On Characterization of Nash Equilibrium Strategy in Bi-Matrix Games with Set Payoffs

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Abstract In this paper, we consider set-valued payoff bi-matrix games where each player's payoffs are given by non-empty sets in n-dimensional Euclidean spaces $\mathbb{R}^n$. First, we define several types of set-orderings on the set of all non-empty subsets in $\mathbb{R}^n$. Second, by using these orderings, we define four kinds of concepts of Nash equilibrium strategies to the games and investigate their properties. Finally, we give sufficient conditions for which there exists these types of Nash equilibrium strategy.

Keywords Set-ordering · Maximal element · Set-valued map · Nonlinear scalarization · Set payoff games · Nash equilibrium strategy · Maximal Nash equilibrium strategy · Pareto Nash equilibrium strategy · Incomplete information game · Weak Pareto Nash equilibrium strategy · Fixed point theorem

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1 Introduction

Since seminal works by Neumann and Morgenstern [26] and Nash [24, 25], Game theory has played an important role in the fields of decision making theory such as economics, management, and operations research, etc.

When we apply the game theory to model some practical problems with which we encounter in real situations, we have to know (1) who are players, (2) what are strategies for each player, and (3) values of payoffs for each player to receive. However it is difficult for us to know the exact values of payoffs and could only know the values of payoffs approximately, or with some imprecise degree in general. In order to model such a situation with game theory, a great number of efforts have been devoted to the developments of game theory from the theoretical and practical points of views.

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For the games where the payoffs are given by random variables, Harsanyi [6] has defined Bayesian games, which is games with incomplete information on players' payoffs, and the concept of Bayesian Nash equilibrium to the games, and investigated the properties.

Campos [4] has considered fuzzy matrix games where the payoffs are given by fuzzy numbers, and proposed some methods to solve fuzzy matrix games based on linear programming, but has not defined explicit concepts of equilibrium strategies.

For fuzzy bi-matrix games with fuzzy payoffs, Maeda [18] has defined three types of Nash equilibrium strategies based on possibility and necessity measures and investigated their properties; Maeda [19] has defined three types of Nash equilibrium strategies by using fuzzy max ordering, and proved that fuzzy bi-matrix games are equivalent to the games with vector payoffs games. While, for fuzzy matrix games, Maeda [20] has defined fuzzy minimax equilibrium strategies based on fuzzy max order and investigated their properties.

Aghassi and Bertsimas [1] have considered matrix games where payoffs are uncertain and players have no information about the probability distributions, and investigated their properties based on robust optimization methods in mathematical programming. Liu and Kao [17], and Li [16] have considered matrix games where the payoffs are compact intervals in $\mathbb{R}$ and proposed some methods to solve the matrix games based on linear programming approaches. However, Liu and Kao [17] and Li [16] have not defined explicit concepts of equilibrium strategies.

In this paper, we consider the bi-matrix games where each player's payoffs are given by non-empty sets in n-dimensional Euclidean spaces $\mathbb{R}^n$, including interval-valued payoffs, which means both players don't know exact values of payoffs but they know their ranges. Namely, we consider the games that payoffs are deterministic uncertainty (See Leitmann [15]). Based on set-valued maps optimization methods (Maeda [21–23]), for set payoff game, we define four kinds of concepts of Nash equilibrium and give sufficient conditions under which there exist these Nash equilibrium strategies.

For those purposes, this paper is organized as follows. In Sect. 2, we introduce several types of set orderings on the set of all non-empty subsets in n-dimensional Euclidean space $\mathbb{R}^n$ and investigate their properties. In Sect. 3, we introduce two types of extended real-valued set functions defined on the set of all non-empty subsets of $\mathbb{R}^n$, which are extensions of the non-convex separation functions (Gerth and Weidner [5], Hamel [7] and Hernández and Rodríguez-Marín [8]) and investigate their properties. In particular, we show that these set functions are monotone and positively homogeneous with respect to the set orderings given in Sect. 2. In Sect. 4, we consider set payoffs bi-matrix games, where the payoffs for each player are compact convex sets in $\mathbb{R}^\ell$. First, we define the concepts of Nash equilibrium strategy to the game; then, associated with the set payoff bi-matrix games, we define the two-person games with scalar payoffs; bi-matrix games; we investigate relationships between set payoff bi-matrix games and scalar payoff two-person games. In Sect. 5, we give sufficient conditions under which there exists at least one Nash equilibrium strategy to set payoff bi-matrix games.

2 Orderings on Sets in $\mathbb{R}^n$ and Set-Valued Maps

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant, respectively. By $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}^n)$, we denote the sets of all non-empty subsets of $\mathbb{R}^n$ and the set of all non-empty compact subsets of $\mathbb{R}^n$, respectively. For any elements $A,\ B \in \mathcal{P}(\mathbb{R}^n)$ and any real number $\lambda \in \mathbb{R}$, we write $A + B := \{z \in \mathbb{R}^n \mid z = x + y,\ x \in A,\ y \in B\}$ and $\lambda A := \{z \in \mathbb{R}^n \mid z = \lambda x,\ x \in A\}$. Whenever $A \in \mathcal{P}(\mathbb{R}^n)$ is a singleton, say $A = \{a\}$, we abuse notations and write a instead of $\{a\}$.

Definition 2.1 For $A,\ B \in \mathcal{P}(\mathbb{R}^n)$, we write

$$A \leqq_L B \quad \text{iff} \quad B \subseteq A + \mathbb{R}^n_+, \tag{1}$$

$$A \leqq_U B \quad \text{iff} \quad A \subseteq B - \mathbb{R}^n_+, \tag{2}$$

$$A \leqq B \quad \text{iff} \quad A \leqq_L B \ \text{ and } \ A \leqq_U B. \tag{3}$$

$$A \prec_L B \quad \text{iff} \quad \operatorname{cl} B \subseteq \operatorname{cl} A + \operatorname{int} \mathbb{R}^n_+, \tag{4}$$

$$A \prec_U B \quad \text{iff} \quad \operatorname{cl} A \subseteq \operatorname{cl} B - \operatorname{int} \mathbb{R}^n_+, \tag{5}$$

$$A \prec B \quad \text{iff} \quad A \prec_L B \ \text{ and } \ A \prec_U B, \tag{6}$$

$$A \preceq B \quad \text{iff} \quad A \prec_L B \ \text{ and } \ A \leqq_U B, \ \text{ or } \ A \leqq_L B \ \text{ and } \ A \prec_U B. \tag{7}$$

where cl A denotes the closure of the set A.

It is easy to see that the binary relations $\leqq_L$, $\leqq_U$, and $\leqq$ are reflexive and transitive, but not antisymmetric. In fact, for any $A, B \in \mathcal{P}(\mathbb{R}^n)$, $A \leqq B$ and $B \leqq A$ implies that $A + \mathbb{R}^n_+ = B + \mathbb{R}^n_+$, in general. Therefore, the binary relations $\leqq_L$, $\leqq_U$, and $\leqq$ are quasi orderings on $\mathcal{P}(\mathbb{R}^n)$. On the other hand, binary relations $\prec_L$, $\prec_U$, $\prec$, and $\preceq$ are strict partial orderings.

The set-orderings $\leqq$ and $\prec$ are introduced by Young [27]. Kuroiwa [12, 13] use the set-orderings $\leqq_L$, $\leqq_U$, $\prec_L$ and $\prec_U$ to study set optimization problems where the objective map is given by set-valued map. By using the set-ordering $\leqq$, Maeda [21] gave the condition that fuzzy mathematical problems are equivalent to set-valued optimization problems. For the relationships among these set orderings and other set orderings, see Jahn and Ha [10].

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^n)$ be any non-empty subset and $A \in \mathcal{A}$ be any set. Then, the set $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set-ordering $\leqq$ iff $A' \in \mathcal{A},\ \ A \leqq A'$ imply $A' \leqq A$. While, $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set-ordering $\preceq$ iff there is no $\bar{A} \in \mathcal{A}$ such that $\mathcal{A} \preceq \bar{A}$, and $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set ordering $\prec$ iff there is no $\bar{A} \in \mathcal{A}$ such that $A \prec \bar{A}$. Similarly, we could define various types of maximal element in $\mathcal{A}$ with respect to other set-orderings given in Definition 2.1

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ be any set-valued map. By $\operatorname{Dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr}(F) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell \mid y \in F(x)\}$, we denote the effective domain and the graph of F, respectively. Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ be any set-valued map and $S \subseteq \operatorname{Dom}(F)$ be any non-empty convex set. Then F is said to be $\leqq$-concave on S if $(1 - \lambda)F(x) + \lambda F(y) \leqq F((1 - \lambda)x + \lambda y)$ holds for $\forall x,\ y \in S$ and $\forall \lambda \in [0, 1]$;

$F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be $\leqq$-convex on S if $F((1-\lambda)x + \lambda y) \leqq (1-\lambda)F(x) + \lambda F(y)$ holds for $\forall x,\ y \in S$ and $\forall \lambda \in [0, 1]$ (See Maeda [23]). While $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be convex-valued if $F(x)$ is convex for $\forall x \in \mathrm{Dom}(F)$; $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be compact-valued if $F(x)$ is compact for $\forall x \in \mathrm{Dom}(F)$.

A set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be upper semi-continuous at $x^o \in \mathrm{Dom}(F)$ if, for any sequences $\{(x^\nu, y^\nu)\}_{\nu=1}^\infty \subseteq \mathrm{Gr}(F)$ converging to $(x^o, y^o) \in \mathbb{R}^n \times \mathbb{R}^\ell$, we have $y^o \in F(x^o)$. While, $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be lower semi-continuous at $x^o \in \mathrm{Dom}(F)$ if, for any $(x^o, y^o) \in \mathrm{Gr}(F)$, and any sequence $\{x^\nu\}_{\nu=1}^\infty \subseteq \mathrm{Dom}(F)$ such that $\{x^\nu\}_{\nu=1}^\infty$ converging to x^o, there exists a subsequence $\{(x^{\nu'}, y^{\nu'})\}_{\nu'=1}^\infty \subseteq \mathrm{Gr}(F)$ such that the sequence $\{y^{\nu'}\}_{\nu'=1}^\infty$ converges to y^o; $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be continuous at $x^o \in \mathbb{R}^n$ if F is upper semi-continuous and lower semi-continuous at x^o. We say F is continuous on $\mathrm{Dom}(F)$ if, for any $x^o \in \mathrm{Dom}(F)$, F is continuous at x^o (see Aubin [3]).

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be any set-valued map, $S \subseteq \mathrm{Dom}(F)$ be any non-empty set and let $x^o \in S$ be any point. Then F is said to be uniformly compact near $x^o \in S$ if there exists a neighborhood $N(x^o)$ of x^o such that $\mathrm{cl} \cup_{x \in N(x^o)} F(x)$ is compact, where cl denotes the closure of the set $\cup_{x \in N(x^o)} F(x)$. F is said to be uniformly compact on S if F is uniformly compact near x for all $x \in S$.

3 Scalarizaion Methods of Set-Valued Maps in $\mathbb{R}^n$

In this section, we define two types of extended real-valued functions defined on $\mathcal{C}(\mathbb{R}^n)$, which are extensions of Gerstewitz's functions and investigate their properties (see [5, 7, 8], Maeda [23] and Araya [2]) .

Let $A \in \mathcal{C}(\mathbb{R}^n)$ be any non-empty compact set and let $k^o \in \mathrm{int}\ \mathbb{R}^n_+$ be any point. We define the real-valued set functions $\phi^i(\cdot; k^o) : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$ by $\phi^i(A; k^o) := \sup\{t \in \mathbb{R} \mid tk^o \leqq_i A\}$, $i = L, U$. Note that, for each $a \in A$, $\phi^L(a; k^o) = \phi^U(a; k^o) = \min\{a_i / k_i \mid i = 1, 2, \cdots, n\}$ and $\phi^L(\cdot; k^o), \phi^U(\cdot; k^o) : \mathbb{R}^n \to \mathbb{R}$ are continuous on $\mathbb{R}^n$ as functions defined on $\mathbb{R}^n$. Then we have the following lemma (see Hamel [7]).

Lemma 3.1 *Let $A \in \mathcal{C}(\mathbb{R}^n)$ be any compact set and let $k^o \in \mathrm{int}\mathbb{R}^n_+$ be any point. Then we have*

$$\phi^L(A; k^o) = \min\{\phi^L(a; k^o) \mid a \in A\}, \tag{8}$$

$$\phi^U(A; k^o) = \max\{\phi^U(a; k^o) \mid a \in A\}, \tag{9}$$

$$A \subseteq \phi^L(A; k^o)k^o + \mathbb{R}^n_+, \tag{10}$$

$$\phi^U(A; k^o)k^o \subseteq A - \mathbb{R}^n_+. \tag{11}$$

The following theorem shows that the set functions $\phi^L(\cdot; k^o)$ and $\phi^U(\cdot; k^o)$ are superadditive and positively homogeneous on $\mathcal{C}(\mathbb{R}^n)$, namely $\phi^i(\cdot, k^o)$, $i = L, U$ are concave set functions.

Theorem 3.1 *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets, $k^o \in \operatorname{int} \mathbb{R}^n_+$ be any point and let $\lambda \in \mathbb{R}_+$ be any real number. Then it holds that*

$$\phi^i(A; k^o) + \phi^i(B; k^o) \leqq \phi^i(A + B; k^o), \quad i = L, U, \tag{12}$$

$$\phi^i(\lambda A; k^o) = \lambda \phi^i(A; k^o) \quad i = L, U. \tag{13}$$

Proof First we show that (12) and (13) hold for $\phi^L(\cdot, k^o)$. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets. From (8), there exist points $\bar{a} \in A$ and $\bar{b} \in B$ such that

$$\begin{aligned}
\phi^L(A + B; k^o) &= \phi^L(\bar{a} + \bar{b}; k^o) \\
&= \min\{\frac{\bar{a}_i + \bar{b}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} \\
&\geqq \min\{\frac{\bar{a}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} + \min\{\frac{\bar{a}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} \\
&= \phi^L(\bar{a}; k^o) + \phi^L(\bar{b}; k^o) \\
&\geqq \phi^L(A; k^o) + \phi^L(B; k^o)
\end{aligned}$$

Next we show that $\phi^L(\cdot; k^o)$ is positively homogeneous. Let $\lambda > 0$ be any real number,

$$\begin{aligned}
\phi^L(\lambda A; k^o) &= \sup\{t \in \mathbb{R} \mid \lambda A \subseteq t k^o + \mathbb{R}^n_+\} \\
&= \sup\{t \in \mathbb{R} \mid A \subseteq (t/\lambda) k^o + \mathbb{R}^n_+\} \\
&= \sup\{\lambda t' \in \mathbb{R} \mid A \subseteq t' k^o + \mathbb{R}^n_+\} \\
&= \lambda \phi^L(A; k^o).
\end{aligned}$$

For $\lambda = 0$, it obvious that (13) holds. By a similar way, we could show that (12) and (13) hold for $\phi^U(\cdot, k^o)$. □

Corollary 3.1 *Let $A, B \in \mathcal{C}(\mathbb{R})$ be any intervals and let $k^o \in \operatorname{int} \mathbb{R}_+$ be any positive real number. Then it holds that*

$$\phi^i(A + B; k^o) = \phi^i(A; k^o) + \phi^i(B; k^o), \quad i = L, U.$$

Proof We omit the proof. □

The following theorem shows that the set functions $\phi^L(\cdot; k^o)$ and $\phi^U(\cdot; k^o)$ are monotone increasing with respect to the set orderings $\leqq_i$, $\prec_i$, $i = L, U$ for any given $k^o \in \operatorname{int} \mathbb{R}^n_+$.

Theorem 3.2 *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets, and let $k^o \in \operatorname{int} \mathbb{R}^n_+$ be any vector and $\lambda \in \mathbb{R}_+$ be any positive number. Then it holds that*

$$\phi^L(A;k^o) \leqq \phi^L(B;k^o) \text{ if } A \leqq_L B, \tag{14}$$
$$\phi^U(A;k^o) \leqq \phi^U(B;k^o) \text{ if } A \leqq_U B, \tag{15}$$
$$\phi^L(A;k^o) < \phi^L(B;k^o) \text{ if } A \prec_L B, \tag{16}$$
$$\phi^U(A;k^o) < \phi^U(B;k^o) \text{ if } A \prec_U B. \tag{17}$$

Proof First we show that (14) holds. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any elements such that $A \leqq_L B$ holds. Since the set orderings $\leqq_L$ is a quasi ordering, from Lemma 3.1, we have $\phi^L(A;k^o)k^o \leqq_L A \leqq_L B$, which implies that $\phi^L(A;k^o) \leqq \phi^L(B;k^o)$. Second we show that (15) holds. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any elements such that $A \leqq_U B$ holds. Since the set ordering $\leqq_U$ is quasi ordering, from Lemma 3.1, we have $\phi^U(A;k^o)k^o \leqq_U A \leqq_U B$, which implies that $\phi^U(A;k^o) \leqq \phi^U(B;k^o)$.

Third, we show that (16) holds. Note that there exists a vector $\bar{b} \in B$ such that $\phi^L(B;k^o) = \phi^L(\bar{b};k^o)$. Since $A \prec_L B$, there exists a vector $\bar{a} \in A$ and a real number $\varepsilon > 0$ such that $\bar{b} = \bar{a} + \varepsilon k^o$ holds. Therefore, we have $\phi^L(B;k^o) = \phi^L(\bar{b};k^o) = \phi^L(\bar{a};k^o) + \varepsilon > \phi^L(A;k^o)$.

Finally, we show (17) holds. Let $\bar{a} \in A$ be any vector such that $\phi^U(A;k^o) = \phi^U(\bar{a};k^o)$. Then there exist a vector $\bar{b} \in B$ and a real number $\varepsilon > 0$ such that $\bar{a} = \bar{b} - \varepsilon k^o$ holds. Hence we have $\phi^U(A;k^o) = \phi^U(\bar{b};k^o) - \varepsilon < \phi^U(\bar{b};k^o) \leqq \phi^U(B;k^o)$. □

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be any set-valued map with compact image, $k^o \in \operatorname{int} \mathbb{R}^m_+$ be any point, and let $S \subseteq \operatorname{Dom}(F)$ be any non-empty set. We define real-valued functions $h^i(\cdot;k^o) : S \to \mathbb{R}$ by $h^i(x;k^o) := \phi^i(F(x);k^o)$, $i = L, U$. Then we have the following theorem.

Theorem 3.3 *Suppose that the set-valued map $F : S \rightsquigarrow \mathbb{R}^m$ is convex-valued and compact-valued and S is a convex set. If F is $\leqq$-concave on S, then real-valued functions $h^i(\cdot;k^o) : S \to \mathbb{R}$, $i = L, U$ are concave on S.*

Proof Let $x, y \in S$ be any elements and $\lambda \in [0, 1]$ be any real number. By assumptions, since the set S is convex and F is $\leqq$-concave, from Theorem 3.1 and 3.2, we have

$$\begin{aligned} h^i((1-\lambda)x + \lambda y; k^o) &= \phi^i(F((1-\lambda)x + \lambda y); k^o) \\ &\geqq \phi^i((1-\lambda)F(x) + \lambda F(y); k^o) \\ &\geqq (1-\lambda)\phi^i(F(x);k^o) + \lambda\phi^i(F(y);k^o) \\ &= (1-\lambda)h^i(x;k^o) + \lambda h^i(y;k^o), \end{aligned}$$

which implies that $h^i(\cdot;k^o)$ is concave on S, $i = L, U$. □

Theorem 3.4 *Suppose that S is compact, the set-valued map $F : S \rightsquigarrow \mathbb{R}^m$ is compact-valued and uniformly compact on S. If the set-valued map F is continuous on S, then functions $h^i(\cdot;k^o)$, $i = L, U$ are continuous on S.*

Proof From Lemma 3.1, it holds that $h^L(x;k^o) = \min\{\phi^L(z;k^o) \mid z \in F(x)\}$ and $h^U(x;k^o) = \max\{\phi^U(z;k^o) \mid z \in F(x)\}$ for $\forall x \in S$. By assumptions, since F is continuous on S and uniformly compact on S, $\phi^i(\cdot;k^o),\ i = L, U$ are continuous as functions defined on $\mathbb{R}^\ell$. Hence, $h^i(\cdot, k^o),\ i = L, U$ are continuous on S (See Hogan [9], Theorems 5 and 6). □

4 Bi-Matrix Game with Set Payoffs and Its Equilibrium Strategy

Let $I,\ J$ denote players and let $M := \{1, 2, \ldots, m\}$ and $N := \{1, 2, \ldots, n\}$ be the sets of all pure strategies available for players I and J, respectively. We denote the sets of all mixed strategies available for players I and J by $S_I := \{x := (x_1, x_2, \ldots, x_m) \in R_+^m \mid x_i \geqq 0,\ i = 1, 2, \ldots, m,\ \sum_{i=1}^m x_i = 1\}$, $S_J := \{y := (y_1, y_2, \ldots, y_n) \in R_+^n \mid y_j \geqq 0,\ j = 1, 2, \ldots, n,\ \sum_{j=1}^n y_j = 1\}$.

By $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell)$, we denote the payoffs that player I receives and J receives when player I plays the pure strategy i and player J plays the pure strategy j, respectively. We set $\mathcal{A} := (A_{ij})$ and $\mathcal{B} := (B_{ij})$, where $\mathcal{A}$ and $\mathcal{B}$ are $m \times n$ matrices whose i, j th elements are A_{ij} and B_{ij}, respectively.

Now we define bi-matrix game with set payoffs by $\Gamma := \langle\{I, J\}, S_I \times S_J, \{\mathcal{A}, \mathcal{B}\}\rangle$ or

		Player J			
		1	2	$\cdots$	n
Player I	1	(A_{11}, B_{11})	(A_{12}, B_{12})	$\cdots$	(A_{1n}, B_{1n})
	2	(A_{21}, B_{21})	(A_{22}, B_{22})	$\cdots$	(A_{2n}, B_{2n})
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	m	(A_{m1}, B_{m1})	(B_{m2}, B_{m2})	$\cdots$	(A_{mn}, B_{mn})

Let $x \in S_I$ and $y \in S_J$ be any mixed strategies. For each player I and J, we define the set-valued payoff maps $F, G : S_I \times S_J \rightsquigarrow \mathbb{R}^\ell$ by $F(x, y) := \sum_{i=1}^m \sum_{j=1}^n x_i A_{ij} y_j$ and $G(x, y) := \sum_{i=1}^m \sum_{j=1}^n x_i B_{ij} y_j$, which are called expected payoffs.

Player I is said to be L-type, U-type, and LU-type if he maximizes his expected payoff $F(x, y)$ with respect to the set-orderings $\leqq_L,\ \leqq_U$, and $\leqq$ for given $y \in S_J$ and player J is said to be L-type, U-type, and LU-type if he maximizes his expected payoff $G(x, y)$ with respect to the set-orderings $\leqq_L, \leqq_U$, and $\leqq$ for given $x \in S_I$. By $\Gamma(\leqq_L, \leqq_U)$, we denote the game where player I is L-type and J is U-type. Then, set-payoff game Γ is classified into the following five set payoff games $\Gamma(\leqq_L, \leqq_L)$, $\Gamma(\leqq_L, \leqq_U)$, $\Gamma(\leqq_U, \leqq_L)$, $\Gamma(\leqq_U, \leqq_U)$, and $\Gamma(\leqq, \leqq)$.

In the above games, each player knows his/her own type, but does not know the other player's type. Therefore, set-payoff games are considered to be incomplete information games and Nash equilibrium strategy is characterized by Bayesian Nash equilibrium strategy (see [6]).

While, from Theorem 3.2, noting that for any $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \leqq B$ implies that $\phi^L(A; k^o) \leqq \phi^L(B; k^o)$ and $\phi^U(A; k^o) \leqq \phi^U(B; k^o)$ hold, we may assume that both players use the set-ordering $\leqq$ to maximize their expected payoffs and this is common knowledge between players. Moreover, from practical point of views, our approach is useful to study bi-matrix with fuzzy vector payoffs. Hence, in the following, we assume that both players I and J are LU type and this is a common knowledge for the players.

For each $(x, y) \in S_I \times S_J$, we set $\mathcal{F}(y) := \{F(x, y) \mid x \in S_I\}$ and $\mathcal{G}(x) := \{G(x, y) \mid y \in S_J\}$ and denote the sets of all maximal elements in $\mathcal{F}(y)$ with respect to the set orderings $\leqq, \preceq$ and $\prec$ by $\mathcal{F}(y)^{\leqq}$, $\mathcal{F}(y)^{\preceq}$, and $\mathcal{F}(y)^{\prec}$, respectively. Similarly, we denote the sets of all maximal elements in $\mathcal{G}(x)$ with respect to the set orderings $\leqq, \preceq$ and $\prec$ by $\mathcal{G}(x)^{\leqq}$, $\mathcal{G}(x)^{\preceq}$, and $\mathcal{G}(x)^{\prec}$. By definitions, for each $(x, y) \in S_I \times S_J$, it holds that $\mathcal{F}(y)^{\leqq} \subseteq \mathcal{F}(y)^{\preceq} \subseteq \mathcal{F}(y)^{\prec}$ and $\mathcal{G}(x)^{\leqq} \subseteq \mathcal{G}(x)^{\preceq} \subseteq \mathcal{G}(x)^{\prec}$.

Now we define the concept of Nash equilibrium strategies to game Γ.

Definition 4.1 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Nash equilibrium strategy to game Γ if it holds that

(i) $F(x, y^*) \leqq F(x^*, y^*), \quad \forall x \in S_I,$
(ii) $G(x^*, y) \leqq G(x^*, x^*), \quad \forall y \in S_J.$

The pair of sets $(F(x^*, y^*), G(x^*, y^*))$ is said to be the value of game Γ.

We define set-valued maps $\mathcal{B}_I : S_J \rightsquigarrow S_I, \mathcal{B}_J : S_I \rightsquigarrow S_J$ and $\mathcal{B} : S_I \times S_J \rightsquigarrow S_I \times S_J$ by $\mathcal{B}_I(y) := \{x \in S_I \mid F(u, y) \leqq F(x, y), \quad \forall u \in S_I\}$, $\mathcal{B}_J(x) := \{y \in S_J \mid G(x, v) \leqq G(x, y), \ \forall v \in S_J\}$ and $\mathcal{B}(x, y) := \mathcal{B}_I(y) \times \mathcal{B}_J(x)$. Then, it is obvious that the pair of strategies $(x, y) \in S_I \times S_J$ is a Nash equilibrium if and only if $(x, y) \in \mathcal{B}(x, y)$ holds.

Example 4.1 We consider the following bi-matrix game with interval-valued payoffs. In Game 1 (Fig. 1), there is no pair of pure strategies such that the pair is a Nash equilibrium. We show that there exists a unique mixed Nash equilibrium in Game 1. Let $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$ be any strategies. Then, by simple calculations, we have

$$F(x, y) = [(1 - 2y_1)x_1 + 3y_1 + 1, (1 - 2y_1)x_1 + 3y_1 + 3], \tag{18}$$
$$G(x, y) = [(2x_1 - 1)y_1 + x_1 + 3, (2x_1 - 1)y_1 + x_1 + 5]. \tag{19}$$

From (18) and (19), we have the the following best response maps:

$$\mathcal{B}_I(y_1, y_2) = \begin{cases} \{(1,0)\} & \text{if } y_1 \in [0, 1/2), \\ S_I & \text{if } y_1 = 1/2, \\ \{(0,1)\} & \text{if } y_1 \in (1/2, 1], \end{cases} \quad \text{and } \mathcal{B}_J(x_1, x_2) = \begin{cases} \{(0,1)\} & \text{if } x_1 \in [0, 1/2), \\ S_J & \text{if } x_1 = 1/2, \\ \{(1,0)\} & \text{if } x_1 \in (1/2, 1]. \end{cases}$$

Then, we have $((0.5, 0.5), (0.5, 0.5)) \in \mathcal{B}_I((0.5, 0.5)) \times \mathcal{B}_J((0.5, 0.5))$, which implies that the pair of strategies $\{(0.5, 0.5), (0.5, 0.5)\}$ is a unique Nash equilibrium strategy in Game 1.

		Player J	
		C	D
Player I	C	$([3,5],[5,7])$	$([2,4],[4,6])$
	D	$([4,6],[2,4])$	$([1,3],[3,5])$

Fig. 1 Game 1

The following example shows that there is no Nash equilibrium strategy in set-payoff games in general.

Example 4.2 We consider the following bi-matrix game with interval-valued payoffs (Fig. 2). For any $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, we have

$$F(x, y) = [(3 - 2y_1)x_1 - 2y_1 + 4, (y_1 - 2)x_1 - 4y_1 + 10].$$

Then, it holds that $\mathcal{B}_I(y) = \emptyset,\ \forall y \in S_J$. Therefore, there is no Nash equilibrium in Game 2.

Based on the above example, we introduce three types of concepts of Nash equilibrium strategies.

Definition 4.2 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a maximal Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\leqq} \times \mathcal{G}(x^*)^{\leqq}$.

Definition 4.3 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Pareto Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\preceq} \times \mathcal{G}(x^*)^{\preceq}$.

		Player J	
		C	D
Player I	C	$([4,5],[5,7])$	$([7,8],[4,6])$
	D	$([2,6],[2,4])$	$([4,10],[3,5])$

Fig. 2 Game 2

Definition 4.4 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a weak Pareto Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\prec} \times \mathcal{G}(x^*)^{\prec}$.

It is easy to see that pairs of pure strategies $\{(C, C)\}$ and $\{(D, D)\}$ are maximal Nash equilibriums in Game 2 (Fig. 2).

Example 4.3 Consider the following bi-matrix game with interval-valued payoffs (Fig. 3).

It is easy to see that the pairs of the pure strategies $\{(C, C)\}$ and $\{(D, D)\}$ are Nash equilibrium strategies in Game 3. For each $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, the set-valued payoff maps for players I and J are given by

$$F(x, y) = [2(5y_1 - 1)x_1 - y_1 + 3, 2x_1y_1 + 5y_1 + 4], \tag{20}$$

$$G(x, y) = [2(5x_1 - 1)y_1 - x_1 + 3, 2x_1y_1 + 5x_1 + 4]. \tag{21}$$

The pair of strategy $\{(1/6, 5/6), (1/6, 5/6)\}$ is a maximal Nash equilibrium strategy in Game 3. But there are infinite number of maximal Nash equilibrium strategies in Game 3, and the set of all maximal Nash equilibrium strategies is given by $\{(x, 1 - x), (y, 1 - y) \in S_I \times S_J \mid 0 < x < 1/5,\ 0 < y < 1/5\} \cup \{(C, C)\} \cup \{(D, D)\}$.

Let $k^o \in \text{int } \mathbb{R}^\ell_+$ be any point and $\lambda_i, \mu_i \in \mathbb{R}_+$, $i = L, U$ be any real numbers such that $\lambda_L + \lambda_U = \mu_L + \mu_U = 1$. Now we define real-valued functions $f, g : S_I \times S_J \to \mathbb{R}$ by

$$f(x, y; k^o, \lambda_L, \lambda_U) := \lambda_L \phi^L(F(x, y); k^o) + \lambda_U \phi^U(F(x, y); k^o),$$
$$g(x, y; k^o, \mu_L, \mu_U) := \mu_L \phi^L(G(x, y); k^o) + \mu_U \phi^U(G(x, y); k^o).$$

Associated with game Γ, we define the following two person non-cooperative game with scalar payoffs $\Gamma(k^o, \lambda_L, \lambda_U\ \mu_L, \mu_U)$ by

$$\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) := \langle\ \{I,\ J\}, S_I \times S_J, \{f(\cdot, \cdot; k^o, \lambda_L, \lambda_U), g(\cdot, \cdot; k^o, \mu_L, \mu_U)\}\ \rangle.$$

Player J

		C	D
Player I	C	([10, 11], [10, 11])	([1, 4], [2, 9])
	D	([2, 9], [1, 4])	([3, 4], [3, 4])

Fig. 3 Game 3

We assume that λ_i and μ_i, $i = L, U$ are common knowledge in $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Definition 4.5 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ if it holds that

(i) $f(x, y^*; k^o, \lambda_L, \lambda_U) \leqq f(x^*, y^*; k^o, \lambda_L, \lambda_U) \quad \forall x \in S_I$,
(ii) $g(x^*, y; k^o, \mu_L, \mu_U) \leqq g(x^*, y^*; k^o, \mu_L, \mu_U) \quad \forall y \in S_J$.

Definition 4.6 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ if it holds that

(i) $f(x, y^*; k^o, \lambda_L, \lambda_U) < f(x^*, y^*; k^o, \lambda_L, \lambda_U) \quad \forall x \in S_I,\ x \neq x^*$,
(ii) $g(x^*, y; k^o, \mu_L, \mu_U) < g(x^*, y^*; k^o, \mu_L, \mu_U) \quad \forall y \in S_J,\ y \neq y^*$.

The following theorem holds between game Γ and game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Theorem 4.1 *Let $(x^*, y^*) \in S_I \times S_J$ be any pair of strategies to game Γ. Then, if the pair of strategies $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, then it is a weak Pareto Nash equilibrium to game Γ.*

Proof Suppose that there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \prec F(\bar{x}, y^*)$ holds. Then, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L) < f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$, which contradicts that (x^*, y^*) is Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \prec G(x^*, \bar{y})$ holds. Then, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) < g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. □

Theorem 4.2 *Let $(x^*, y^*) \in S_I \times S_J$ be any pair of strategies and suppose that $\lambda_i, \mu_i \in \operatorname{int}\mathbb{R}_+$, $i = L, U$ are positive numbers in game $\Gamma(k^o, \lambda_L, \lambda_U\, \mu_L, \mu_U)$. Then we have the following:*

(i) *If the pair of strategies (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it is a Pareto Nash equilibrium to game Γ.*
(ii) *If the pair of strategies (x^*, y^*) is a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it is a maximal Nash equilibrium to game Γ.*

Proof First, we show that (i) holds. On the contrary, we suppose that the pair of strategies (x^*, y^*) is not a Pareto Nash equilibrium to game Γ. Then there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \preceq F(\bar{x}, y^*)$ holds. Since $\lambda_i > 0$, $i = L, U$, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L) < f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \preceq G(x^*, \bar{y})$ holds. Since $\mu_i > 0$, $i = L, U$, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) < g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Next we show that (ii) holds. Suppose that there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \leqq F(\bar{x}, y^*)$ holds. Since $\lambda_i > 0$, $i = L, U$, and (x^*, y^*) is a strict

Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U) = f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$ and $x^* = \bar{x}$. Therefore, we have $F(\bar{x}, y^*) \leqq F(x^*, y^*)$.

Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \leqq G(x^*, \bar{y})$ holds. Then, since $\mu_i > 0$, $i = L, U$ and (x^*, y^*) is a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) = g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$ and $y^* = \bar{y}$. Therefore, we have $G(\bar{x}, y^*) \leqq G(x^*, y^*)$. □

From Theorems 4.1 and 4.2, by varying parameters λ_i, μ_i, $i = L, U$, we could obtain another maximal Nash, Pareto Nash and weak Pareto Nash equilibrium strategies to game Γ.

We consider Game 3 given in Example 4.3 again. Let $k^o = 1 \in \mathbb{R}$ and $\lambda_i, \mu_i \in \operatorname{int}\mathbb{R}_+$, $i = L, U$ be any positive numbers such that $\lambda_L + \lambda_U = \mu_L + \mu_U = 1$. Then, for each $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, real-valued payoff functions $f, g : S_I \times S_J \to \mathbb{R}$ for each player I and J are given by

$$f(x, y; k^o, \lambda_L, \lambda_U) := 2\{(5\lambda_L + \lambda_U)y_1 - 2\lambda_L\}x_1 - (\lambda_L - 5\lambda_U)y_1 + (3\lambda_L + 4\lambda_U),$$
$$g(x, y; k^o, \mu_L, \mu_U) := 2\{(5\mu_L + \mu_U)x_1 - 2\mu_L\}y_1 - (\mu_L - 5\mu_U)x_1 + (3\mu_L + 4\mu_U).$$

We set $x_1^* := \mu_L/(5\mu_L + \mu_U) \in (0, 1/5)$ and $y_1^*; = \lambda_L/(5\lambda_L + \lambda_U) \in (0, 1/5)$. Then, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a Nash equilibrium in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. From Theorem 4.2, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a Pareto Nash equilibrium in game Γ. We show that the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a maximal Nash equilibrium in game Γ. Suppose that there exists a strategy $(x_1, x_2) \in S_I$ such that $F((x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)) \leqq F((x_1, x_2), (y_1^*, 1 - y_2^*))$. Then, by simple calculations, we have $(x_1, x_2) = (x_1^*, 1 - x_1^*)$. Similarly, we could show that $G((x_1^*, 1 - x^*), (y_1^*, 1 - y_1^*)) \leqq G((x_1^*, 1 - x_1^*), (y_1, y_2))$ implies $(y_1, y_2) = (y_1^*, 1 - y_1^*)$. Therefore, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a maximal Nash equilibrium in game Γ.

Note that, the scalar-payoff game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ induced from set-payoff game Γ, is the game with incomplete informations. Because in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, player I does not know μ_L and μ_U, while player J does not know the value of λ_L and λ_U which are necessary for each player to find best response strategies. Moreover, each player may choose different k^o in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Therefore, the Nash equilibrium strategy in the scalar-payoff game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is a Bayesian Nash equilibrium (See Harsanyi [6]).

5 Existence of Nash Equilibrium Strategy to Game Γ

In the previous section, for any given set payoff bi-matrix games, we define two person games with scalar-valued payoff functions, and investigate relationships between these games. In this section, we shall give some conditions under which there exists at least one maximal Nash, Pareto Nash and weak Pareto Nash equilibrium strategies to game Γ.

Lemma 5.1 *In game* Γ, *suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets. Then, it holds that*

(i) $(1-\lambda)F(x^1, y) + \lambda F(x^2, y) = F((1-\lambda)x^1 + \lambda x^2, y) \ \ \forall x^1, x^2 \in S_I,\ \forall y \in S_J,\ \ \forall \lambda \in [0, 1]$,
(ii) *The set-valued map* $F : S_I \times S_J \rightsquigarrow \mathbb{R}^\ell$ *is continuous on* $S_I \times S_J$,
(iii) F *is uniformly compact on* $S_I \times S_J$,
(iv) $(1-\lambda)G(x, y^1) + \lambda G(x, y^2) = G(x, (1-\lambda)y^1 + \lambda y^2) \ \ \forall x \in S_I,\ \forall y^1, y^2 \in S_J,\ \ \forall \lambda \in [0, 1]$,
(v) *The set-valued map* $G : \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ *is continuous on* $S_I \times S_J$,
(vi) G *is uniformly compact on* $S_I \times S_J$.

Proof We shall show that (i), (ii) and (iii) hold. Let $(x^1, y), (x^2, y) \in S_I \times S_J$ be any strategies and $\lambda \in [0, 1]$ be any real number. Then by simple calculations, we have

$$\begin{aligned}(1-\lambda)F(x^1, y) + \lambda F(x^2, y) &= (1-\lambda)\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^1 A_{ij} y_j + \lambda \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^2 A_{ij} y_j \\ &= (1-\lambda)\sum_{i=1}^{m} x_i^1 \sum_{j=1}^{n} A_{ij} y_j + \lambda \sum_{i=1}^{m} x_i^2 \sum_{j=1}^{n} A_{ij} y_j \\ &= \sum_{i=1}^{m} \{(1-\lambda)x_i^1 + \lambda x_i^2\} \sum_{j=1}^{n} A_{ij} y_j \\ &= F((1-\lambda)x^1 + \lambda x^2, y).\end{aligned}$$

Next we shall show that (ii) holds. First, we shall show that $F(\cdot, \cdot)$ is upper semi-continuous on $S_I \times S_J$. Let $\{(x^\nu, y^\nu, z^\nu)\}_{\nu=1}^\infty \subseteq \mathrm{Gr}(F)$ be any sequence converging to $(x^o, y^o, z^o) \in S_I \times S_J \times \mathbb{R}^\ell$. By Definition, for each ν, there exits an $a_{ij}^\nu \in A_{ij},\ i \in M,\ j \in N$ such that $z^\nu = \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^\nu y_j^\nu$. Since A_{ij} is compact, without loss of any generality we assume that $\{a_{ij}^\nu\}_{\nu=1}^\infty$ converges to some point $a_{ij}^o \in A_{ij}$. Therefore, we have $z^o \in F(x^o, y^o)$.

Second we shall show that $F(\cdot, \cdot)$ is lower semi-continuous on $S_I \times S_J$. Let $\{(x^\nu, y^\nu)\}_{\nu=1}^\infty \subseteq S_I \times S_J$ be any sequence converging to $(x^o, y^o) \in S_I \times S_J$ and $z^o \in F(x^o, y^o)$ be any point. Then there exists $a_{ij}^o \in A_{ij}$ such that $z^o = \sum_{i=1}^m \sum_{j=1}^n x_i^o a_{ij}^o y_j^o$. By setting $z^\nu := \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^o y_j^\nu$, we have $z^\nu \in F(x^\nu, y^\nu)$ and $z^\nu \to z^o$, which implies that $F(\cdot, \cdot)$ is lower semi-continuous on $S_I \times S_J$. From the above, $F(\cdot, \cdot)$ is continuous on $S_I \times S_J$.

Finally we show that (iii) holds. In order to show that F is uniformly compact on $S_I \times S_J$, it suffices to show that $F(S_I, S_J)$ is compact. Let $\{z^\nu\}_{\nu=1}^\infty \subseteq F(S_I, S_J)$ be any sequence. By definition, for each ν, there exist points $a_{ij}^\nu \in A_{ij}, x_i^\nu \in S_I$ and $y_j^\nu \in S_J,\ i \in M,\ j \in N$ such that $z^\nu = \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^\nu y_j^\nu$. By assumptions, A_{ij}, S_I and S_J are compact, without loss of any generality, we may assume that $x_i^\nu \to x_i^o,\ y_j^\nu \to y_j^o$, and $a_{ij}^\nu \to a_{ij}^o,\ i \in M,\ j \in N$, which implies that $z^\nu \to z^o \in$

$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^o a_{ij}^o y_j^o \in F(S_I, S_J)$. By a similar way, we could show that (iv), (v), and (vi) hold. □

From Theorems 3.3, 3.4 and Lemma 5.1, we have the following lemma.

Lemma 5.2 *Suppose that $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M, j \in N$ are compact convex sets in game Γ. Then, it holds that*

(i) *$f(\cdot, y; k^o, \lambda_L, \lambda_U) : S_I \to \mathbb{R}$ is concave and $f(\cdot, \cdot; k^o, \lambda_L, \lambda_U)$ is continuous on $S_I \times S_J$,*
(ii) *$g(x, \cdot; k^o, \mu_L, \mu_U) : S_J \to \mathbb{R}$ is concave and $g(\cdot, \cdot; k^o, \mu_L, \mu_U)$ is continuous on $S_I \times S_J$.*

Let $(x, y) \in S_I \times S_J$ be any pair of strategies in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. We define the set-valued maps $B_I(\cdot; k^o, \lambda_L, \lambda_U) : S_J \rightsquigarrow S_I$ and $B_J(\cdot; k^o, \mu_L, \mu_U) : S_I \rightsquigarrow S_J$ by $B_I(y; k^o; \lambda_L, \lambda_U) := \{u \in S_I \mid f(u, y; k^o, \lambda_L, \lambda_U) \geqq f(x, y; k^o, \lambda_L, \lambda_U)\ \forall x \in S_I\}$ and $B_J(x; k^o, \mu_L, \mu_U) := \{v \in S_J \mid g(x, v; k^o, \mu_L, \mu_U) \geqq g(x, y; k^o, \mu_L, \mu_U)\ \forall y \in S_J\}$, which are called the best response maps for players I and J, respectively.

From Lemma 5.2, we have the following lemmas.

Lemma 5.3 *Suppose that $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M, j \in N$ are compact convex sets in game Γ. Then, it holds that*

(i) *$B_I(y; k^o, \lambda_L, \lambda_U)$ and $B_J(x; k^o, \mu_L, \mu_U)$ are non-empty, compact and convex set for each $(x, y) \in S_I \times S_J$.*
(ii) *$B_I(\cdot; k^o, \lambda_L, \lambda_U) : S_J \rightsquigarrow S_I$ and $B_J(\cdot; k^o, \mu_L, \mu_U) : S_I \rightsquigarrow S_J$ are upper semi-continuous on S_J and S_I respectively.*

Proof First we show that (i) holds. From Lemma 5.2, for each $y \in S_J$, $f(\cdot, y; k^o, \lambda_L, \lambda_U)$ is concave and continuous on S_I. Since S_I is compact and convex, it holds that $B_I(y; k^o, \lambda_L, \lambda_U)$ is non-empty, compact and convex for all $y \in S_J$. Similarly, we could show that (i) holds for $B_J(x; k^o, \mu_L, \mu_U)$.

Next, we prove that (ii) holds for $B_I(\cdot; k^o, \lambda_L, \lambda_U)$. Let $\{(x^\nu, y^\nu)\} \subseteq \mathrm{Gr}(B_I(\cdot; k^o, \lambda_L, \lambda_U))$ be any sequence converging to $(x^o, y^o) \in S_I \times S_J$. Then it holds that $f(x^\nu, y^\nu; k^o, \lambda_L, \lambda_U) \geqq f(u, y^\nu; k^o, \lambda_L, \lambda_U)$ for $\forall u \in S_I$. From Lemma 5.2, since $f(\cdot, \cdot; k^o, \lambda_L, \lambda_U)$ is continuous on $S_I \times S_J$, we have $f(x^o, y^o; k^o, \lambda_L, \lambda_U) \geqq f(u, y^o; k^o, \lambda_L, \lambda_U)$ for $\forall u \in S_I$, which implies that $x^o \in B_I(y^o; k^o, \lambda_L, \lambda_U)$. Similarly, we could show that (ii) holds for $B_J(\cdot; k^o, \mu_L, \mu_U)$. □

Now we define the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) : S_I \times S_J \rightsquigarrow S_I \times S_J$ by $B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) := B_I(y; k^o, \lambda_L, \lambda_U) \times B_J(x; k^o, \mu_L, \mu_U)$. Then from Lemma 5.3, we have the following lemma.

Lemma 5.4 *Suppose that $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M, j \in N$ are compact convex sets in game Γ. Then, it holds that*

(i) *$B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is non-empty, compact and convex for each $(x, y) \in S_I \times S_J$.*

(ii) *The set-valued map* $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) : S_I \times S_J \rightsquigarrow S_I \times S_J$ *is upper semi-continuous on* $S_I \times S_J$.

Lemma 5.5 *The pair of strategies* $(x^*, y^*) \in S_I \times S_J$ *is a Nash equilibrium to game* $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ *if and only if* $(x^*, y^*) \in B(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ *holds (See Kakutani [11]).*

Lemma 5.5 shows that a pair of strategies (x^*, y^*) is a Nash equilibrium if and only if it is a fixed point of the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. From the above lemmas, we have the following theorem.

Theorem 5.1 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M, \ j \in N$ *are compact convex sets in game* Γ. *Then, there exists at least one Pareto Nash equilibrium strategy in game* Γ.

Proof Let $k^o \in \operatorname{int} \mathbb{R}^\ell_+$ be any point and $\lambda_i, \ \mu_i \in \mathbb{R}_+, \ i = L, U$ be any positive numbers. Then, from Lemma 5.3, $B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is non-empty, compact and convex for each $(x, y) \in S_I \times S_J$ and the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is upper semi-continuous on $S_I \times S_J$. Therefore, from Kakutani's fixed point theorem [11], there exists at least one point $(x^*, y^*) \in S_I \times S_J$ such that $(x^*, y^*) \in B(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Therefore, from Lemma 5.5 and Theorem 4.2, the point (x^*, y^*) is a Pareto Nash equilibrium strategy in game Γ. □

Theorem 5.2 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}), i \in M, \ j \in N$ *are compact convex sets in game* Γ. *Then there exists at least one maximal Nash equilibrium strategy in game* Γ.

Proof Without loss of any generality, we assume that $k^o = 1, \ \lambda_i = \mu_i = 1, i = L, U$ and $A_{ij} = [a^L_{ij}, a^U_{ij}]$ and $B_{ij} = [b^L_{ij}, b^U_{ij}], \ i \in M, \ j \in N$. Then, from Corollary 3.1,

$$
\begin{aligned}
f(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) &= \sum_{i=1}^m \sum_{j=1}^n x_i (\phi^L(A_{ij}; k^o) + \phi^U(A_{ij}; k^o)) y_j, \\
&= \sum_{i=1}^m \sum_{j=1}^n x_i (a^L_i + a^U_{ij}) y_j, \qquad (22) \\
g(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) &= \sum_{i=1}^m \sum_{j=1}^n x_i (\phi^L(B_{ij}; k^o) + \phi^U(B_{ij}; k^o)) y_j \\
&= \sum_{i=1}^m \sum_{j=1}^n x_i (b^L_{ij} + b^U_{ij}) y_j, \qquad (23)
\end{aligned}
$$

Namely, game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is a bi-matrix game with scalar valued payoffs. Let $(x^*, y^*) \in S_I \times S_J$ be any Nash equilibrium strategy to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Suppose that there exists a strategy $\bar{x} \in S_I$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*[a_{ij}^L, a_{ij}^U]y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i[a_{ij}^L, a_{ij}^U]y_j^*. \tag{24}$$

Then by Definition 2.1, it holds

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^L y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^L y_j^*, \tag{25}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^U y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^U y_j^*. \tag{26}$$

Since (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it must hold that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*(a_{ij}^L + a_{ij}^U)y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i(a_{ij}^L + a_{ijj}^U)y_j^*. \tag{27}$$

From (25), (26) and (27), it follows that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^L y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^L y_j^*$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^U y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^U y_j^*,$$

which implies that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i[a_{ij}^L, a_{ij}^U]y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*[a_{ij}^L, a_{ij}^U]y_j^*.$$

Namely, $F(x^*, y^*)$ is a maximal element in $\mathcal{F}(y^*)^{\leqq}$. By a similar way, we could show that $G(x^*, y^*)$ is a maximal element in $\mathcal{G}(x^*)^{\leqq}$. Hence (x^*, y^*) is a maximal Nash equilibrium strategy in game Γ. □

Theorem 5.3 *Suppose that $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ are compact convex sets and that $\mathcal{B}(x, y) \neq \emptyset$ for each $(x, y) \in S_I \times S_J$ holds in game Γ. Then, there exists a Nash equilibrium in game Γ.*

Proof From Kakutani's fixed point theorem, it suffices to show that the set-valued map $\mathcal{B}$ is convex-valued and upper semi-continuous on $S_I \times S_J$.

First we show that set-valued map $\mathcal{B}_I$ is convex-valued. Let $y \in S_J$ be any element. Then from Lemma 5.1, for each $x^1, x^2 \in \mathcal{B}_I(y)$ and any $\lambda \in [0, 1]$, it holds that

$$(1-\lambda)F(x^1, y) + \lambda F(x^2, y) = F((1-\lambda)x^1 + \lambda x^2, y) \subseteq F(u, y) + \mathbb{R}^\ell_+ \quad \forall u \in S_I,$$
$$F(u, y) \subseteq (1-\lambda)F(x^1, y) + \lambda F(x^2, y) = F((1-\lambda)x^1 + \lambda x^2, y) - \mathbb{R}^\ell_+ \quad \forall u \in S_I,$$

which implies $(1-\lambda)x^1 + \lambda x^2 \in \mathcal{B}_I(y)$. Namely, the set-valued map $\mathcal{B}_I$ is convex-valued. Similarly, we could prove that the set -valued map $\mathcal{B}_J$ is convex-valued.

Next, we show that set-valued map $\mathcal{B}_I$ is upper semi-continuous on S_J. Let $\{(y^\nu, x^\nu)\} \subseteq S_I \times S_J$ be any sequence converging to (y^o, x^o) such that $x^\nu \in \mathcal{B}_I(y^\nu)$ for $\forall \nu$. It suffices to show that $F(u, y^o) \leqq F(x^o, y^o)$ holds for $\forall u \in S_I$. Let $z^o \in F(x^o, y^o) \subseteq F(u, y^o) + \mathbb{R}^\ell_+$ be any element. From Lemma 5.1, since F is continuous on $S_I \times S_J$, there exists a sequence $\{z^\nu\}$ converging to z^o such that $z^\nu \in F(x^\nu, y^\nu) \subseteq F(u, y^\nu) + \mathbb{R}^\ell_+$. By Definition, it holds that $F(u, y^\nu) \leqq_L F(x^\nu, y^\nu)$ for $\forall u \in S_I$ and for $\forall \nu$. Again, from Lemma 5.1, $F(u, y^\nu)$ is compact, we have $z^o \in F(u, y^o) + \mathbb{R}^\ell_+$, $\forall u \in S_I$, which implies that $F(u, y^o) \leqq_L F(x^o, y^o)$ holds for $\forall u \in S_I$.

Finally we show that $F(u, y^o) \leqq_U F(x^o, y^o)$ holds for $\forall u \in S_I$. Let $u \in S_I$ be any element. From the continuity of F, for any $z^o \in F(u, y^o)$, there exists $z^\nu \in F(u, y^\nu) \subseteq F(x^\nu, y^\nu) - \mathbb{R}^\ell_+$ such that $z^o \in F(u, y^o)$. Since F is compact-valued and continuous on $S_I \times S_J$, we have $z^o \in F(x^o, y^o) - \mathbb{R}^\ell$, which implies that $F(u, y^o) \leqq_U F(x^o, y^o)$ holds for $\forall u \in S_I$. By a similar way we could show that $\mathcal{B}_J$ is upper semi-continuous on S_I. Hence $\mathcal{B}$ is upper semi-continuous on $S_I \times S_J$. □

6 Conclusion

In this paper, we considered set payoff bi-matrix games where payoffs for each player are given by compact convex sets in $\mathbb{R}^\ell$, namely, players don't know the values of payoffs but the ranges of the payoffs. We call this environment deterministic uncertainty. This type of game may encompass interval payoff games, fuzzy payoff games and robust games. First, we define several types of set orderings on the set of all non-empty subsets in n-dimensional Euclidean space $\mathbb{R}^n$. Second, by using these orderings, we define four kinds of concepts of Nash equilibrium strategies, that is, Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium strategies to the games and investigate their properties. In particular, we investigate the relationships between set-payoff games and incomplete information games. Finally, we give sufficient conditions under which there exists these Nash equilibrium strategies in bi-matrix games with set-valued payoffs and necessary condition under which there exists Nash equilibrium strategies in bi-matrix games with interval-valued payoffs.

In this paper, we use the set-orderings $\leqq$, $\preceq$ and $\prec$ to define the concepts of Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium to the games with set payoff. However, it is easy to use other types of set-orderings, say $\leqq_i$, $i = L, U$ etc. to define the concepts of Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium to the games with set payoff and we could derive similar results to the games with set payoffs. Moreover, we could define the incomplete information game with set payoffs, where each player chooses a set-orderings among set-ordering given in Definition 2.1, after that, each players plays the game with set payoff, without knowing the set-ordering which the other player chooses each other. This means that there are deep relationships between set-payoff games and incomplete information games.

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