Generalized Minimality in Set Optimization

Daishi Kuroiwa

Abstract In this paper, we propose a generalized minimality in set optimization. At first, we introduce parametrized embedding functions, which includes the embedding function in the previous literatures. By using the embedding functions, we generalize notions of minimal solutions for set optimization, and give existence results of the generalized minimal solutions. Also we introduce parametrized scalarizing functions which are generalizations of scalarizing functions defined in the previous literatures, and we characterize the generalized minimal solutions by using the scalarizing functions.

Keywords Set optimization · Embedding approach · Unification and generalization of minimal solution · Existence of minimal solution · Generalized scalarizing function

1 Introduction

We study the following optimization problem (SP):

$$\begin{array}{ll}\text{(SP)} & \text{Minimize} \quad F(x) \\ & \text{subject to} \quad x \in X,\end{array}$$

where X is a nonempty set, F is a set-valued map from X to an ordered vector space E. Notions of minimal solutions of (SP) are defined in accordance with *set relations*, which are binary relations on the power set of E, e.g., see [12]. Such optimization problem (SP) is called *set optimization*.

For every set relation, notions of minimal solutions of (SP) can be defined. For example, l-minimal and u-minimal solutions are given by using set relations $\preceq_K^l$ and

D. Kuroiwa (✉)
Department of Mathematics, Shimane University, Shimane 690-8504, Japan
e-mail: kuroiwa@math.shimane-u.ac.jp

A.H. Hamel et al. (eds.), *Set Optimization and Applications - The State of the Art*,
Springer Proceedings in Mathematics & Statistics 151,
DOI 10.1007/978-3-662-48670-2_10

$\preceq_K^u$, respectively. We studied various notions and properties in each set relation, that is, notions of weak and proper minimal solutions of (SP), conditions for the existence for such minimal notions, duality results for such minimal notions, notions of convex functions for set-valued maps, and notions of derivatives for set-valued maps in each set relation. Therefore, it sometimes takes much time to observe them.

In this paper, we propose a unified approach to study set optimization which covers the study with respect to set relations $\preceq_K^l$ and $\preceq_K^u$, and we define a notion of minimality which is a generalization of l and u-minimality but also s-minimality, see [6]. In Sects. 2 and 3, we give preliminaries about vector and set optimization. In Sect. 4, we introduce parametrized embedding functions by observing behavior of a singleton, which is a generalization of the previous embedding function defined by the author, and we study properties of the parametrized embedding functions. By using the parametrized embedding functions, we define generalized minimal solutions for set optimization, and show existence theorems of the generalized minimal solutions. In Sect. 5, we introduce parametrized scalarizing functions which are generalizations of scalarizing functions defined in the previous literatures. By using the scalarizing functions, we characterize the generalized minimal solutions.

2 Preliminaries—Vector Optimization

Let C be a closed convex cone of a topological vector space E over $\mathbb{R}$ satisfying $C \cap (-C) = \{0\}$ and $\text{int}C \neq \emptyset$, where 0 is the null vector and $\text{int}C$ is the set of all interior points of C. The partial order $\leq_C$ is given by

$$x \leq_C y \text{ if and only if } y - x \in C,$$

and binary relation $<_C$ by

$$x <_C y \text{ if and only if } y - x \in \text{int}C.$$

For any subset A of E, the set of all minimal elements of A with respect to C is written by

$$\begin{aligned} \text{Min}(A \mid C) &= \{a \in A \mid (a - C) \cap A = \{a\}\} \\ &= \{a \in A \mid a' \in A, a' \leq_C a \Rightarrow a \leq_C a'\}, \end{aligned}$$

and the set of all weak minimal elements of A with respect to C is written by

$$\begin{aligned} w\text{Min}(A \mid C) &= \{a \in A \mid (a - \text{int}C) \cap A = \emptyset\} \\ &= \{a \in A \mid \nexists a' \in A \text{ such that } a' <_C a\}. \end{aligned}$$

The positive polar cone of C is given by

$$C^+ = \{c^* \in E^* \mid \langle c^*, c\rangle \geq 0, \forall c \in C\},$$

where E^* is the continuous dual space of E, and it is well known that C^{++}, the second positive polar cone of C, which is given by

$$C^{++} = \{c \in E \mid \langle c^*, c\rangle \geq 0, \forall c^* \in C^*\},$$

coincides with C.

For a nonempty convex subset A of E, $x_0 \in w\text{Min}(A \mid C)$, that is, $A \cap (x_0 - \text{int}C) = \emptyset$ if and only if there exists $c^* \in C^+$ such that $\langle c^*, x_0\rangle = \min_{x \in A} \langle c^*, x\rangle$ by using a separation theorem.

In the nonconvex case, nonlinear scalarization is a well-known tool to study minimal and weak minimal elements. Such scalarizing functions are given as follows:

$$z(x) = \inf\{t \in \mathbb{R} \mid x \in te - C\},$$

or

$$f(x) = \inf\{t \in \mathbb{R} \mid x \in te + a - \text{int}C\},$$

for fixed $e \in C$ and $a \in E$, see [2, 14]. These two scalarizing functions, which are essentially the same because $z(x - a) = f(x)$ under the assumptions of this section, play very important roles to study vector optimization.

3 Preliminaries—Set Optimization

In the rest of the paper, let E be a normed vector space, and $\mathcal{C} := \mathcal{C}(E)$ be the family of all nonempty compact convex subsets of E. For each $A, B \in \mathcal{C}$ and $\lambda \in \mathbb{R}$,

$$A + B = \{x + y \mid x \in A, y \in B\} \text{ and } \lambda A = \{\lambda x \mid x \in A\},$$

and also $A - B = A + (-B)$. It is clear that $\mathcal{C}$ is not a vector space under these operators, because there does not exist $C \in \mathcal{C}$ satisfying $A + C = \{0\}$ for given $A \in \mathcal{C}$ which has at least two points.

Set relations are binary relations on $\mathcal{C}$ based on an ordering cone and these are the most important notions to consider set optimization problems. Throughout the paper, let K be a closed convex cone of E satisfying $K \cap (-K) = \{0\}$ and $\text{int}K \neq \emptyset$. We introduce set relations $\preceq^l_K$ and $\preceq^u_K$ on $\mathcal{C}$: for each $A, B \in \mathcal{C}$,

$$A \preceq^l_K B \text{ if and only if } A + K \supset B,$$

$$A \preceq^u_K B \text{ if and only if } A \subset B - K,$$

and weak set relations $\prec_K^l$ and $\prec_K^u$ on $\mathcal{C}$: for each $A, B \in \mathcal{C}$,

$$A \prec_K^l B \text{ if and only if } A + \operatorname{int} K \supset B,$$

$$A \prec_K^u B \text{ if and only if } A \subset B - \operatorname{int} K.$$

Also we define $A \sim_K^l B$ if $A \preceq_K^l B$ and $B \preceq_K^l A$. In the previous literatures [12], above set relations are called type (iii) or type (v), and then, these are written by $A \preceq_K^{(\text{iii})} B$, $A \preceq_K^{(\text{v})} B$, $A \prec_K^{(\text{iii})} B$, and $A \prec_K^{(\text{v})} B$, respectively.

Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$. By using these set relations, notions of minimality of $\mathcal{A}$ with respect to K are defined as follows: a set $A \in \mathcal{A}$ is said to be an l-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\, B \preceq_K^l A \Rightarrow A \preceq_K^l B,$$

and a set $A \in \mathcal{A}$ is said to be a weak l-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\, B \prec_K^l A \Rightarrow A \prec_K^l B,$$

or equivalently,

$$\nexists B \in \mathcal{A} \text{ such that } B \prec_K^l A.$$

Replacing l by u, notions of u-minimality and weak u-minimality of $\mathcal{A}$ are given.

Consider the following set-valued optimization problem:

$$\begin{array}{ll}\text{(SP)} \ \text{Minimize} & F(x) \\ \quad\quad\ \ \text{subject to} & x \in X,\end{array}$$

where X is a nonempty set and $F : X \to \mathcal{C}$. By using the notions of minimality defined above, we define notions of solutions of (SP) with respect to K. An element $x_0 \in X$ is said to be an l-minimal solution of (SP) if and only if $F(x_0)$ is an l-minimal element of $\{F(x) \mid x \in X\}$, and is said to be a weak l-minimal solution of (SP) if and only if $F(x_0)$ is a weak l-minimal element of $\{F(x) \mid x \in X\}$. In similar way, u-minimal solutions and weak u-minimal solutions are defined.

To study set-valued optimization problem (SP), many researchers have proposed several generalizations of scalarizing function which is given in the last section, see [1, 4, 5, 13]. In these literature, such scalarizing functions are classified broadly into the following four types:

$$\begin{aligned}
I_e^l(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^l te + B\},\\
I_e^u(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^u te + B\},\\
S_e^l(A; B) &= \sup\{t \in \mathbb{R} \mid A \preceq_K^l te + B\}, \text{ and}\\
S_e^u(A; B) &= \sup\{t \in \mathbb{R} \mid A \preceq_K^u te + B\}.
\end{aligned}$$

In this paper, we propose an idea of a unification of the above minimalities, and a unification of the above scalarizing functions. For this purpose, we introduce a partially ordered normed vector space in which the family $\mathcal{C}$ is embedded in the next section.

4 An Embedding Space and an Embedding Function

In this study, we provide a vector space in which the class $\mathcal{C}$ is embedded, in order to reformulate set optimization problem (SP) as a vector optimization problem. There are several literatures with respect to the construction of a vector space in which a family of convex sets is embedded, for example, see [15, 16]. In this section, we introduce a specialized embedding vector space $\mathcal{C}^2/\equiv$ and an embedding function ψ to observe l-minimal solutions of (SP). All definitions and results are based on the previous literatures, see [10, 11].

Let $\equiv$ be a binary relation on $\mathcal{C}^2$ defined by

$$(A, B) \equiv (C, D) \text{ if and only if } A + D + K = B + C + K,$$

then $\equiv$ is an equivalence relation on $\mathcal{C}^2$. To show this, the following cancellation law is used: for each $A, B, C \in \mathcal{C}$,

$$A + C + K = B + C + K \Rightarrow A + K = B + K.$$

Denote the equivalence class of $(A, B) \in \mathcal{C}$ as $[A, B] = \{(C, D) \in \mathcal{C}^2 \mid (A, B) \equiv (C, D)\}$, and the quotient space of $\mathcal{C}^2$ by $\equiv$ as $\mathcal{C}^2/\equiv\; = \{[A, B] \mid (A, B) \in \mathcal{C}^2\}$. On the quotient space, we define addition and scalar multiplication as follows:

$$[A, B] + [C, D] = [A + C, B + D],$$
$$\lambda \cdot [A, B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{C}^2/\equiv, +, \cdot)$ becomes a vector space over $\mathbb{R}$ with the null vector $[\{0\}, \{0\}](=: \theta)$. Clearly, $[A, A] = \theta$ for each $A \in \mathcal{C}$ by using the cancellation law. Next we can define a norm on $\mathcal{C}^2/\equiv$ for a given bounded base W of K^+, that is $\cup_{\lambda \geq 0} \lambda W = K^+$, whose closure does not contain 0. The existence of such W is guaranteed by $\text{int} K \neq \emptyset$, for example, see [7]. Define

$$\|[A, B]\| = \sup_{w \in W} \left| \inf \langle w, A \rangle - \inf \langle w, B \rangle \right|,$$

for every $[A, B] \in \mathcal{C}^2/\equiv$, then $\| \cdot \|$ is a norm on $\mathcal{C}^2/\equiv$, and we equip the vector space $\mathcal{C}^2/\equiv$ with the topology which is induced by the norm. Let $\mathcal{K}$ be defined as

$$\mathcal{K} = \{[A, B] \in \mathcal{C}^2/\equiv \mid B \preceq^l_K A\}.$$

Then $\mathcal{K}$ is a closed convex cone with nonempty interior, $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$ and moreover

$$\mathrm{int}\mathcal{K} = \{[A, B] \in \mathcal{C}^2/\equiv \mid B \prec_K A\}.$$

From this, we can define the following partial order $\leq_{\mathcal{K}}$ and binary relation $<_{\mathcal{K}}$ on $\mathcal{C}^2/\equiv$ in the same manner to vector optimization: for each $[A, B], [C, D] \in \mathcal{C}^2/\equiv$,

$$[A, B] \leq_{\mathcal{K}} [C, D] \text{ if and only if } [C, D] - [A, B] \in \mathcal{K},$$

and

$$[A, B] <_{\mathcal{K}} [C, D] \text{ if and only if } [C, D] - [A, B] \in \mathrm{int}\mathcal{K}.$$

Let $(\mathcal{C}^2/\equiv)^*$ be the continuous dual space of $\mathcal{C}^2/\equiv$. The positive polar cone of $\mathcal{K}$ is given by

$$\mathcal{K}^+ = \{T \in (\mathcal{C}^2/\equiv)^* \mid \langle T, [A, B]\rangle \geq 0, \forall [A, B] \in \mathcal{K}\},$$

and the second positive polar cone of $\mathcal{K}$ is given by

$$\mathcal{K}^{++} = \{[A, B] \in \mathcal{C}^2/\equiv \mid \langle T, [A, B]\rangle \geq 0, \forall T \in \mathcal{K}^+\}.$$

Also we have $\mathcal{K} = \mathcal{K}^{++}$ from the closedness of convex cone $\mathcal{K}$.

Define an embedding function $\psi : \mathcal{C} \to \mathcal{C}^2/\equiv$ by

$$\psi(A) = [A, \{0\}]$$

for all $A \in \mathcal{C}$. Because $\mathcal{C}^2/\equiv$ is an ordered normed vector space with convex cone $\mathcal{K}$, we reconsider notions of minimality with respect to $\preceq^l_K$ by using the embedding function. For a subfamily $\mathcal{A}$ of $\mathcal{C}$, $A \in \mathcal{A}$ is l-minimal of $\mathcal{A}$ with respect to K

$$\begin{aligned}
&\iff B \in \mathcal{A}, B \preceq^l_K A \Rightarrow A \preceq^l_K B \\
&\iff B \in \mathcal{A}, [A, B] \in \mathcal{K} \Rightarrow [B, A] \in \mathcal{K} \\
&\iff B \in \mathcal{A}, [A, \{0\}] - [B, \{0\}] \in \mathcal{K} \Rightarrow [B, \{0\}] - [A, \{0\}] \in \mathcal{K} \\
&\iff B \in \mathcal{A}, \psi(A) - \psi(B) \in \mathcal{K} \Rightarrow \psi(B) - \psi(A) \in \mathcal{K} \\
&\iff B \in \mathcal{A}, \psi(A) - \psi(B) \in \mathcal{K} \Rightarrow \psi(B) - \psi(A) = \theta \\
&\iff B \in \mathcal{A}, \psi(B) \in \psi(A) - \mathcal{K} \Rightarrow \psi(B) = \psi(A) \\
&\iff \psi(\mathcal{A}) \cap (\psi(A) - \mathcal{K}) \subset \{\psi(A)\} \\
&\iff \psi(A) \in \mathrm{Min}(\psi(\mathcal{A}) \mid \mathcal{K}) \\
&\iff \psi(A) \text{ is a minimal element of } \psi(\mathcal{A}) \text{ with respect to } \mathcal{K}.
\end{aligned}$$

Therefore l-minimality is represented by minimality of vector optimization. Also, $A \in \mathcal{A}$ is weak l-minimal of $\mathcal{A}$ with respect to K if and only if $\psi(A) \in w\mathrm{Min}(\psi(\mathcal{A}) \mid \mathcal{K})$, that is, $\psi(A)$ is a weak minimal element of $\psi(\mathcal{A})$ with respect to $\mathcal{K}$. In the same way, set optimization

$$\text{(SP)}\ \ \text{Minimize}\ \ F(x) \\ \text{subject to}\ \ x \in X,$$

can be regarded as the following vector optimization:

$$\text{(VP)}\ \ \text{Minimize}\ \ \psi \circ F(x) \\ \text{subject to}\ \ x \in X.$$

An element $x_0 \in X$ is an l-minimal solution of (SP) in the last section if and only if $\psi \circ F(x_0) \in \text{Min}(\psi \circ F(X) \mid \mathcal{K})$ and $x_0 \in X$ is a weak l-minimal solution of (SP) in the last section if and only if $\psi \circ F(x_0) \in w\text{Min}(\psi \circ F(X) \mid \mathcal{K})$, where $\psi \circ F(X) = \{\psi(F(x)) \mid x \in X\}$.

The embedding space $\mathcal{C}^2/\equiv$ and the embedding function ψ play very important role to study l-minimal solutions and weak l-minimal solutions of set optimization problems. In the rest of this paper, we propose parameterized embedding functions ψ_λ, which include the previous embedding function ψ. By using the parametrized embedding functions, we define notions of generalized minimal solutions, and we characterize such solutions by using given parametrized scalarizing functions.

5 Parameterized Embedding Functions

At first, we give an important observation of a singleton $\{a\} \subset E$ as follows:

$$[\{a\}, \{0\}] = [\{0\}, -\{a\}] = [(1-\lambda)\{a\}, -\lambda\{a\}],$$

for each $\lambda \in \mathbb{R}$. Indeed, the first equality follows from $\{a\} + (-\{a\}) = \{0\} + \{0\}$ and the second equality follows from $\{0\} - \lambda\{a\} = -\{a\} + (1-\lambda)\{a\}$. From the observation, we define new embedding functions $\psi_\lambda : \mathcal{C} \to \mathcal{C}^2/\equiv$ as follows:

$$\psi_\lambda(A) = [(1-\lambda)A, -\lambda A]$$

for each $A \in \mathcal{C}$. Clearly ψ_0 is the same to ψ, which was given previously. By using the embedding function, we have the following remarkable proposition:

Proposition 1 *For each $A, B \in \mathcal{C}$, the following are satisfied:*

(i) $\psi_0(A) \leq_{\mathcal{K}} \psi_0(B)$ *if and only if* $A \preceq^l_K B$,
(ii) $\psi_0(A) <_{\mathcal{K}} \psi_0(B)$ *if and only if* $A \prec^l_K B$,
(iii) $\psi_0(A) = \psi_0(B)$ *if and only if* $A \sim^l_K B$,
(iv) $\psi_1(A) \leq_{\mathcal{K}} \psi_1(B)$ *if and only if* $A \preceq^u_K B$,
(v) $\psi_1(A) <_{\mathcal{K}} \psi_1(B)$ *if and only if* $A \prec^u_K B$, *and*
(vi) $\psi_1(A) = \psi_1(B)$ *if and only if* $A \sim^u_K B$.

Proof Proof of (i) is shown as follows:

$$\psi_0(A) \le_{\mathcal{K}} \psi_0(B) \iff [A, \{0\}] \le_{\mathcal{K}} [B, \{0\}] \iff \theta \le_{\mathcal{K}} [B, A] \iff A \preceq_K^l B.$$

From this and $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$, (iii) is given immediately. Proof of (iv) is given in similar way:

$$\begin{aligned}\psi_1(A) \le_{\mathcal{K}} \psi_1(B) &\iff [\{0\}, -A] \le_{\mathcal{K}} [\{0\}, -B] \iff \theta \le_{\mathcal{K}} [-A, -B] \\ &\iff -A \preceq_K^l -B \iff B \preceq_K^u A.\end{aligned}$$

Proofs of (ii), (v) and (vi) are similar and omitted. □

Motivated by Proposition 1, we give the following notations:

$$\begin{aligned}&A \preceq_K^\lambda B \textit{ if and only if } \psi_\lambda(A) \le_{\mathcal{K}} \psi_\lambda(B),\\ &A \prec_K^\lambda B \textit{ if and only if } \psi_\lambda(A) <_{\mathcal{K}} \psi_\lambda(B), \quad \text{and}\\ &A \sim_K^\lambda B \textit{ if and only if } \psi_\lambda(A) = \psi_\lambda(B).\end{aligned}$$

Clearly these include binary relations $\preceq_K^l$, $\preceq_K^u$, $\prec_K^l$, $\prec_K^u$, $\sim_K^l$ and $\sim_K^u$.

Now we observe properties of the parametrized embedding functions.

Proposition 2 *For each $A \in \mathcal{C}$, the following are satisfied:*

(i) for each $\alpha, \beta \in [0, \infty)$, $\alpha A + \beta A = (\alpha + \beta)A$;
(ii) for each $\alpha, \beta \in [0, \infty)$, $[\alpha A, \beta A] = (\alpha - \beta)[A, \{0\}]$;
(iii) if $\lambda \le 0$ then $\psi_\lambda(A) = \psi_0(A)$;
(iv) if $1 \le \lambda$ then $\psi_\lambda(A) = \psi_1(A)$.

Proof Let $A \in \mathcal{C}$ and $\alpha, \beta \in [0, \infty)$. (i) is shown from the convexity of A. Indeed, we may assume that $\alpha + \beta > 0$. Then

$$\alpha A + \beta A = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}A + \frac{\beta}{\alpha + \beta}A\right) = (\alpha + \beta)A.$$

Next we show (ii). When $\alpha > \beta$, since $\alpha = (\alpha - \beta) + \beta$ and $\alpha - \beta, \beta \ge 0$, we have

$$\begin{aligned}[\alpha A, \beta A] &= [(\alpha - \beta)A + \beta A, \beta A] = [(\alpha - \beta)A, \{0\}] + [\beta A, \beta A]\\ &= [(\alpha - \beta)A, \{0\}] = (\alpha - \beta)[A, \{0\}].\end{aligned}$$

The first equality is shown from (i). In similar way, when $\alpha \le \beta$, since $\beta = \alpha + (\beta - \alpha)$ and $\alpha, \beta - \alpha \ge 0$, we have

$$\begin{aligned}[\alpha A, \beta A] &= [\alpha A, \alpha A + (\beta - \alpha)A] = [\alpha A, \alpha A] + [\{0\}, (\beta - \alpha)A]\\ &= [\{0\}, (\beta - \alpha)A] = (\beta - \alpha)[\{0\}, A] = (\alpha - \beta)[A, \{0\}].\end{aligned}$$

Next we show (iii). Let $\lambda \leq 0$. Since $0 \leq -\lambda \leq 1-\lambda$, by using (ii), we have

$$\psi_\lambda(A) = [(1-\lambda)A, -\lambda A] = ((1-\lambda)-(-\lambda))[A, \{0\}] = [A, \{0\}] = \psi_0(A).$$

The proof of (iv) is similar to (iii) and omitted. □

The next proposition is about monotonicity of the embedding functions with respect to variable λ for a given $A \in C$.

Proposition 3 *Let $A \in \mathcal{C}$ and $0 \leq \lambda_0 \leq \lambda_1 \leq 1$. Then the following are satisfied:*

(i) $\psi_{\lambda_1}(A) - \psi_{\lambda_0}(A) = (\lambda_1 - \lambda_0)[\{0\}, A - A]$;
(ii) $\psi_{(1-t)\lambda_0+t\lambda_1}(A) = (1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A)$ *for each* $t \in [0,1]$;
(iii) $\psi_{\lambda_0}(A) \leq_{\mathcal{K}} \psi_{\lambda_1}(A)$;
(iv) $\lambda_0 < \lambda_1$ *and* $A - A \prec^l_K \{0\}$ *if and only if* $\psi_{\lambda_0}(A) <_{\mathcal{K}} \psi_{\lambda_1}(A)$;
(v) if $\lambda_0 < \lambda_1$ *and A is not a singleton, then* $\psi_{\lambda_0}(A) \neq \psi_{\lambda_1}(A)$.

Proof Let $A \in \mathcal{C}$ and $0 \leq \lambda_0 \leq \lambda_1 \leq 1$. The proof of (i) is as follows:

$$\begin{aligned}
\psi_{\lambda_1}(A) - \psi_{\lambda_0}(A) &= [(1-\lambda_1)A, -\lambda_1 A] - [(1-\lambda_0)A, -\lambda_0 A] \\
&= [(1-\lambda_1)A - \lambda_0 A, (1-\lambda_0)A - \lambda_1 A] \\
&= [(1-\lambda_1)A, (1-\lambda_0)A] + [\lambda_0(-A), \lambda_1(-A)] \\
&= (\lambda_0 - \lambda_1)[A, \{0\}] + (\lambda_0 - \lambda_1)[-A, \{0\}] \\
&= (\lambda_0 - \lambda_1)[A - A, \{0\}] \\
&= (\lambda_1 - \lambda_0)[\{0\}, A - A].
\end{aligned}$$

The fourth equality is shown by using Proposition 2 (ii). Next we show (ii). For each $t \in [0,1]$, by using (i),

$$\begin{aligned}
&\psi_{(1-t)\lambda_0+t\lambda_1}(A) - \psi_{\lambda_0}(A) = t(\lambda_1 - \lambda_0)[\{0\}, A - A], \quad \text{and} \\
&\psi_{\lambda_1}(A) - \psi_{(1-t)\lambda_0+t\lambda_1}(A) = (1-t)(\lambda_1 - \lambda_0)[\{0\}, A - A],
\end{aligned}$$

and then, we have the following equality, which is equivalent to (ii):

$$(1-t)(\psi_{(1-t)\lambda_0+t\lambda_1}(A) - \psi_{\lambda_0}(A)) = t(\psi_{\lambda_1}(A) - \psi_{(1-t)\lambda_0+t\lambda_1}(A)).$$

We show (iii). Since $A - A \ni 0$, it is clear that $A - A + K \supset \{0\}$, that is, $A - A \preceq^l_K \{0\}$, or equivalently $[\{0\}, A - A] \in \mathcal{K}$, and then we have $(\lambda_1 - \lambda_0)[\{0\}, A - A] \in \mathcal{K}$ because $\mathcal{K}$ is a cone and $\lambda_1 - \lambda_0 \geq 0$. The proof of (iv) is similar to (iii). Finally we show (v). Assume that $\lambda_0 < \lambda_1$, A is not a singleton, and $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$. From (i) and $\lambda_1 - \lambda_0 > 0$, we have $[\{0\}, A - A] = \theta$, or equivalently, $A - A + K = K$. Since A is not a singleton, there exist different two elements $a, a' \in A$. Since $A - A \subset K$, $a - a' \in K$ and $a' - a \in K$, therefore $a - a' \in K \cap (-K) = \{0\}$. This is a contradiction. □

We have observed that if A is a singleton then all embedding functions ψ_λ have the same image at the beginning of this section. The inverse implication holds from the following proposition:

Proposition 4 *For each $A \in \mathcal{C}$, the following are equivalent:*

(i) $A = \{a\}$ *for some* $a \in E$*;*
(ii) *there exist different* $\lambda_0, \lambda_1 \in [0, 1]$ *such that* $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$*;*
(iii) *for each* $\lambda_0, \lambda_1 \in [0, 1]$, $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$.

Proof It is clear that (i) implies (iii), which is the first observation of this section, and (iii) implies (ii). Also (ii) implies (i) from (v) of Proposition 3. □

The following property is essential to define generalized minimality of (SP):

Proposition 5 *Let* $A, B \in \mathcal{C}$ *and* $0 \le \lambda_0 < \lambda_1 \le 1$. *The following are satisfied:*

(i) *both* $A \preceq_K^{\lambda_0} B$ *and* $A \preceq_K^{\lambda_1} B$ *if and only if* $A \preceq_K^{\lambda} B$ *for every* $\lambda \in (\lambda_0, \lambda_1)$*;*
(ii) *both* $A \prec_K^{\lambda_0} B$ *and* $A \prec_K^{\lambda_1} B$ *if and only if* $A \prec_K^{\lambda} B$ *for every* $\lambda \in [\lambda_0, \lambda_1]$*;*
(iii) $\{\lambda \in [0, 1] \mid A \preceq_K^{\lambda} B\}$ *is a closed interval, a singleton or empty;*
(iv) $\{\lambda \in [0, 1] \mid A \prec_K^{\lambda} B\}$ *is an interval which is open in* $[0, 1]$ *or empty.*

Proof Let $A, B \in \mathcal{C}$ and $0 \le \lambda_0 < \lambda_1 \le 1$. We show (i). Assume that $A \preceq_K^{\lambda_0} B$ and $A \preceq_K^{\lambda_1} B$, that is, both $\psi_{\lambda_0}(A) \le_{\mathcal{K}} \psi_{\lambda_0}(B)$ and $\psi_{\lambda_1}(A) \le_{\mathcal{K}} \psi_{\lambda_1}(B)$. For any $\lambda \in (\lambda_0, \lambda_1)$, $\lambda = (1-t)\lambda_0 + t\lambda_1$ for some $t \in (0, 1)$. From (ii) of Proposition 3,

$$\psi_\lambda(A) = (1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A) \text{ and } \psi_\lambda(B) = (1-t)\psi_{\lambda_0}(B) + t\psi_{\lambda_1}(B).$$

This implies $\psi_\lambda(A) \le_{\mathcal{K}} \psi_\lambda(B)$, that is, $A \preceq_K^{\lambda} B$. Conversely, assume that $A \preceq_K^{\lambda} B$, that is, $\psi_\lambda(A) \le_{\mathcal{K}} \psi_\lambda(B)$ for every $\lambda \in (\lambda_0, \lambda_1)$. This is equivalent to

$$(1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A) \le_{\mathcal{K}} (1-t)\psi_{\lambda_0}(B) + t\psi_{\lambda_1}(B)$$

for every $t \in (0, 1)$ by using (ii) of Proposition 3. From the closedness of $\mathcal{K}$, we have $\psi_{\lambda_0}(A) \le_{\mathcal{K}} \psi_{\lambda_0}(B)$ and $\psi_{\lambda_1}(A) \le_{\mathcal{K}} \psi_{\lambda_1}(B)$ by considering the cases $t \searrow 0$ and $t \nearrow 1$. The proof of (ii) is similar to (i) and omitted. We show (iii). Put $\Lambda = \{\lambda \in [0, 1] \mid A \preceq_K^{\lambda} B\}$. We may assume that $|\Lambda| > 1$. For any $\lambda_0, \lambda_1 \in \Lambda$ such that $\lambda_0 < \lambda_1$, we have $(\lambda_0, \lambda_1) \subset \Lambda$ from (i). This shows that Λ is an interval in $[0, 1]$. To prove that Λ is closed, choose a sequence $\{\lambda_n\} \subset \Lambda$ converges to λ_0. We will show that $A \preceq_K^{\lambda_0} B$, that is,

$$(1-\lambda_0)A - \lambda_0 B + K \supset -\lambda_0 A + (1-\lambda_0)B.$$

For any $a \in A$ and $b \in B$, since

$$(1-\lambda_n)A - \lambda_n B + K \supset -\lambda_n A + (1-\lambda_n)B$$

for every $n \in \mathbb{N}$, there exist $\{a_n\} \subset A$, $\{b_n\} \subset B$ and $\{k_n\} \subset K$ such that

$$(1-\lambda_n)a_n - \lambda_n b_n + k_n = -\lambda_n a + (1-\lambda_n)b$$

for every $n \in \mathbb{N}$. From the compactness of A and B, we can choose a subsequence $\{n'\}$ of $\{n\}$ such that $\{a_{n'}\}$ converges to some $a_0 \in A$ and $\{b_{n'}\}$ converges to some $b_0 \in B$. Therefore $\{k_{n'}\}$ converges to $k_0 = (1-\lambda_0)(b-a_0) + \lambda_0(b_0 - a)$, which is an element of K because K is closed, and

$$(1-\lambda_0)A - \lambda_0 B + K \ni (1-\lambda_0)a_0 - \lambda_0 b_0 + k_0 = -\lambda_0 a + (1-\lambda_0)b.$$

Finally we show (iv). Put $\Lambda = \{\lambda \in [0,1] \mid A \prec_K^\lambda B\}$. In similar way to (iii), Λ is an interval. We show Λ is open in $[0,1]$. Let $\lambda_0 \in \Lambda$. Since $(1-\lambda_0)A - \lambda_0 B + \mathrm{int} K \supset -\lambda_0 A + (1-\lambda_0)B$, there exists $r > 0$ such that

$$(1-\lambda_0)A - \lambda_0 B + K \supset -\lambda_0 A + (1-\lambda_0)B + 3rU,$$

where U is the unit closed ball of E. Put $\varepsilon = r \inf \|W\| / \max\{\|[-A,B]\|, \|[A,-B]\|\}$. We will show that $\{\lambda \in [0,1] \mid |\lambda - \lambda_0| \le \varepsilon\} \subset \Lambda$. For any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \le \varepsilon$,

$$|\lambda - \lambda_0| \|[-A,B]\| \le r \inf \|W\| \ \text{ and } \ |\lambda - \lambda_0| \|[A,-B]\| \le r \inf \|W\|,$$

then for any $w \in W$,

$$\begin{aligned}
&(\lambda_0 - \lambda)\left(\inf \langle w, -A\rangle - \inf \langle w, B\rangle\right) \le r\|w\|, \quad \text{and} \\
&(\lambda - \lambda_0)\left(\inf \langle w, A\rangle - \inf \langle w, -B\rangle\right) \le r\|w\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&\inf \langle w, -\lambda A + (1-\lambda)B\rangle \ge \inf \langle w, -\lambda_0 A + (1-\lambda_0)B\rangle - r\|w\|, \quad \text{and} \\
&\inf \langle w, (1-\lambda_0)A - \lambda_0 B\rangle + r\|w\| \ge \inf \langle w, (1-\lambda)A - \lambda B\rangle.
\end{aligned}$$

Therefore, for any $w \in W$,

$$\begin{aligned}
\inf \langle w, -\lambda A + (1-\lambda)B + rU\rangle &= \inf \langle w, -\lambda A + (1-\lambda)B\rangle - r\|w\| \\
&\ge \inf \langle w, -\lambda_0 A + (1-\lambda_0)B\rangle - 2r\|w\| \\
&= \inf \langle w, -\lambda_0 A + (1-\lambda_0)B + 3rU\rangle + r\|w\| \\
&\ge \inf \langle w, (1-\lambda_0)A - \lambda_0 B\rangle + r\|w\| \\
&\ge \inf \langle w, (1-\lambda)A - \lambda B\rangle.
\end{aligned}$$

This shows

$$(1-\lambda)A - \lambda B + K \supset -\lambda A + (1-\lambda)B + rU,$$

that is, $A \prec_K^\lambda B$. This completes the proof. □

Motivated by Proposition 5, we define Λ-minimality as follows:

Definition 1 Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$, $A \in \mathcal{A}$, and Λ be a nonempty subset of $[0, 1]$. The set A is said to be a Λ-minimal element of $\mathcal{A}$ with respect to $\mathcal{K}$ if and only if

$$B \in \mathcal{A},\ B \preceq_K^\lambda A \text{ for any } \lambda \in \Lambda \Rightarrow A \preceq_K^\lambda B \text{ for any } \lambda \in \Lambda,$$

or equivalently,

$$\nexists B \in \mathcal{A} \text{ s.t. } \forall \lambda \in \Lambda,\ B \preceq_K^\lambda A \text{ and } \exists \lambda_0 \in \Lambda \text{ s.t. } A \npreceq_K^{\lambda_0} B,$$

and A is said to be a weak Λ-minimal element of $\mathcal{A}$ with respect to $\mathcal{K}$ if and only if

$$\nexists B \in \mathcal{A} \text{ s.t. } \forall \lambda \in \Lambda,\ B \prec_K^\lambda A \text{ and } \exists \lambda_0 \in \Lambda \text{ s.t. } A \nprec_K^{\lambda_0} B,$$

When $\Lambda = \{\lambda\}$, λ-minimality and weak λ-minimality mean Λ-minimality and weak Λ-minimality respectively.

Clearly, $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if

$$\psi_\lambda(A) \in \text{Min}(\psi_\lambda(\mathcal{A}) \mid \mathcal{K})$$

and $A \in \mathcal{A}$ is a weak λ-minimal element of $\mathcal{A}$ if and only if

$$\psi_\lambda(A) \in w\text{Min}(\psi_\lambda(\mathcal{A}) \mid \mathcal{K}).$$

The notion of Λ-minimality includes not only the notions of l and u-minimality, but also the notion of s-minimality, which was introduced in [6]. Indeed, 0-minimality, weak 0-minimality, 1-minimality, and weak 1-minimality are equivalent to l-minimality, weak l-minimality, u-minimality, and weak u-minimality, respectively. For a given family $\mathcal{A} \subset \mathcal{C}$, remember that $A \in \mathcal{A}$ is said to be an s-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\ B \preceq_K^s A \Rightarrow A \preceq_K^s B,$$

where set relation $A \preceq_K^s B$ is defined by $A \preceq_K^l B$ and $A \preceq_K^u B$. From Proposition 5,

$$A \preceq_K^s B \iff A \preceq_K^0 B \text{ and } A \preceq_K^1 B \iff A \preceq_K^\lambda B \text{ for all } \lambda \in [0, 1],$$

and this shows the equivalence of s-minimality and $[0, 1]$-minimality. More generally, Λ-minimality is equivalent to $(\text{co}\,\Lambda)$-minimality, where $\text{co}\,\Lambda$ is the convex hull of Λ.

Proposition 6 *Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$, $A \in \mathcal{A}$, and Λ, Λ' be nonempty subsets of* $[0, 1]$. *The following are satisfied:*

(i) A is a Λ-minimal element of $\mathcal{A}$ if and only if A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$;
(ii) A is a weak Λ-minimal element of $\mathcal{A}$ if and only if A is a weak $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$;
(iii) if A is a Λ-minimal element of $\mathcal{A}$ and A is a Λ'-minimal element of $\mathcal{A}$, then A is a $\Lambda \cup \Lambda'$-minimal element of $\mathcal{A}$;
(iv) if A is a weak Λ-minimal element of $\mathcal{A}$ and A is a weak Λ'-minimal element of $\mathcal{A}$, then A is a weak $\Lambda \cup \Lambda'$-minimal element of $\mathcal{A}$.

Proof We show (i). Assume that A is a Λ-minimal element of $\mathcal{A}$, $B \in \mathcal{A}$ and $B \preceq_K^{\lambda} A$ for all $\lambda \in \text{co}(\Lambda)$. Since $\Lambda \subset \text{co}(\Lambda)$ and A is a Λ-minimal element of $\mathcal{A}$, $A \preceq_K^{\lambda} B$ for all $\lambda \in \Lambda$. For any $\lambda \in \text{co}(\Lambda) \setminus \Lambda$, there exist $\lambda_0, \lambda_1 \in \Lambda$ such that $\lambda \in (\lambda_0, \lambda_1)$. Since $A \preceq_K^{\lambda_0} B$ and $A \preceq_K^{\lambda_1} B$, $A \preceq_K^{\lambda} B$ holds by using (i) of Proposition 5. This shows A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$. Conversely, Assume that A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$, $B \in \mathcal{A}$ and $B \preceq_K^{\lambda} A$ for all $\lambda \in \Lambda$. By using (i) of Proposition 5, we have $B \preceq_K^{\lambda} A$ for all $\lambda \in \text{co}(\Lambda)$. Since A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$, $A \preceq_K^{\lambda} B$ hold for all $\lambda \in \text{co}(\Lambda)$, and from $\Lambda \subset \text{co}(\Lambda)$, we have A is a Λ-minimal element of $\mathcal{A}$. To prove (ii), we use (ii) of Proposition 5. The proof is similar to (i) and left to the reader. Proofs of (iii) and (iv) are easy and omitted. □

We define notions of Λ-minimal solutions of (SP) with respect to K by using the notions of Λ-minimality defined above. Remember

$$\begin{array}{ll} \text{(SP)} & \text{Minimize } F(x) \\ & \text{subject to } x \in X, \end{array}$$

where X is a nonempty set, and $F : X \to \mathcal{C}$. An element $x_0 \in X$ is said to be a Λ-minimal solution of (SP) if and only if $F(x_0)$ is a Λ-minimal element of $\{F(x) \mid x \in X\}$, and is said to be a weak Λ-minimal solution of (SP) if and only if $F(x_0)$ is a weak Λ-minimal element of $\{F(x) \mid x \in X\}$. Next we give examples of Λ-minimality.

Example 1 Let $A = \{(0, 0)\}$, $B = \text{co}\{(1, 1), (-1, -1), (0, -2), (2, 0)\}$, $\mathcal{A} = \{A, B\}$ and $K = \{(x, y) \mid x, y \geq 0\}$. For any $\lambda \in [0, 1]$,

$$A \preceq_K^{\lambda} B \iff -\lambda B + K \supset (1-\lambda)B \iff \frac{2}{3} \leq \lambda, \quad \text{and}$$
$$B \preceq_K^{\lambda} A \iff (1-\lambda)B + K \supset -\lambda B \iff \lambda \leq \frac{1}{3}.$$

So, A is a $[\frac{2}{3}, 1]$-minimal element of $\mathcal{A}$, and B is a $[0, \frac{1}{3}]$-minimal element of $\mathcal{A}$. Clearly, A and B are u and l-minimal elements of $\mathcal{A}$ respectively. The notion of Λ-minimality shows attributes characteristic of A and B in $\mathcal{A}$.

Example 2 Let $F : [0, 2] \to 2^{\mathbb{R}}$ be a set-valued map defined by $F(x) = [f(x), g(x)]$, where $f(x) = x - 2$ and $g(x) = \frac{1}{2}|x-1| - x + \frac{3}{2}$, for each $x \in [0, 2]$. Consider

$$\text{(SP)} \quad \text{Minimize} \quad F(x) \\ \text{subject to} \quad x \in [0, 2],$$

with $K = [0, +\infty)$. For fixed $\lambda \in [0, 1]$,

$$\begin{aligned} &F(x) \preceq_K^{\lambda} F(y) \\ &\iff [(1-\lambda)[f(x), g(x)], -\lambda[f(x), g(x)]] \leq_{\mathcal{K}} [(1-\lambda)[f(y), g(y)], -\lambda[f(y), g(y)]] \\ &\iff [(1-\lambda)[f(y), g(y)] - \lambda[f(x), g(x)], (1-\lambda)[f(x), g(x)] - \lambda[f(y), g(y)]] \in \mathcal{K} \\ &\iff (1-\lambda)[f(x), g(x)] - \lambda[f(y), g(y)] + K \supset (1-\lambda)[f(y), g(y)] - \lambda[f(x), g(x)] \\ &\iff (1-\lambda)f(x) - \lambda g(y) \leq (1-\lambda)f(y) - \lambda g(x) \\ &\iff (1-\lambda)f(x) + \lambda g(x) \leq (1-\lambda)f(y) + \lambda g(y). \end{aligned}$$

Then $\bar{x} \in [0, 2]$ is a Λ-minimal solution of (SP) if and only if $(1-\lambda)f(\bar{x}) + \lambda g(\bar{x}) \leq (1-\lambda)f(x) + \lambda g(x)$ for any $x \in [0, 2]$ and $\lambda \in \Lambda$. Therefore 0 is a $[0, \frac{2}{5}]$-minimal solution, each element of $(0, 1)$ is a $\frac{2}{5}$-minimal solution, 1 is a $[\frac{2}{5}, \frac{2}{3}]$-minimal solution, each element of $(1, 2)$ is a $\frac{2}{3}$-minimal solution, and 2 is a $[\frac{2}{3}, 1]$-minimal solution.

At the end of this section, we study the existence of λ-minimal solutions of set optimization problem (SP) because λ_0 and λ_1-minimality implies $[\lambda_0, \lambda_1]$-minimality from Proposition 6. We give proofs of the existence theorems in similar ways to the previous existence theorems of l-minimal solutions of (SP) in [8, 9].

Theorem 1 *Let F be a function from a compact topological space X to $\mathcal{C}$. Assume that the following property: if $\{x_\alpha\}_{\alpha \in T}$ is a totally ordered λ-decreasing net in X, that is, T is totally ordered, and $\alpha < \alpha'$ implies $F(x_{\alpha'}) \preceq_{\mathcal{K}}^{\lambda} F(x_\alpha)$, and if $\{x_\alpha\}_{\alpha \in T}$ converges x_0, then $\psi_\lambda(F(x_0)) \in \bigcap_{\alpha \in T}(\psi_\lambda(F(x_\alpha)) - \mathcal{K})$. Then there exists a λ-minimal solution of (SP).*

Proof Let $\{\psi_\lambda(F(x))\}_{x \in T}$ be a totally ordered set of $\{\psi_\lambda(F(x))\}_{x \in X}$. Define a reflexive and transitive binary relation $<$ on T by $x < x'$ if $\psi_\lambda(F(x')) \leq_{\mathcal{K}} \psi_\lambda(F(x))$ for each $x, x' \in T$, then $(T, <)$ is a directed set. Since X is compact set, we can choose a subnet T' of T such that T' converges to some element x_0 of X. From the assumption of the theorem, $\psi_\lambda(F(x_0)) \in \bigcap_{x \in T'}(\psi_\lambda(F(x)) - \mathcal{K})$.

Now, we will show that $\psi_\lambda(F(x_0)) \leq_{\mathcal{K}} \psi_\lambda(F(x))$ for each $x \in T$. If not, there exists $\hat{x} \in T$ such that $\psi_\lambda(F(x_0)) \not\leq_{\mathcal{K}} \psi_\lambda(F(\hat{x}))$. For each $x \in T'$ satisfying $\hat{x} < x$, $\psi_\lambda(F(x)) \leq_{\mathcal{K}} \psi_\lambda(F(\hat{x}))$, therefore $\bigcap_{x \in T', \hat{x} < x}(\psi_\lambda(F(x)) - \mathcal{K}) \subset \psi_\lambda(F(\hat{x})) - \mathcal{K}$.

Clearly $\bigcap_{x\in T'}(\psi_\lambda(F(x))-\mathcal{K})\subset\bigcap_{x\in T',\hat{x}<x}(\psi_\lambda(F(x))-\mathcal{K})$, we have $\psi_\lambda(F(x_0))\in\psi_\lambda(F(\hat{x}))-\mathcal{K}$, or equivalently $\psi_\lambda(F(x_0))\leq_{\mathcal{K}}\psi_\lambda(F(\hat{x}))$. This is a contradiction. Hence, we have that $\psi_\lambda(F(x_0))$ is a lower bound of $\{\psi_\lambda(F(x))\}_{x\in T}$ with respect to $\leq_{\mathcal{K}}$. From Zorn's lemma, there exists a minimal element of $\{\psi_\lambda(F(x))\}_{x\in X}$, that is, there exists a λ-minimal solution of (SP). □

When $\lambda=0$, the condition of F in Theorem 1 is weaker than the notion of Hausdorff cone-upper continuity; F is Hausdorff K-upper continuous at x_0 if for any neighborhood V of the origin in E, there is a neighborhood U of x_0 in X such that $F(x)\subset F(x_0)+V+K$ for all $x\in U\cap X$, for example, see [3]. From this fact and Theorem 1, the following result is shown, the proof is left to the reader:

Corollary 1 *Let F be a function from a compact topological space X to $\mathcal{C}$. If F is Hausdorff K-upper continuous at every $x\in X$, then there exists an l-minimal solution of (SP). If F is Hausdorff $(-K)$-lower continuous at every $x\in X$, that is, for every $x\in X$ and for any neighborhood V of the origin in E, there is a neighborhood U of x in X such that $F(x)\subset F(x')+V-K$ for all $x'\in U\cap X$, then there exists an u-minimal solution of (SP).*

Define λ-level sets of F by

$$\mathrm{Lev}_\lambda(A)=\{x\in X\mid F(x)\leq^\lambda_{\mathcal{K}}A\},$$

where $A\in\mathcal{C}$.

Theorem 2 *If (X,d) is a complete metric space, $\mathrm{Lev}_\lambda(F(x))$ is closed for each $x\in X$, and the following condition is satisfied:*

> *there exists a function $l:X\to[0,+\infty)$ such that for each $x_1,x_2\in X$, $F(x_1)\preceq^\lambda_K F(x_2)$ implies $d(x_2,x_1)\leq l(x_2)-l(x_1)$.*

Then, there exists a λ-minimal solution of (SP).

Proof Let $x_0\in X$. We construct a sequence $\{x_k\}\subset X$ by induction as follows:

(i) if $\mathrm{Lev}_\lambda(F(x_k))\neq\{x_k\}$, since $\psi_\lambda(F(x'))\leq_{\mathcal{K}}\psi_\lambda(F(x_k))$ for some $x'\neq x_k$,

$$0<d(x_k,x')\leq l(x_k)-l(x')\leq l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x).$$

Since $l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)>0$, we can choose $x_{k+1}\in\mathrm{Lev}_\lambda(F(x_k))$ such that

$$l(x_{k+1})\leq\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)+\frac{1}{2}\left\{l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)\right\}.$$

(ii) if $\mathrm{Lev}_\lambda(F(x_k))=\{x_k\}$, let $x_{k+1}:=x_k$.

In each case, we can check easily that $\mathrm{Lev}_\lambda(F(x_{k+1}))\subset\mathrm{Lev}_\lambda(F(x_k))$ and

$$l(x_{k+1})-\inf_{x\in\mathrm{Lev}_\lambda(F(x_{k+1}))}l(x)\leq\frac{1}{2}\left\{l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)\right\}.$$

Now we show that $\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \to 0$ as $k \to +\infty$. Indeed, let $u \in \mathrm{Lev}_\lambda(F(x_k))$. From the assumption and $\psi_\lambda(F(u)) \leq_{\mathcal{K}} \psi_\lambda(F(x_k))$, we have $d(x_k, u) \leq l(x_k) - l(u)$. Hence

$$\begin{aligned} d(x_k, u) &\leq l(x_k) - l(u) \\ &\leq l(x_k) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_k))} l(x) \\ &\leq \frac{1}{2}\left\{ l(x_{k-1}) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_{k-1}))} l(x) \right\} \\ &\leq \cdots \leq \frac{1}{2^k}\left\{ l(x_0) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_0))} l(x) \right\} \leq \cdots \leq \frac{1}{2^k} l(x_0). \end{aligned}$$

This shows us

$$\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \leq \frac{1}{2^{k-1}} l(x_0),$$

therefore, we have $\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \to 0$ as $k \to +\infty$. Since $\mathrm{Lev}_\lambda(F(x_k))$ is nonempty closed, $\mathrm{Lev}_\lambda(F(x_{k+1})) \subset \mathrm{Lev}_\lambda(F(x_k))$, and (X, d) is complete, we conclude $\bigcap_{k \in \mathbb{N}} \mathrm{Lev}_\lambda(F(x_k)) = \{\hat{x}\}$ for some $\hat{x} \in X$. This implies $\mathrm{Lev}_\lambda(F(\hat{x})) = \{\hat{x}\}$ and, it follows that $\hat{x}$ is a λ-minimal solution of (SP). □

6 A Generalized Scalarizing Function on $\mathcal{C}$

Since $\mathcal{C}^2/\equiv$ is an ordered vector space with convex cone $\mathcal{K}$, the scalarizing function from $\mathcal{C}^2/\equiv$ to $\mathbb{R}$ is given in this way:

$$\varphi_{[P,Q]}([A, B]) = \inf\{t \in \mathbb{R} \mid [A, B] \in t[P, Q] - \mathcal{K}\},$$

for fixed $[P, Q] \in \mathcal{C}^2/\equiv$. From the definition, it is clear that $\varphi_{[P,Q]}([A, B] + r[P, Q]) = \varphi_{[P,Q]}([A, B]) + r$. When $[P, Q] \in \mathrm{int}\mathcal{K}$, this function $\varphi_{[P,Q]}$ has the following properties: it is a special case of vector-valued version in [3], and the proof of the following theorem is omitted.

Theorem 3 *If* $[P, Q] \in \mathrm{int}\mathcal{K}$*, then* $\varphi_{[P,Q]} : \mathcal{C}^2/\equiv \to \mathbb{R}$ *is a well-defined continuous function, and for each* $[A, B], [C, D] \in \mathcal{C}^2/\equiv$*, we have*

(i) $\varphi_{[P,Q]}([A, B]) \leq r$ *if and only if* $[A, B] \in r[P, Q] - \mathcal{K}$*;*
(ii) $\varphi_{[P,Q]}([A, B]) < r$ *if and only if* $[A, B] \in r[P, Q] - \mathrm{int}\mathcal{K}$*;*
(iii) $\varphi_{[P,Q]}([A, B]) > r$ *if and only if* $[A, B] \notin r[P, Q] - \mathcal{K}$*;*
(iv) $\varphi_{[P,Q]}([A, B]) \geq r$ *if and only if* $[A, B] \notin r[P, Q] - \mathrm{int}\mathcal{K}$*;*
(v) $[A, B] \leq_{\mathcal{K}} [C, D]$ *implies* $\varphi_{[P,Q]}([A, B]) \leq \varphi_{[P,Q]}([C, D])$*;*
(vi) $[A, B] <_{\mathcal{K}} [C, D]$ *implies* $\varphi_{[P,Q]}([A, B]) < \varphi_{[P,Q]}([C, D])$*.*

Now we characterize solutions of (SP) by using the scalarizing function. At first we observe λ-minimal elements of a subfamily $\mathcal{A} \subset \mathcal{C}$ with respect to K:

Theorem 4 *Let $[P, Q] \in \operatorname{int}\mathcal{K}$ and $\mathcal{A}$ be a subfamily of $\mathcal{C}$. The set $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) > 0$ or $B \sim_K^\lambda A$. The set $A \in \mathcal{A}$ is a weak λ-minimal element of $\mathcal{A}$ if and only if for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) \geq 0$.*

Proof The set $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if $B \in \mathcal{A}$, $\psi_\lambda(B) \leq_\mathcal{K} \psi_\lambda(A)$ implies $\psi_\lambda(A) \leq_\mathcal{K} \psi_\lambda(B)$, that is, for each $B \in \mathcal{A}$, $\psi_\lambda(B) \not\leq_\mathcal{K} \psi_\lambda(A)$ or else $\psi_\lambda(B) = \psi_\lambda(A)$. By using Theorem 3, this is equivalent to for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) > 0$ or $B \sim_K^\lambda A$. The latter is shown in the similar way. □

From this theorem, we may choose any $[P, Q] \in \operatorname{int}\mathcal{K}$ to observe λ-minimal elements and weak λ-minimal elements. When $e \in \operatorname{int}K$, we can check that $[\{e\}, \{0\}] \in \operatorname{int}\mathcal{K}$, and embedding function $\psi_{[P,Q]}$ is a generalization of $I_e^l(A; B)$ and $I_e^u(A; B)$, indeed,

$$\begin{aligned} I_e^l(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^l te + B\} \\ &= \inf\{t \in \mathbb{R} \mid [A, \{0\}] \leq_\mathcal{K} t[\{e\}, \{0\}] + [B, \{0\}]\} \\ &= \varphi_{\psi_0(\{e\})}(\psi_0(A) - \psi_0(B)), \text{ and} \end{aligned}$$

$$\begin{aligned} I_e^u(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^u te + B\} \\ &= \inf\{t \in \mathbb{R} \mid -B \preceq_K^l te - A\} \\ &= \inf\{t \in \mathbb{R} \mid [-B, \{0\}] \leq_\mathcal{K} t[\{e\}, \{0\}] + [-A, \{0\}]\} \\ &= \varphi_{\psi_1(\{e\})}(\psi_1(A) - \psi_1(B)). \end{aligned}$$

Also $S_e^l(A; B)$ and $S_e^u(A; B)$ can be written by using φ because $S_e^l(A; B) = -I_{-e}^l(A; B)$ and $S_e^u(A; B) = -I_{-e}^u(A; B)$. Motivated by the observation, we give the following simple notation $\varphi_e^\lambda(A, B)$ as follows: for each $\lambda \in [0, 1]$,

$$\varphi_e^\lambda(A, B) = \varphi_{\psi_\lambda(\{e\})}(\psi_\lambda(A) - \psi_\lambda(B)).$$

Clearly we have

$$\begin{aligned} &\varphi_e^0(A, B) = I_e^l(A; B), \quad \varphi_e^1(A, B) = I_e^u(A; B), \\ &\varphi_e^0(A, B) = -S_{-e}^l(A; B), \text{ and } \varphi_e^1(A, B) = -S_{-e}^u(A; B), \end{aligned}$$

and we can characterize solutions of (SP) by using the function:

Corollary 2 *Let X be a nonempty set, $F : X \to \mathcal{C}$, and $e \in \operatorname{int}K$. The element $x_0 \in X$ is a λ-minimal solution of (SP) if and only if for each $x \in X$, $\varphi_e^\lambda(F(x), F(x_0)) > 0$ or $F(x) \sim_K^\lambda F(x_0)$. The element $x_0 \in X$ is a weak λ-minimal solution of (SP) if and only if for each $x \in X$, $\varphi_e^\lambda(F(x), F(x_0)) \geq 0$ or $F(x) \sim_K^\lambda F(x_0)$.*

The above characterizations are generalizations of the previous ones of set optimization problems. Finally, we observe the following example:

Example 3 Let $F : X \to 2^{\mathbb{R}^n}$ be a set-valued map defined by

$$F(x) = (f_0(x) + K) \cap (f_1(x) - K)$$

where functions $f_0, f_1 : X \to \mathbb{R}^n$ satisfy $f_0(x) \leq_K f_1(x)$ for each $x \in X$, and consider a set optimization problem

$$\begin{aligned} \text{(SP)}\ \ \text{Minimize}\ \ & F(x) \\ \text{subject to}\ \ & x \in X. \end{aligned}$$

For given $e \in \text{int}K$ and for any $\lambda \in [0, 1]$, we can check that

$$\varphi_e^\lambda(F(x), F(y)) = \inf\{t \in \mathbb{R} \mid f_\lambda(x) \leq_K f_\lambda(y) + te\},$$

in the similar way to Example 2, where $f_\lambda = (1 - \lambda) f_0 + \lambda f_1$. The right-hand side of the above equality can be regarded as a convolution of f_λ and the scalarizing function in Sect. 2. Then the λ-minimal solutions of (SP) is characterized by the K-minimal solutions of the following vector optimization (VP):

$$\begin{aligned} \text{(VP)}\ \ \text{Minimize}\ \ & f_\lambda(x) \\ \text{subject to}\ \ & x \in X. \end{aligned}$$

Acknowledgments The author is grateful to anonymous referees for many comments and suggestions improved the quality of the paper. The author's work was partially supported by JSPS KAKENHI Grant Number 25400205.

References

1. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**, 3821–3835 (2012)
2. Gerth, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
3. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational methods in partially ordered spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 17. Springer, New York (2003)
4. Hamel, A., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)
5. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
6. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
7. Jameson, G.: Ordered linear spaces. Lecture Notes in Mathematics, vol. 141. Springer, Berlin (1970)
8. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**, 73–84 (2003)

9. Kuroiwa, D.: Existence of efficient points of set optimization with weighted criteria. J. Nonlinear Convex Anal. **4**, 117–123 (2003)
10. Kuroiwa, D.: On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. **10**, 41–50 (2009)
11. Kuroiwa, D., Nuriya, T.: A generalized embedding vector space in set optimization. Nonlinear analysis and convex analysis, pp. 297–303. Yokohama Publisher, Yokohama (2007)
12. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Anal. **30**, 1487–1496 (1997)
13. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets and set-valued Ky Fan minimax inequality. J. Nonlinear Convex Anal. **11**, 513–525 (2010)
14. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
15. Pallaschke, D., Urbański, R.: Pairs of Compact Convex Sets, Fractional Arithmetic with Convex Sets, Mathematics and its Applications, vol. 548. Kluwer Academic Publishers, Dordrecht (2002)
16. Rådström, H.: An embedding theorem for spaces of convex sets. Proc. Am. Math. Soc. **3**, 165–169 (1952)