather simple problem in terms of the the objective map. In other words, solving (VOP) is equivalent find the solutions of the following vector problem:

$$(\mathrm{V}_1) \quad \begin{cases} \mathrm{Min}\ \Pi_Y(x, y) \\ (x, y) \in \mathrm{Gra}(F), \end{cases}$$

where $\mathrm{Gra}(F) = \{(x, y) \in X \times Y : x \in M,\ y \in F(x)\}$ and Π_Y is the projection of $\mathrm{Gra}(F) \subset X \times Y$ on the second space.

Thus, $z_0 = (x_0, y_0) \in \mathrm{Gra}(F)$ is an solution of (V_1) if and only if z_0 is a minimizer of (VOP).

The above result is a peculiar characteristic of (VOP) since if we consider other level of complexity for the objective map, we know that in order to solve a vector optimization problem (P) via a scalar optimization problem we have to apply some technique of scalarization which is not always possible, in general.

In terms of optimality conditions for (SOP) we could consider solutions which image sets are not related with the boundary line of the image set $F(M)$. It is a geometric property of the set solutions which must be overcame in order to give necessary conditions via separation theorems.

One advantage of the set criterion over the vector criterion is the possibility of considering preference relations on 2^Y. On the contrary, the main disadvantage of set criterion over vector criterion is the loss of structure lineal. Hamel [29] studied the structure of $\wp_0(Y)$ introducing a conlineal space.

In order to avoid such a problem several authors have considered specializations of F or tools to study the problem (SOP) via a structure well-known or simpler than a conlineal space. For instance, in Hernández [33] solutions of (SOP) are characterized via nonlineal scalarization, see also [8, 34, 72]. Nuriya and Kuroiwa [58, 77] construct an embedding vector space. Maeda [71], working on n-dimensional Euclidean spaces shows that whenever set-valued map is rectangle-valued, (SOP) is equivalent to a pair of vector-valued optimization problems. Recently Jahn [47] states that a certain vector optimization problem can be associated to (SOP) when $\preceq$ is defined by some complicated set relations.

In general, there is no any relationship between solutions obtained by vector criterion (solutions of (VOP)) and solutions obtained by set criterion (solutions of (SOP)). Moreover, the existence of solutions of one type does not imply the existence of solutions of the other type. See, for instance, [27, 40].

On the other hand, it is natural to pose questions about the relationships between solutions obtained by each criterion. Hernández and Rodríguez-Marín [42], under certain assumptions for the set-valued map, show that to solve (SOP) it is possible to reduce the feasible set through the set criterion.

Even though both criteria are different, they extend (V) in the following sense. If we consider a pre-order $\preceq$ on $\wp_0(Y)$ compatible with $\leq$ and F is replaced by a vector-valued map, then (SOP) and (VOP) are equivalent to (V). If, in addition, we consider weakly solutions of (VOP) and (SOP) it is possible to prove that each weakly vector solution is a weakly set solution, see [34, Proposition 2.10] and [71, Theorem 5].

See also [5, 42, 71] to find more relationships between vector solutions (VOP) and set solutions (SOP). Thus, for a certain classes of set-valued maps and pre-order on $\wp_0(Y)$ the set criterion is equivalent to the vector criterion. A particular case is obtained when $Y = \mathbb{R}$ since each image set has a strongly minimal point.

To end this section we refer the reader to [19, 65] for a deeper discussion of the above approaches of solutions for a optimization problem.

3 Set Relations Considered in the Literature

Now, we introduce the main preferences considered in the existing papers devoted to study solutions of (SOP). In addition, we focus on pre-order relations defined on $\wp_0(Y)$ which generalizes the ordering defined by K on Y.

The first systematic treatment of set relations in the context of ordered vector spaces and its power sets is due to Kuroiwa, Tanaka and Ha [59] in 1997.

Definition 3.1 [59] Let $A, B \in \wp_0(Y)$.

$$\begin{array}{lll}
A \leq^i B & \Leftrightarrow & B - A \subset K \quad \Leftrightarrow \quad a \leq b \quad \text{for all} \quad b \in B, a \in A \\
A \leq^{ii} B & \Leftrightarrow & \text{there exists} \quad a \in A \quad \text{such that} \quad a \leq b \quad \text{for all} \quad b \in B \\
A \leq^{iii} B & \Leftrightarrow & B \subset A + K \\
A \leq^{iv} B & \Leftrightarrow & \text{there exists} \quad b \in B \quad \text{such that} \quad a \leq b \quad \text{for all} \quad a \in A \\
A \leq^{v} B & \Leftrightarrow & A \subset B - K \\
A \leq^{vi} B & \Leftrightarrow & \text{there exist} \quad b \in B, a \in A \quad \text{such that} \quad a \leq b.
\end{array}$$

It is easy to check that $\leq^k$ with $k \in \{i, ii, iv\}$ are preferences such that the anti-symmetric and reflexive properties do not hold while $\leq^k$ with $k \in \{iii, v, vi\}$ are pre-orders on $\wp_0(Y)$. In addition,

$$A \leq^i_K B \Rightarrow A \leq^{ii} B \Rightarrow A \leq^{iii}_K B \Rightarrow A \leq^{vi} B$$

$$A \leq^i_K B \Rightarrow A \leq^{iv} B \Rightarrow A \leq^{v}_K B \Rightarrow A \leq^{vi} B.$$

In general, two nonempty sets A and B could not be related by $\leq^k$ for any k. Indeed, let $E = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ be. Then $A = \{(x+2)^2 + (y-2)^2 = 1\}$ and $B = \{(x-3)^2 + (y+3)^2 = 1\}$ satisfy that $A \not\leq^{vi} B$ y $B \not\leq^{vi} A$.

The authors defined the above set-relations on $\wp_0(Y)$ to study generalized convexity of a set-valued map. Since then, relations of a similar type have been proposed for other authors, in several papers.

The most important property of set relations introduced in Definition 3.1 is that all of them generalize the ordering defined by K on Y in the sense of Definition 1.3. We emphasize that, in terms of optimality conditions, the set relations $\leq^{iii}$ and $\leq^{v}$ are called lower and upper set-relations (denoted by $\leq^{l}$ and $\leq^{u}$) respectively. It is clear that $A \leq^{l} B$ is equivalent to $-B \leq^{u} -A$. In addition, it is possible to rewrite them via $-K$ instead of K.

So, Kuroiwa, Tanaka and Ha started developing a new approach to set-valued optimization which is based on comparison among values of the set-valued map from a set into a ordered vector space. At the same year, 1997, Kuroiwa [52] introduced solutions for (SOP) in the sense of Definition 1.2 by using $\leq^{l}$ and $\leq^{u}$. Due to this fact, the set criterion is also called in the literature Kuroiwa's criterion. In order to illustrate the set criterion we give a example.

Example 3.1 Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ be. 1. Consider $F\colon M = [0, \infty) \longrightarrow 2^{\mathbb{R}^2}$ such that

$$F(x) = \begin{cases} \{(0,0)\} & x = 0 \\ \left[(0,0), \left(-x, \frac{1}{x}\right)\right] & x \neq 0 \end{cases}$$

Then $\leq^{l} -\mathrm{Eff}(F) = \emptyset$ and $\mathrm{Eff}(F) = M$.

2. Consider $F\colon [-1, 0] \longrightarrow 2^{\mathbb{R}^2}$ such that

$$F(\lambda) = \begin{cases} \{(x, -x^2) \in \mathbb{R}^2\colon\ -1 < x \leq 0\} & \lambda = -1 \\ \left[(\lambda, 0), (\lambda, -\lambda^2)\right] & \lambda \neq -1 \end{cases}$$

Then $\leq^{l} -\mathrm{Eff}(F) = \{-1\}$ and $\mathrm{Eff}(F) = \emptyset$.

However, the set relations given in Definition 3.1 have just been considered in different frameworks many years ago, as is remarked in [19, 29, 46]. Firstly, Young in 1931, [84] considered the above ret relations, among others, in terms of algebraic structures. Fifty years before, Nishnianidze [75] studied theory of fixed points of monotonic operators. That is the reason why Jahn in [46] called KNY order relations to refer to set relations presented in Definition 3.1.

Alonso and Rodríguez-Marín [5] proved that the study of the set optimization problems: $\leq^{l} -\mathrm{Min}\, F$, $\leq^{u} -\mathrm{Min}\, F$ $\leq^{l} -\mathrm{Max}\, F$ and $\leq^{u} -\mathrm{Max}\, F$ is reduced to the study of the following ones: $\leq^{l} -\mathrm{Min}\, F$ and $\leq^{u} -\mathrm{Min}\, F$. Moreover, one can be solved by the other with a suitable definition of the objective map.

To end this section, we recall other set relations defined in set optimization theory. In 2003, Kuroiwa [56, Definition 2.1] introduced new binary relations on $\wp_0(Y)$ which are weaker than $\leq^{l}$ and $\leq^{u}$ by using elements of the positive polar cone of K. See also, [45].

On the other hand, by combining the set relations $\leq^{l}$ and $\leq^{u}$ we obtain the following pre-order on $\wp_0(Y)$

$$A \preceq^l_u B \Leftrightarrow A \preceq^l B \quad \text{and} \quad A \preceq^u B. \tag{1}$$

In 2010, Maeda [71], working on n-dimensional Euclidean spaces, defined preferences on $\wp_0(Y)$ via the strong minimal and maximal elements of sets. The same author, in [72] defined Pareto optimal solutions, semi-weak Pareto optimal solution, and weak Pareto optimal solutions of (SOP) by considering $\preceq^l_u$.

Janh and Ha [48], in a more general framework, introduced new set relations motivated by analysis interval and related with the above ones which generalize those given by Maeda [72] and seem be more appropriate to set optimization theory according to their applications.

In Löhne and Tammer [64] several set relations are presented on the family of all subsets A of $Y = \mathbb{R}^n$ with $\text{cl}(A + K) = A$ (where cl denotes the topological closure) to construct a pre-ordered conlinear space. See also, [62]. See also set relations given in [58].

4 Classification of the Literature

In this section we recover the main results presented in the literatures of set-optimization by using set approach. Several existence theorems for solutions of (SOP) will be presented under a unified framework.

In the pioneering paper [52], Kuroiwa introduced the definitions of l-type solutions or u-type solutions (by using $\preceq^l$ or $\preceq^u$ respectively) of (SOP) and a motivation for the study of set optimization problems is given by means non academic examples.

Since 1997, when set criterion was introduced, the optimality conditions for solutions of (SOP) are divided into two categories: those following results from the vector case (using continuity, properties of a set, differentiability, scalarization, Lagrangian duality, well-posedness and approximate solutions) and those obtained by applying new results or tools.

In the sequel we list the main results related with the existence of solutions of (SOP) from the beginning to up to now.

The first optimality conditions of l-type solutions of (SOP) were presented in [54, Theorem 4] by considering M compact and F a set-valued map with level sets closed ([54, Definition 2]) and, in addition, K closed and pointed. See also [53, Theorem 3.1].

In 2005, Alonso and Rodríguez-Marín [5] extended the definitions of cone-semicompactness and domination property from a set to a family of sets. In addition, in Proposition 22, gave a sufficient condition of u-type solution by using a notion of cone-regularity defined by subcovers of a family of subsets. See also [5, Corollary 24] to obtain a existence condition of u-type solution under M compact and F lower cone-semicontinuous. A sufficient condition of l-type solution under M compact and F upper cone-semicontinuous was given in [5, Propositions 29 and 30].

Following the subcovers introduced in [5, Definition 27], Hernández and Rodríguez-Marín [35] introduced the notion of strongly some-compactness and cone-completeness for a family of sets to extend optimality conditions from the vector case to the lower set relation case in Theorems 4.1 and 4.6. Such results generalize those presented in [69] for the vector case. In [35, Sect. 5], several optimality conditions presented in [5, 55] are slightly improved. On the other hand, under assumptions of generalized continuity, not only the existence of solutions is proven but also the domination property of the family $\{F(x) \colon x \in M\}$ ([35, Definition 4.3]) was established in Corollaries 5.5, 5.6 and 5.7 and Theorems 5.8 and 5.9.

Other definitions of semicompactness, completeness and semicontinuity and related general theoretical properties with respect to a pre-order $\preceq$ on $\wp_0(Y)$ was given in [48] in a more general framework.

Also, assuming $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ and considering the combined set-relation $\leq^l_u$ defined in (1), Maeda [72, Theorem 4.1] gave a sufficient condition for a $\leq^l_u$-solution under compactness and generalized continuity.

Remark 4.1 The above results allow to state that the existence conditions of solutions of set type are, in general, weaker that those of vector type and, in addition, several existence results in vector optimization do not depend on linearity of the image space.

In terms of duality theory, Kuroiwa introduced a generalized Lagrangian as follows $L(x, y, T) = F(x) + T(y)$ (where T is a linear map from X to Y and $y \in F(x)$) and the dual problem associates to a constrained set optimization problem. He established conditions of saddle points in [53, Theorem 4.1] and [54, Theorem 9]. Hernández and Rodríguez-Marín [36, 37] generalized the above Lagrangian map by defining $L(x, T) = F(x) + T(F(x))$ (where T is an affine map from X to Y) and gave weak and strong duality theorems and saddle points results which extend those known in the vector case. In [36, Sect. 3] and [37, Sect. 4], some multiplier rules by means of an affine linear map under generalized convexity assumptions were given by considering l-type solutions of (SOP).

By using $\leq^i$ Lin and Chen [68] gave weak solutions and strong solutions of set equilibrium problems and [43, Theorem 5.5] established a Lagrange multiplier rule.

Alonso and Rodríguez-Marín [5, Theorems 35 and 38] gave optimality conditions for existence of strict solutions of (SOP) in terms of continuous selections of set-valued maps. The same authors, in [6, Theorem 25] established a necessary and sufficient condition for the existence of weakly l-type solutions of (SOP) under generalized convexity assumptions and contingent derivative of F.

Rodríguez-Marín and Sama in [79] gave a notion of following graphical derivative of a set-valued map.

Definition 4.1 [79] Let X, Y be real normed spaces. Assume K is closed, strongly minihedral and regular. The (Λ, C)-lower contingent derivative of F at $\overline{x}$ is the set-valued map $\underline{D}_\Lambda F(\overline{x}) : X \to 2^Y$ defined by

$$\text{Gra}\, \underline{D}_\Lambda F(\overline{x}) = \text{Limsup}_t\, T(\text{Gra}\, \varphi_{\Lambda,t}, (\overline{x}, \varphi_{\Lambda,t}(\overline{x}))),$$

where $T(A, z)$ with $A \subset Y$ denotes the contingent cone to A at $z \in A$. Based on ordered spaces techniques, the authors defined two types of contingent derivatives to set-valued maps and gave optimality conditions in terms of the contingent derivatives for local l-type solutions of (SOP) in Theorems 5.1, 5.2, 5.6 and 5.7. The obtained results prove that the above derivative is suitable for the formulation of necessary and sufficient conditions for set-valued optimization problems following the set approach.

In 2009, Kuroiwa [57] also presented directional derivatives based on an embedding idea to establish necessary and sufficient conditions for a weakly minimal and minimal solutions of (SOP).

Considering constrained optimization problems, Maeda [71, Theorems 6 and 7] established existence conditions for weakly $\leq_u^l$-solutions by radial Dini derivatives and lower and upper Dini derivatives of F.

Hernández and Rodríguez-Marín [39, Sect. 5] obtained optimality conditions for the existence of solutions l-type solutions via weak and strong subgradients for a set-valued map.

Remark 4.2 From the above results, it is clear that even notions on optimality conditions in terms of differentiabily notions for set-valued maps is still an open issue in set optimization.

In 2007, Hernández and Rodríguez-Marín [34, Sect. 4], by considering the preorder was defined by $\leq^l$, gave results on scalarization for (SOP) and characterized its solutions without convexity assumptions for F a K-closed and K-bounded valued. Hamel and Löhne [31] one year before had introduced a similar generalization to give minimal element theorems. In this context, see also results given in [50, 61, 76, 81, 82]. Recently, Araya [8] presented new nonconvex separation theorems to apply to set optimization by using $\leq^l$ and $\leq^u$ preferences. So, existence theorems of weakly l-type minimal and weakly u-type minimal solutions via scalarizations were given in Sect. 5.1 and a Takahashi's minimization theorem in Sect. 5.2 was presented in terms of set optimization.

Maeda [72] studied constrained set optimization problems with various types of set solutions and, via scalarization, gave necessary and sufficient conditions under compactness assumptions (Theorems 4.2 and 4.3) and a characterization under convexity assumptions (Theorem 4.4).

Remark 4.3 We emphasize that in all the above papers devoted to scalarization, the scalarizing function considered was a generalization of the Gerstewitz's nonconvex separation function introduced in [23] and extensively studied in [24].

Ha [27, Theorem 3.1], by using strict l-type solutions, established a variant of EVP for F (where X is a complete metric space). In Sects. 4 and 5, other variants of the EVP by using conical extensions and the concept of cone extension and the Mordukhovich coderivative (see [73, 74]) were established.

In Kuroiwa [56, Theorem 3.5], via weight criteria, problem (SOP) was embedded to a complete metric space to obtain an existence theorem for weakly efficient solutions based on the Ekeland variational principle.

Also, in a framework more general, Hamel and Löhne [31] obtained two existence results for minimal of a family of subsets of the product space $X \times 2^Y$ (where X is a separated uniform space) with respect to appropriate ordering relations on 2^Y. As application, the authors established a variant of Ekeland's principle for a set-valued optimization problem via generalizations of the functionals introduced in [23]. In [13, Theorem 3.5] by using $\leq^u$ the authors established a variational principle for set-valued maps.

See also [25, Theorems 5.1 and 5.2] for approximate variants of the EVP given in [27, 31] and [20, Theorem 6.2 and 6.3] for Ekeland variational principles on quasi ordered spaces in a framework more abstract. In addition, considering relations between values of F and pre-orders generated by set-valued maps, in [67, Theorems 3.4 and 3.5] the authors directly expressed the Ekeland principle but in terms of values of F.

Others generalizations of EVP for set-valued mappings and set approach are given in [49] (via generalized distances) and [78, Theorem 3.1] by considering strict solutions of (SOP). By using a perturbed map (stability) of F see also, [34, Sect. 5].

In 2009, Zhang, Li and Teo [87] introduced three kinds of well-posedness for a set optimization problem called k_0-well posedness at a minimizer, generalized k_0-well posedness and extended k_0-well-posedness (where $k_0 \in \text{int}(K)$) and extended some basic results of well-posedness of scalar optimization to set optimization by using a generalization of the Gerstewitz's function given in [34].

Compare the above results and the tools used with those presented in [26].

In [27, Theorem 4.2] Ha defined properly positive efficient points in the framework of set approach and gave a sufficient condition of approximated solution of the perturbed map. Approximate solutions for problem (SOP) were also introduced in [25, Defintion 3.2] and [7, Definitions 17 and 19].

About locating set solutions, in [38] using polyhedral cones Hernández and Rodríguez-Marín extended the first theorem for locating the set of all efficient points of a set through ordinary mathematical programming introduced by Yu [85]. Recently, in Löhne and Schrage [66] an algorithm which computes a solution of a set optimization problem was provided.

5 Related Theories or Applications

In this section several papers related with the previous set relations are enumerated to show its applicability.

On the one hand, to give optimality conditions in set-valued optimization theory, several set relations have been used to generalize the convexity of a set-valued map in [59], to give scalar representations of a vector optimization problem in [83], to study conjugate duality in [62], to show continuity for set-valued maps under some convexity assumptions in [60], to establish alternative theorems in [76] or to find vector solutions in [41] where the set approach is used to reduce the feasible set of (VOP). In addition, as we have shown in Sect. 4, several Ekeland-type principles are developed in different frameworks by using pre-order or preferences on $\wp_0(Y)$. In

[13] the authors present minimax methods in variational analysis, exactly, a mountain pass-type theorem.

Hamel, Löhne, Heyde and their collaborators have developed a new research line in terms of infimum and supremum by using set-relations (which pre-serve the structure of complete lattices) and definitions understood in the sense of solutions like those defined in Remark 1.2 (in which both criteria are implicitly considered). Their results can be appropriate in terms of risk function in Finance. An overview of such results can be found in [63] and references therein.

On the other hand, the preferences between nonempty sets have been also considered in other theories different to Optimization. For instance, to present existence results for inclusion problems in [32] and to obtain fixed point theorems in [16, 17].

The relation defined in (1), among others, seems to be more suitable in practice, for instance, in the framework of interval analysis according to the basic investigations of Chiriaev and Walster [15]. In addition, such set relations are widely used in theoretical computer sciences, see for example Brink [14] in where a study of power structures in a universal-algebraic context is presented. For more details see [19, 48].

Since the seminal paper [86], fuzzy sets theory has been applied to various fields of decision making theory including economics, management science and engineering widely. In [70] the notion of non-dominated solution is related with Kuroiwa's solution by using a both set relations, $\leq^l$ and $\leq^u$, in terms of fuzzy mathematical programming.

6 Conclusions

According to the previous sections, we can conclude that the analysis of Kuroiwa's concept deserves an exhaustive treatment. In addition, set optimization theory can be considered as an area which is beginning since it is possible to identify future lines of research from the existing literature.

The set-valued optimization theory by using the set criterion is a natural extension of vector optimization theory. It is due to the published results allow to extend those given the vector case. In addition, the research on set optimization has proven that several existence results do not rely on linearity of the image space and therefore they can be extended to set relations.

There is no doubt that many frameworks of optimization theory can be related with set relations. However, a result which is worthy of being studied is an academic example in where a solution criterion in terms of set relations is considered. Maybe the main problem is to know what is the optimal alternative. It is clear that the set criterion seems a natural extension of vector optimization theory and seems to have the potential to become an important tool for many areas in optimization. In the same direction, Jahn [46] asserts that such set relations open a new and wide field of research and turn out to be promising in set optimization.

On the other hand, the set relations proposed in Kuroiwa, Tanaka and Ha [59], among others, have been used as tools not only in optimization but also in others fields. Probably, fuzzy programming and analysis interval are areas in which the practical point of view of the set approach is developed.

To summarize, as mentioned in Sect. 1, new relations of more general types should be explored to find adequate applications. A goal of the present survey is to motivate such a study.

References

1. Aubin, J-P.: Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. Mathematical Analysis and Applications, Part A. Adv. in Math. Suppl. Stud., **7a**, 159–229, Academic Press, New York-London (1981)
2. Aubin, J-P., Ekeland I.: Applied nonlinear analysis. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. Wiley, New York (1984)
3. Aubin, J-P., Frankowska, H.: Set-valued Analysis. Systems and Control: Foundations and Applications, vol. 2. Birkhäuser, Boston (1990)
4. Aubin, J.-P., Cellina, A.: Differential Inclusions. Set-Valued Maps and Viability Theory. Grundlehren Math. Wiss. Springer, Berlin (1984)
5. Alonso, M., Rodríguez-Marín, L.: Set-relations and optimality conditions in set-valued maps. Nonlinear Anal. **63**, 1167–1179 (2005)
6. Alonso, M., Rodríguez-Marín, L.: Optimality conditions for set-valued maps with set optimization. Nonlinear Anal. **70**, 3057–3064 (2009)
7. Alonso-Durán, M., Rodríguez-Marín, L.: On approximate solutions in set-valued optimization problems. J. Comput. Appl. Math. **236**, 4421–4427 (2012)
8. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**, 3821–3835 (2012)
9. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers in multiobjective optimization: existence and optimality conditions. Math. Prog. **122**, 301–347 (2010)
10. Bao, T.Q., Mordukhovich, B.S.: Set-valued optimization in welfare economics. Adv. Math. Econ. **13**, 113–153 (2010)
11. Bao, T.Q., Mordukhovich, B.S.: Extended pareto optimality in multiobjective problem. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Advances in Vector Optimization, pp. 379–417, Springer, Berlin (2011)
12. Bao, T.Q., Mordukhovich, B.S.: Necessary optimality conditions and applications to welfare economics. Dyn. Syst. **31**, 1069–1096 (2011)
13. Bednarczuk, E.M., Miglierina, E., Molho, E.: A mountain pass-type theorem for vector-valued functions. Set-Valued Var. Anal. **19**, 569–587 (2011)
14. Brink, C.: Power structures. Algebr. Univers. **30**, 177–216 (1993)
15. Chiriaev, A., Walster, G.W.: Interval arithmetic specification. Technical Report (1998)
16. Dhage, B.C.: A functional integral inclusion involving Caratheodories. Electron. J. Qual. Theory Differ. Equ. **14**, 1–18 (2003)
17. Dhage, B.C.: A general multi-valued hybrid fixed point theorem and perturbed differential inclusions. Nonlinear Anal. **64**, 2747–2772 (2006)
18. Dauer, J.P., Stadler, W.: A survey of vector optimization in infinite-dimensional spaces. II. J. Optim. Theory Appl. **51**, 205–241 (1986)
19. Eichfelder, G., Jahn, J.: Vector optimization problems and their solution concepts. Recent Developments in Vector Optimization. Vector Optim. Springer, Berlin (2012)
20. Flores-Bazán, F., Gutiérrez, C., Novo, V.: A Brézis-Browder principle on partially ordered spaces and related ordering theorems. J. Math. Anal. Appl. **375**, 245–260 (2011)
21. Flores-Bazán, F., Hernández, E.: A unified vector optimization problem: complete scalarizations and applications. Optimization **60**, 1399–1419 (2011)

22. Flores-Bazán, F., Hernández, E.: Optimality conditions for a unified vector optimization problem with not necessarily preordering relations. J. Global Optim. **56**, 299–315 (2013)
23. Gerth, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
24. Göpfert, A., Riahi, H., Tammer, T., Zalinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
25. Gutiérrez, C., Jiménez, B., Novo, V., Thibault, L.: Strict approximate solutions in set-valued optimization with applications to the approximate Ekeland variational principle. Nonlinear Anal. **73**, 3842–3855 (2010)
26. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**, 1822–1833 (2012)
27. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**, 187–206 (2005)
28. Ha, T.X.D.: Optimality conditions for several types of efficient solutions of set-valued optimization problems. In: Pardalos, P., Rassias, Th.M., Khan, A.A. (eds.) Nonlinear Analysis and Variational Problems, pp. 305–324. Springer, Heidelberg (2010)
29. Hamel, A.: Variational principles on metric and uniform spaces. Habilitation thesis, University of Halle-Wittenberg, Germany (2005)
30. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**, 66–95 (2010)
31. Hamel, A., Löhne, A.: Minimal element theorems and Ekelands principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)
32. Heikkilä, S.: Inclusion problems in ordered topological vector spaces and applications. J. Math. Anal. Appl. **298**, 94–105 (2004)
33. Hernández, E.: Problemas de optimizacin en anlisis de multifunciones. Ph.D. thesis Doctoral. UNED, Madrid (2005)
34. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
35. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**, 1726–1736 (2007)
36. Hernández, E., Rodríguez-Marín, L.: Lagrangian duality in set-valued optimization. J. Optim. Theory Appl. **134**, 119–134 (2007)
37. Hernández, E., Rodríguez-Marín, L.: Duality in set optimization with set-valued maps. Pac. J. Optim. **3**, 245–255 (2007)
38. Hernández, E., Rodríguez-Marín, L.: Locating minimal sets using polyhedral cones. Oper. Res. Lett. **39**, 466–470 (2011)
39. Hernández, E., Rodríguez-Marín, L.: Weak and strong subgradients of set-valued maps. J. Optim. Theory Appl. **149**, 352–365 (2011)
40. Hernández, E., Rodríguez-Marín, L., Sama, M.: Criteria of solution in set-valued optimization, In: Proceedings of Kosen Workshop MTE2008, Ibaraki national college of technology, Hitachinaka, Ibaraki (2008)
41. Hernández, E., Rodríguez-Marín, L., Sama, M.: Some equivalent problems in set optimization. Oper. Res. Lett. **37**, 61–64 (2009)
42. Hernández, E., Rodríguez-Marín, L., Sama, M.: On solutions of set-valued optimization problems. Comput. Math. Appl. **60**, 1401–1408 (2010)
43. Hernández, E., Rodríguez-Marín, L., Sama, M.: About Hahn-Banach extension theorems and applications to set-valued optimization. Comput. Math. Appl. **64**, 1778–1788 (2012)
44. Heyde, F., Löhne, A., Tammer, Chr: Set-valued duality theory for multiple objective linear programs and application to mathematical finance, Math. Meth. Oper. Res. **69**, 159–179 (2009)
45. Hu, R., Fang, Y.P., Huang, N.J.: Optimality conditions for set optimization with weighted criteria. Acta Math. Sinica (Chin. Ser.) **52**, 989–1000 (2009)
46. Jahn, J.: Vector Optimization. Theory, Applications, and Extensions. Springer, Berlin (2004)
47. Janh, J.: Vectorization in set optimization. Manuscript (2013)

48. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
49. Khanh, P.Q., Quy, D.N.: On Ekelands variational principle for pareto minima of set-valued mappings. J. Optim. Theory Appl. **153**, 280–297 (2012)
50. Kimura, K., Shimizu, A.: Minimal element theorems for set-optimizations. J. Nonlinear Convex Anal., Yokohama Publ., Yokohama, 105–113 (2010)
51. Klein, E., Thompson, A.C.: Theory of Correspondences. Including Applications to Mathematical Economics. Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York (1984)
52. Kuroiwa, D.: Some criteria in set-valued optimization. RIMS Kokyuroku **985**, 171–176 (1997)
53. Kuroiwa, D.: Some duality theorems of set-valued optimizacion with natural criteria. In: Proccedings of the International Conference on Nonlinear Analysis and Convex Analysis, World Scientific, pp. 221–228 (1999)
54. Kuroiwa, D.: On set-valued optimization. In: Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000). Nonlinear Anal. 472(2), 1395–1400 (2001)
55. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**, 73–84 (2003)
56. Kuroiwa, D.: Existence of efficient points of set optimization with weighted criteria. J. Nonlinear Convex Anal. **4**, 117–123 (2003)
57. Kuroiwa, D.: On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. **10**, 41–50 (2009)
58. Kuroiwa, D., Nuriya, T.: A generalized embedding vector space in set optimization. J. Nonlinear Convex Anal. 297–303 (2007)
59. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 3 (Athens, 1996). Nonlinear Anal. **30**, 1487–1496 (1997)
60. Kuwano, I., Tanaka, T.: Continuity of cone-convex functions. Optim. Lett. **6**, 1847–1853 (2012)
61. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets in set-valued optimization. RIMS Kokyuroku **1685**, 270–280 (2010)
62. Löhne, A.: Optimization with set relations: conjugate duality. Optimization **54**, 265–282 (2005)
63. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
64. Löhne, A., Tammer, C.: A new approach to duality in vector optimization. Optimization **56**, 221–239 (2007)
65. Löhne, A., Frank, H.: Solution concepts in vector optimization, a fresh look at an old story. Optimization **60**, 1421–1440 (2011)
66. Löhne, A., Schrage, C.: An algorithm to solve polyhedral convex set optimization problems. Optimization **62**, 131–141 (2013)
67. Liu, C.G., Ng, K.F.: Ekeland's variational principle for set-valued functions. SIAM J. Optim. **21**, 41–56 (2011)
68. Lin, Y.-C., Chen, H.-J.: Solving the set equilibrium problems. Fixed Point Theory Appl. (2011). doi:10.1155/2011/945413
69. Luc, D.T.: Theory of vector optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
70. Maeda, T.: On characterization of fuzzy vectors and its applications to fuzzy mathematical programming problems. Fuzzy Sets Syst. **159**, 3333–3346 (2008)
71. Maeda, T.: On optimization problems with set-valued objective maps. Appl. Math. Comput. **217**, 1150–1157 (2010)
72. J. Optim. Theory Appl. On Optimization Problems with Set-Valued Objective Maps: Existence and Optimality. **153**, 263–279 (2012)
73. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory. Grundlehren Series 330. Springer, Berlin (2006)
74. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, II: Applications. Grundlehren Series 331. Springer, Berlin (2006)
75. Nishnianidze, Z.G.: Fixed points of monotonic multiple-valued operators. Bull. Georgian Acad. Sci. **114**, 489–491 (1984)

76. Nishizawa, S., Onodsuka, M., Tanaka, T.: Alternative theorems for set-valued maps based on a nonlinear scalarization. Pac. J. Optim. **1**, 147–159 (2005)
77. Nuriya, T., Kuroiwa, D.: A topology on the embedding space in set optimization without compactness, J. Nonlinear Convex Anal. 497–503 (2007)
78. Qiu, J.H.: On has version of set-valued Ekelands variational principle. Acta Mathematica Sinica, Engl. Ser. **28**, 717–726 (2012)
79. Rodríguez-Marín, L., Sama, M.: (Λ, C)-contingent derivatives of set-valued maps. J. Math. Anal. Appl. **335**, 974–989 (2007)
80. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of multiobjective optimization. Mathematics in Science and Engineering, vol. 176. Academic Press, Inc., Orlando (1985)
81. Shimizu, A., Nishizawa, S., Tanaka, T., *Optimality conditions in set-valued optimization using nonlinear scalarization methods*, Nonlinear Anal. Convex Anal. 565–574, Yokohama Publ., Yokohama (2007)
82. Shimizu, A., Tanaka, T.: Minimal element theorem with set-relations. J. Nonlinear Convex Anal. **9**, 249–253 (2008)
83. Truong, X.D.H.: Cones admitting strictly positive functionals and scalarization of some vector optimization problems. J. Optim. Theory Appl. **93**, 355–372 (1997)
84. Young, R.C.: The algebra of many-valued quantities. Math. Ann. **104**, 260–290 (1931)
85. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)
86. Zadeh, L.A.: Fuzzy sets. Inform. Control **8**, 338–353 (1965)
87. Zhang, W.Y., Li, S.J., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**, 3769–3778 (2009)