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Set Optimization and Applications - The State of the Art

From Set Relations to Set-Valued Risk Measures

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Foreword

I finished an Oxford D.Phil. on multi-criteria optimisation in 1974 and so have been observing and participating in the field for over four decades.

My research interests have included efficiency and tangency, lattice structure, set-valued mappings, convex analysis and convex programming. So it is a delight to see these old friends given a new life and a new purpose in this volume.

Let me turn to some remarks on the specifics of this valuable collection. Early in my research career, I discovered the power of set-valued analysis, whether for more efficient proofs of classical results such as the open mapping theorem, or for new approaches to current research as with the heading results of this volume.

Set Optimization connotes the study of an optimization problem with a set-valued objective. Why should one do this? What are the prospects?

1. A theory for set optimization problems can only be developed if it is accompanied by a convex analysis for set-valued functions; concepts like subdifferentials, Legendre–Fenchel conjugates, dual optimization problems are just too important and too relevant for all kinds of applications to be ignored; however, a "canonical convex analysis" for vector-valued functions did not exist so far (not to speak of set-valued ones), for example, there are many different "conjugates" for vector-valued functions which work under different—more or less restrictive—assumptions; recent developments summarized in this volume may fill this gap by means of set-valued approaches to duality, leading in particular to conjugation, for vector- as well as set-valued functions;
2. the concepts "infimum" and "supremum" are not relevant in the overwhelming part of the existing literature on vector- and set-valued optimization problems; the reason: infima/suprema with respect to vector orders may not exist for many sets (e.g., if the ordering cone does not generate a lattice) or do not provide useful solution concepts whence non-dominated (minimal or maximal) points are usually looked for in multi-criteria decision-making; set relations (order relations on the power set of an ordered set) not only open the way for a revival of the infimum/supremum but also trigger investigation of new solution concepts in vector/set optimization which show a split into infimum/supremum

attainment on the one side and minimality/maximality on the other side (a totally new idea);

3. this new solution concept (Heyde, Löhne 2011, see already Hamel 2004 and also Löhne 2011) provides “a fresh look” (title of Heyde, Löhne 2011) to multi-criteria decision-making problems: the infimum/supremum attainment property provides the decision maker with overall information about what can be potentially achieved whereas the minimal/maximal solutions provide non-dominated outcomes; in this way it is not necessary to look at all “efficient” points but only at enough of them to make a well-informed decision; this helps to answer the question of what is actually understood by a solution of a vector-valued optimization problem. The latter question is rarely answered in a satisfactory way in the many papers on such problems (one sees discussion of one or all “weak” or “efficient” or “properly efficient” minimizers/maximizers, one or all ‘minimal/maximal’ solutions or non-dominated image points; an evenly distributed subset of minimal/maximal image points, etc.);
4. recent developments in mathematical finance produced sets of superhedging portfolios and, more generally, set-valued risk measures which turned out to be appropriate tools for risk evaluation in markets with “frictions” (bid-ask price spreads, transaction costs etc.); dual representation results (such as Kabanov’s 1999 superhedging theorem) have been identified as special cases of general set-valued duality theorems; optimization problems involving set-valued risk measures (optimal risk allocation, risk minimization, and hedging under constraints) are highly desirable subjects of study; the overall question—in statistics, math finance, as well as the mathematics of insurance—how to deal with multivariate risks—is a topical one which may profit from set-valued approaches;
5. the role of scalarization procedures has been clarified; as it is immediately clear and already known for decades, a convex vector- or set-valued function has an equivalent representation by a family of extended real-valued functions (take the collection of support functions of the images, for example). The set-valued approach can be seen as just another (efficient and elegant) calculus for such families; compared to the scalar case additional dual variables appear along with a new dependence of the classical dual variable (“Lagrange multipliers”) on this new one which captures the order structure in the image space. This helps answer the question of what to choose as dual variables in vector/set optimization problems which is usually not answered clearly by the many papers on such problems (linear operators, some special type of nonlinear functions, etc. are possible answers); in mathematical finance applications these new dual variables could be interpreted as “consistent price processes,“ exactly as had been obtained earlier in finance papers (Kabanov 1999, Schachermayer 2004, among others); Set-valued duality in terms of scalarizations also paves the way to efficient algorithms for set-/vector optimization problems–along with a new ‘geometric duality’ which turned out to be extremely useful in particular for linear and polyhedral vector optimization problems; for such problems,

algorithms could be constructed which produce a solutions in the set-valued sense (Löhne, Schrage 2013).

Finally, consideration of a set-valued function as a family of extended real-valued functions provides a link between set optimization theory and generalized convexity; this is an area which needs further exploration. An important application in economics might be utility maximization for incomplete preferences (that is, non-total reflexive and transitive relations); for such relations, multi-utility representations are available (due to Aumann, Evren, Kannai, Marinacci, Ok, and a few others). That is, families of scalar functions which represent the preference; however, the problem of maximizing (expected) utility for such preferences has not yet been addressed. The current set-valued approach could well provide the missing tools.

Newcastle NSW
July 2015

Jonathan Borwein

Preface

In 1998, a special issue of the journal Mathematical Methods of Operations Research was published, edited by Guang-Ya Chen and Johannes Jahn. It was devoted solely to optimization problems with set-valued objective functions.

Since then, major breakthroughs have been made including new "set relations," new solution concepts for set optimization problems and a new framework for a set-valued convex analysis.

The area has been pushed further by the discovery of its relevance for financial mathematics: risk evaluation in markets with "frictions" such as transaction costs or illiquidity effects is best done using set-valued functions. It turns out that results such as the superhedging theorem of Y. Kabanov from 1999 are essentially set-valued duality results, and the dual variables in this superhedging theorem are precisely what the recent theory expects them to be.

Finally, development of algorithms was initiated that can deal with the sometimes scaring complexity of a set-valued objective function and can deliver results which are useful in applications. As a side effect, the theory of vector optimization is not what it used to be: set-valued approaches produced new insides, extensions and in many cases provide methods for repairing unsatisfactory "vector results." Examples of the latter include duality for linear vector optimization problems and Benson's now famous algorithm. The latter method was designed for (linear) vector optimization problems, but appropriate extensions allow the computation of infima and even solutions of set optimization problems.

All of this gave rise to a need to summarize the development. This is what motivated the compilation of this volume. The reader may find both surveys with extended bibliographies and original research articles, which provide evidence for the claims above, as well as open questions. The area of "set optimization" is under rapid development, and it is the opinion of the editors that it is becoming a field in its own right: new tools for example from lattice theory (residuation) and new algebraic structures (conlinear spaces of sets) enter the picture. These even shed new light on scalar optimization theory (the objective function is extended real-valued).

Looking at a bigger picture, there are two common denominators in many of the relevant developments in optimization theory. The first is the departure from linear structures in particular on the "image" side. Conlinear spaces of sets are not linear since there is no inverse addition, a feature that is already shared by the extended reals. Modules over L^0 turn out to be fundamental for capturing features of conditional risk measurement in a dynamic framework. We are, therefore, happy to include a contribution from this new field. The second is the utilization of order-complete lattices which leads to a comeback of the notions "infimum" and "supremum"—in particular in vector optimization where the infimum with respect to a vector order is not very useful or does not even exist. This "complete lattice approach" to set optimization complements the "set relation approach" initiated by D. Kuroiwa in the 1990s.

The editors joined this development at an early stage: two of us (Hamel, Löhne) started working on "set relations" in 2001, and were soon followed by the others. A workshop at Humboldt University Berlin, organized by A. Hamel and R. Henrion in 2003, witnessed the first talk about set-valued risk measures from a set optimization perspective, and two theses were completed in 2005 (Hamel's habilitation, Löhne's Ph.D.) at Martin Luther University Halle-Wittenberg which paved the way for the "lattice approach" to set optimization.

A regular conference is now devoted to set optimization and finance, see www.set-optimization.org. The first edition took place in Lutherstadt-Wittenberg, Germany, 2012 the second one in Brunico-Bruneck, Italy, 2014. The third one is planned for 2016.

We thank all contributors of this volume for their effort and their patience. We thank Springer for publishing this volume. We thank all referees who decisively contributed to the scientific quality of the articles. Last but not least, we thank Prof. J. Jahn because he not only contributed to the editorial work of this volume, but already in 2003[1] shared our vision of a new area in optimization emerging, and also gave us the opportunity to publish and to present our results whenever possible and appropriate.

Brunico — Andreas H. Hamel
Freiberg — Frank Heyde
Halle — Andreas Löhne
Princeton — Birgit Rudloff
Bolzano — Carola Schrage
December 2014

[1] J. Jahn, "Grundlagen der Mengenoptimierung" (in German). Multi-Criteria-und Fuzzy-Systeme in Theorie und Praxis. Deutscher Universitätsverlag, 2003, 37–71.

Contents

Part I
Surveys

A Comparison of Techniques for Dynamic Multivariate Risk Measures

Zachary Feinstein and Birgit Rudloff

Abstract This paper contains an overview of results for dynamic multivariate risk measures. We provide the main results of four different approaches. We will prove under which assumptions results within these approaches coincide, and how properties like primal and dual representation and time consistency in the different approaches compare to each other.

Keywords Dynamic risk measures · Transaction costs · Set-valued risk measures · Multivariate risk

Mathematics Subject Classification (2010): 91B30 · 46N10 · 26E25

1 Introduction

The concept of coherent risk measures was introduced in an axiomatic way in [3, 4] to find the minimal capital required to cover the risk of a portfolio. The notion was relaxed by introducing convex risk measures in [24, 25]. In these papers the risk was measured only at time zero, in a frictionless market, for univariate claims, and with only a single eligible asset that can be used for the capital requirements and serves as the numéraire. We call this the static scalar framework. In this paper these four assumptions will be removed and different methods compared.

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The static assumptions were relaxed by considering dynamic risk measures, where the risk evaluation of a portfolio is updated as time progresses and new information become available. In the dynamic framework time consistency plays an important role and has been studied for example in [8, 14, 17, 45, 48].

Eliminating the assumption that the financial markets are frictionless required a new framework. Since the 'value' of a portfolio is not uniquely determined anymore when bid and ask prices or market illiquidity exist, it is natural to consider portfolios as vectors in physical units instead, i.e. a portfolio is specified by the number of each of the asset which is held as opposed to their value. But even in the absence of transaction costs multivariate claims might be of interest, e.g. when assets are denoted in different currencies with fluctuating exchange rates, or different business lines with no direct exchange or different regularity rules are considered, see [13]. In contrast to frictionless univariate models also the choice of the numéraire assets matters, which leads to different approaches: pick a numéraire and allow capital requirements to be in this numéraire, which allows a risk manager to work with scalar risk measures again (see e.g. [5, 18, 26, 39, 49]); or use the more general numéraire-free approach and allow risk compensation to be made in a basket of assets which leads to risk measures that are set-valued. This approach was first studied in Jouini et al. [36] in the coherent case. Several extensions have been made. In this paper we will introduce four approaches to deal with dynamic multivariate risk measures, and compare and relate them by giving conditions under which the results obtained in each approach coincide. The four approaches we discuss are

1. a set-optimization approach;
2. a measurable selector approach;
3. an approach utilizing set-valued portfolios; and
4. a family of multiple asset scalar risk measures.

The first three approaches correspond to the numéraire-free framework, whereas the last approach includes scalar risk measures where a numéraire asset is chosen.

In [28–31] the results of [36] were extended to the convex case and a stochastic market model. The extension of the dual representation results were made possible by an application of convex analysis for set-valued functions (set-optimization), see Hamel [27]. The dynamic case and time consistency was studied in [21, 23]. We will call this approach the set-optimization approach. The values of risk measures and its minimal elements in this framework have been studied and computed in [22, 32, 33, 41, 42] via Benson's algorithm for linear vector optimization (see e.g. [40]) in the coherent and polyhedral convex case, respectively via an approximation algorithm in the convex case, see [22, 42].

Tahar and Lépinette [52] extended the results of [36] for coherent risk measures to the dynamic case. We will call this the measurable selector approach as it considers the value of a risk measures as a random set, and then provides a primal and dual representation for the measurable selectors in that set. Time consistency properties were also introduced and some equivalent characterizations discussed.

Most recently, in [13], set-valued coherent risk measures were considered as functions from random sets into the upper sets. The transaction costs model, and other financial considerations like trading constraints, or illiquidity, are then embedded into the construction of "set-valued portfolios". A subclass of risk measures in this framework can be constructed using a vector of scalar risk measures and [13] gives upper and lower bounds as well as dual representations for this subclass. We will present here the dynamic extension of this approach. Time consistency properties have not yet been studied within this framework. However, by comparing and relating the different approaches we will see that a larger subclass can be obtained by using the set-valued risk measures of the set-optimization approach, which provides already a link to dual representations and time consistency properties for this larger subclass.

The fourth approach is to consider a family of dynamic scalar risk measures to evaluate the risk of a multivariate claim. This approach has not been studied so far in the dynamic case. In the special case of frictionless markets, the family of scalar risk measures coincides with scalar risk measures using multiple eligible assets as discussed in [5, 18, 26, 39, 49]. Also the scalar static risk measure of multivariate claims with a single eligible asset studied in [11]; the scalar liquidity adjusted risk measures in market with frictions as studied in [53]; and the scalar superhedging price in markets with transaction costs, see [7, 10, 35, 41, 44, 46, 47], are special cases of this approach. Thus, the family of dynamic scalar risk measures for portfolio vectors generalizes these special cases in a unified way to allow for frictions, multiple eligible assets, and multivariate portfolios in a dynamic framework. The connection to the set-optimization approach allows to utilize the dual representation and time consistency results deduced there.

Other papers in the context of set-valued risk measures are [51], where an extension of the tail conditional expectation to the set-valued framework of [36] was presented and a numerical approximation for calculation was given; and [12], where set-valued risk measures in a more abstract setting were studied and a consistent structure for scalar-valued, vector-valued, and set-valued risk measures (but for constant solvency cones) was created. Furthermore, in [12] distribution based risk measures were extended to the set-valued framework via depth-trimmed regions. More recently, vector-valued risk measures were studied in [6].

Section 2 introduces the four approaches mentioned above. In Sect. 3 these four approaches are compared by showing how the set-optimization approach corresponds to each of the other three. For each comparison, assumptions are given under which there is a one-to-one relationship between the approaches. These relations allow generalizations in most of the different approaches that go beyond the results obtained so far.

2 Dynamic Risk Measures

Consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P}\right)$ satisfying the usual conditions with $\mathcal{F}_0$ being the completed trivial sigma algebra and $\mathcal{F}_T = \mathcal{F}$. Let $|\cdot|$ be an arbitrary norm in $\mathbb{R}^d$. Denote $L_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ for $p \in [0, +\infty]$ (with $L^p := L_T^p$). If $p = 0$, L_t^0 is the linear space of the equivalence classes of $\mathcal{F}_t$-measurable functions $X : \Omega \to \mathbb{R}^d$. For $p > 0$, L_t^p denotes the linear space of $\mathcal{F}_t$-measurable functions $X : \Omega \to \mathbb{R}^d$ such that $\|X\|_p = \left(\int_\Omega |X(\omega)|^p d\mathbb{P}\right)^{1/p} < +\infty$ for $p \in (0, +\infty)$, and $\|X\|_\infty = \text{ess sup}_{\omega \in \Omega} |X(\omega)| < +\infty$ for $p = +\infty$. For $p \in [1, +\infty]$ we will consider the dual pair $\left(L_t^p, L_t^q\right)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = +\infty$ when $p = 1$ and $q = 1$ when $p = +\infty$), and endow it with the norm topology, respectively the weak* topology (that is the $\sigma\left(L_t^\infty, L_t^1\right)$-topology on L_t^∞) in the case $p = +\infty$ unless otherwise noted.

We write $L_{t,+}^p = \left\{X \in L_t^p : X \in \mathbb{R}_+^d \ \mathbb{P}\text{-a.s.}\right\}$ for the closed convex cone of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random vectors with non-negative components. Similarly define $L_+^p := L_{T,+}^p$. We denote by $L_t^p(D_t)$ those random vectors in L_t^p that take $\mathbb{P}$-a.s. values in D_t. Let $1_D : \Omega \to \{0, 1\}$ be the indicator function of $D \in \mathcal{F}$ defined by $1_D(\omega) = 1$ if $\omega \in D$ and 0 otherwise. Throughout we will consider the summation of sets by Minkowski addition. To distinguish the spaces of random vectors from those of random variables, we will write $L_t^p(\mathbb{R}) := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ for the linear space of the equivalence classes of p integrable $\mathcal{F}_t$-measurable random variables $X : \Omega \to \mathbb{R}$. Note that an element $X \in L_t^p$ has components $X_1, \ldots, X_d$ in $L_t^p(\mathbb{R})$.

(In-)equalities between random vectors are always understood componentwise in the $\mathbb{P}$-a.s. sense. The multiplication between a random variable $\lambda \in L_t^\infty(\mathbb{R})$ and a set of random vectors $D \subseteq L^p$ is understood in the elementwise sense, i.e. $\lambda D = \{\lambda Y : Y \in D\} \subseteq L^p$ with $(\lambda Y)(\omega) = \lambda(\omega)Y(\omega)$. The multiplication and division between (random) vectors is understood in the componentwise sense, i.e. $xy := (x_1 y_1, \ldots, x_d y_d)^\mathsf{T}$ and $x/y := (x_1/y_1, \ldots, x_d/y_d)^\mathsf{T}$ for $x, y \in \mathbb{R}^d$ ($x, y \in L_t^p$) and with $y_i \neq 0$ (almost surely) for every index $i \in \{1, \ldots, d\}$ for division.

As in [37] and discussed in [38, 50], the portfolios in this paper are in "physical units" of an asset rather than the value in a fixed numéraire, except where otherwise mentioned. That is, for a portfolio $X \in L_t^p$, the values of X_i (for $1 \leq i \leq d$) are the number of units of asset i in the portfolio at time t.

Let $\tilde{M}_t[\omega]$ denote the set of eligible portfolios, i.e. those portfolios which can be used to compensate for the risk of a portfolio, at time t and state ω. We assume $\tilde{M}_t[\omega]$ is a linear subspace of $\mathbb{R}^d$ for almost every $\omega \in \Omega$. It then follows that $M_t := L_t^p(\tilde{M}_t)$ is a closed (and additionally weak* closed if $p = +\infty$) linear subspace of L_t^p, see Sect. 5.4 and Proposition 5.5.1 in [38]. For example, $\tilde{M}_t[\omega]$ could specify a certain ratio of Euros and Dollars to be used for risk compensations. Another typical example is the case where a subset of assets are used for capital requirements, i.e. $\tilde{M}_t^n[\omega] = \left\{m \in \mathbb{R}^d : \forall i \in \{n+1, \ldots, d\} : m_i = 0\right\}$ and $M_t^n = L_t^p(\tilde{M}_t^n)$. We will denote $M_{t,+} := M_t \cap L_{t,+}^p$ to be the nonnegative elements of M_t. We will assume that $M_{t,+} \neq \{0\}$, i.e. $M_{t,+}$ is nontrivial.

In the first three methods discussed below the risk measures have set-valued images. In the set-optimization approach (Sect. 2.1) and the set-valued portfolio approach (Sect. 2.3) the image space is explicitly given by the upper sets, i.e. $\mathcal{P}\left(M_t; M_{t,+}\right)$ where $\mathcal{P}(\mathcal{Z}; C) := \{D \subseteq \mathcal{Z} : D = D + C\}$ for some vector space $\mathcal{Z}$ and an ordering cone $C \subset \mathcal{Z}$. Additionally, let $\mathcal{G}(\mathcal{Z}; C) := \{D \subseteq \mathcal{Z} : D = \text{cl co}(D + C)\} \subseteq \mathcal{P}(\mathcal{Z}; C)$ be the upper closed convex subsets. It seems natural to use upper sets as the values of risk measures since if one portfolio can cover the risk then any larger portfolio should also cover this risk. Alternatively, one could consider the set of "minimal elements" of the risk compensating portfolios. However, in contrast to the upper sets, the set of "minimal elements" is in general not a convex set when convex risk measure are considered.

2.1 Set-Optimization Approach

The set-optimization approach to dynamic risk measures is studied in [21, 23], where set-valued risk measures [29, 31] were extended to the dynamic case. A benefit of this method is that dual representations are obtained by a direct application of the set-valued duality developed in [27], which allowed for the first time to study not only conditional coherent, but also convex set-valued risk measures.

In this setting we consider risk measures that map a portfolio vector into the complete lattice $\mathcal{P}\left(M_t; M_{t,+}\right)$ of upper sets.

Set-valued conditional risk measures have been defined in [21]. Here we give a stronger property for finiteness at zero than in [21] to ease the comparison to the other approaches.

Definition 2.1 A **conditional risk measure** is a mapping $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ which satisfies:

1. L^p_+-monotonicity: if $Y - X \in L^p_+$ then $R_t(Y) \supseteq R_t(X)$;
2. M_t-translativity: $R_t(X + m) = R_t(X) - m$ for any $X \in L^p$ and $m \in M_t$;
3. finiteness at zero: $R_t(0) \neq \emptyset$ and $R_t(0)[\omega] \neq \tilde{M}_t[\omega]$ for almost every $\omega \in \Omega$, where $R_t(0)[\omega] := \{u(\omega) : u \in R_t(0)\}$.

For finiteness at zero, and elsewhere in later sections, we consider the ω projection of the risk compensating set $R_t(X)$. We point out that $R_t(X)$ is a collection of random vectors and is *not* a random set; therefore $R_t(X)[\omega] := \{u(\omega) : u \in R_t(X)\}$ is the collection of risk covering portfolios at state ω. As $R_t(X)$ is not a random set, it is generally the case that $R_t(X) \neq L^p_t(R_t(X)) := \{u \in M_t : \mathbb{P}(\omega \in \Omega : u(\omega) \in R_t(X)[\omega]) = 1\}$.

Below we consider additional properties for conditional risk measures that have useful financial and mathematical interpretations. Note that the definition for K-compatibility below is more general than the one given in [21], and corresponds to the definition in [32]. A conditional risk measure R_t at time t is

- ***convex (conditionally convex)*** if for all $X, Y \in L^p$ and any $\lambda \in [0, 1]$ (respectively $\lambda \in L^\infty_t(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$R_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y);$$

- ***positive homogeneous (conditionally positive homogeneous)*** if for all $X \in L^p$ and any $\lambda \in \mathbb{R}_{++}$ (respectively $\lambda \in L^\infty_t(\mathbb{R}_{++})$)

$$R_t(\lambda X) = \lambda R_t(X);$$

- ***coherent (conditionally coherent)*** if it is convex and positive homogeneous (respectively conditionally convex and conditionally positive homogeneous);
- ***normalized*** if $R_t(X) = R_t(X) + R_t(0)$ for every $X \in L^p$;
- ***local*** if for every $D \in \mathcal{F}_t$ and every $X \in L^p$, $1_D R_t(X) = 1_D R_t(1_D X)$;
- K***-compatible*** for some convex cone $K \subseteq L^p$ if $R_t(X) = \bigcup_{k \in K} R_t(X - k)$;
- ***closed*** if the graph of the risk measure

$$\operatorname{graph}(R_t) = \left\{(X, u) \in L^p \times M_t : u \in R_t(X)\right\}$$

is closed in the product topology (with the weak* topology if $p = +\infty$);
- ***convex upper continuous*** if

$$R_t^{-1}(D) := \left\{X \in L^p : R_t(X) \cap D \neq \emptyset\right\}$$

is closed (weak* closed if $p = +\infty$) for any closed convex set $D \in \mathcal{G}(M_t; M_{t,-})$.

(Conditional) convexity and coherence for a risk measure define a regulatory framework which promotes diversification. Set-valued normalization is a generalization of the scalar normalization (zero capital needed to compensate the risk of the 0 portfolio). The local property means that the risks at some state (in $\mathcal{F}_t$) only depend on the possible future values of the portfolio reachable from that state. K-compatibility is closely related to a market model; assume for the moment an investor can trade the initial portfolio 0 into any random vector in $-K$ by the terminal time T, then K-compatibility means considering the (minimal) risk of a portfolio when all possible trades are taken into account. The closure is the set-valued version of lower semicontinuity and is necessary for the dual representation to hold. Convex upper continuity is a stronger property than closure and is useful when characterizing or creating multi-portfolio time consistent risk measures, the details will be given below.

A ***dynamic risk measure*** is a sequence $(R_t)_{t=0}^T$ of conditional risk measures. A dynamic risk measure is said to have a certain property if R_t has that property for all times t.

A static risk measure in the sense of [31] is a conditional risk measure at time 0. Note that for static risk measures convexity (positive homogeneity) coincides with conditional convexity (conditional positive homogeneity).

Any conditionally convex risk measure $R_t : L^p \to \mathcal{P}\left(M_t; M_{t,+}\right)$ is local, see Proposition 2.8 in [21].

Definition 2.2 A set $A_t \subseteq L^p$ is a **conditional acceptance set** at time t if it satisfies $A_t + L^p_+ \subseteq A_t$, $M_t \cap A_t \neq \emptyset$, and $\tilde{M}_t[\omega] \cap (\mathbb{R}^d \backslash A_t[\omega]) \neq \emptyset$ for almost every $\omega \in \Omega$, where $A_t[\omega] = \{X(\omega) : X \in A_t\}$.

The acceptance set of a conditional risk measure R_t is given by $A_t = \left\{X \in L^p : 0 \in R_t(X)\right\}$, which is the collection of "risk free" portfolios. For any conditional acceptance set A_t, the function defined by $R_t(X) = \{u \in M_t : X + u \in A_t\}$ is a conditional risk measure. This is the primal representation for conditional risk measures via acceptance sets, see [21]. This relation is one-to-one, i.e. we can consider an (R_t, A_t) pair or equivalently just one of the two. Given a risk measure and acceptance set pair (R_t, A_t) then the following properties hold, see Proposition 2.11 in [21].

- R_t is normalized if and only if $A_t + A_t \cap M_t = A_t$;
- R_t is (conditionally) convex if and only if A_t is (conditionally) convex;
- R_t is (conditionally) positive homogeneous if and only if A_t is a (conditional) cone;
- R_t has a closed graph if and only if A_t is closed.

For the duality results below we will consider $p \in [1, +\infty]$. Let $\mathcal{M}$ denote the set of d-dimensional probability measures absolutely continuous with respect to $\mathbb{P}$, and let $\mathcal{M}^e$ denote the set of d-dimensional probability measures equivalent to $\mathbb{P}$. We will say $\mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}$ for vector probability measures $\mathbb{Q}$ and some time $t \in [0, T]$ if for every $D \in \mathcal{F}_t$ it follows that $\mathbb{Q}_i(D) = \mathbb{P}(D)$ for all $i = 1, \ldots, d$. Consider $\mathbb{Q} \in \mathcal{M}$. We will use a $\mathbb{P}$-almost sure version of the $\mathbb{Q}$-conditional expectation of $X \in L^p$ given by

$$\mathbb{E}^{\mathbb{Q}}\left[X|\,\mathcal{F}_t\right] := \mathbb{E}\left[\xi_{t,T}(\mathbb{Q})X\middle|\,\mathcal{F}_t\right],$$

where $\xi_{r,s}(\mathbb{Q}) = \left(\bar{\xi}_{r,s}(\mathbb{Q}_1), \ldots, \bar{\xi}_{r,s}(\mathbb{Q}_d)\right)^{\mathsf{T}}$ for any times $0 \le r \le s \le T$ with

$$\bar{\xi}_{r,s}(\mathbb{Q}_i)[\omega] := \begin{cases} \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right](\omega)}{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_r\right](\omega)} & \text{on } \mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\,\mathcal{F}_r\right](\omega) > 0 \\ 1 & \text{else} \end{cases}$$

for every $\omega \in \Omega$, see e.g. [15, 21]. For any probability measure $\mathbb{Q}_i \ll \mathbb{P}$ and any times $0 \le r \le s \le t \le T$, it follows that $\frac{d\mathbb{Q}_i}{d\mathbb{P}} = \bar{\xi}_{0,T}(\mathbb{Q}_i)$, $\bar{\xi}_{t,s}(\mathbb{Q}_i) = \bar{\xi}_{t,r}(\mathbb{Q}_i)\bar{\xi}_{r,s}(\mathbb{Q}_i)$, and $\mathbb{E}\left[\bar{\xi}_{r,s}(\mathbb{Q}_i)\middle|\,\mathcal{F}_r\right] = 1$ almost surely. The halfspace and the conditional "halfspace" in L_t^p with normal direction $w \in L_t^q\backslash\{0\}$ are denoted by

$$G_t(w) := \left\{u \in L_t^p : 0 \le \mathbb{E}\left[w^{\mathsf{T}}u\right]\right\}, \qquad \Gamma_t(w) := \left\{u \in L_t^p : 0 \le w^{\mathsf{T}}u \ \mathbb{P}\text{-a.s.}\right\}.$$

We will define the set of dual variables to be

$$\mathcal{W}_t := \left\{(\mathbb{Q}, w) \in \mathcal{M} \times \left(M_{t,+}^{+}\backslash M_t^{\perp}\right) : w_t^T(\mathbb{Q}, w) \in L_+^q, \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}\right\},$$

where for any $0 \le t \le s \le T$

$$w_t^s(\mathbb{Q}, w) = w\xi_{t,s}(\mathbb{Q}),$$

$M_t^{\perp} = \left\{v \in L_t^q : \mathbb{E}\left[v^{\mathsf{T}}u\right] = 0 \ \forall u \in M_t\right\}$ and $C^+ = \left\{v \in L_t^q : \mathbb{E}\left[v^{\mathsf{T}}u\right] \ge 0 \ \forall u \in C\right\}$ denotes the positive dual cone of a cone $C \subseteq L_t^p$.

The set of dual variables $\mathcal{W}_t$ consists of two elements. The first component is a vector probability measure absolutely continuous to the physical measure $\mathbb{P}$ and corresponds to the dual element in the traditional scalar theory. The second component reflects the order relation in the image space as the w's are the collection of possible relative weights between the eligible portfolios. This component is not needed in the scalar case. The coupling condition $w_t^T(\mathbb{Q}, w) \in L_+^q$ guarantees that the probability measure $\mathbb{Q}$ and the ordering vector w are "consistent" in the following sense. If a portfolio X is component-wise ($\mathbb{P}$-)almost surely greater than or equal to another portfolio Y, then the $\mathbb{Q}$-conditional expectation keeps that relationship with respect to the order relation defined by w, that is $w^{\mathsf{T}}\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t] \geq w^{\mathsf{T}}\mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_t]$ ($\mathbb{P}$-)almost surely. In the following, we review the duality results from [23]. Note that since we are only considering closed (conditionally) convex risk measures we can restrict the image space to $\mathcal{G}(M_t; M_{t,+}) := \left\{D \subseteq M_t : D = \operatorname{cl} \operatorname{co}\left(D + M_{t,+}\right)\right\}$.

Corollary 2.3 (Corollary 2.4 of [23]) *A conditional risk measure $R_t : L^p \to \mathcal{G}(M_t; M_{t,+})$ is closed and conditionally convex if and only if*

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[-\alpha_t^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}}[-X|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t\right], \tag{2.1}$$

where $-\alpha_t^{\min}$ is the minimal conditional penalty function given by

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \operatorname{cl} \bigcup_{Z\in A_t} \left(\mathbb{E}^{\mathbb{Q}}[Z|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t. \tag{2.2}$$

R_t is additionally conditionally coherent if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}[-X|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t \tag{2.3}$$

with

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_t : w_t^T(\mathbb{Q}, w) \in A_t^+\right\}. \tag{2.4}$$

The more general convex and coherent case reads analogously to Corollary 2.3, just with $\Gamma_t(w)$ replaced by $G_t(w)$ in Eqs. (2.1), (2.2) and (2.3), see Theorem 2.3 in [23]. As shown in [21, 31], the G_t-version of the minimal penalty function $-\alpha_t^{\min}$ is the set-valued (negative) convex conjugate in the sense of [27] and the dual representation is the biconjugate, both with infimum and supremum defined for the image space $\mathcal{G}(M_t; M_{t,+})$.

Remark 2.4 The dual representation given in [36] for the static coherent case (and in [52] for the dynamic case, see Sect. 2.2 below) uses a single dual variable. This set of dual variables from [36] is equivalent to $\left\{w_t^T(\mathbb{Q}, w) : (\mathbb{Q}, w) \in \mathcal{W}_t\right\}$, and as discussed in the Proof of theorem 2.3 in [23], the dual representation (2.1) and (2.3) can be given by this set alone. This means, the results presented in this section include the previously known dual representation results.

We conclude this section by giving a brief description and equivalent characterizations of a time consistency property for set-valued risk measures in the set-optimization approach. The property we will discuss is multi-portfolio time consistency, which was proposed in [21] and further studied in [23]. We also return to the general case with $p \in [0, +\infty]$.

Definition 2.5 A dynamic risk measure $(R_t)_{t=0}^T$ is called **multi-portfolio time consistent** if for all times $t, s \in [0, T]$ with $t < s$, all portfolios $X \in L^p$ and all sets $\mathbf{Y} \subseteq L^p$ the implication

$$R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y) \tag{2.5}$$

is satisfied.

Multi-portfolio time consistency means that if at some time any risk compensating portfolio for X also compensates the risk of some portfolio Y in the set $\mathbf{Y}$, then at any prior time the same relation should hold true. Implicitly within the definition, the choice of eligible portfolios can have an impact on the multi-portfolio time consistency of a risk measure.

In [21], (set-valued) time consistency was also introduced. This property is defined by

$$R_s(X) \subseteq R_s(Y) \Rightarrow R_t(X) \subseteq R_t(Y)$$

for any time $t, s \in [0, T]$ with $t < s$ and any portfolios $X, Y \in L^p$. It is weaker than multi-portfolio time consistency, though in the scalar case both properties coincide.

Before we give some equivalent characterizations for multi-portfolio time consistency, we must give a few additional definitions. These definitions are used for defining the stepped risk measures $R_{t,s} : M_s \to \mathcal{P}(M_t; M_{t,+})$ for $t \leq s$, as discussed in [23, Appendix C]. We denote and define the stepped acceptance set by $A_{t,s} := A_t \cap M_s$. And akin to Corollary 2.3, for the closed conditionally convex and closed (conditionally) coherent stepped risk measures we will define the minimal stepped penalty function (for the conditionally convex case with $M_t \subseteq M_s$) by $-\alpha_{t,s}^{\min}(\mathbb{Q}, w) := \operatorname{cl} \bigcup_{X \in A_{t,s}} \left(\mathbb{E}^{\mathbb{Q}}\left[X \middle| \mathcal{F}_t\right] + \Gamma_t(w)\right) \cap M_t$ for every $(\mathbb{Q}, w) \in \mathcal{W}_{t,s}$ and the maximal stepped dual set (for the (conditionally) coherent case with $M_t \subseteq M_s$) by $\mathcal{W}_{t,s}^{\max} := \left\{(\mathbb{Q}, w) \in \mathcal{W}_{t,s} : w_t^s(\mathbb{Q}, w) \in A_{t,s}^+\right\}$. As can be seen, both the stepped penalty function and the stepped maximal dual set are with respect to dual elements $\mathcal{W}_{t,s}$, which in general differ from $\mathcal{W}_t$. In the case that $\tilde{M}_t = M_0$ almost surely, it holds $\mathcal{W}_{t,s} \supseteq \mathcal{W}_t$ for all times $t \leq s \leq T$; if $\tilde{M}_s = \mathbb{R}^d$ almost surely then $\mathcal{W}_{t,s} = \mathcal{W}_t$.

In the below theorem, for the convex upper continuous (conditionally) coherent case we introduce two more definitions. We define the mapping $H_t^s : 2^{\mathcal{W}_s} \to 2^{\mathcal{W}_t}$ for times $t \leq s \leq T$ by $H_t^s(D) := \left\{(\mathbb{Q}, w) \in \mathcal{W}_t : \left(\mathbb{Q}, w_t^s(\mathbb{Q}, w)\right) \in D\right\}$ for $D \subseteq \mathcal{W}_s$.

Additionally, for $\mathbb{Q}, \mathbb{R} \in \mathcal{M}$ we denote by $\mathbb{Q} \oplus^s \mathbb{R}$ the pasting of $\mathbb{Q}$ and $\mathbb{R}$ in s, i.e. the vector probability measures $\mathbb{S} \in \mathcal{M}$ defined via

$$\frac{d\mathbb{S}}{d\mathbb{P}} = \xi_{0,s}(\mathbb{Q})\xi_{s,T}(\mathbb{R}).$$

The following theorem gives equivalent characterizations of multi-portfolio time consistency: a recursion in the spirit of Bellman's principle (property 2 below), an additive property for the acceptance sets (property 3), the so called cocyclical property (property 4) and stability (property 6). The properties are important for the construction of multi-portfolio time consistent risk measures.

Theorem 2.6 (Theorem 3.4 of [21], Corollaries 3.5 and 4.3 and Theorem 4.6 of [23]) *For a normalized dynamic risk measure $(R_t)_{t=0}^T$ the following are equivalent:*

1. *$(R_t)_{t=0}^T$ is multi-portfolio time consistent,*
2. *R_t is recursive, that is for every time $t, s \in [0, T]$ with $t < s$*

$$R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X)). \tag{2.6}$$

If additionally $M_t \subseteq M_s$ for every time $t, s \in [0, T]$ with $t < s$ then all of the above is also equivalent to

3. *for every time $t, s \in [0, T]$ with $t < s$*

$$A_t = A_s + A_{t,s}. \tag{2.7}$$

If additionally $p \in [1, +\infty]$, $\tilde{M}_t = \mathbb{R}^n \times \{0\}^{d-n}$ almost surely for some $n \leq d$ for every time $t \in [0, T]$, $(R_t)_{t=0}^T$ is a c.u.c. conditionally convex risk measure and

$$R_t(X) = \bigcap_{(\mathbb{Q},w) \in \mathcal{W}_t^e} \left[-\alpha_t^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}}[-X|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t\right]$$

for every $X \in L_T^p$ where $\mathcal{W}_t^e = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \mathbb{Q} \in \mathcal{M}^e\}$, then all of the above is also equivalent to

4. *for every time $t, s \in [0, T]$ with $t < s$*

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \operatorname{cl}\left(-\alpha_{t,s}^{\min}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[-\alpha_s^{\min}(\mathbb{Q}, w_t^s(\mathbb{Q}, w))\middle| \mathcal{F}_t\right]\right) \tag{2.8}$$

   for every $(\mathbb{Q}, w) \in \mathcal{W}_t^e$.

If additionally $p \in [1, +\infty]$, $\tilde{M}_t = \mathbb{R}^n \times \{0\}^{d-n}$ almost surely for some $n \leq d$ for every time $t \in [0, T]$ and $(R_t)_{t=0}^T$ is a c.u.c. (conditionally) coherent risk measure then all of the above is also equivalent to

5. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$\mathcal{W}_t^{\max} = \mathcal{W}_{t,s}^{\max} \cap H_t^s\left(\mathcal{W}_s^{\max}\right), \tag{2.9}$$

which in turn is equivalent to

6. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q} \oplus^s \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,s}^{\max}, (\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in \mathcal{W}_s^{\max}\right\}. \tag{2.10}$$

2.2 Measurable Selector Approach

The measurable selector approach was proposed in [52] and is an extension of [36] to the dynamic framework. Only coherent risk measures are considered in this approach as the technique used to deduce the dual representation relies on coherency. The risk measures are assumed to be compatible to a conical market model at the final time T, i.e. portfolios are compared based on the final "values". In so doing, a new pre-image space denoted by $B_{K_T,n}$ is introduced, which will be defined below and is discuss in Remark 3.1. In [52], the space of eligible assets is $M_t^n = L_t^0(\tilde{M}_t^n)$ with $\tilde{M}_t^n[\omega] = \left\{m \in \mathbb{R}^d : \forall i \in \{n+1, \ldots, d\} : m_i = 0\right\}$, i.e. $n \leq d$ of the d assets can be used to cover risk.

Let $\mathcal{S}_t^d$ be the set of $\mathcal{F}_t$-measurable random sets in $\mathbb{R}^d$. Recall that a mapping $\Gamma : \Omega \to 2^{\mathbb{R}^d}$ is an $\mathcal{F}_t$-measurable random set if

$$\text{graph } \Gamma = \left\{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \Gamma(\omega)\right\}$$

is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra). The random set Γ is closed (convex, conical) if $\Gamma(\omega)$ is closed (convex, conical) for almost every $\omega \in \Omega$.

Let $K_T \in \mathcal{S}_T^d$ satisfy the following assumptions:

k1. for almost every $\omega \in \Omega$: $K_T(\omega)$ is a closed convex cone in $\mathbb{R}^d$;
k2. for almost every $\omega \in \Omega$: $\mathbb{R}_+^d \subseteq K_T(\omega) \neq \mathbb{R}^d$;
k3. for almost every $\omega \in \Omega$: $K_T(\omega)$ is a proper cone, i.e. $K_T(\omega) \cap -K_T(\omega) = \{0\}$.

It is then possible to create a partial ordering in L^0 defined by K_T such that $X \geq_{K_T} Y$ if and only if $\mathbb{P}(X - Y \in K_T) = 1$. The solvency cones with friction, see e.g. [37, 38, 50], satisfy the conditions given above for K_T.

Let $n \leq d$, then we define $B_{K_T,n} := \left\{X \in L^0 : \exists c \in \mathbb{R}_+ : c1_{d,n} \geq_{K_T} X \geq_{K_T} -c1_{d,n}\right\}$ where the i-th component of $1_{d,n} \in \mathbb{R}^d$ is $1_{d,n}^i = \begin{cases} 1 & \text{if } i \in \{1, \ldots, n\} \\ 0 & \text{else} \end{cases}$. Then we can define a norm on $B_{K_T,n}$ by $\|X\|_{K_T,n} := \inf\left\{c \in \mathbb{R}_+ : c1_{d,n} \geq_{K_T} X \geq_{K_T} -c1_{d,n}\right\}$, and $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ defines a Banach space.

Let $\mathcal{S}_t^{d,n} \subseteq \mathcal{S}_t^d$ be such that $\Gamma \in \mathcal{S}_t^{d,n}$ if $\Gamma \in \mathcal{S}_t^d$ and $\Gamma(\omega) \subseteq \tilde{M}_t^n[\omega]$ for almost every $\omega \in \Omega$.

Definition 2.7 A **risk process** is a sequence $(\tilde{R}_t)_{t=0}^T$ of mappings $\tilde{R}_t : B_{K_T,n} \to \mathcal{S}_t^{d,n}$ satisfying

1. $\tilde{R}_t(X)$ is a closed $\mathcal{F}_t$-measurable random set for any $X \in B_{K_T,n}$, $\tilde{R}_t(0) \neq \emptyset$, and $\tilde{R}_t(0)[\omega] \neq \tilde{M}_t^n[\omega]$ for almost every $\omega \in \Omega$.
2. For any $X, Y \in B_{K_T,n}$ with $Y \geq_{K_T} X$ it holds $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$.
3. $\tilde{R}_t(X + m) = \tilde{R}_t(X) - m$ for any $X \in B_{K_T,n}$ and $m \in M_t^n$.

A risk process is **conditionally convex** at time t if for all $X, Y \in B_{K_T,n}$ and $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \leq \lambda \leq 1$ almost surely it holds $\lambda \tilde{R}_t(X) + (1-\lambda)\tilde{R}_t(Y) \subseteq \tilde{R}_t(\lambda X + (1-\lambda)Y)$.

A risk process is **conditionally positive homogeneous** at time t if for all $X \in B_{K_T,n}$ and $\lambda \in L_t^0(\mathbb{R}_{++})$ with $\lambda X \in B_{K_T,n}$ it holds $\tilde{R}_t(\lambda X) = \lambda \tilde{R}_t(X)$.

A risk process is **conditionally coherent** at time t if it is both conditionally convex and conditionally positive homogeneous at time t.

A risk process is **normalized** at time t if $\tilde{R}_t(X) + \tilde{R}_t(0) = \tilde{R}_t(X)$ for every $X \in B_{K_T,n}$.

Thus, the values $\tilde{R}_t(X)$ of a risk process are $\mathcal{F}_t$-measurable random sets in $\mathbb{R}^d$. Primal and dual representations can be provided for the measurable selectors of this set. Recall that γ is a $\mathcal{F}_t$-measurable selector of a $\mathcal{F}_t$-random set Γ if $\gamma(\omega) \in \Gamma(\omega)$ for almost every $\omega \in \Omega$. Then using the notation from above, the measurable selectors in L^p are given by $L_t^p(\Gamma) = \left\{\gamma \in L_t^p : \mathbb{P}(\gamma \in \Gamma) = 1\right\}$.

Definition 2.8 Given a risk process $(\tilde{R}_t)_{t=0}^T$, then $S_{\tilde{R}} : [0, T] \times B_{K_T,n} \to 2^{M_t^n}$ is a **selector risk measure** if $S_{\tilde{R}}(t, X) := L_t^0(\tilde{R}_t(X))$ for every time t and portfolio $X \in B_{K_T,n}$. The **bounded selector risk measure** is defined by $S_{\tilde{R}}^\infty(t, X) := S_{\tilde{R}}(t, X) \cap B_{K_T,n}$.

Definition 2.9 A set $A_t \subseteq B_{K_T,n}$ is a **conditional acceptance set** at time t if:

1. A_t is closed in the $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ topology.
2. If $X \in B_{K_T,n}$ such that $X \geq_{K_T} 0$ then $X \in A_t$.
3. $B_{K_T,n} \cap M_t^n \not\subseteq A_t$.
4. A_t is $\mathcal{F}_t$-decomposable, i.e. if for any finite partition $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$ of Ω and any family $(X_n)_{n=1}^N \subseteq A_t$, then $\sum_{n=1}^N 1_{\Omega_t^n} X_n \in A_t$.
5. A_t is a conditionally convex cone.

Remark 2.10 Note that the definition for $\mathcal{F}_t$-decomposability above differs from that in [52], as in that paper $\mathcal{F}_t$-decomposability is considered with respect to countable rather than finite partitions. We weakened the condition by adapting the Proof of theorem 1.6 of Chap. 2 from [43] when $p = +\infty$ to the space $B_{K_T,n}$.

Proposition 2.11 (Proposition 3.4 of [52]) *Given a conditionally coherent risk process $\tilde{R}_t$ at time t, then $A_t := \left\{X \in B_{K_T,n} : 0 \in \tilde{R}_t(X)\right\}$ is a conditional acceptance set at time t.*

A primal representation of the selector risk measure is given as follows.

Theorem 2.12 (Theorem 3.3 in [52]) *Let A_t be a closed subset of $(B_{K_T,n}, \|\cdot\|_{K_T,n})$. Then A_t is a conditional acceptance set if and only if there exists some conditionally coherent risk process $\tilde{R}_t$ at time t such that the associated bounded selector risk measure $S^{\infty}_{\tilde{R}}$ satisfies $S^{\infty}_{\tilde{R}}(t, X) = \left\{m \in M^n_t : X + m \in A_t\right\}$ for all $X \in B_{K_T,n}$.*

Below, we give the dual representation for coherent selector risk measures as done in Theorems 4.1 and 4.2 of [52]. This dual representation can be viewed as the intersection of supporting halfspaces for the selector risk measure, which is the reason that coherence is needed in this approach.

From [52], it is known that $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ is a Banach space, we will let $ba_{K_T,n}$ be the topological dual of $B_{K_T,n}$, and let $ba^+_{K_T,n}$ denote the positive linear forms, that is

$$ba^+_{K_T,n} := \left\{\phi \in ba_{K_T,n} : \phi(X) \geq 0 \;\forall X \geq_{K_T} 0\right\}.$$

Definition 2.13 (*Definition 4.1 of* [52]) A set $\Lambda \subseteq ba_{K_T,n}$ is called $\mathcal{F}_t$**-stable** if for all $\lambda \in L^{\infty}_t(\mathbb{R}_+)$ and $\phi \in \Lambda$, the linear form $\phi^{\lambda} : X \ni B_{K_T,n} \mapsto \phi(\lambda X)$ is an element of Λ.

Theorem 2.14 (Theorem 4.1 of [52]) *Let $(\tilde{R}_t)_{t=0}^T$ be a sequence of $(\mathcal{S}^{d,n}_t)_{t=0}^T$-valued mappings on $B_{K_T,n}$. Then the following are equivalent:*

1. *$(\tilde{R}_t)_{t=0}^T$ is a conditionally coherent risk process.*
2. *There exists a nonempty $\sigma(ba_{K_T,n}, B_{K_T,n})$-closed subset $\mathcal{Q}_t \neq \{0\}$ of $ba^+_{K_T,n}$ which is $\mathcal{F}_t$-stable and satisfies the equality*

$$S^{\infty}_{\tilde{R}}(t, X) = \left\{u \in M^n_t \cap B_{K_T,n} : \phi(X + u) \geq 0 \;\forall \phi \in \mathcal{Q}_t\right\}. \tag{2.11}$$

We finish the discussion of the dual representation by considering the case when the risk process additionally satisfies a "Fatou property" as defined below.

Definition 2.15 A sequence $(\tilde{R}_t)_{t=0}^T$ of $(\mathcal{S}^{d,n}_t)_{t=0}^T$-valued mappings on $B_{K_T,n}$ is said to satisfy the **Fatou property** if for all $X \in B_{K_T,n}$ and all times t

$$\limsup_{n \to +\infty} S^{\infty}_{\tilde{R}}(t, X_n) \subseteq S^{\infty}_{\tilde{R}}(t, X)$$

for any bounded sequence $(X_m)_{m \in \mathbb{N}} \subseteq B_{K_T,n}$ which converges to X in probability.

Note that in the above definition the limit superior is defined to be $\limsup_{n \to +\infty} B_n = \operatorname{cl} \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} B_m$ for a sequence of sets $(B_n)_{n \in \mathbb{N}}$.

For the following theorem we assume two additional properties on the convex cone K_T:

k4. for almost every $\omega \in \Omega$: $\mathbb{R}^d_+ \backslash \{0\} \subseteq \operatorname{int}[K_T(\omega)]$ or equivalently $K_T(\omega)^+ \backslash \{0\} \subseteq \operatorname{int}[\mathbb{R}^d_+]$;
k5. K_T and K_T^+ are both generated by a finite number of linearly independent and bounded generators denoted respectively by $(\xi_i)_{i=1}^N$ and $(\xi_i^+)_{i=1}^{N^+}$.

Let $L^{1,n}(K_T^+) := \left\{Z \in L^0(K_T^+) : 1_{d,n}^\mathsf{T} Z \in L^1(\mathbb{R})\right\}$. In the following theorem we will use $L^{1,n}(K_T^+)$ as a dual space for $B_{K_T,n}$. For $Z \in L^{1,n}(K_T^+)$, the linear form $\phi_Z(X) := \mathbb{E}\left[Z^\mathsf{T} X\right]$ belongs to $ba^+_{K_T,n}$. The norm $\|Z\|_{d,n} := \sup\left\{|\mathbb{E}\left[Z^\mathsf{T} X\right]| : X \in B_{K_T,n}, \|X\|_{K_T,n} \leq 1\right\}$ is the dual norm for any $Z \in L^{1,n}(K_T^+)$.

Theorem 2.16 (Theorem 4.2 of [52]) *Let $(\tilde{R}_t)_{t=0}^T$ be a conditionally coherent risk process on $B_{K_T,n}$ and let K_T satisfy property $k1 - k5$. The following are equivalent:*

1. *For every time $t \in [0, T]$, there exists a closed conditional cone $\{0\} \neq \mathcal{Q}_t^1 \subseteq L^{1,n}(K_T^+)$ (in the norm topology, with norm $\|\cdot\|_{d,n}$) such that for any $X \in B_{K_T,n}$*

$$S_{\tilde{R}}^\infty(t, X) = \left\{u \in M_t^n \cap B_{K_T,n} : \forall Z \in \mathcal{Q}_t^1 : \mathbb{E}\left[Z^\mathsf{T}(X + u)\right] \geq 0\right\}. \quad (2.12)$$

2. *$(\tilde{R}_t)_{t=0}^T$ satisfies the Fatou property.*
3. *$C_t := \left\{X \in B_{K_T,n} : 0 \in \tilde{R}_t(X)\right\}$ is $\sigma(B_{K_T,n}, L^{1,n}(K_T^+))$-closed.*

We conclude this section by discussing time consistency properties as they were defined in the measurable selector approach in [52]. As in the set-optimization approach in the previous section one would like to define a property that is equivalent to a recursive form. For this reason we will extend the risk process to be a function of a set. For a set $\mathbf{X} \subseteq B_{K_T,n}$, let us define $\tilde{R}_t(\mathbf{X}) \in \mathcal{S}_t^{d,n}$ via its selectors, that is

$$L_t^0(\tilde{R}_t(\mathbf{X})) \cap B_{K_T,n} = \operatorname{cl} \operatorname{env}_{\mathcal{F}_t} \bigcup_{X \in \mathbf{X}} S_{\tilde{R}}^\infty(t, X) =: S_{\tilde{R}}^\infty(t, \mathbf{X}),$$

where, for any $\Gamma \subseteq B_{K_T,n}$, $\operatorname{env}_{\mathcal{F}_t}\Gamma$ denotes the smallest $\mathcal{F}_t$-decomposable set (see Definition 2.9) which contains Γ. This means that the measurable selectors of the risk process of a set are defined by the closed and $\mathcal{F}_t$-decomposable version of the pointwise union. Note that if $\mathbf{X} = \{X\}$ then this reduces to the prior definition on portfolios. The risk process of a set is defined in this way because the selection risk measure must be closed and $\mathcal{F}_t$-decomposable-valued to guarantee the existence of an $\mathcal{F}_t$-measurable random set $\tilde{R}_t(\mathbf{X})$ such that $S_{\tilde{R}}^\infty(t, \mathbf{X}) = L_t^0(\tilde{R}_t(\mathbf{X})) \cap B_{K_T,n}$.

Definition 2.17 A risk process $(\tilde{R}_t)_{t=0}^T$ is called **consistent in time** if for any $t, s \in [0, T]$ with $t < s$ and $X \in B_{K_T,n}$, $\mathbf{Y} \subseteq B_{K_T,n}$

$$\tilde{R}_s(X) \subseteq \tilde{R}_s(\mathbf{Y}) \Rightarrow \tilde{R}_t(X) \subseteq \tilde{R}_t(\mathbf{Y}).$$

The following theorem gives equivalent characterizations of consistency in time, the last one being a recursion in the spirit of Bellman's principle.

Theorem 2.18 (Theorem 5.1 of [52]) *A normalized risk process* $(\tilde{R}_t)_{t=0}^T$ *on* $B_{K_T,n}$ *is consistent in time if any of the following equivalent conditions hold:*

1. *If* $\tilde{R}_s(X) \subseteq \tilde{R}_s(\mathbf{Y})$ *for* $X \in B_{K_T,n}$ *and* $\mathbf{Y} \subseteq B_{K_T,n}$*, then* $\tilde{R}_t(X) \subseteq \tilde{R}_t(\mathbf{Y})$ *for* $t \leq s \leq T$.
2. *If* $\tilde{R}_s(X) = \tilde{R}_s(\mathbf{Y})$ *for* $X \in B_{K_T,n}$ *and* $\mathbf{Y} \subseteq B_{K_T,n}$*, then* $\tilde{R}_t(X) = \tilde{R}_t(\mathbf{Y})$ *for* $t \leq s \leq T$.
3. *For all* $X \in B_{K_T,n}$*,* $S^{\infty}_{\tilde{R}}(t, X) = S^{\infty}_{\tilde{R}}(t, -S^{\infty}_{\tilde{R}}(s, X))$ *for* $t \leq s \leq T$.

2.3 Set-Valued Portfolio Approach

The approach for considering sets of portfolios, so called set-valued portfolios, as the argument of a set-valued risk measure was proposed in [13]. The reasoning for considering set-valued portfolios is to take the risk, not only of a portfolio X, but of every possible portfolio that X can be traded for in the market, into account. We will denote by $\mathbf{X}$ the random set of portfolios for which $X \in L^p$ can be exchanged. The concept of set-valued portfolios appears naturally when trading opportunities in the market are taken into account. Below we provide two examples, one in which no trading is allowed and another in which any possible trade can be used. There are other examples provided in [13] on how a set-valued portfolio can be obtained, and the definition of the risk measure is independent of the method used to construct set-valued portfolios.

Example 2.19 The random mapping $\mathbf{X} = X + \mathbb{R}^d_-$ for a random vector $X \in L^p$ describes the case when no exchanges are allowed.

Example 2.20 (Example 2.2 of [13]) The random mapping $\mathbf{X} = X + \mathbf{K}$ for a random vector $X \in L^p$ and a lower convex (random) set $\mathbf{K}$, such that $L^p(\mathbf{K})$ is closed, defines the set-valued portfolios related to the exchanges defined by $\mathbf{K}$. If K is a solvency cone (see e.g. [37, 38, 50]) or the sum of solvency cones at different time points, then $\mathbf{K} = -K$ is an exchange cone, and the associated random mapping defines a set-valued portfolio. The setting of Example 2.19 corresponds to the case where $\mathbf{K} = \mathbb{R}^d_-$.

We will slightly adjust the definitions given in [13] to include the dynamic extension of such risk measures, to incorporate the set of eligible portfolios M_t, and go beyond the coherent case.

Let $\mathcal{S}^d_T$ denote the set of $\mathcal{F}$-random sets in $\mathbb{R}^d$ (as in Sect. 2.2 above). Let $\bar{\mathcal{S}}^d_T \subseteq \mathcal{S}^d_T$ be those random sets that are nonempty, closed, convex and lower, that is for $X \in \mathbf{X}$ also $Y \in \mathbf{X}$ whenever $X - Y \in \mathbb{R}^d_+$ $\mathbb{P}$-a.s. As in [13], we will consider set-valued portfolios $\mathbf{X} \in \bar{\mathcal{S}}^d_T$. By Proposition 2.1.5 and Theorem 2.1.6 in [43], the collection of p-integrable selectors of $\mathbf{X}$, that is $L^p(\mathbf{X})$, is a nonempty, closed, ($\mathcal{F}$-)conditionally convex, lower and $\mathcal{F}$-decomposable set, which is an element of $\mathcal{G}(L^p; L^p_-)$. In [13], $\bar{\mathcal{S}}^d_T$ is used as the pre-image set, one could also use the family of sets of selectors $\{L^p(\mathbf{X}) : \mathbf{X} \in \bar{\mathcal{S}}^d_T\} \subset \mathcal{G}(L^p; L^p_-)$ as the pre-image set, which is particular useful when dynamic risk measures are considered and recursions due to multi-portfolio time consistency become important. Recall that $\mathcal{P}(M_t; M_{t,+}) := \left\{D \subseteq M_t : D = D + M_{t,+}\right\}$ denotes the set of upper sets, which will be used as the image space for the risk measures. Closed (conditionally) convex risk measures map into $\mathcal{G}(M_t; M_{t,+})$.

In the following definition for convex risk measures we consider a modified version of set-addition used in [13] which is denoted by $\oplus$. For two random sets $\mathbf{X}, \mathbf{Y} \in \mathcal{S}^d_T$, $\mathbf{X} \oplus \mathbf{Y} \in \mathcal{S}^d_T$ is the random set defined by the closure of $\mathbf{X}[\omega] + \mathbf{Y}[\omega]$ for all $\omega \in \Omega$. Note that, by Proposition 2.1.4 in [43], if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic and $p \in [1, +\infty)$ then $L^p(\mathbf{X} \oplus \mathbf{Y}) = \operatorname{cl}\left[L^p(\mathbf{X}) + L^p(\mathbf{Y})\right]$.

Definition 2.21 (*Definition 2.9 of* [13]) A function $\mathbf{R}_t : \bar{\mathcal{S}}^d_T \to \mathcal{P}(M_t; M_{t,+})$ is called a **set-valued conditional risk measure** if it satisfies the following conditions.

1. Cash invariance: $\mathbf{R}_t(\mathbf{X} + m) = \mathbf{R}_t(\mathbf{X}) - m$ for any $\mathbf{X}$ and $m \in M_t$.
2. Monotonicity: Let $\mathbf{X} \subseteq \mathbf{Y}$ almost surely, then $\mathbf{R}_t(\mathbf{Y}) \supseteq \mathbf{R}_t(\mathbf{X})$.

The risk measure $\mathbf{R}_t$ is said to be **closed-valued** if its values are closed sets.

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) convex** if for every set-valued portfolio $\mathbf{X}, \mathbf{Y}$ and $\lambda \in [0, 1]$ (respectively $\lambda \in L^\infty_t(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$\mathbf{R}_t(\lambda\mathbf{X} \oplus (1-\lambda)\mathbf{Y}) \supseteq \lambda\mathbf{R}_t(\mathbf{X}) + (1-\lambda)\mathbf{R}_t(\mathbf{Y}).$$

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) positive homogeneous** if for every $\mathbf{X}$ and $\lambda > 0$ (respectively $\lambda \in L^\infty_t(\mathbb{R}_{++})$)

$$\mathbf{R}_t(\lambda\mathbf{X}) = \lambda\mathbf{R}_t(\mathbf{X}).$$

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) coherent** if it is (conditionally) convex and (conditionally) positive homogeneous.

The closed-valued variant of R_t is denoted by $\bar{\mathbf{R}}_t(\mathbf{X}) = \operatorname{cl}(\mathbf{R}_t(\mathbf{X}))$ for every set-valued portfolio $\mathbf{X} \in \bar{\mathcal{S}}^d_T$.

A set-valued portfolio $\mathbf{X}$ is acceptable if $0 \in \mathbf{R}_t(\mathbf{X})$, i.e. we can define the acceptance set $\mathbf{A}_t \subseteq \bar{\mathcal{S}}^d_T$ by $\mathbf{A}_t := \{\mathbf{X} : 0 \in \mathbf{R}_t(\mathbf{X})\}$. And a primal representation for the risk measures can be given by the usual definition $\mathbf{R}_t(\mathbf{X}) = \{u \in M_t : \mathbf{X} + u \in \mathbf{A}_t\}$ due to cash invariance.

We will now consider a subclass of set-valued conditional risk measures presented in [13, Sect. 3] that are constructed using a scalar dynamic risk measure for each component. For the remainder of this section we will consider the case when $M_t = L^p_t$. In [13], only (scalar) law invariant coherent risk measures were considered for this approach, we will consider the more general case.

Let $\rho_t^1, \ldots, \rho_t^d$ be dynamic risk measures defined on $L^p(\mathbb{R})$ with values in $L_t^p(\mathbb{R} \cup \{+\infty\})$. For a random vector $X = (X_1, \ldots, X_d)^\mathsf{T} \in L^p$ we define

$$\boldsymbol{\rho}_t(X) = \left(\rho_t^1(X_1), \ldots, \rho_t^d(X_d)\right)^\mathsf{T}.$$

We say the vector $X \in L^p$ is **acceptable** if $\boldsymbol{\rho}_t(X) \leq 0$, i.e. $\rho_t^i(X_i) \leq 0$ for all $i = 1, \ldots, d$. We say the set-valued portfolio $\mathbf{X}$ is **acceptable** if there exists a $Z \in L^p(\mathbf{X})$ such that $\boldsymbol{\rho}_t(Z) \leq 0$.

Definition 2.22 (*Definition 3.3 of* [13]) The **constructive conditional risk measure** $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(L_t^p; L_{t,+}^p)$ is defined for any set-valued portfolio $\mathbf{X}$ by

$$\mathbf{R}_t(\mathbf{X}) = \left\{u \in L_t^p : \mathbf{X} + u \text{ is acceptable}\right\},$$

which is equivalently to

$$\mathbf{R}_t(\mathbf{X}) = \bigcup_{Z \in L^p(\mathbf{X})} (\boldsymbol{\rho}_t(Z) + L_{t,+}^p). \tag{2.13}$$

The closed-valued variant is defined by $\bar{\mathbf{R}}_t(\mathbf{X}) := \mathrm{cl}(\mathbf{R}_t(\mathbf{X}))$ for every $\mathbf{X} \in \bar{\mathcal{S}}_T^d$.

In [13], the constructive (static) risk measures have been called selection risk measures, we modified the name here in accordance to the title of the paper [13] to avoid confusion with the measurable selector approach from Sect. 2.2.

Example 2.23 Consider the no-exchange set-valued portfolios from Example 2.19. Then the constructive conditional risk measure associated with any vector of scalar conditional risk measures is given by

$$\mathbf{R}_t(\mathbf{X}) = \boldsymbol{\rho}_t(X) + L_{t,+}^p.$$

Theorem 2.24 (Theorem 3.4 of [13]) *Let $\boldsymbol{\rho}_t$ be a vector of dynamic risk measures, then $\mathbf{R}_t$ and $\bar{\mathbf{R}}_t$ given in Definition 2.22 are both set-valued conditional risk measures.*

If $\boldsymbol{\rho}_t$ is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an non-atomic probability space), then $\mathbf{R}_t$ and $\bar{\mathbf{R}}_t$ are convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an non-atomic probability space).

Furthermore, [13] gives conditions under which the constructive (static) risk measure $\mathbf{R}_0$ defined in (2.13) in the coherent case is closed, or Lipschitz and deduces upper and lower bounds for it and dual representations in certain special cases. Numerical examples for the calculation of upper and lower bounds are given.

2.4 Family of Scalar Risk Measures

Consider L^p, $p \in [1, +\infty]$ with the norm topology for $p \in [1, +\infty)$ and the weak* topology for $p = +\infty$. Recall from Definition 2.2 that a set $A_t \subseteq L^p$ is a ***conditional acceptance set*** at time t if it satisfies $M_t \cap A_t \neq \emptyset$, $\tilde{M}_t[\omega] \cap (\mathbb{R}^d \backslash A_t[\omega]) \neq \emptyset$ for almost every $\omega \in \Omega$, and $A_t + L^p_+ \subseteq A_t$.

We will define a family of scalar conditional risk measures ρ_t^w with parameter $w \in M_{t,+}^+ \backslash M_t^\perp$ via their primal representation. The scalar risk measures map into the random variables with values in the extended real line, that is, into the space $L_t^0(\bar{\mathbb{R}})$ with $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition 2.25 A function $\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}})$ satisfying

$$\rho_t^w(X) = \operatorname{ess\,inf}\left\{w^\mathsf{T} u : u \in M_t, X + u \in A_t\right\} \tag{2.14}$$

for a parameter $w \in M_{t,+}^+ \backslash M_t^\perp$ and a conditional acceptance set A_t is called a ***multiple asset conditional risk measure*** at time t.

Clearly, the scalar risk measures defined above are scalarizations of a set-valued risk measure from the set-optimization approach (see Sect. 2.1) defined by $R_t := \{u \in M_t : X + u \in A_t\}$, where the scalarizations are taken with respect to vectors $w \in M_{t,+}^+ \backslash M_t^\perp$, that is

$$\rho_t^w(X) = \operatorname*{ess\,inf}_{u \in R_t(X)} w^\mathsf{T} u = \operatorname{ess\,inf}\left\{w^\mathsf{T} u : u \in M_t, X + u \in A_t\right\}. \tag{2.15}$$

Note, that when R_t is K-compatible (that is $A_t = A_t + L_t^p(K)$) for some $\mathcal{F}_t$-measurable random cone $K \subseteq \tilde{M}_t$, then $\rho_t^w(X)[\omega] = -\infty$ on $w(\omega) \notin K[\omega]^+$ for any $X \in L^p$. Thus, one can restrict oneself in this case to parameters w in the basis of $L_t^q(K^+) \backslash M_t^\perp$.

We will give some examples from the literature of scalar risk measures of form (2.14).

Example 2.26 In [5, 18, 26, 39, 49] risk measures of form (2.14) have been studied in the static case.

In a frictionless market let the time t prices be given by the (random) vector S_t. In this case the solvency cones (see [37, 38, 50]) $(K_t)_{t=0}^T$ are given by $K_t[\omega] = \left\{x \in \mathbb{R}^d : S_t(\omega)^\mathsf{T} x \geq 0\right\}$, where the normal vector $S_t(\omega)$ is the unique vector in the basis of $K_t[\omega]^+$. Let $A_t = A_t + L_t^p(K_t \cap \tilde{M}_t) + L^p(K_T)$,

$$\rho_t^{S_t}(X) = \operatorname{ess\,inf}\left\{S_t^\mathsf{T} u : u \in M_t, X + u \in A_t\right\} = \tilde{\rho}_t^{S_t}(S_T^\mathsf{T} X)$$

for any $X \in L^p$ (since $L_t^q((K_t \cap \tilde{M}_t)^+) := L_t^q(K_t^+) + M_t^\perp$). It can be seen that $\tilde{\rho}_t^{S_t}(Z) = \operatorname{ess\,inf}\left\{S_t^\mathsf{T} u : u \in M_t, Z + S_T^\mathsf{T} u \in \tilde{A}_t\right\}$ with $\tilde{A}_t = \left\{S_T^\mathsf{T} X : X \in A_t\right\}$ is the dynamic version of the risk measures with multiple eligible assets defined in [5, 18, 26, 39, 49] (and with single eligible assets (which is not necessarily the original numéraire) defined in [19, 20]). $\tilde{A}_t$ satisfies Definition 1 of [18] for an acceptance set.

Example 2.27 Burgert and Rüschendorf [11] discusses scalar static risk measure of multivariate claims, when only a single eligible asset is considered, that is

$$\rho(X) = \inf\{m \in \mathbb{R} : X + me_1 \in A\}$$

for $X \in L^\infty$, where $A \subseteq L^\infty$ is an acceptance set. We can see that this has the form $\rho(X) = \inf\left\{e_1^\top u : u \in \mathbb{R} \times \{0\}^{d-1}, X + u \in A\right\}$, i.e. the scalarization of a set-valued risk measure with $M_0 = \mathbb{R} \times \{0\}^{d-1}$ and $w = e_1$.

Example 2.28 In [53] so called liquidity-adjusted risk measure $\rho^V : L^\infty \to \mathbb{R}$, which are scalar static risk measure of multivariate claims in markets with frictions, are studied, when only a single eligible asset is considered. The primal representation

$$\rho^V(X) = \inf\{k \in \mathbb{R} : X + ke_1 \in A^V\}$$

for $A^V := \{X \in L^\infty : V(X) \in A\}$, where V is a real valued function providing the value of a portfolio X under liquidity and portfolio constraints and $A \subseteq L^\infty(\mathbb{R})$ is the acceptance set of a scalar convex risk measure in the sense of [24]. Clearly, $\rho^V(X)$ is of form (2.14).

Example 2.29 In [7, 10, 35, 44, 46] (and many other papers) the scalar superhedging price in a market with two assets and transaction costs has been studied. The d asset case is treated in [41, 47]. Let $(K_t)_{t=0}^T$ be the sequence of solvency cones modeling the market with proportional transaction costs.

The d dimensional version of the dual representation of the scalar superhedging price given in Jouini, Kallal [35] reads as follows. Let $X \in L^p$ be a payoff in physical units. Under an appropriate robust no arbitrage condition, the scalar superhedging price $\pi_i^a(X)$ in units of asset $i \in \{1, \ldots, d\}$ at time $t = 0$ is given by

$$\pi_i^a(X) = \sup_{(S_t, \mathbb{Q}) \in \mathcal{Q}^i} \mathbb{E}^{\mathbb{Q}}\left[S_T^\top X\right], \tag{2.16}$$

where $\mathcal{Q}^i$ is the set of all processes $(S_t)_{t=0}^T$ and their equivalent martingale measures $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\mathcal{F}_T)$, $S_t^i \equiv 1$, $\mathbb{E}\left[\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right| \mathcal{F}_t\right] S_t \in L_d^q(\mathcal{F}_t; K_t^+)$ for all t. Theorem 6.1 in [41] shows that (2.16) can be obtained by scalarizing the coherent set-valued risk measure with acceptance set $A_0 = \sum_{s=0}^T L_s^p(K_s)$ and single eligible asset (asset i, which is also the numéraire asset, i.e. $M_0 = \left\{m \in \mathbb{R}^d : m_j = 0\ \forall j \neq i\right\}$) w.r.t. the unit vector $w = e_i \in (K_0 \cap M_0)^+$. Thus, π_i^a is a special case of (2.14).

Of course any standard scalar risk measure in a frictionless markets with single eligible asset as in [4, 24] is also special cases of (2.14), but in that case there is no advantage to explore the relationship with a set-valued risk measure via (2.15). In any other case, i.e. if one of the following is considered: multiple eligible assets, multivariate claims, transaction costs or other market frictions, it can be advantageous to explore (2.15) as the dual representation of the corresponding set-valued risk measure given in Sect. 2.1 can lead to a dual representation of the scalarization as demonstrated in (2.16). Furthermore, even if one is interested in only one particular scalarization (as it is the case in all the examples above), the dual representation of the scalar risk measure might involve the whole family of scalarizations (as in Example 2.29, where the constraints $S_t \in K_t^+$ a.s. for all t enter the scalar problem in (2.16)). This is related to time consistency properties of the scalar risk measure and multi-portfolio time consistency of the corresponding set-valued risk measure (see Definition 2.5). In this paper we are only concerned with the connection between a family of scalar risk measures and a set-valued risk measure. Lemma 3.18 below gives very mild conditions under which a set-valued risk measures can be equivalently represented by a family of scalar risk measures. Results about dual representations and the study of time consistency properties of the family of scalar risk measures are left for further research.

The main motivation to study a family of scalar risk measures in this section is that it allows to generalize all of the examples given above in a unified way by allowing multiple eligible assets, multivariate claims and frictions in the form of transaction costs, as well as considering a dynamic setting. As Example 2.29 suggests, viewing a scalar risk measure in a market with frictions as being a scalarization of a set-valued risk measure has the advantage of obtaining dual representations and conditions on time consistency by using the corresponding results of the set-valued risk measure.

A different approach concerning a family of scalar risk measures and multiple eligible assets in a frictionless market was taken in [34]. In that paper, given a set of eligible assets (with values S_T^i for $i = 1, \dots, n$), the risk of the portfolio X is the set of values $\left\{\sum_{i=1}^n \rho_t^{S_T^i}(X_i) S_T^i : X = \sum_{i=1}^n X_i\right\}$ where $\rho_t^{S_T^i}$ is a risk measure in asset i (with change of numéraire). However, we will not discuss this approach further since lemma 4.10 of that paper demonstrates that $\rho_t^{S_T^0}(X) \leq \rho_t^{S_T^0}(-\sum_{i=1}^n \rho_t^{S_T^i}(X_i) S_T^i)$ for any choice of numéraire 0 and any allocation of $X = \sum_{i=1}^n X_i$, i.e. the family of risks (as a portfolio) has a risk bounded below by the risk of the initial portfolio no matter the numéraire chosen.

In the following proposition we show that the multiple asset conditional scalar risk measures satisfy monotonicity and a translative property. These properties are usually given as the definition of a risk measure in the literature given in the above examples. However, here we consider the primal representation Definition 2.25 as the starting point.

Proposition 2.30 *Let $\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}})$ be a multiple asset conditional scalar risk measure at time t for pricing vector $w \in M_{t,+}^+ \backslash M_t^\perp$. Then ρ_t^w satisfies the following conditions.*

1. *If $Y - X \in L^p_+$ for $X, Y \in L^p$, then $\rho_t^w(Y) \leq \rho_t^w(X)$.*
2. *$\rho_t^w(X + m) = \rho_t^w(X) - w^\mathsf{T} m$ for all $X \in L^p$ and $m \in M_t$.*

Further, if we consider the family of such risk measures over all pricing vectors $w \in M_{t,+}^+ \backslash M_t^\perp$ then we have the following finiteness properties.

3. *$\rho_t^w(0) < +\infty$ for every $w \in M_{t,+}^+ \backslash M_t^\perp$.*
4. *$\rho_t^w(0) > -\infty$ for some $w \in M_{t,+}^+ \backslash M_t^\perp$.*

Proof Let $\rho_t^w(X) := \text{ess inf}\left\{w^\mathsf{T} u : u \in M_t, X + u \in A_t\right\}$ for every $X \in L^p$, every $w \in M_{t,+}^+ \backslash M_t^\perp$, and some conditional acceptance set A_t.

1. Let $X, Y \in L^p$ such that $Y - X \in L^p_+$. Let $w \in M_{t,+}^+ \backslash M_t^\perp$.

$$\begin{aligned}\rho_t^w(Y) &= \text{ess inf}\left\{w^\mathsf{T} u : u \in M_t, Y + u \in A_t\right\} \\ &= \text{ess inf}\left\{w^\mathsf{T} u : u \in M_t, X + (Y - X) + u \in A_t\right\} \\ &\leq \text{ess inf}\left\{w^\mathsf{T} u : u \in M_t, X + u \in A_t\right\} = \rho_t^w(X).\end{aligned}$$

2. Let $X \in L^p$ and $m \in M_t$. Let $w \in M_{t,+}^+ \backslash M_t^\perp$.

$$\begin{aligned}\rho_t^w(X + m) &= \text{ess inf}\left\{w^\mathsf{T} u : u \in M_t : X + m + u \in A_t\right\} \\ &= \text{ess inf}\left\{w^\mathsf{T}(u - m) : u \in M_t, X + u \in A_t\right\} = \rho_t^w(X) - w^\mathsf{T} m.\end{aligned}$$

3. Fix some $\omega \in \Omega$. $\rho_t^w(0)[\omega] = +\infty$ for some $w \in M_{t,+}^+ \backslash M_t^\perp$ if and only if $A_t[\omega] \cap \tilde{M}_t[\omega] = \emptyset$, which by $A_t \cap M_t \neq \emptyset$ is false.
4. Fix some $\omega \in \Omega$. $\rho_t^w(0)[\omega] = -\infty$ for every $w \in M_{t,+}^+ \backslash M_t^\perp$ if and only if $(\mathbb{R}^d \backslash A_t[\omega]) \cap \tilde{M}_t[\omega] = \emptyset$, which by definition is false.

□

3 Relation Between Approaches

In this section we compare the properties for each of the techniques for dynamic multivariate risk measures. It will be shown that the set-valued portfolio approach to dynamic risk measures is the most general model into which every other approach can be embedded. It will be shown in Sect. 3.2 that under weak assumptions on the construction of the set-valued portfolios, the set-optimization approach is equivalent to the set-valued portfolio approach. Because additional properties for dynamic risk measures have been studied previously for the set-optimization approach and due to the (often) one-to-one relation with the set-portfolio approach, we will present the relations in this section as comparisons with the *set-optimization approach*.

3.1 Set-Optimization Approach Versus Measurable Selectors

In order to compare these two approaches, one first needs to agree on the same preimage and image space. One possibility would be to define the risk measures of Sect. 2.1 on the space $B_{K_T,n}$. This can be done as the theory involved (set-optimization) works for any locally convex space as the preimage space. The other possibility is to consider the measurable selectors approach of Sect. 2.2 on L^p spaces. This in not a problem for the definition of risk processes given in Definition 2.7, but could pose a problem for primal and dual representations, see discussion in Remark 3.1 for more details. However, since for the comparison results we just work with the definitions, we will follow this path here. Thus, consider L^p spaces for $p \in [0, +\infty]$ endowed with the metric topology (that is the norm topology for $p \geq 1$), even for $p = +\infty$ which is in contrast to [21, 23] where the weak* topology is used for $p = +\infty$. Also, as the definition of the risk process does not rely on the space of eligible portfolios to be M_t^n, we will use a general space of eligible portfolios M_t. We will show that when the dynamic risk measure has closed and conditionally convex images, the set-optimization and the measurable selectors approach coincide.

Remark 3.1 While the space $B_{K_T,n}$ shares many properties with L^∞, the two do not coincide in general. If $n = d$ or additional assumptions (e.g. substitutability from [36]) are satisfied, then $L^\infty \subseteq B_{K_T,n}$. If $n = d$ and $K_T = \mathbb{R}^d_+$ almost surely, then $B_{K_T,n} = L^\infty$. However, in general the two spaces are not comparable in the set-inclusion relation. Therefore, without additional assumptions, it is not trivial to use the representation results from [52] for the space L^∞. Furthermore, the assumptions for the Fatou duality (Theorem 2.16) exclude the special case $K_T = \mathbb{R}^d_+$ and thus exclude the case $B_{K_T,n} = L^\infty$ when $n = d$. However, the definition for risk process can be given for L^p spaces (and this is used in this section). But complications arise in both, the primal and dual definition, as e.g. boundedness is used in the proofs in [52].

The definition for $\mathcal{F}_t$-decomposability given below can be found in [43, p. 148] or [38, p. 260].

Definition 3.2 A set $D \subseteq L^p$ is said to be $\mathcal{F}_t$**-decomposable** if for any finite partition $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$ of Ω and any family $(X_n)_{n=1}^N \subseteq D$ for $N \in \mathbb{N}$, we have $\sum_{n=1}^N 1_{\Omega_t^n} X_n \in D$.

The following theorem and Corollary 3.4 below state that there is a one-to-one relation between conditional risk measures R_t with closed and $\mathcal{F}_t$-decomposable images and closed risk processes $\tilde{R}_t$. In Corollary 3.9 we demonstrate that any conditional risk measure with closed and conditionally convex images also has $\mathcal{F}_t$-decomposable images.

For notational purposes, let $\mathcal{S}_t := \{\Gamma \in \mathcal{S}_t^d : \Gamma(\omega) \subseteq \tilde{M}_t[\omega] \text{ a.s.}\} \subseteq \mathcal{S}_t^d$.

Theorem 3.3 *Let $\tilde{R}_t : L^p \to \mathcal{S}_t$ be a risk process at time t (see Definition 2.7), then $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$, defined by $R_t(X) := L_t^p(\tilde{R}_t(X))$ for any $X \in L^p$, is a conditional risk measure at time t (see Definition 2.1) with $\mathcal{F}_t$-decomposable images.*

Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time t (see Definition 2.1) with closed and $\mathcal{F}_t$-decomposable images, then there exists a risk process $\tilde{R}_t : L^p \to \mathcal{S}_t$ (see Definition 2.7) such that $R_t(X) = L_t^p(\tilde{R}_t(X))$ for any $X \in L^p$.

Proof 1. Let $\tilde{R}_t : L^p \to \mathcal{S}_t$ be a risk process at time t. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be defined by $R_t(X) := L_t^p(\tilde{R}_t(X))$ for any $X \in L^p$. It remains to show that R_t is a conditional risk measure at time t. L_+^p-monotonicity: let $X, Y \in L^p$ such that $Y - X \in L_+^p$, then $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$, and thus $R_t(X) \supseteq R_t(X)$. M_t-translativity: let $X \in L^p$ and $m \in M_t$, then $R_t(X + m) = L_t^p(\tilde{R}_t(X + m)) = L_t^p(\tilde{R}_t(X) - m) = L_t^p(\tilde{R}_t(X)) - m = R_t(X) - m$. Finiteness at zero: By $\tilde{R}_t(0) \neq \emptyset$ almost surely then trivially $R_t(0) = L_t^p(\tilde{R}_t(0)) \neq \emptyset$. By $\tilde{R}_t(0) \neq \tilde{M}_t$ almost surely then if $u(\omega) \in \tilde{M}_t[\omega] \backslash \tilde{R}_t(0)[\omega]$ for almost every $\omega \in \Omega$ such that $u \in M_t$, then $u(\omega) \notin R_t(0)[\omega]$ for almost every $\omega \in \Omega$. $\mathcal{F}_t$-decomposable images: Let $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$ for some $N \in \mathbb{N}$ be a finite partition of Ω and let $(u_n)_{n=1}^N \subseteq R_t(X)$ then $\sum_{n=1}^N 1_{\Omega_t^n} u_n \in M_t$, then since $R_t(X)$ are the measurable selectors of $\tilde{R}_t(X)$ it immediately follows that $\sum_{n=1}^N 1_{\Omega_t^n} u_n \in R_t(X)$.

2. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time t with closed and $\mathcal{F}_t$-decomposable images. By Proposition 5.4.3 in [38] (for $p \in [0, +\infty)$) and Theorem 1.6 of Chap. 2 from [43] (for $p = +\infty$), it follows that $R_t(X) = L_t^p(\tilde{R}_t(X))$ for some almost surely closed random set $\tilde{R}_t(X)$ for every $X \in L^p$. Trivially, we can see that $\tilde{R}_t(X) \subseteq \tilde{M}_t$ almost surely. It remains to show that $\tilde{R}_t$ is a risk process at time t. Let $X \in L^p$, then $\tilde{R}_t(X)$ is a closed $\mathcal{F}_t$-measurable random set [38, Proposition 5.4.3] and [43, Chap. 2, Theorem 1.6]. Finiteness at zero of R_t implies finiteness at zero of $\tilde{R}_t$. Consider $X, Y \in L^p$ with $Y - X \in L_+^p$, then $R_t(Y) \supseteq R_t(X)$, which implies that $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$. Let $X \in L^p$ and $m \in M_t$, then $R_t(X + m) = R_t(X) - m$. This implies $L_t^p(\tilde{R}_t(X + m)) = L_t^p(\tilde{R}_t(X)) - m = L_t^p(\tilde{R}_t(X) - m)$, i.e. $\tilde{R}_t(X + m) = \tilde{R}_t(X) - m$ almost surely. □

In the below corollaries the conditional risk measure associated with the risk process (and vice versa) is defined as in Theorem 3.3 above.

Corollary 3.4 *Let $\tilde{R}_t : L^p \to \mathcal{S}_t$ be a conditionally convex (conditionally positive homogeneous, normalized) risk process at time t then the associated conditional risk measure is conditionally convex (conditionally positive homogeneous, normalized).*

Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditionally convex (conditionally positive homogeneous, normalized) conditional risk measure at time t with closed and $\mathcal{F}_t$-decomposable images, then the associated risk process is conditionally convex (conditionally positive homogeneous, normalized).

Proof 1. Let $\tilde{R}_t : L^p \to \mathcal{S}_t$ be a risk process at time t and R_t be the associated conditional risk measure. Let $\tilde{R}_t$ be conditionally convex. Take $X, Y \in L^p$, $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \leq \lambda \leq 1$. Then,

$$\begin{aligned}\lambda R_t(X) + (1-\lambda)R_t(Y) &= \lambda L_t^p(\tilde{R}_t(X)) + (1-\lambda)L_t^p(\tilde{R}_t(Y)) \\ &= L_t^p(\lambda \tilde{R}_t(X) + (1-\lambda)\tilde{R}_t(Y)) \\ &\subseteq L_t^p(\tilde{R}_t(\lambda X + (1-\lambda)Y)) = R_t(\lambda X + (1-\lambda)Y).\end{aligned}$$

Let $\tilde{R}_t$ be conditionally positive homogeneous. Take $X \in L^p$ and $\lambda \in L_t^\infty(\mathbb{R}_{++})$. Then, $\lambda R_t(X) = \lambda L_t^p(\tilde{R}_t(X)) = L_t^p(\lambda \tilde{R}_t(X)) = L_t^p(\tilde{R}_t(\lambda X)) = R_t(\lambda X)$. Let $\tilde{R}_t$ be normalized and let $X \in L^p$. Then, $R_t(X) + R_t(0) = L_t^p(\tilde{R}_t(X)) + L_t^p(\tilde{R}_t(0)) = L_t^p(\tilde{R}_t(X) + \tilde{R}_t(0)) = L_t^p(\tilde{R}_t(X)) = R_t(X)$.

2. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time t and let $\tilde{R}_t$ be the associated risk process. Let R_t be conditionally convex. Take $X, Y \in L^p$ and $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \leq \lambda \leq 1$. Then,

$$\begin{aligned}L_t^p(\lambda \tilde{R}_t(X) + (1-\lambda)\tilde{R}_t(Y)) &= \lambda R_t(X) + (1-\lambda)R_t(Y) \\ &\subseteq R_t(\lambda X + (1-\lambda)Y) = L_t^p(\tilde{R}_t(\lambda X + (1-\lambda)Y)).\end{aligned}$$

By [43, Chap. 2, Proposition 1.2 (iii)] it holds $\lambda \tilde{R}_t(X) + (1-\lambda)\tilde{R}_t(Y) \subseteq \tilde{R}_t(\lambda X + (1-\lambda)Y)$ almost surely. The proof for conditional positive homogeneity and normalization is analog.

□

As discussed in Sects. 2.1 and 2.2, we have time consistency properties for both the set-optimization and measurable selector approach to risk measures. Therefore, we would like to be able to compare multi-portfolio time consistency (Definition 2.5) and consistency in time (Definition 2.17). These properties coincide in their notation, however as we will show below the two properties only coincide under additional assumptions.

Corollary 3.5 *Let $(\tilde{R}_t)_{t=0}^T$ be a normalized conditionally convex consistent in time risk process, then the associated dynamic risk measure is multi-portfolio time consistent if it is convex upper continuous.*

Let $(R_t)_{t=0}^T$ be a normalized multi-portfolio time consistent dynamic risk measure with closed and $\mathcal{F}_t$-decomposable images for all times t, then the associated risk process is consistent in time.

Proof 1. Let $(\tilde{R}_t)_{t=0}^T$ be a normalized conditionally convex risk process which is consistent in time such that the associated dynamic risk measure $(R_t)_{t=0}^T$ is convex upper continuous. By Theorem 2.18, it follows that $R_t(X) = \operatorname{cl}\operatorname{env}_{\mathcal{F}_t} \bigcup_{Z \in R_s(X)} R_t(-Z)$ for any $X \in L^p$ and any times $t, s \in [0, T]$ such that $t \leq s$. By Corollary 3.4 above, $(R_t)_{t=0}^T$ is conditionally convex.

We will show that the recursive form $\bigcup_{Z \in R_s(X)} R_t(-Z)$ is $\mathcal{F}_t$-decomposable. Let $N \in \mathbb{N}$, $(u_n)_{n=1}^N \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z)$ and $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$ is a partition of Ω. Denote by $Z_n \in R_s(X)$ the element such that $u_n \in R_t(-Z_n)$ for every $n \in$

$\{1, \ldots, N\}$. By Lemma 3.6, it follows that $\sum_{m=1}^{N} 1_{\Omega_t^m} Z_m \in R_s(X)$. Then we can see

$$\begin{aligned}
\sum_{n=1}^{N} 1_{\Omega_t^n} u_n &\in \sum_{n=1}^{N} 1_{\Omega_t^n} R_t(Z_n) = \sum_{n=1}^{N} 1_{\Omega_t^n} R_t(1_{\Omega_t^n} Z_n) \\
&= \sum_{n=1}^{N} 1_{\Omega_t^n} R_t(1_{\Omega_t^n} \sum_{m=1}^{N} 1_{\Omega_t^m} Z_m) = \sum_{n=1}^{N} 1_{\Omega_t^n} R_t(\sum_{m=1}^{N} 1_{\Omega_t^m} Z_m) \\
&\subseteq \Bigg\{ u \in M_t : \exists J \subseteq \{1, \ldots, N\} : \mathbb{P}(\cup_{j \in J} A_j) = 1, \\
&\qquad \forall j \in J : 1_{\Omega_t^j} u \in 1_{\Omega_t^j} R_t(\sum_{m=1}^{N} 1_{\Omega_t^m} Z_m) \Bigg\} \\
&= R_t(\sum_{m=1}^{N} 1_{\Omega_t^m} Z_m) \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z).
\end{aligned}$$

In the above we use the local property for conditionally convex risk measures ([21, Proposition 2.8]) and Lemma 3.6. Therefore, $\bigcup_{Z \in R_s(X)} R_t(-Z)$ is $\mathcal{F}_t$-decomposable, and thus $R_t(X) = \operatorname{cl} \bigcup_{Z \in R_s(X)} R_t(-Z)$. And as seen in [23, Appendix B], if $(R_t)_{t=0}^T$ is convex upper continuous then $\bigcup_{Z \in R_s(X)} R_t(-Z)$ is closed for any $X \in L^p$. Therefore, $R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z)$, i.e. $R_t(X)$ multi-portfolio time consistent.

2. Let $(R_t)_{t=0}^T$ be a normalized multi-portfolio time consistent dynamic risk measure with closed and $\mathcal{F}_t$-decomposable images for all time t. Let $(\tilde{R}_t)_{t=0}^T$ be the associated risk process. By Theorem 2.6, it follows that $R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z)$ for any $X \in L^p$ and any times $t, s \in [0, T]$ such that $t \leq s$. Since R_t has closed and $\mathcal{F}_t$-decomposable images then it additionally follows that $\bigcup_{Z \in R_s(X)} R_t(-Z) = \operatorname{cl} \operatorname{env}_{\mathcal{F}_t} \bigcup_{Z \in R_s(X)} R_t(-Z)$ for any $X \in L^p$. Therefore, $L_t^p(\tilde{R}_t(X)) = L_t^p(\tilde{R}_t(-R_s(X)))$ and thus, by Theorem 2.18, it follows that $(\tilde{R}_t)_{t=0}^T$ is consistent in time.

□

The convex upper continuity in the first part of the above theorem could we weakened as one only needs $\bigcup_{Z \in R_s(X)} R_t(-Z)$ is closed for any $X \in L^p$ and $t \leq s$.

Up to this point we have made the additional assumption for conditional risk measures of Sect. 2.1 to be $\mathcal{F}_t$-decomposable. The following results (Lemma 3.6 and Corollary 3.9 below) demonstrate that a conditional risk measure with closed and conditionally convex images satisfies a property stronger than $\mathcal{F}_t$-decomposable images as the property remains true for any (possibly uncountable) partition as well.

Lemma 3.6 *Let $(R_t)_{t=0}^T$ be a dynamic risk measure with closed and conditionally convex images. Let $(A_i)_{i\in I} \subseteq \mathcal{F}_t$ be a partition of Ω. Then*

$$R_t(X) = \left\{u \in M_t : \exists J \subseteq I \text{ with } \mathbb{P}(\cup_{j\in J} A_j) = 1 \text{ such that } 1_{A_j} u \in 1_{A_j} R_t(X)\ \forall j \in J\right\}$$

for any $X \in L^p$ and any time t.

Before giving the proof we give a remark on the uncountable summation as it will be used in part 2 (b) of the proof.

Remark 3.7 As given in [9, Chap. 3, Sect. 5] and [16, Chap. 3, Sect. 3.9], the arbitrary summation on a Hausdorff commutative topological group is given by $\sum_{j\in J} f_j = \lim_{K\in\mathcal{J}} \sum_{k\in K} f_k$, for any $\{f_j \in \mathcal{X} : j \in J\}$ where $\mathcal{X}$ is a Hausdorff commutative topological group, such that $\mathcal{J} = \{K \subseteq J : \#K < +\infty\}$, i.e. $\mathcal{J}$ are the finite subsets of J. Note that $\mathcal{J}$ is a net with order given by set inclusion and join given by the union.

In particular, for our concerns, the metric topologies for L_t^p for $p \in [0, +\infty]$ are all Hausdorff commutative topological groups. (If $p = 0$ then we consider convergence in measure, which is equivalent to a metric space with metric $d(f, g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mathbb{P}$ (lemma 13.40 in [1])).

Proof of lemma 3.6 Note that $1_D R_t(X) = \{1_D u : u \in R_t(X)\}$ for any $D \in \mathcal{F}_t$. For notational convenience let $\hat{R}_t(X) := \left\{u \in M_t : \exists J \subseteq I \text{ with } \mathbb{P}(\cup_{j\in J} A_j) = 1 \text{ such that} 1_{A_j} u \in 1_{A_j} R_t(X)\ \forall j \in J\right\}$.

1. The inclusion $R_t \subseteq \hat{R}_t$ follows straight forward: Let $u \in R_t(X)$, then by definition $1_D u \in 1_D R_t(X)$ for any $D \in \mathcal{F}_t$, and in particular this is true for $D = A_i$ for any $i \in I$. Therefore it follows that $u \in \hat{R}_t(X)$.
2. To prove $\hat{R}_t \subseteq R_t$ we will consider the two case: finite and infinite partitions. Let $u \in \hat{R}_t(X)$ and $J \subseteq I$ the underlying subindex. Then $u = \sum_{j\in J} 1_{A_j} u$ almost surely, therefore $u \in R_t(X)$ if and only if $\sum_{j\in J} 1_{A_j} u \in R_t(X)$ since they are in the same equivalence class. Let $\#J$ denote the cardinality of the set J. Note that by definition $1_{A_j} u \in 1_{A_j} R_t(X)$ for every $j \in J$.

   (a) If $\#J < +\infty$, i.e. if J is a finite set, then trivially

   $$\sum_{j\in J} 1_{A_j} u \in \sum_{j\in J} 1_{A_j} R_t(X) \subseteq R_t(X)$$

   by closedness and conditional convexity of $R_t(X)$ as shown in Proposition 3.8 below. And thus $u \in R_t(X)$.

   (b) Consider the case $\#J = +\infty$, i.e. if J is not a finite set. Let $u \in \hat{R}_t(X)$, that is there exists $J \subseteq I$ with $\mathbb{P}(\cup_{j\in J} A_j) = 1$ such that $1_{A_j} u \in 1_{A_j} R_t(X)$ for all $j \in J$, or equivalently $1_{A_j}(u - m) \in 1_{A_j} R_t(X + m)$ for all $j \in J$ for some $m \in R_t(X)$ by using the translation property of R_t. We want to show $u \in R_t(X)$, respectively $u - m \in R_t(X + m)$. Recall the summation as given in Remark 3.7, and the notation $\mathcal{J} = \{K \subseteq J : \#K < +\infty\}$.

$$
\begin{aligned}
u-m &= \sum_{j\in J} 1_{A_j}(u-m) \in \sum_{j\in J} 1_{A_j} R_t(X+m) \\
&= \left\{ \sum_{j\in J} 1_{A_j} Z_j : \forall j\in J : Z_j \in R_t(X+m) \right\} \\
&= \left\{ \lim_{K\in\mathcal{J}} \sum_{k\in K} 1_{A_k} Z_k : \forall j\in J : Z_j \in R_t(X+m) \right\} \qquad (3.1)\\
&= \left\{ \lim_{K\in\mathcal{J}} \left(\sum_{k\in K} 1_{A_k} Z_k + 1_{(\cup_{j\in J\setminus K} A_j)} 0 \right) : \forall j\in J : Z_j \in R_t(X+m) \right\} \\
&\subseteq \left\{ \lim_{K\in\mathcal{J}} \left(\sum_{k\in K} 1_{A_k} Z_k + 1_{(\cup_{j\in J\setminus K} A_j)} \bar{Z} \right) : \forall j\in J : Z_j, \bar{Z} \in R_t(X+m) \right\} \\
& \qquad (3.2)\\
&\subseteq \liminf_{K\in\mathcal{J}} \left\{ \sum_{k\in K} 1_{A_k} Z_k + 1_{(\cup_{j\in J\setminus K} A_j)} \bar{Z} : \forall k\in K : Z_k, \bar{Z} \in R_t(X+m) \right\} \\
&= \liminf_{K\in\mathcal{J}} \left(\sum_{k\in K} 1_{A_k} R_t(X+m) + 1_{(\cup_{j\in J\setminus K} A_j)} R_t(X+m) \right) \\
&= \liminf_{K\in\mathcal{J}} R_t(X+m) = R_t(X+m). \qquad (3.3)
\end{aligned}
$$

Equation (3.1) follows from the definition of an arbitrary summation as given in [9, 16], see Remark 3.7. Inclusion (3.2) follows from $0 \in R_t(X+m)$ since $m \in R_t(X)$. Equation (3.3) follows from the finite case given above applied to the partition $((A_k)_{k\in K}, \cup_{j\in J\setminus K} A_j)$. Note that $\cup_{j\in J\setminus K} A_j \in \mathcal{F}_t$ by $(\mathcal{F}_t)_{t=0}^T$ a filtration satisfying the usual conditions (and $\mathcal{F}_t$ is a sigma algebra). Furthermore, note that we define the limit inferior as in [40] to be $\liminf_{n\in N} B_n = \bigcap_{n\in N} \operatorname{cl} \bigcup_{m\geq n} B_m$ for a net of sets $(B_n)_{n\in N}$.

□

The following proposition is used in the proof of Lemma 3.6.

Proposition 3.8 *A closed set $D \subseteq L_t^p$ is conditionally convex if and only if for any $N \in \mathbb{N}$ where $N \geq 2$*

$$\sum_{n=1}^{N} \lambda_n D \subseteq D \qquad (3.4)$$

for every $(\lambda_n)_{n=1}^N \in \Lambda_N := \left\{ (x_n)_{n=1}^N : \sum_{n=1}^N x_n = 1 \text{ a.s.}, x_n \in L_t^\infty(\mathbb{R}_+) \forall n \in \{1, \ldots, N\} \right\}$.

Proof $\Leftarrow$ If $N = 2$ then this is the definition of conditional convexity. If $N > 2$ then choose $(\lambda_n)_{n=1}^N$ such that $\lambda_n = 0$ almost surely for every $n > 2$, this then reduces to the case when $N = 2$ and thus D is conditionally convex.

$\Rightarrow$ We will first define a set of multipliers for strict convex combinations

$$\Lambda_N^{>} = \left\{ (x_n)_{n=1}^N : \sum_{n=1}^N x_n = 1 \text{ a.s.}, x_n \in L_t^\infty(\mathbb{R}_{++}) \; \forall n \in \{1, \ldots, N\} \right\}.$$

Then the result for $\Lambda_N^{>}$ for any $N \in \mathbb{N}$ follows as in the static case (i.e. when $x_n \in \mathbb{R}_{++}$) by induction.

Let $(\lambda_n)_{n=1}^N \in \Lambda_N$. Then there exists a sequence of $((\lambda_n^m)_{n=1}^N)_{m=0}^{+\infty} \subseteq \Lambda_N^{>}$ which converges almost surely to $(\lambda_n)_{n=1}^N$ (i.e. for any $n \in \{1, \ldots, N\}$, $(\lambda_n^m)_{m=0}^{+\infty}$ converges almost surely to λ_n, and for every m it holds $\sum_{n=1}^N \lambda_n^m = 1$ almost surely). By the dominated convergence theorem, it follows that $\lambda_n^m X$ converges to $\lambda_n X$ in the metric topology for any $X \in L_t^p$. Therefore for any $(X_n)_{n=1}^N \subseteq D$ (and let $\bar{X}_m = \sum_{n=1}^N \lambda_n^m X_n \in D$ for any m)

$$\sum_{n=1}^N \lambda_n X_n = \sum_{n=1}^N \lim_{m \to +\infty} \lambda_n^m X_n = \lim_{m \to +\infty} \sum_{n=1}^N \lambda_n^m X_n = \lim_{m \to +\infty} \bar{X}_m \in D$$

by $\bar{X}_m$ convergent (since it is the finite sum of converging series) and D closed. □

Corollary 3.9 *Any conditional risk measure R_t with closed and conditionally convex images has $\mathcal{F}_t$-decomposable images.*

Proof Let R_t be a conditional risk measure with closed and conditionally convex images, and let $X \in L^p$. Let $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$, for some $N \in \mathbb{N}$, be a finite partition of Ω. By Lemma 3.6,

$$R_t(X) = \left\{ u \in M_t : \exists J \subseteq \{1, \ldots, N\} : \mathbb{P}(\cup_{j \in J} \Omega_t^j) = 1, \forall j \in J : 1_{\Omega_t^j} u \in 1_{\Omega_t^j} R_t(X) \right\}.$$

Therefore, if $(u_n)_{n=1}^N \subseteq R_t(X)$, then $1_{\Omega_t^m} \sum_{n=1}^N 1_{\Omega_t^n} u_n = 1_{\Omega_t^m} u_m \in 1_{\Omega_t^m} R_t(X)$ for every $m \in \{1, \ldots, N\}$, and thus $\sum_{n=1}^N 1_{\Omega_t^n} u_n \in R_t(X)$. □

We showed that when the dynamic risk measure has closed and conditionally convex images, the set-optimization approach of Sect. 2.1 and the measurable selector approach of Sect. 2.2 coincide. As a conclusion, the set-optimization approach which is using convex analysis results for set-valued functions, i.e. set-optimization, seems to be the richer approach as it allows to handle primal and dual representations for L^p spaces ($p \in [1, +\infty]$) as well as for the space $B_{K_T,n}$ (or any other locally convex preimage space). Furthermore, it allows to consider conditionally convex (and not necessarily conditionally coherent) risk measures as well as convex risk measures, whereas the measurable selectors approach relies heavily on the conditional coherency assumption.

3.2 Set-Optimization Approach Versus Set-Valued Portfolios

As in the prior sections, consider L^p spaces with $p \in [0, +\infty]$.

Theorem 3.10 *Given a conditional risk measure $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ (see Definition 2.1), then the function $\mathbf{R}_t : \mathcal{S}_T^d \to \mathcal{P}(M_t; M_{t,+})$ defined by*

$$\mathbf{R}_t(\mathbf{X}) := \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z) \tag{3.5}$$

for any set-valued portfolio $\mathbf{X}$ is a set-valued conditional risk measure (see Definition 2.21).

Given a set-valued conditional risk measure $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(M_t; M_{t,+})$ (see Definition 2.21) and a mapping $\mathbf{X} : L^p \to \bar{\mathcal{S}}_T^d$ of the set-valued portfolio associated with a (random) portfolio vector such that $\mathbf{X}$ is monotone and translative, i.e. $\mathbf{X}(X) \subseteq \mathbf{X}(Y)$ if $Y - X \in L_+^p$ and $\mathbf{X}(X + u) = \mathbf{X}(X) + u$ for any $X, Y \in L^p$ and $u \in M_t$, then the function $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ defined by

$$R_t(X) := \mathbf{R}_t(\mathbf{X}(X)) \tag{3.6}$$

for any $X \in L^p$ is a conditional risk measure (see Definition 2.1) which might not be finite at zero.

Proof 1. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure as in Definition 2.1. Let $\mathbf{R}_t(\mathbf{X}) := \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z)$ for any set-valued portfolio $\mathbf{X}$. We wish to show that $\mathbf{R}_t$ satisfies Definition 2.21.

(a) Trivially $\mathbf{R}_t(\mathbf{X}) \in \mathcal{P}(M_t; M_{t,+})$ for any set-valued portfolio $\mathbf{X}$.
(b) Cash invariance: let $\mathbf{X}$ be a set-valued portfolio and let $m \in M_t$, then

$$\begin{aligned}\mathbf{R}_t(\mathbf{X} + m) &= \bigcup_{Z \in L^p(\mathbf{X}+m)} R_t(Z) = \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z + m) \\ &= \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z) - m = \mathbf{R}_t(\mathbf{X}) - m.\end{aligned}$$

(c) Monotonicity: Let $\mathbf{X} \subseteq \mathbf{Y}$ almost surely, then

$$\mathbf{R}_t(\mathbf{X}) = \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z) \subseteq \bigcup_{Z \in L^p(\mathbf{Y})} R_t(Z) = \mathbf{R}_t(\mathbf{Y}).$$

2. Let $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(M_t; M_{t,+})$ be a set-valued conditional risk measure as in Definition 2.21. Let $\mathbf{X} : L^p \to \bar{\mathcal{S}}_T^d$ be a mapping of portfolio vectors to set-valued portfolios that is monotone and translative. Let $R_t(X) := \mathbf{R}_t(\mathbf{X}(X))$ for any $X \in L^p$. We wish to show that R_t satisfies Definition 2.1.

(a) L^p_+-monotonicity: Let $X, Y \in L^p$ such that $Y - X \in L^p_+$. Then $\mathbf{X}(Y) \supseteq \mathbf{X}(X)$, and thus $R_t(X) = \mathbf{R}_t(\mathbf{X}(X)) \subseteq \mathbf{R}_t(\mathbf{X}(Y)) = R_t(Y)$.
(b) M_t-translativity: Let $X \in L^p$ and $m \in M_t$, then

$$R_t(X+m) = \mathbf{R}_t(\mathbf{X}(X+m)) = \mathbf{R}_t(\mathbf{X}(X)+m) = \mathbf{R}_t(\mathbf{X}(X)) - m = R_t(X) - m.$$

□

The above theorem states that conditional risk measures as in Definition 2.1 can be used to construct set-valued conditional risk measure (see Definition 2.21). This is in analogy to construction (2.13), but yields a larger class of risk measures. If one restricts oneself to set-valued portfolios $\mathbf{X} : L^p \to \bar{\mathcal{S}}^d_T$ which are monotonic and with $\mathbf{X}(X+m) = \mathbf{X}(X) + m$ for any $X \in L^p$ and $m \in M_t$, then conditional risk measures as in Definition 2.1 are one-to-one to set-valued conditional risk measure as in Definition 2.21. This is the case whenever the set of portfolios $\mathbf{X}$ represents the set of portfolios that can be obtained from $X \in L^p$ following certain exchange rules (including transaction costs, trading constraints, illiquidity). The advantage of considering $\mathbf{R}_t$ as a function of the set $\mathbf{X}(X)$ as opposed to a function of X as in (3.6) is that $\mathbf{R}_t$ might be law invariant (see Theorem 2.24), whereas R_t is in general not law invariant.

Example 3.11 If $\mathbf{X}(X) := X + K$ for some (almost surely) closed convex lower set K such that $L^p(K)$ is closed, then trivially $\mathbf{X}(X)$ is a set-valued portfolio and satisfies monotonicity and translativity.

If $\mathbf{X}(X)$ is as in Example 3.11 and K is additionally a convex cone, then for a given set-valued conditional risk measure $\mathbf{R}_t$, the associated conditional risk measure R_t defined by (3.6) is $L^p(K)$-compatible.

Note, that constructions very similar to (3.5) appear (a) in [2, 32] to define the market extension (that is a $C_{t,T}$-compatible version) of a risk measures R_t by

$$R_t^{mar}(X) := \bigcup_{Z \in X + C_{t,T}} R_t(Z),$$

where $C_{t,T} = -\sum_{s=t}^T L^p_s(K_s)$ and $(K_t)_{t=0}^T$ is a sequence of solvency cones modeling the bid-ask prices of the d assets, and b) in [21, 23] to define a multi-portfolio time consistent risk measure $(\tilde{R}_t)_{t=0}^T$ by backward recursion of a discrete time dynamic risk measure $(R_t)_{t=0}^T$ via $\tilde{R}_T(X) = R_T(X)$ and

$$\tilde{R}_t(X) := \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z)$$

for $t \in \{T-1, \ldots, 0\}$.

The following two corollaries provide additional relations between the conditional risk measures of the set-optimization approach and the set-valued portfolio conditional risk measures. Specifically, they provide sufficient conditions for (conditional) convexity and coherence of one type of risk measure to be associated with a (conditionally) convex and coherent risk measure of the other type.

Corollary 3.12 *Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a convex (conditionally convex, positive homogeneous, conditionally positive homogeneous) conditional risk measure (see Definition 2.1) at time t, then the associated set-valued conditional risk measure (see Definition 2.21) $\mathbf{R}_t$ defined by (3.5) is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous).*

Proof Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure and let $\mathbf{R}_t(\mathbf{X}) := \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z)$ for any $\mathbf{X} \in \bar{\mathcal{S}}_T^d$.

1. Let R_t be convex. Consider $\mathbf{X}, \mathbf{Y} \in \bar{\mathcal{S}}_T^d$ and $\lambda \in [0, 1]$. Then,

$$
\begin{aligned}
\mathbf{R}_t(\lambda\mathbf{X} \oplus (1-\lambda)\mathbf{Y}) &= \bigcup_{Z \in L^p(\lambda\mathbf{X} \oplus (1-\lambda)\mathbf{Y})} R_t(Z) \supseteq \bigcup_{Z \in \mathrm{cl}\left(\lambda L^p(\mathbf{X}) + (1-\lambda) L^p(\mathbf{Y})\right)} R_t(Z) \\
&\supseteq \bigcup_{\substack{Z_X \in L^p(\mathbf{X}) \\ Z_Y \in L^p(\mathbf{Y})}} R_t(\lambda Z_X + (1-\lambda) Z_Y) \\
&\supseteq \bigcup_{\substack{Z_X \in L^p(\mathbf{X}) \\ Z_Y \in L^p(\mathbf{Y})}} [\lambda R_t(Z_X) + (1-\lambda) R_t(Z_Y)] \\
&= \lambda \bigcup_{Z_X \in L^p(\mathbf{X})} R_t(Z_X) + (1-\lambda) \bigcup_{Z_Y \in L^p(\mathbf{Y})} R_t(Z_Y) \\
&= \lambda \mathbf{R}_t(\mathbf{X}) + (1-\lambda) \mathbf{R}_t(\mathbf{Y}).
\end{aligned}
$$

   The inclusion on the first line follows from $\mathrm{cl}\left(L^p(\mathbf{Z}_1) + L^p(\mathbf{Z}_2)\right) \subseteq L^p(\mathbf{Z}_1 \oplus \mathbf{Z}_2)$ for any random sets $\mathbf{Z}_1, \mathbf{Z}_2$ (with the norm topology on $p \in [1, +\infty]$, the metric topology on $p \in (0, 1)$, and the topology generated by convergence in probability for $p = 0$); for $p \in [1, +\infty)$ equality holds.
2. Let R_t be conditionally convex. Then the proof is analogous to the convex case above.
3. Let R_t be positive homogeneous. Consider $\mathbf{X} \in \bar{\mathcal{S}}_T^d$ and $\lambda > 0$. It holds

$$
\mathbf{R}_t(\lambda\mathbf{X}) = \bigcup_{Z \in L^p(\lambda\mathbf{X})} R_t(Z) = \bigcup_{Z \in L^p(\mathbf{X})} R_t(\lambda Z) = \lambda \bigcup_{Z \in L^p(\mathbf{X})} R_t(Z) = \lambda \mathbf{R}_t(\mathbf{X}).
$$

4. Let R_t be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.

□

Corollary 3.13 *Let $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(M_t; M_{t,+})$ be a set-valued conditional risk measure (see Definition 2.21) at time t, and let $\mathbf{X} : L^p \to \bar{\mathcal{S}}_T^d$ of the set-valued portfolio* *associated with a (random) portfolio vector be monotonic and translative. Let R_t be the associated conditional risk measure (see Definition 2.1).*

1. *If $\mathbf{R}_t$ is convex and $\mathbf{X}(\lambda X + (1-\lambda)Y) \supseteq \lambda \mathbf{X}(X) \oplus (1-\lambda)\mathbf{X}(Y)$ for every $X, Y \in L^p$ and $\lambda \in [0,1]$ ($\mathbf{X}$ is closed-convex), then R_t is convex.*
2. *If $\mathbf{R}_t$ is conditionally convex and $\mathbf{X}(\lambda X + (1-\lambda)Y) \supseteq \lambda \mathbf{X}(X) \oplus (1-\lambda)\mathbf{X}(Y)$ for every $X, Y \in L^p$ and $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \le \lambda \le 1$ ($\mathbf{X}$ is conditionally closed-convex), then R_t is conditionally convex.*
3. *If $\mathbf{R}_t$ is positive homogeneous and $\mathbf{X}(\lambda X) = \lambda \mathbf{X}(X)$ for every $X \in L^p$ and $\lambda > 0$ ($\mathbf{X}$ is positive homogeneous), then R_t is positive homogeneous.*
4. *If $\mathbf{R}_t$ is conditionally positive homogeneous and $\mathbf{X}(\lambda X) = \lambda \mathbf{X}(X)$ for every $X \in L^p$ and $\lambda \in L_t^\infty(\mathbb{R}_{++})$ ($\mathbf{X}$ is conditionally positive homogeneous), then R_t is conditionally positive homogeneous.*

Proof Let $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(M_t; M_{t,+})$ be a set-valued conditional risk measure, let $\mathbf{X}$ be as above and let $R_t(X) := \mathbf{R}_t(\mathbf{X}(X))$ for every portfolio vector $X \in L^p$.

1. Let $\mathbf{R}_t$ be convex and $\mathbf{X}$ be closed-convex. Let $X, Y \in L^p$ and $\lambda \in [0,1]$.

$$\begin{aligned} R_t(\lambda X + (1-\lambda)Y) &= \mathbf{R}_t(\mathbf{X}(\lambda X + (1-\lambda)Y)) \supseteq \mathbf{R}_t(\lambda \mathbf{X}(X) \oplus (1-\lambda)\mathbf{X}(Y)) \\ &\supseteq \lambda \mathbf{R}_t(\mathbf{X}(X)) + (1-\lambda)\mathbf{R}_t(\mathbf{X}(Y)) = \lambda R_t(X) + (1-\lambda)R_t(Y). \end{aligned}$$

2. Let $\mathbf{R}_t$ be conditionally convex and $\mathbf{X}$ be conditionally closed-convex. Then the proof is analogous to the convex case above.
3. Let $\mathbf{R}_t$ and $\mathbf{X}$ be positive homogeneous. Let $X \in L^p$ and $\lambda > 0$.

$$R_t(\lambda X) = \mathbf{R}_t(\mathbf{X}(\lambda X)) = \mathbf{R}_t(\lambda \mathbf{X}(X)) = \lambda \mathbf{R}_t(\mathbf{X}(X)) = \lambda R_t(X).$$

4. Let $\mathbf{R}_t$ and $\mathbf{X}$ be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.

□

Example 3.14 (Example 3.11 continued) Let $\mathbf{X}(X) := X + K$ for every $X \in L^p$ for some random set K. If K is (almost surely) convex and closed then $\mathbf{X}$ is ($\mathcal{F}$-)conditionally closed-convex (and thus closed-convex as well). If K is (almost surely) a cone then $\mathbf{X}$ is ($\mathcal{F}$-)conditionally positive homogeneous (and thus positive homogeneous as well).

In light of Theorem 3.10, Eq. (3.6) and Corollary 3.13 for set-valued portfolios of the form $\mathbf{X}(X) := X + K$ for all $X \in L^p$ and some random closed convex cone K, one obtains the following. The dual representation of a constructive risk measure $\mathbf{R}_0$ with coherent components $\rho^1, \ldots, \rho^d$ given in Eq. (5.2) in [13] coincides with a special case of the dual representation of a K_T-compatible risk measure R_0 given in Theorem 4.2 in [31], by choosing $A = \times_{i=1}^d A_i$ (A_i being the acceptance set of ρ_i), $M_0 = R^d$, $K_I = R_+^d$ and $K_T = -K$:

$$R_0(X) = \mathbf{R}_0(X+K) = \bigcap_{w \in \mathbb{R}^d_+ \backslash \{0\}, \mathbb{Q} \in \mathcal{Q}, w \frac{d\mathbb{Q}}{d\mathbb{P}} \in (-K)^+ \text{ a.s.}} \{u \in \mathbb{R}^d : w^\mathsf{T} \mathbb{E}^\mathbb{Q}[X] \leq w^\mathsf{T} u\},$$

where $\mathcal{Q} = \times_{i=1}^d \mathcal{Q}_i$ and $\mathcal{Q}_i$ denotes the set of probability measures in the dual representation of ρ_i. This also follows from Corollary 2.3, where the set of dual variables is

$$\mathcal{W}_0^{\max} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_0 : w_t^T(\mathbb{Q}, w) \in A_t^+\right\} = \{(\mathbb{Q}, w) \in \mathcal{W}_0 : \mathbb{Q} \in \mathcal{Q}\},$$

with

$$\mathcal{W}_0 := \left\{(\mathbb{Q}, w) \in \mathcal{M} \times \mathbb{R}^d_+ \backslash \{0\} : w_0^T(\mathbb{Q}, w) \in L^q_d(\mathcal{F}_T; K_T^+)\right\}$$

due do K_T-compatibility of R_0.

Additional to dual representations for constructive risk measure, Theorem 3.10 allows to deduce dual representations of a larger class of conditional risk measure for set-valued portfolios (Definition 2.21) by using Eq. (3.5) and the duality results for set-valued risk measures of the set-optimization approach.

3.3 Set-Optimization Approach Versus Family of Scalar Risk Measures

For this section consider $p \in [1, +\infty]$, where L^p_t has the norm topology for any $p \in [1, +\infty)$ and the weak* topology for $p = +\infty$. In the static setting, the relation between set-valued risk measures and multiple asset scalar risk measures has been studied in [18, 29, 31].

Theorem 3.15 *Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time t (see Definition 2.1), then $\rho_t^w : L^p \to L^0_t(\bar{\mathbb{R}})$, defined by*

$$\rho_t^w(X) := \operatorname*{\mathit{ess\ inf}}_{u \in R_t(X)} w^\mathsf{T} u$$

for any $X \in L^p$, is a family of multiple asset scalar risk measures indexed by $w \in M_{t,+}^+ \backslash M_t^\perp$ at time t (see Definition 2.25).

Let $\left\{\rho_t^w : L^p \to L^0_t(\bar{\mathbb{R}}) : w \in M_{t,+}^+ \backslash M_t^\perp\right\}$ be a family of multiple asset scalar risk measures at time t indexed by $w \in M_{t,+}^+ \backslash M_t^\perp$ (see Definition 2.25), then $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$, defined by

$$R_t(X) := \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-}a.s.\right\}$$

for any $X \in L^p$, is a conditional risk measure at time t (see Definition 2.1).

Proof 1. This follows form Definition 2.25 and (2.15).

2. We will show that $R_t(X) := \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\}$ is a conditional risk measure. We use the properties of ρ_t^w given in Proposition 2.30.

   (a) L_+^p-monotonicity: let $X, Y \in L^p$ such that $Y - X \in L_+^p$, then $\rho_t^w(Y) \leq \rho_t^w(X)$ almost surely for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Therefore $R_t(Y) \supseteq R_t(X)$.

   (b) M_t-translativity: let $X \in L^p$ and $m \in M_t$, then

$$\begin{aligned} R_t(X+m) &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X+m) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\} \\ &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) - w^\mathsf{T} m \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\} \\ &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} (u+m) \ \mathbb{P}\text{-a.s.}\right\} \\ &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\} - m = R_t(X) - m. \end{aligned}$$

   (c) Finiteness at zero: $R_t(0) \neq \emptyset$ since $\rho_t^w(0) < +\infty$ for every $w \in M_{t,+}^+ \backslash M_t^\perp$, and $R_t(0)[\omega] \neq \tilde{M}_t[\omega]$ since there exists a $v \in M_{t,+}^+ \backslash M_t^\perp$ such that $\rho_t^v(0) > -\infty$.

□

Remark 3.16 If R_t is normalized, with closed and conditionally convex images, and $w \in R_t(0)^+ \backslash M_t^\perp$ then $\rho_t^w(0) = 0$, i.e. ρ_t^w is normalized in the scalar framework.

Apart from closedness, many properties are one-to-one for conditional risk measures R_t and the corresponding family of scalarizations. The corresponding results for the static case can be found in lemma 5.1 and lemma 6.1 of [29]. An example showing that closedness of R_t does not necessarily imply closedness of all scalarizations can be found in the beginning of Sect. 5 in [29] for the case $t = 0$.

Corollary 3.17 *Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a convex (conditionally convex, positive homogeneous, conditionally positive homogeneous) conditional risk measure at time t with closed and $\mathcal{F}_t$-decomposable images, then the associated family of multiple asset scalar risk measures is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous).*

Let $\left\{\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}}) : w \in M_{t,+}^+ \backslash M_t^\perp\right\}$ be a family of convex (positive homogeneous, conditionally positive homogeneous, lower semicontinuous) multiple asset scalar risk measures at time t indexed by $w \in M_{t,+}^+ \backslash M_t^\perp$ then the associated conditional risk measure is convex (positive homogeneous, conditionally positive homogeneous, closed). Additionally, if $\left\{\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}}) : w \in M_{t,+}^+ \backslash M_t^\perp\right\}$ is a family of lower semicontinuous conditionally convex risk measures then the associated conditional risk measure is conditionally convex.

Proof 1. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time t. Let $\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}})$ be defined by $\rho_t^w(X) := \operatorname{ess\,inf}_{u \in R_t(X)} w^\mathsf{T} u$ for every $X \in L^p$.

(a) Let R_t be convex. Let $X, Y \in L^p$, $\lambda \in [0, 1]$, and $w \in M_{t,+}^+ \backslash M_t^\perp$.

$$\begin{aligned}\rho_t^w(\lambda X + (1-\lambda)Y) &= \operatorname*{ess\,inf}_{u \in R_t(\lambda X + (1-\lambda)Y)} w^\mathsf{T} u \\ &\leq \operatorname*{ess\,inf}_{u \in \lambda R_t(X) + (1-\lambda) R_t(Y)} w^\mathsf{T} u \\ &= \lambda \operatorname*{ess\,inf}_{u_X \in R_t(X)} w^\mathsf{T} u_X + (1-\lambda) \operatorname*{ess\,inf}_{u_Y \in R_t(Y)} w^\mathsf{T} u_Y \\ &= \lambda \rho_t^w(X) + (1-\lambda)\rho_t^w(Y).\end{aligned}$$

(b) Let R_t be conditionally convex. Then the proof is analogous to the convex case above.

(c) Let R_t be positive homogeneous. Let $X \in L^p$, $\lambda > 0$, and $w \in M_{t,+}^+ \backslash M_t^\perp$.

$$\rho_t^w(\lambda X) = \operatorname*{ess\,inf}_{u \in R_t(\lambda X)} w^\mathsf{T} u = \operatorname*{ess\,inf}_{u \in \lambda R_t(X)} w^\mathsf{T} u = \lambda \operatorname*{ess\,inf}_{u \in R_t(X)} w^\mathsf{T} u = \lambda \rho_t^w(X).$$

(d) Let R_t be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.

2. Let $\left\{\rho_t^w : L^p \to L_t^0(\bar{\mathbb{R}}) : w \in M_{t,+}^+ \backslash M_t^\perp\right\}$ be a family of multiple asset scalar risk measures at time t indexed by $w \in M_{t,+}^+ \backslash M_t^\perp$. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be defined by $R_t(X) := \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\}$ for every $X \in L^p$.

(a) Let ρ_t^w be convex for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Let $X, Y \in L^p$ and $\lambda \in (0, 1)$.

$$\begin{aligned}R_t(\lambda X + (1-\lambda)Y) &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \rho_t^w(\lambda X + (1-\lambda)Y) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\} \\ &\supseteq \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u \in M_t : \lambda\rho_t^w(X) + (1-\lambda)\rho_t^w(Y) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.}\right\} \\ &\supseteq \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left[\left\{\lambda u_X : u_X \in M_t, \rho_t^w(X) \leq w^\mathsf{T} u_X \ \mathbb{P}\text{-a.s.}\right\}\right. \\ &\qquad \left. + \left\{(1-\lambda) u_Y : u_Y \in M_t, \rho_t^w(Y) \leq w^\mathsf{T} u_Y \ \mathbb{P}\text{-a.s.}\right\}\right] \\ &\supseteq \lambda \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u_X \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u_X \ \mathbb{P}\text{-a.s.}\right\} \\ &\qquad + (1-\lambda) \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{u_Y \in M_t : \rho_t^w(Y) \leq w^\mathsf{T} u_Y \ \mathbb{P}\text{-a.s.}\right\} \\ &= \lambda R_t(X) + (1-\lambda) R_t(Y).\end{aligned}$$

Let $\lambda = 0$ (the case for $\lambda = 1$ is analogous), then $R_t(\lambda X + (1-\lambda)Y) = \lambda R_t(X) + (1-\lambda)R_t(Y)$ for any conditional risk measure and the result follows.

(b) Let ρ_t^w be positive homogeneous for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Let $X \in L^p$ and $\lambda > 0$.

$$\begin{aligned} R_t(\lambda X) &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{ u \in M_t : \rho_t^w(\lambda X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.} \right\} \\ &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{ u \in M_t : \lambda \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.} \right\} \\ &= \bigcap_{w \in M_{t,+}^+ \backslash M_t^\perp} \left\{ \lambda u : u \in M_t, \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.} \right\} \\ &= \lambda R_t(X). \end{aligned}$$

(c) Let ρ_t^w be conditionally positive homogeneous for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Then the proof is analogous to the positive homogeneous case above.

(d) Let ρ_t^w be lower semicontinuous for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Consider a sequence $(X_n, u_n)_{n \in \mathbb{N}} \subseteq \operatorname{graph} R_t$ (respectively a net if $p = +\infty$) with $\lim_{n \to +\infty}(X_n, u_n) = (X, u)$. Note that $(X_n, u_n) \in \operatorname{graph} R_t$ if and only if $\rho_t^v(X_n) \leq v^\mathsf{T} u_n$ for every $v \in M_{t,+}^+ \backslash M_t^\perp$.

$$\rho_t^w(X) \leq \liminf_{n \to +\infty} \rho_t^w(X_n) \leq \liminf_{n \to +\infty} w^\mathsf{T} u_n = w^\mathsf{T} u.$$

The last equality above follows from $u_n \to u$ in L_t^p implies $w^\mathsf{T} u_n \to w^\mathsf{T} u$ in $L_t^1(\mathbb{R})$ (by Hölder's inequality). Thus, $(X, u) \in \operatorname{graph} R_t$.

(e) Let ρ_t^w be lower semicontinuous and conditionally convex for every $w \in M_{t,+}^+ \backslash M_t^\perp$. Let $X, Y \in L^p$ and $\lambda \in L_t^\infty(\mathbb{R})$ with $0 < \lambda < 1$, then the proof is analogous to the convex case above.
We will now extend conditional convexity to the case for $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \leq \lambda \leq 1$ in the same way as was accomplished in the proof of [21, Corollary 4.9], noting that R_t is closed by ρ_t^w lower semicontinuous. Take a sequence $(\lambda_n)_{n=0}^{+\infty} \subseteq L_t^\infty(\mathbb{R})$ such that $0 < \lambda_n < 1$ for every $n \in \mathbb{N}$ which converges almost surely to λ. Then by dominated convergence $\lambda_n X$ converges to λX in the norm topology (weak* topology if $p = +\infty$) for any $X \in L^p$. Therefore, for any $X, Y \in L^p$

$$\begin{aligned} R_t(\lambda X + (1-\lambda)Y) &= R_t(\lim_{n \to +\infty}(\lambda_n X + (1-\lambda_n)Y)) \\ &\supseteq \liminf_{n \to +\infty} R_t(\lambda_n X + (1-\lambda_n)Y) \\ &\supseteq \liminf_{n \to +\infty} [\lambda_n R_t(X) + (1-\lambda_n)R_t(Y)] \\ &\supseteq \lambda R_t(X) + (1-\lambda)R_t(Y) \end{aligned}$$

by R_t closed (see proposition 2.34 in [40]) and conditionally convex on the interval $0 < \lambda_n < 1$. Note that we use the convention from [40] that the limit inferior of a sequence of sets $(B_i)_{i \in \mathbb{N}}$ is given by $\liminf_{i \to +\infty} B_i = \bigcap_{i \in \mathbb{N}} \operatorname{cl} \bigcup_{j \geq i} B_j$.

□

In the following lemma we will show that when the conditional risk measure has closed and conditionally convex images, the family of scalarizations can be used to recover the conditional risk measure.

Lemma 3.18 *Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a dynamic risk measure with closed and conditionally convex images. Then, for any $X \in L^p$*

$$R_t(X) = \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \left\{ u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.} \right\} \tag{3.7}$$

where $\rho_t^w(X) := \operatorname{ess\,inf}_{u \in R_t(X)} w^\mathsf{T} u$ is the multiple asset scalar risk measure associated with R_t.

Proof $\subseteq$: By definition it is easy to see that $u \in R_t(X)$ implies that $w^\mathsf{T} u \geq \rho_t^w(X)$ for any $w \in M_{t,+}^+ \setminus M_t^\perp$.

$\supseteq$: Let $u \in \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \left\{ u \in M_t : \rho_t^w(X) \leq w^\mathsf{T} u \ \mathbb{P}\text{-a.s.} \right\}$. Assume $u \notin R_t(X)$. Then since $R_t(X)$ is closed and convex we can apply the separating hyperplane theorem. In particular, there exists a $v \in M_{t,+}^+ \setminus M_t^\perp$ such that $\mathbb{E}\left[v^\mathsf{T} u\right] < \inf_{\hat{u} \in R_t(X)} \mathbb{E}\left[v^\mathsf{T} \hat{u}\right]$ (if $v \notin M_{t,+}^+ \setminus M_t^\perp$ then $\inf_{\hat{u} \in R_t(X)} \mathbb{E}\left[v^\mathsf{T} \hat{u}\right] = -\infty$ by $R_t(X) = R_t(X) + M_{t,+}$). This implies that $\mathbb{E}\left[\rho_t^v(X)\right] = \mathbb{E}\left[\operatorname{ess\,inf}_{\hat{u} \in R_t(X)} v^\mathsf{T} \hat{u}\right] \leq \mathbb{E}\left[v^\mathsf{T} u\right] < \inf_{\hat{u} \in R_t(X)} \mathbb{E}\left[v^\mathsf{T} \hat{u}\right]$.

By Corollary 3.9, $R_t(X)$ is $\mathcal{F}_t$-decomposable. Therefore by Theorem 1 of [54] (and $\left\{v^\mathsf{T} u : u \in R_t(X)\right\} \subseteq L_t^1(\mathbb{R})$), it follows that $\mathbb{E}\left[\operatorname{ess\,inf}_{\hat{u} \in R_t(X)} v^\mathsf{T} \hat{u}\right] = \inf_{\hat{u} \in R_t(X)} \mathbb{E}\left[v^\mathsf{T} \hat{u}\right]$. This is a contradiction and thus $u \in R_t(X)$.

□

References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer, Berlin (2007)
2. Ararat, C., Hamel, A.H., Rudloff, B.: Set-valued shortfall and divergence risk measures (2014) (submitted for publication)
3. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Thinking coherently. Risk **10**, 68–71 (1997)
4. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. Math. Financ. **9**(3), 203–228 (1999)
5. Artzner, P., Delbaen, F., Koch-Medina, P.: Risk measures and efficient use of capital. ASTIN Bull. **39**(1), 101–116 (2009)

6. Bálbas, A., Bálbas, R., Jiménez-Guerra, P.: Vector risk functions. Mediterr. J. Math. **9**(4), 563–574 (2012)
7. Bensaid, B., Lesne, J.-P., Pagès, H., Scheinkman, J.: Derivative asset pricing with transaction costs. Math. Financ. **2**(2), 63–86 (1992)
8. Bion-Nadal, J.: Conditional risk measures and robust representation of convex risk measures. Ecole Polytechnique, CMAP, preprint no. 557 (2004)
9. Bourbaki, N.: General Topology: Chapters 1–4. General Topology. Springer, Berlin (1998)
10. Boyle, P.P., Vorst, T.: Option replication in discrete time with transaction costs. J. Financ. **47**(1), 271–293 (1992)
11. Burgert, C., Rüschendorf, L.: Consistent risk measures for portfolio vectors. Insur.: Math. Econ. **38**(2), 289–297 (2006)
12. Cascos, I., Molchanov, I.: Multivariate risks and depth-trimmed regions. Financ. Stoch. **11**(3), 373–397 (2007)
13. Cascos, I., Molchanov, I.: Multivariate risk measures: a constructive approach based on selections. Math. Financ. (2014), doi:10.1111/mafi.12078
14. Cheridito, P., Delbaen, F., Kupper, M.: Dynamic monetary risk measures for bounded discrete-time processes. Electron. J. Probab. **11**(3), 57–106 (2006)
15. Cheridito, P., Kupper, M.: Composition of time-consistent dynamic monetary risk measures in discrete time. Int. J. Theor. Appl. Financ. **14**(1), 137–162 (2011)
16. Choquet, G.: Topology. Pure and Applied Mathematics Series. Academic Press, New York (1966)
17. Detlefsen, K., Scandolo, G.: Conditional and dynamic convex risk measures. Financ. Stoch. **9**(4), 539–561 (2005)
18. Farkas, W., Koch-Medina, P., Munari, C.: Measuring risk with multiple eligible assets. Math. Financ. Econ. 1–25 (2014)
19. Farkas, W., Koch-Medina, P., Munari, C.-A.: Beyond cash-additive risk measures: when changing the numéraire fails. Financ. Stoch. **18**(1), 145–173 (2014)
20. Farkas, W., Koch-Medina, P., Munari, C.-A.: Capital requirements with defaultable securities. Insur.: Math. Econ. **55**, 58–67 (2014)
21. Feinstein, Z., Rudloff, B.: Time consistency of dynamic risk measures in markets with transaction costs. Quant. Financ. **13**(9), 1473–1489 (2013)
22. Feinstein, Z., Rudloff, B.: A recursive algorithm for multivariate risk measures and a set-valued Bellman's principle (2015, submitted for publication)
23. Feinstein, Z., Rudloff, B.: Multi-portfolio time consistency for set-valued convex and coherent risk measures. Financ. Stoch. **19**(1), 67–107 (2015)
24. Föllmer, H., Schied, A.: Convex measures of risk and trading constraints. Financ. Stoch. **6**(4), 429–447 (2002)
25. Föllmer, H., Schied, A.: Stochastic Finance, 3rd edn. Walter de Gruyter & Co., Berlin (2011)
26. Frittelli, M., Scandolo, G.: Risk measures and capital requirements for processes. Math. Financ. **16**(4), 589–612 (2006)
27. Hamel, Andreas H.: A duality theory for set-valued functions I: Fenchel conjugation theory. Set-Valued Var. Anal. **17**(2), 153–182 (2009)
28. Hamel, A.H., Heyde, F., Höhne, M.: *Set Valued Measures of Risk*. Report. Inst. für Math. (2007)
29. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**(1), 66–95 (2010)
30. Hamel, A.H, Rudloff, B.: Continuity and finite-valuedness of set-valued risk measures. In: Tammer, C., Heyde, F. (eds.), *Festschrift in Celebration of Prof. Dr. Wilfried Grecksch's 60th Birthday*, pp. 46–64. Shaker Verlag (2008)
31. Hamel, A.H., Heyde, F., Rudloff, B.: Set-valued risk measures for conical market models. Math. Financ. Econ. **5**(1), 1–28 (2011)
32. Hamel, A.H., Rudloff, B., Yankova, M.: Set-valued average value at risk and its computation. Math. Financ. Econ. **7**(2), 229–246 (2013)
33. Hamel, A.H., Löhne, A., Rudloff, B.: Benson type algorithms for linear vector optimization and applications. J. Glob. Optim. **59**(4), 811–836 (2014)

34. Jacka, S., Berkaoui, A.: On representing claims for coherent risk measures, August (2007) (ArXiv e-prints)
35. Jouini, E., Kallal, H.: Martingales and arbitrage in securities markets with transaction costs. J. Econ. Theory **66**(1), 178–197 (1995)
36. Jouini, E., Meddeb, M., Touzi, N.: Vector-valued coherent risk measures. Financ. Stoch. **8**(4), 531–552 (2004)
37. Kabanov, Y.M.: Hedging and liquidation under transaction costs in currency markets. Financ. Stoch. **3**(2), 237–248 (1999)
38. Kabanov, Y.M., Safarian, M.: Markets with Transaction Costs: Mathematical Theory. Springer Finance. Springer, Berlin (2009)
39. Kountzakis, C.E.: Generalized coherent risk measures. Appl. Math. Sci. **3**(49), 2437–2451 (2009)
40. Löhne, A.: Vector Optimization with Infimum and Supremum. Vector Optimization. Springer, Berlin (2011)
41. Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. Int. J. Theor. Appl. Financ. **17**(2), 1450012 (2014)
42. Löhne, A., Rudloff, B., Ulus, F.: Primal and dual approximation algorithms for convex vector optimization problems. J. Glob. Optim. **60**(4), 713–736 (2014)
43. Molchanov, I.: Theory of Random Sets. Probability and Its Applications. Springer, Berlin (2005)
44. Perrakis, S., Lefoll, J.: Derivative asset pricing with transaction costs: an extension. Comput. Econ. **10**(4), 359–376 (1997)
45. Riedel, F.: Dynamic coherent risk measures. Stoch. Process. Appl. **112**(2), 185–200 (2004)
46. Roux, A.: *Options Under Transaction Costs: Algorithms for Pricing and Hedging of European and American Options Under Proportional Transaction Costs and Different Borrowing and Lending Rates*. VDM Verlag (2008)
47. Roux, A., Zastawniak, T.: American and Bermudan options in currency markets under proportional transaction costs, Acta Appl. Math. (2015), doi:10.1007/s10440-015-0010-9
48. Ruszczynski, A., Shapiro, A.: Conditional risk mappings. Math. Oper. Res. **31**(3), 544–561 (2006)
49. Scandolo, G.: Models of capital requirements in static and dynamic settings. Econ. Notes **33**(3), 415–435 (2004)
50. Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Math. Financ. **14**(1), 19–48 (2004)
51. Tahar, I.B.: Tail conditional expectation for vector-valued risks. SFB 649 Discussion Papers 2006-029, Humboldt University (2006)
52. Tahar, I.B., Lépinette, E.: Vector-valued coherent risk measure processes. Int. J. Theor. Appl. Financ. **17**(2), 1450011 (2014)
53. Weber, S., Anderson, W., Hamm, A.-M., Knispel, T., Liese, M., Salfeld, T.: Liquidity-adjusted risk measures. Math. Financ. Econ. **7**(1), 69–91 (2013)
54. Yan, J.: On the commutability of essential infimum and conditional expectation operations. Chin. Sci. Bull. (Kexue Tongbao) **30**(8), 1013–1018 (1985)

Nonlinear Scalarizations of Set Optimization Problems with Set Orderings

César Gutiérrez, Bienvenido Jiménez and Vicente Novo

Abstract This paper concerns with scalarization processes of set-valued optimization problems, whose objective space is a Hausdorff locally convex topological linear space and the preferences between the objective values are stated through set orderings. To be precise, general necessary and sufficient optimality conditions for minimal and weak minimal solutions of these optimization problems are obtained by dealing with abstract scalarization mappings that satisfy certain order preserving and order representing properties. Then these conditions are applied to well-known scalarization mappings in set optimization. This approach extends and unifies the main nonlinear scalarization results of the literature on set optimization problems with set orderings.

Keywords Set-valued optimization · Set relations · Nonlinear scalarization

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1 Introduction

Scalarization methods are one of the most powerful mathematical tools to study set-valued optimization problems with set orderings (see [1, 7, 13–15, 17–19, 31–34, 36, 38–41, 43]). As an example, let us observe that they have been successfully used to obtain minimal element theorems (see [13, 17–19, 41]), Ekeland variational principles (see [7, 13, 15, 17, 18]), well-posedness properties (see [14, 43]), stability results (see [15]), scalar representations without convexity assumptions (see [19]), nonconvex separation type theorems and alternative theorems (see [1, 38]), Takahashi type minimization theorems (see [1]) and optimality conditions through solutions of associated scalar optimization problems (see [1, 11, 19, 34, 36, 40]).

To the best of our knowledge, the first scalarization mappings for set-valued optimization problems with set orderings were introduced by Hamel and Löhne [17] (see also [18]), Nishizawa et al. [39] and Ha [15]. The mappings defined by Hamel and Löhne, and Nishizawa et al. extend the so-called Gerstewitz's nonconvex separation functional (see [8–10]) in order to deal with set orderings through the usual order in $\mathbb{R} \cup \{\pm\infty\}$. The approach due to Hamel and Löhne is more general, since they consider a fixed set that plays the role of "parameter". This fact is crucial in order to characterize minimal solutions of set-valued optimization problems with set orderings through solutions of associated scalar optimization problems (see [1, 34]). The scalarization mappings due to Nishizawa et al. do not consider this parameter and they characterize nondominated solutions (see [40]), a particular type of minimal solutions. On the other hand, Ha [15] generalized the well-known weighting scalarization method, extensively used in convex vector optimization problems.

The ideas, concepts and mathematical tools introduced by Hamel and Löhne in [17, 18] have motivated a lot of new contributions for scalarizing set-valued optimization problems with set orderings. In [19], Hernández and Rodríguez-Marín introduced a nonlinear scalarization mapping, studied its properties in deep and, for the first time in the literature, they characterized minimal and weak minimal solutions of set-valued optimization problems with set orderings via solutions of associated scalar optimization problems. Some new properties of this scalarization mapping have been stated in [43], where it has been used to derive well-posedness properties of set-valued optimization problems with set orderings.

Recently and inspired by this approach, Gutiérrez et al. [14] derived new properties of the scalarization mappings due to Hamel and Löhne, generalized the well-posedness properties obtained in [43], and characterized minimal and strict minimal solutions of set-valued optimization problems with set orderings via scalarization. Also, Gutiérrez et al. [13] defined a sup-inf type scalarization mapping and via this mapping they derived approximate strict minimal element theorems and approximate versions of the Ekeland variational principle for set orderings.

On the other hand, Kuwano et al. [31] and Araya [1] introduced scalarization schemes that unify several nonlinear scalarization mappings introduced in the literature and allow to characterize via scalarization minimal solutions of set-valued optimization problems by considering different set orderings (see [34]). Moreover, Maeda [36] characterized via the scalarization mappings due to Hamel and Löhne new concepts of solution based on set orderings motivated by fuzzy mathematical programming problems (see [35]).

This work is structured as follows. In Sect. 2 we introduce the set-valued optimization problem and the basic notations. Moreover, some technical results on topological properties of the conic extension of a set and about the set orderings induced by an open ordering cone are stated. In Sect. 3 we introduce the order representing and monotonicity properties for mappings defined in the power set of a vector space. This kind of properties have been widely used in vector optimization to characterize minimal solutions through scalarization. Moreover, we study in deep some properties of the nonlinear scalarization mappings introduced by Hamel and Löhne [17, 18] and Gutiérrez et al. [13]. In particular, we analyze when these scalarization mappings satisfy the mentioned order representing and monotonicity properties, and we prove that the scalarization mapping due to Hernández and Rodríguez-Marín [19] coincides with the scalarization mapping due to Hamel and Löhne. Finally, in Sect. 4, we characterize the minimal and weak minimal solutions of set-valued optimization problems with set orderings by solutions of scalar optimization problems defined via generic scalarization mappings that satisfy order representing and monotonicity properties. We show that these "implicit" characterizations can be done "explicit" by using Hamel and Löhne scalarization mapping and, in general, by considering any scalarization mapping that satisfies the required order representing and monotonicity properties. The results obtained in Sects. 3 and 4 extend and clarify some results of the literature.

2 Preliminaries

Let Y be a Hausdorff locally convex topological linear space. The topological dual space of Y is denoted by Y^*, and the duality pairing by $\langle y^*, y\rangle$, $y^* \in Y^*$, $y \in Y$. We denote by int M, cl M and cone M the interior, the closure and the cone generated by a set $M \subset Y$, and we say that M is solid if int $M \neq \emptyset$. The ordering cone of Y is denoted by D, which is assumed to be proper (i.e., $D \neq Y$), closed, solid and convex. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

The positive polar cone of D is denoted by D^+, i.e.,

$$D^+ := \{\lambda \in Y^* : \langle \lambda, d\rangle \geq 0, \forall\, d \in D\}.$$

For each $q \in \operatorname{int} D$ we denote

$$D^+(q) := \{\lambda \in D^+ : \langle \lambda, q\rangle = 1\}.$$

It is well-known (see, for instance, [10]) that $D^+(q)$ is compact in the weak star topology, convex and cone $D^+(q) = D^+$.

We denote the Minkowski sum of two nonempty sets $M_1, M_2 \subset Y$ by $M_1 + M_2$, i.e.,

$$M_1 + M_2 := \{y_1 + y_2 : y_1 \in M_1, y_2 \in M_2\}.$$

Moreover, we assume that $M + \emptyset = \emptyset + M = \emptyset$, for all $M \subset Y$, and for each $y \in Y$, $y + M$ (resp. $M + y$) denotes $\{y\} + M$ (resp. $M + \{y\}$). The following topological properties on the conic extension of a set will be used in the paper. Part (a) was stated in [4, Lemma 2.5] and so its proof is omitted.

Proposition 2.1 *Consider a nonempty set $M \subset Y$. We have that*

(a) $\operatorname{int}\operatorname{cl}(M + D) = M + \operatorname{int} D$,
(b) $\operatorname{cl}(M + D) + \operatorname{int} D \subset M + \operatorname{int} D$.

Proof Let us proof part (b). It is clear that

$$\operatorname{cl}(M + D) + \operatorname{int} D \subset \operatorname{cl}(M + D) + D = \operatorname{cl}(M + D).$$

Then, by part (a) it follows that

$$\operatorname{int}(\operatorname{cl}(M + D) + \operatorname{int} D) \subset M + \operatorname{int} D$$

and the proof is completed. □

Recall that a set $M \subset Y$ is D-bounded if for each neighborhood U of zero in Y there exists $\alpha > 0$ such that $M \subset \alpha U + D$. M is D-compact if any cover of M of the form $\{U_i + D : U_i \text{ is open}\}$ admits a finite subcover. Observe that the family $\{V_i + \operatorname{int} D : V_i \subset Y\}$ fits with this form, since $V_i + \operatorname{int} D = (V_i + \operatorname{int} D) + D$, and the sets $U_i := V_i + \operatorname{int} D$ are open. Analogously, we say that M is D-closed if $M + D$ is closed.

On the other hand, M is D-proper if $M + D \neq Y$. Cone properness is a kind of boundedness weaker than the cone boundedness (see [14]). Next we recall an important characterization of this property.

Lemma 2.2 *[14, Theorem 3.6] A nonempty set $M \subset Y$ is D-proper if and only if there is not an element $e \in \operatorname{int} D$ such that $-e + M \subset M + D$.*

In this paper we study the following set optimization problem:

$$\operatorname{Min}\{F(x) : x \in S\}, \tag{P}$$

where the objective mapping $F : X \to 2^Y$ is set-valued, the decision space X is an arbitrary set, and the feasible set $S \subset X$ is nonempty. In [2, 3, 16, 20, 24] the reader can find some practical problems which are modeled by set optimization problems.

We denote

$$\operatorname{Dom} F := \{x \in X : F(x) \neq \emptyset\}$$

and we suppose that F is proper in S, i.e., $\mathrm{Dom} F \cap S \neq \emptyset$. We say that F is D-proper (resp. D-compact, D-closed) valued in S if $F(x)$ is D-proper (resp. D-compact, D-closed), for all $x \in S$.

To solve this problem one needs to discriminate between the objective values $F(x)$, $x \in S$. We model this task via the following well-known set orderings (see [25–30]), where $K \in \{D, \mathrm{int}\, D\}$:

$$
\begin{aligned}
M_1, M_2 \subset Y, \quad & M_1 \precsim^l_K M_2 \iff M_2 \subset M_1 + K,\\
& M_1 \precsim^u_K M_2 \iff M_1 \subset M_2 - K,\\
& M_1 \backsim^j_K M_2 \iff M_1 \precsim^j_K M_2 \text{ and } M_2 \precsim^j_K M_1,\\
& M_1 \prec^j_K M_2 \iff M_1 \precsim^j_K M_2 \text{ and } M_1 \not\backsim^j_K M_2 \quad (j \in \{l, u\}).
\end{aligned}
$$

Remark 2.3 (a) Let $K \in \{D, \mathrm{int}\, D\}$. The following equivalences are clear:

$$
\begin{aligned}
M_1 \backsim^l_K M_2 &\iff M_2 \backsim^l_K M_1,\\
M_1 \precsim^u_K M_2 &\iff M_2 \precsim^l_{-K} M_1,\\
M_1 \backsim^u_K M_2 &\iff M_1 \backsim^l_{-K} M_2,\\
M_1 \prec^u_K M_2 &\iff M_2 \prec^l_{-K} M_1,\\
M_1 \prec^l_K M_2 &\iff M_2 \subset M_1 + K \text{ and } M_1 \not\subset M_2 + K,\\
M_1 \prec^u_K M_2 &\iff M_1 \subset M_2 - K \text{ and } M_2 \not\subset M_1 - K.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
M_1 \backsim^l_D M_2 &\iff M_1 + D = M_2 + D,\\
M_1 \backsim^u_D M_2 &\iff M_1 - D = M_2 - D.
\end{aligned}
$$

These two statements could be false for the relations $\backsim^l_{\mathrm{int}\, D}$ and $\backsim^u_{\mathrm{int}\, D}$. For example, consider the following data: $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $M_1 = \mathrm{int}\, \mathbb{R}^2_+$, $M_2 = \mathbb{R}^2_+$. It is easy to check that $M_1 + \mathrm{int}\, D = M_2 + \mathrm{int}\, D$ and $M_2 \prec^l_{\mathrm{int}\, D} M_1$.

(b) Let us observe that there exist in the literature a lot of set relations from which one can model the preferences between the objective values of the problem (P) (see [6, 23]).

In the next lemma we state two technical properties of the relation $\precsim^l_{\mathrm{int}\, D}$, which will be used in the paper.

Lemma 2.4 *Consider $q \in \mathrm{int}\, D$ and two nonempty sets $A, M \subset Y$. The following statements hold:*

(a) If A is D-bounded, then there exists $t \in \mathbb{R}$ such that $M + tq \precsim^l_{\mathrm{int}\, D} A$.
(b) If A is D-compact and $M \precsim^l_{\mathrm{int}\, D} A$, then there exists $t > 0$ such that $M + tq \precsim^l_{\mathrm{int}\, D} A$.

Proof (a) Let us consider an arbitrary point $y \in M$. It is obvious that $A - y$ is D-bounded and $-q + \operatorname{int} D$ is a neighborhood of zero in Y. Then there exists $\alpha > 0$ such that $A - y \subset \alpha(-q + \operatorname{int} D) + D = -\alpha q + \operatorname{int} D$. Therefore, $A \subset M - \alpha q + \operatorname{int} D$ and the proof of part (a) is finished.

(b) As $A \subset M + \operatorname{int} D$ and $M + \operatorname{int} D$ is an open set, for each $y \in A$ there exists $t_y > 0$ such that $y - t_y q \in M + \operatorname{int} D$, and it follows that

$$A \subset \bigcup_{y \in A} (t_y q + M + \operatorname{int} D).$$

Since A is D-compact we deduce that there exist $\{y_1, y_2, \ldots, y_n\} \subset A$ such that

$$A \subset \bigcup_{i=1}^{n} (t_{y_i} q + M + \operatorname{int} D).$$

By considering $t := \min\{t_{y_i} : i = 1, 2, \ldots, n\} > 0$ we obtain

$$A \subset tq + M + \operatorname{int} D,$$

and the proof is completed. □

The set orderings $\precsim^l$ and $\precsim^u$ define the preferences on the feasible points via their objective values. The concepts of solution of problem (P) were introduced according with these preferences (see [19, 25–29]) and the classical minimality notion in the theory of ordered sets.

Definition 2.5 Consider $j \in \{l, u\}$. A point $x_0 \in S$ is a j-minimal (resp. weak j-minimal) solution of problem (P), denoted by $x_0 \in \mathrm{M}^j(F, S)$ (resp. $x_0 \in \mathrm{WM}^j(F, S)$), if

$$x \in S,\ F(x) \precsim_D^j F(x_0) \Rightarrow F(x_0) \precsim_D^j F(x)$$
$$(\text{resp. } x \in S,\ F(x) \precsim_{\operatorname{int} D}^j F(x_0) \Rightarrow F(x_0) \precsim_{\operatorname{int} D}^j F(x)).$$

Remark 2.6 (a) Let $G : X \to 2^Y$ be such that $F(x) = G(x) + D$ (resp. $F(x) = G(x) - D$), for all $x \in S$. It is easy to check that $\mathrm{M}^l(F, S) = \mathrm{M}^l(G, S)$ and $\mathrm{WM}^l(F, S) = \mathrm{WM}^l(G, S)$ (resp. $\mathrm{M}^u(F, S) = \mathrm{M}^u(G, S)$ and $\mathrm{WM}^u(F, S) = \mathrm{WM}^u(G, S)$).

(b) Let us recall that the first concepts of solution of problem (P) introduced in the literature did not use set relations. They consider feasible points whose images contain minimal points with respect to the whole image of the objective mapping and the ordering induced by the cone D (see [5, 21]).

As F is proper in S, it follows that $\mathrm{M}^l(F, S) \subset \mathrm{Dom} F \cap S$ and $\mathrm{WM}^l(F, S) \subset \mathrm{Dom} F \cap S$. Analogously, if $S \backslash \mathrm{Dom} F \neq \emptyset$ then $\mathrm{M}^u(F, S) = \mathrm{WM}^u(F, S) = S \backslash \mathrm{Dom} F$. Therefore, without losing generality, in the sequel we assume that $S \subset \mathrm{Dom} F$. Moreover, we denote $\mathcal{Y} := 2^Y \backslash \{\emptyset\}$ and $\mathcal{F} := \{F(x) : x \in S\}$.

Let us observe that if there exists $x \in S$ such that $F(x)$ is not D-proper, then

$$\mathrm{M}^l(F, S) = \mathrm{WM}^l(F, S) = \{z \in S : F(z) + D = Y\}.$$

On the other hand, if there exists $x \in S$ such that $F(x)$ is $-D$-proper, then $F(z)$ is $-D$-proper, for all $z \in \mathrm{M}^u(F, S)$ and for all $z \in \mathrm{WM}^u(F, S)$. Therefore, for solving problem (P) in the sense of the l-minimality or weak l-minimality (resp. u-minimality or weak u-minimality), one could assume without loss of generality that the objective mapping F is D-proper valued (resp. $-D$-proper valued) in S.

3 Scalarization Processes

The scalarization processes are among the most important techniques to study problem (P). They relate the solutions of problem (P) with solutions of associated scalar optimization problems. Usually, these associated scalar optimization problems are defined by the composition of the objective mapping F with the elements of a parametric family $\{\varphi_p\}_{p \in \mathcal{P}}$ of extended real-valued mappings $\varphi_p : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$, where $\mathcal{P}$ is an index set (see [11, 19]). Then, the scalarization processes relate the minimal and weak minimal solutions of problem (P) with the solutions of the following scalar optimization problems:

$$\mathrm{Min}\{(\varphi_p \circ F)(x) : x \in S\}. \qquad (\mathrm{P}_{\varphi_p})$$

We denote the set of solutions of problem (P_{φ_p}) by $\mathrm{S}(\varphi_p \circ F, S)$. Let us recall that a point $x_0 \in S$ is a strict solution of problem (P_{φ_p}) if $\varphi_p(F(x_0)) < \varphi_p(F(x))$, $\forall\, x \in S \backslash \{x_0\}$, i.e., if $\mathrm{S}(\varphi_p \circ F, S) = \{x_0\}$.

Let $\mathcal{M} \subset \mathcal{Y}$, we say that an extended real-valued mapping $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ is proper if $\varphi(M) > -\infty$, for all $M \in \mathcal{M}$, and

$$\mathrm{Dom}\varphi := \{M \in \mathcal{M} : \varphi(M) < +\infty\} \neq \emptyset.$$

It is well-known between practitioners and researchers in vector optimization that a scalarization mapping is useful to characterize the solutions of a vector optimization problem through the solutions of the associated scalar optimization problem whenever it is monotone and satisfies the so-called order representing property (see [12, 37, 42]). Next we extend these properties to problem (P).

Definition 3.1 Let $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$, $A \in \mathcal{M}$ and $j \in \{l, u\}$.

(a) φ is order $\precsim^l_D$-representing (resp. $\precsim^u_D$-representing) at A if

$$\{M \in \mathcal{M} : \varphi(M) \leq \varphi(A)\} \subset \{M \in \mathcal{M} : M \precsim^l_D A\}$$
$$(\text{resp. } \{M \in \mathcal{M} : \varphi(A) \leq \varphi(M)\} \subset \{M \in \mathcal{M} : A \precsim^u_D M\}).$$

(b) φ is strictly order $\precsim^l_D$-representing (resp. strictly $\precsim^u_D$-representing) at A if

$$\{M \in \mathcal{M} : \varphi(M) < \varphi(A)\} \subset \{M \in \mathcal{M} : M \prec^l_D A\}$$
$$(\text{resp. } \{M \in \mathcal{M} : \varphi(A) < \varphi(M)\} \subset \{M \in \mathcal{M} : A \prec^u_D M\}).$$

(c) φ is $\precsim^l_D$-monotone (resp. $\precsim^u_D$-monotone) at A if

$$M \in \mathcal{M}, M \precsim^l_D A \quad \Rightarrow \quad \varphi(M) \leq \varphi(A)$$
$$(\text{resp. } M \in \mathcal{M}, A \precsim^u_D M \quad \Rightarrow \quad \varphi(A) \leq \varphi(M)).$$

(d) φ is strictly $\precsim^l_D$-monotone (resp. strictly $\precsim^u_D$-monotone) at A if

$$M \in \mathcal{M}, M \prec^l_D A \quad \Rightarrow \quad \varphi(M) < \varphi(A)$$
$$(\text{resp. } M \in \mathcal{M}, A \prec^u_D M \quad \Rightarrow \quad \varphi(A) < \varphi(M)).$$

(e) φ is $\precsim^j_D$-monotone (resp. strictly $\precsim^j_D$-monotone) on $\mathcal{M}$ if φ is $\precsim^j_D$-monotone (resp. strictly $\precsim^j_D$-monotone) at A, for all $A \in \mathcal{M}$.

In the literature one can find several scalarization processes to deal with problem (P) without assuming any convexity assumption (see [1, 13, 14, 17–19, 31, 33, 34, 36, 39, 40, 43]). All of them generalize the so-called Gerstewitz scalarization mapping $s_q : Y \to \mathbb{R}$ (see [8–10]):

$$s_q(y) := \inf\{t \in \mathbb{R} : y \in tq - D\}, \quad \forall\, y \in Y,$$

where q is an arbitrary point in int D. This mapping has been extensively used for scalarizing vector optimization problems (see [5, 10] and the references therein). In particular, it follows that (see [5, Proposition 1.53]):

$$s_q(y) = \max\{\langle \lambda, y\rangle : \lambda \in D^+(q)\}, \quad \forall\, y \in Y. \tag{1}$$

Motivated by the scalarization processes due to Hamel and Löhne, we consider the following families $\{\Phi^{j,D}_{q,F(x)}\}_{x\in S}$, $j \in \{l, u\}$, $q \in \text{int}\, D$, of scalarization mappings of problem (P) (see [1, 14, 17, 18, 34]). For each $A \in \mathcal{Y}$, $\Phi^{j,D}_{q,A} : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$,

$$\Phi^{l,D}_{q,A}(M) := \inf \Lambda^{l,D}_{q,A}(M)$$

$$\Phi^{u,D}_{q,A}(M) := \sup \Lambda^{u,D}_{q,A}(M)$$

where

$$\Lambda^{l,D}_{q,A}(M) := \{t \in \mathbb{R} : M \precsim^l_D tq + A\},$$
$$\Lambda^{u,D}_{q,A}(M) := \{t \in \mathbb{R} : tq + A \precsim^u_D M\},$$

and we assume the usual conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. It is clear that

$$t \in \Lambda^{l,D}_{q,A}(M) \Rightarrow [t, \infty) \subset \Lambda^{l,D}_{q,A}(M)$$

and then, one of the following cases happens: $\Lambda^{l,D}_{q,A}(M) = \emptyset$, $\Lambda^{l,D}_{q,A}(M) = \mathbb{R}$ (i.e., $\Phi^{l,D}_{q,A}(M) = -\infty$) or $\Lambda^{l,D}_{q,A}(M) \neq \emptyset$, $\Lambda^{l,D}_{q,A}(M) \neq \mathbb{R}$ (i.e., $\Phi^{l,D}_{q,A}(M) \in \mathbb{R}$ and $\Lambda^{l,D}_{q,A}(M) = [\Phi^{l,D}_{q,A}(M), +\infty)$ or $\Lambda^{l,D}_{q,A}(M) = (\Phi^{l,D}_{q,A}(M), +\infty)$).

On the other hand, it is easy to check that

$$-\Lambda^{u,D}_{q,A}(M) = \Lambda^{l,-D}_{-q,A}(M) = \{t \in \mathbb{R} : M \precsim^{l}_{-D} t(-q) + A\}, \quad \forall\, M \in \mathcal{Y},$$

(observe that $\Lambda^{l,-D}_{-q,A}(M)$ is defined by $-D$ instead of D) and so we have that

$$\Phi^{u,D}_{q,A}(M) = -\Phi^{l,-D}_{-q,A}(M), \quad \forall\, M \in \mathcal{Y}. \tag{2}$$

Thus, in the sequel, the statements on the scalarization process $\Phi^{u,D}_{q,A}$ are not proved, since they follow easily from the corresponding statements on the scalarization process $\Phi^{l,D}_{q,A}$ by the equivalences of Remark 2.3(*a*) and relation (2). Let us observe that $\Phi^{l,D}_{q,A}$ and $\Phi^{u,D}_{q,A}$ reduce to the scalarization mappings introduced by Nishizawa et al. in [39] by taking $A = \{0\}$, i.e., by removing the "parameter" A.

In the next four results we collect some of the main properties of these scalarization processes. Proposition 3.2 and Theorem 3.5 extend Proposition 4.1 and Theorem 4.2 of [14]. In order to simplify the notation we write $\Lambda^{l}_{q,A}$, $\Lambda^{u}_{q,A}$, $\Phi^{l}_{q,x}$ and $\Phi^{u}_{q,x}$ instead of $\Lambda^{l,D}_{q,A}$, $\Lambda^{u,D}_{q,A}$, $\Phi^{l,D}_{q,F(x)}$ and $\Phi^{u,D}_{q,F(x)}$, respectively. Let us recall that mappings $\Phi^{l}_{q,x}$ and $\Phi^{u}_{q,x}$ are defined on $\mathcal{Y} = 2^{Y}\backslash\{\emptyset\}$.

Proposition 3.2 *Let $q \in \operatorname{int} D$, $x \in S$ and $M \in \mathcal{Y}$.*

(a) $\Phi^{l}_{q,x}(M) = -\infty$ *if and only if M is not D-proper.*
(b) If $F(x)$ is not D-proper, then

$$\Phi^{l}_{q,x}(M) = \begin{cases} +\infty & \text{if } M \text{ is } D\text{-proper}, \\ -\infty & \text{otherwise}. \end{cases}$$

(c) Suppose that $F(x)$ is D-bounded. Then $\Phi^{l}_{q,x}(M) < +\infty$.

Proof (*a*) If $\Phi^{l}_{q,x}(M) = -\infty$, then $\Lambda^{l}_{q,F(x)}(M) = \mathbb{R}$ and $M \precsim^{l}_{D} tq + F(x)$, for all $t \in \mathbb{R}$. In other words,

$$\bigcup_{t\in\mathbb{R}} (tq + F(x)) \subset M + D. \tag{3}$$

As $q \in \operatorname{int} D$, it is easy to check that $\bigcup_{t\in\mathbb{R}}(tq + D) = Y$. Then, by (3) we see that

$$Y = Y + F(x) = \bigcup_{t\in\mathbb{R}}(tq + D) + F(x) \subset M + D + D = M + D$$

and M is not D-proper. Reciprocally, if M is not D-proper, then it is obvious that $M \precsim^l_D tq + F(x)$, for all $t \in \mathbb{R}$, and so $\Phi^l_{q,x}(M) = -\infty$.

(b) Suppose that $F(x)$ is not D-proper and let $M \in \mathcal{Y}$. If M is not D-proper, then by part (a) we deduce that $\Phi^l_{q,x}(M) = -\infty$. If M is D-proper and $\Lambda^l_{q,F(x)}(M) \neq \emptyset$, then there exists $t \in \mathbb{R}$ such that $tq + F(x) \subset M + D$, and so

$$Y = tq + F(x) + D \subset M + D + D = M + D,$$

that is a contradiction. Thus $\Lambda^l_{q,F(x)}(M) = \emptyset$ and so $\Phi^l_{q,x}(M) = +\infty$, which finishes the proof of part (b).

(c) As $F(x)$ is D-bounded and $q \in \operatorname{int} D$, by Lemma 2.4($a$), there exists $t \in \mathbb{R}$ such that $F(x) + tq \subset M + D$. Then $t \in \Lambda^l_{q,F(x)}(M)$, and as $\Lambda^l_{q,F(x)}(M) \neq \emptyset$ it follows that $\Phi^l_{q,x}(M) < +\infty$, which finishes the proof. □

The next similar properties on $\Phi^u_{q,x}$ are direct consequences of Proposition 3.2 and relation (2).

Proposition 3.3 *Let $q \in \operatorname{int} D$, $x \in S$ and $M \in \mathcal{Y}$. Then:*

(a) $\Phi^u_{q,x}(M) = +\infty$ *if and only if M is not $-D$-proper.*
(b) *If $F(x)$ is not $-D$-proper, then*

$$\Phi^u_{q,x}(M) = \begin{cases} -\infty & \text{if } M \text{ is } -D\text{-proper}, \\ +\infty & \text{otherwise}. \end{cases}$$

(c) *Suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}(M) > -\infty$.*

Remark 3.4 The sufficient condition of Proposition 3.2(a) and Proposition 3.2(c) reduce to [36, Theorem 3.1(i),(ii)] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that M is D-bounded. Analogously, the sufficient condition of Proposition 3.3(a) and Proposition 3.3(c) reduce to [36, Theorem 3.1(iv), (iii)] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that M is $-D$-bounded.

Theorem 3.5 *Consider $q \in \operatorname{int} D$, $x \in S$, and suppose that $F(x)$ is D-proper.*

(a) $\Phi^l_{q,x}(F(x)) = 0$.
(b) *If F is D-proper valued in S, then $\Phi^l_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is proper.*
(c) $\Phi^l_{q,x}(M + tq) = \Phi^l_{q,x}(M) + t$, *for all $M \in \mathcal{Y}$ and $t \in \mathbb{R}$.*
(d) $\Phi^l_{q,x}$ *is $\precsim^l_D$-monotone on $\mathcal{Y}$.*
(e) *Let $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It follows that*

$$\Phi^l_{q,x}(M) \le t \iff F(x) \subset -tq + \operatorname{cl}(M + D). \tag{4}$$

(f) $\Phi^l_{q,x}$ *is strictly order $\precsim^l_{\operatorname{int} D}$-representing at $F(x)$.*

(g) *Consider $Q, M \in \mathcal{Y}$ such that Q is D-compact and $M \precsim^l_{\operatorname{int} D} Q$, and suppose that $F(x)$ is D-bounded. Then $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(Q)$.*

(h) *Let $\mathcal{M} \subset \mathcal{Y}$ be a family of D-compact sets and suppose that $F(x)$ is D-bounded. Then $\Phi^l_{q,x}$ is strictly $\precsim^l_{\operatorname{int} D}$-monotone on $\mathcal{M}$.*

Proof (a) It is clear that $\Phi^l_{q,x}(F(x)) \leq 0$, since $0 \in \Lambda^l_{q,F(x)}(F(x))$. If $\Phi^l_{q,x}(F(x)) < 0$, then there exists $t > 0$ such that $F(x) \precsim^l_D -tq + F(x)$, i.e., $-tq + F(x) \subset F(x) + D$. By Lemma 2.2 it follows that $F(x)$ is not D-proper, which is a contradiction. Thus $\Phi^l_{q,x}(F(x)) = 0$ and the proof of part (a) is completed.

(b) As $F(z)$ is D-proper for all $z \in S$, by Proposition 3.2(a) we deduce that $\Phi^l_{q,x}(F(z)) > -\infty$, for all $z \in S$. Moreover, by part (a) we have that $F(x) \in \operatorname{Dom}\Phi^l_{q,x}$. Then $\Phi^l_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is proper.

(c) Consider $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It is easy to check that $\Lambda^l_{q,F(x)}(M + tq) = \Lambda^l_{q,F(x)}(M) + t$. Then, $\Phi^l_{q,x}(M + tq) = +\infty$ if and only if $\Phi^l_{q,x}(M) = +\infty$, since $\Lambda^l_{q,F(x)}(M + tq) = \emptyset$ if and only if $\Lambda^l_{q,F(x)}(M) = \emptyset$. On the other hand, if $\Phi^l_{q,x}(M) < +\infty$, then

$$\Phi^l_{q,x}(M + tq) = \inf \Lambda^l_{q,F(x)}(M + tq) = \inf \Lambda^l_{q,F(x)}(M) + t = \Phi^l_{q,x}(M) + t$$

and part (c) is proved.

(d) Consider $A, M \in \mathcal{Y}$ such that $M \precsim^l_D A$. As the relation $\precsim^l_D$ is transitive, it is easy to check that $\Lambda^l_{q,F(x)}(A) \subset \Lambda^l_{q,F(x)}(M)$ and so $\Phi^l_{q,x}(M) \leq \Phi^l_{q,x}(A)$. Thus, $\Phi^l_{q,x}$ is $\precsim^l_D$-monotone at A, for all $A \in \mathcal{Y}$.

(e) It follows that

$$\begin{aligned} \Phi^l_{q,x}(M) \leq t &\Rightarrow t + \varepsilon \in \Lambda^l_{q,F(x)}(M), \quad \forall\, \varepsilon > 0 \\ &\Rightarrow F(x) + (t + \varepsilon)q \subset M + D, \quad \forall\, \varepsilon > 0 \\ &\Rightarrow y + \varepsilon q \in -tq + M + D, \quad \forall\, \varepsilon > 0, \forall\, y \in F(x). \end{aligned}$$

Then, by taking the limit when $\varepsilon \downarrow 0$ we have that

$$\Phi^l_{q,x}(M) \leq t \Rightarrow y \in \operatorname{cl}(-tq + M + D) = -tq + \operatorname{cl}(M + D), \quad \forall\, y \in F(x),$$

and the necessary condition of part (e) is proved. Reciprocally, if $y \in -tq + \operatorname{cl}(M + D)$ for all $y \in F(x)$, then by Proposition 2.1(b) we have

$$\begin{aligned} y + \varepsilon q \in -tq + \operatorname{cl}(M + D) + \varepsilon q &\subset -tq + \operatorname{cl}(M + D) + \operatorname{int} D \\ &\subset -tq + M + \operatorname{int} D, \quad \forall\, \varepsilon > 0, \forall\, y \in F(x). \end{aligned}$$

Therefore, $t + \varepsilon \in \Lambda^l_{q,F(x)}(M)$ for all $\varepsilon > 0$, and so $\Phi^l_{q,x}(M) \leq t$.

(f) Let us consider $M \in \mathcal{Y}$ such that $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(F(x))$. By part (a) we see that $\Phi^l_{q,x}(M) < 0$ and there exists $t > 0$ such that $-tq + F(x) \subset M + D$. Thus,

$$F(x) \subset M + (tq + D) \subset M + \operatorname{int} D$$

and we see that $M \precsim^l_{\operatorname{int} D} F(x)$. If $F(x) \precsim^l_{\operatorname{int} D} M$, then it is obvious that $F(x) \precsim^l_D M$ and as $\Phi^l_{q,x}$ is $\precsim^l_D$-monotone on $\mathcal{Y}$ we deduce that $\Phi^l_{q,x}(F(x)) \leq \Phi^l_{q,x}(M)$, which is a contradiction. Therefore $M \prec^l_{\operatorname{int} D} F(x)$, which finishes the proof of part (*f*).

(*g*) As Q is D-compact it follows that Q is D-proper, and by Proposition 3.2(*a*) we have that $\Phi^l_{q,x}(Q) > -\infty$. If M is not D-proper, then we see that $\Phi^l_{q,x}(M) = -\infty$, and so $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(Q)$.

Suppose that M is D-proper. By Lemma 2.4(*b*) it follows that there exists $t > 0$ such that $M + tq \precsim^l_D Q$. As M is D-proper and $F(x)$ is D-bounded, by Proposition 3.2(*a*), (*c*) we see that $\Phi^l_{q,x}(M) \in \mathbb{R}$. Then, by parts (*c*) and (*d*) we deduce that

$$\Phi^l_{q,x}(M) < t + \Phi^l_{q,x}(M) = \Phi^l_{q,x}(M + tq) \leq \Phi^l_{q,x}(Q).$$

Part (*h*) is a direct consequence of part (*g*). □

Similar results for $\Phi^u_{q,x}$ are collected without proof in the next theorem.

Theorem 3.6 *Consider $q \in \operatorname{int} D$, $x \in S$, and suppose that $F(x)$ is $-D$-proper.*

(a) $\Phi^u_{q,x}(F(x)) = 0$.
(b) *If F is $-D$-proper valued in S and $F(x)$ is $-D$-bounded, then $\Phi^u_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is finite valued.*
(c) $\Phi^u_{q,x}(M + tq) = \Phi^u_{q,x}(M) + t$, *for all $M \in \mathcal{Y}$ and $t \in \mathbb{R}$.*
(d) $\Phi^u_{q,x}$ *is $\precsim^u_D$-monotone on $\mathcal{Y}$.*
(e) *Let $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It follows that*

$$\Phi^u_{q,x}(M) \geq t \iff F(x) \subset -tq + \operatorname{cl}(M - D).$$

(f) $\Phi^u_{q,x}$ *is strictly order $\precsim^u_{\operatorname{int} D}$-representing at $F(x)$.*
(g) *Consider $Q, M \in \mathcal{Y}$ such that Q is $-D$-compact and $Q \precsim^u_{\operatorname{int} D} M$, and suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}(Q) < \Phi^u_{q,x}(M)$.*
(h) *Let $\mathcal{M} \subset \mathcal{Y}$ be a family of $-D$-compact sets and suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}$ is strictly $\precsim^u_{\operatorname{int} D}$-monotone on $\mathcal{M}$.*

Remark 3.7 (*a*) Parts (*c*) and (*d*) in Theorem 3.5 have been stated in different papers (see [18, Theorem 3.1(ii), (iii)], [1, Theorem 3.2(iii), (v)], [34, Proposition 3.2]). Theorem 3.5(*d*) reduces to [36, Theorem 3.2, statement (14)] by considering $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$. Analogously, parts (*c*) and (*d*) in Theorem 3.6 have been obtained in [18, Corollary 3.2], [1, Theorem 3.7(iii), (v)], [34, Proposition 3.2], and Theorem 3.6(*d*) reduces to [36, Theorem 3.2, statement (13)] by considering $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$.

On the other hand, Theorems 3.5(*g*) and 3.6(*g*) reduce to statements (18) and (17) of [36, Theorem 3.3], respectively, by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that Q and M are compact. Analogously, Theorems 3.5(*a*), (*f*), (*g*) and 3.6(*a*), (*f*), (*g*) reduce to statements (i)–(v) of [36, Theorem 3.4] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and the class of all nonempty and compact sets of $\mathbb{R}^n$.

(b) Proposition 3.2 and Theorem 3.5 extend [18, Theorem 3.1]. Analogously, Proposition 3.3 and Theorem 3.6 extend [18, Corollary 3.2].

In [18, Theorem 3.1], the authors prove that $\Phi^l_{q,x}(M) < +\infty$, for all $M \in \mathcal{Y}$, $M \precsim^l_D F(x)$. In Theorem 3.5 we obtain $\Phi^l_{q,x}(M) < +\infty$, for all $M \in \mathcal{Y}$ whenever $F(x)$ is D-bounded. On the other hand, in [18, Theorem 3.1], the authors prove that $\Phi^l_{q,x}$ is lower bounded on $\mathcal{M} \subset \mathcal{Y}$ whenever the elements of $\mathcal{M}$ are lower $\precsim^l_D$-bounded by a bounded set A, i.e., whenever there exists a (topological) bounded set A such that $A \precsim^l_D M$, for all $M \in \mathcal{M}$. Moreover, let us observe that [18, Theorem 3.1] can be applied if the ordering cone D is not solid whenever $D\backslash(-D) \neq \emptyset$.

(c) Proposition 3.2 and Theorem 3.5 (resp. Proposition 3.3 and Theorem 3.6) extend and clarify [1, Theorem 3.2] (resp. [1, Theorem 3.7]). To be precise, in [1, Theorem 3.2] the mapping $\Phi^l_{q,A}$ is denoted by $h^l_{\inf}(\cdot\,; A)$ and it is defined by a point $q \in D\backslash(-D)$ instead of $q \in \operatorname{int} D$. Under these assumptions, Theorem 3.2(i) of [1] states that $h^l_{\inf}(M; A) > -\infty$, for all D-proper sets $M \in \mathcal{Y}$. However, this statement could be wrong if $q \notin \operatorname{int} D$ (compare with Proposition 3.2(a)), as it is showed in the following example. Consider $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 0)$ $A = \{(0, 0)\}$ and $M = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$. It is clear that M is D-proper and $h^l_{\inf}(M; A) = -\infty$.

On the other hand, for each $t \in \mathbb{R}$, Theorem 3.2(ii) of [1] states that

$$h^l_{\inf}(M; A) \leq t \iff A + tq \subset M + D,$$

but this equivalence is true whenever M is D-closed (compare with (4)), as the following example shows. Let $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 1)$, $A = \{(0, 0)\}$ and $M = \operatorname{int} \mathbb{R}^2_+$. It is clear that $h^l_{\inf}(M; A) = 0$ and $A \not\subset M + D$. Analogously, the sufficient condition of [1, Theorem 3.2(xi)] and [1, Theorem 3.2(xii)] could not be true for non-D-compact sets (compare with parts (g) and (h) of Theorem 3.5). Indeed, consider $Y = \mathbb{R}^2$, $D = M = \mathbb{R}^2_+$, $q = (1, 1)$ and $A = \operatorname{int} \mathbb{R}^2_+$. It is obvious that $M \precsim^l_{\operatorname{int} D} A$, but $h^l_{\inf}(M; A) = h^l_{\inf}(A; A) = 0$. Moreover, the necessary condition of [1, Theorem 3.2(xi)] can be generalized as follows:

$$A, M \in \mathcal{Y}, t \in \mathbb{R}, \quad h^l_{\inf}(M; A) < t \Rightarrow tq + A \subset M + \operatorname{int} D, \tag{5}$$

i.e., the assumptions on the D-properness and D-closedness of M can be removed. Let us check (5). If $h^l_{\inf}(M; A) < t$ there exists $\varepsilon > 0$ such that $h^l_{\inf}(M; A) < t - \varepsilon$. Thus

$$tq + A \subset (t - \varepsilon)q + A + \varepsilon q \subset M + (\varepsilon q + D) \subset M + \operatorname{int} D$$

and statement (5) is proved.

(d) Theorem 3.5(a) (resp. Theorem 3.6(a)) has been stated in [34, Proposition 3.3] for any nonempty set $F(x) \subset Y$, i.e., without assuming that $F(x)$ is D-proper (resp. $-D$-proper). However, Proposition 3.2 (resp. Proposition 3.3) shows that this assumption cannot be removed.

Analogously, Theorem 3.5(g) (resp. Theorem 3.6(g)) has been derived in [34, Proposition 3.6] by assuming that Q is D-closed (resp. $-D$-closed) instead of D-compact (resp. $-D$-compact). The following data show that the D-closedness is not sufficient to satisfy Theorem 3.5(g) (a similar example can be proposed in order to show that the $-D$-closedness is not sufficient to satisfy Theorem 3.6(g)). Consider $Y = \mathbb{R}^2$, $D = M = \mathbb{R}^2_+$, $q = (1, 1)$ and $F(x) = Q = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_2 = 1/y_1\}$. It is easy to check that Q is D-closed, $M \precsim^l_{\text{int}\, D} Q$ and $\Phi^l_{q,x}(M) = \Phi^l_{q,x}(Q) = 0$.

Next we show an equivalent representation of the mapping $\Phi^l_{q,x}$.

Theorem 3.8 *Consider $x \in S$. We have that*

$$\Phi^l_{q,x}(M) = \sup_{y \in F(x)} \Phi^l_{q,\{y\}}(M), \quad \forall\, M \in \mathcal{Y}. \tag{6}$$

Proof Let us define the mapping $H_q : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$,

$$H_q(M_1, M_2) := \sup_{y \in M_2} \Phi^l_{q,\{y\}}(M_1), \quad \forall\, M_1, M_2 \in \mathcal{Y},$$

and consider $r \in \mathbb{R}$. Let us prove

$$H_q(M, F(x)) \le r \iff \Phi^l_{q,x}(M) \le r. \tag{7}$$

Indeed, by (4) we have that

$$\begin{aligned} H_q(M, F(x)) \le r &\iff \Phi^l_{q,\{y\}}(M) \le r, \quad \forall\, y \in F(x), &(8)\\ &\iff y \in -rq + \text{cl}\,(M + D), \quad \forall\, y \in F(x),\\ &\iff F(x) + rq \subset \text{cl}\,(M + D). &(9) \end{aligned}$$

Suppose that $H_q(M, F(x)) \le r$ and consider an arbitrary $\varepsilon > 0$. Then by (9) and Proposition 2.1(b) we see that

$$F(x) + (r + \varepsilon)q \subset \text{cl}\,(M + D) + \text{int}\, D \subset M + D, \quad \forall\, \varepsilon > 0,$$

and so $r + \varepsilon \in \Lambda^l_{q,F(x)}(M)$, for all $\varepsilon > 0$. Thus $\Phi^l_{q,x}(M) \le r$.

Reciprocally, if $\Phi^l_{q,x}(M) \le r$ then $r + \varepsilon \in \Lambda^l_{q,F(x)}(M)$, for all $\varepsilon > 0$ and we have $F(x) + (r + \varepsilon)q \subset M + D$, for all $\varepsilon > 0$. By (8) and (9) we deduce that $H_q(M, F(x)) \le r + \varepsilon$, for all $\varepsilon > 0$ and it follows that $H_q(M, F(x)) \le r$.

By (7) we have that

$$\Phi^l_{q,x}(M) = -\infty \iff H_q(M, F(x)) = -\infty,$$
$$\Phi^l_{q,x}(M) = +\infty \iff H_q(M, F(x)) = +\infty,$$

and also $\Phi^l_{q,x}(M) = H_q(M, F(x))$ whenever $\Phi^l_{q,x}(M) \in \mathbb{R}$ and $H_q(M, F(x)) \in \mathbb{R}$, which finishes the proof. □

Remark 3.9 The scalarization mapping H_q was introduced as G_{-q} in [19] by assuming that the ordering cone D is pointed. Theorem 3.8 shows that it coincides with the scalarization mapping $\Phi^l_{q,A}$ due to Hamel and Löhne (see [17, 18]).

Let us observe that some properties stated in Theorem 3.5 were obtained implicitly in different results of [19] via the formulation H_q and by assuming additional hypotheses. To be precise, part (*a*) of Theorem 3.5 reduces to [19, Theorem 3.10(i)] by assuming that $F(x)$ is D-closed; Part (*d*) extends [19, Theorem 3.8(v)] to nonempty sets which are not D-proper; Parts (*f*) and (*g*) extend [19, Corollary 3.11(i)] to sets $A, B \in \mathcal{Y}$, where A could not be D-compact.

The following scalarization process $\{\Psi^l_{q,x}\}_{x\in S}$, $\Psi^l_{q,x} : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$ for all $x \in S$, was introduced in [13] to study approximate versions of the Ekeland variational principle in set-valued optimization problems. Consider $q \in \operatorname{int} D$ and the mapping $\xi_q : \mathcal{Y} \times Y \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\xi_q(M, y) := \inf_{z\in M}\{\max\{\langle\lambda, z - y\rangle : \lambda \in D^+(q)\}\}, \quad \forall\, M \in \mathcal{Y}, y \in Y.$$

Then, for each $x \in S$,

$$\Psi^l_{q,x}(M) := \sup\{\xi_q(M, y) : y \in F(x)\}, \quad \forall\, M \in \mathcal{Y}.$$

If $F = f$, where $f : X \to Y$ (i.e., F is single-valued), then $\Psi^l_{q,x} : Y \to \mathbb{R}$ reduces to the Gerstewitz scalarization mapping. Indeed, it is clear that

$$\xi_q(\{z\}, y) := \max\{\langle\lambda, z - y\rangle : \lambda \in D^+(q)\}, \quad \forall\, z, y \in Y,$$

and by (1) we see that

$$\begin{aligned}\Psi^l_{q,x}(\{z\}) = \xi_q(\{z\}, f(x)) &= \max\{\langle\lambda, z - f(x)\rangle : \lambda \in D^+(q)\}\\ &= s_q(z - f(x)), \quad \forall\, z \in Y.\end{aligned}$$

Moreover, in view of the definition it is clear that

$$\Psi^l_{q,x}(M) = \sup_{y\in F(x)} \inf_{z\in M} s_q(z - y), \quad \forall\, M \in \mathcal{Y}. \tag{10}$$

The following theorem shows that $\Psi^l_{q,x}$ is a reformulation of $\Phi^l_{q,x}$.

Theorem 3.10 *Consider* $q \in \operatorname{int} D$ *and* $x \in S$. *Then the mappings* $\Psi^l_{q,x}$ *and* $\Phi^l_{q,x}$ *coincide.*

Proof For each $y \in Y$ and $M \in \mathcal{Y}$ we have that:

$$\begin{aligned}\Phi^l_{q,\{y\}}(M) &= \inf\{t \in \mathbb{R} : M \precsim^l_D tq + y\} \\ &= \inf\{t \in \mathbb{R} : tq + y \in M + D\} \\ &= \inf_{z \in M} \inf\{t \in \mathbb{R} : z - y \in tq - D\} \\ &= \inf_{z \in M} s_q(z - y)\end{aligned}$$

and the proof follows by (6) and (10). □

By Remark 3.9 and Theorem 3.10 we see that the mapping G_{-q} by Hernández and Rodríguez-Marín, the mapping $\Phi^l_{q,x}$ due to Hamel and Löhne and the mapping $\Psi^l_{q,x}$ introduced by ourselves are the same function.

4 Minimality Conditions Through Scalarization

Next we obtain necessary and sufficient conditions for weak l-minimal and l-minimal solutions of problem (P) by scalarization processes $\{\varphi_x\}_{x\in S}$ such that for each $x \in S$, the mapping $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ satisfies certain order representing and monotonicity properties at $F(x)$. For each $x_0 \in S$ we denote

$$\begin{aligned}E(x_0) &= \{x \in S : F(x) \sim^l_D F(x_0)\}, \\ S(x_0) &= (S \backslash E(x_0)) \cup \{x_0\}.\end{aligned}$$

First we derive necessary l-minimality conditions by using order representing mappings.

Theorem 4.1 *Let $\{\varphi_x\}_{x\in S}$ be a scalarization process, $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$, for all $x \in S$.*

(a) Let $x_0 \in S$ and suppose that φ_{x_0} is strictly order $\precsim^l_{\operatorname{int} D}$-representing at $F(x_0)$. If $x_0 \in WM^l(F, S)$, then $x_0 \in S(\varphi_{x_0} \circ F, S)$.

(b) Let $x_0 \in S$ and suppose that φ_{x_0} is order $\precsim^l_D$-representing at $F(x_0)$. If $x_0 \in M^l(F, S)$, then $S(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$. If additionally φ_{x_0} is $\precsim^l_D$-monotone on $\mathcal{F}$, then $S(\varphi_{x_0} \circ F, S) = E(x_0)$.

Proof (*a*) Suppose that $x_0 \notin \mathrm{S}(\varphi_{x_0} \circ F, S)$. Then there exists $x \in S$ such that $\varphi_{x_0}(F(x)) < \varphi_{x_0}(F(x_0))$. As φ_{x_0} is strictly order $\precsim^l_{\operatorname{int} D}$-representing at $F(x_0)$, it follows that $F(x) \prec^l_{\operatorname{int} D} F(x_0)$, which is a contradiction since $x_0 \in \mathrm{WM}^l(F, S)$.

(*b*) Consider $x \in S(x_0)$, $x \neq x_0$. It follows that $x \in S$, $F(x) \nsim^l_D F(x_0)$ and since $x_0 \in \mathrm{M}^l(F, S)$ we have $F(x) \not\precsim^l_D F(x_0)$. Thus, as φ_{x_0} is order $\precsim^l_D$-representing at $F(x_0)$ we deduce that $\varphi_{x_0}(F(x)) > \varphi_{x_0}(F(x_0))$ and so $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$.

If additionally φ_{x_0} is $\precsim^l_D$-monotone, then

$$\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\} \iff \mathrm{S}(\varphi_{x_0} \circ F, S) = E(x_0) \tag{11}$$

and so the last statement of part (b) is a direct consequence of the first one. Let us prove equivalence (11). The sufficient condition is trivial. On the other hand, suppose that $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ and consider $x \in E(x_0)$. Since $F(x) \precsim^l_D F(x_0)$, $F(x_0) \precsim^l_D F(x)$ and φ_{x_0} is $\precsim^l_D$-monotone on $\mathcal{F}$, we have that $\varphi_{x_0}(F(x)) = \varphi_{x_0}(F(x_0))$ and so $\mathrm{S}(\varphi_{x_0} \circ F, S) = E(x_0)$, which finishes the proof. $\square$

In the following theorem we obtain sufficient conditions for weak l-minimal and l-minimal solutions of problem (P) by solutions and strict solutions of scalarization processes $\{\varphi_x\}_{x \in S}$ such that for each $x \in S$, the mapping $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone and $\precsim^l_D$-monotone at $F(x)$, respectively.

Theorem 4.2 *Let $\{\varphi_x\}_{x \in S}$ be a scalarization process, where $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$, for all $x \in S$.*

(a) Let $x_0 \in S$ be such that φ_{x_0} is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone at $F(x_0)$. If $x_0 \in S(\varphi_{x_0} \circ F, S)$ then $x_0 \in WM^l(F, S)$.

(b) Suppose that φ_x is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone on $\mathcal{F}$, for all $x \in S$. Then

$$\bigcup_{x \in S} S(\varphi_x \circ F, S) \subset WM^l(F, S).$$

(c) Let $x_0 \in S$ be such that φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$. If $S(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ then $x_0 \in M^l(F, S)$.

Proof Let us prove parts (a) and (c), since the proof of part (b) is similar to the proof of part (a).

(a) Suppose that $x_0 \notin \mathrm{WM}^l(F, S)$. Then there exists $x \in S$ such that $F(x) \prec^l_{\mathrm{int}\, D} F(x_0)$. As φ_{x_0} is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone at $F(x_0)$ we deduce that $\varphi_{x_0}(F(x)) < \varphi_{x_0}(F(x_0))$, which is a contradiction.

(c) Suppose that $x_0 \notin \mathrm{M}^l(F, S)$. Then there exists $x \in S$ such that $F(x) \prec^l_D F(x_0)$, i.e., $F(x) \precsim^l_D F(x_0)$ and $F(x) \not\sim^l_D F(x_0)$, and so $x \in S(x_0)\backslash\{x_0\}$. We have that $\varphi_{x_0}(F(x)) \le \varphi_{x_0}(F(x_0))$, since φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$. As $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$, it follows that $x = x_0$, that is a contradiction and the proof is completed. $\square$

Remark 4.3 Let us observe that if φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$ and $\mathrm{S}(\varphi_{x_0} \circ F, S) = \{x_0\}$ (i.e., x_0 is a strict solution of problem (P_{φ_p}) with $\varphi_p = \varphi_{x_0}$), then $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ and by Theorem 4.2 it follows that $x_0 \in \mathrm{M}^l(F, S)$.

By applying Theorems 4.1 and 4.2 to the mappings $\{\Phi^l_{q,x}\}_{x \in S}$ one obtains the following characterizations for weak l-minimal and l-minimal solutions of problem (P). Moreover, the same approach can be done to characterize weak u-minimal and u-minimal solutions of problem (P) by considering the scalarization mapping $\Phi^u_{q,x}$ and the set orderings $\precsim^u_D$ and $\precsim^u_{\mathrm{int}\, D}$ instead of $\precsim^l_D$ and $\precsim^l_{\mathrm{int}\, D}$, respectively. In fact, this approach can be extended to other set orderings (see [11]).

Corollary 4.4 *Let $q \in \mathrm{int}\, D$. The following statements hold:*

(a) *Assume that F is D-compact valued in S. Then, for each $x_0 \in X$ it follows that*

$$x_0 \in WM^l(F, S) \iff x_0 \in S(\Phi^l_{q,x_0} \circ F, S).$$

Moreover,

$$WM^l(F, S) = \bigcup_{x \in S} S(\Phi^l_{q,x} \circ F, S). \quad (12)$$

(b) *Let $x_0 \in S$ such that $F(x_0)$ is D-proper, and assume that F is D-closed valued in S. Then*

$$x_0 \in M^l(F, S) \iff S(\Phi^l_{q,x_0} \circ F, S) = E(x_0).$$

Proof (a) By Theorem 3.5 it follows that $\Phi^l_{q,x}$ is strictly order $\precsim^l_{\text{int } D}$-representing at $F(x)$, and strictly $\precsim^l_{\text{int } D}$-monotone on $\mathcal{F}$, for all $x \in S$. Then part (a) is a consequence of part (a) of Theorem 4.1 and parts (a) and (b) of Theorem 4.2.

(b) As $F(x_0)$ is D-proper, by Theorem 3.5(d) we have that $\Phi^l_{q,x_0} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is $\precsim^l_D$-monotone on $\mathcal{F}$, and using Theorem 3.5(a), (e) it is easy to check that Φ^l_{q,x_0} is order $\precsim^l_D$-representing at $F(x_0)$, since F is D-closed valued in S. Then the result follows by Theorem 4.1(b), statement (11) and Theorem 4.2(c). □

Remark 4.5 (a) Corollary 4.4(a) improves [19, Corollary 4.11], since it generalizes the sufficient condition of [19, Corollary 4.11] to the whole solution set of the scalarized problem and so the scalar representation (12) holds. Analogously, Corollary 4.4(b) reduces to [19, Corollary 4.8] by assuming that $F(x_0)$ is D-bounded instead of D-proper.

Let us observe that Corollaries 4.8 and 4.11 of [19] characterize also weak l-maximal and l-maximal solutions of problem (P).

On the other hand, the sufficient condition of Corollary 4.4(b) reduces to [34, Theorem 4.3] by assuming that $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = \{x_0\}$ (i.e., x_0 is a strict solution), since in this case it follows that $E(x_0) = \{x_0\}$ and so $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = E(x_0)$. Indeed, suppose that x_0 is a strict solution and consider $x \in E(x_0)$. As Φ^l_{q,x_0} is $\precsim^l_D$-monotone on $\mathcal{Y}$ we deduce that $\Phi^l_{q,x_0}(F(x)) = \Phi^l_{q,x_0}(F(x_0))$. Then $x = x_0$, since $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = \{x_0\}$ and we have that $E(x_0) = \{x_0\}$.

(b) Corollary 4.4(a) has been stated in [1, Theorem 5.2] and [34, Theorem 4.2] by assuming that F is D-bounded and D-closed in S. The following data show that these assumptions do not guarantee both results. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 1)$, $F(x) = \mathbb{R}^2_+$, for all $x \in \mathbb{R}$, $x \neq 0$, and $F(0) = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_2 = 1/y_1\}$. It is easy to check that F is D-closed and D-bounded in X and $\Phi^l_{q,0}(F(x)) = 0$ for all $x \in \mathbb{R}$, but $0 \notin \mathrm{WM}^l(F, X)$, since $F(x) \prec^l_{\text{int } D} F(0)$, for all $x \in X \backslash \{0\}$.

Analogously, Corollary 4.4(b) clarifies Theorem 5.2 of [1], which could not be true. For example, if F is constant in the feasible set S then $\mathrm{M}^l(F, S) = S$ and for each $x_0 \in S$ it is clear that $h^l_{\inf}(F(x); F(x_0)) = 0$, for all $x \in S$.

5 Conclusions

In this paper, a detailed study on the scalarization of set optimization problems has been carried out. Several concepts of solution based on set relations have been characterized via generic scalarization mappings that satisfy suitable properties. Then these characterizations have been specified by using well-known scalarization mappings, whose properties have previously been studied. An useful research direction motivated by the results of this paper is to derive from them numerical procedures to solve this kind of optimization problems.

Recently, another approach called vectorization has been proposed to characterize solutions of set optimization problems based on set relations (see [22]). In this approach, a suitable vector optimization problem is defined whose solutions are related with the solutions of the set optimization problem. It would be interesting to relate this approach with the results of this paper. In particular, one can analyze if some of these results can be derived by combining the vectorization approach with some of the well-known scalarization processes used in vector optimization.

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References

1. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**(9), 3821–3835 (2012)
2. Bao, T.Q., Mordukhovich, B.S.: Set-valued optimization in welfare economics. Adv. Math. Econ. **13**, 113–153 (2010)
3. Bao, T.Q., Mordukhovich, B.S., Soubeyran, A.: Variational analysis in psychological modeling. J. Optim. Theory Appl. **164**(1), 290–315 (2015). doi:10.1007/s10957-014-0569-8
4. Breckner, W.W., Kassay, G.: A systematization of convexity concepts for sets and functions. J. Convex Anal. **4**(1), 109–127 (1997)
5. Chen, G.-Y., Huang, X., Yang, X.: Vector Optimization. Set-valued and variational analysis. Lecture Notes in Economics and Mathematical Systems, vol. 541. Springer, Berlin (2005)
6. Eichfelder, G., Jahn, J.: Vector optimization problems and their solution concepts. In: Ansari, Q.H., Yao, J.C. (eds.) Recent Developments in Vector Optimization, pp. 1–27. Springer, Berlin (2012). Vector Optim
7. Flores-Bazán, F., Gutiérrez, C., Novo, V.: A Brézis-Browder principle on partially ordered spaces and related ordering theorems. J. Math. Anal. Appl. **375**(1), 245–260 (2011)
8. Gerstewitz (Tammer), C.: Nichtkonvexe dualität in der vektoroptimierung. Wiss. Z. Tech. Hochsch. Leuna-Mersebg. **25**(3), 357–364 (1983)
9. Gerth (Tammer), C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**(2), 297–320 (1990)
10. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
11. Gutiérrez, C., Jiménez, B., Miglierina, E., Molho, E.: Scalarization in set optimization with solid and nonsolid ordering cones. J. Global Optim. **61**(3), 525–552 (2015). doi:10.1007/s10898-014-0179-x

12. Gutiérrez, C., Jiménez, B., Novo, V.: Optimality conditions via scalarization for a new ε-efficiency concept in vector optimization problems. European J. Oper. Res. **201**(1), 11–22 (2010)
13. Gutiérrez, C., Jiménez, B., Novo, V., Thibault, L.: Strict approximate solutions in set-valued optimization with applications to the approximate Ekeland variational principle. Nonlinear Anal. **73**(12), 3842–3855 (2010)
14. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**(4), 1822–1833 (2012)
15. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**(1), 187–206 (2005)
16. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**(1), 66–95 (2010)
17. Hamel, A., Löhne, A.: Minimal set theorems. Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik. Report, 11 (2002)
18. Hamel, A., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**(1), 19–37 (2006)
19. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**(1), 1–18 (2007)
20. Heyde, F., Löhne, A., Tammer, C.: Set-valued duality theory for multiple objective linear programs and application to mathematical finance. Math. Methods Oper. Res. **69**(1), 159–179 (2009)
21. Jahn, J.: Vector Optimization. Theory, Applications, and Extensions. Springer, Berlin (2011)
22. Jahn, J.: Vectorization in set optimization. J. Optim. Theory Appl. (2013). doi:10.1007/s10957-013-0363-z
23. Jahn, J., Ha, T.H.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**(2), 209–236 (2011)
24. Khan, A.A., Tammer, C., Zălinescu, C.: Set-valued Optimization. An Introduction with Applications. Springer, Berlin (2014)
25. Kuroiwa, D.: Some criteria in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **985**, 171–176 (1997)
26. Kuroiwa, D.: The natural criteria in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **85–90**, 1998 (1031)
27. Kuroiwa, D.: On natural criteria in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **86–92**, 1998 (1048)
28. Kuroiwa, D.: Some duality theorems of set-valued optimization with natural criteria. In: Nonlinear Analysis and Convex Analysis (Niigata, 1998), pp. 221–228. World Scientific Publishing, River Edge (1999)
29. Kuroiwa, D.: On set-valued optimization. In: Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000), vol. 47, pp. 1395–1400 (2001)
30. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 3 (Athens, 1996), vol. 30, pp. 1487–1496 (1997)
31. Kuwano, I., Tanaka, T., Yamada, S.: Characterization of nonlinear scalarizing functions for set-valued maps. In: Nonlinear Analysis and Optimization, pp. 193–204. Yokohama Publisher, Yokohama (2009)
32. Kuwano, I., Tanaka, T., Yamada, S.: Inherited properties of nonlinear scalarizing functions for set-valued maps. In: Nonlinear Analysis and Convex Analysis, pp. 161–177. Yokohama Publisher, Yokohama (2010)
33. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets and set-valued Ky Fan minimax inequality. J. Nonlinear Convex Anal. **11**(3), 513–525 (2010)
34. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **1685**, 270–280 (2010)
35. Maeda, T.: On characterization of fuzzy vectors and its applications to fuzzy mathematical programming problems. Fuzzy Sets Syst. **159**(24), 3333–3346 (2008)

36. Maeda, T.: On optimization problems with set-valued objective maps: existence and optimality. J. Optim. Theory Appl. **153**(2), 263–279 (2012)
37. Miglierina, E., Molho, E.: Scalarization and stability in vector optimization. J. Optim. Theory Appl. **114**(3), 657–670 (2002)
38. Nishizawa, S., Onodsuka, M., Tanaka, T.: Alternative theorems for set-valued maps based on a nonlinear scalarization. Pac. J. Optim. **1**(1), 147–159 (2005)
39. Nishizawa, S., Tanaka, T., Georgiev, P.Gr.: On inherited properties of set-valued maps. In: Takahashi, W., Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis, pp. 341–350. Yokohama Publishers, Yokohama (2003)
40. Shimizu, A., Nishizawa, S., Tanaka, T.: Optimality conditions in set-valued optimization using nonlinear scalarization methods. In: Takahashi, W.,Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis, pp. 565–574. Yokohama Publisher, Yokohama (2007)
41. Shimizu, A., Tanaka, T.: Minimal element theorem with set-relations. J. Nonlinear Convex Anal. **9**(2), 249–253 (2008)
42. Wierzbicki, A.P.: On the completeness and constructiveness of parametric characterizations to vector optimization problems. OR Spektrum **8**(2), 73–87 (1986)
43. Zhang, W.Y., Li, S.J., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**(9), 3769–3778 (2009)

Set Optimization—A Rather Short Introduction

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Abstract Recent developments in set optimization are surveyed and extended including various set relations as well as fundamental constructions of a convex analysis for set- and vector-valued functions, and duality for set optimization problems. Extensive sections with bibliographical comments summarize the state of the art. Applications to vector optimization and financial risk measures are discussed along with algorithmic approaches to set optimization problems.

Keywords Set relation · Conlinear space · Infimizer · Scalarization · Set-valued function · Duality · Subdifferential · Vector optimization · Risk measure · Benson's algorithm

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1 Introduction

In his book [114, p. 378], J. Jahn states that the set relation approach 'opens a new and wide field of research' and the so-called set relations 'turn out to be very promising in set optimization.' We share this opinion, and this note aims at a (partial) fulfillment of this promise.

What is "set optimization?" The answer given in this note concerns minimization problems with set-valued objective functions and is based on a twofold solution concept: Look for a set of arguments each of which has a function value which is minimal in some sense, and all those values generate the infimum of the function. Thus, infimum attainment and minimality are the two, no longer equivalent requirements for a solution of a set optimization problem. It turns out that the set relation infimum is a useful concept in contrast to the vector order infimum which may not exist, and even if it does, it is of no practical use.

What is a motivation to consider set optimization problems? The heart of the problem is the question of how to deal with a non-total order relation, i.e. when there are non-comparable alternatives. The "complete lattice approach" based on set relations re-gains meaningful and applicable notions of infimum and supremum even if the departing pre-ordered vector space does not have the least upper bound property, is not even a lattice, its positivity cone is not pointed, not normal or has an empty interior. The theory presented in this survey suggests that even vector-valued optimization problems should be treated as set-valued ones. This point of view has already been emphasized in [151] for problems with a solid ordering cone.

According to an old theorem by Szpilrajn [215], which is well-known in mathematical economics, but less so in vector and set optimization, a partial order (preorder) is the intersection of all linear orders (total preorders) including it. In the same spirit, dual descriptions of objects related to a preorder such as convex functions can be given in terms of half spaces generating total orders, hence dual objects are naturally halfspace- or hyperplane-valued.[1] Since the simplest dual object is a linear functional, set-valued replacements for them should be halfspace- or hyperplane-valued as well and "as linear as possible." This basic idea leads to a new type of duality which is not only strong enough to provide set-valued analogs of the Fenchel-Moreau and the Lagrange duality theorem, but also implies known duality results in vector optimization which are usually stated under much stronger assumptions.

It turns out that convex analysis, in particular duality, does not rely on linearity of functionals or image spaces, but rather on "conlinearity." The structure of a conlinear space as introduced in [77] is precisely the part of the structure of a linear space which remains invariant under passing to the power set (with Minkowski addition) or order completion (add a least and greatest element to an ordered vector space). Thus, $\mathbb{R} \cup \{-\infty, +\infty\}$ is the prototype of a conlinear space. A particular feature is the resulting bifurcation: The extended reals can be provided with inf-addition or sup-

[1] In contrast to many duality results in vector optimization, this can bee seen as a realization of one of the many 'duality principles in optimization theory that relate a problem expressed in terms of vectors in a space to a problem expressed in terms of hyperplanes in the space,' see [160, p. 8].

addition (see [197, p. 15], but already introduced by Moreau [175]), which produces two different conlinear spaces. A preoder on a linear space can be extended in two different ways to the power set of this space, thus producing two different ordered conlinear spaces. Below, it should become clear why this happens and how to deal with this ambiguity.

Finally, "set optimization" is motivated by problems in Economics and Mathematical Finance. The classic books [209] (first edition from 1953) and [125] contain many examples of set-valued functions which naturally occur in economic models, among them 'production technologies' [209, p. 13] which are basically monotone lattice-valued functions in the sense of this survey. In finance, market models with transaction costs provide plenty of examples for the theory discussed in this survey; for example the superhedging theorems in [121, 205] can be identified as particular cases of the set-valued Fenchel-Moreau theorem stated below, and the theory of set-valued risk measures, initiated in [120], was particularly motivating for the development for the set-valued duality in Sect. 5 below.

This survey aims at developing ideas and structures and providing a framework for principal results. Full proofs are only given for new or unpublished results, or if they illustrate an important idea particularly nicely. Sections with bibliographical remarks conclude each part with the goal to put the presented material into perspective with variational analysis and vector optimization theory in view.

Several results are new, mostly complementing those obtained by the authors in several recent publications. For example, Proposition 2.17 discusses the totalness of set relations, Sect. 4.2 relies on an improved general scheme for scalarizations and includes several new observations such as the supermodularity of the scalarizations given in Corollary 4.15, Theorem 5.2 characterizes set-valued dual variables parallel to results for continuous linear functions and convex processes, Sect. 5.5 contains a new general framework for directionally translative functions and Proposition 6.7 provides a new sufficient optimality condition including a complementary slackness condition for set optimization.

2 Set Relations and Lattices of Sets

2.1 *Extending Preorders from Linear Spaces to their Power Sets*

Let Z be a non-trivial real linear space and $C \subseteq Z$ a convex cone with $0 \in C \neq Z$. In particular, $C = \{0\}$ is allowed. Here, C is said to be a cone if $z \in C$, $t > 0$ imply $tz \in C$. By

$$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 - z_1 \in C$$

a reflexive and transitive relation $\leq_C$ is defined on Z; such a relation is usually called a preorder. It is compatible with the linear structure of Z in the usual sense, i.e. $z_1, z_2, z \in Z$, $t \geq 0$ and $z_1 \leq_C z_2$ imply $z_1 + z \leq_C z_2 + z$ as well as $tz_1 \leq tz_2$. Obviously,

$$z_1 \leq_C z_2 \;\Leftrightarrow\; z_2 - z_1 \in C \;\Leftrightarrow\; z_2 \in z_1 + C \;\Leftrightarrow\; z_1 \in z_2 - C.$$

The last two relationships can be used to extend $\leq_C$ from Z to $\mathcal{P}(Z)$, the set of all subsets of Z including the empty set $\emptyset$. Take $A, B \in \mathcal{P}(Z)$ and define

$$\begin{aligned} A \preccurlyeq_C B \quad &\Leftrightarrow \quad B \subseteq A + C, \\ A \curlyeqprec_C B \quad &\Leftrightarrow \quad A \subseteq B - C. \end{aligned}$$

Here and in the following, we use $+$ to denote the usual Minkowski (element-wise) addition for sets with the conventions $A + \emptyset = \emptyset + A = \emptyset$ for all $A \in \mathcal{P}(Z)$ and $A - C = A + (-C)$, $-C = \{-c \mid c \in C\}$. The following facts are immediate.

Proposition 2.1 *(a) Both $\preccurlyeq_C$ and $\curlyeqprec_C$ are reflexive and transitive relations on $\mathcal{P}(Z)$. Moreover, they are not antisymmetric in general, and they do not coincide.*
(b) $A \preccurlyeq_C B \Leftrightarrow -B \curlyeqprec_C -A \Leftrightarrow B \preccurlyeq_{-C} A$.
(c) $A \preccurlyeq_C B \Leftrightarrow A + C \supseteq B + C$; $A \curlyeqprec_C B \Leftrightarrow A - C \subseteq B - C$.

Proof Left as exercise. □

The property (c) above gives rise to define the set

$$\mathcal{P}(Z, C) = \{A \in \mathcal{P}(Z) \mid A = A + C\}$$

and to observe that it can be identified with the set of equivalence classes with respect to the equivalence relation on $\mathcal{P}(Z)$ defined by

$$A \sim_C B \quad \Leftrightarrow \quad (A \preccurlyeq_C B \;\wedge\; B \preccurlyeq_C A) \quad \Leftrightarrow \quad A + C = B + C, \tag{2.1}$$

i.e. $\sim_C$ is the symmetric part of $\preccurlyeq_C$. Likewise,

$$\mathcal{P}(Z, -C) = \{A \in \mathcal{P}(Z) \mid A = A - C\}$$

can be identified with the set of equivalence classes with respect to the equivalence relation

$$A \sim_{(-C)} B \quad \Leftrightarrow \quad (A \curlyeqprec_C B \;\wedge\; B \curlyeqprec_C A) \quad \Leftrightarrow \quad A - C = B - C.$$

Below, we will mainly discuss the relation $\preccurlyeq_C$ which is the appropriate one when it comes to minimization; however, the theory becomes completely symmetric since every statement for the $\preccurlyeq_C$ relation (and minimization) has a counterpart for $\curlyeqprec_C$ (and maximization).

The following proposition relies on (c) of Proposition 2.1. We recall that the infimum of a subset $V \subseteq W$ of a partially ordered set $(W, \preceq)$ is an element $\bar{w} \in W$ (unique if it exists) satisfying $\bar{w} \preceq v$ for all $v \in V$ and $w \preceq \bar{w}$ whenever $w \preceq v$ for all $v \in V$. This means that the infimum is the greatest lower bound of V in W. The infimum of V is denoted by $\inf V$. Likewise, the supremum $\sup V$ is defined as the least upper bound of V. A partially ordered set $(W, \preceq)$ is called a lattice if $\inf \{w_1, w_2\}$ and $\sup \{w_1, w_2\}$ exist in W for any two elements $w_1, w_2 \in W$. A lattice $(W, \preceq)$ is called (order) complete if each subset has an infimum and a supremum in W.

Proposition 2.2 *The pair* $(\mathcal{P}(Z, C), \supseteq)$ *is a complete lattice. Moreover, for a subset* $\mathcal{A} \subseteq \mathcal{P}(Z, C)$*, the infimum and the supremum of* $\mathcal{A}$ *are given by*

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A \tag{2.2}$$

where it is understood that $\inf \mathcal{A} = \emptyset$ *and* $\sup \mathcal{A} = Z$ *whenever* $\mathcal{A} = \emptyset$*. The greatest (top) element of* $\mathcal{P}(Z, C)$ *with respect to* $\supseteq$ *is* $\emptyset$*, the least (bottom) element is* Z*.*

In particular, $\supseteq$ is a partial order on $\mathcal{P}(Z, C)$. This property does not depend on the cone C: It can be a trivial cone, i.e. $C = \{0\}$, or a half space, i.e. $C = \{z \in Z \mid \xi(z) \geq 0\}$ where ξ is a (non-zero) linear function on Z (an element of the algebraic dual of Z), i.e. $\leq_C$ is not antisymmetric in the latter case in general. Of course, a parallel result holds for $(\mathcal{P}(Z, -C), \subseteq)$.

Note that the convention $\inf \emptyset = \emptyset$ and $\sup \emptyset = Z$ is in accordance with the following monotonicity property: If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then $\inf \mathcal{A}_1 \subseteq \inf \mathcal{A}_2$ and $\sup \mathcal{A}_1 \supseteq \sup \mathcal{A}_2$ in $\mathcal{P}(Z, C)$.

Proof To show the first formula in (2.2) one has to prove two facts: First,

$$\forall A' \in \mathcal{A}\colon \bigcup_{A \in \mathcal{A}} A \supseteq A',$$

and secondly, for $B \in \mathcal{P}(Z, C)$

$$(\forall A \in \mathcal{A}\colon B \supseteq A) \;\Rightarrow\; B \supseteq \bigcup_{A \in \mathcal{A}} A.$$

Both claims are obvious. The second formula of (2.2) also follows from the definition of the supremum with respect to $\supseteq$. The lattice property is a consequence of (2.2). □

Remark 2.3 One could also use other representatives of the equivalence classes defined by (2.1)

$$\left\{A' \in \mathcal{P}(Z) \mid A \preccurlyeq_C A' \;\wedge\; A' \preccurlyeq_C A\right\}.$$

As as rule, one has to impose additional assumptions, for example a non-empty interior of the cone C. An example is the infimal set approach of Nieuwenhuis [179] which has been extended in [149, 155] (compare also [151, 216]). This approach is summarized in Example 2.12 below.

2.2 Comments on Set Relations

In the (vector and set) optimization community, D. Kuroiwa is credited for the introduction of the two "set relations" $\preccurlyeq_C$ and $\curlyeqprec_C$ above and, indeed, he was the first who used them for defining optimality notions for optimization problems with a set-valued objective function, compare [128, 138] and several reports [129–134] published by RIMS Kokyuroku 1996–1999. However, it should be noted that these "set relations" were in use much earlier in different contexts.

In the 1993 paper [22], Brink describes an algebraically motivated approach to power structures where the two relations $\curlyeqprec$ and $\preccurlyeq$ (analog extensions of a preorder on a general set, not necessarily a vector space) are denoted by R_0^+ and R_1^+, respectively. These and similar structures mostly defined on finite or countable sets are widely used in theoretical computer science as becomes clear from the reference list of [22]. For example, in [188, Definition 1] the following definition is used: A set A 'can be reduced to' another set B if for all $a \in A$ there is $b \in B$ such that $a \leq b$ for some partial order $\leq$, thus $A \curlyeqprec B$, which is parallel to the definition of $\curlyeqprec_C$ above.

Nishianidze [180] also used the relations $\preccurlyeq$ and $\curlyeqprec$ in the context of fixed point theory. This reference was brought to our attention by J. Jahn. Constructions mainly motivated by applications in economics and social choice theory can be found e.g. in [18, 177]. Compare also the references therein, especially [122]. In [56], set relations (on finite sets) and corresponding best choice problems are motivated by committee selection, governing coalition formation, product line formation and similar problems.

As pointed out in [77, 83], the earliest reference known to us is the paper [228] by Young. It already contains the definitions of $\preccurlyeq_C$ and $\curlyeqprec_C$ implicitly and presents applications to the analysis of upper and lower limits of sequences of real numbers.

Another field of application for set relations is interval mathematics. In the survey [178, Sect. 2.2] from 1975, an order relation is investigated which is defined on the set of order intervals of a partially ordered set M. This relation coincides with $\preccurlyeq_C \cap \curlyeqprec_C$ if $M = Z$ and $\leq_C$ is a partial order on Z. It has also been discussed, for example, in [115, 117]. Jahn [118] applies it in fixed point theory for interval functions, and Schmidt [206] relates it to general ordered convex cones. Later, the "set-less-or-equal relation" became a standard part of the FORTRAN 95 specification for interval arithmetic, see [30]. We point out that the "set less" relation in [115] actually is the "set-less-or-equal" relation in [30, Sect. 12.8] and also coincides with $\preccurlyeq_C \cap \curlyeqprec_C$.

In [138], one can find a systematic investigation of six extensions of a preorder $\leq_C$ on a topological linear space generated by a convex ordering cone C with nonempty interior to its power set; the relations $\preccurlyeq_C$ and $\curlyeqprec_C$ are proven to be the only such relations which are reflexive and transitive; definitions for the convexity of set-valued functions with respect to several order relations are given. Subsequent papers of the three authors of [138] contain applications to optimization problems with a set-valued objective, see for example [73, 135, 136]. For this topic, compare also the book [114], especially Chap. 5. The recent paper [117] contains even more set relations.

After 2005, many authors adopted the concepts related to $\preccurlyeq_C$ and $\curlyeqprec_C$, see, among an increasing number of others, [1, 85, 94–96, 140, 163, 164]. Quite recently, robustness for vector optimization problems has been linked to the two (and other) set relations, see [107–109].

In [77, 149, 155], it has been realized that the set relations unfold their full potential in the framework of complete lattices; Propositions 2.2 above and infimal set versions of it such as [151, Proposition 1.52] may serve as a blueprint for this idea. Because of Proposition 2.2 (which can be found, even in a more general set-up, in [77, Theorem 6] and, for a different image space, [149, Proposition 1.2.3]) we call this approach the "complete lattice approach" to set optimization.[2]

2.3 Inf-Residuated Conlinear Spaces of Sets

We start with a definition which provides the algebraic framework for the image space analysis. It is taken from [77] where references and more material about structural properties of conlinear spaces can be found.

Definition 2.4 A nonempty set W together with two algebraic operations $+ \colon W \times W \to W$ and $\cdot \colon \mathbb{R}_+ \times W \to W$ is called a conlinear space provided that

(C1) $(W, +)$ is a commutative semigroup with neutral element θ,
(C2) (i) $\forall w_1, w_2 \in W,\ \forall r \in \mathbb{R}_+\colon\ r \cdot (w_1 + w_2) = r \cdot w_1 + r \cdot w_2$, (ii) $\forall w \in W$, $\forall r, s \in \mathbb{R}_+\colon s \cdot (r \cdot w) = (sr) \cdot w$, (iii) $\forall w \in W\colon 1 \cdot w = w$, (iv) $0 \cdot \theta = \theta$.

An element $w \in W$ is called a convex element of the conlinear space W if

$$\forall s, t \geq 0\colon\ (s + t) \cdot w = s \cdot w + t \cdot w.$$

A conlinear space $(W, +, \cdot)$ together with a partial order $\preceq$ on W (a reflexive, antisymmetric, transitive relation) is called ordered conlinear space provided that (v) $w, w_1, w_2 \in W$, $w_1 \preceq w_2$ imply $w_1 + w \preceq w_2 + w$, (vi) $w_1, w_2 \in W$, $w_1 \preceq w_2$, $r \in \mathbb{R}_+$ imply $r \cdot w_1 \preceq r \cdot w_2$.

A non-empty subset $V \subseteq W$ of the conlinear space $(W, +, \cdot)$ is called a conlinear subspace of W if (vii) $v_1, v_2 \in V$ implies $v_1 + v_2 \in V$ and (viii) $v \in V$ and $t \geq 0$ imply $t \cdot v \in V$.

[2] For apparent reasons, we would like to call this just "set optimization," but this term is currently used for just too many other purposes.

It can easily be checked that a conlinear subspace of a conlinear space again is a conlinear space. Note that an important feature of the above definition is the missing second distributivity law which is used to define convex elements.

Example 2.5 (a) The Minkowski addition $+$ has already been extended to $\mathcal{P}(Z, C)$ and $\mathcal{P}(Z, -C)$ (see the paragraph before Proposition 2.1). The multiplication with non-negative numbers is extended to $\mathcal{P}(Z, C)$ by defining $t \cdot A = \{ta \mid a \in A\}$ for $A \in \mathcal{P}(Z, C) \setminus \{\emptyset\}$, $t > 0$ and

$$0 \cdot A = C, \quad t \cdot \emptyset = \emptyset$$

for all $A \in \mathcal{P}(Z, C)$ and $t > 0$. In particular, $0 \cdot \emptyset = C$ by definition, and we will drop the $\cdot$ in most cases. Since the same can be done for $\mathcal{P}(Z, -C)$, the triples $(\mathcal{P}(Z, C), +, \cdot)$ and $(\mathcal{P}(Z, -C), +, \cdot)$ are conlinear spaces in the sense of Definition 2.4.

Note that it does not hold:

$$\forall s, t \geq 0, \ \forall A \in \mathcal{P}(Z, C): \ (s + t) \cdot A = s \cdot A + t \cdot A.$$

Counterexamples are provided by non-convex sets $A \subseteq Z$. Therefore, $(\mathcal{P}(Z, C), +, \cdot, \supseteq)$ is neither an ordered semilinear space [202] nor a semi-module over the semi-ring $\mathbb{R}_+$ [230].

(b) The extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ provide two more examples. Supplied with the inf-addition $\dot{+}$ and the sup-addition $\underset{\cdot}{+}$, respectively, one obtains two (different!) conlinear spaces. For terminology and references, see [89, 197].

The next result connects the conlinear structure on $(\mathcal{P}(Z, C), +, \cdot)$ with the order structure of $(\mathcal{P}(Z, C), \supseteq)$.

Proposition 2.6 *(a)* $A, B, D, E \in \mathcal{P}(Z, C)$*,* $A \supseteq B$*,* $D \supseteq E \Rightarrow A + D \supseteq B + E$*,*
(b) $A, B \in \mathcal{P}(Z, C)$*,* $A \supseteq B$*,* $s \geq 0 \Rightarrow sA \supseteq sB$*,*
(c) $\mathcal{A} \subseteq \mathcal{P}(Z, C)$*,* $B \in \mathcal{P}(Z, C) \Rightarrow$

$$\inf(\mathcal{A} + B) = (\inf \mathcal{A}) + B \tag{2.3}$$
$$\sup(\mathcal{A} + B) \supseteq (\sup \mathcal{A}) + B. \tag{2.4}$$

where $\mathcal{A} + B = \{A + B \mid A \in \mathcal{A}\}$*.*

Proof Exercise. □

The following example shows that (2.4) does not hold with equality in general.

Example 2.7 Let $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\mathcal{A} = \{[t, \infty) \mid t \geq 0\}$, $B = \mathbb{R}$. Then,

$$\forall t \geq 0\colon [t, \infty) + \mathbb{R} = \mathbb{R} \quad \text{and} \quad \sup \mathcal{A} = \bigcap_{t \geq 0} [t, \infty) = \emptyset,$$

so $\sup(\mathcal{A} + B) = \mathbb{R} \neq \emptyset = (\sup \mathcal{A}) + B$.

Items (a) and (b) of the previous proposition show that $(\mathcal{P}(Z, C), +, \cdot, \supseteq)$ (as well as $(\mathcal{P}(Z, -C), +, \cdot, \subseteq)$) carries the structure of an ordered conlinear space. Moreover, Proposition 2.2 shows that they are also complete lattices. The innocent looking Eq. (2.3) has far reaching consequences. In lattice theoretical terms, it means that $(\mathcal{P}(Z, C), +, \cdot, \supseteq)$ is *inf-residuated* (but not *sup-residuated* in general). The opposite is true for $(\mathcal{P}(Z, -C), +, \cdot, \subseteq)$: it is sup-, but not inf-residuated. The following proposition is an explanation for the word "inf-residuated."

Proposition 2.8 *The relationship* (2.3) *given in (c) of Proposition 2.6 is equivalent to: For each $A, B \in \mathcal{P}(Z, C)$, the set*

$$\{D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\}$$

has a least element (with respect to $\supseteq$).

Proof Assume (2.3) and fix $A, B \in \mathcal{P}(Z, C)$. Define

$$\hat{D} = \inf\{D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\}.$$

By (2.3) and (2.2),

$$\begin{aligned} B + \hat{D} &= B + \inf\{D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\} \\ &= \inf\{B + D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\} \\ &= \bigcup\{B + D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\} \subseteq A \end{aligned}$$

which means $\hat{D} \in \{D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\}$, so $\hat{D}$ is the desired least element. The converse direction is left as an exercise. □

The inf-residuation of $A, B \in \mathcal{P}(Z, C)$ is denoted

$$A \mathbin{-^{\!\!\cdot}} B = \inf\{D \in \mathcal{P}(Z, C) \mid A \supseteq B + D\}. \tag{2.5}$$

This operation will serve as a substitute for the difference in linear spaces. Indeed, for $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $A = a + \mathbb{R}_+$, $B = b + \mathbb{R}_+$, $a, b \in \mathbb{R}$ one obtains

$$A \mathbin{-^{\!\!\cdot}} B = \{r \in \mathbb{R} \mid b + r + \mathbb{R}_+ \subseteq a + \mathbb{R}_+\} = \{r \in \mathbb{R} \mid b - a + r + \mathbb{R}_+ \subseteq \mathbb{R}_+\} = a - b + \mathbb{R}_+.$$

Compare Example 2.15 below for more about the extended reals. The following proposition states two elementary properties of $\mathbin{-^{\!\!\cdot}}$. A full calculus exists for $\mathbin{-^{\!\!\cdot}}$ which to a large extend can be derived from known results in lattice/residuation theory. One example is Proposition 4.17 below which can be understood as a residuation version of "the negative of the infimum is the supremum of the negative."

Proposition 2.9 *Let $A, B \in \mathcal{P}(Z, C)$. Then*

$$A -^{\!\centerdot} B = \{z \in Z \mid B + z \subseteq A\}, \tag{2.6}$$

and if A is closed (convex) then $A -^{\!\centerdot} B$ is closed (convex) where Z is required to be a topological linear space if closedness is involved.

Proof The proof of the equation is immediate from (2.2) and (2.5), and the second claim follows from the first and

$$\{z \in Z \mid B + z \subseteq A\} = \bigcap_{b \in B} \{z \in Z \mid z \in A + \{-b\}\}.$$

Of course, $A + \{-b\}$ is closed (convex) if A is closed (convex), and these properties are stable under intersection. □

Remark 2.10 We would like to draw the reader's attention to the fact that the structure of an ordered conlinear space which also is an inf-residuated complete lattice is "rich enough" to serve as an image space in convex analysis. In fact, this structure is shared by $\overline{\mathrm{I\!R}}$ with inf-addition and $(\mathcal{P}(Z, C), +, \cdot, \supseteq)$ (as well as others, see below). Completely symmetric counterparts are provided by $\overline{\mathrm{I\!R}}$ with sup-addition and $(\mathcal{P}(Z, -C), +, \cdot, \subseteq)$ which are sup-residuated complete lattices. The transition from one to the other, provided by multiplication with -1, is a 'duality' in the sense of [210]. Although elements of this structure were well-known and have been used before (see the comments section below), the development of this framework for a "set-valued" convex/variational analysis and optimization theory is one contribution of the authors of this survey. A nice feature is that this structure admits to establish many results for vector/set-valued functions in the same way as for extended real-valued functions—not surprising after one realizes the similarities between the extended reals and conlinear spaces of sets.

We conclude this section by providing more examples of inf-residuated complete lattices of sets which will be used later on.

Example 2.11 Let Z be a topological linear space and $C \subseteq Z$ a convex cone with $0 \in C$. The set

$$\mathcal{F}(Z, C) = \{A \subseteq Z \mid A = \mathrm{cl}\,(A + C)\}$$

clearly is a subset of $\mathcal{P}(Z, C)$, but not closed under (Minkowski) addition. Therefore, we define an associative and commutative binary operation $\oplus\colon \mathcal{F}(Z, C) \times \mathcal{F}(Z, C) \to \mathcal{F}(Z, C)$ by

$$A \oplus B = \mathrm{cl}\,(A + B) \tag{2.7}$$

for $A, B \in \mathcal{F}(Z, C)$. The element-wise multiplication with non-negative real numbers is extended by

$$0 \odot A = \operatorname{cl} C, \quad t \odot \emptyset = \emptyset$$

for all $A \in \mathcal{F}(Z, C)$ and $t > 0$. In particular, $0 \odot \emptyset = \operatorname{cl} C$ by definition, and we will drop $\odot$ in most cases. The triple $(\mathcal{F}(C), \oplus, \odot)$ is a conlinear space with neutral element $\operatorname{cl} C$.

On $\mathcal{F}(Z, C)$, $\supseteq$ is a partial order which is compatible with the algebraic operations just introduced. Thus, $(\mathcal{F}(Z, C), \oplus, \odot, \supseteq)$ is a partially ordered, conlinear space.

Moreover, the pair $(\mathcal{F}(Z, C), \supseteq)$ is a complete lattice, and if $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ then

$$\inf_{(\mathcal{F}(Z,C),\supseteq)} \mathcal{A} = \operatorname{cl} \bigcup_{A \in \mathcal{A}} A, \qquad \sup_{(\mathcal{F}(Z,C),\supseteq)} \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

where again $\inf_{(\mathcal{F}(Z,C),\supseteq)} \mathcal{A} = \emptyset$ and $\sup_{(\mathcal{F}(Z,C),\supseteq)} \mathcal{A} = Z$ whenever $\mathcal{A} = \emptyset$.

The inf-residuation in $\mathcal{F}(Z, C)$ is defined as follows: For $A, B \in \mathcal{F}(Z, C)$, set

$$A \dot{-} B = \inf_{(\mathcal{F}(Z,C),\supseteq)} \{D \in \mathcal{F}(Z, C) \mid B + D \subseteq A\} = \{z \in Z \mid B + z \subseteq A\}. \quad (2.8)$$

Note that, for $A \in \mathcal{F}(Z, C)$, the set on the right hand side of (2.8) is indeed closed by Proposition 2.9.

Example 2.12 Let Z be a topological linear space and $C \subsetneq Z$ be a closed convex cone with $\emptyset \neq \operatorname{int} C \neq Z$. The set of *weakly minimal* points of a subset $A \subseteq Z$ (with respect to C) is defined by

$$\operatorname{wMin} A = \{y \in A \mid (\{y\} - \operatorname{int} C) \cap A = \emptyset\}.$$

For $A \in \mathcal{F}(Z, C)$, it can be shown ([151, Proposition 1.40 and Corollary 1.44]) that $\operatorname{wMin} A \neq \emptyset$ if and only if $A \notin \{Z, \emptyset\}$. This justifies the following construction. For $A \in \mathcal{F}(Z, C)$, set

$$\operatorname{Inf} A = \begin{cases} \operatorname{wMin} A & : \ A \notin \{Z, \emptyset\} \\ \{-\infty\} & : \ A = Z \\ \{+\infty\} & : \ A = \emptyset. \end{cases}$$

Then $\operatorname{Inf} A \subseteq Z \cup \{\pm\infty\}$, and $\operatorname{Inf} A$ is never empty. The set A can be reconstructed from $\operatorname{Inf} A$ by

$$A = \begin{cases} \operatorname{Inf} A \oplus C & : \ \operatorname{Inf} A \notin \{\{-\infty\}, \{+\infty\}\} \\ Z & : \ \operatorname{Inf} A = \{-\infty\} \\ \emptyset & : \ \operatorname{Inf} A = \{+\infty\}. \end{cases}$$

Defining the set $\mathcal{I}(Z, C) = \{\operatorname{Inf} A \mid A \in \mathcal{F}(Z, C)\}$ (the collection of 'self-infimal' subsets of $Z \cup \{\pm\infty\}$, [151, Definition 1.50]) and appropriate algebraic operations as well as an order one obtains $\mathcal{F}(Z, C) = \{B \oplus C \mid B \in \mathcal{I}(Z, C)\}$. Moreover, $\mathcal{I}(Z, C)$ and $\mathcal{F}(Z, C)$ are algebraically and order isomorphic ordered conlinear spaces (compare Proposition 1.52 of [151]). The reader is referred to [151, 155, 179, 216] for more details concerning infimal (and supremal) sets.

Example 2.13 Let Z, C be as in Example 2.11. The set

$$\mathcal{G}(Z, C) = \{A \subseteq Z \mid A = \mathrm{cl\,co}\,(A + C)\} \subseteq \mathcal{F}(Z, C)$$

together with the operations $\oplus$ and $\odot$ introduced in Example 2.11 is a conlinear subspace of $(\mathcal{F}(C), \oplus, \odot)$. In fact, $\mathcal{G}(Z, C)$ precisely contains the convex elements of $\mathcal{F}(Z, C)$. Moreover, the pair $(\mathcal{G}(Z, C), \supseteq)$ is a complete lattice, and for $\emptyset \neq \mathcal{A} \subseteq \mathcal{G}(Z, C)$

$$\inf_{(\mathcal{G}(Z,C),\supseteq)} \mathcal{A} = \mathrm{cl\,co} \bigcup_{A \in \mathcal{A}} A$$

whereas the formula for the supremum is the same as in $\mathcal{F}(Z, C)$. The inf-residuation in $(\mathcal{G}(Z, C), \oplus, \odot, \supseteq)$ is the same as in $(\mathcal{F}(Z, C), \oplus, \odot, \supseteq)$ which is a consequence of (2.8) and Proposition 2.9.

Example 2.14 If in Example 2.12 and under the assumptions therein, $\mathcal{F}(Z, C)$ is replaced by $\mathcal{G}(Z, C)$, we obtain a conlinear space $\mathcal{I}_{co}(Z, C)$, which is a subspace of $\mathcal{I}(Z, C)$ that is algebraically and order isomorphic to $\mathcal{G}(Z, C)$. For further details, the reader is referred to [151, Sect. 1.6].

Note that parallel results are obtained for $\mathcal{F}(Z, -C)$, $\mathcal{G}(Z, -C)$ with the same algebraic operations as in $\mathcal{F}(Z, C)$, $\mathcal{G}(Z, C)$ and the order relation $\subseteq$.

Example 2.15 Let us consider $Z = \mathbb{R}$, $C = \mathbb{R}_+$. Then

$$\mathcal{F}(\mathbb{R}, \mathbb{R}_+) = \mathcal{G}(\mathbb{R}, \mathbb{R}_+) = \{[r, +\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\emptyset\}.$$

Moreover, by

$$a = \inf_{(\mathbb{R},\leq)} A \quad \text{for} \quad A \in \mathcal{G}(\mathbb{R}, \mathbb{R}_+)$$

and

$$A = \begin{cases} \mathbb{R} & : \; a = -\infty \\ [a, +\infty) & : \; a \in \mathbb{R} \\ \emptyset & : \; a = +\infty \end{cases}$$

we obtain an algebraic and order isomorphism between $(\mathcal{G}(\mathbb{R}, \mathbb{R}_+), \oplus, \odot, \supseteq)$ and $\left(\overline{\mathbb{R}}, \dot{+}, \cdot, \leq\right)$ where $\dot{+}$ is the inf-addition (see [197]) on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with $(+\infty) \dot{+} r = r \dot{+} (+\infty) = +\infty$ for all $r \in \overline{\mathbb{R}}$, and $\cdot$ is an extension of the multiplication of non-negative real numbers by elements of $\overline{\mathbb{R}}$. Note that $0 \cdot (-\infty) = 0 \cdot (+\infty) = 0$ since otherwise $\left(\overline{\mathbb{R}}, \dot{+}, \cdot\right)$ is not a conlinear space. Of course, $A \supseteq B$ if, and only if, $\inf_{(\mathbb{R},\leq)} A \leq \inf_{(\mathbb{R},\leq)} B$. Thus, $\left(\overline{\mathbb{R}}, \dot{+}, \cdot, \leq\right)$ is an ordered conlinear space which is a complete lattice with respect to $\leq$. Moreover,

$$\forall M \subseteq \overline{\mathbb{R}},\ \forall r \in \overline{\mathbb{R}}\colon \inf_{(\overline{\mathbb{R}}, \leq)} \left(M \dotplus \{r\}\right) = r \dotplus \inf_{(\overline{\mathbb{R}}, \leq)} M,$$

which admits the introduction of the inf-residuation in $\overline{\mathbb{R}}$ [89]. Here, $M \dotplus \{r\} = \left\{m \dotplus r \mid m \in M\right\}$. We have

$$r \dot{-} s = \inf \left\{t \in \mathbb{R} \mid r \leq s \dotplus t\right\}$$

for all $r, s \in \overline{\mathbb{R}}$ with some strange properties. For examples, expressions like $(+\infty) \dot{-} (-\infty)$ are well-defined and even useful as shown in [89, 90].

Remark 2.16 As a simple, but instructive exercise the reader should try to establish the isomorphism between $(\mathcal{G}(\mathbb{R}, -\mathbb{R}_+), \oplus, \cdot, \subseteq)$ and $\left(\overline{\mathbb{R}}, \underset{\cdot}{+}, \cdot, \leq\right)$ where $\underset{\cdot}{+}$ denotes the “sup-addition” [197]. This shows that the reason why ‘there’s no single symmetric way of handling $(+\infty) + (-\infty)$’ is basically the same as the one for having two “canonical” extensions of a vector pre-order to the power set of the vector space.[3]

The image space $\mathcal{G}(Z, C)$ will feature prominently in duality theories for set-valued functions/optimization problems. The last example shows that it shares almost all properties with the extended reals provided with the inf-addition. The notable exception is that the order $\supseteq$ is not total in general. The following result clarifies this question.

Proposition 2.17 *Let Z be a locally convex space and $C \subseteq Z$ a convex cone with $0 \in C$ and $Z \neq \operatorname{cl} C$. Then $\supseteq$ is total on $\mathcal{F}(Z, C)$ if, and only if,* $\operatorname{cl} C$ *coincides with a half-space $H^+(z^*) := \{z \in Z \mid z^*(z) \geq 0\}$ for some $z^* \in C^+ \backslash \{0\}$.*

Proof The “if” part is immediate. For the “only if” part, assume $\supseteq$ is total on $\mathcal{F}(Z, C)$ and $\operatorname{cl} C$ is not a half space. Then, there are $z^* \in C^+ \backslash \{0\}$ and $\bar{z} \in Z$ such that

$$\operatorname{cl} C \subseteq H^+(z^*) \quad \text{and} \quad \bar{z} \in H^+(z^*) \backslash \operatorname{cl} C.$$

Indeed, the existence of $z^* \in C^+ \backslash \{0\}$ and the first inclusion follow from a separation argument, the second from the assumption. We claim that

$$\forall s \in \mathbb{R}\colon H(z^*, s) := \left\{z \in Z \mid z^*(z) \geq s\right\} \not\subseteq \operatorname{cl} C.$$

In order to verify the claim, assume $H(z^*, s) \subseteq \operatorname{cl} C$ for some $s \in \mathbb{R}$. Then, there is $z_s \in Z$ such that $H(z^*, s) = z_s + H^+(z^*)$ and $z^*(z_s) = s$. This implies

[3] R. T. Rockafellar and R.-B. Wets also remark on p. 15 of [197] that the second distributivity law does not extend to all of $\overline{\mathbb{R}}$ which is another motivation for the concept of “conlinear” spaces. Finally, it is interesting to note that the authors of [197] consider it a matter of cause to associate minimization with inf-addition (see p. 15). In the set optimization community, there is no clear consensus yet about which relation to use in what context and for what purpose. However, this note makes a clear point towards [197]: associate $\preccurlyeq_C$ with minimization and $\curlyeqprec_C$ with maximization because the theory works for these cases. One should have a very strong reason for doing otherwise and be advised that in this case many standard mathematical tools just don’t work.

$$\forall t > 0 \colon z_s + t\bar{z} \in H(z^*, s)$$

since $z^*(z_s + t\bar{z}) = s + tz^*(\bar{z}) \geq s$. By assumption, $z_s + t\bar{z} \in \operatorname{cl} C$ for all $t > 0$, hence

$$\forall t > 0 \colon \frac{1}{t} z_s + \bar{z} \in \operatorname{cl} C$$

which in turn gives $\bar{z} \in \operatorname{cl} C$, a contradiction. This proves the claim, i.e. $\operatorname{cl} C \not\supseteq H(z^*, s)$ for all $s \in \mathrm{I\!R}$. Since $\supseteq$ is total,

$$\operatorname{cl} C \subseteq \bigcap_{s \in \mathrm{I\!R}} H(z^*, s) = \emptyset,$$

a contradiction. □

2.4 Comments on Conlinear Spaces and Residuation

The term 'conlinear space' was coined in [77] because of the analogies to 'convex cones' on the one hand and to linear spaces on the other hand.

A related concept is the one of quasilinear spaces or almost linear spaces as defined in, for example, [66, 168], respectively. A quasilinear (or almost linear) space satisfies all the axioms of a linear space, but the second distributivity law which is required only for non-negative reals. Hence $(\mathcal{P}(Z), +, \cdot)$, $(\mathcal{P}(Z, C), +, \cdot)$ and $(\mathcal{F}(Z, C), \oplus, \cdot)$ are not quasilinear spaces in general. With respect to interval mathematics, Schmidt [206, Sect. 4] observed '[...] it seems to be convenient to generalize one step further and to restrict the multiplication by scalars to positive scalars alone.' Keeping all the other requirements for a quasilinear space we obtain an abstract convex cone in the sense of B. Fuchssteiner, W. Lusky [60]. In [124], the same concept is the basic one, sometimes a convex cone even without a zero element. Abstract convex cones also coincide with semilinear spaces as probably introduced by A. M. Rubinov [202] (he refers to a 1975 joint paper with S.S. Kutateladze) and recalled, for example, in [65, Definition 2.6].

We remark that convex cones in the sense of [60] and semilinear spaces (with a "zero") in the sense of [65, Definition 2.6] are also semi-modules over $\mathrm{I\!R}_+$ (and even semivector spaces) as defined by U. Zimmermann in [230, Sect. 5]. Finally, another close relative of a conlinear space is a semivector space in the sense of [192]. Prakash and Sertel (see also [193]) defined this structure in the early Seventies and observed that the collections of non-empty and non-empty convex sets of a vector space form a semivector space. In a semivector space, the existence of a neutral element with respect to the addition is not required. Therefore, it might be considered as the "weakest" algebraic concept discussed here.

Dedekind [43] already introduced the residuation concept and used it in order to construct the real numbers as 'Dedekind sections' of rational numbers. Among others, R. P. Dilworth and M. Ward turned it into a standard tool in abstract lattice theory, see [46, 223–225] and many followers.

Sometimes, the right hand side of (2.8) is called the geometric difference [189], star-difference (for example in [220]), or Minkowski difference [76] of the two sets A and B, and H. Hadwiger should probably be credited for its introduction. The relationship to residuation theory (see, for instance, [13, 61]) has been established in [89]. At least, we do not know an earlier reference. In the context of abstract duality, residuation has been used, for example, in [64, 167] and also in idempotent analysis (see [62, Sect. 3.3], for example). Note that in [64], the set $\overline{\mathrm{IR}}$ is supplied both with $\dot{+}$ and $\underset{\cdot}{+}$ at the same time, and this idea is extended to 'the canonical enlargement' of a 'boundedly complete lattice ordered group' (see [64, Sect. 3]) which is different from the point of view of this survey. On the other hand, $\mathcal{F}(Z, C)$ and $\mathcal{G}(Z, C)$ are special cases of $(A, \preceq)$ in [64, Sect. 2], but therein the conlinear structure is not used.

3 Minimality Notions

3.1 Basic Concepts

This section is concerned with the question of how to define "infimum attainment" and "minimality." We shall focus on the relation $\supseteq$ on $\mathcal{F}(Z, C)$ and $\mathcal{G}(Z, C)$ noting that there are parallel concepts and results for $\subseteq$ on $\mathcal{F}(Z, -C)$, $\mathcal{G}(Z, -C)$. In the remainder of the paper, the infimum or supremum is always taken in the corresponding space of elements, for example

$$\inf \mathcal{A} = \mathrm{cl} \bigcup_{A \in \mathcal{A}} A$$

whenever $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ whereas for $\mathcal{A} \subseteq \mathcal{G}(Z, C)$

$$\inf \mathcal{A} = \mathrm{cl}\,\mathrm{co} \bigcup_{A \in \mathcal{A}} A.$$

With the constructions from the previous section in view, we have at least two possibilities for a minimality notion. Given a set $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ or $\mathcal{A} \subseteq \mathcal{G}(Z, C)$, look for

(I) $\inf \mathcal{A} = \mathrm{cl} \bigcup_{A \in \mathcal{A}} A$ or $\inf \mathcal{A} = \mathrm{cl}\,\mathrm{co} \bigcup_{A \in \mathcal{A}} A$, respectively, or
(II) minimal elements with respect to $\supseteq$, i.e. $B \in \mathcal{A}$ satisfying

$$A \in \mathcal{A},\ A \supseteq B \quad \Rightarrow \quad A = B.$$

Note that the second possibility corresponds to the so-called set criterion which became popular due to the work of D. Kuroiwa and collaborators: One looks for minimal elements of $\mathcal{A} \subseteq \mathcal{P}(Z)$ with respect to $\preccurlyeq_C$. Since $\preccurlyeq_C$ is not antisymmetric in general one has to look for $B \in \mathcal{A}$ satisfying

(IIa)
$$A \in \mathcal{A},\ A \preccurlyeq_C B \quad \Rightarrow \quad B \preccurlyeq_C A.$$

Neither of the two possibilities above has been considered first. Rather, the following problem has been studied since the 1980s by Corley [32], Dinh The Luc [156] and others, and it is still popular.

(III) Find minimal elements of $\bigcup_{A \in \mathcal{A}} A$ with respect to $\leq_C$, i.e. find $b \in \bigcup_{A \in \mathcal{A}} A$ satisfying
$$a \in \bigcup_{A \in \mathcal{A}} A,\ a \leq_C b \quad \Rightarrow \quad b \leq_C a.$$

In this way, a set optimization problem is reduced to a vector optimization problem, and sometimes this problem is referred to as the vector criterion in set optimization. Note that, in some way, it involves the infimum of $\mathcal{A}$ in $\mathcal{P}(Z, C)$.

Example 3.1 Let $Z = \mathbb{R}^2$, $C = \{0\} \times \mathbb{R}_+$ and $\mathcal{A} = \{A_t \mid t \in [0, 1]\}$ where

$$A_t = [-1 + t, t] \times \mathbb{R}_+.$$

Then each A_t is minimal with respect to $\supseteq$ and

$$\inf\ \mathcal{A} = A_0 \cup A_1 = [-1, 1] \times \mathbb{R}_+.$$

This shows that not all minimal elements are required to generate the infimum which prepares the following definition.

Definition 3.2 Let $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ or $\mathcal{A} \subseteq \mathcal{G}(Z, C)$.

(a) A set $\mathcal{B} \subseteq \mathcal{A}$ is said to generate the infimum of $\mathcal{A}$ if
$$\inf\ \mathcal{B} = \inf\ \mathcal{A}.$$

(b) An element $\bar{A} \in \mathcal{A}$ is called minimal for $\mathcal{A}$ if it satisfies
$$A \in \mathcal{A},\ A \supseteq \bar{A} \quad \Rightarrow \quad A = \bar{A}.$$

The set of all minimal elements of $\mathcal{A}$ is denoted by Min $\mathcal{A}$.

Parallel definitions apply to generators of the supremum and maximal elements. Of course, $\mathcal{A}$ always generates the infimum of $\mathcal{A}$. On the other hand, a set of minimal elements of $\mathcal{A}$ does not necessarily generate the infimum of $\mathcal{A}$. In Example 3.1, every subset of $\mathcal{A}$ including A_0 and A_1 consists of minimal elements and generates the infimum, i.e., in general, sets of minimal elements generating the infimum are not singletons. If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}, \mathbb{R}_+)$, then a single element $A \in \mathcal{A}$ generates the infimum of $\mathcal{A}$ if, and only if, it is a minimal one. Definition 3.2 leads to the following "complete lattice approach." Given a set $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ or $\mathcal{A} \subseteq \mathcal{G}(Z, C)$ look for

(IV) a set $\mathcal{B} \subseteq \mathcal{A}$ such that

$$\inf \mathcal{B} = \inf \mathcal{A} \quad \text{and} \quad \mathcal{B} \subseteq \operatorname{Min} \mathcal{A}.$$

Hence, the minimality notion of the "complete lattice approach" consists of looking for sets of minimal elements which generate the infimum. We turn these notions into a solution concept for set optimization problems. The following definition is a special case of the general one given in [102].

Definition 3.3 Let X be a non-empty set, $f : X \to \mathcal{F}(Z, C)$ (or $f : X \to \mathcal{G}(Z, C)$) a function and $f[X] = \{f(x) \mid x \in X\}$.

(a) A set $M \subseteq X$ is called an infimizer for f if

$$\inf f[M] = \inf f[X].$$

(b) An element $\bar{x} \in X$ is called a minimizer of f if $f(\bar{x})$ is minimal for $f[X]$.
(c) A set $M \subseteq X$ is called a solution of the problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in X \tag{P}$$

if M is an infimizer for f, and each $\bar{x} \in M$ is a minimizer of f. It is called a full solution if the set $f[M]$ includes all minimal elements of $f[X]$.

Thus, solutions of set minimization problems in the "complete lattice" sense are infimizers consisting only of minimizers. Again, parallel definitions apply to solutions of maximization problems which will later appear in duality results. One more concept is needed for a Weierstraß type theorem.

Definition 3.4 A set $\mathcal{A} \subseteq \mathcal{F}(Z, C)$ (or $\mathcal{A} \subseteq \mathcal{G}(Z, C)$) is said to satisfy the domination property if

$$\forall A \in \mathcal{A},\ \exists \bar{A} \in \operatorname{Min} \mathcal{A} \colon \bar{A} \supseteq A.$$

Proposition 3.5 *Let $f : X \to \mathcal{F}(Z, C)$ (or $f : X \to \mathcal{G}(Z, C)$) be a function and $f[X]$ satisfy the domination property. Then*

$$M = \{x \in X \mid f(x) \in \operatorname{Min} f[X]\}$$

is a full solution of (P).

Proof The domination property yields the first of the following inclusions while the second one follows from $M \subseteq X$:

$$\inf_{x \in M} f(x) \supseteq \inf_{x \in X} f(x) \supseteq \inf_{x \in M} f(x).$$

This already completes the proof since M comprises all minimizers of $f[X]$. □

3.2 Comments on Solution Concepts in Set Optimization

The appearance of set-valued functions in optimization theory was mainly motivated by unifying different forms of constraints, see [14] and also [184, 185]. Problem (P) in [16, p. 196] seems to be the first explicit set-valued optimization problem. J. M. Borwein defines its optimal value as the infimum with respect to the underlying vector order and assumes that the image space is conditional order complete, i.e. every subset which is bounded from below (above) has an infimum (supremum) in the space. Clearly, a necessary condition for this is that the image space is a vector lattice. This restricts the applicability of such results considerably and besides, the vector infimum/supremum does not produce solution concepts which are useful in applications.

In [190, 191], Postolica formulates an optimization problem with a set-valued objective and uses the minimality concept (III) above.

Corley [32, 33] defined 'the maximization of a set-valued function with respect to a cone in possibly infinite dimensions' mainly motivated by the fact that, in duality theories for multiobjective problems as established by Tanino and Sawaragi [218], 'dual problems took this form' (quotes from [32, p. 489]). The same motivation can be found in Dinh The Luc's book [156] in which vector optimization problems with a set-valued objective are investigated using the approach (III).

Both authors considered optimality in the sense of (III) above: Take the union of all objective values and then look for (weakly, properly etc.) minimal points in this union with respect to the vector order. This approach has been the leading idea ever since, among the many followers are [29, 54, 55, 57, 141, 145–147] (just to mention a few), the book [28], and even the more recent [34, 35, 74, 105, 173, 174, 203], [19, Sects. 7.1.3., 7.4.2.], [58, 199] and many more. We call this approach the vector approach to set optimization.

The picture changed when the set relations were popularized by Kuroiwa and his co-authors [130, 132, 133, 136, 138]. Still, it took several years until the idea to use (II) above as a solution concept for set-valued optimization problems became more popular, see [1, 75, 83, 85, 93–96, 229] and also Chap. 5 of Jahn's book [114]. The basic idea is, of course, to "lift" the concept of minimal (=non-dominated) image points from elements of a vector space to elements of the power set of the vector space. Therefore, we call this approach the set relation approach to set optimization. A comparison of the vector and the set relation approach can be found in [97].

Roughly another ten years later, it has been realized that the so-called set relations can be utilized in a more subtle manner which is described in the previous section: Via equivalence classes with respect to the two pre-orders and hull operations one defines (conlinear) spaces of sets which enjoy rich algebraic and order structures. The set relations somehow disappear from the final picture since they serve as a tool to construct the image spaces in which the subset or superset inclusion appears as a partial order. This approach, which we call the "complete lattice approach" to set optimization has been developed in the two theses [77, 149] and led to the solution concept in [102, 151] which is the basis for the Definitions 3.2 and 3.3 above (see already [91] for a precursor). One may realize that the complete lattice approach (IV) absorbs both of (I) and (II) as well as (IIa).

It might be interesting to note that Ekeland's variational principle became one of the first major results in (nonlinear, nonconvex) functional analysis that was generalized to set-valued functions via the set relation approach. While [26], [27, Theorem 4.1], [84, Theorem 5.1], [106, Theorems 2.3 and 2.4] still follow the vector approach, first set relation versions were independently established in [75, 85] (with precursor [83] already from 2002). Note that the results in [85] are more general (weaker assumptions like boundedness from below, more general pre-image spaces) and more complete (both set relations are involved, minimal "set"theorems, not only Ekeland's principle). In particular, the main result [75, Theorem 4.1] is a special case of [85, Theorem 6.1].

4 Set-Valued Functions

4.1 Basic Concepts

Let X be another linear space and $f\colon X \to \mathcal{P}(Z, C)$ a function. The goal is to develop a convex analysis for such functions f. We start by recalling a popular definition. A function $\hat{f}\colon X \to \mathcal{P}(Z)$ is called C-convex (see e.g. [14, Definition 1.1]) if

$$t \in (0,1)\,,\; x_1, x_2 \in X \;\Rightarrow\; \hat{f}(tx_1 + (1-t)x_2) + C \supseteq t\hat{f}(x_1) + (1-t)\hat{f}(x_2)\,, \tag{4.1}$$

and it is called C-concave ([156, p. 117]) if

$$t \in (0,1)\,,\; x_1, x_2 \in X \;\Rightarrow\; t\hat{f}(x_1) + (1-t)\hat{f}(x_2) \subseteq \hat{f}(tx_1 + (1-t)x_2) - C. \tag{4.2}$$

Of course, the C-convexity inequality is just

$$\hat{f}(tx_1 + (1-t)x_2) \preccurlyeq_C t\hat{f}(x_1) + (1-t)\hat{f}(x_2)\,,$$

and the C-concavity inequality

$$t\hat{f}(x_1) + (1-t)\,f(x_2) \preccurlyeq_C \hat{f}(tx_1 + (1-t)\,x_2)\,.$$

Here is another interesting feature of the set-valued framework. If f maps into $\mathcal{P}(Z, C)$, then the cone C can be dropped from (4.1) whereas (4.2) becomes meaningless for many interesting cones C (for example, for generating cones, i.e. $C - C = Z$). The opposite is true for $\mathcal{P}(Z, -C)$-valued functions. This gives a hint why convexity (and minimization) is related to $\mathcal{P}(Z, C)$-valued functions and concavity (and maximization) to $\mathcal{P}(Z, -C)$-valued ones.

The graph of a function $\hat{f} \colon X \to \mathcal{P}(Z)$ is the set

$$\operatorname{graph} \hat{f} = \left\{(x, z) \in X \times Z \mid z \in \hat{f}(x)\right\},$$

and the domain is the set

$$\operatorname{dom} \hat{f} = \{x \in X \mid f(x) \neq \emptyset\}\,.$$

Definition 4.1 A function $f \colon X \to \mathcal{P}(Z, C)$ is called

(a) convex if graph f is a convex subset of $X \times Z$,
(b) positively homogeneous if graph f is a cone in $X \times Z$,
(c) sublinear if graph f is a convex cone $X \times Z$,
(d) proper if dom $f \neq \emptyset$ and $f(x) \neq Z$ for all $x \in X$.

Proposition 4.2 *A function $f \colon X \to \mathcal{P}(Z, C)$ is convex if, and only if,*

$$t \in (0, 1)\,,\ x_1, x_2 \in X \ \Rightarrow\ f(tx_1 + (1-t)\,x_2) \supseteq tf(x_1) + (1-t)\,f(x_2)\,. \tag{4.3}$$

It is positively homogeneous if, and only if,

$$t > 0,\ x \in X \ \Rightarrow\ f(tx) \supseteq tf(x)\,, \tag{4.4}$$

and it is sublinear if, and only if,

$$s, t > 0,\ x_1, x_2 \in X \ \Rightarrow\ f(sx_1 + tx_2) \supseteq sf(x_1) + tf(x_2)\,. \tag{4.5}$$

Proof Exercise. □

A parallel result for concave $\mathcal{P}(Z, -C)$-valued functions can be established. As a straightforward consequence of Proposition 4.2 we obtain the following facts.

Proposition 4.3 *Let* $f: X \to \mathcal{P}(Z, C)$ *be a convex function. Then*

(a) $f(x)$ *is convex for all* $x \in X$*, i.e.* f *is convex-valued,*
(b) $\{x \in X \mid z \in f(x)\}$ *is convex for all* $z \in Z$*,*
(c) $\operatorname{dom} f$ *is convex.*

Proof Another exercise. □

In the remainder of this subsection, let X and Z be topological linear spaces. We shall denote by $\mathcal{N}_X$ and $\mathcal{N}_Z$ a neighborhood base of $0 \in X$ and $0 \in Z$, respectively.

Definition 4.4 A function $f: X \to \mathcal{P}(Z, C)$ is called

(a) closed-valued if $f(x)$ is a closed set for all $x \in X$,
(b) level-closed if $\{x \in X \mid z \in f(x)\}$ is closed for all $z \in Z$,
(c) closed if $\operatorname{graph} f$ is a closed subset of $X \times Z$ with respect to the product topology.

Remark 4.5 A function $f: X \to \mathcal{F}(Z, C)$ is level-closed if, and only if, $\{x \in X \mid f(x) \supseteq A\}$ is closed for all $A \in \mathcal{F}(Z, C)$ which may justify the term "level-closed." Indeed, this follows from $\{z\} \oplus C \in \mathcal{F}(Z, C)$ and

$$\forall A \in \mathcal{F}(Z, C)\colon\ \{x \in X \mid f(x) \supseteq A\} = \bigcap_{a \in A} \{x \in X \mid a \in f(x)\}.$$

Level-closedness is even equivalent to closedness if $\operatorname{int} C \neq \emptyset$, see [151, Proposition 2.38], even for functions mapping into a completely distributive lattice as in [148], but not in general.

Example 4.6 This example is taken from [176, Example 3.1]. Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$, $C = \left\{(0, t)^T \mid t \geq 0\right\}$ and consider the function

$$f(x) = \begin{cases} \begin{pmatrix} x \\ x+1 \end{pmatrix} + C & : \ 0 \leq x < 1 \\ \begin{pmatrix} 1 \\ 4 \end{pmatrix} + C & : \ x = 1 \\ \emptyset & : \ \text{otherwise.} \end{cases}$$

Defining sequences by

$$x^k = 1 - \frac{1}{k} \quad \text{and} \quad z^k = \begin{pmatrix} 1 - \frac{1}{k} \\ 2 - \frac{1}{k} \end{pmatrix}$$

we obtain $z^k \in f(x^k)$ for all $k = 1, 2, \ldots$, $x^k \to 1$, $z^k \to (1, 2)^T$ and $(1, 2)^T \notin f(1)$, thus $\operatorname{graph} f$ is not closed. On the other hand,

$$\{x \in X \mid z \in f(x)\} = \begin{cases} \{z_1\} & : \ 0 \leq z_1 < 1 \text{ and } z_2 \geq z_1 + 1 \\ \{1\} & : \ z_1 = 1 \text{ and } z_2 \geq 4 \\ \emptyset & : \ \text{otherwise} \end{cases}$$

thus f is level-closed.

The following result is immediate.

Proposition 4.7 *Let $f \colon X \to \mathcal{P}(Z, C)$ be a closed function. Then f is closed-valued and level-closed.*

Proof Yet another exercise. □

Proposition 4.7 shows that a closed $\mathcal{P}(Z, C)$-valued function actually maps into $\mathcal{F}(Z, C)$. Therefore, we can restrict the discussion of lower semicontinuity and closedness to $\mathcal{F}(Z, C)$-valued functions. The following definition introduces two more related notions.

Definition 4.8 A function $f \colon X \to \mathcal{F}(Z, C)$ is called lattice-lower semicontinuous (lattice-l.s.c.) at $\bar{x} \in X$ iff

$$f(x) \supseteq \liminf_{x \to \bar{x}} f(x) = \sup_{U \in \mathcal{N}_X} \inf_{x \in \bar{x}+U} f(x) = \bigcap_{U \in \mathcal{N}_X} \operatorname{cl} \bigcup_{x \in \bar{x}+U} f(x). \tag{4.6}$$

It is called lattice-lower semicontinuous iff it is lattice-l.s.c. at every $\bar{x} \in X$.

Parallel definitions apply for $\mathcal{G}(Z, C)$-valued functions. The next result shows the equivalence of lattice-lower semicontinuity and closedness for $\mathcal{F}(Z, C)$-valued functions.

Proposition 4.9 *A function $f \colon X \to \mathcal{F}(Z, C)$ is lattice-l.s.c. if, and only if, it is closed.*

Proof The proof of Proposition 2.34 in [151] also applies to this case as already discussed in [151, p. 59]. □

The following result contains the heart of the argument for the Weierstraß type theorem.

Proposition 4.10 *Let $f \colon X \to \mathcal{F}(Z, C)$ be a level-closed function such that* dom f *is compact. Then $f[X]$ satisfies the domination property.*

Proof This is a special case of Proposition 2.38 in [151]. □

Theorem 4.11 *Let $f \colon X \to \mathcal{F}(Z, C)$ be a level-closed function such that* dom f *is compact. Then (P) has a full solution.*

Proof This directly follows from Propositions 3.5 and 4.10. □

Because of Propositions 4.7 and 4.9, lattice-lower semicontinuity or closedness are sufficient conditions for level-closedness.

We turn to upper semi-continuity type properties which will mainly be used to establish sufficient conditions for convex duality results.

Definition 4.12 A function $f\colon X \to \mathcal{F}(Z, C)$ is called lattice-upper semicontinuous (lattice-u.s.c.) at $\bar{x} \in X$ if

$$\limsup_{x \to \bar{x}} f(x) = \inf_{U \in \mathcal{N}_X} \sup_{x \in \bar{x}+U} f(x) = \operatorname{cl} \bigcup_{U \in \mathcal{N}_X} \bigcap_{x \in \bar{x}+U} f(x) \supseteq f(\bar{x}).$$

It is called lattice-upper semicontinuous (lattice-u.s.c.) if it is lattice-u.s.c. at every $x \in X$.

Because of Proposition 4.3, we only need to consider $\mathcal{G}(Z, C)$-valued functions in the following result.

Proposition 4.13 *Let X be a locally convex topological linear space and $\mathcal{N}_X$ a neighborhood base of $0 \in X$ consisting of convex sets. Let $f\colon X \to (\mathcal{F}(Z, C), \supseteq)$ be convex. Then, f is lattice-l.s.c. (lattics-u.s.c.) at $\bar{x} \in X$ if, and only if, it is lattice-l.s.c. (lattice-u.s.c.) as a function into $(\mathcal{G}(Z, C), \supseteq)$ at $\bar{x}$.*

Proof It is easy to prove that if f is convex, then for all $x \in X$ and all $U \in \mathcal{N}_X$ the set $\bigcup_{x \in \bar{x}+U} f(x)$ is convex, hence

$$\bigcap_{U \in \mathcal{N}_X} \operatorname{cl} \bigcup_{x \in \bar{x}+U} f(x) = \bigcap_{U \in \mathcal{N}_X} \operatorname{cl} \operatorname{co} \bigcup_{x \in \bar{x}+U} f(x).$$

With the definition of lim inf in view, the case of lattice-lower semi-continuity follows.

Concerning lattice upper semi-continuity, take

$$z_1,\ z_2 \in \bigcup_{U \in \mathcal{N}_X} \bigcap_{x \in \bar{x}+U} f(x).$$

Then, there are $U_1, U_2 \in \mathcal{N}_X$ such that $z_i \in \bigcap_{x \in \bar{x}+U_i} f(x)$ for $i = 1, 2$. Since $\mathcal{N}_X$ is a neighborhood base of $0 \in X$ there is $V \in \mathcal{N}_X$ such that $V \subseteq U_1 \cap U_2$. Hence

$$\forall x \in \bar{x} + V : z_1, z_2 \in f(x).$$

Since $f(x)$ is a convex set, this implies

$$\forall t \in (0, 1), \forall x \in \bar{x} + V : t z_1 + (1 - t) z_2 \in f(x),$$

hence $t z_1 + (1 - t) z_2 \in \bigcup_{U \in \mathcal{U}} \bigcap_{x \in \bar{x}+U} f(x)$. This shows that the latter is a convex set. Consequently,

$$\operatorname{cl}\operatorname{co}\bigcup_{U\in\mathcal{U}}\bigcap_{x\in\bar{x}+U} f(x) = \operatorname{cl}\bigcup_{U\in\mathcal{U}}\bigcap_{x\in\bar{x}+U} f(x).$$

The claim for lattice-upper semi-continuity follows from the definition of lim sup. □

4.2 *Scalarization of $\mathcal{G}(Z, C)$-Valued Functions*

In the following, we assume that Z is a non-trivial locally convex linear space with topological dual Z^*. For $A \subseteq Z$, define the extended real-valued functions $\sigma^{\vartriangle}_A : Z^* \to \overline{\mathbb{R}}$ and $\sigma^{\triangledown}_A : Z^* \to \overline{\mathbb{R}}$ by

$$\sigma^{\vartriangle}_A\left(z^*\right) = \inf_{a\in A} z^*(a) \quad \text{and} \quad \sigma^{\triangledown}_A\left(z^*\right) = \sup_{a\in A} z^*(a),$$

respectively. Of course, $\sigma^{\triangledown}_A$ is the classical support function of A and $\sigma^{\vartriangle}_A(z^*) = -\sigma^{\triangledown}_A(-z^*)$ a version of it. It is well-known (and a consequence of a separation argument) that $A \in \mathcal{G}(Z, C)$ if, and only if,

$$A = \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \sigma^{\vartriangle}_A\left(z^*\right) \le z^*(z)\right\}. \tag{4.7}$$

Moreover, one easily checks for $A, B \in \mathcal{G}(Z, C)$,

$$\forall z^* \in C^+\backslash\{0\} \colon \sigma^{\vartriangle}_{A\oplus B}\left(z^*\right) = \sigma^{\vartriangle}_A\left(z^*\right) \dot{+} \sigma^{\vartriangle}_B\left(z^*\right). \tag{4.8}$$

Lemma 4.14 *If $\mathcal{A} \subseteq \mathcal{G}(Z, C)$ then*

$$\forall z^* \in C^+\backslash\{0\} \colon \sigma^{\vartriangle}_{\inf\mathcal{A}}\left(z^*\right) = \inf\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\}, \tag{4.9}$$

$$\forall z^* \in C^+\backslash\{0\} \colon \sigma^{\vartriangle}_{\sup\mathcal{A}}\left(z^*\right) \ge \sup\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\}. \tag{4.10}$$

Moreover,

$$\inf\mathcal{A} = \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \inf\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\} \le z^*(z)\right\}, \tag{4.11}$$

$$\sup\mathcal{A} = \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \sup\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\} \le z^*(z)\right\} \tag{4.12}$$

Proof If $\mathcal{A} \subseteq \{\emptyset\}$ then there is nothing to prove. Otherwise,

$$\forall A \in \mathcal{A} \colon \sigma^{\vartriangle}_{\inf\mathcal{A}}\left(z^*\right) = \inf_{z\in\inf\mathcal{A}} z^*(z) \le \sigma^{\vartriangle}_A\left(z^*\right),$$

hence $\sigma^{\vartriangle}_{\inf \mathcal{A}}(z^*) \le \inf\left\{\sigma^{\vartriangle}_A(z^*) \mid A \in \mathcal{A}\right\}$. Conversely,

$$\forall z \in \bigcup_{A\in\mathcal{A}} A\colon z^*(z) \ge \inf\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\},$$

hence $\sigma^{\vartriangle}_{\inf \mathcal{A}}(z^*) = \inf\left\{z^*(z) \mid z \in \bigcup_{A\in\mathcal{A}} A\right\} \ge \inf\left\{\sigma^{\vartriangle}_A(z^*) \mid A \in \mathcal{A}\right\}$ since the support function of a set coincides with the support function of its closed convex hull. This proves (4.9) which in turn immediately implies (4.11).

Moreover, if $z \in \sup \mathcal{A} = \bigcap_{A\in\mathcal{A}} A$ then

$$\forall A \in \mathcal{A}\colon z^*(z) \ge \inf_{a\in A} z^*(a) = \sigma^{\vartriangle}_A\left(z^*\right)$$

which already proves (4.10). Finally, for all $z^* \in C^+\backslash\{0\}$

$$\left\{z \in Z \mid z^*(z) \ge \sup\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\}\right\} = \bigcap_{A\in\mathcal{A}} \left\{z \in Z \mid z^*(z) \ge \sigma^{\vartriangle}_A\left(z^*\right)\right\},$$

hence

$$\bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid z^*(z) \ge \sup\left\{\sigma^{\vartriangle}_A\left(z^*\right) \mid A \in \mathcal{A}\right\}\right\} = \bigcap_{z^*\in C^+\backslash\{0\}} \bigcap_{A\in\mathcal{A}} \left\{z \in Z \mid z^*(z) \ge \sigma^{\vartriangle}_A\left(z^*\right)\right\}$$
$$= \bigcap_{A\in\mathcal{A}} \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid z^*(z) \ge \sigma^{\vartriangle}_A\left(z^*\right)\right\} = \bigcap_{A\in\mathcal{A}} A = \sup \mathcal{A}$$

according to (4.7), and this is just (4.12). □

The following example shows that the inequality in (4.10) can be strict. Consider $\mathcal{A} = \left\{\{a\} + \mathbb{R}^2_+ \mid a = (a_1, a_2)^T \in \mathbb{R}^2,\ a_1 \ge 0,\ a_2 \ge 0,\ a_1 + a_2 = 1\right\} \subseteq \mathcal{G}\left(\mathbb{R}^2, \mathbb{R}^2_+\right)$ and $z^* = (1, 1)^T$. Then $\sigma^{\vartriangle}_A(z^*) = 1$ for all $A \in \mathcal{A}$ and $\sigma^{\vartriangle}_{\sup \mathcal{A}}(z^*) = 2$.

As an immediate consequence, the sub/supermodularity[4] of the scalarization functions $\sigma^{\triangledown}_A, \sigma^{\vartriangle}_A$ as functions of $A \in \mathcal{G}(Z, C)$ can be established. This property is fundamental in the theory of Choquet integrals [44]. For $z^* \in Z^*$ define

$$\psi^{\vartriangle}_{z^*}(A) = \inf_{a\in A} z^*(a) \quad \text{and} \quad \psi^{\triangledown}_{z^*}(A) = \sup_{a\in A} z^*(a)$$

which are functions $\psi^{\vartriangle}_{z^*}, \psi^{\triangledown}_{z^*}\colon \mathcal{G}(Z, C) \to \mathbb{R} \cup \{\pm\infty\}$.

Corollary 4.15 *If $z^* \in C^+\backslash\{0\}$, then $\psi^{\vartriangle}_{z^*}$ is a supermodular function on the complete lattice $(\mathcal{G}(Z, C), \supseteq)$, i.e.*

$$\psi^{\vartriangle}_{z^*}(A) + \psi^{\vartriangle}_{z^*}(B) \le \psi^{\vartriangle}_{z^*}(A \cap B) + \psi^{\vartriangle}_{z^*}(A \cup B).$$

Likewise, $\psi^{\triangledown}_{z^*}$ is a submodular function on $(\mathcal{G}(Z, C), \supseteq)$, i.e.

[4] Sophie Qingzhen Wang provided the hint to this observation.

$$\psi^{\nabla}_{z^*}(A) + \psi^{\nabla}_{z^*}(B) \geq \psi^{\nabla}_{z^*}(A \cap B) + \psi^{\nabla}_{z^*}(A \cup B).$$

Proof This follows from the definition of the functions $\psi^{\vartriangle}_{z^*}$, $\psi^{\nabla}_{z^*}$ and Lemma 4.14. □

The inf-residuation in $\mathcal{G}(Z, C)$ can also be represented via scalarization.

Proposition 4.16 *For all $A, B \in \mathcal{G}(Z, C)$,*

$$A \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} B = \bigcap_{z^* \in C^+ \setminus \{0\}} \left\{ z \in Z \mid \sigma^{\vartriangle}_A\left(z^*\right) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \sigma^{\vartriangle}_B\left(z^*\right) \leq z^*(z) \right\}.$$

In particular, if $A = \left\{ z \in Z \mid \sigma^{\vartriangle}_A(z^*) \leq z^*(z) \right\} \left(= A \oplus H^+(z^*)\right)$ for $z^* \in C^+ \setminus \{0\}$, then

$$A \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} B = \left\{ z \in Z \mid \sigma^{\vartriangle}_A\left(z^*\right) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \sigma^{\vartriangle}_B\left(z^*\right) \leq z^*(z) \right\}.$$

Moreover,

$$\forall z^* \in C^+ \setminus \{0\} \colon \sigma^{\vartriangle}_{A \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} B}\left(z^*\right) \geq \sigma^{\vartriangle}_A\left(z^*\right) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \sigma^{\vartriangle}_B\left(z^*\right)$$

with equality if $A = \left\{ z \in Z \mid \sigma^{\vartriangle}_A(z^*) \leq z^*(z) \right\} \left(= A \oplus H^+(z^*)\right)$.

Proof See [89, Proposition 5.20] while recalling $H^+(z^*) = \{z \in Z \mid z^*(z) \geq 0\}$ for $z^* \in Z^*$. □

The following result can be seen as a "$-inf = sup-$" rule for the inf-residuation in $\mathcal{G}(Z, C)$. It turns out to be useful later on.

Proposition 4.17 *Let $\mathcal{A} \subseteq \mathcal{G}(Z, C)$, $z^* \in C^+ \setminus \{0\}$ and $H^+(z^*) = \left\{ z \in Z \mid z^*(z) \geq 0 \right\}$. Then*

$$H^+(z^*) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \inf \mathcal{A} = \sup_{A \in \mathcal{A}} \left[H^+(z^*) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} A \right], \tag{4.13}$$

$$H^+(z^*) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \sup \mathcal{A} \supseteq \inf_{A \in \mathcal{A}} \left[H^+(z^*) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} A \right]. \tag{4.14}$$

If $A \oplus H^+(z^*) = A$ for all $A \in \mathcal{A}$ then (4.14) is satisfied as an equation.

Proof Formula (4.13) directly follows from

$$\begin{aligned} H^+(z^*) \mathbin{-\!\!\raisebox{0.5ex}{$\cdot$}} \inf \mathcal{A} &= \left\{ z \in Z \mid \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A + z \subseteq H^+(z^*) \right\} \\ &= \left\{ z \in Z \mid \forall A \in \mathcal{A} \colon A + z \subseteq H^+(z^*) \right\} = \bigcap_{A \in \mathcal{A}} \left\{ z \in Z \mid A + z \subseteq H^+(z^*) \right\}. \end{aligned}$$

The proof of (4.14) makes use of the fact $B_1 \subseteq B_2 \Leftrightarrow H^+(z^*) \mathrel{-\!\!\cdot} B_2 \subseteq H^+(z^*) \mathrel{-\!\!\cdot} B_1$. Applying it to $B_1 = \bigcap_{A \in \mathcal{A}} A$ and $B_2 = A$ we obtain (4.14). The equality case can be proven with the help of Lemma 4.14 and Proposition 4.16. □

A simple counterexample for equality in (4.14) is as follows: $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = (1,0)^T + \mathbb{R}^2_+$, $A_2 = (0,1)^T + \mathbb{R}^2_+$ and $z^* = (1,1)^T$. Both (4.13) and (4.14) are valid for more general sets than $H^+(z^*)$, but this is not needed in the following.

The previous results establish a one-to-one relationship between $\mathcal{G}(Z, C)$ and the set

$$\Gamma\left(Z^*, C^+\right) = \left\{\sigma \colon C^+ \to \overline{\mathbb{R}} \mid \ \sigma \text{ is superlinear and has a closed hypograph}\right\}.$$

On $\Gamma\left(Z^*, C^+\right)$, we consider the pointwise addition $\dot{+}$ and the pointwise multiplication with non-negative numbers. Finally, two elements of $\Gamma\left(Z^*, C^+\right)$ are compared pointwise, and we write $\sigma \le \gamma$ whenever

$$\forall z^* \in C^+ \colon \sigma\left(z^*\right) \le \gamma\left(z^*\right).$$

The one-to-one relationship includes the algebraic structure as well as the order structure.

Proposition 4.18 *The quadrupel* $\left(\Gamma\left(Z^*, C^+\right), \le, \dot{+}, \cdot\right)$ *is an inf-residuated conlinear space which is algebraically and order isomorphic to* $(\mathcal{G}(Z, C), \supseteq, \oplus, \odot)$.

Proof The formulas

$$\sigma_A^\vartriangle\left(z^*\right) = \inf_{a \in A} z^*(a), \quad A_\sigma^\vartriangle = \bigcap_{z^* \in C^+ \setminus \{0\}} \left\{z \in Z \mid \sigma\left(z^*\right) \le z^*(z)\right\}$$

and

$$\sigma_{A_\sigma^\vartriangle}^\vartriangle = \sigma, \quad A_{\sigma_A^\vartriangle}^\vartriangle = A \tag{4.15}$$

provide the relationship; the algebraic isomorphism is provided by

$$\sigma_{A \oplus B}^\vartriangle = \sigma_A^\vartriangle \dot{+} \sigma_B^\vartriangle, \quad A_\sigma^\vartriangle \oplus A_\gamma^\vartriangle = A_{\sigma \dot{+} \gamma}^\vartriangle$$

and for $t \ge 0$

$$\sigma_{tA}^\vartriangle = t\sigma_A^\vartriangle, \quad tA_\sigma^\vartriangle = A_{t\sigma}^\vartriangle;$$

the order isomorphism is provided by

$$A \supseteq B \quad \Leftrightarrow \quad \sigma_A^\vartriangle \le \sigma_B^\vartriangle$$

and (4.15). □

Corollary 4.19 *Let $\mathcal{A} \subseteq \mathcal{G}(Z, C)$. Then:*

(a) A set $\mathcal{B} \subseteq \mathcal{A}$ generates the infimum of $\mathcal{A}$ if, and only if,

$$\sigma^{\triangle}_{\inf\ \mathcal{B}} = \sigma^{\triangle}_{\inf\ \mathcal{A}}.$$

(b) $\bar{A} \in \mathcal{A}$ is minimal for $\mathcal{A}$ if, and only if, $\sigma^{\triangle}_{\bar{A}}$ is a minimal element of

$$\left\{\sigma^{\triangle}_{A} \mid A \in \mathcal{A}\right\}$$

with respect to the point-wise order in $\Gamma\left(Z^*, C^+\right)$,

Proof This is an obvious consequence of the previous results. □

One may think that this straightforward result reduces $\mathcal{G}(Z, C)$-valued (= set-valued) problems to vector optimization problem since the functions $\sigma^{\triangle}_{A}$ could be considered as elements of some function space with point-wise order. Such an approach can be found in [115]. The problem with this point of view is that the functions $\sigma^{\triangle}_{A}$ may attain (and frequently do) the values $-\infty$ and/or $+\infty$. Therefore, the difficulty is conserved by passing from $\mathcal{G}(Z, C)$ to $\Gamma\left(Z^*, C^+\right)$ since the latter is an ordered conlinear space which, in general, cannot be embedded into a linear space of functions.

We turn the above ideas into a scalarization concept for set-valued functions. Let X be a topological linear space and $f \colon X \to \mathcal{P}(Z)$, $z^* \in C^+$ be given. Define an extended real-valued function $\varphi_{f,z^*} \colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi_{f,z^*}(x) = \sigma^{\triangle}_{f(x)}\left(z^*\right) = \inf_{z \in f(x)} z^*(z). \tag{4.16}$$

The new symbol φ_{f,z^*} is justified by the fact that we want to emphasize the dependence on x rather than on z^*. From (4.7) we obtain the following "setification" formula: If $f \colon X \to \mathcal{G}(Z, C)$ then

$$\forall x \in X \colon f(x) = \bigcap_{z^* \in C^+ \setminus \{0\}} \left\{z \in Z \mid \varphi_{f,z^*}(x) \leq z^*(z)\right\}. \tag{4.17}$$

Several important properties of $\mathcal{G}(Z, C)$-valued functions can equivalently be expressed using the family of its scalarizations $\left\{\varphi_{f,z^*}\right\}_{z^* \in C^+ \setminus \{0\}}$. One may say that, according to formula (4.16), a $\mathcal{G}(Z, C)$-valued function is, as a mathematical object, equivalent to this family of extended real-valued functions.

Topological properties like closedness pose difficulties in this context since scalarizations of a closed $\mathcal{G}(Z, C)$-valued function are not necessarily closed. A simple example is as follows: The function $f \colon \mathbb{R} \to \mathcal{G}(\mathbb{R}^2, \mathbb{R}^2_+)$ defined by $f(x) = \left\{\left(\frac{1}{x}, 0\right)^T\right\} + \mathbb{R}^2_+$ for $x > 0$ and $f(x) = \emptyset$ for $x \leq 0$ is closed and convex, but φ_{f,z^*} for $z^* = (0, 1)^T$ is convex, but not closed. Below, we will deal with this issue.

Lemma 4.20 *Let* $f : X \to \mathcal{G}(Z, C)$ *be a function. Then:*

(a) f *is convex if, and only if,* $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ *is convex for all* $z^* \in C^+\backslash\{0\}$.
(b) f *is positively homogeneous if, and only if,* $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ *is positively homogeneous for all* $z^* \in C^+\backslash\{0\}$.
(c) f *is (sub)additive if, and only if,* $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ *is (sub)additive for all* $z^* \in C^+\backslash\{0\}$.
(d) f *is proper if, and only if, there is* $z^* \in C^+\backslash\{0\}$ *such that* $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ *is proper.*
(e) $\operatorname{dom} f = \operatorname{dom} \varphi_{f,z^*}$ *for all* $z^* \in C^+\backslash\{0\}$.

Proof (a) "⇒" Take $t \in (0, 1)$, $x, y \in X$ and $z^* \in C^+\backslash\{0\}$. Then

$$\begin{aligned}\varphi_{f,z^*}(tx + (1-t)\,y) &= \inf_{z \in f(tx+(1-t)y)} z^*(z) \le \inf_{z \in tf(tx)+(1-t)f(y)} z^*(z) \\ &= \inf_{u \in tf(x)} z^*(u) + \inf_{v \in (1-t)f(y)} z^*(v) \\ &= t \inf_{\frac{u}{t} \in f(x)} \frac{z^*(u)}{t} + (1-t) \inf_{\frac{v}{(1-t)} \in f(y)} \frac{z^*(v)}{(1-t)} \\ &= t\varphi_{f,z^*}(x) + (1-t)\,\varphi_{f,z^*}(y)\end{aligned}$$

where the inequality is a consequence of the convexity of f.

"⇐" By the way of contradiction, assume that f is not convex. Then there are $t \in (0, 1)$, $x, y \in X$, $z \in Z$ satisfying

$$z \in tf(x) + (1-t)\,f(y), \quad z \notin f(tx + (1-t)\,y).$$

Since the values of f are closed convex sets we can apply a separation theorem and obtain $z^* \in C^+\backslash\{0\}$ such that

$$z^*(z) < \varphi_{f,z^*}(tx + (1-t)\,y) \le t\varphi_{f,z^*}(x) + (1-t)\,\varphi_{f,z^*}(y)$$

where the second inequality is a consequence of the convexity of the scalarizations. Since f maps into $\mathcal{G}(Z, C)$, $z^* \in C^+\ \{0\}$. Since $z \in tf(x) + (1-t)\,f(y)$ there are $u \in f(x)$ and $v \in f(y)$ such that $z = tu + (1-t)v$. Hence

$$z^*(z) = tz^*(u) + (1-t)z^*(v) \ge t\varphi_{f,z^*}(x) + (1-t)\varphi_{f,z^*}(y)$$

by definition of the scalarization. This contradicts the strict inequality above.

(c) If f is (sub)additive, then (sub)additivity of the scalarizations φ_{f,z^*} follows from (4.8). The converse can be proven using the same separation idea as in the proof of (a).

The remaining claims are straightforward. □

Finally, we link closedness and semicontinuity of $\mathcal{G}(Z, C)$-valued functions to corresponding properties of their scalarizations. The main result is Theorem 4.22

below which shows that a proper closed and convex set-valued function and the family of its proper closed and convex scalarizations are equivalent as mathematical objects. We start with a characterization of the lattice-limit inferior in terms of scalarizations.

Corollary 4.21 *Let $f\colon X \to \mathcal{G}(Z,C)$ and $\bar{x} \in \operatorname{dom} f$ such that f is lattice-l.s.c. at $\bar{x}$. Then*

$$\liminf_{x\to\bar{x}} f(x) = \left\{z \in Z \mid \forall z^* \in C^+\backslash\{0\}\colon\ \liminf_{x\to\bar{x}} \varphi_{f,z^*}(x) \leq z^*(z)\right\}.$$

Proof Observing $\varphi_{f,z^*}(x) = \sigma^{\Delta}_{f(x)}(z^*)$ for each $x \in X$ and applying Lemma 4.14 we obtain

$$\begin{aligned}
\sup_{U\in\mathcal{N}_X}\inf_{x\in\bar{x}+U} f(x) &= \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \sup_{U\in\mathcal{U}} \sigma^{\Delta}_{\inf\limits_{x\in\bar{x}+U} f(x)}(z^*) \leq z^*(z)\right\} \\
&= \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \sup_{U\in\mathcal{U}}\inf_{x\in\bar{x}+U} \sigma^{\Delta}_{f(x)}(z^*) \leq z^*(z)\right\} \\
&= \bigcap_{z^*\in C^+\backslash\{0\}} \left\{z \in Z \mid \liminf_{x\to\bar{x}} \varphi_{f,z^*}(x) \leq z^*(z)\right\}.
\end{aligned}$$

Indeed, the first equality follows from the last equation in Lemma 4.14 applied to $\mathcal{A} = \{\inf_{x\in\bar{x}+U} f(x) \mid U \in \mathcal{U}\}$ whereas the second follows from the first equation in Lemma 4.14 applied to $\mathcal{A} = \{f(x) \mid x \in \bar{x}+U\}$ for $U \in \mathcal{U}$. □

Theorem 4.22 *Let $f\colon X \to \mathcal{F}(Z,C)$ be a function and $\operatorname{dom} f \neq \emptyset$. Then f is closed, convex and either constant Z or proper, if and only if,*

$$\forall x \in X\colon\ f(x) = \bigcap_{\substack{z^*\in C^+\backslash\{0\} \\ \operatorname{cl}\operatorname{co}\varphi_{f,z^*}\colon X\to\overline{\mathbb{R}} \text{ is proper}}} \left\{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{f,z^*}(x) \leq z^*(z)\right\}, \tag{4.18}$$

where $\operatorname{cl}\operatorname{co}\varphi_{f,z^*}$ denotes the lower semi-continuous convex hull of φ_{f,z^*} defined by

$$\operatorname{epi}\left(\operatorname{cl}\operatorname{co}\varphi_{f,z^*}\right) = \operatorname{cl}\operatorname{co}\left(\operatorname{epi}\varphi_{f,z^*}\right).$$

Proof If the set $\left\{z^* \in C^+\backslash\{0\} \mid \operatorname{cl}\operatorname{co}\varphi_{f,z^*}\colon X \to \overline{\mathbb{R}} \text{ is proper}\right\}$ is empty, then (4.18) produces $f(x) = Z$ for all $x \in X$ since $\operatorname{dom} f \neq \emptyset$. On the other hand, $f(x) = Z$ for all $x \in X$ implies the emptyness of the same set, hence (4.18) is satisfied in this case.

The graphs of $x \mapsto \left\{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{f,z^*}(x) \leq z^*(z)\right\}$ are closed convex sets in $X \times Z$, and $\left\{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{f,z^*}(x) \leq z^*(z)\right\} \neq Z$ for all $x \in X$ is true whenever $\operatorname{cl}\operatorname{co}\varphi_{f,z^*}$ is proper. Thus, (4.18) implies f is closed, convex and either proper or constantly equal to Z.

On the other hand, assume f is closed, convex and proper. Then

$$\forall x \in X\colon\ f(x) = \liminf_{y \to x} f(y) \neq Z,$$

and all scalarizations are convex. Corollary 4.21 yields

$$f(x) = \sup_{U \in \mathcal{U}} \inf_{x \in \bar{x}+U} f(x) = \bigcap_{z^* \in C^+ \backslash \{0\}} \left\{ z \in Z \mid \operatorname{cl} \operatorname{co} \varphi_{f,z^*}(x) \leq z^*(z) \right\}.$$

If $\operatorname{cl}\operatorname{co}\varphi_{f,z^*}$ is improper, then $\left\{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{f,z^*}(x) \leq z^*(z)\right\} = Z$ for all $x \in \operatorname{dom} f = \operatorname{dom} \varphi_{f,z^*}$ (see [235, Proposition 2.2.5]), hence these scalarizations can be omitted from the intersection. This completes the proof. □

We state a few more facts about relationships between semicontinuity properties of set-valued functions and their scalarizations.

Proposition 4.23 *(a) If $f\colon X \to \mathcal{F}(Z, C)$ is lattice-u.s.c. at $\bar{x} \in X$, then $\varphi_{f,z^*}\colon X \to \overline{\mathrm{IR}}$ is u.s.c. at $\bar{x}$ for all $z^* \in C^+\backslash\{0\}$.*

(b) If $f\colon X \to \mathcal{G}(Z, C)$ is such that $\varphi_{f,z^*}\colon X \to \overline{\mathrm{IR}}$ is l.s.c. at $\bar{x} \in X$ for all $z^* \in C^+\backslash\{0\}$, then f is lattice-l.s.c. at $\bar{x}$.

Proof (a) Define $A(\bar{x}) = \limsup_{x \to \bar{x}} f(x) = \operatorname{cl} \bigcup_{U \in \mathcal{N}_X} \bigcap_{x \in \bar{x}+U} f(x)$ and take $z^* \in C^+\backslash\{0\}$. By assumption,

$$\varphi_{f,z^*}(\bar{x}) \geq \sigma^{\vartriangle}_{A(\bar{x})}\left(z^*\right).$$

By a successive application of the first and the second relation of Lemma 4.14,

$$\sigma^{\vartriangle}_{A(\bar{x})}\left(z^*\right) \geq \inf_{U \in \mathcal{N}_X} \sup_{x \in \bar{x}+U} \varphi_{f,z^*}(x).$$

This verifies the upper semicontinuity of the scalarizations.

(b) From Corollary 4.21, the lower semicontinuity of the φ_{f,z^*}'s and (4.17) we obtain

$$\begin{aligned}\liminf_{x \to \bar{x}} f(x) &= \bigcap_{z^* \in C^+ \backslash \{0\}} \left\{ z \in Z \mid \liminf_{x \to \bar{x}} \varphi_{f,z^*}(x) \leq z^*(z) \right\} \\ &\subseteq \bigcap_{z^* \in C^+ \backslash \{0\}} \left\{ z \in Z \mid \varphi_{f,z^*}(\bar{x}) \leq z^*(z) \right\} = f(\bar{x})\end{aligned}$$

which means that f is lattice-l.s.c. at $\bar{x}$.

□

Corollary 4.24 *If $f\colon X \to \mathcal{F}(Z, C)$ is convex and lattice-u.s.c. at $\bar{x} \in \operatorname{dom} f$, then each scalarization φ_{f,z^*} is continuous at $\bar{x}$ and f is lattice-l.s.c. at $\bar{x}$. Moreover, in this case f also is lattice-u.s.c. and -l.s.c. at $\bar{x}$ as a function into $\mathcal{G}(Z, C)$.*

Proof By Proposition 4.23 (a), φ_{f,z^*} is u.s.c. at $\bar{x}$ for each $z^* \in C^+\backslash\{0\}$ (and also convex by Lemma 4.20 (a)). Well-known results about extended real-valued convex functions [235, Theorem 2.2.9] imply that φ_{f,z^*} for each $z^* \in C^+\backslash\{0\}$ is continuous which in turn yields that f is lattice-l.s.c. at $\bar{x}$ by 4.23 (b). The last claim follows from Proposition 4.13. □

Corollary 4.25 *Let* $f : X \to \mathcal{G}(Z, C)$ *be a convex function and* $\bar{x} \in X$ *such that there exists a* $\bar{z} \in Z$ *with* $(\bar{x}, \bar{z}) \in \operatorname{int}(\operatorname{graph} f)$. *Then* $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ *is continuous on* $\emptyset \neq \operatorname{int}(\operatorname{dom} f)$ *for all* $z^* \in C^+\backslash\{0\}$.

Proof If $(\bar{x}, \bar{z}) \in \operatorname{int}(\operatorname{graph} f)$ then φ_{f,z^*} is bounded from above by $z^*(\bar{z})$ on a neighborhood of $\bar{x}$, thus continuous on $\emptyset \neq \operatorname{int}(\operatorname{dom} f)$ for all $z^* \in C^+\backslash\{0\}$ again by [235, Theorem 2.2.9]. □

4.3 Comments on Convexity, Semicontinuity and Scalarization

The properties which are called lattice-lower and lattice-upper semicontinuity can already be found in the 1978 paper [142]. Note that in this survey, for obvious reasons, 'upper' and 'lower' are swapped compared to [142]. Therein, the result of Proposition 4.9 is even referenced to a paper by Choquet from 1947.

Level-closedness features in [54, 55] as '*D*-lower semi-continuity' and '*C*-lower semi-continuity', respectively: Proposition 2.3 in [54] states the equivalence of (epi)closedness and level-closedness whenever the cone has a non-empty interior. The assumption "pointedness of the cone" and a compactness assumption used in [54] are not necessary, the latter already removed in [55, Proposition 3.1]. Compare also [176].

Of course, the lattice semicontinuity concepts of this survey differ from the definitions of lower and upper semicontinuity as used, for example, in [4, Definitions 1.4.1 and 1.4.2]. This is one reason why lower and upper continuity replace lower and upper semicontinuity, respectively, in [67]. We refer to Sect. 2.5 of [67] for a survey about continuity concepts of set-valued functions and also a few bibliographical remarks at the end of the section.

For a more detailed discussion of (semi)continuity concepts for set-valued functions, compare [102, 104, 151]: Whereas Corollary 4.21 seems to be new in this form, Proposition 4.23 appears in [104] with a (slightly) different proof.

The scalarization approach via (4.16) (and Lemma 4.20) has many contributors. Motivated by economic applications, Shephard used it in [209], compare, for example, the definition of the 'factor minimal cost function' [209, p. 226, (97)] and Proposition 72 on the following page where essentially Lemma 4.20 (a) is stated. Moreover, the first part of Proposition 4.2 corresponds to [209, Appendix 2, Proposition 3]. Rockafellar baptized these functions *Kuhn-Tucker functions* in his 1967 monograph [195, Definition 2 on p. 17] where they were used as an auxiliary tool for representing the then new "convex processes." Compare also the relationship to the orders $\supseteq$ for sets of 'convex type' and $\subseteq$ for sets of 'concave type,' [195, p. 16].

Pshenichnyi [194, Lemma 1] also used the functions φ_{f,z^*} and proved Lemma 4.20 (a), see also [12]. Another reference is [45, Proposition 1.6]. In [169] as well as in [170] continuity concepts for set-valued functions are discussed using the φ_{f,z^*}-functions as essential tool. See also [9, Proposition 2.1] and the more recent [68, p. 188] (see also the references therein).

Theorem 4.22 has been established in [207, 208] and is the basis for the scalarization approach to convex duality results for set-valued functions. Together with the "setification" formula (4.17) it basically tells us that one can either deal with the $\mathcal{G}(Z, C)$-valued function or a whole family of scalar functions, and both approaches are equivalent in the sense that major (convex duality) results can be expressed and proven either way: Using the "set calculus" or "scalarizations." The reader may compare the two different proofs for the Lagrange duality theorem in [86].

Finally, we mention that an alternative scalarization approach to (convex as well as non-convex) problems is based on directionally translative extended real-valued functions which are used in many areas of mathematics and prominently in vector optimization, see [67, Sect. 2.3]. To the best of our knowledge, [83] (eventually published as [85]) was the first generalization to set-valued problems. Subsequent applications of this construction include [2, 72, 96, 164, 181–183, 229].

5 Set-Valued Convex Analysis

What is convex analysis? A core content of this theory could be described as follows: Define affine minorants, directional derivatives, (Fenchel) conjugates and subdifferentials for convex functions and relate them by means of a Fenchel-Moreau type theorem, a max-formula and Young-Fenchel's inequality as an equation. How can one establish such a theory for set-valued convex functions? In this section, we will define appropriate "dual variables" for the set-valued framework, define "affine minorants" of set-valued functions and introduce corresponding Fenchel conjugates, directional derivatives and subdifferentials. The difference in expressions involved in these constructions for scalar functions will be replaced by a residuation.

In the following, we assume that X and Z are non-trivial, locally convex, topological linear spaces with topological duals X^* and Z^*, respectively. As before, $C \subseteq Z$ is a convex cone with $0 \in C$, and $C^+ = \{z^* \in Z^* \mid \forall z \in C : z^*(z) \geq 0\}$ is its positive (topological) dual.

5.1 Conlinear Functions

What is an appropriate replacement for the dual variables $x^* : X \to \mathrm{I\!R}$ in scalar convex analysis? A good guess might be to use linear operators $T : X \to Z$ instead of linear functionals in expressions like

$$f^*\left(x^*\right) = \sup_{x \in X} \left\{x^*\left(x\right) - f\left(x\right)\right\}.$$

This has been done in most references about duality for vector/set optimization problems. A notable exception is the definition of the coderivative of set-valued functions due to B. S. Mordukhovich which goes back to [171] and can be found in [172, Sect. 2]. Coderivatives at points of the graph are defined as sets of x^*'s depending on an element $z^* \in Z^*$. Another exception is the use of "rank one" operators of the form $\hat{z}x^*$ whose existence can be proven using classical separation results, compare [32, Proof of Theorem 4.1] and [98, Theorem 4.1] for an older and a more recent example. The constructions in [231] are also based on this idea.

Another attempt to find set-valued analogues of linear functions is the theory of convex processes. See [4, p. 55] in which the authors state that 'it is quite natural to regard set-valued maps, with closed convex cones as their graphs, as these set-valued analogues.'

In our approach, a class of set-valued functions will be utilized the members of which almost behave like linear functions. In some sense (see Proposition 8 in [78]), they are more general than linear operators and also than linear processes as defined in [4, p. 55], and on the other hand, they form a particular class of convex processes. In fact, these functions are characterized by the fact that their graphs are homogeneous closed half spaces in $X \times Z$.

Let $x^* \in X^*$ and $z^* \in Z^*$ be given. Define a function $S_{(x^*,z^*)} \colon X \to \mathcal{P}(Z)$ by

$$S_{(x^*,z^*)}\left(x\right) = \left\{z \in Z \mid x^*\left(x\right) \le z^*\left(z\right)\right\}.$$

The next result shows that these functions are indeed as "linear" as one can hope for.

Proposition 5.1 *Let* $(x^*, z^*) \in X^* \times Z^*\backslash\{0\}$. *Then*

(a) for all $x \in X$ *and for all* $t > 0$

$$S_{(x^*,z^*)}\left(tx\right) = tS_{(x^*,z^*)}\left(x\right);$$

(b) for all $x_1, x_2 \in X$

$$S_{(x^*,z^*)}\left(x_1 + x_2\right) = S_{(x^*,z^*)}\left(x_1\right) + S_{(x^*,z^*)}\left(x_2\right),$$

in particular

$$S_{(x^*,z^*)}\left(x\right) + S_{(x^*,z^*)}\left(-x\right) = S_{(x^*,z^*)}\left(0\right) = H^+(z^*);$$

(c) $S_{(x^*,z^*)}$ *maps into* $\mathcal{G}(Z, C)$, *hence in particular into* $\mathcal{P}(Z, C)$, *if, and only if,* $z^* \in C^+$;

(d) $S_{(x^*,z^*)}\left(x\right)$ *is a closed half space with normal* z^* *if, and only if,* $z^* \neq 0$; *and* $S_{(x^*,0)}\left(x\right) \in \{Z, \emptyset\}$;

(e) if $\widehat{z} \in Z$ *such that* $z^*\left(\widehat{z}\right) = -1$ *then*

$$\forall x \in X\colon S_{(x^*,z^*)}(x) = x^*(x)\widehat{z} + S_{(x^*,z^*)}(0) = x^*(x)\widehat{z} + H^+(z^*). \qquad (5.1)$$

Proof Elementary, see, for instance, [78]. □

A function of the type $S_{(x^*,z^*)}$ is called *conlinear*. It will turn out that convex analysis is a "conlinear" theory–not because convex functions are not linear, but because the image space of a convex function is a conlinear space and all properties of linear functions necessary for the theory are only the "conlinear" ones from the previous proposition. The following result gives a characterization of the class of conlinear functions in the class of all positively homogeneous and additive set-valued functions.

Theorem 5.2 *Let $f\colon X \to \mathcal{G}(Z, C)$ be a function. Then, the following are equivalent:*

(a) $\exists\,(x^*, z^*) \in X^* \times C^+\backslash\{0\}$, $\forall x \in X$*:* $f(x) = S_{(x^*,z^*)}(x)$.
(b) graph f *is a closed homogeneous half-space of* $X \times Z$ *and* $f(0) \neq Z$.
(c) f *is positively homogeneous, additive, lattice-l.s.c. at* $0 \in X$ *and* $f(0) \subseteq Z$ *is a non-trivial, closed homogeneous half-space.*

Proof (a) ⇒ (b), (c): Straightforward.

(b) ⇒ (a): graph f is a closed homogenous half-space if, and only if,

$$\exists\left(x^*, z^*\right) \in X^* \times Z^*\backslash\{(0,0)\}\colon \text{graph}\, F = \left\{(x,z) \in X \times Z \mid x^*(x) - z^*(z) \leq 0\right\}.$$

This implies

$$\forall x \in X\colon f(x) = \left\{z \in Z \mid x^*(x) \leq z^*(z)\right\} = S_{(x^*,z^*)}(x).$$

Since $f(0) \neq Z$ and f maps into $\mathcal{G}(Z, C)$, $z^* \in C^+\backslash\{0\}$. By Proposition 5.1 (b), f is additive.

(c) ⇒ (a): By assumption, $f(0) = H^+(z_0^*) = \left\{z \in Z \mid z_0^*(z) \geq 0\right\}$ for some $z_0^* \in C^+\backslash\{0\}$. By additivity, $f(0) = H^+(z_0^*) = f(x) \oplus f(-x)$ for all $x \in X$, hence $f(x)$ is never $\emptyset$ nor Z. Moreover, additivity implies $f(x) = f(x+0) = f(x) \oplus f(0) = f(x) \oplus H^+(z_0^*)$ for each $x \in X$. This means that every value $f(x)$ is a closed half space with normal z_0^*.

Next, we use (4.17) which reads

$$\forall x \in X\colon f(x) = \bigcap_{z^* \in C^+\backslash\{0\}} \left\{z \in Z \mid \varphi_{f,z^*}(x) \leq z^*(z)\right\}.$$

Since every value $f(x)$ is a half space with normal z_0^* the intersection in the above formula can be replaced just by $\left\{z \in Z \mid \varphi_{f,z_0^*}(x) \leq z_0^*(z)\right\}$.

We shall show that φ_{f,z_0^*} is linear. By Lemma 4.20 (b) and (c) it is additive because f is additive, and $\varphi_{f,z_0^*}(tx) = t\varphi_{f,z^*}$ for $t \geq 0$, so it remains to show this for $t < 0$ in order to prove homogeneity. Indeed,

$$0 = \varphi_{f,z_0^*}(0) = \inf_{z \in f(x) \oplus f(-x)} z_0^*(z) = \inf_{z_1 \in f(x)} z_0^*(z_1) + \inf_{z_2 \in f(-x)} z_0^*(z_2) = \varphi_{f,z_0^*}(x) + \varphi_{f,z_0^*}(-x),$$

which gives us

$$\forall t < 0 \colon \varphi_{f,z_0^*}(tx) = \varphi_{f,z_0^*}(-|t|x) = |t|\, \varphi_{f,z_0^*}(-x) = -\,|t|\, \varphi_{f,z_0^*}(x) = t\varphi_{f,z_0^*}(x).$$

Therefore, φ_{f,z_0^*} is a linear function and can be identified with some $x' \in X'$, the algebraic dual of X. Since f is lower semicontinuous at $0 \in X$, Corollary 4.21 with $\bar{x} = 0$ yields

$$\liminf_{x \to 0} f(x) = \left\{ z \in Z \mid \forall z^* \in C^+ \backslash \{0\} \colon \liminf_{x \to 0} x'(x) \leq z^*(z) \right\}.$$

If x' is not continuous then it is not bounded (from below) on every neighborhood $U \in \mathcal{N}_X$. Thus,

$$\forall U \in \mathcal{N}_X \colon \inf_{x \in U} x'(x) = -\infty,$$

hence

$$\liminf_{x \to 0} x'(x) = \sup_{U \in \mathcal{U}} \inf_{x \in U} x'(x) = -\infty$$

and consequently $Z = \liminf_{x \to 0} f(x)$ which contradicts $f(0) = H^+(z_0^*) \supseteq \liminf_{x \to 0} f(x)$. Hence, there is $x^* \in X^*$ such that $x^*(x) = \varphi_{f,z_0^*}(x)$ for all $x \in X$. □

The basic idea for the development of a set-valued convex analysis simply is as follows: Replace the extended reals by $\mathcal{G}(Z, C)$, $\leq$ by $\supseteq$, use the inf/sup-formulas from Proposition 2.2, replace continuous linear functionals by conlinear functions and the difference by inf-residuation. We start the program with conjugates.

5.2 *Fenchel Conjugates of Set-Valued Functions*

A crucial observation concerning Fenchel conjugates for extended real-valued functions $\varphi \colon X \to \mathbb{R} \cup \{\pm\infty\}$ is as follows:

$$r \geq \varphi^*\left(x^*\right) \quad \Leftrightarrow \quad \forall x \in X \colon x^*(x) - r \leq \varphi(x).$$

This means, x^* belongs to the domain of φ^* precisely if there is an affine minorant of φ with "slope" x^*. Replacing x^* by $S_{(x^*,z^*)}$, $\leq$ by $\supseteq$ and recalling (2.5) we obtain

$$\begin{aligned}\forall x \in X\colon S_{(x^*,z^*)}(x) - z \supseteq f(x) \quad &\Leftrightarrow \quad \forall x \in X\colon f(x) + z \subseteq S_{(x^*,z^*)}(x) \\ &\Leftrightarrow \quad \forall x \in X\colon z \in S_{(x^*,z^*)}(x) \dot{-} f(x) \\ &\Leftrightarrow \quad z \in \bigcap_{x\in X} \left\{S_{(x^*,z^*)}(x) \dot{-} f(x)\right\}.\end{aligned}$$

The function $x \mapsto S_{(x^*,z^*)}(x) - z$ is called an affine minorant of f precisely if the above (equivalent) conditions are satisfied. This discussion may justify the following definition.

Definition 5.3 The Fenchel conjugate of the function $f\colon X \to \mathcal{P}(Z, C)$ is $f^*\colon X^* \times C^+\backslash\{0\} \to \mathcal{P}(Z, C)$ defined by

$$f^*\left(x^*, z^*\right) = \sup_{x\in X}\left\{S_{(x^*,z^*)}(x) \dot{-} f(x)\right\} = \bigcap_{x\in X}\left\{S_{(x^*,z^*)}(x) \dot{-} f(x)\right\}.$$

The biconjugate of f is $f^{**}\colon X \to \mathcal{P}(Z, C)$ defined by

$$\begin{aligned}f^{**}(x) &= \sup_{x^*\in X^*,\, z^*\in C^+\backslash\{0\}}\left\{S_{(x^*,z^*)}(x) \dot{-} f^*\left(x^*, z^*\right)\right\} \\ &= \bigcap_{x^*\in X^*,\, z^*\in C^+\backslash\{0\}}\left(S_{(x^*,z^*)}(x) \dot{-} f^*\left(x^*, z^*\right)\right).\end{aligned}$$

The Fenchel conjugate defined above shares most properties with her scalar little sister.

Proposition 5.4 *Let $f, g\colon X \to \mathcal{P}(Z, C)$ be two functions. Then*

(a) $f \supseteq g \Rightarrow g^* \supseteq f^*$.
(b) *f^* maps into $\mathcal{G}(Z, C)$, and each value of f^* is a closed half space with normal z^*, or $\emptyset$, or Z.*
(c) *$f^{**} \supseteq f$ and f^{**} is a proper closed convex function into $\mathcal{G}(Z, C)$, or $\equiv Z$, or $\equiv \emptyset$.*
(d) $(f^{**})^* = f^*$.
(e) *For all $x \in X$, $x^* \in X^*$, $z^* \in C^+\backslash\{0\}$,*

$$f^*\left(x^*, z^*\right) \subseteq S_{(x^*,z^*)}(x) \dot{-} f(x) \quad \Leftrightarrow \quad f^*\left(x^*, z^*\right) + f(x) \subseteq S_{(x^*,z^*)}(x).$$

Proof The equivalence in (e) follows from the definition of $\dot{-}$. The other relationships can be found in [78, 207, 208]. □

Remark 5.5 In [78], the "negative conjugate"

$$(-f^*)\left(x^*, z^*\right) = \inf_{x\in X}\left\{f(x) \oplus S_{(x^*,z^*)}(-x)\right\} = \text{cl}\bigcup_{x\in X}\left\{f(x) \oplus S_{(x^*,z^*)}(-x)\right\}$$

has been introduced which avoids the residuation. The transition from f^* to $-f^*$ and vice versa can be done via

$$(-f^*)\left(x^*, z^*\right) = H^+\left(z^*\right) \mathbin{-^{\!\centerdot}} f^*\left(x^*, z^*\right), \quad f^*\left(x^*, z^*\right) = H^+\left(z^*\right) \mathbin{-^{\!\centerdot}} (-f^*)\left(x^*, z^*\right)$$

using Proposition 4.17. Sometimes, it even seems to be more natural to work with $-f^*$, for example, when it comes to Fenchel-Rockafellar duality results as presented in [79].

Set-valued conjugates can be expressed using the (scalar) conjugates of the scalarizing functions.

Lemma 5.6 *If $f \colon X \to \mathcal{P}(Z, C)$, then*

$$\forall x^* \in X^*,\ \forall z^* \in C^+\backslash\{0\} \colon f^*\left(x^*, z^*\right) = \left\{z \in Z \mid \left(\varphi_{f,z^*}\right)^*\left(x^*\right) \le z^*(z)\right\}, \quad (5.2)$$

$$\forall x \in X \colon f^{**}(x) = \bigcap_{z^* \in C^+\backslash\{0\}} \left\{z \in Z \mid \left(\varphi_{f,z^*}\right)^{**}(x) \le z^*(z)\right\}. \quad (5.3)$$

Proof The first formula is a consequence of the definitions, the second follows from $\left(\varphi_{f,z^*}\right)^{**} = \left(\varphi_{f^{**},z^*}\right)^{**}$ and Theorem 4.22. □

Remark 5.7 Conversely, $\varphi_{f^*(\cdot,z^*),z^*} = \left(\varphi_{f,z^*}\right)^*$ is true (see [208, Proposition 4.2] and [86, Lemma 5.1]. On the other hand, φ_{f^{**},z^*} does not always coincide with $\left(\varphi_{f,z^*}\right)^{**}$ since the latter is a closed function which is not true for the former even if f is proper closed convex (see the example before Lemma 4.20).

The following result is a set-valued version of the famous Fenchel-Moreau theorem. Note that the additional dual variable z^* disappears via the definition of the biconjugate.

Theorem 5.8 *Let $f \colon X \to \mathcal{P}(Z, C)$ be a function. Then $f = f^{**}$ if, and only if, f is proper closed and convex, or identically Z, or identically $\emptyset$.*

Proof This follows from Theorem 4.22, Lemma 5.6 and the classical Fenchel-Moreau theorem for scalar functions, see, for example, [235, Theorem 2.3.3]. □

Remark 5.9 Another, more direct way to prove Theorem 5.8 consists in applying the basic convex duality relationship 'every closed convex set is the intersection of closed half spaces containing it' to the graph of f (such half spaces are generated by pairs $(x^*, z^*) \in X^* \times C^+$), making sure that one can do without $z^* = 0$ and converting the result into formulas involving the $S_{(x^*,z^*)}$-functions. In this way, the scalar Fenchel-Moreau theorem is obtained as a special case. See [78] for details.

To conclude this section, we point out that the Fenchel conjugate does not distinct between a function $f \colon X \to \mathcal{P}(Z, C)$ and the function

$$\tilde{f}(x) = \operatorname{cl}\operatorname{co} f(x);$$

we have $\tilde{f}^* = f^*$ since (compare Proposition 2.9)

$$\forall x \in X\colon S_{(x^*,z^*)}(x) \mathbin{-^{\!\cdot}} f(x) = \left\{z \in Z \mid f(x) + z \subseteq S_{(x^*,z^*)}(x)\right\}$$
$$= \left\{z \in Z \mid \mathrm{cl\,co}\, f(x) + z \subseteq S_{(x^*,z^*)}(x)\right\} = S_{(x^*,z^*)}(x) \mathbin{-^{\!\cdot}} \tilde{f}(x).$$

The function $\tilde{f}$ maps into $\mathcal{G}(Z, C)$. The above relationship means that when it comes to Fenchel conjugates it does not make a difference to start with a $\mathcal{G}(Z, C)$-valued function.

Under additional assumptions, the formulas for (bi)conjugates can be simplified. One such assumption is as follows: There is an element $\hat{z} \in C\backslash\{0\}$ such that

$$\forall z^* \in C^+\backslash\{0\}\colon z^*\left(\hat{z}\right) > 0.$$

In this case, the set $B^+(\hat{z}) = \left\{z^* \in C^+ \mid z^*\left(\hat{z}\right) = 1\right\}$ is a base of C^+ with $0 \notin \mathrm{cl}\, B^+(\hat{z})$. That is, for each $z^* \in C^+\backslash\{0\}$ there is a unique representation $z^* = tz_0^*$ with $t > 0$ and $z_0^* \in B^+(\hat{z})$. Compare [67], Definition 2.1.14, Theorems 2.1.15 and 2.2.12 applied to C^+ instead of C. Clearly, a pointed closed convex cone with non-empty interior has a base, but, for example, the cone L^2_+ has an empty interior, but a base is generated by the constant 1 function.

The very definition of the functions $S_{(x^*,z^*)}$ gives

$$\left\{S_{(x^*,z^*)} \mid x^* \in X^*,\ z^* \in C^+\backslash\{0\}\right\} = \left\{S_{(x^*,z^*)} \mid x^* \in X^*,\ z^* \in B^+(\hat{z})\right\}.$$

Therefore, it is sufficient to run an intersection like in the definition of f^{**} over $x^* \in X^*$ and $z^* \in B^+(\hat{z})$. Moreover, one easily checks (see also Proposition 5.1 (e)) for $z^* \in B^+(\hat{z})$

$$\forall x \in X\colon S_{(x^*,z^*)}(x) = \left\{x^*(x)\,\hat{z}\right\} + H^+(z^*).$$

Thus, the negative conjugate of a function $f\colon X \to \mathcal{P}(Z, C)$ can be written as

$$(-f^*)\left(x^*, z^*\right) = \mathrm{cl} \bigcup_{x \in X} \left[f(x) - x^*(x)\,\hat{z} + H^+(z^*)\right] = \mathrm{cl} \bigcup_{x \in X} \left[f(x) - x^*(x)\,\hat{z}\right] \oplus H^+(z^*).$$

The part which does not depend on z^* (remember $\hat{z}$ defines a base of C^+ and is the same for all $z^* \in C^+\backslash\{0\}$) has been used in [150, 155] for the definition of another set-valued conjugate, namely

$$(-f^*_{\hat{z}})\left(x^*\right) = \mathrm{cl} \bigcup_{x \in X} \left[f(x) - x^*(x)\,\hat{z}\right].$$

In particular, if $Z = \mathbb{R}$, $C = \mathbb{R}_+$, then $C^+ = \mathbb{R}_+$, and $\{1\}$ is a base of C^+, thus the intersection over the z^*'s disappears from the definition of f^{**} and formulas like (5.3).

5.3 Directional Derivatives

Usually, derivatives for set-valued functions are defined at points of their graphs as for example in [4, Chap. 5] and [114, Chap. 5]. Here, we use the inf-residuation in order to define a "difference quotient" (which could be called "residuation quotient") and take "lattice limits." This leads to the concept of a lower Dini directional derivative for $\mathcal{G}(Z, C)$-valued functions as introduced in [36].

Definition 5.10 The lower Dini directional derivative of a function $f: X \to \mathcal{G}(Z, C)$ with respect to $z^* \in C^+ \backslash \{0\}$ at $\bar{x} \in X$ in direction $x \in X$ is defined to be

$$f'_{z^*}(\bar{x}, x) = \liminf_{t \downarrow 0} \frac{1}{t} \left[\left(f(\bar{x} + tx) \oplus H^+(z^*) \right) -^{\!\cdot} f(\bar{x}) \right]$$
$$= \bigcap_{s>0} \text{cl} \bigcup_{0<t<s} \frac{1}{t} \left[\left(f(\bar{x} + tx) \oplus H^+(z^*) \right) -^{\!\cdot} f(\bar{x}) \right].$$

Obviously, $f'_{z^*} = f'_{tz^*}$ for $t > 0$. Hence, if C^+ has a base one only gets "as many" directional derivatives as there are elements in the basis.

One may ask why the set $H^+(z^*)$ appears in the definition of the difference quotient. The reason is that frequently the sets $f(\bar{x} + tx) -^{\!\cdot} f(\bar{x})$ and also corresponding "lattice limits" are empty.

Example 5.11 Let $X = \mathrm{I\!R}$, $Z = \mathrm{I\!R}^2$, $C = \left\{(0, 1)^T s \mid s \geq 0\right\}$ and the function $f: X \to \mathcal{G}(Z, C)$ be defined by

$$f(x) = \begin{cases} [-x, x] \times \mathrm{I\!R}_+ & : \ x \in [0, 1] \\ \emptyset & : \ \text{otherwise} \end{cases}.$$

Then, f is convex and $f(1) = \inf_{x \in X} f(x) \neq Z$. However, $f(1 + tx) -^{\!\cdot} f(1) = \emptyset$ whenever $x < 0$ and $t < -\frac{1}{x}$, or $x > 0$ and $t > 0$. This means that the directional derivative of f at $\bar{x} = 1$ (defined without $H^+(z^*)$) would be identically $\emptyset$. On the other hand, $f'_{z^*}(1, x)$ is never empty for $z^* \in C^+ \backslash \{0\}$ and provides much better information about the local behavior of f at $\bar{x} = 1$.

For scalar functions, the standard definition of the lower Dini directional derivative can be adapted.

Definition 5.12 The lower Dini directional derivative of a function $\varphi: X \to \overline{\mathrm{I\!R}}$ at $\bar{x}$ in direction x is

$$\varphi^{\downarrow}(\bar{x}, x) = \liminf_{t \downarrow 0} \frac{1}{t} \left[\varphi(\bar{x} + tx) -^{\!\cdot} \varphi(\bar{x}) \right].$$

In Definition 5.12, it is neither assumed $\bar{x} \in \text{dom}\, \varphi$, nor φ be a proper function. This is possible since the difference operator is replaced by the residual operator. For $\mathcal{G}(Z, C)$-valued functions, the lower Dini directional derivative can be expressed by corresponding derivatives of scalarizations.

Proposition 5.13 *(a) For all $\bar{x} \in X$, for all $x \in X$,*

$$f^{\downarrow}_{z^*}(\bar{x}, x) = \left\{z \in Z \mid \varphi^{\downarrow}_{f,z^*}(\bar{x}, x) \leq -z^*(z)\right\} \tag{5.4}$$

$$\varphi^{\downarrow}_{f,z^*}(\bar{x}, x) = \varphi_{f^{\downarrow}_{z^*}(\bar{x},\cdot),z^*}(x). \tag{5.5}$$

Proof See [36, Proposition 3.4]. □

The next result is familiar in the scalar case for proper functions, see [235, Theorem 2.1.14].

Lemma 5.14 *Let $f : X \to \mathcal{G}(Z, C)$ be convex, $\bar{x} \in X$ and $z^* \in C^+\backslash\{0\}$. Then*

$$\forall x \in X : f'_{z^*}(\bar{x}, x) = \inf_{t>0} \frac{1}{t}\left[\left(f(\bar{x} + tx) \oplus H^+(z^*)\right) -^{\!\cdot} f(\bar{x})\right], \tag{5.6}$$

and the function

$$x \mapsto f'_{z^*}(x_0, x)$$

is sublinear as a function from X into $\mathcal{G}(Z, C)$. If $\bar{x} \in \operatorname{dom} f$, then $\operatorname{dom} f'_{z^*}(\bar{x}, \cdot) = \operatorname{cone}(\operatorname{dom} f - \bar{x})$. *Moreover,*

$$f'_{z^*}(\bar{x}, 0) = \begin{cases} H^+(z^*) & : \ f(\bar{x}) \oplus H^+(z^*) \notin \{Z, \emptyset\} \\ Z & : \ f(\bar{x}) \oplus H^+(z^*) \in \{Z, \emptyset\} \end{cases}.$$

Proof It relies on the monotonicity of the "residuation quotient"

$$\frac{1}{t}\left[\left(f(\bar{x} + tx) \oplus H^+(z^*)\right) -^{\!\cdot} f(\bar{x})\right]$$

which in turn is proven using a calculus for the inf-residuation and the convexity of f. For details, compare [90]. □

The following result tells us when the directional derivative has only "finite" values. As usual, we denote by core M the algebraic interior of a set $M \subseteq X$.

Theorem 5.15 *Let $f : X \to \mathcal{G}(Z, C)$ be convex and $\bar{x} \in \operatorname{core}(\operatorname{dom} f)$. If f is proper, then there exists $z^* \in C^+\backslash\{0\}$ such that $f'_{z^*}(\bar{x}, x) \notin \{Z, \emptyset\}$ for all $x \in X$.*

Proof See [90]. □

5.4 The Subdifferential

For convex functions, we define elements of the subdifferential using conlinear minorants of the sublinear directional derivative.

Definition 5.16 Let $f\colon X \to \mathcal{G}(Z, C)$ be convex, $\bar{x} \in X$ and $z^* \in C^+\backslash\{0\}$. The set

$$\partial f_{z^*}(\bar{x}) = \left\{x^* \in X^* \mid \forall x \in X\colon S_{(x^*,z^*)}(x) \supseteq f'_{z^*}(\bar{x}, x)\right\}$$

is called the z^*-subdifferential of f at $\bar{x}$.

Again, the basic idea is to replace a continuous linear functional x^* by $S_{(x^*,z^*)}$. An alternative characterization of the subdifferential is provided in the following result.

Proposition 5.17 *Let $f\colon X \to \mathcal{G}(Z, C)$ be convex and $\bar{x} \in X$. The following statements are equivalent for $x^* \in X^*$, $z^* \in C^+\backslash\{0\}$:*

(a) $\forall x \in X$: $S_{(x^*,z^*)}(x) \supseteq f'_{z^*}(\bar{x}, x)$,
(b) $\forall x \in X$: $S_{(x^*,z^*)}(x - \bar{x}) \supseteq \left(f(x) \oplus H^+(z^*)\right) \mathbin{-^{\!\cdot}} f(\bar{x})$.
(c) $x^* \in \partial\varphi_{f,z^*}(\bar{x})$.

Proof See [90]. □

Some extra care is necessary for defining the subdifferential $\partial\varphi_{f,z^*}$ of the extended real-valued function φ_{f,z^*} in the previous proposition since its is not necessarily proper. The reader may compare [89, 90]. Condition (c) opens the path to a subdifferential calculus: With some effort, one can transform the subdifferential rules from the scalar to the set-valued case obtaining corresponding "z^*-wise" rules, see [207].

Under some "regularity", the directional derivative can be reconstructed from the subdifferential. This result is known as the max-formula. Here is a set-valued version.

Theorem 5.18 *Let $f\colon X \to \mathcal{G}(Z, C)$ be a convex function, $\bar{x} \in \operatorname{dom} f$ and $z^* \in C^+\backslash\{0\}$ such that the function $x \mapsto f(x) \oplus H^+(z^*)$ is proper and the function $\varphi_{f,z^*}\colon X \to \mathrm{I\!R} \cup \{+\infty\}$ is upper semi-continuous at $\bar{x}$. Then $\partial f_{z^*}(\bar{x}) \neq \emptyset$ and it holds*

$$\forall x \in X\colon f'_{z^*}(\bar{x}, x) = \bigcap_{x^* \in \partial f_{z^*}(\bar{x})} S_{(x^*,z^*)}(x). \tag{5.7}$$

Moreover, for each $x \in X$ there exists $\bar{x}^* \in \partial f_{z^*}(\bar{x})$ such that

$$f'_{z^*}(\bar{x}, x) = S_{(\bar{x}^*,z^*)}(x). \tag{5.8}$$

Proof See [90]. □

Next, we link the subdifferential and the Fenchel conjugate.

Proposition 5.19 *Let $f\colon X \to \mathcal{G}(Z, C)$ be convex, $\bar{x} \in X$, $\operatorname{dom} f \neq \emptyset$ and $f(\bar{x}) \oplus H^+(z^*) \neq Z$. Then, the following statements are equivalent for $x^* \in X^*$, $z^* \in C^+\backslash\{0\}$:*

(a) $x^* \in \partial f_{z^*}(\bar{x})$,
(b) $\forall x \in X$: $S_{(x^*,z^*)}(x) \mathbin{-^{\!\cdot}} f(x) \supseteq S_{(x^*,z^*)}(\bar{x}) \mathbin{-^{\!\cdot}} f(\bar{x})$.

Proof See [90]. □

This results basically says that $x^* \in \partial f_{z^*}(\bar{x})$ if the supremum in the definition of the conjugate is attained at $\bar{x}$ since from the Young-Fenchel inequality we have

$$S_{(x^*,z^*)}(\bar{x}) \mathbin{-^{\!\cdot}} f(\bar{x}) \supseteq f^*\left(x^*, z^*\right)$$

whereas (b) above produces

$$f^*\left(x^*, z^*\right) = \bigcap_{x \in X} \left\{S_{(x^*,z^*)}(x) \mathbin{-^{\!\cdot}} f(x)\right\} \supseteq S_{(x^*,z^*)}(\bar{x}) \mathbin{-^{\!\cdot}} f(\bar{x}).$$

This means: In the sense of Definition 3.3 adapted to maximization, the set $\{\bar{x}\}$ is a solution of the problem

$$\text{maximize} \quad S_{(x^*,z^*)}(x) \mathbin{-^{\!\cdot}} f(x) \quad \text{over} \quad x \in X.$$

Finally, we want to describe the set of points satisfying the condition $0 \in \partial_{z^*} f(\bar{x})$.

Proposition 5.20 *Let $f : X \to \mathcal{G}(Z, C)$ be convex, $z^* \in C^+\backslash\{0\}$ and $\bar{x} \in \operatorname{dom} f$ such that $f(\bar{x}) \oplus H^+(z^*) \neq Z$. Then, the following statements are equivalent:*

(a) $H^+(z^*) \supseteq f'_{z^*}(\bar{x}, x)$ *for all* $x \in X$,
(b) $0 \in \partial f_{z^*}(\bar{x})$,
(c) $f(\bar{x}) \oplus H^+(z^*) = [\inf_{x \in X} f(x)] \oplus H^+(z^*)$,
(d) $\varphi_{f,z^*}(\bar{x}) \leq \varphi_{f,z^*}(x)$ *for all* $x \in X$.

Proof This is immediate from the previous results. □

We will call an $\bar{x} \in X$ for which there is $z^* \in C^+\backslash\{0\}$ satisfying (c) in Proposition 5.20 a C^+-minimizer of problem (P) in Definition 3.3. The question arises if there is a (full) solution of (P) consisting of C^+-minimizers and how such a solution can be characterized.

We conclude this section by noting that a calculus for the z^*-subdifferential can be derived from corresponding calculus rules for extended real-valued convex functions. The additional feature in the set-valued case is the dependence of $\partial f_{z^*}(\bar{x})$ on z^*, i.e. properties of the mapping $z^* \mapsto \partial f_{z^*}(\bar{x})$. It turns out that this is an adjoint process type relationship as pointed out in [90].

5.5 *A Case Study: Set-Valued Translative Functions*

Let X, Z be two topological linear spaces and $T : Z \to X$ an injective continuous linear operator. A function $f : X \to \mathcal{P}(Z, C)$ is called **translative** with respect to T (or just T**-translative**) if

$$\forall x \in X,\ \forall z \in Z\colon f(x + Tz) = f(x) + \{z\}.$$

A special case of interest will be $Z = \mathbb{R}^m$, $\left\{h^1, \ldots, h^m\right\} \subseteq X$ a set of m linearly independent elements and $T\colon \mathbb{R}^m \to X$ defined by $Tz = \sum_{k=1}^m z_k h^k$. This construction is very close to (and motivated by) set-valued risk measures as shown below.

It is an easy exercise to show that a T-translative function f can be represented as follows:

$$\forall x \in X\colon f(x) = \left\{z \in \mathbb{R}^m \mid x - Tz \in A_f\right\} \tag{5.9}$$

where $A_f = \{x \in X \mid 0 \in f(x)\}$ is the zero sublevel set of f. This set satisfies

$$\forall z \in C\colon A_f - Tz \subseteq A_f$$

since f maps into $\mathcal{P}(Z, C)$. The latter property is called (T, C)-translativity of A_f.

The representation (5.9) can be written as

$$\forall x \in X\colon f(x) = \left(I_{A_f} \Box \alpha_T\right)(x) = \inf\left\{I_{A_f}(x_1) + \alpha_T(x_2) \mid x_1 + x_2 = x\right\}$$

where $\alpha_T\colon X \to \mathcal{P}(Z, C)$ is given by

$$\alpha_T(x) = \begin{cases} \{z\} + C & : \quad x = Tz \\ \emptyset & : \quad \text{otherwise} \end{cases}$$

and I_A is the set-valued indicator function of A: $I_A(x) = C$ if $x \in A$ and $I_A(x) = \emptyset$ if $x \notin A$. Note that the function α_T is well-defined since T is assumed to be injective.

We start the investigation of set-valued translative functions with their conjugates and make use of the fact that the conjugate of the infimal convolution of two functions is the sum of the two conjugates. For set-valued functions, this has been established in [78, Lemma 2]. The conjugate of the indicator function is indeed the set-valued support function as shown in [78]:

$$I^*_{A_f}(x^*, z^*) = \bigcap_{x \in A_f} S_{(x^*, z^*)}(x).$$

Moreover,

$$\begin{aligned}
\alpha^*_T\left(x^*, z^*\right) &= \bigcap_{x \in X}\left(S_{(x^*,z^*)}(x) \,\dot{-}\, \alpha_T(x)\right) = \bigcap_{u \in Z}\left(S_{(x^*,z^*)}(Tu) \,\dot{-}\, (\{u\} + C)\right) \\
&= \bigcap_{u \in Z}\left\{z \in Z \mid z + u + C \subseteq S_{(x^*,z^*)}(Tu)\right\} \\
&= \left\{z \in Z \mid \forall u \in Z\colon z^*(z + u) \geq x^*(Tu)\right\} \\
&= \left\{z \in Z \mid z^*(z) \geq \sup_{u \in Z}(T^*x^* - z^*)(u)\right\} = \begin{cases} H^+(z^*) & : \quad z^* = T^*x^* \\ \emptyset & : \quad z^* \neq T^*x^* \end{cases}
\end{aligned}$$

Hence, for a T-translative function f we get

$$f^*(x^*, z^*) = I^*_{A_f}(x^*, z^*) + \alpha^*_T\left(x^*, z^*\right) = \begin{cases} \bigcap\limits_{x \in A_f} S_{(x^*,z^*)}(x) & : \ z^* = T^*x^* \\ \emptyset & : \ z^* \neq T^*x^* \end{cases} \quad (5.10)$$

and (see Remark 5.5)

$$(-f^*)(x^*, z^*) = H^+(z^*) -^{\cdot} f^*(x^*, z^*) = \begin{cases} \operatorname{cl} \bigcup\limits_{x \in A_f} S_{(x^*,z^*)}(-x) & : \ z^* = T^*x^* \\ Z & : \ z^* \neq T^*x^* \end{cases}$$

since $H^+(z^*) -^{\cdot} \emptyset = Z$ and $H^+(z^*) -^{\cdot} \bigcap_{x \in A_f} S_{(x^*,z^*)}(x) = \operatorname{cl} \bigcup_{x \in A_f} \left[H^+(z^*) -^{\cdot} S_{(x^*,z^*)}(x)\right] = S_{(x^*,z^*)}(-x)$ according to Proposition 4.17.

If the function f additionally maps into $\mathcal{G}(Z, C)$ and is proper, closed and convex, then the biconjugation theorem applies, and the following dual representation is obtained:

$$\forall x \in X \colon f(x) = \bigcap_{\substack{x^* \in X^* \\ T^*x^* \in C^+ \setminus \{0\}}} \left[S_{(x^*,T^*x^*)}(x) -^{\cdot} I^*_{A_f}(x^*, T^*x^*)\right]. \quad (5.11)$$

If f is additionally sublinear, then A_f is a closed convex cone and (5.11) simplifies to

$$\forall x \in X \colon f(x) = \bigcap_{\substack{x^* \in A_f^- \\ T^*x^* \in C^+ \setminus \{0\}}} S_{(x^*,T^*x^*)}(x) \quad (5.12)$$

since in this case

$$I^*_{A_f}(x^*, z^*) = \begin{cases} H^+(z^*) & : \ x^* \in A_f^- \\ \emptyset & : \ \text{otherwise} \end{cases}.$$

Of course, $A_f^- = -\left(A_f\right)^+$.

The value of these formulas depends on how the dual data x^*, T^* and $I^*_{A_f}$ can be interpreted in terms of the application at hand. We will show in Sect. 7.4 below that this can be done very nicely.

Example 5.21 $Z = \mathbb{R}^m$, $T \colon \mathbb{R}^m \to X$ defined by $Tz = \sum_{k=1}^m z_k h^k$. Then

$$\forall z \in \mathbb{R}^m \colon \left(T^*x^*\right)(z) = \sum_{k=1}^m x^*(h^k) z_k,$$

thus T^*x^* can be identified with $\left(x^*\left(h^1\right), \ldots, x^*\left(h^m\right)\right)^T \in \mathbb{R}^m$.

We turn to the subdifferential of T-translative functions. The result reads as follows.

Corollary 5.22 *Let $f\colon X \to \mathcal{G}(Z, C)$ be convex, T-translative and $z^* \in C^+\backslash\{0\}$. If $\partial f_{z^*}(\bar{x}) \neq \emptyset$ then*

$$\partial f_{z^*}(\bar{x}) = \left\{x^* \in X^* \mid z^* = T^*x^* \text{ and } \forall x \in A_f\colon S_{(x^*,T^*x^*)}(x) \supseteq S_{(x^*,T^*x^*)}(\bar{x}) -^{\!\cdot} f(\bar{x})\right\}. \tag{5.13}$$

Proof First, we show "$\subseteq$". The assumption $\partial f_{z^*}(\bar{x}) \neq \emptyset$ in conjunction with Proposition 5.17 implies $f(\bar{x}) \oplus H^+(z^*) \notin \{Z, \emptyset\}$. Hence $S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x}) \notin \{Z, \emptyset\}$, and Proposition 5.19 produces $f^*(x^*, z^*) \notin \{Z, \emptyset\}$. Take $x^* \in \partial f_{z^*}(\bar{x})$. From (5.10) we now obtain

$$z^* = T^*x^* \quad \text{and} \quad f^*(x^*, z^*) = I^*_{A_f}(x^*, z^*).$$

The definition of the set-valued support function yields that x^* belongs to the right hand side of (5.14).

Conversely, assume that $x^* \in X^*$ satisfies $z^* = T^*x^*$ as well as

$$\forall x \in A_f\colon S_{(x^*,z^*)}(x) \supseteq S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x}).$$

Take $x \in \operatorname{dom} f$. Then

$$\forall z \in f(x)\colon x - Tz \in A_f$$

by T-translativity and hence

$$\forall z \in f(x)\colon S_{(x^*,z^*)}(x - Tz) \supseteq S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x}).$$

Since $z^* = T^*x^*$ we have

$$S_{(x^*,z^*)}(x - Tz) = S_{(x^*,z^*)}(x) + \{-z\}$$

and therefore

$$\forall z \in f(x)\colon S_{(x^*,z^*)}(x) + \{-z\} \supseteq S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x}).$$

This means that any $\eta \in S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x})$ satisfies

$$\forall z \in f(x)\colon z + \eta \in S_{(x^*,z^*)}(x),$$

thus $\eta \in S_{(x^*,z^*)}(x) -^{\!\cdot} f(x)$. Hence

$$\forall x \in \operatorname{dom} f\colon S_{(x^*,z^*)}(x) -^{\!\cdot} f(x) \supseteq S_{(x^*,z^*)}(\bar{x}) -^{\!\cdot} f(\bar{x})$$

which is, according to Proposition 5.19, equivalent to $x^* \in \partial f_{z^*}(\bar{x})$. □

The above corollary tells us that the knowledge of ∂f_{z^*} can be obtained by knowledge about A_f and T^*. This becomes even more clear in the sublinear case.

Corollary 5.23 *Let* $f\colon X \to \mathcal{G}(Z, C)$ *be sublinear, T-translative and* $z^* \in C^+\backslash\{0\}$. *If* $\partial f_{z^*}(\bar{x}) \neq \emptyset$ *then*

$$\partial f_{z^*}(\bar{x}) = \left\{x^* \in X^* \mid z^* = T^*x^*,\ x^* \in A_f^-,\ S_{(x^*,z^*)}(\bar{x}) = f(\bar{x}) \oplus H^+(z^*)\right\}. \tag{5.14}$$

Proof As observed above, in this case A_f is a convex cone and I^* can only attain the two values $H^+(z^*)$ for $x^* \in A_f^-$ and $\emptyset$ otherwise. Finally,

$$S_{(x^*,z^*)}(\bar{x}) \dot{-} f(\bar{x}) = H^+(z^*) \quad \Leftrightarrow \quad S_{(x^*,z^*)}(\bar{x}) = f(\bar{x}) \oplus H^+(z^*).$$

The result now follows from Corollary 5.22. □

5.6 Comments on Vector- and Set-Valued Convex Analysis

The history of convex analysis for scalar functions is a continuing success story, and this area of mathematics is the theoretical basis for linear and nonlinear, in particular non-smooth, optimization and optimal control theory: compare [196, p. 3][5] or the preface of [8, p. xii].[6]

Surprisingly, the gap between theory and applications (in optimization and multi-criteria decision making) is much wider for vector- or even set-valued functions. For example, there is no canonical (Fenchel) conjugate of a vector-valued function, but rather a whole bunch of different definitions which work under different assumptions (see below for references).

If one ignores for a moment scalarization approaches, then there are basically two different paths to a "vector-valued" convex analysis.

The first one simply consists in an extended interpretation of the infimum and the supremum in formulas like the definition of the Fenchel conjugate: Under the assumption that the function maps into a conditional complete vector lattice (this means that every set which is bounded from below with respect to the vector order has an infimum in the space) one considers infima/suprema with respect to the vector order. This approach has been followed by Zowe [232, 233], Elster and Nehse [23, 50], Borwein [16], Zalinescu [234], Kutateladze [139] and others. One may compare [17] for the state of the art in the mid 1980s and more references. This approach has the advantage that a corresponding version of the Hahn-Banach theorem is available which is due to L.V. Kantorovich, see for example Day's book [42]. Disadvantages are, of course, the strong assumptions to the image space and, even worth for applications, the fact that a vector infimum/supremum is hardly an appropriate concept when it comes to vector optimization and multi-criteria decision making.

[5] 'In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity'.

[6] 'Theoretically, what modern optimization can solve well are *convex optimization problems*'.

In the second approach, infima and suprema are therefore replaced by sets of minimal and maximal points, respectively, with respect to the vector order. This worked for (and was motivated by) applications of vector optimization, but made the task of developing a corresponding vector-valued convex analysis incredibly harder: It turns out that "dual constructions" like conjugates or dual optimization problems become set-valued: 'for a vector problem, its dual constructed by several means, is a problem whose objective function is set-valued, whatever the objective of the primal problem be' ([156, p. 57]). Set-valued Legendre–Fenchel conjugates with maximal points replacing the supremum appear in [156, 204, 219], with weakly maximal points in [166, 204], with (weakly) supremal points in [123, 190, 191, 212, 214, 217] and an even more general construction involving "non-submitted" points is used in [47], for example.

A major difficulty for this approach is the lack of an appropriate Hahn-Banach theorem which is at the heart of convex analysis: One has to turn to scalarizations in order to apply the "usual" Hahn-Banach argument. Zowe's paper [231] shows how difficult it is to get back to vector-valued concepts after a scalarization.

In both approaches, continuous linear operators were used as dual variables. One way to avoid this again is a scalarization approach: An early attempt is Jahn's work [113] (compare also [114, Chap. 8]). This approach leads to peculiar difficulties even if the problem at hand is linear: Compare [113, Conclusions] and the discussion at the ends of [114, Sects. 8.2 and 8.3]. A modern account is given in [19] which leads to dual problems with a, in general, non-convex feasibility set even if the original problem is convex (or linear).

Let us mention that there are at least two quite different attempts to answer the duality question for vector problems: In [5, 6] as well as in [21] Fenchel conjugates of vector- or set-valued functions are defined in terms of scalar functions depending on an additional dual variable. Although in both attempts quite strong assumptions are imposed, they seem to be only a few steps short of the constructions in this section.

The approach summarized in [151] is also based on scalarization via support functions, but it involves a set infimum/supremum which admits to obtain stronger results.

The concepts presented in this survey go without the usual assumptions to the ordering cone C (non-empty interior, pointedness, generating a lattice order etc.), and they basically produce set-valued versions of all the known (duality) formulas for scalar convex functions, and this includes the case of vector-valued functions. A crucial observation is the theoretical equivalence of a convex $\mathcal{G}(Z, C)$-valued function f and the family $\left\{\varphi_{f,z^*}\right\}_{z^* \in C^+ \setminus \{0\}}$. Formula (5.11) is an example for how the set-valued theory tells us what kind of scalarizations should be taken into consideration. New insights can be obtained by investigating relationships between the two components of the dual variable (x^*, z^*) which is essentially of adjoint process duality type (see [90, Sect. 4]). Set-valued functions satisfying (a) and (b) of Proposition 5.1 are sometimes called linear, e.g. in [186]. On the other hand, Proposition 5.1 and Theorem 5.2 show that the "conlinear" functions $S_{(x^*,z^*)}$ are in some sense one-sided versions of linear multivalued operators (linear relations) as surveyed e.g. in [37]. The reader may check the relationships in the case of a linear subspace C with $C^+ = C^\perp$ its orthogonal complement.

Directional derivatives for set-valued functions are usually defined at points of its graph, thus fixing (only) one element in the image set along with a point in the pre-image set. The standard reference is [4], and Mordukhovich's coderivative [172] is of the same type. Compare also [227]. Quite a different path is the attempt to embed certain subsets of $\mathcal{P}(Z)$ into a linear space and then use the usual "linear" constructions, see [137] for an example. This, of course, only works under strong assumptions since, in general, $\mathcal{G}(Z, C)$ cannot be embedded into a linear space even if one drops $\emptyset$ and Z.

Concerning subgradients for set-valued functions, the paper [98] presents an overview over the existing concepts each of which is afflicted with a peculiar difficulty: for example, the 'weak subgradient' of [29] (see also [28, Definition 2.53]) leaves the realm of convexity, the 'strong subgradient' introduced in [98] needs an artificial exclusion condition in its definition. Both require rather strong assumptions for their existence: compare [236, Corollary 9] with respect to weak subgradients while observing that the space Z therein has the least upper bound property, and with respect to strong subgradients compare [98, Definition 3.2 and Theorem 4.1].

One should note that the application of Yang's Hahn-Banach type theorem [226, 236] also suffers the "non-convexity issue:" since it relies on the "not strictly greater than" relation: inequalities cannot be added whenever the relation is non-complete. This means that the weak subdifferentials of convex set-valued function obtained for example via [236, Corollary 9] are not convex in general.

Another way of defining subgradients is to do it at points of the graph of a set-valued mapping rather than at points of its domain, see [16, 204], [7, Definition 2.1] and also the 'positive subgradients' defined in [143, Definition 2.5], [92, Definition 3.1] and the 'k-subgradients' of [19, Definition 7.1.9] among many others.

Most of those concepts use linear operators as dual variables, but when it comes to existence very often operators of rank 1 show up, see, for example, [28, Theorem 2.55], [98, Theorem 4.1]. The (straightforward) relationships are discussed in [19, p. 331] and [92, Sect. 4].

We interpret this as evidence that, unless the image space is a (conditional) complete vector lattice and the Hahn-Banach-Kantorovitch theorem is available, the dual variables should involve linear functionals rather than linear operators. Using "conlinear functions" generated by pairs of linear functionals, the constructions in Sects. 5.3 and 5.4 offer a way to obtain results which are very close in shape to the scalar case and avoid strong assumptions to the ordering cone in Z. Moreover, in contrast to most of the "vectorial" constructions in the literature (for example, see the discussion in [19, p. 313]), our set-valued results reproduce the ones for scalar extended real-valued functions as special cases; this includes e.g. existence of subgradients and strong duality with attainment of the supremum for the dual problem.

The subdifferential as given in Definition 5.16 is exactly the same set which is called the 'conjugate to' f in [194, Definition 2 and the remark thereafter] provided one assumes that every expression in B. N. Pshenichnyi's definition is finite. Section 5.4 should make it clear why we call it a subdifferential; the relationship to convex process duality can be found in [90]. It should be pointed out that the complete lattice approach of this survey also adds new insights to scalar convex analysis: the improper case, in particular the function value $-\infty$, can be dealt with using the residuation. We refer to [89].

Scalar translative functions appear in many areas of applied mathematics, for example probability (quantile functions and lower previsions [221]), insurance and finance (constant additive insurance premiums [222] and cash additive risk measures, introduced in [3] and reviewed in [59]), mathematical economics (benefit and shortage functions [161, 162]), vector optimization (nonlinear scalarization functions, compare [63] also for earlier references and [67] for an overview) and idempotent analysis (compare the survey [126]) as well as in max-plus algebra (see e.g. [31]). A relationship between vector optimization and risk measures in finance is pointed out in [99].

Following an idea of [120], in [80, 82] cash additive risk measures have been generalized to set-valued risk measures for multivariate positions which turned out to be T-translative for some special T. Thus, such functions are important in applications, and they provide examples for the set optimization theory of this survey.

6 Set-valued Optimization

6.1 Unconstrained Problems

Within the set-up of the previous section, the basic problem again is

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in X. \tag{P}$$

The difficulty with the solution concept given in Definition 3.3 is that solutions are, in general, sets rather than single points. Thus, optimality conditions such as "zero belongs to the subdifferential of some function" should actually be taken "at sets" rather than "at points." Of course, this does not sound very attractive. The following construction provides a remedy.

Definition 6.1 Let $f\colon X \to \mathcal{G}(Z, C)$ be a function and $M \subseteq X$ a non-empty set. The function $\hat{f}(\cdot; M) : X \to \mathcal{G}(Z, C)$ defined by

$$\hat{f}(x; M) = \inf_{u \in M} f(x+u) = \operatorname{cl}\operatorname{co} \bigcup_{u \in M} f(x+u) \tag{6.1}$$

is called the inf-translation of f by M.

The function $\hat{f}(\cdot; M)$ coincides with the canonical extension of f at $M + \{x\}$ as defined in [102]. A few elementary properties of the inf-translation are collected in the following lemma.

Lemma 6.2 *Let $M \subseteq X$ be non-empty and $f : X \to \mathcal{G}(Z, C)$ a function.*

(a) *If $M \subseteq N \subseteq X$ then $\hat{f}(x; M) \subseteq \hat{f}(x; N)$ for all $x \in X$.*
(b) $\inf_{x \in X} f(x) = \inf_{x \in X} \hat{f}(x; M)$.
(c) *If f and M are convex, so is $\hat{f}(\cdot; M) : X \to \mathcal{G}(Z, C)$, and in this case $\hat{f}(x; M) = \operatorname{cl} \bigcup_{u \in M} f(u + x)$.*

Proof The proof can be found in [90]. □

Proposition 6.3 *Let $f : X \to \mathcal{G}(Z, C)$ be a convex function and $\emptyset \neq M \subseteq \operatorname{dom} f$. The following statements are equivalent:*

(a) *M is an infimizer for f;*
(b) *$\{0\} \subseteq X$ is an infimizer for $\hat{f}(\cdot; M)$;*
(c) *$\{0\}$ is an infimizer for $\hat{f}(\cdot; \operatorname{co} M)$ and $\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$.*

Proof The equivalence of (a) and (b) is immediate from $\hat{f}(0; M) = \inf_{u \in M} f(u)$ and Lemma 6.2, (b). The equivalence of (a) and (c) follows from $\hat{f}(0; \operatorname{co} M) = \inf_{u \in \operatorname{co} M} f(u)$ and Lemma 6.2, (b). □

The previous proposition makes clear that by an inf-translation an infimizer (set) can be reduced to a single point, namely just $0 \in X$. Moreover, it should be apparent that we need to consider $\hat{f}(\cdot; \operatorname{co} M)$: Since we want to characterize infimizers via directional derivatives and subdifferentials, a convex function is needed, and $\hat{f}(\cdot; M)$ is not convex in general even if f is convex (find a counterexample!). Obviously, an infimizer is not necessarily a convex set; on the contrary, sometimes one prefers a nonconvex one, for example a collection of vertices of a polyhedral set instead of higher dimensional faces.

Theorem 6.4 *Let $f : X \to \mathcal{G}(Z, C)$ be a convex function satisfying*

$$I(f) = \inf_{x \in X} f(x) \notin \{Z, \emptyset\}.$$

Then f is proper, and the set $\Gamma^+(f) = \left\{z^* \in C^+ \backslash \{0\} \mid I(f) \oplus H^+(z^*) \neq Z\right\}$ is non-empty. Moreover, a set $M \subseteq X$ is an infimizer for f if, and only if, $\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$ and

$$0 \in \bigcap_{z^* \in \Gamma^+(f)} \partial \hat{f}_{z^*}(\cdot; \operatorname{co} M)(0).$$

Proof Since $\{0\}$ is a singleton infimizer of the function $x \mapsto \hat{f}(x; M)$, $\bar{x} = 0 \in X$ satisfies (c) of Proposition 5.20 with f replaced by $\hat{f}(\cdot; M)$ for each $z^* \in \Gamma^+(f)$. Now, the result follows from Proposition 5.20 and Proposition 6.3. □

Theorem 6.4 highlights the use of the "z^*-wise" defined directional derivatives and subdifferentials. One needs to take into consideration all reasonable (= proper) scalarizations at the same time in order to characterize infimizers.

6.2 Constrained Problems and Lagrange Duality

Let Y be another locally convex spaces with topological dual Y^*, and $D \subseteq Y$ a convex cone. The set $\mathcal{G}(Y, D)$ is defined in the same way as $\mathcal{G}(Z, C)$. Finally, let $f : X \to \mathcal{G}(Z, C)$ and $g : X \to \mathcal{G}(Y, D)$ be two functions. We are interested in the problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad 0 \in g(x). \tag{PC}$$

The set

$$\mathcal{X} = \{x \in X \mid 0 \in g(x)\}$$

is called the feasible set for (PC) and $I(f, g) = \inf\{f(x) \mid x \in \mathcal{X}\}$ is the optimal value of the problem. With Definition 3.2 in view we define a solution of (PC) as follows.

Definition 6.5 A set $M \subseteq \mathcal{X}$ is called a solution of (PC) if

(a) $\inf\{f(x) \mid x \in M\} = I(f, g)$,
(b) $\bar{x} \in M, x \in \mathcal{X}, f(x) \supseteq f(\bar{x})$ imply $f(x) = f(\bar{x})$.

Clearly, $M \subseteq X$ is a solution of (PC) if, and only if $f[M]$ generates the infimum of $f[\mathcal{X}] = \{f(x) \mid x \in \mathcal{X}\}$ and each $f(\bar{x})$ for $\bar{x} \in M$ is minimal in $f[\mathcal{X}]$ with respect to $\supseteq$.

We define the Lagrangian $L : X \times Y^* \times C^+ \backslash \{0\} \to \mathcal{G}(Z, C)$ of problem (PC) by

$$L\left(x, y^*, z^*\right) = f(x) \oplus \bigcup_{y \in g(x)} S_{(y^*, z^*)}(y) = f(x) \oplus \inf\left\{S_{(y^*, z^*)}(y) \mid y \in g(x)\right\}. \tag{6.2}$$

Under a mild condition, the primal problem can be reconstructed from the Lagrangian.

Proposition 6.6 *If $f(x) \neq Z$ for each $x \in \mathcal{X}$, then*

$$\sup_{(y^*, z^*) \in Y^* \times C^+ \backslash \{0\}} L\left(x, y^*, z^*\right) = \bigcap_{(y^*, z^*) \in D^+ \times C^+ \backslash \{0\}} L\left(x, y^*, z^*\right) = \begin{cases} f(x) & : \quad 0 \in g(x) \\ \emptyset & : \quad 0 \notin g(x). \end{cases}$$

Proof The proof is based on the assumption that the values of f and g are closed convex sets. See [86] for details. □

The next proposition provides a Lagrange sufficient condition which is a simple, but important result with an algorithmic character since it admits to test if a given set is an infimizer of (PC).

Proposition 6.7 *Let $M \subseteq \mathcal{X}$ be a non-empty set of feasible points for (PC). Assume that for each $z^* \in C^+ \backslash \{0\}$ there is $y^* \in D^+$ satisfying*

$$\hat{f}(0; M) \oplus \inf_{y \in \hat{g}(0;M)} S_{(y^*,z^*)}(y) = \inf_{x \in X} L\left(x, y^*, z^*\right) \tag{6.3}$$

and

$$\inf_{y \in \hat{g}(0;M)} S_{(y^*,z^*)}(y) = H^+(z^*). \tag{6.4}$$

Then, M is an infimizer for (PC).

Proof Using (6.4) and (6.3) we obtain

$$\begin{aligned}\hat{f}(0; M) \oplus H^+(z^*) &= \hat{f}(0; M) \oplus \inf_{y \in \hat{g}(0;M)} S_{(y^*,z^*)}(y) = \inf_{x' \in X} L\left(x', y^*, z^*\right) \\ &\supseteq f(x) \oplus \inf_{y \in g(x)} S_{(y^*,z^*)}(y) \supseteq f(x) \oplus H^+(z^*)\end{aligned}$$

for all $x \in \mathcal{X}$ since $S_{(y^*,z^*)}(0) = H^+(z^*)$. Taking the infimum over the feasible x on the right hand side and then the intersection over $z^* \in C^+\backslash\{0\}$ on both sides while observing $\hat{f}(0; M) = \inf_{u \in M} f(u)$ we obtain that M indeed is an infimizer for (PC). □

Condition (6.4) serves as set-valued complementary slackness condition. If one considers the Lagrange function $(x, y^*, z^*) \mapsto \hat{L}(x, y^*, z^*; M)$ for the "inf-translated" problem

$$\text{minimize} \quad \hat{f}(x; M) \quad \text{subject to} \quad 0 \in \hat{g}(x; M)$$

then condition (6.3) means that the infimum of the Lagrange function for the original problem coincides with $\hat{L}(0, y^*, z^*; M)$. Finally, if $z^* \in C^+\backslash\{0\}$ and $y^* \in D^+$ satisfy (6.4) and (6.3) then y^* is nothing else than a Lagrange multiplier for the by z^* scalarized problem. One may therefore expect that strong duality is something like "strong duality for all reasonable scalarized problems." This idea works as shown in the following.

Define the function $h\colon Y^* \times C^+\backslash\{0\} \to \mathcal{G}(Z, C)$ by

$$h\left(y^*, z^*\right) = \inf_{x \in X} L\left(x, y^*, z^*\right) = \text{cl} \bigcup_{x \in X} L\left(x, y^*, z^*\right).$$

Since the values of $L(\cdot, y^*, z^*)$ are closed half spaces with the same normal z^*, the convex hull can be dropped in the infimum. The dual problem,

$$\text{maximize} \quad h\left(y^*, z^*\right) \quad \text{subject to} \quad y^* \in Y^*, \ z^* \in C^+\backslash\{0\}, \tag{DC}$$

thus consists in finding

$$d = \sup_{y^* \in Y^*,\, z^* \in C^+\backslash\{0\}} h\left(y^*, z^*\right) = \bigcap_{y^* \in Y^*,\, z^* \in C^+\backslash\{0\}} h\left(y^*, z^*\right)$$

and corresponding (full) solutions. The following weak duality result is immediate.

Proposition 6.8 *Let* $f\colon X \to \mathcal{F}(Z, C)$ *and* $g\colon X \to \mathcal{F}(Y, D)$. *Then*

$$\sup\left\{h\left(y^*, z^*\right) \mid y^* \in Y^*,\ z^* \in C^+\backslash\{0\}\right\} \supseteq \inf\left\{f(x) \mid x \in X,\ 0 \in g(x)\right\}.$$

Proof This is true since for $(y^*, z^*) \in Y^* \times C^+\backslash\{0\}$ and $x \in X$ satisfying $0 \in g(x)$ we have

$$h\left(y^*, z^*\right) \supseteq f(x) \oplus \operatorname{cl} \bigcup_{y \in g(x)} S_{(y^*,z^*)}(y) \supseteq f(x) \oplus S_{(y^*,z^*)}(0) = f(x) \oplus H^+\left(z^*\right).$$

□

As usual, a constraint qualification condition is needed as part of sufficient conditions for strong duality. The following condition is called the *Slater condition* for problem (PC):

$$\exists \bar{x} \in \operatorname{dom} f\colon g(\bar{x}) \cap \operatorname{int}(-D) \neq \emptyset. \tag{6.5}$$

The implicit assumption is $\operatorname{int} D \neq \emptyset$.

Theorem 6.9 *Assume* $p = \inf\{f(x) \mid x \in \mathcal{X}\} \neq Z$. *If* $f\colon X \to \mathcal{G}(Z, C)$ *and* $g\colon X \to \mathcal{G}(Y, D)$ *are convex and the Slater condition for problem (PC) is satisfied then strong duality holds for (PC), that is*

$$\inf\left\{f(x) \mid 0 \in g(x)\right\} = \sup\left\{h\left(y^*, z^*\right) \mid y^* \in Y^*,\ z^* \in C^+\backslash\{0\}\right\}, \tag{6.6}$$

$$z^* \in C^+\backslash\{0\},\ p \oplus H^+\left(z^*\right) \neq Z \quad \Rightarrow \quad \exists y^* \in Y^*\colon p \oplus H^+\left(z^*\right) = h\left(y^*, z^*\right). \tag{6.7}$$

Proof Hamel and Löhne [86]. □

Note that the assumption $p \neq Z$ implies the existence of $z^* \in C^+\backslash\{0\}$ with $p \oplus H^+(z^*) \neq Z$. Thus, (6.7) is attainment of the supremum for the dual problem "z^*-wise."

Corollary 6.10 *Under the assumptions of the strong duality theorem, the set*

$$\Delta = \left\{\left(y^*, z^*\right) \in Y^* \times C^+\backslash\{0\} \mid Z \neq p \oplus H^+(z^*) = h\left(y^*, z^*\right)\right\}$$

is non-empty and a full solution of the dual problem (DC).

Proof See [86]. □

6.3 Comments on Set Optimization Duality

Among the first papers in which optimization problems with a set-valued constraint have been systematically studied are [15, 16, 185]. It is, for example, instructive to realize that the Lagrange function in (6.2) is nothing else, but a set-valued version of the one in [185, p. 197]. Compare also [151, Theorem 3.28].

Whereas in [16, Problem (P) in (3.1)] the vector infimum serves as the building block for optimality, in [15, Theorem 3] a Lagrange duality result is established for properly efficient points of vector optimization problems. The dual variables are rank one linear operators. Similarly, in [211, Theorem 3.3] and also [213, Theorem 3.3], rank one linear operators and a set-valued Lagrange function (see equation (6.8) below) are used under strong assumptions (cones with weakly compact base). A similar idea can be found in the proof of the Lagrangian duality theorem, [156, Theorem 1.6 on p. 113] under the assumption that the ordering cone in Z has non-empty interior. These examples may suffice with respect to vector optimization problems in view although the literature is huge.

In [131, 134] the same type of set-valued Lagrangian has been used (without giving proofs) in connection with set relations, i.e., basically the solution concept IIa of Sect. 3.1. The more recent [93, 95] proceed similarly: Theorem 3.3 in [95] (basically the same as Theorem 4.2 in [93]) is a Lagrange duality result for weakly $\preccurlyeq_C$-minimal solutions with Lagrange function

$$f(x) + (T \circ g)(x) = f(x) + \{Ty \mid y \in g(x)\} \tag{6.8}$$

where $T \in \mathcal{L}(Y, Z)$, the set of continuous linear operators from Y to Z. It is again based on rank one operators, an idea which at least dates back to [32, Theorem 4.1]. The same set of dual variables is used in [81] for a Lagrangian approach to linear vector optimization. However, the Lagrange function, even for a vector-valued problem, already is a set-valued one.

A thorough discussion of optimality conditions of Fermat and Lagrange type for (non-convex) set-valued optimization problems based on the minimality concept III can be found in [48] (compare also the references therein). These conditions are formulated in terms of the Mordukhovich subdifferential. It might be worth noting that the use of $\mathcal{F}(Z, C)$-valued functions 'gives better conclusions' [48, Remark 3.10].

A complete lattice approach based on infimal and supremal sets was developed in [92, 151]. The Lagrange function for a vector-valued function f and a set-valued G has the form

$$f(x) + \operatorname{Inf}\left\{y^*(y)c \mid y \in G(x)\right\}$$

where Inf stands for the infimal set and $c \in \operatorname{int} C$ is a (fixed) element. Assumptions, of course, include $\operatorname{int} C \neq \emptyset$. The same assumption also is crucial in [144]; Theorems 3.2 and 3.3 therein are probably as far as one get in terms of conjugate duality based on "suprema" of a set, i.e. the elements which belong to the closure of the set, but are not dominated with respect to the relation which is generated by the interior of the ordering cone.

Other approaches rely on other set-valued derivatives, for example on contingent epiderivatives [69] or coderivatives [74, 173].

In virtually all approaches for set/vector optimization problems known to the authors, the strong duality assertion is based on the assumption of the existence of a (weakly, properly etc.) minimal element of the primal problem either with respect to the vector order (see [32], [156, Theorem 1.6 on p. 113, Theorem 2.7 on p. 119], [214, Theorems 3.4 and 3.5], [19, Theorems 5.2.4 and 5.2.6]) or with respect to a set relation (see [95, Theorem 3.3], [93, Theorem 4.2]). The two exceptions are the approaches in [86, 151] where the primal problems only have finite values in some sense and still existence for the dual problems is obtained–which is standard in the scalar case. In [151, p. 98] (see also Open Problem 3.6 therein with respect to Fenchel duality) and [92] it is discussed that the approach based on infimal/supremal sets indeed yields strong duality, but it is not clear whether the existence of the dual solution can be guaranteed without the z^*-component of the dual variable.

By means of the "complete lattice approach" surveyed here, the type of results which is known from the scalar case can be transferred to a "set level." Strong duality then indeed means "inf equals sup" and includes the existence of dual solutions: compare [86, 151] for Lagrange duality and [79] for Fenchel-Rockafellar duality. The Lagrange function as defined in (6.2) basically is the composition of the two set-valued functions $S_{(x^*,z^*)}$ and g, compare, for example, [125, Definition 6.3.2] and for scalar problems with a set-valued constraint already [185, p. 197].

The reduction of a "set solution" in the sense of Definition 6.5 to a "point solution" via an inf-translation (see Definition 6.1) is due to [90]. The exploitation of this construction seems to be very promising for obtaining optimality conditions and algorithms.

The complementary slackness condition given in Proposition 6.7 seems to be new although it clearly is in the spirit of [14, formulae (10), (12)].

7 Applications

7.1 Vector Optimization

In this section, let X and Z be as in Sect. 5 and $C \subseteq Z$ a closed, convex, pointed (i.e. $C \cap -C = \{0\}$) and non-trivial cone. Then, $\leq_C$ is a partial order (i.e. also anti-symmetric). Moreover, let a function $F \colon X \to Z \cup \{-\infty, +\infty\}$ be given. Defining a function $f \colon X \to \mathcal{G}(Z, C)$ by

$$f(x) = \begin{cases} F(x) + C & : \quad F(x) \in Z \\ Z & : \quad F(x) = -\infty \\ \emptyset & : \quad F(x) = +\infty \end{cases}$$

we observe

$$f(x_1) \supseteq f(x_2) \quad \Leftrightarrow \quad F(x_1) \leq_C F(x_2),$$

where it is understood that $-\infty \leq_C z \leq_C +\infty$ for all $z \in Z \cup \{-\infty, +\infty\}$. Hence the two problems

$$\text{find minimizers w.r.t. } \leq_C \text{ of } F(x) \text{ subject to } 0 \in g(x), \tag{VOP}$$

$$\text{find minimizers w.r.t. } \supseteq \text{ of } f(x) \text{ subject to } 0 \in g(x) \tag{SOP}$$

have the same feasible elements and the same minimizers. The minimizers of (VOP) are called'minimal solutions' [114, Definition 7.1] or 'efficient solutions' [19, Definition 2.5.1]. In most cases, it does not make sense to look for the infimum in (VOP) with respect to $\leq_C$: It may not exist (not even for simple polyhedral cones C, see e.g. [151, Example 1.9]), and even if it does, it is not useful in practice at it refers to so-called utopia points which are typically not realizable by feasible points (i.e. "decisions").

The (PC) version of (SOP) considered as an $\mathcal{F}(Z, C)$- or $\mathcal{G}(Z, C)$-valued problem is called the lattice extension of (VOP), and a solution of (VOP) is defined to be a solution of its lattice extension (see [102], compare Definition 6.5). In this way, the notion of an "infimum" makes a strong comeback, and the infimum attainment becomes a new feature in vector optimization, which is useful for theory and applications: It ensures that the decision maker possesses a sufficient amount of information about the problem if (s)he knows a solution. For a detailed discussion see [151, Chap. 2]. Note that one possibly obtains different solutions depending on the choice of $\mathcal{F}(Z, C)$ or $\mathcal{G}(Z, C)$ as image space. Since the infimum in $\mathcal{G}(Z, C)$ involves the convex hull, solutions of $\mathcal{G}(Z, C)$-valued problems may include "fewer" elements, and this is in particular preferable for convex problems.

If f is the "lattice extension" of a vector-valued function F as given above, the Lagrange function for (PC) takes the form

$$\begin{aligned} L\left(x, y^*, z^*\right) &= f(x) \oplus \inf_{y \in g(x)} S_{(y^*,z^*)}(y) = F(x) + \inf_{y \in g(x)} S_{(y^*,z^*)}(y) \\ &= \inf_{y \in g(x)} \left\{z + F(x) \in Z \mid y^*(y) \leq z^*(z)\right\} \\ &= \left\{z \in Z \mid \inf_{y \in g(x)} y^*(y) + z^*\left(F(x)\right) \leq z^*(z)\right\} \end{aligned}$$

whenever $F(x) \in Z$, $L(x, y^*, z^*) = \emptyset$ whenever $F(x) = +\infty$ or $g(x) = \emptyset$, and $L(x, y^*, z^*) = Z$ whenever $F(x) = -\infty$ and $g(x) \neq \emptyset$. The function $\Lambda_{z^*}(x, y^*) := z^*(F(x)) + \inf_{y \in g(x)} y^*(y)$ (with the convention $z^*(\pm\infty) = \pm\infty$) is the (classical)

Lagrange function of the (scalar) problem

$$\inf \left\{z^* \left(F(x)\right) \mid 0 \in g(x)\right\}$$

(see, for example, already [185, p. 197]). Moreover, if g is generated by a vector-valued function $G\colon X \to Y \cup \{-\infty, +\infty\}$ in the same way as f by F, then

$$\inf_{y \in g(x)} y^*(y) = \begin{cases} y^*\left(G(x)\right) & : \quad G(y) \in Z, \ y^* \in D^+ \\ -\infty & : \quad G(y) = -\infty, \text{ or } G(y) \in Z \text{ and } y^* \notin D^+ \\ +\infty & : \quad G(y) = +\infty. \end{cases}$$

Thus, $\Lambda_{z^*}\left(x, y^*\right) = z^*\left(F(x)\right) + y^*\left(G(y)\right)$ whenever $F(x) \in Z, G(x) \in Y$ and $y^* \in D^+$. The dual objective becomes

$$h(y^*, z^*) = \inf_{x \in X} L\left(x, y^*, z^*\right) = \left\{z \in Z \mid \inf_{x \in X} \Lambda_{z^*}\left(x, y^*\right) \leq z^*(z)\right\}.$$

Corollary 7.1 *Let F be C-convex, f its lattice extension and $g\colon X \to \mathcal{G}(Y, D)$ convex such that the Slater condition (6.5) is satisfied. If $I(f, g) = \inf\{f(x) \mid 0 \in g(x)\} \notin \{Z, \emptyset\}$, then $\Gamma^+(f, g) = \left\{z^* \in C^+\backslash\{0\} \mid I(f, g) \oplus H^+(z^*) \neq Z\right\}$ is non-empty and*

$$I(f, g) = \operatorname{cl} \bigcup \{F(x) \mid 0 \in g(x)\} = \bigcap_{y^* \in D^*,\, z^* \in \Gamma^+(f,g)} \left\{z \in Z \mid \Lambda_{z^*}\left(x, y^*\right) \leq z^*(z)\right\}, \tag{7.1}$$

$$\forall z^* \in \Gamma^+(f, g)\ \exists y^* \in Y^*\colon I(f, g) \oplus H^+\left(z^*\right) = h\left(y^*, z^*\right). \tag{7.2}$$

Proof Of course, f is convex if, and only if, F is C-convex (see [156, Definition 1.6 on p. 29] for a definition). Theorem 6.9 and the above discussion produce the result. □

It might be worth to compare Corollary 7.1 with standard duality results in vector optimization. First, there is no assumption about the existence of (weakly, properly) minimal solutions: This is in contrast to most results in vector optimization such as [67, Theorems 3.7.4 and 3.7.7], [114, Theorem 8.7], [19, Theorems 4.1.2 and 4.1.4]. Secondly, there are no interior point assumptions to the cone C. Thirdly, with Corollary 6.10 in view, the existence of a dual solution in a set-valued sense is provided in the sense of the "maximization" version of Definition 3.3. Finally, classical duality results in vector optimization can be obtained from Corollary 7.1 as it is described in [151, Sect. 3.5].

7.2 A Case Study: Linear Vector Optimization

We proceed with an exemplary application of the set-valued theory to linear vector optimization problems and show that we obtain what we expect in view of scalar linear programming duality: a dual program of the same type. In this section, we will write $\leq$ and $\geq$ for $\leq_{\mathbb{R}^m_+}$ and $\geq_{\mathbb{R}^m_+}$, respectively, for any $m \in \{1, 2, \ldots\}$.

Consider the linear vector optimization problem

$$\min{}_C \quad Px \quad \text{subject to} \quad Ax \geq b, \tag{P_L}$$

where $P \in \mathbb{R}^{q\times n}$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and the cone C is polyhedral convex with nonempty interior. A representation $C = \left\{z \in \mathbb{R}^q \mid W^T z \geq 0\right\}$ by a matrix $W \in \mathbb{R}^{q\times k}$ is given. The feasible set is denoted by $S := \{x \in \mathbb{R}^n \mid Ax \geq b\}$.

With (PC) in view we define $f(x) = Px + C$ and $g(x) = b - Ax + \mathbb{R}^m_+$. Then, the set $\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^q \mid z \in f(x),\ 0 \in g(x)\}$ is polyhedral convex. We modify the solution concept in Definition 6.5 by adding the requirement that a solution is a finite set of vectors and directions, see [151] and also [87], the latter also including "ε-variants." The reason is that every polyhedral set can be expressed as the generalized convex hull of finitely many vectors and directions. Such a solution is called *finitely generated solution*, but we call it just *solution* if the context of polyhedral convex set-valued problems or the subclass of linear vector optimization problems is clear. To keep the notation simple, we only consider *bounded* problems here, that is, we assume

$$\exists \bar{z} \in \mathbb{R}^q : \forall x \in S : \bar{z} \leq_C Px. \tag{7.3}$$

Under this assumption, a solution consists of finitely many vectors only. For the general case, see [151, Chap. 4]. A solution to (P_L) is a nonempty finite set $\bar{S} \subseteq S$ of minimizers ('efficient solutions' in the most textbooks) such that $P[S] \subseteq P[\bar{S}] + C$, where the latter condition refers to infimum attainment in $\bar{S}$ with respect to the lattice extension (compare Definition 3.3).

Considering the lattice extension of (P_L) we show that the Lagrange technique from Sect. 6.2 leads to a dual problem, which enjoys nice properties and is useful for applications and algorithms. Re-labeling the dual variables by $u = y^*$, $w = z^*$ we obtain the Lagrangian

$$\begin{aligned} L(x, u, w) &= Px + C + \operatorname{cl} \bigcup_{z \geq b - Ax} \left\{z \in \mathbb{R}^q \mid u^T y \leq w^T z\right\} \\ &= Px + C + \operatorname{cl} \bigcup_{r \in \mathbb{R}^m_+} \left\{z \in \mathbb{R}^q \mid u^T (r - Ax + b) \leq w^T z\right\}. \end{aligned}$$

The dual objective is

$$\begin{aligned} h(u,w) &= \text{cl} \bigcup_{x\in\mathbb{R}^n} L(x,u,w) \\ &= \text{cl} \bigcup_{r\in\mathbb{R}^m_+,\, x\in\mathbb{R}^n,\, v\in C} \left\{ z\in\mathbb{R}^q \mid u^T(r-Ax+b) \leq w^T(z-Px-v) \right\} \\ &= \text{cl} \bigcup_{r\in\mathbb{R}^m_+,\, x\in\mathbb{R}^n,\, v\in C} \left\{ z\in\mathbb{R}^q \mid (w^TP-u^TA)x \leq w^T(z-v)-u^T(b+r) \right\} \\ &= \begin{cases} \left\{ z\in\mathbb{R}^q \mid 0 \leq w^Tz-u^Tb \right\} & : \ A^Tu=P^Tw,\ u\geq 0,\ w\in C^+\setminus\{0\} \\ \mathbb{R}^q & : \ \text{otherwise.} \end{cases} \end{aligned}$$

Let $C^+ = \left\{ w\in\mathbb{R}^q \mid V^Tw \geq 0 \right\}$ be a representation of C^+ by a matrix $V\in\mathbb{R}^{q\times l}$. Note that a basis of C^+ is already sufficient to cover all values of the dual objective h (see the end of Sect. 5.2). If we fix some $c\in\text{int}\, C$, we obtain the (set-valued) dual problem

$$\text{maximize } D(u,w) \quad \text{subject to} \quad (u,w)\in T \qquad (D_L)$$

with objective function

$$D\colon \mathbb{R}^m\times C^+ \to \mathcal{G}(\mathbb{R}^q, C), \quad D(u,w) := \left\{ z\in\mathbb{R}^q \mid u^Tb \leq w^Tz \right\}$$

and feasible set

$$T := \left\{ (u,w)\in\mathbb{R}^m\times\mathbb{R}^q \mid A^Tu=P^Tw,\ u\geq 0,\ V^Tw\geq 0,\ c^Tw=1 \right\}.$$

This dual problem has a very simple structure: linear constraints, a halfspace-valued objective function and maximization means to take the intersection over these halfspaces. The objective function is conlinear in b and in u, i.e., $D(u,w) = S_{(u,w)}(b) = S_{(b,w)}(u)$, and therefore a natural replacement of the dual objective "b^Tu" in (scalar) linear programming. A (finitely generated) solution of (D_L) is a nonempty set $\bar{T}\subseteq T$ of maximizers with respect to the ordering $\supseteq$ satisfying $\bigcap_{(u,w)\in\bar{T}} D(u,w) = \bigcap_{(u,w)\in T} D(u,w)$, where the latter conditions means supremum attainment in $\bar{T}$.

Remark 7.2 Using the construction of Example 2.12, we obtain an equivalent problem with a hyperplane-valued objective. This shows that we indeed have a very natural generalization of scalar linear programs to the vectorial case because in $\mathbb{R}$, a real number and a hyperplane are the same object. In more general linear spaces, vectors and half-spaces are dual in some sense. Compare the footnote on p. 2.

Weak duality (see Proposition 6.8) means that $x\in S$ and $(u,w)\in T$ imply $D(u,w)\supseteq Px+C$. As a consequence, for every subset $\tilde{T}\subseteq T$ of feasible points, the set $\bigcap_{(u,w)\in\tilde{T}} D(u,w)$ is a superset ("outer approximation") of the set $\mathcal{P} := \{Px \mid Ax\geq b\}+C$, which is just the optimal value (the infimum) of the lattice extension. Likewise, for every subset $\tilde{S} \subseteq S$ of feasible points of (P_L), the set $\operatorname{cl}\operatorname{co} \bigcup_{x \in \tilde{S}} Px + C$ is a subset ("inner approximation") of $\mathcal{P}$.

Strong duality means that $\bigcap_{(u,w) \in T} D(u, w) = \mathcal{P}$. A constraint qualification is not needed as in the case of linear constraints in (scalar) convex programming. Note further that, if $\emptyset \neq \bar{S} \subseteq S$ such that $P[\bar{S}]$ is the set of vertices of $\mathcal{P}$, then $\bar{S}$ is a solution to (P_L). Likewise, a set $\emptyset \neq \bar{T} \subseteq T$ such that $\left\{D(u, w) \mid (u, w) \in \bar{T}\right\}$ is the family of half-spaces supporting $\mathcal{P}$ in facets, then $\bar{T}$ is a solution of (D_L).

Remark 7.3 In the vector optimization literature one can observe the longstanding paradigm that the dual of a vector optimization problem should be a vector optimization problem with the same ordering cone. To fulfill this requirement, problems of the type

$$\max\nolimits_C \quad z \quad \text{subject to} \quad z \in D(u, w), \;\; (u, w) \in T \tag{7.4}$$

have been considered, see e.g. [19, Sect. 4.5.1] and in the linear case [20]. The price is high. In general, important properties like linearity of the constraints and convexity of the feasible set get lost by such a transformation.

To emphasize the "linear" character of problem (D_L), we transform it into an equivalent linear vector optimization problem:

$$\max\nolimits_K \quad D^*(u, w) \quad \text{subject to} \quad (u, w) \in T, \qquad (D_L^*)$$

where the objective function $D^* \colon \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q$, given by

$$D^*(u, w) := (w_1, \ldots, w_{q-1}, b^T u)^T,$$

is linear and vector-valued, and the ordering cone is $K := \{\, z \in \mathbb{R}^q \mid z_1 = \cdots = z_{q-1} = 0, z_q \geq 0 \,\}$. A (finitely generated) solution of (D_L^*) is a nonempty set $\bar{T} \subseteq T$ of maximizers with respect to $\leq_K$ in $\mathbb{R}^q$ satisfying $D^*[T] \subseteq \operatorname{co} D^*[\bar{T}] - K$, where the latter condition refers to supremum attainment in $\bar{T}$ (with respect to the lattice extension with image space $\mathcal{G}(\mathbb{R}^q, K)$).

Proposition 7.4 *The problems (D_L) and (D_L^*) have the same solutions.*

Proof See [151, Theorems 4.38 and 4.57]. □

In the sense of the previous proposition, (D_L) and (D_L^*) are equivalent. This means that the set-valued dual problem (D_L) can be expressed as a linear vector optimization problem, however, with a different ordering cone K and an interpretation of the duality relation which differs from the one in standard references.

Of course, we can derive a set-valued dual problem to (D_L^*) by an analogous procedure. This leads to outer and inner approximations and different representations of $\mathcal{D} := \{D^*(u, w) - K \mid (u, w) \in T\}$, i.e., the optimal value of the lattice extension of (D_L^*).

Problem (D^*_L) is called the *geometric dual problem*, and there is a further duality relation called *geometric duality* [101] between (P_L) and (D^*_L): There is an inclusion-reversing one-to-one map between the proper faces of $\mathcal{P}$ and the proper K-maximal faces of $\mathcal{D}$. This means, for instance, that a vertex of one set can be used to describe a facet of the other set and vice versa. For a detailed explanation of geometric duality see [101, 151]. Geometric duality has been extended to convex vector optimization problems, see [100]. The paper [157] is in the same spirit.

7.3 Approximate Solutions and Algorithms

In this section, we assume that C is a closed convex cone. Let $f\colon X \to \mathcal{G}(Z, C)$ be a function. The starting point for constructing algorithms for solving the problem (P) (see Sect. 3.1), i.e.

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in X \tag{P}$$

should be Definition 3.3: It involves minimal values of f as well as the infimum taken in $\mathcal{G}(Z, C)$. In order to make algorithms reasonable, both notions should be replaced by appropriate approximate versions.

Recall $I(f) = \inf_{x \in X} f(x)$. Two sets $A, B \in \mathcal{G}(Z, C)$ are called an outer approximation and an inner approximation of $I(f)$, respectively, if $A \supseteq I(f) \supseteq B$. Outer and inner approximations of $I(f)$ could be generated by sets $M \subseteq \operatorname{dom} f$ or by dual admissible elements.

Definition 7.5 Let $D\colon \mathbb{R}_+ \to \mathcal{G}(Z, C)$ be a function satisfying

(i) $D(\varepsilon_2) \supseteq D(\varepsilon_1)$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ with $0 < \varepsilon_1 \leq \varepsilon_2$, and
(ii) $C = D(0) = \bigcap_{\varepsilon > 0} D(\varepsilon)$.

A set $M \subseteq \operatorname{dom} f$ is called a (D, ε)-solution of (P) if

$$\inf f[M] \oplus D(\varepsilon) \supseteq I(f),$$

and each $x \in M$ is a minimizer of f.

A similar concept applies to supremum problems which can be useful in connection with duality. If M is a (D, ε)-solution of (P), then

$$\inf f[M] \oplus D(\varepsilon) \supseteq I(f) \supseteq \inf f[M],$$

i.e., $\inf f[M]$ trivially is an inner approximation of $I(f)$.

The condition that elements of M be minimizers for f might be relaxed to any type of approximate minimizers, thus producing sets of (D, ε)-solutions consisting of approximate minimizers. Similarly, the intersection in (ii) might be replaced by any type of set convergence which is sometimes useful if $C \subseteq D(\varepsilon)$ is not satisfied for some (or all) $\varepsilon > 0$.

It turned out that effective algorithms for vector and set optimization problems generate (D, ε)-solutions, for example with

$$D(\varepsilon) = C - \varepsilon c$$

with some $c \in C \backslash (-C)$, even $c \in \operatorname{int} C$ under the assumption that the latter set is non-empty. This idea has been exploited with Benson's outer approximation algorithm as the building block, see [87, Remark 4.10] and [153, Proposition 4.8]. The obtained algorithms indeed produce approximations of the set-valued infimum for (linear, convex) vector optimization problems. In [154], it is shown that the same idea can be used for minimizing a polyhedral set-valued function (i.e., a $\mathcal{G}(\mathbb{R}^q, C)$-valued function whose graph is a polyhedral set): The corresponding algorithm produces solutions in the sense of Definition 3.3 and might be considered as the first "true set-valued" algorithm. Its extension to non-polyhedral problems is highly desirable and another challenge for the future.

We note that a different algorithmic approach for producing minimizers with respect to a set relation can be found in [116]. In particular, it provides a numerical test if two (compact) sets $A, B \subseteq Z$ are in relation with respect to $\preccurlyeq_C \cap \curlyeqprec_C$ (compare the closely related Sect. 4.2 of this survey and [115]). In the polyhedral case, this test can be implemented on a computer. An algorithm is given which produces minimizers of a set-valued function if the set of feasible points is finite, and a descent method [116, Algorithm 4.1] for problem (P) generates feasible points which are minimal with respect to some finite subset of the set of feasible points.

7.4 Set-valued Risk Measures

Set-valued risk measures shall serve as a prominent example of set-valued translative functions as discussed in Sect. 5.5. The framework will be the following. By a slight abuse of notation, X in this section does not denote a linear space, but rather a random variable etc.

Let $(\Omega, \mathcal{F}_T, P)$ be a probability space. A multivariate random variable is a P-measurable function $X \colon \Omega \to \mathbb{R}^d$ for some positive integer $d \geq 2$. If $d = 1$, the random variable is called univariate. Let us denote by $L^0_d = L^0_d(\Omega, \mathcal{F}_T, P)$ the linear space of the equivalence classes of all $\mathbb{R}^d$-valued random variables which coincide up to sets of P-measure zero (P-almost surely). As usual, we write

$$\left(L^0_d\right)_+ = \left\{X \in L^0_d \mid P\left(\left\{\omega \in \Omega \mid X(\omega) \in \mathbb{R}^d_+\right\}\right) = 1\right\}$$

for the closed convex cone of $\mathbb{R}^d$-valued random vectors with P-almost surely non-negative components. An element $X \in L_d^0$ has components $X_1, \dots, X_d$ in $L^0 = L_1^0$. In a similar way, we use L_d^p for the spaces of equivalence classes of d-dimensional random variables whose components are to the pth power integrable (if $0 < p < \infty$) and essentially bounded (if $p = \infty$). The symbol $1\!\!1$ denotes the random variable in L_1^0 which has P-almost surely the value 1.

Let $M \subseteq \mathbb{R}^d$ be a linear subspace. We set $M_+ = M \cap \mathbb{R}^d_+$ and assume $M_+ \neq \{0\}$ in the following.

Definition 7.6 ([82]) A function $R\colon L_d^p \to \mathcal{P}(M, M_+)$ is called a risk measure if it is

(R0) finite at $0 \in L_d^p$: $R(0) \neq \emptyset$, $R(0) \neq M$;
(R1) M-translative:

$$\forall X \in L_d^p,\ \forall u \in M\colon R(X + u1\!\!1) = R(X) - u; \tag{7.5}$$

(R2) $\left(L_d^p\right)_+$-monotone: $X^2 - X^1 \in \left(L_d^p\right)_+ \Rightarrow R\left(X^2\right) \supseteq R\left(X^1\right)$.

Set-valued risk measures are indeed recognized as T-translative if, within the notation of Sect. 5.5, $X = L_d^p$, $Z = M$, $C = M_+ \subseteq M$ and the linear operator $T\colon M \to L_d^p$ is defined by $Tu = -u1\!\!1$. This means that T assigns to each $u \in M$ the random vector being constantly equal to $-u$.

A financial interpretation is as follows. A multivariate random variable is understood as a model for an unknown future portfolio or payoff of d assets where each component indicates the number of units of the corresponding asset in the portfolio. The elements of $R(X)$ are understood as deposits, to be given at initial time, which compensate for the risk of X. The collection of all such risk compensating initial portfolios is understood as a measure of the risk associated to X. Such deposits usually involve fewer assets than the original portfolio, for example cash in a few currencies. This motivates the introduction of the space M which is called the space of eligible portfolios. A typical example is $M = \mathbb{R}^m \times \{0\}^{d-m}$ for $1 \leq m \leq d$ with $m \ll d$.

The axiom (R1) roughly means that the risk of $X + u1\!\!1$ is the risk of X reduced by u whenever $u \in M$. Axiom (R2) also has a clear interpretation: if a random vector $Y \in L_d^p$ dominates another random vector $X \in L_d^p$, then there should be more possibilities to compensate for the risk of Y (in particular cheaper ones) than for X. Finiteness at zero means that there is an eligible portfolio which covers the risk of the zero payoff, but not all eligible portfolios do. Convexity is an important property as it allows to invoke diversification effects.

From M-translativity and $\left(L_d^p\right)_+$-monotonicity it follows that R maps into $\mathcal{P}(M, M_+)$. Clearly, the image space of a closed convex risk measure is $\mathcal{G}(M, M_+)$.

If trading is allowed a market model has to be incorporated. Here, a one-period market with proportional transaction costs as in [121, 205] is considered. It is given by closed convex cones K_0 and $K_T = K_T(\omega)$ with $\mathbb{R}^d_+ \subseteq K_t(\omega) \subsetneq \mathbb{R}^d$ for all $\omega \in \Omega$ and $t \in \{0, T\}$ such that $\omega \mapsto K_T(\omega)$ is $\mathcal{F}_T$-measurable. These cones, called solvency cones, include precisely the set of positions which can be exchanged into a nonnegative portfolio at time 0 and T, respectively, by trading according to the prevailing exchange rates. We set $K_0^M := M \cap K_0 \subseteq M$ which is the cone containing the "solvent" eligible portfolios. The set

$$L_d^p(K_T) = \left\{X \in L_d^p \mid P(\{\omega \in \Omega \mid X(\omega) \in K_T(\omega)\}) = 1\right\}$$

is a closed convex cone in L_d^p.

Definition 7.7 ([82]) A risk measure $R\colon L_d^p \to \mathcal{P}(M, M_+)$ is called market-compatible if it maps into $\mathcal{P}\left(M, K_0^M\right)$ and is $L_d^p(K_T)$-monotone, that is $X^2 - X^1 \in L_d^p(K_T)$ implies $R\left(X^2\right) \supseteq R\left(X^1\right)$.

Let $1 \leq p \leq \infty$. We consider the dual pairs (L_d^p, L_d^q) with $\frac{1}{p} + \frac{1}{q} = 1$ and endow them with the norm topology if $p < \infty$ and the $\sigma\left(L_d^\infty, L_d^1\right)$-topology on L_d^∞ in the case $p = +\infty$, respectively. The duality pairing is given by $(X, Y) \mapsto E[Y^T X]$ for $X \in L_d^p, Y \in L_d^q$. The adjoint operator $T^*\colon L_d^q \to M$ is given by $T^*Y = \Pr_M E[-Y]$ where $\Pr_M$ denotes the projection operator onto the linear subspace M.

The biconjugation theorem, Theorem 5.8, can be used to obtain a dual description of a closed convex market-compatible set-valued risk measure of the form

$$R(X) = R^{**}(X) = \bigcap_{Y \in L_d^q,\, v \in \left(K_0^M\right)^+ \setminus \{0\}} \left(S_{(Y,v)}(X) + (-R^*)(Y, v)\right) \tag{7.6}$$

with

$$S_{(Y,v)}(X) = \left\{u \in M \mid v^T u \geq E\left[Y^T X\right]\right\}$$

and $\left(K_0^M\right)^+ = \left\{v \in M \mid \forall u \in K_0^M \colon v^T u \geq 0\right\}$.

Using the considerations of Sect. 5.5 and taking into account that $L_d^p(K_T)$-monotonicity implies $(-R^*)(Y, v) = M$ if $-Y \notin L_d^q\left(K_T^+\right)$ we get

$$(-R^*)(Y, v) = \begin{cases} \operatorname{cl} \bigcup\limits_{X \in A_R} S_{(-Y,v)}(X) & : \quad -Y \in L_d^q\left(K_T^+\right),\ v = \Pr_M E[-Y] \\ M & : \quad \text{else.} \end{cases} \tag{7.7}$$

Recall $A_R = \left\{X \in L_d^p \mid 0 \in R(X)\right\}$ from Sect. 5.5.

The next lemma admits a change of variables from vector densities Y to vector probability measures Q. This allows a formulation of the dual representation result in terms of probability measures as it is common in the scalar case.

In the following, $\operatorname{diag}(w)$ with $w \in \mathbb{R}^d$ denotes the diagonal matrix with the components of w as entries in its main diagonal and zero elsewhere. Moreover, $\mathcal{M}_d^P =$

$\mathcal{M}_d^P(\Omega, \mathcal{F}_T)$ denotes the set of all vector probability measures with components being absolutely continuous with respect to P, i.e. $Q_i : \mathcal{F}_T \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F}_T)$ such that $\frac{dQ_i}{dP} \in L^1$ for $i = 1, \ldots, d$.

Lemma 7.8 *(a) Let* $Y \in L_d^q\left(K_T^+\right)$, $v = \mathrm{Pr}_M\, E[Y] \in \left(K_0^M\right)^+ \setminus\{0\}$. *Then there are* $Q \in \mathcal{M}_d^P$, $w \in K_0^+ \setminus M^\perp + M^\perp$ *such that* $\mathrm{diag}(w)\frac{dQ}{dP} \in L_d^q\left(K_T^+\right)$ *and* $S_{(Y,v)} = F_{(Q,w)}^M$ *with*

$$F_{(Q,w)}^M[X] = \left\{z \in M \mid w^T E^Q[X] \leq w^T z\right\} = \left(E^Q[X] + H^+(w)\right) \cap M. \tag{7.8}$$

(b) Vice versa, if $Q \in \mathcal{M}_d^P$, $w \in K_0^+ \setminus M^\perp + M^\perp$ *such that* $\mathrm{diag}(w)\frac{dQ}{dP} \in L_d^q\left(K_T^+\right)$ *then there is* $Y \in L_d^q\left(K_T^+\right)$ *such that* $v := \mathrm{Pr}_M\, E[Y] \in \left(K_0^M\right)^+ \setminus\{0\}$ *and* $F_{(Q,w)}^M = S_{(Y,v)}$.

Proof See [82]. □

Let us denote the set of dual variables by

$$\mathcal{W}^q = \left\{(Q, w) \in \mathcal{M}_d^P \times \mathbb{R}^d \mid w \in K_0^+ \setminus M^\perp + M^\perp,\ \mathrm{diag}(w)\frac{dQ}{dP} \in L_d^q\left(K_T^+\right)\right\}.$$

The preceding considerations lead to the following dual representation result.

Theorem 7.9 *A function* $R\colon L_d^p \to \mathcal{G}\left(M, K_0^M\right)$ *is a market-compatible closed* ($\sigma\left(L_d^\infty, L_d^1\right)$*-closed if* $p = \infty$) *convex risk measure if, and only if, there is a set* $\mathcal{W}_R^q \subseteq \mathcal{W}^q$ *such that*

$$\forall X \in L_d^p\colon R(X) = \bigcap_{(Q,w)\in\mathcal{W}^q} \left[(-\alpha_R)(Q, w) + \left(E^Q[-X] + H^+(w)\right) \cap M\right], \tag{7.9}$$

where the function $-\alpha_R\colon \mathcal{W}^q \to \mathcal{G}(M, M_+)$ *is defined by*

$$\forall (Q, w) \in \mathcal{W}_R^q\colon (-\alpha_R)(Q, w) = \mathrm{cl} \bigcup_{X' \in A_R} \left(E^Q\left[X'\right] + H^+(w)\right) \cap M$$

and $(-\alpha_R)(Q, w) = M$ *whenever* $(Q, w) \in \mathcal{W}^q \setminus \mathcal{W}_R^q$.

Proof See [82]. □

Lemma 7.8 shows that the set $\mathcal{W}^\infty$ for $M = \mathbb{R}^d$ coincides with the set of so-called consistent price systems (or processes). Strictly consistent price systems are crucial for market models with proportional transaction cost: In finite discrete time, the existence of such a price system is equivalent to the fundamental robust-no-arbitrage condition (see [205] for conical and [187] for convex market models). Therefore, results like Theorem 7.9, derived with set-valued duality tools, fit nicely into the mathematical finance background: They produce the correct dual variables, and they yield formulas which look like the corresponding scalar ones.

7.5 Comments on Applications

Duality for vector optimization problems is already discussed in Sect. 6.3. We add a few remarks about the linear case. It is an astounding fact that there still is no consensus on what to consider as the "canonical" dual of a linear vector optimization problem. After early contributions by Kornbluth [127], Isermann [110, 111] and Rödder [198], Ivanov and Nehse [112] discuss five different duals for a given linear vector optimization problem which illustrates the ambiguity even in the "simplest", i.e. linear, case. The difficulty is further illustrated by means of the examples in [24] and [114, Discussion after Theorem 8.13]. A set-valued approach has been presented in [81] and later compared to several "vector-valued" duals in [20]. Compare also [103] and Dinh The Luc [157]. We believe that this ambiguity and the mathematical difficulties that come with it are rooted in the non-totalness of the order: A two-player matrix game with vector payoffs is hardly in equilibrium since the decisions of the players also depend on their "vertical preferences" (as well as on their guesses about the vertical preference of the opponent), i.e. the weight they put on the components of the payoff vectors. This topic, essentially the link between set-valued convex duality and games with vector payoffs (more general, payoffs which are not totally ordered), seems to be one of the most interesting open questions that can be derived from the material presented in this survey.

One advantage of the complete lattice approach is that the set-valued calculus deals with all "vertical preferences", i.e. all reasonable scalarizations at the same time. This admits to re-discover scalar duality results on a "set level."

In 1998, Benson [10, 11] proposed an "outer approximation algorithm" to solve linear vector optimization problems "in the outcome space." Benson motivated this by three practical reasons: First, the set of minimal elements in the outcome space $\mathbb{R}^q$ has a simpler structure than the set of minimizers in the decision space $\mathbb{R}^n$, because one usually has $q \ll n$. The same argument motivated [40, 41], which already contain similar algorithms based on the analysis given in [39]. The second reason is that a decision maker prefers to base her decision on objectives rather than directly on a set of efficient decisions. The third argument is that many feasible points are mapped on a single image point which may lead to redundant information.

Later it turned out that Benson's algorithm just computes solutions to (P_L) and (D_L) as defined above, see [87, 151]. Therefore Benson's arguments motivate the solution concepts introduced in Sect. 3.1 from an application oriented viewpoint, compare also [154]. The geometric duality theory [100, 101] briefly discussed in

Sect. 7.2 is a fundamental tool to develop dual algorithms for solving linear and convex vector optimization problems, see [49, preprint already from 2007] [87, 153]. Compare also [158, 159] for the convex case and [70, 71], even for nonconvex variants.

Set-valued risk measures have been introduced in [120]. It contains a dual representation result for the sublinear case, basically a combination of the formulae (5.12) and (7.6). A more systematic development including the extension to the general convex case has been presented in [80] while market compatibility is due to [82]. A link to depth-trimmed regions, yet another set-valued object from statistics, can be found in [25]. Currently, the set-valued approach for evaluating multivariate risks is gaining more and more attention, see for example [53, 119, 165] and also [51, 52]. Applications of Benson's algorithm and its variants to financial problems can be found in [87, 88, 152] and related approaches in [200, 201] as well as in [38].

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References

1. Alonso, M., Rodríguez-Marín, L.: Set-relations and optimality conditions in set-valued maps. Nonlinear Anal. **63**(8), 1167–1179 (2005)
2. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**(9), 3821–3835 (2012)
3. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. Math. Finance **9**, 203–228 (1999)
4. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis Modern Birkhäuser Classics. Birkhäuser, Boston (2009). Reprint of the 1990 edition
5. Azimov, A.Y.: Duality of multiobjective problems. Math. USSR-Sb. **59**(2), 515 (1988)
6. Azimov, A.Y.: Duality for set-valued multiobjective optimization problems, part 1: mathematical programming. J. Optim. Theory Appl. **137**(1), 61–74 (2008)
7. Baier, J., Jahn, J.: On subdifferentials of set-valued maps. J. Optim. Theory Appl. **100**(1), 233–240 (1999)
8. Ben-Tal, A., Nemirovskiĭ, A .S.: Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPS & SIAM, Philadelphia (2001)
9. Benoist, J., Popovici, N.: Characterizations of convex and quasiconvex set-valued maps. Math. Methods Oper. Res. **57**(3), 427–435 (2003)
10. Benson, H.P.: Further analysis of an outcome set-based algorithm for multiple-objective linear programming. J. Optim. Theory Appl. **97**(1), 1–10 (1998)
11. Benson, H.P.: An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. J. Glob. Optim. **13**(1), 1–24 (1998)
12. Beresnev, V.V.: Mappings conjugate to convex many-valued mappings. Cybern. Syst. Anal. **9**(5), 813–819 (1973)
13. Blyth, T.S., Janowitz, M.F.: Residuation Theory. Pergamon Press, Oxford (1972)
14. Borwein, J.M.: Multivalued convexity and optimization: a unified approach to inequality and equality constraints. Math. Program. **13**(2), 183–199 (1977)

15. Borwein, J.M.: Proper efficient points for maximizations with respect to cones. SIAM J. Control Optim. **15**(1), 57–63 (1977)
16. Borwein, J.M.: A Lagrange multiplier theorem and a sandwich theorem for convex relations. Math. Scand. **48**(2), 189–204 (1981)
17. Borwein, J.M., Penot, J.P., Thera, M.: Conjugate convex operators. J. Math. Anal. Appl. **102**(2), 399–414 (1984)
18. Bossert, W.: On the extension of preferences over a set to the power set: an axiomatic characterization of a quasi-ordering. J. Econ. Theory **49**(1), 84–92 (1989)
19. Boţ, R.I., Grad, S.-M., Wanka, G.: Duality in Vector Optimization. Vector Optimization. Springer, Berlin (2009)
20. Boţ, R.I., Grad, S.-M., Wanka, G.: Classical linear vector optimization duality revisited. Optim. Lett. **6**(1), 199–210 (2012)
21. Breckner, W.W., Kolumbán, I.: Konvexe Optimierungsaufgaben in topologischen Vektorräumen. Mathematica Scandinavica **25**, 227–247 (1969)
22. Brink, C.: Power structures. Algebra Univers. **30**(2), 177–216 (1993)
23. Brumelle, S.L.: Convex operators and supports. Math. Oper. Res. **3**(2), 171–175 (1978)
24. Brumelle, S.L.: Duality for multiple objective convex programs. Math. Oper. Res. **6**(2), 159–172 (1981)
25. Cascos, I., Molchanov, I.: Multivariate risks and depth-trimmed regions. Finance. Stoch. **11**, 373–397 (2007)
26. Chen, G.Y., Huang, X.X.: Ekeland's ε-variational principle for set-valued mappings. Math. Methods Oper. Res. **48**(2), 181–186 (1998)
27. Chen, G.Y., Huang, X.X., Hou, S.H.: General Ekeland's variational principle for set-valued mappings. J. Optim. Theory Appl. **106**(1), 151–164 (2000)
28. Chen, G.Y., Huang, X.X., Yang, X.: Vector Optimization: Set-Valued and Variational Analysis. LNEMS, vol. 541. Springer, Berlin (2006)
29. Chen, G.Y., Jahn, J.: Optimality conditions for set-valued optimization problems. Math. Methods Oper. Res. **48**(2), 187–200 (1998)
30. Chiriaev, D., Walster. G.W.: Interval arithmetic specification (1998). Available online at http://www.mscs.mu.edu/~globsol/walster-papers.html [69]
31. Cohen, G., Gaubert, S., Quadrat, J.-P., Singer, I.: Max-plus convex sets and functions. In: Litvinov, G.L., Maslov, V.P. (eds.) Idempotent Mathematics and Mathematical Physics. Contemporary Mathematics, vol. 377, pp. 105–129, Am. Math. Soc., Providence (2005)
32. Corley, H.W.: Existence and Lagrangian duality for maximizations of set-valued functions. J. Optim. Theory Appl. **54**(3), 489–501 (1987)
33. Corley, H.W.: Optimality conditions for maximizations of set-valued functions. J. Optim. Theory Appl. **58**(1), 1–10 (1988)
34. Crespi, G.P., Ginchev, I., Rocca, M.: First-order optimality conditions for constrained set-valued optimization. Pac. J. Optim. **2**, 225–239 (2006)
35. Crespi, G.P., Ginchev, I., Rocca, M.: First-order optimality conditions in set-valued optimization. Math. Methods Oper. Res. **63**(1), 87–106 (2006)
36. Crespi, G.P., Hamel, A.H., Schrage, C.: A Minty variational principle for set optimization. J. Math. Anal. Appl. **423**, 770–796 (2014)
37. Cross, R.: Multivalued Linear Operators. Pure and Applied Mathematics, vol. 213. Marcel Dekker, Inc., New York (1998)
38. Csirmaz, L.: Using multiobjective optimization to map the entropy region of four random variables Comput. Optim. Appl. (2015). doi:10.1007/s10589-015-9760-6
39. Dauer, J.P.: Analysis of the objective space in multiple objective linear programming. J. Math. Anal. Appl. **126**(2), 579–593 (1987)
40. Dauer, J.P., Liu, Y.-H.: Solving multiple objective linear programs in objective space. Eur. J. Oper. Res. **46**(3), 350–357 (1990)
41. Dauer, J.P., Saleh, O.A.: Constructing the set of efficient objective values in multiple objective linear programs. Eur. J. Oper. Res. **46**(3), 358–365 (1990)

42. Day, M.M.: Normed Linear Spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete Band, vol. 21. Springer, Berlin (1958)
43. Dedekind, R.: Stetigkeit und irrationale Zahlen. 6te unveränderte Aufl. Friedr. Vieweg & Sohn, Braunschweig (1960)
44. Denneberg, D.: Non-Additive Measure and Integral. Springer, Netherlands (1994)
45. Dien, P.H.: Locally Lipschitzian set-valued maps and generalized extremal problems with inclusion constraints. Acta Math. Vietnam. **8**(1), 109–122 (1983)
46. Dilworth, R.P.: Abstract residuation over lattices. Bull. Am. Math. Soc. **44**(4), 262–268 (1938)
47. Dolecki, S., Malivert, C.: General duality in vector optimization. Optimization **27**(1–2), 97–119 (1993)
48. Durea, M., Strugariu, R.: On some fermat rules for set-valued optimization problems. Optimization **60**(5), 575–591 (2011)
49. Ehrgott, M., Löhne, A., Shao, L.: A dual variant of Bensons outer approximation algorithm for multiple objective linear programming. J. Glob. Optim. **52**(4), 757–778 (2012)
50. Elster, K.-H., Nehse, R.: Konjugierte Operatoren und Subdifferentiale. Math. Operationsforsch. u. Statist. **6**(4), 641–657 (1975)
51. Farkas, W., Koch-Medina, P., Munari, C.: Measuring risk with multiple eligible assets, Math. Finance Econ. **9** (1), 3–27 (2015)
52. Farkas, W., Koch-Medina, P., Munari, C.-A.: Beyond cash-additive risk measures: when changing the numéraire fails. Finance. Stoch. **18**(1), 145–173 (2014)
53. Feinstein, Z., Rudloff, B.: Multi-portfolio time consistency for set-valued convex and coherent risk measures. Finance Stoch. **19** (1), 67–107 (2015)
54. Ferro, F.: An optimization result for set-valued mappings and a stability property in vector problems with constraints. J. Optim. Theory Appl. **90**(1), 63–77 (1996)
55. Ferro, F.: Optimization and stability results through cone lower semicontinuity. Set-valued Anal. **5**(4), 365–375 (1997)
56. Fishburn, P.C.: Signed orders and power set extensions. J. Econ. Theory **56**(1), 1–19 (1992)
57. Flores-Bazán, F.: Optimality conditions in non-convex set-valued optimization. Math. Methods Oper. Res. **53**(3), 403–417 (2001)
58. Flores-Bazán, F., Jiménez, B.: Strict efficiency in set-valued optimization. SIAM J. Control Optim. **48**(2), 881–908 (2009)
59. Föllmer, H., Schied, A.: Stochastic Finance: An Introduction in Discrete Time, 3rd extended edn. Walter de Gruyter & Co., Berlin (2011)
60. Fuchssteiner, B., Lusky, W.: Convex Cones: North-Holland Mathematics Studies, vol. 56. North-Holland Publishing Co., Amsterdam (1981)
61. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Elsevier B. V, Amsterdam (2007)
62. Gaubert, S., Plus, M.: Methods and applications of (max,+) linear algebra. STACS 97, pp. 261–282. Springer, Berlin (1997)
63. Gerth (Tammer), C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**(2), 297–320 (1990)
64. Getán, J., Martínez-Legaz, J.-E., Singer, I.: $(*, s)$-dualities. J. Math. Sci. **115**(4), 2506–2541 (2003)
65. Glover, B.M., Jeyakumar, V., Rubinov, A.M.: A general approach to dual characterizations of solvability of inequality systems with applications. J. Convex Anal. **2**(1–2), 309–344 (1995)
66. Godini, G.: A framework for best simultaneous approximation: normed almost linear spaces. J. Approx. Theory **43**(4), 338–358 (1985)
67. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. CMS Books in Mathematics 17. Springer, New York (2003)
68. Gorokhovik, V.V.: Representations of affine multifunctions by affine selections. Set-valued Anal. **16**(2–3), 185–198 (2008)
69. Götz, A., Jahn, J.: The Lagrange multiplier rule in set-valued optimization. SIAM J. Optim. **10**(2), 331–344 (2000)

70. Gourion, D., Luc, Dinh The: Generating the weakly efficient set of nonconvex multiobjective problems. J. Glob. Optim. **41**(4), 517–538 (2008)
71. Gourion, D., Luc, Dinh The: Finding efficient solutions by free disposal outer approximation. SIAM J. Optim. **20**(6), 2939–2958 (2010)
72. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**(4), 1822–1833 (2012)
73. Ha, T.X.D.: The Ekeland variational principle for set-valued maps involving coderivatives. J. Math. Anal. Appl. **286**(2), 509–523 (2003)
74. Ha, T.X.D.: Lagrange multipliers for set-valued optimization problems associated with coderivatives. J. Math. Anal. Appl. **311**(2), 647–663 (2005)
75. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**(1), 187–206 (2005)
76. Hadwiger, H.: Minkowskische Addition und Substraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt. Math. Z. **53**, 210–218 (1950)
77. Hamel, A.H.: Variational Principles on Metric and Uniform Spaces. Habilitation thesis, Martin-Luther-Universität Halle-Wittenberg (2005)
78. Hamel, A.H.: A duality theory for set-valued functions I: Fenchel conjugation theory. Set-valued Var. Anal. **17**(2), 153–182 (2009)
79. Hamel, A.H.: A Fenchel-Rockafellar duality theorem for set-valued optimization. Optimization **60**(8–9), 1023–1043 (2011)
80. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**(1), 66–95 (2010)
81. Hamel, A.H., Heyde, F., Löhne, A., Tammer, C., Winkler, K.: Closing the duality gap in linear vector optimization. J. Convex Anal. **11**(1), 163–178 (2004)
82. Hamel, A.H., Heyde, F., Rudloff, B.: Set-valued risk measures for conical market models. Math. Financ. Econ. **5**(1), 1–28 (2011)
83. Hamel, A.H., Löhne, A.: Minimal set theorems. Preprint #11, Martin-Luther-Universität Halle-Wittenberg (2002)
84. Hamel, A.H., Löhne, A.: A minimal point theorem in uniform spaces. In Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. vol. 1, 2, pp. 577–593. Kluwer Academic Publishers, Dordrecht (2003)
85. Hamel, A.H., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**(1), 19–37 (2006)
86. Hamel, A.H., Löhne, A.: Lagrange duality in set optimization. J. Optim. Theory Appl. **161**(2), 368–397 (2014)
87. Hamel, A.H., Löhne, A., Rudloff, B.: A Benson type algorithm for linear vector optimization and applications. J. Glob. Optim. **59**(4), 811–836 (2014)
88. Hamel, A.H., Rudloff, B., Yankova, M.: Set-valued average value at risk and its computation. Math. Financ. Econ. **7**(2), 229–246 (2013)
89. Hamel, A.H., Schrage, C.: Notes on extended real- and set-valued functions. J. Convex Anal. **19**(2), 355–384 (2012)
90. Hamel, A.H., Schrage, C.: Directional derivatives and subdifferentials of set-valued convex functions. Pac. J. Optim., **10**(4), 667–689 (2014)
91. Hamel, A.H.: Optimal control with set-valued objective function. In: Proceedings of the 6th Portuguese Conference on Automatic Control–Controlo, pp. 648–652. Faro, Portugal (2004)
92. Hernández, E., Löhne, A., Rodríguez-Marín, L., Tammer, C.: Lagrange duality, stability and subdifferentials in vector optimization. Optimization **62**(3), 415–428 (2013)
93. Hernández, E., Rodríguez-Marín, L.: Duality in set optimization with set-valued maps. Pac. J. Optim. **3**, 245–255 (2007)
94. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**(6), 1726–1736 (2007)
95. Hernández, E., Rodríguez-Marín, L.: Lagrangian duality in set-valued optimization. J. Optim. Theory Appl. **134**(1), 119–134 (2007)

96. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**(1), 1–18 (2007)
97. Hernández, E., Rodríguez-Marín, L., Sama, M.: Some equivalent problems in set optimization. Pac. J. Optim. **37**(1), 61–64 (2009)
98. Hernández, L., Rodríguez-Marín, E.: Weak and strong subgradients of set-valued maps. J. Optim. Theory Appl. **149**(2), 352–365 (2011)
99. Heyde, F.: Coherent risk measures and vector optimization. In: Küfer, K.-H., Rommelfanger, H., Tammer, C., Winkler, K. (eds.) Multicriteria Decision Making and Fuzzy Systems, pp. 3–12. Shaker Verlag, Aachen (2006)
100. Heyde, F.: Geometric duality for convex vector optimization problems. J. Convex Anal. **20**(3), 813–832 (2013)
101. Heyde, F., Löhne, A.: Geometric duality in multiple objective linear programming. SIAM J. Optim. **19**(2), 836–845 (2008)
102. Heyde, F., Löhne, A.: Solution concepts in vector optimization: a fresh look at an old story. Optimization **60**(12), 1421–1440 (2011)
103. Heyde, F., Löhne, A., Tammer, C.: Set-valued duality theory for multiple objective linear programs and application to mathematical finance. Math. Methods Oper. Res. **69**(1), 159–179 (2009)
104. Heyde, F., Schrage, C.: Continuity concepts for set-valued functions and a fundamental duality formula for set-valued optimization. J. Math. Anal. Appl. **397**(2), 772–784 (2012)
105. Hu, R., Fang, Y.-P.: Set-valued increasing-along-rays maps and well-posed set-valued star-shaped optimization. J. Math. Anal. Appl. (2007)
106. Huang, X.X.: A new variant of Ekeland's variational principle for set-valued maps. Optimization **52**(1), 53–63 (2003)
107. Ide, J., Köbis, E.: Concepts of robustness for multi-objective optimization problems based on set order relations. Preprint #24, Georg-August-Universität Göttingen, Institut f. Numerische u. Angewandte Mathematik (2013)
108. Ide, J., Köbis, E., Kuroiwa, D., Schöbel, A., Tammer, C.: The relationship between multi-objective robustness concepts and set-valued optimization. Fixed Point Theory Appl. **83** (2014)
109. Ide, J., Schöbel, A.: Robustness for uncertain multi-objective optimization. Preprint #27, Georg-August-Universität Göttingen, Institut f. Numerische u. Angewandte Mathematik (2013)
110. Isermann, H.: Duality in multiple objective linear programming. In: Zionts, S. (ed.) Multiple Criteria Problem Solving, pp. 274–285, Springer, Berlin (1978)
111. Isermann, H.: On some relations between a dual pair of multiple objective linear programs. Zeitschrift für Oper. Res. **22**(1), 33–41 (1978)
112. Ivanov, E.H., Nehse, R.: Some results on dual vector optimization problems. Optimization **16**(4), 505–517 (1985)
113. Jahn, J.: Duality in vector optimization. Math. Program. **25**(3), 343–353 (1983)
114. Jahn, J.: Vector Optimization. Springer, Berlin (2004)
115. Jahn, J.: Vectorization in set optimization. J. Optim. Theory Appl. (2013). online first. doi:10.1007/s10957-013-0363-z
116. Jahn, J.: A derivative-free descent method in set optimization. Comput. Optim. Appl. **60**(2), 393–411 (2015)
117. Jahn, J., Truong, X.D.H.: New order relations in set optimization. J. Optim. Theory Appl. **148**(2), 209–236 (2011)
118. Jahn, K.-U.: Maximale Fixpunkte von Intervallfunktionen. Computing **33**(2), 141–151 (1984)
119. Joe, H., Li, H.: Tail risk of multivariate regular variation. Methodol. Comput. Appl. Probab. **13**(4), 671–693 (2011)
120. Jouini, E., Meddeb, M., Touzi, N.: Vector-valued coherent risk measures. Financ. Stoch. **8**(4), 531–552 (2004)

121. Kabanov, Y.M.: Hedging and liquidation under transaction costs in currency markets. Financ. Stoch. **3**(2), 237–248 (1999)
122. Kannai, Y., Peleg, B.: A note to the extension of an order on a set to the power set. J. Econ. Theory **32**, 172–175 (1984)
123. Kawasaki, H.: Conjugate relations and weak subdifferentials of relations. Math. Oper. Res. **6**(4), 593–607 (1981)
124. Keimel, K., Roth, W.: Ordered Cones and Approximation. Lecture Notes in Mathematics, vol. 1517. Springer, Berlin (1992)
125. Klein, E., Thompson, A.: Theory of Correspondences. Wiley-Interscience, New York (1984)
126. Kolokoltsov, V.N.: On linear, additive, and homogeneous operators in idempotent analysis. In: Maslov, V.P., Samborskii, S.N. (eds.) Idempotent Analysis. Advances in Soviet Mathematics, vol. 13, pp. 87–101, American Mathematical Society (1992)
127. Kornbluth, J.S.H.: Duality, indifference and sensitivity analysis in multiple objective linear programming. Oper. Res. Q. **25**(4), 599–614 (1974)
128. Kuroiwa, D.: Convexity for set-valued maps. Appl. Math. Lett. **9**(2), 97–101 (1996)
129. Kuroiwa, D.: Some notions of convexity for set-valued maps and their relations. RIMS Kokyuroku **939**, 168–172 (1996)
130. Kuroiwa, D.: Some criteria in set-valued optimization. RIMS Kokyuroku **985**, 171–176 (1997)
131. Kuroiwa, D.: Lagrange duality of set-valued optimization with natural criteria. RIMS Kokyuroku **1068**, 164–170 (1998)
132. Kuroiwa, D.: On natural criteria in set-valued optimization. RIMS Kokyuroku **1048**, 86–92 (1998)
133. Kuroiwa, D.: The natural criteria in set-valued optimization. RIMS Kokyuroku **1031**, 85–90 (1998)
134. Kuroiwa, D.: Some duality theorems of set-valued optimization. RIMS Kokyuroku **1079**, 15–19 (1999)
135. Kuroiwa, D.: On set-valued optimization. Nonlinear Anal. **47**, 1395–1400 (2001)
136. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**(1), 73–84 (2003)
137. Kuroiwa, D.: On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. **10**(1), 41–50 (2009)
138. Kuroiwa, D., Tanaka, T., Truong, X.D.H.: On cone convexity of set-valued maps. Nonlinear Anal. **30**(3), 1487–1496 (1997)
139. Kutateladze, S.S.: Convex operators. Russ. Math. Surv. **34**(1), 181–214 (1979)
140. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets and set-valued ky Fan minimax inequality. J. Nonlinear Convex Anal. **11**, 1–13 (2010)
141. Lalitha, C.S., Dutta, J., Govil, M.G.: Optimality criteria in set-valued optimization. J. Aust. Math. Soc. **75**(2), 221–231 (2003)
142. Lechicki, A.A.: On bounded and subcontinuous multifunctions. Pac. J. Math. **75**(1), 191–197 (1978)
143. Li, S.J., Chen, C.R., Wu, S.-Y.: Conjugate dual problems in constrained set-valued optimization and applications. Eur. J. Oper. Res. **196**(1), 21–32 (2009)
144. Li, S.J., Sun, X.K., Liu, H.M., Yao, S.F., Teo, K.L.: Conjugate duality in constrained set-valued vector optimization problems. Numer. Funct. Anal. Optim. **32**(1), 65–82 (2011)
145. Li, Z.: The optimality conditions for vector optimization of set-valued maps. J. Math. Anal. Appl. **237**(2), 413–424 (1999)
146. Li, Z.-F., Chen, G.-Y.: Lagrangian multipliers, saddle points, and duality in vector optimization of set-valued maps. J. Math. Anal. Appl. **215**(2), 297–316 (1997)
147. Lin, L.-J.: Optimization of set-valued functions. J. Math. Anal. Appl. **186**(1), 30–51 (1994)
148. Liu, Y.M., Luo, M.K.: Lattice-valued mappings, completely distributive law and induced spaces. Fuzzy Sets Syst. **42**(1), 43–56 (1991)
149. Löhne, A.: Optimization with Set Relations. Ph.D. thesis, Martin-Luther-Universität Halle-Wittenberg (2005)

150. Löhne, A.: Optimization with set relations: conjugate duality. Optimization **54**(3), 265–282 (2005)
151. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
152. Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. Int. J. Theory Appl. Finance. **17**(2) (2014)
153. Löhne, A., Rudloff, B., Ulus, F.: Primal and dual approximation algorithms for convex vector optimization problems. J. Glob. Optim. **60**(4), 713–736 (2014)
154. Löhne, A., Schrage, C.: An algorithm to solve polyhedral convex set optimization problems. Optimization **62**(1), 131–141 (2013)
155. Löhne, A., Tammer, C.: A new approach to duality in vector optimization. Optimization **56**(1–2), 221–239 (2007)
156. Luc, D.T.: Theory of Vector Optimization. LNEMS, vol. 319. Springer, Berlin (1989)
157. Luc, D.T.: On duality in multiple objective linear programming. Eur. J. Oper. Res. **210**(2), 158–168 (2011)
158. Luc, D.T., Phong, T.Q., Volle, M.: Scalarizing functions for generating the weakly efficient solution set in convex multiobjective problems. SIAM. J. Optim. **15**(4), 987–1001 (2005)
159. Luc, D.T., Phong, T.Q., Volle, M.: A new duality approach to solving concave vector maximization problems. J. Glob. Optim. **36**(3), 401–423 (2006)
160. Luenberger, D.G.: Optimization by Vector Space Methods. John Wiley & Sons (1968)
161. Luenberger, D.G.: Benefit functions and duality. J. Math. Econ. **21**, 461–481 (1992)
162. Luenberger, D.G.: New optimality principles for economic efficiency and equilibrium. J. Optim. Theory Appl. **75**(2), 221–264 (1992)
163. Maeda, T.: On optimization problems with set-valued objective maps. Appl. Math. Comput. **217**(3), 1150–1157 (2010)
164. Maeda, T.: On optimization problems with set-valued objective maps: existence and optimality. J. Optim. Theory Appl. **153**(2), 263–279 (2012)
165. Maggis, M., La Torre, D.: A goal programming model with satisfaction function for risk management and optimal portfolio diversification. INFOR: Inf. Syst. Oper. Res. **50**(3), 117–126 (2012)
166. Malivert, C.: Fenchel duality in vector optimization. Advances in Optimization (Lambrecht 1991). LNEMS, vol. 382, pp. 420–438. Springer, Berlin (1992)
167. Martínez-Legaz, J.-E., Singer, I.: Dualities associated to binary operations on $\mathbb{R}$. J. Convex Anal. **2**(1–2), 185–209 (1995)
168. Mayer, O.: Algebraische und metrische Strukturen in der Intervallrechnung und einige Anwendungen. Comput. (Arch. Elektron. Rechnen) **5**, 144–162 (1970)
169. Minh, N.B., Tan, N.X.: On the C-lipschitz continuities and C-approximations of multivalued mappings. Vietnam J. Math. **30**(4), 343–363 (2002)
170. Minh, N.B., Tan, N.X.: On the continuity of vector convex multivalued functions. Acta Math. Vietnam. **27**(1), 13–25 (2002)
171. Mordukhovich, B.S.: Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems. Soviet Math. Dokl. **22**, 526–530 (1980)
172. Mordukhovich, B.S.: Coderivatives of set-valued mappings: calculus and applications. Nonlinear Anal. **30**(5), 3059–3070 (1997)
173. Mordukhovich, B.S., Bao, T.Q.: Existence of minimizers and necessary conditions in set-valued optimization with equilibrium constraints. Appl. Math. **52**(6), 453–472 (2007)
174. Mordukhovich, B.S., Bao, T.Q.: Variational principles for set-valued mappings with applications to multiobjective optimization. Control Cybern. **36**(3), 531–562 (2007)
175. Moreau, J.J.: Fonctions à valeurs dans $-\infty, +\infty$: notions algebriques. Université de Montpellier, Seminaires de Mathematiques (1963)
176. Muselli, E.: Upper and lower semicontinuity for set-valued mappings involving constraints. J. Optim. Theory Appl. **106**(3), 527–550 (2000)
177. Nehring, K., Puppe, C.: Continuous extensions of an order on a set to the power set. J. Econ. Theory **68**, 456–479 (1996)

178. Nickel, K.: Verbandstheoretische Grundlagen der Intervall-Mathematik. Interval Mathematics, pp. 251–262. Springer, Berlin (1975)
179. Nieuwenhuis, J.W.: Supremal points and generalized duality. Math. Operationsforsch. Statist. Ser. Optim. **11**(1), 41–59 (1980)
180. Nishianidze, Z.G.: Fixed points of monotonic multiple-valued operators. Bull. Georgian Acad. Sci. (in Russian) **114**, 489–491 (1984)
181. Nishizawa, S., Onodsuka, M., Tanaka, T.: Alternative theorems for set-valued maps based on a nonlinear scalarization. Pac. J. Optim. **1**, 147–159 (2005)
182. Nishizawa, S., Shimizu, A., Tanaka, T.: On nonlinear scalarization methods in set-valued optimization. RIMS Kôkyûroku **1415**, 20–28 (2005)
183. Nishizawa, S., Shimizu, A., Tanaka, T.: Optimality conditions in set-valued optimization using nonlinear scalarization methods. In: Takahashi, W., Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis, pp. 565–574. Yokohama Publishers, Yokohama (2007)
184. Oettli, W.: Optimality conditions involving generalized convex mappings. In: Schaible, S., Ziemba, W.T. (eds.) Generalized Concavity in Optimization and Economics, pp. 227–238. Academic Press Inc., New York (1981)
185. Oettli, W.: Optimality conditions for programming problems involving multivalued mappings. Mod. Appl. Math., 195–226, Amsterdam (1982)
186. Olko, J.: Selections of an iteration semigroup of linear set-valued functions. Aequ. Math. **56**(1–2), 157–168 (1998)
187. Pennanen, T., Penner, I.: Hedging of claims with physical delivery under convex transaction costs. SIAM J. Financ. Math. **1**(1), 158–178 (2010)
188. Pollák, G.: Infima in the power set of free semigroups. Semigroups Theory and Applications, pp. 281–286. Springer, Berlin (1988)
189. Pontrjagin, L.S.: Linear differential pursuit games. Mat. Sb. (N.S.), **112(154)**(3(7)) 307–330, 475 (1980)
190. Postolică, V.: A generalization of Fenchel's duality theorem. Ann. Sci. Math. Québec **10**(2), 199–206 (1986)
191. Postolică, V.: Vectorial optimization programs with multifunctions and duality. Ann. Sci. Math. Québec **10**(1), 85–102 (1986)
192. Prakash, P., Sertel, M.R.: Topological semivector spaces: convexity and fixed point theory. Semigroup Forum, **9**(2): 117–138 (1974/75)
193. Prakash, P., Sertel, M.R.: Hyperspaces of topological vector spaces: their embedding in topological vector spaces. Proc. AMS **61**(1), 163–168 (1976)
194. Pshenichnyi, B.N.: Convex multivalued mappings and their conjugates. Cybern. Syst. Anal. **8**(3), 453–464 (1972)
195. Rockafellar, R.T.: Monotone Processes of Convex and Concave Type. American Mathematical Society, Providence (1967)
196. Rockafellar, R.T.: Lagrange multipliers and optimality. SIAM Rev. **35**(2), 183–238 (1993)
197. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Grundlehren der Mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998)
198. Rödder, W.: A generalized saddlepoint theory: Its application to duality theory for linear vector optimum problems. Eur. J. Oper. Res. **1**(1), 55–59 (1977)
199. Rodríguez-Marín, L., Sama, M.: Scalar Lagrange multiplier rules for set-valued problems in infinite-dimensional spaces. J. Optim. Theory Appl. **156**(3), 683–700 (2013)
200. Roux, A., Tokarz, K., Zastawniak, T.: Options under proportional transaction costs: An algorithmic approach to pricing and hedging. Acta Applicandae Mathematicae **103**(2), 201–219 (2008)
201. Roux, A., Zastawniak, T.: American and Bermudan options in currency markets under proportional transaction costs. Acta Appl. Math. (2015). doi:10.1007/s10440-015-0010-9
202. Rubinov, A.M.: Sublinear operators and their applications. Rus. Math. Surv. **32**(4), 115–175 (1977)

203. Sach, P.H.: New generalized convexity notion for set-valued maps and application to vector optimization. J. Optim. Theory Appl. **125**(1), 157–179 (2005)
204. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization, vol. 176. Academic Press, New York (1985)
205. Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Math. Financ. **14**(1), 19–48 (2004)
206. Schmidt, K.D.: Embedding theorems for cones and applications to classes of convex sets occurring in interval mathematics. Interval Mathematics. Lecture Notes in Computer Science, vol. 212, pp. 159–173. Springer, Berlin (1986)
207. Schrage, C.: Set-valued Convex Analysis. Ph.D. thesis, Martin-Luther-Universität Halle-Wittenberg (2009)
208. Schrage, C.: Scalar representation and conjugation of set-valued functions. Optim. **64**(2), 197–223 (2015)
209. Shephard, R.W.: Theory of Cost and Production Functions. Princeton University Press, Princeton (1970)
210. Singer, I.: Abstract Convex Analysis. Wiley, New York (1997)
211. Song, W.: Duality for vector optimization of set-valued functions. J. Math. Anal. Appl. **201**(1), 212–225 (1996)
212. Song, W.: Conjugate duality in set-valued vector optimization. J. Math. Anal. Appl. **216**(1), 265–283 (1997)
213. Song, W.: Lagrangian duality for minimization of nonconvex multifunctions. J. Optim. Theory Appl. **93**(1), 167–182 (1997)
214. Song, W.: Duality in set-valued optimization. Dissertationes Mathematicae (Rozprawy Mat.) **375**, 1–69 (1998)
215. Szpilrajn, E.: Sur l'extension de l'ordre partiel. Fundamenta Mathematicae **16**(1), 386–389 (1930)
216. Tanino, T.: On supremum of a set in a multidimensional space. J. Math. Anal. Appl. **130**(2), 386–397 (1988)
217. Tanino, T.: Conjugate duality in vector optimization. J. Math. Anal. Appl. **167**(1), 84–97 (1992)
218. Tanino, T., Sawaragi, Y.: Duality theory in multiobjective programming. J. Optim. Theory Appl. **27**(4), 509–529 (1979)
219. Tanino, T., Sawaragi, Y.: Conjugate maps and duality in multiobjective optimization. J. Optim. Theory Appl. **31**(4), 473–499 (1980)
220. Volle, M.: Concave duality: application to problems dealing with difference of functions. Math. Program. **41**(1–3), 261–278 (1988)
221. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Monographs on Statistics and Applied Probability, vol. 42. Chapman and Hall Ltd., London (1991)
222. Wang, S.S., Young, V.R., Panjer, H.H.: Axiomatic characterization of insurance prices. Insur. Math. Econ. **21**(2), 173–183 (1997)
223. Ward, M.: Some arithmetical applications of residuation. Am. J. Math. **59**(4), 921–926 (1937)
224. Ward, M., Dilworth, R.P.: Residuated lattices. Proc. Natl. Acad. Sci. **24**(3), 162 (1938)
225. Ward, M., Dilworth, R.P.: Residuated lattices. Trans. Am. Math. Soc. **45**(3), 335–354 (1939)
226. Yang, X.Q.: A Hahn-Banach theorem in ordered linear spaces and its applications. Optimization **25**(1), 1–9 (1992)
227. Yang, X.Q.: Directional derivatives for set-valued mappings and applications. Math. Methods Oper. Res. **48**(2), 273–285 (1998)
228. Young, R.C.: The algebra of many-valued quantities. Math. Ann. **104**, 260–290 (1931)
229. Zhang, W.Y., Li, S.J., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**(9), 3769–3778 (2009)
230. Zimmermann, U.: Linear and Combinatorial Optimization in Ordered Algebraic Structures. Annals of Discrete Mathematics, vol. 10. North-Holland Publishing Co., New York (1981)

231. Zowe, J.: Subdifferentiability of convex functions with values in an ordered vector space. Math. Scand. **34**(1), 69–83 (1974)
232. Zowe, J.: A duality theorem for a convex programming problem in order complete vector lattices. J. Math. Anal. Appl. **50**(2), 273–287 (1975)
233. Zowe, J.: Sandwich theorems for convex operators with values in an ordered vector space. J. Math. Anal. Appl. **66**(2), 282–296 (1978)
234. Zălinescu, C.: Duality for vectorial nonconvex optimization by convexification and applications. An. Stiint. Univ. Al. I. Cuza Iasi. Sect. I a Mat.(NS), **39**(1), 16–34 (1983)
235. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2002)
236. Zălinescu, C.: Hahn-Banach extension theorems for multifunctions revisited. Math. Methods Oper. Res. **68**(3), 493–508 (2008)

A Survey of Set Optimization Problems with Set Solutions

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Dedicated to the memory of Professor Luis Rodríguez-Marín

Abstract This paper presents a state-of-the-art survey on set-valued optimization problems whose solutions are defined by set criteria. It provides a general framework that allows to give an overview about set-valued optimization problems according to decision concepts based on certain set relations. The first part of this paper (Sects. 1 and 2) motivates and describes the set-valued optimization problem (in short, SVOP). The present survey deals with general problems of set-valued optimization and recall its main properties in order to establish the differences between vector set-valued optimization problems (VOP) and set optimization problems (SOP). In this context, in the second part (Sects. 3–5) we focus on those results existing in the literature related with optimality conditions by using a set approach. We list and quote references devoted to (SOP) from the beginning up to now. In Sect. 5, a particular attention is paid to applications of the set relations considered in other fields as fixed point theory. The last section provides some conclusions and suggestions for further study.

Keywords Set-valued maps · Set optimization · Optimality conditions · Set approach · Solutions of set type

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1 Introduction

The set-valued maps receive great attention from more and more authors. This is partly due to its wide applications in diverse fields as for example: Control theory, Optimization, Economics or Game theory, to name a few. See, for instance, [4, 51] and references therein.

On the other hand, set-valued optimization problems are very known in Optimization theory and Economics as for example equilibrium theorems for Economies. See, [1–3, 10, 12, 30, 46].

Throughout this paper, we consider preference relations generated by a pre-order (a binary relation which is reflexive and transitive). In the sequel M denotes a nonempty subset of a set X, Y a linear space and $K \subset Y$ a convex cone. If $y,\ y' \in Y$ we denote by $y \leq y'$ if and only if $y' - y \in K$. This relation $\leq$ is obviously a pre-order on Y. Thus, the pair (Y, K) is called a pre-ordered linear space (or partially ordered space) with the ordering $\leq$ induced by K. To consider weakly efficient solutions, we also assume, in addition, that Y is a topological space and K is solid, that is, its topological interior is nonempty, int $K \neq \emptyset$.

Remark 1.1
1. If K is pointed, $K \cap (-K) = \{0\}$, the preference $\leq$ is also antisymmetric and (Y, K) is an ordered linear space.
2. In spite of the most optimization theory is based on a pre-order on the criteria space, other preferences (non-reflexitivy or non-transitivity) are very important from the practical point of view, for instance, in Economic. Some references and results can be found in [11] where the variational approach developed in this paper allows to obtain new necessary conditions for various types of solutions and to apply it to nonconvex models of welfare economics with finite-dimensional and infinite-dimensional commodity spaces.
3. In this paper the discussion is based on a general framework. Note that it is possible to avoid the topological structure on Y to consider weakly efficient solutions via the algebraic interior of K, that is, core(K). See also [19].

Given a nonempty set $A \subset Y$, we denote by $\text{Min}\, A = \{\bar{y} \in A\colon\ y \in A,\ y \leq \bar{y} \text{ imply } \bar{y} \leq y\}$ the set of minimal points of A. In particular, if K is pointed, $\text{Min}\, A = \{\bar{y} \in A\colon\ (\bar{y} - K) \cap A = \bar{y}\}$. It is said that $\bar{y} \in A$ is a strongly or ideal minimal point of A, $\bar{y} \in \text{IMin}\, A$, if $A \subset \bar{y} + K$. By replacing K by $-K$, we can define maximal and ideal maximal point of A.

We denote by (V) the following vector optimization problem:

$$\text{(V)} \quad \begin{cases} \text{Min } f(x) \\ \text{subject to } x \in M, \end{cases}$$

where $f\colon M \to Y$. An element $x_0 \in M$ is said to be an efficient solution of (V), $x_0 \in \text{Eff}(f)$, if $f(x_0) \in \text{Min} \bigcup\limits_{x \in M} f(x)$.

The above solution is defined via Edgeworth-Pareto solution. However, for a vector optimization problem there are various solution concepts as for instance proper solutions or strong solutions. For more references about this problem, see [18, 46, 69, 80] and references therein. It is well-known that there exist other solution concepts (different to Edgeworth-Pareto notions) which have been investigated by many authors. We remark that according to [21, 22, 28] it is possible to study a unified vector problem which includes other efficient notions.

The present survey deals with optimization problems where the objective map is more complex than that given in (V). In addition, our presented results address the notions of extended Edgeworth-Pareto optimality.

The general formulation of a set-valued optimization problem is as follows:

$$\text{(SVOP)} \quad \begin{cases} \text{Min } F(x) \\ \text{subject to } x \in M, \end{cases}$$

where $F\colon M \longrightarrow 2^Y$ is a set-valued map with $F(x) \neq \emptyset$ for all $x \in M$.

Unlike the vector optimization problem (V), for the above problem there is not a only one approach of solution associated to it. The solutions of (SVOP) are categorized into

(i) *vector solutions*; when the problem, denoted by (VOP) and called vector set-valued optimization problem, is a vector optimization problem with set-valued maps.
(ii) *set solutions*; when the problem, denoted by (SOP), is a set optimization problem.

Now, we present the above problems to establish the differences between them. For this, we introduce some notations and define theirs solutions.

The general vector set-valued optimization problem is denoted as follows:

$$\text{(VOP)} \quad \begin{cases} \text{Min } F(x) \\ \text{subject to } x \in M. \end{cases}$$

We denote $F(M) = \bigcup_{x \in M} F(x)$ the image set under F on M. To define the solutions of vector type we consider the pre-order $\leq$ defined on Y by the convex cone K. Roughly speaking, the solutions of (VOP) are introduced by means the minimal elements of $F(M)$.

Definition 1.1 We say that $\bar{x} \in M$ is a solution of (VOP), $\bar{x} \in \text{Eff}(F)$, if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in \text{Min } F(M)$. The pair $(\bar{x}, \bar{y})$ is called minimizer of (VOP).

On the contrary, the solutions of set-type are defined via a preference, $\preceq$, on the family of nonempty subsets of Y, $\wp_0(Y)$. We denote a set optimization problem as follows:

$$\text{(SOP)} \quad \begin{cases} \preceq -\text{Min } F(x) \\ \text{subject to } x \in M. \end{cases}$$

The essence of set approach consists in considering the whole set as a solution, not just one point of the image. Following the vector case, the solutions of (SOP) with respect to $\preceq$ are defined by the more preferred sets of $\{F(x)\colon x \in M\}$ as follows:

Definition 1.2 We say that $\bar{x} \in M$ is a $\preceq$-solution of (SOP), $\bar{x} \in \preceq -\text{Eff } F$, if $x \in M$ and $F(x) \preceq F(\bar{x})$ imply $F(\bar{x}) \preceq F(x)$.

A natural extension of problem (V) is when $\preceq$ is compatible with the ordering defined by K in the following sense:

Definition 1.3 Let $a,\ b \in Y$ be. We say that $\preceq$ is compatible with $\leq$ if $\{a\} \preceq \{b\}$ is equivalent to $a \leq b$.

Remark 1.2 We point out that we can combine both approach (vector and set) to define new preferences and solutions for a set-valued optimization problem. For instance, $A \subseteq M$ is a set solution of (SVOP) if $A \subseteq \preceq -\text{Eff } F$ and $\text{Min} \bigcup_{x \in M} F(x) = \text{Min} \bigcup_{x \in A} F(x)$. Such preferences could be related with Finance according to [44].

Remark 1.3 1. It is clear that it is possible to define different vector solutions of (VOP) from those presented in Definition 1.1 like weak, strong or proper minimizer. See also [9]. Similarly for set solutions of (SOP).

2. Definitions 1.1 and 1.2 are given in a natural way. Both seem to be the most appropriated to generalize the Edgeworth-Pareto notions.
3. A decision maker considers (VOP) or (SOP) depending on his preferences are given on elements of Y or on elements of $\wp_0(Y)$.

In terms of existing literature, we point out that it is usual to call set-valued optimization problem or set-optimization problem to refer to (VOP) or (SOP). In this paper we establish such a difference. On the other hand, about solutions for a set-valued optimization problem, the vector criterion is the most well-known and investigated in the branch of set-valued optimization. Thus, the vast majority of publications on (SVOP) is about optimality conditions for (VOP).

The set approach was introduced by Kuroiwa [52] in 1997 by using set-relations which generalize that given by the ordering cone (Sect. 3). Since the notion of set solution was introduced, there has been rapid growth in the field about it. In this survey, our claim is to show several bibliographic collections reported about the set approach to give a comprehensive listing and to analyse the research covering its first 16 years of history which is not available, as far we know.

This paper is decomposed into six sections. The second one is devoted to establish the main differences between (VOP) and (SOP). In Sect. 3 we introduce the main preferences defined on $\wp_0(Y)$ which have been explored in the literature in terms of (SVOP). In the next section, an extensive listing of set optimization research that covers theoretical developments from the beginning to the year 2013 is given. In Sect. 5, we present several areas different to optimization in which the set-relations have been used implicitly. Finally, in Sect. 6, several remarks and conclusions are presented for new research.

2 Vector Optimization Problem Versus Set Optimization Problem

In this section, firstly we show the main (geometric and analytic) aspects of (VOP) and (SOP) and secondly, the immediate relationships between their solutions.

It is clear that solving a vector set-valued optimization problem is equivalent to solve a rather simple problem in terms of the the objective map. In other words, solving (VOP) is equivalent find the solutions of the following vector problem:

$$(\mathrm{V}_1) \quad \begin{cases} \mathrm{Min}\ \Pi_Y(x, y) \\ (x, y) \in \mathrm{Gra}(F), \end{cases}$$

where $\mathrm{Gra}(F) = \{(x, y) \in X \times Y : x \in M,\ y \in F(x)\}$ and Π_Y is the projection of $\mathrm{Gra}(F) \subset X \times Y$ on the second space.

Thus, $z_0 = (x_0, y_0) \in \mathrm{Gra}(F)$ is an solution of (V_1) if and only if z_0 is a minimizer of (VOP).

The above result is a peculiar characteristic of (VOP) since if we consider other level of complexity for the objective map, we know that in order to solve a vector optimization problem (P) via a scalar optimization problem we have to apply some technique of scalarization which is not always possible, in general.

In terms of optimality conditions for (SOP) we could consider solutions which image sets are not related with the boundary line of the image set $F(M)$. It is a geometric property of the set solutions which must be overcame in order to give necessary conditions via separation theorems.

One advantage of the set criterion over the vector criterion is the possibility of considering preference relations on 2^Y. On the contrary, the main disadvantage of set criterion over vector criterion is the loss of structure lineal. Hamel [29] studied the structure of $\wp_0(Y)$ introducing a conlineal space.

In order to avoid such a problem several authors have considered specializations of F or tools to study the problem (SOP) via a structure well-known or simpler than a conlineal space. For instance, in Hernández [33] solutions of (SOP) are characterized via nonlineal scalarization, see also [8, 34, 72]. Nuriya and Kuroiwa [58, 77] construct an embedding vector space. Maeda [71], working on n-dimensional Euclidean spaces shows that whenever set-valued map is rectangle-valued, (SOP) is equivalent to a pair of vector-valued optimization problems. Recently Jahn [47] states that a certain vector optimization problem can be associated to (SOP) when $\preceq$ is defined by some complicated set relations.

In general, there is no any relationship between solutions obtained by vector criterion (solutions of (VOP)) and solutions obtained by set criterion (solutions of (SOP)). Moreover, the existence of solutions of one type does not imply the existence of solutions of the other type. See, for instance, [27, 40].

On the other hand, it is natural to pose questions about the relationships between solutions obtained by each criterion. Hernández and Rodríguez-Marín [42], under certain assumptions for the set-valued map, show that to solve (SOP) it is possible to reduce the feasible set through the set criterion.

Even though both criteria are different, they extend (V) in the following sense. If we consider a pre-order $\preceq$ on $\wp_0(Y)$ compatible with $\leq$ and F is replaced by a vector-valued map, then (SOP) and (VOP) are equivalent to (V). If, in addition, we consider weakly solutions of (VOP) and (SOP) it is possible to prove that each weakly vector solution is a weakly set solution, see [34, Proposition 2.10] and [71, Theorem 5].

See also [5, 42, 71] to find more relationships between vector solutions (VOP) and set solutions (SOP). Thus, for a certain classes of set-valued maps and pre-order on $\wp_0(Y)$ the set criterion is equivalent to the vector criterion. A particular case is obtained when $Y = \mathbb{R}$ since each image set has a strongly minimal point.

To end this section we refer the reader to [19, 65] for a deeper discussion of the above approaches of solutions for a optimization problem.

3 Set Relations Considered in the Literature

Now, we introduce the main preferences considered in the existing papers devoted to study solutions of (SOP). In addition, we focus on pre-order relations defined on $\wp_0(Y)$ which generalizes the ordering defined by K on Y.

The first systematic treatment of set relations in the context of ordered vector spaces and its power sets is due to Kuroiwa, Tanaka and Ha [59] in 1997.

Definition 3.1 [59] Let $A,\ B \in \wp_0(Y)$.

$$\begin{array}{lll}
A \leq^{i} B & \Leftrightarrow & B - A \subset K \quad \Leftrightarrow \quad a \leq b \quad \text{for all} \quad b \in B,\, a \in A \\
A \leq^{ii} B & \Leftrightarrow & \text{there exists} \quad a \in A \quad \text{such that} \quad a \leq b \quad \text{for all} \quad b \in B \\
A \leq^{iii} B & \Leftrightarrow & B \subset A + K \\
A \leq^{iv} B & \Leftrightarrow & \text{there exists} \quad b \in B \quad \text{such that} \quad a \leq b \quad \text{for all} \quad a \in A \\
A \leq^{v} B & \Leftrightarrow & A \subset B - K \\
A \leq^{vi} B & \Leftrightarrow & \text{there exist} \quad b \in B,\, a \in A \quad \text{such that} \quad a \leq b.
\end{array}$$

It is easy to check that $\leq^k$ with $k \in \{i, ii, iv\}$ are preferences such that the anti-symmetric and reflexive properties do not hold while $\leq^k$ with $k \in \{iii, v, vi\}$ are pre-orders on $\wp_0(Y)$. In addition,

$$A \leq^{i}_{K} B \Rightarrow A \leq^{ii} B \Rightarrow A \leq^{iii}_{K} B \Rightarrow A \leq^{vi} B$$

$$A \leq^{i}_{K} B \Rightarrow A \leq^{iv} B \Rightarrow A \leq^{v}_{K} B \Rightarrow A \leq^{vi} B.$$

In general, two nonempty sets A and B could not be related by $\leq^k$ for any k. Indeed, let $E = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ be. Then $A = \{(x+2)^2 + (y-2)^2 = 1\}$ and $B = \{(x-3)^2 + (y+3)^2 = 1\}$ satisfy that $A \not\leq^{vi} B$ y $B \not\leq^{vi} A$.

The authors defined the above set-relations on $\wp_0(Y)$ to study generalized convexity of a set-valued map. Since then, relations of a similar type have been proposed for other authors, in several papers.

The most important property of set relations introduced in Definition 3.1 is that all of them generalize the ordering defined by K on Y in the sense of Definition 1.3. We emphasize that, in terms of optimality conditions, the set relations $\leq^{iii}$ and $\leq^{v}$ are called lower and upper set-relations (denoted by $\leq^{l}$ and $\leq^{u}$) respectively. It is clear that $A \leq^{l} B$ is equivalent to $-B \leq^{u} -A$. In addition, it is possible to rewrite them via $-K$ instead of K.

So, Kuroiwa, Tanaka and Ha started developing a new approach to set-valued optimization which is based on comparison among values of the set-valued map from a set into a ordered vector space. At the same year, 1997, Kuroiwa [52] introduced solutions for (SOP) in the sense of Definition 1.2 by using $\leq^{l}$ and $\leq^{u}$. Due to this fact, the set criterion is also called in the literature Kuroiwa's criterion. In order to illustrate the set criterion we give a example.

Example 3.1 Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ be. 1. Consider $F\colon M = [0, \infty) \longrightarrow 2^{\mathbb{R}^2}$ such that

$$F(x) = \begin{cases} \{(0,0)\} & x = 0 \\ \left[(0,0), \left(-x, \frac{1}{x}\right)\right] & x \neq 0 \end{cases}$$

Then $\leq^{l} - \mathrm{Eff}(F) = \emptyset$ and $\mathrm{Eff}(F) = M$.

2. Consider $F\colon [-1, 0] \longrightarrow 2^{\mathbb{R}^2}$ such that

$$F(\lambda) = \begin{cases} \{(x, -x^2) \in \mathbb{R}^2\colon -1 < x \leq 0\} & \lambda = -1 \\ \left[(\lambda, 0), (\lambda, -\lambda^2)\right] & \lambda \neq -1 \end{cases}$$

Then $\leq^{l} - \mathrm{Eff}(F) = \{-1\}$ and $\mathrm{Eff}(F) = \emptyset$.

However, the set relations given in Definition 3.1 have just been considered in different frameworks many years ago, as is remarked in [19, 29, 46]. Firstly, Young in 1931, [84] considered the above ret relations, among others, in terms of algebraic structures. Fifty years before, Nishnianidze [75] studied theory of fixed points of monotonic operators. That is the reason why Jahn in [46] called KNY order relations to refer to set relations presented in Definition 3.1.

Alonso and Rodríguez-Marín [5] proved that the study of the set optimization problems: $\leq^{l} - \mathrm{Min}\, F$, $\leq^{u} - \mathrm{Min}\, F$ $\leq^{l} - \mathrm{Max}\, F$ and $\leq^{u} - \mathrm{Max}\, F$ is reduced to the study of the following ones: $\leq^{l} - \mathrm{Min}\, F$ and $\leq^{u} - \mathrm{Min}\, F$. Moreover, one can be solved by the other with a suitable definition of the objective map.

To end this section, we recall other set relations defined in set optimization theory. In 2003, Kuroiwa [56, Definition 2.1] introduced new binary relations on $\wp_0(Y)$ which are weaker than $\leq^{l}$ and $\leq^{u}$ by using elements of the positive polar cone of K. See also, [45].

On the other hand, by combining the set relations $\leq^{l}$ and $\leq^{u}$ we obtain the following pre-order on $\wp_0(Y)$

$$A \preceq_u^l B \Leftrightarrow A \preceq^l B \quad \text{and} \quad A \preceq^u B. \tag{1}$$

In 2010, Maeda [71], working on n-dimensional Euclidean spaces, defined preferences on $\wp_0(Y)$ via the strong minimal and maximal elements of sets. The same author, in [72] defined Pareto optimal solutions, semi-weak Pareto optimal solution, and weak Pareto optimal solutions of (SOP) by considering $\preceq_u^l$.

Janh and Ha [48], in a more general framework, introduced new set relations motivated by analysis interval and related with the above ones which generalize those given by Maeda [72] and seem be more appropriate to set optimization theory according to their applications.

In Löhne and Tammer [64] several set relations are presented on the family of all subsets A of $Y = \mathbb{R}^n$ with $\text{cl}(A + K) = A$ (where cl denotes the topological closure) to construct a pre-ordered conlinear space. See also, [62]. See also set relations given in [58].

4 Classification of the Literature

In this section we recover the main results presented in the literatures of set-optimization by using set approach. Several existence theorems for solutions of (SOP) will be presented under a unified framework.

In the pioneering paper [52], Kuroiwa introduced the definitions of l-type solutions or u-type solutions (by using $\preceq^l$ or $\preceq^u$ respectively) of (SOP) and a motivation for the study of set optimization problems is given by means non academic examples.

Since 1997, when set criterion was introduced, the optimality conditions for solutions of (SOP) are divided into two categories: those following results from the vector case (using continuity, properties of a set, differentiability, scalarization, Lagrangian duality, well-posedness and approximate solutions) and those obtained by applying new results or tools.

In the sequel we list the main results related with the existence of solutions of (SOP) from the beginning to up to now.

The first optimality conditions of l-type solutions of (SOP) were presented in [54, Theorem 4] by considering M compact and F a set-valued map with level sets closed ([54, Definition 2]) and, in addition, K closed and pointed. See also [53, Theorem 3.1].

In 2005, Alonso and Rodríguez-Marín [5] extended the definitions of cone-semicompactness and domination property from a set to a family of sets. In addition, in Proposition 22, gave a sufficient condition of u-type solution by using a notion of cone-regularity defined by subcovers of a family of subsets. See also [5, Corollary 24] to obtain a existence condition of u-type solution under M compact and F lower cone-semicontinuous. A sufficient condition of l-type solution under M compact and F upper cone-semicontinuous was given in [5, Propositions 29 and 30].

Following the subcovers introduced in [5, Definition 27], Hernández and Rodríguez-Marín [35] introduced the notion of strongly some-compactness and cone-completeness for a family of sets to extend optimality conditions from the vector case to the lower set relation case in Theorems 4.1 and 4.6. Such results generalize those presented in [69] for the vector case. In [35, Sect. 5], several optimality conditions presented in [5, 55] are slightly improved. On the other hand, under assumptions of generalized continuity, not only the existence of solutions is proven but also the domination property of the family $\{F(x) \colon x \in M\}$ ([35, Definition 4.3]) was established in Corollaries 5.5, 5.6 and 5.7 and Theorems 5.8 and 5.9.

Other definitions of semicompactness, completeness and semicontinuity and related general theoretical properties with respect to a pre-order $\preceq$ on $\wp_0(Y)$ was given in [48] in a more general framework.

Also, assuming $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ and considering the combined set-relation $\leq^l_u$ defined in (1), Maeda [72, Theorem 4.1] gave a sufficient condition for a $\leq^l_u$-solution under compactness and generalized continuity.

Remark 4.1 The above results allow to state that the existence conditions of solutions of set type are, in general, weaker that those of vector type and, in addition, several existence results in vector optimization do not depend on linearity of the image space.

In terms of duality theory, Kuroiwa introduced a generalized Lagrangian as follows $L(x, y, T) = F(x) + T(y)$ (where T is a linear map from X to Y and $y \in F(x)$) and the dual problem associates to a constrained set optimization problem. He established conditions of saddle points in [53, Theorem 4.1] and [54, Theorem 9]. Hernández and Rodríguez-Marín [36, 37] generalized the above Lagrangian map by defining $L(x, T) = F(x) + T(F(x))$ (where T is an affine map from X to Y) and gave weak and strong duality theorems and saddle points results which extend those known in the vector case. In [36, Sect. 3] and [37, Sect. 4], some multiplier rules by means of an affine linear map under generalized convexity assumptions were given by considering l-type solutions of (SOP).

By using $\leq^i$ Lin and Chen [68] gave weak solutions and strong solutions of set equilibrium problems and [43, Theorem 5.5] established a Lagrange multiplier rule.

Alonso and Rodríguez-Marín [5, Theorems 35 and 38] gave optimality conditions for existence of strict solutions of (SOP) in terms of continuous selections of set-valued maps. The same authors, in [6, Theorem 25] established a necessary and sufficient condition for the existence of weakly l-type solutions of (SOP) under generalized convexity assumptions and contingent derivative of F.

Rodríguez-Marín and Sama in [79] gave a notion of following graphical derivative of a set-valued map.

Definition 4.1 [79] Let X, Y be real normed spaces. Assume K is closed, strongly minihedral and regular. The (Λ, C)-lower contingent derivative of F at $\overline{x}$ is the set-valued map $\underline{D}_\Lambda F(\overline{x}) : X \to 2^Y$ defined by

$$\text{Gra}\, \underline{D}_\Lambda F(\overline{x}) = \text{Limsup}_t\, T(\text{Gra}\, \varphi_{\Lambda,t}, (\overline{x}, \varphi_{\Lambda,t}(\overline{x}))),$$

where $T(A, z)$ with $A \subset Y$ denotes the contingent cone to A at $z \in A$. Based on ordered spaces techniques, the authors defined two types of contingent derivatives to set-valued maps and gave optimality conditions in terms of the contingent derivatives for local l-type solutions of (SOP) in Theorems 5.1, 5.2, 5.6 and 5.7. The obtained results prove that the above derivative is suitable for the formulation of necessary and sufficient conditions for set-valued optimization problems following the set approach.

In 2009, Kuroiwa [57] also presented directional derivatives based on an embedding idea to establish necessary and sufficient conditions for a weakly minimal and minimal solutions of (SOP).

Considering constrained optimization problems, Maeda [71, Theorems 6 and 7] established existence conditions for weakly $\leq_u^l$-solutions by radial Dini derivatives and lower and upper Dini derivatives of F.

Hernández and Rodríguez-Marín [39, Sect. 5] obtained optimality conditions for the existence of solutions l-type solutions via weak and strong subgradients for a set-valued map.

Remark 4.2 From the above results, it is clear that even notions on optimality conditions in terms of differentiabily notions for set-valued maps is still an open issue in set optimization.

In 2007, Hernández and Rodríguez-Marín [34, Sect. 4], by considering the preorder was defined by $\leq^l$, gave results on scalarization for (SOP) and characterized its solutions without convexity assumptions for F a K-closed and K-bounded valued. Hamel and Löhne [31] one year before had introduced a similar generalization to give minimal element theorems. In this context, see also results given in [50, 61, 76, 81, 82]. Recently, Araya [8] presented new nonconvex separation theorems to apply to set optimization by using $\leq^l$ and $\leq^u$ preferences. So, existence theorems of weakly l-type minimal and weakly u-type minimal solutions via scalarizations were given in Sect. 5.1 and a Takahashi's minimization theorem in Sect. 5.2 was presented in terms of set optimization.

Maeda [72] studied constrained set optimization problems with various types of set solutions and, via scalarization, gave necessary and sufficient conditions under compactness assumptions (Theorems 4.2 and 4.3) and a characterization under convexity assumptions (Theorem 4.4).

Remark 4.3 We emphasize that in all the above papers devoted to scalarization, the scalarizing function considered was a generalization of the Gerstewitz's nonconvex separation function introduced in [23] and extensively studied in [24].

Ha [27, Theorem 3.1], by using strict l-type solutions, established a variant of EVP for F (where X is a complete metric space). In Sects. 4 and 5, other variants of the EVP by using conical extensions and the concept of cone extension and the Mordukhovich coderivative (see [73, 74]) were established.

In Kuroiwa [56, Theorem 3.5], via weight criteria, problem (SOP) was embedded to a complete metric space to obtain an existence theorem for weakly efficient solutions based on the Ekeland variational principle.

Also, in a framework more general, Hamel and Löhne [31] obtained two existence results for minimal of a family of subsets of the product space $X \times 2^Y$ (where X is a separated uniform space) with respect to appropriate ordering relations on 2^Y. As application, the authors established a variant of Ekeland's principle for a set-valued optimization problem via generalizations of the functionals introduced in [23]. In [13, Theorem 3.5] by using $\leq^u$ the authors established a variational principle for set-valued maps.

See also [25, Theorems 5.1 and 5.2] for approximate variants of the EVP given in [27, 31] and [20, Theorem 6.2 and 6.3] for Ekeland variational principles on quasi ordered spaces in a framework more abstract. In addition, considering relations between values of F and pre-orders generated by set-valued maps, in [67, Theorems 3.4 and 3.5] the authors directly expressed the Ekeland principle but in terms of values of F.

Others generalizations of EVP for set-valued mappings and set approach are given in [49] (via generalized distances) and [78, Theorem 3.1] by considering strict solutions of (SOP). By using a perturbed map (stability) of F see also, [34, Sect. 5].

In 2009, Zhang, Li and Teo [87] introduced three kinds of well-posedness for a set optimization problem called k_0-well posedness at a minimizer, generalized k_0-well posedness and extended k_0-well-posedness (where $k_0 \in \text{int}(K)$) and extended some basic results of well-posedness of scalar optimization to set optimization by using a generalization of the Gerstewitz's function given in [34].

Compare the above results and the tools used with those presented in [26].

In [27, Theorem 4.2] Ha defined properly positive efficient points in the framework of set approach and gave a sufficient condition of approximated solution of the perturbed map. Approximate solutions for problem (SOP) were also introduced in [25, Defintion 3.2] and [7, Definitions 17 and 19].

About locating set solutions, in [38] using polyhedral cones Hernández and Rodríguez-Marín extended the first theorem for locating the set of all efficient points of a set through ordinary mathematical programming introduced by Yu [85]. Recently, in Löhne and Schrage [66] an algorithm which computes a solution of a set optimization problem was provided.

5 Related Theories or Applications

In this section several papers related with the previous set relations are enumerated to show its applicability.

On the one hand, to give optimality conditions in set-valued optimization theory, several set relations have been used to generalize the convexity of a set-valued map in [59], to give scalar representations of a vector optimization problem in [83], to study conjugate duality in [62], to show continuity for set-valued maps under some convexity assumptions in [60], to establish alternative theorems in [76] or to find vector solutions in [41] where the set approach is used to reduce the feasible set of (VOP). In addition, as we have shown in Sect. 4, several Ekeland-type principles are developed in different frameworks by using pre-order or preferences on $\wp_0(Y)$. In

[13] the authors present minimax methods in variational analysis, exactly, a mountain pass-type theorem.

Hamel, Löhne, Heyde and their collaborators have developed a new research line in terms of infimum and supremum by using set-relations (which pre-serve the structure of complete lattices) and definitions understood in the sense of solutions like those defined in Remark 1.2 (in which both criteria are implicitly considered). Their results can be appropriate in terms of risk function in Finance. An overview of such results can be found in [63] and references therein.

On the other hand, the preferences between nonempty sets have been also considered in other theories different to Optimization. For instance, to present existence results for inclusion problems in [32] and to obtain fixed point theorems in [16, 17].

The relation defined in (1), among others, seems to be more suitable in practice, for instance, in the framework of interval analysis according to the basic investigations of Chiriaev and Walster [15]. In addition, such set relations are widely used in theoretical computer sciences, see for example Brink [14] in where a study of power structures in a universal-algebraic context is presented. For more details see [19, 48].

Since the seminal paper [86], fuzzy sets theory has been applied to various fields of decision making theory including economics, management science and engineering widely. In [70] the notion of non-dominated solution is related with Kuroiwa's solution by using a both set relations, $\leq^l$ and $\leq^u$, in terms of fuzzy mathematical programming.

6 Conclusions

According to the previous sections, we can conclude that the analysis of Kuroiwa's concept deserves an exhaustive treatment. In addition, set optimization theory can be considered as an area which is beginning since it is possible to identify future lines of research from the existing literature.

The set-valued optimization theory by using the set criterion is a natural extension of vector optimization theory. It is due to the published results allow to extend those given the vector case. In addition, the research on set optimization has proven that several existence results do not rely on linearity of the image space and therefore they can be extended to set relations.

There is no doubt that many frameworks of optimization theory can be related with set relations. However, a result which is worthy of being studied is an academic example in where a solution criterion in terms of set relations is considered. Maybe the main problem is to know what is the optimal alternative. It is clear that the set criterion seems a natural extension of vector optimization theory and seems to have the potential to become an important tool for many areas in optimization. In the same direction, Jahn [46] asserts that such set relations open a new and wide field of research and turn out to be promising in set optimization.

On the other hand, the set relations proposed in Kuroiwa, Tanaka and Ha [59], among others, have been used as tools not only in optimization but also in others fields. Probably, fuzzy programming and analysis interval are areas in which the practical point of view of the set approach is developed.

To summarize, as mentioned in Sect. 1, new relations of more general types should be explored to find adequate applications. A goal of the present survey is to motivate such a study.

References

1. Aubin, J-P.: Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. Mathematical Analysis and Applications, Part A. Adv. in Math. Suppl. Stud., **7a**, 159–229, Academic Press, New York-London (1981)
2. Aubin, J-P., Ekeland I.: Applied nonlinear analysis. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. Wiley, New York (1984)
3. Aubin, J-P., Frankowska, H.: Set-valued Analysis. Systems and Control: Foundations and Applications, vol. 2. Birkhäuser, Boston (1990)
4. Aubin, J.-P., Cellina, A.: Differential Inclusions. Set-Valued Maps and Viability Theory. Grundlehren Math. Wiss. Springer, Berlin (1984)
5. Alonso, M., Rodríguez-Marín, L.: Set-relations and optimality conditions in set-valued maps. Nonlinear Anal. **63**, 1167–1179 (2005)
6. Alonso, M., Rodríguez-Marín, L.: Optimality conditions for set-valued maps with set optimization. Nonlinear Anal. **70**, 3057–3064 (2009)
7. Alonso-Durán, M., Rodríguez-Marín, L.: On approximate solutions in set-valued optimization problems. J. Comput. Appl. Math. **236**, 4421–4427 (2012)
8. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**, 3821–3835 (2012)
9. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers in multiobjective optimization: existence and optimality conditions. Math. Prog. **122**, 301–347 (2010)
10. Bao, T.Q., Mordukhovich, B.S.: Set-valued optimization in welfare economics. Adv. Math. Econ. **13**, 113–153 (2010)
11. Bao, T.Q., Mordukhovich, B.S.: Extended pareto optimality in multiobjective problem. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Advances in Vector Optimization, pp. 379–417, Springer, Berlin (2011)
12. Bao, T.Q., Mordukhovich, B.S.: Necessary optimality conditions and applications to welfare economics. Dyn. Syst. **31**, 1069–1096 (2011)
13. Bednarczuk, E.M., Miglierina, E., Molho, E.: A mountain pass-type theorem for vector-valued functions. Set-Valued Var. Anal. **19**, 569–587 (2011)
14. Brink, C.: Power structures. Algebr. Univers. **30**, 177–216 (1993)
15. Chiriaev, A., Walster, G.W.: Interval arithmetic specification. Technical Report (1998)
16. Dhage, B.C.: A functional integral inclusion involving Caratheodories. Electron. J. Qual. Theory Differ. Equ. **14**, 1–18 (2003)
17. Dhage, B.C.: A general multi-valued hybrid fixed point theorem and perturbed differential inclusions. Nonlinear Anal. **64**, 2747–2772 (2006)
18. Dauer, J.P., Stadler, W.: A survey of vector optimization in infinite-dimensional spaces. II. J. Optim. Theory Appl. **51**, 205–241 (1986)
19. Eichfelder, G., Jahn, J.: Vector optimization problems and their solution concepts. Recent Developments in Vector Optimization. Vector Optim. Springer, Berlin (2012)
20. Flores-Bazán, F., Gutiérrez, C., Novo, V.: A Brézis-Browder principle on partially ordered spaces and related ordering theorems. J. Math. Anal. Appl. **375**, 245–260 (2011)
21. Flores-Bazán, F., Hernández, E.: A unified vector optimization problem: complete scalarizations and applications. Optimization **60**, 1399–1419 (2011)

22. Flores-Bazán, F., Hernández, E.: Optimality conditions for a unified vector optimization problem with not necessarily preordering relations. J. Global Optim. **56**, 299–315 (2013)
23. Gerth, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
24. Göpfert, A., Riahi, H., Tammer, T., Zalinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
25. Gutiérrez, C., Jiménez, B., Novo, V., Thibault, L.: Strict approximate solutions in set-valued optimization with applications to the approximate Ekeland variational principle. Nonlinear Anal. **73**, 3842–3855 (2010)
26. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**, 1822–1833 (2012)
27. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**, 187–206 (2005)
28. Ha, T.X.D.: Optimality conditions for several types of efficient solutions of set-valued optimization problems. In: Pardalos, P., Rassias, Th.M., Khan, A.A. (eds.) Nonlinear Analysis and Variational Problems, pp. 305–324. Springer, Heidelberg (2010)
29. Hamel, A.: Variational principles on metric and uniform spaces. Habilitation thesis, University of Halle-Wittenberg, Germany (2005)
30. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**, 66–95 (2010)
31. Hamel, A., Löhne, A.: Minimal element theorems and Ekelands principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)
32. Heikkilä, S.: Inclusion problems in ordered topological vector spaces and applications. J. Math. Anal. Appl. **298**, 94–105 (2004)
33. Hernández, E.: Problemas de optimizacin en anlisis de multifunciones. Ph.D. thesis Doctoral. UNED, Madrid (2005)
34. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
35. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**, 1726–1736 (2007)
36. Hernández, E., Rodríguez-Marín, L.: Lagrangian duality in set-valued optimization. J. Optim. Theory Appl. **134**, 119–134 (2007)
37. Hernández, E., Rodríguez-Marín, L.: Duality in set optimization with set-valued maps. Pac. J. Optim. **3**, 245–255 (2007)
38. Hernández, E., Rodríguez-Marín, L.: Locating minimal sets using polyhedral cones. Oper. Res. Lett. **39**, 466–470 (2011)
39. Hernández, E., Rodríguez-Marín, L.: Weak and strong subgradients of set-valued maps. J. Optim. Theory Appl. **149**, 352–365 (2011)
40. Hernández, E., Rodríguez-Marín, L., Sama, M.: Criteria of solution in set-valued optimization, In: Proceedings of Kosen Workshop MTE2008, Ibaraki national college of technology, Hitachinaka, Ibaraki (2008)
41. Hernández, E., Rodríguez-Marín, L., Sama, M.: Some equivalent problems in set optimization. Oper. Res. Lett. **37**, 61–64 (2009)
42. Hernández, E., Rodríguez-Marín, L., Sama, M.: On solutions of set-valued optimization problems. Comput. Math. Appl. **60**, 1401–1408 (2010)
43. Hernández, E., Rodríguez-Marín, L., Sama, M.: About Hahn-Banach extension theorems and applications to set-valued optimization. Comput. Math. Appl. **64**, 1778–1788 (2012)
44. Heyde, F., Löhne, A., Tammer, Chr: Set-valued duality theory for multiple objective linear programs and application to mathematical finance, Math. Meth. Oper. Res. **69**, 159–179 (2009)
45. Hu, R., Fang, Y.P., Huang, N.J.: Optimality conditions for set optimization with weighted criteria. Acta Math. Sinica (Chin. Ser.) **52**, 989–1000 (2009)
46. Jahn, J.: Vector Optimization. Theory, Applications, and Extensions. Springer, Berlin (2004)
47. Janh, J.: Vectorization in set optimization. Manuscript (2013)

48. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
49. Khanh, P.Q., Quy, D.N.: On Ekelands variational principle for pareto minima of set-valued mappings. J. Optim. Theory Appl. **153**, 280–297 (2012)
50. Kimura, K., Shimizu, A.: Minimal element theorems for set-optimizations. J. Nonlinear Convex Anal., Yokohama Publ., Yokohama, 105–113 (2010)
51. Klein, E., Thompson, A.C.: Theory of Correspondences. Including Applications to Mathematical Economics. Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York (1984)
52. Kuroiwa, D.: Some criteria in set-valued optimization. RIMS Kokyuroku **985**, 171–176 (1997)
53. Kuroiwa, D.: Some duality theorems of set-valued optimizacion with natural criteria. In: Proccedings of the International Conference on Nonlinear Analysis and Convex Analysis, World Scientific, pp. 221–228 (1999)
54. Kuroiwa, D.: On set-valued optimization. In: Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000). Nonlinear Anal. 472(2), 1395–1400 (2001)
55. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**, 73–84 (2003)
56. Kuroiwa, D.: Existence of efficient points of set optimization with weighted criteria. J. Nonlinear Convex Anal. **4**, 117–123 (2003)
57. Kuroiwa, D.: On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. **10**, 41–50 (2009)
58. Kuroiwa, D., Nuriya, T.: A generalized embedding vector space in set optimization. J. Nonlinear Convex Anal. 297–303 (2007)
59. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 3 (Athens, 1996). Nonlinear Anal. **30**, 1487–1496 (1997)
60. Kuwano, I., Tanaka, T.: Continuity of cone-convex functions. Optim. Lett. **6**, 1847–1853 (2012)
61. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets in set-valued optimization. RIMS Kokyuroku **1685**, 270–280 (2010)
62. Löhne, A.: Optimization with set relations: conjugate duality. Optimization **54**, 265–282 (2005)
63. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
64. Löhne, A., Tammer, C.: A new approach to duality in vector optimization. Optimization **56**, 221–239 (2007)
65. Löhne, A., Frank, H.: Solution concepts in vector optimization, a fresh look at an old story. Optimization **60**, 1421–1440 (2011)
66. Löhne, A., Schrage, C.: An algorithm to solve polyhedral convex set optimization problems. Optimization **62**, 131–141 (2013)
67. Liu, C.G., Ng, K.F.: Ekeland's variational principle for set-valued functions. SIAM J. Optim. **21**, 41–56 (2011)
68. Lin, Y.-C., Chen, H.-J.: Solving the set equilibrium problems. Fixed Point Theory Appl. (2011). doi:10.1155/2011/945413
69. Luc, D.T.: Theory of vector optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
70. Maeda, T.: On characterization of fuzzy vectors and its applications to fuzzy mathematical programming problems. Fuzzy Sets Syst. **159**, 3333–3346 (2008)
71. Maeda, T.: On optimization problems with set-valued objective maps. Appl. Math. Comput. **217**, 1150–1157 (2010)
72. J. Optim. Theory Appl. On Optimization Problems with Set-Valued Objective Maps: Existence and Optimality. **153**, 263–279 (2012)
73. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory. Grundlehren Series 330. Springer, Berlin (2006)
74. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, II: Applications. Grundlehren Series 331. Springer, Berlin (2006)
75. Nishnianidze, Z.G.: Fixed points of monotonic multiple-valued operators. Bull. Georgian Acad. Sci. **114**, 489–491 (1984)

76. Nishizawa, S., Onodsuka, M., Tanaka, T.: Alternative theorems for set-valued maps based on a nonlinear scalarization. Pac. J. Optim. **1**, 147–159 (2005)
77. Nuriya, T., Kuroiwa, D.: A topology on the embedding space in set optimization without compactness, J. Nonlinear Convex Anal. 497–503 (2007)
78. Qiu, J.H.: On has version of set-valued Ekelands variational principle. Acta Mathematica Sinica, Engl. Ser. **28**, 717–726 (2012)
79. Rodríguez-Marín, L., Sama, M.: (Λ, C)-contingent derivatives of set-valued maps. J. Math. Anal. Appl. **335**, 974–989 (2007)
80. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of multiobjective optimization. Mathematics in Science and Engineering, vol. 176. Academic Press, Inc., Orlando (1985)
81. Shimizu, A., Nishizawa, S., Tanaka, T., *Optimality conditions in set-valued optimization using nonlinear scalarization methods*, Nonlinear Anal. Convex Anal. 565–574, Yokohama Publ., Yokohama (2007)
82. Shimizu, A., Tanaka, T.: Minimal element theorem with set-relations. J. Nonlinear Convex Anal. **9**, 249–253 (2008)
83. Truong, X.D.H.: Cones admitting strictly positive functionals and scalarization of some vector optimization problems. J. Optim. Theory Appl. **93**, 355–372 (1997)
84. Young, R.C.: The algebra of many-valued quantities. Math. Ann. **104**, 260–290 (1931)
85. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)
86. Zadeh, L.A.: Fuzzy sets. Inform. Control **8**, 338–353 (1965)
87. Zhang, W.Y., Li, S.J., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**, 3769–3778 (2009)

Linear Vector Optimization and European Option Pricing Under Proportional Transaction Costs

Alet Roux and Tomasz Zastawniak

Abstract A method for pricing and superhedging European options under proportional transaction costs based on linear vector optimisation and geometric duality developed by Löhne and Rudloff (Int. J. Theor. Appl. Finance 17(2): 1450012–1–1450012–33, 2014) is compared to a special case of the algorithms for American type derivatives due to Roux and Zastawniak (Acta Applicandae Mathematicae, published online 2015). An equivalence between these two approaches is established by means of a general result linking the support function of the upper image of a linear vector optimisation problem with the lower image of the dual linear optimisation problem.

Keywords Option pricing · Superhedging · Transaction costs · Linear vector optimisation

1 Introduction

We compare two existing methods for the computational pricing and superhedging of European options in the presence of proportional transaction costs, and investigate the relationships between them, highlighting their similarities, differences and relative strengths.and dual constructions stated in Sect. 3.3, goes back to [20, 21], where it was developed for the much more general class of American type derivative securities, of which European options are a special case. The other method, which relies on linear vector optimisation and geometric duality, was proposed by [17] and named the SHP-algorithm by them; see Sect. 3.4.

As a by-product, we prove a general result establishing one-to-one correspondence between the support function of the upper image of a linear vector optimisation

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problem on the one hand, and the lower image of the dual linear vector optimisation problem on the other hand; see Proposition 2.1. This result provides a link between the two methods for pricing and superhedging European options, and it is also interesting in its own right.

We work within the general model of a currency exchange market of [9], with proportional transaction costs included in the form of exchange rate bid ask spreads. This model has been extensively studied, for example, by [10, 11, 22].

All three algorithms, the primal construction, the dual construction and the SHP-algorithm lend themselves well to computer implementation. For the primal and dual constructions this has been done by [21] with the aid of the *Maple* package *Convex* developed by [3]. To implement the SHP-algorithm [17] used Benson's linear vector optimisation technique; see [2, 4]. We illustrate the results by a numerical example computed by means of the primal and dual constructions and compare this with a similar example presented by [17], who employed the SHP-algorithm.

We conclude by suggesting a possible extension of the SHP-algorithm to hedge and price the seller's (short) position in an American option, and pointing out an inherent difficulty in hedging and pricing the buyer's (long) position in an American option due to the essential non-convexity of the problem.

2 A General Duality Result

In this section we present a simple observation that links support functions with duality in linear vector optimization. The related work of [18] provides further insight on the connection between support functions and duality. This result will prove useful in comparing the various pricing and hedging algorithms in the following sections.

For a cone $C \subseteq \mathbb{R}^q$ we define a partial ordering $\leq_C$ on $\mathbb{R}^q$ by

$$y \leq_C z \iff z - y \in C$$

and denote by C^+ the dual (or positive polar) cone of C, i.e.

$$C^+ = \left\{x \in \mathbb{R}^q : x^T y \geq 0 \forall y \in C\right\}.$$

In what follows we assume that C is a polyhedral cone with non-empty interior, and there exists some $c \in \operatorname{int} C$ with $c_q = 1$. Suppose that matrices $P \in \mathbb{R}^{q \times d}$ and $B \in \mathbb{R}^{m \times d}$ and a vector $b \in \mathbb{R}^m$ are given, and consider the linear vector optimization problem

$$\text{minimize } Px \text{ with respect to } \leq_C \text{ over } x \in S, \tag{P}$$

with feasible set

$$S = \{x \in \mathbb{R}^d : Bx \geq b\}.$$

The *upper image* of problem (P) is the set

$$\mathcal{P} = P[S] + C.$$

The dual problem to (P) is

$$\text{maximize } D^*(u, w) \text{ with respect to } \leq_K \text{ over } (u, w) \in T, \tag{D*}$$

where the linear operator $D^* : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^q$ is defined as

$$D^*(u, w) = (w_1, \dots, w_{q-1}, b^T u)^T \text{ for } (u, w) \in \mathbb{R}^m \times \mathbb{R}^q,$$

with $K = \text{cone}\{e^q\}$ for $e^q = (0, \dots, 0, 1) \in \mathbb{R}^q$, and with

$$T = \{(u, w) \in \mathbb{R}^m \times \mathbb{R}^q : u \geq 0, B^T u = P^T w, c^T w = 1, w \in C^+\}.$$

The *lower image* of problem (D*) is the set

$$\mathcal{D}^* = D^*[T] - K.$$

We now state and prove a general result that links the lower image $\mathcal{D}^*$ of (D*) with the support function of $-\mathcal{P}$, where $\mathcal{P}$ is the upper image of (P). The support function $Z : \mathbb{R}^q \to \mathbb{R}$ of $-\mathcal{P}$ is defined as (see e.g. [19] p. 28)

$$Z(x) = \sup \left\{ x^T z : z \in -\mathcal{P} \right\} \text{ for all } x \in \mathbb{R}^q.$$

Note that $Z(x)$ is the negative of a scalarization of $\mathcal{P}$ with respect to the weighting vector x (see e.g. [15] Sect. 4.1.1). Thus the following result can be regarded as a reformulation of strong geometric duality (see [15] Theorems 4.40, 4.41) by means of the family of scalarizations of $\mathcal{P}$.

Proposition 2.1 *If C contains no lines, i.e. if $C \cap (-C) = \{0\}$, then*

$$\mathcal{D}^* = \left\{ w \in \mathbb{R}^q : -w_q \geq Z\left(w_1, \dots w_{q-1}, 1 - \sum_{i=1}^{q-1} c_i w_i \right) \right\}, \tag{2.1}$$

$$Z(w) = \begin{cases} -\sup \left\{ y \in \mathbb{R} : \frac{1}{c^T w} \left(w_1, \dots, w_{q-1}, y \right) \in \mathcal{D}^* \right\} & \text{if } c^T w > 0, \\ 0 & \text{if } w = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Proof If C contains no lines, then Theorems 4.40 and 4.41 of [15] (see also [4] Remark 3.7) give

$$\mathcal{D}^* = \left\{ w \in \mathbb{R}^q : \varphi(y, w) \geq 0 \forall y \in \mathcal{P} \right\},$$

where the bi-affine coupling function $\varphi : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ is defined as

$$\varphi(y, w) = \sum_{i=1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) - w_q \text{ for } (y, w) \in \mathbb{R}^q \times \mathbb{R}^q.$$

The function φ was first introduced for the special case $c = (1, \dots, 1)^T$ by [7] and for general c by [17].

Observe that $\varphi(y, w) \geq 0$ for all $y \in \mathcal{P}$ if and only if

$$-w_q \geq \sum_{i-1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) \quad \text{for all } y \in -\mathcal{P},$$

that is, if and only if

$$\begin{aligned} -w_q &\geq \sup \left\{ \sum_{i-1}^{q-1} y_i w_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i w_i\right) : y \in -\mathcal{P} \right\} \\ &= Z\left(w_1, \dots w_{q-1}, 1 - \sum_{i=1}^{q-1} c_i w_i\right). \end{aligned}$$

This proves (2.1).

Now take any $w \in \mathbb{R}^d$ such that $c^T w > 0$. Then $-y \geq Z(w)$ is equivalent to $-\frac{y}{c^T w} \geq Z\left(\frac{w}{c^T w}\right)$ since the support function is positively homogeneous. By (2.1), the last inequality is in turn equivalent to $\frac{1}{c^T w}\left(w_1, \dots, w_{q-1}, y\right) \in \mathcal{D}^*$. This shows that

$$\begin{aligned} Z(w) &= -\sup\{y \in \mathbb{R} : -y \geq Z(w)\} \\ &= -\sup\left\{y \in \mathbb{R} : \frac{1}{c^T w}\left(w_1, \dots, w_{q-1}, y\right) \in \mathcal{D}^*\right\} \end{aligned}$$

when $c^T w > 0$. If $w = 0$, then $Z(w) = 0$ by the definition of the support function. Finally, take any $w \neq 0$ such that $c^T w \leq 0$. Since $c \in \operatorname{int} C$, there is an $\varepsilon > 0$ such that $c - \varepsilon w \in C$. It follows that $(c - \varepsilon w)^T w = c^T w - \varepsilon w^T w < 0$ because $w^T w > 0$. As $\mathcal{P} = \mathcal{P} + C$, for any fixed $x \in \mathcal{P}$ and for each $\lambda > 0$ we have $x + \lambda(c - \varepsilon w) \in \mathcal{P}$. Hence, by the definition of the support function,

$$Z(w) \geq -(x + \lambda(c - \varepsilon w))^T w = -x^T w - \lambda(c - \varepsilon w)^T w$$

for each $\lambda > 0$. Since $(c - \varepsilon w)^T w < 0$, this means that $Z(w) = \infty$, completing the proof of (2.2). □

Remark 2.2 According to Proposition 2.1,

$$\mathcal{D}^* = \left\{(w_1, \dots, w_{q-1}, y) \in \mathbb{R}^q : (w, y) \in -\operatorname{epi} Z, c^T w = 1\right\}, \qquad (2.3)$$

so $\mathcal{D}^*$ can be identified with the section of the cone $-\operatorname{epi} Z$ by the hyperplane $\{(w, y) \in \mathbb{R}^q \times \mathbb{R} : c^T w = 1\}$ in $\mathbb{R}^{q+1}$. The convex set $\mathcal{D}^*$ (which depends on c) captures the same information as the support function Z. This is remarkable given that Z is independent of the arbitrary choice of c. Also note the similarity between (2.3) and the representation by [6, p. 828] of the dual image in a more general setting.

This section concludes with a simple example.

Example 2.3 Suppose that

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \quad C = \operatorname{cone}\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$

and fix $c = (0, 1)^T \in \operatorname{int} C$. For this data we have

$$\mathcal{P} = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq \tfrac{1}{3} z_1 + 4, z_2 \geq z_1, z_2 \geq -\tfrac{1}{3} z_1 + 4\},$$
$$\mathcal{D}^* = \{(w_1, y) \in \mathbb{R}^2 : -1 \leq w_1 \leq \tfrac{1}{3}, y \leq 4, y - 6w_1 \leq 6\}$$

(full details in [16] Example 6.4]. The sets $\mathcal{P}$ and $\mathcal{D}^*$ are represented graphically in Fig. 1.

The support function Z is finite on its effective domain, which consists of vectors $w \in \mathbb{R}^2$ such that $x^T w \leq 0$ for each $x \in -\mathcal{P}$, so

$$\operatorname{dom} Z = \{w \in \mathbb{R}^2 : Z(w) < \infty\} = \{(w_1, w_2) \in \mathbb{R}^2 : w_2 \geq -w_1, w_2 \geq 3w_1\}.$$

For each $w \in \operatorname{dom} Z$ the linear function $x \mapsto x^T w$ takes a maximum at one of the extreme points $(0, -4)$, $(-6, -6)$ of the convex set $-\mathcal{P}$, hence

$$Z(w) = \sup\{x^T w : x \in -\mathcal{P}\} = \max\{-4w_2, -6w_1 - 6w_2\}.$$

This means that

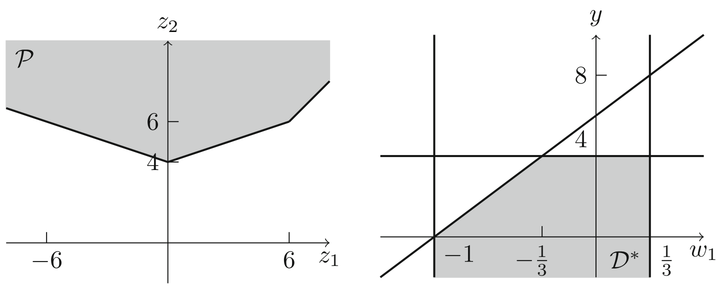


Fig. 1 Upper and lower images in Example 2.3

$$\begin{aligned}\{(w_1, y) \in \mathbb{R}^2 : (w, y) \in -\operatorname{epi} Z, c^T w = 1\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \leq -Z(w_1, w_2), (w_1, w_2) \in \operatorname{dom} Z, w_2 = 1\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \leq -Z(w_1, 1), -1 \leq w_1 \leq \tfrac{1}{3}\} \\ &= \{(w_1, y) \in \mathbb{R}^2 : y \leq 4, y \leq 6w_1 + 6, -1 \leq w_1 \leq \tfrac{1}{3}\} = \mathcal{D}^*.\end{aligned}$$

This identifies $\mathcal{D}^*$ with the section of $-\operatorname{epi} Z$ by the hyperplane

$$\{(w, y) \in \mathbb{R}^2 \times \mathbb{R} : c^T w = 1\} = \{(w_1, w_2, y) \in \mathbb{R}^3 : w_2 = 1\}.$$

3 Pricing and Hedging European Options Under Proportional Transaction costs

3.1 Currency Model

The model is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t=0}^T)$. We assume that Ω is finite, that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, that $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For each t denote by Ω_t the collection of atoms of $\mathcal{F}_t$, called the time t *nodes* of the associated stock price tree model. Note that $\Omega_0 = \{\Omega\}$ and $\Omega_T = \{\{w\} : \omega \in \Omega\}$. For every $t < T$ a node $\nu \in \Omega_{t+1}$ is said to be a *successor* of a node $\mu \in \Omega_t$ if $\nu \subseteq \mu$. We denote for all $\mu \in \Omega_t$

$$\operatorname{succ} \mu = \{\nu \in \Omega_{t+1} : \nu \text{ a successor of } \mu\}.$$

For each t let $\mathcal{L}_t = \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}_t)$ be the collection of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables. We identify elements of $\mathcal{L}_t$ with functions on Ω_t whenever convenient.

We consider the discrete-time currency model introduced by [9] and studied by others. The model contains d assets or currencies. At each trading date $t = 0, 1, \dots, T$ one unit of each asset $k = 1, \dots, d$ can be obtained by exchanging $\pi_t^{jk} > 0$ units of asset $j = 1, \dots, d$. We assume that the exchange rates π_t^{jk} are $\mathcal{F}_t$-measurable and $\pi_t^{jj} = 1$ for all t and j, k.

We say that a portfolio $x \in \mathcal{L}_t$ can be *exchanged* into a portfolio $y \in \mathcal{L}_t$ at time t whenever there are $\mathcal{F}_t$-measurable random variables $\beta^{jk} \geq 0$, $j, k = 1, \ldots, d$ such that for all $k = 1, \ldots, d$

$$y^k = x^k + \sum_{j=1}^{d} \beta^{jk} - \sum_{j=1}^{d} \beta^{kj} \pi_t^{kj},$$

where β^{jk} represents the number of units of asset k received as a result of exchanging some units of asset j.

The *solvency cone* $\mathcal{K}_t \subseteq \mathcal{L}_t$ is the set of portfolios that are *solvent* at time t, i.e. those portfolios at time t that can be exchanged into portfolios with non-negative holdings in all d assets. It is straightforward to show that $\mathcal{K}_t$ is the convex cone generated by the canonical basis $e^1, \ldots, e^d$ of $\mathbb{R}^d$ and the vectors $\pi_t^{jk} e^j - e^k$ for $j, k = 1, \ldots, d$, and so $\mathcal{K}_t$ is a polyhedral cone. Note that $\mathcal{K}_t$ contains all the non-negative elements of $\mathcal{L}_t$.

A *self-financing strategy* $y = (y_t)_{t=0}^T$ is a predictable $\mathbb{R}^d$-valued process (i.e. $y_0 \in \mathcal{L}_0$ and $y_t \in \mathcal{L}_{t-1}$ for $t = 1, \ldots, T$) such that

$$y_t - y_{t+1} \in \mathcal{K}_t \quad \text{for all } t = 0, \ldots, T-1$$

Here $y_0 \in \mathcal{L}_0$ is the initial endowment, and $y_t \in \mathcal{L}_{t-1}$ for each $t = 1, \ldots, T$ is the portfolio held from time $t-1$ to time t. Let Φ be the set of self-financing strategies.

A self-financing strategy $y = (y_t) \in \Phi$ is called an *arbitrage opportunity* if $y_0 = 0$ and there is a portfolio $x \in \mathcal{L}_T \setminus \{0\}$ with non-negative holdings in all d assets such that $y_T - x \in \mathcal{K}_T$. This notion of arbitrage was considered by [22], and its absence is formally different but equivalent to the weak no-arbitrage condition introduced by [11].

Theorem 3.1 ([11, 22]) *The model admits no arbitrage opportunity if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an $\mathbb{R}^d$-valued $\mathbb{Q}$-martingale $S = (S_t)$ such that*

$$S_t \in \mathcal{K}_t^+ \setminus \{0\} \text{ for all } t, \tag{3.1}$$

where $\mathcal{K}_t^+$ is the dual cone of $\mathcal{K}_t$.

Remark 3.2 A pair $(\mathbb{Q}, S)$ satisfying the conditions in Theorem 3.1 is called a *consistent pricing pair*. In place of such a pair $(\mathbb{Q}, S)$ one can equivalently use the so-called *consistent price process* $S_t \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t)$; see [22].

3.2 European Options

A *European option* with expiry time $T > 0$ and payoff $\xi \in \mathcal{L}_T$ is a contract that gives its holder (i.e. the option buyer) the right to receive a portfolio ξ of currencies at time T. On the other hand, the writer (seller) of the option is obliged to deliver this portfolio to the buyer.

To hedge against this liability the writer can follow a self-financing strategy $y \in \Phi$ such that $y_T - \xi \in \mathcal{K}_T$. The initial endowment y_0 of such a strategy y is called a *superhedging portfolio*, and the strategy y itself is called a *superhedging strategy* for the European option ξ.

The *ask price* (*seller's price*, *superhedging price*) $\pi_i^a(\xi)$ of the European option in currency $i = 1, \ldots, d$ can be understood as the lowest value x such that the portfolio consisting of x units of currency i and no other currency is a superhedging portfolio for ξ. In other words,

$$\pi_i^a(\xi) = \min \left\{ x \in \mathbb{R} : xe^i \text{ is a superhedging portfolio for } \xi \right\}.$$

On the other hand, to hedge his position the option buyer would like to follow a self-financing strategy $y \in \Phi$ such that $y_T + \xi \in \mathcal{K}_T$. Here $-y_0$ is a portfolio of currencies which the option buyer could borrow at time 0 and would be able to settle later by following the strategy y and using the payoff ξ to be received on exercising the option at time T. We call $-y_0$ a *subhedging portfolio* and $-y$ a *subhedging strategy* for the European option ξ.

The *bid price* (*buyer's price*, *subhedging price*) $\pi_i^b(\xi)$ of the European option in currency $i = 1, \ldots, d$ can be understood as the highest value x such that the portfolio consisting of x units of currency i and no other currency is a subhedging portfolio for ξ,

$$\pi_i^b(\xi) = \max \left\{ x \in \mathbb{R} : xe^i \text{ is a subhedging portfolio for } \xi \right\}.$$

It is the highest amount in currency i that an option holder could raise by using the option as collateral.

Observe that $-y$ is a subhedging strategy for a European option ξ if and only if y is a superhedging strategy for $-\xi$. It follows immediately that

$$\pi_i^b(\xi) = -\pi_i^a(-\xi).$$

Because of these relationships it is sufficient to develop algorithms for hedging and pricing the seller's (short) position in a European option.

3.3 Primal and Dual Constructions

The constructions presented here for European options are a special case of those developed by [21] to hedge and price the much wider class of American type options under proportional transaction costs. Construction 4.2 in [21], which produces the set of superhedging portfolios, takes a particularly simple form in this special case:

- For each $\omega \in \Omega_T$ put

$$\mathcal{Z}_T^\omega = \xi^\omega + \mathcal{K}_T^\omega.$$

- If $\mathcal{Z}_{t+1}$ has already been constructed for some $t = 0, 1, \dots, T-1$, then for each $\omega \in \Omega_t$ put

$$\mathcal{W}_t^\omega = \bigcap_{\omega' \in \operatorname{succ} \omega} \mathcal{Z}_{t+1}^{\omega'},$$
$$\mathcal{Z}_t^\omega = \mathcal{W}_t^\omega + \mathcal{K}_t^\omega$$

(To link this with Construction 4.2 in [21] observe that the formula for $\mathcal{W}_t$ can be written concisely as $\mathcal{W}_t = \mathcal{Z}_{t+1} \cap \mathcal{L}_t$.)

For each t the set $\mathcal{Z}_t$ consists of all portfolios that allow the seller to hedge the option by following a self-financing strategy between times t and T. In particular, $\mathcal{Z}_0$ is the set of superhedging portfolios. The ask price of the option can be expressed in terms of $\mathcal{Z}_0$ as

$$\pi_i^a(\xi) = \min\left\{x \in \mathbb{R} : xe^i \in \mathcal{Z}_0\right\}. \tag{3.2}$$

The above construction involves two standard operations on polyhedral convex sets, namely the intersection of finitely many such sets and the algebraic sum of such a set and a polyhedral convex cone. Both operations can be implemented using standard geometric methods in existing software libraries, for example, *Parma Polyhedra Library* [1] and *PolyLib* [8, 13, 14, 23, among others]. As soon as the set $\mathcal{Z}_0$ of superhedging portfolios has been computed in this manner, it becomes a routine task to evaluate the option price $\pi_i^a(\xi)$ using (3.2). Roux and Zastawniak [21] provided a numerical implementation of this procedure for hedging and pricing European options (and much more generally, American type options) in currency markets with transaction costs by using the *Maple* package *Convex* [3].

Moreover, once the $\mathcal{Z}_t$ have been constructed, it is straightforward to compute a superhedging strategy starting from any superhedging portfolio $y_0 \in \mathcal{Z}_0$. Namely, if $y_t \in \mathcal{Z}_t$ has already been computed for some $t = 0, 1, \dots, T-1$, we can take $y_{t+1} \in (y_t - \mathcal{K}_t) \cap \mathcal{W}_t$. The intersection is non-empty since $\mathcal{Z}_t = \mathcal{W}_t + \mathcal{K}_t$, so it is always possible to find such y_{t+1}, though it may be non-unique. The self-financing condition $y_t - y_{t+1} \in \mathcal{K}_t$ is clearly satisfied. Moreover, since $\mathcal{W}_t = \mathcal{Z}_{t+1} \cap \mathcal{L}_t$, it follows that y_{t+1} is $\mathcal{F}_t$-measurable, so y constructed in this manner will be a predictable process. It also follows that $y_{t+1} \in \mathcal{Z}_{t+1}$, which makes it possible to iterate the procedure.

It is also possible to follow the construction using convex dual objects to the $\mathcal{Z}_t$. We introduce the support functions

$$Z_t(x) = \sup\left\{x^T z : z \in -\mathcal{Z}_t\right\}, \quad W_t(x) = \sup\left\{x^T z : z \in -\mathcal{W}_t\right\}$$

and the linear function

$$U(x) = -x^T \xi$$

defined for all $x \in \mathbb{R}^d$. If we need to make the dependence on $\omega \in \Omega$ explicit in these functions, we shall write Z_t^ω, W_t^ω, U^ω. The above construction (we call it the *primal construction*) can now be written in the following equivalent form (called the *dual construction*); see Lemma 5.5 in [21]:

- For each $\omega \in \Omega_T$

$$Z_T^\omega = \begin{cases} U^\omega & \text{on } \mathcal{K}_T^{+\omega}, \\ \infty & \text{otherwise.} \end{cases}$$

  This is the linear function U^ω restricted to the domain $\mathcal{K}_T^{+\omega}$.
- Suppose that Z_{t+1} has been constructed for some $t = 0, 1, \dots, T-1$. Then, for each node $\omega \in \Omega_t$ let W_t^ω be the convex hull of the family of convex functions $Z_{t+1}^{\omega'}$ indexed by $\omega' \in \operatorname{succ} \omega$, and let Z_t^ω be the restriction of W_t^ω to the domain $\mathcal{K}_t^{+\omega}$:

$$W_t^\omega = \operatorname{conv}\left\{ Z_{t+1}^{\omega'} : \omega' \in \operatorname{succ} \omega \right\},$$
$$Z_t^\omega = \begin{cases} W_t^\omega & \text{on } \mathcal{K}_t^{+\omega}, \\ \infty & \text{otherwise.} \end{cases}$$

Once Z_0 has been computed, the ask price of the option can be obtained as (see Theorem 4.4 in [21])

$$\pi_i^a(\xi) = -\min\left\{ Z_0(x) : x \in \mathbb{R}^d, x_i = 1 \right\}.$$

This dual construction also lends itself well to computer implementation. Taking the convex hull of finitely many polyhedral convex functions and restricting the domain of such a function to a given polyhedral convex cone are operations equivalent to some standard operations on polyhedral convex sets, which are widely available in computer packages such as the *Convex* library in *Maple* used by [21].

Observe that the dual construction, which follows from Lemma 5.5 in [21] specialised to the case of European options, is equivalent to the construction in Corollary 6.3 of [17]. The only difference is that the dual construction is expressed in terms of the support functions Z_t and W_t, whereas [17] use $\widetilde{V}_t(x) = -Z_t(x)$ and $V_t(x) = -W_t(x)$ defined for all x's on the hyperplane in $\mathbb{R}^d$ given by the condition $x^i = 1$. Both are a straightforward extension to d assets of the construction stated in Algorithm 4.1 of [20] in the case of 2 assets.

3.4 SHP-Algorithm

Löhne and Rudloff [17] consider the same problem of pricing and hedging European options (though not options of American type). In particular, the same sets as in the primal construction above are denoted by [17] as

$$SHP_t(\xi) = \mathcal{Z}_t.$$

These authors propose a different construction of the $\mathcal{Z}_t$ based on linear vector optimisation methods and geometric duality.

From this perspective, $S = \mathcal{W}_t$ can be viewed as the feasible set of a linear vector optimisation problem (P). If the solvency cone $\mathcal{K}_t$ contains no lines, which means that there are non-zero transaction costs between any two currencies, then the matrix P in (P) is just the $d \times d$ unit matrix, and the ordering cone is $C = \mathcal{K}_t$. The upper image of the linear vector optimisation problem (P) is

$$\mathcal{P} = P[S] + C = \mathcal{W}_t + \mathcal{K}_t = \mathcal{Z}_t.$$

Because C contains no lines, Benson's algorithm, see [2] or [4], can be applied to compute a solution to the dual problem (D*) and hence the corresponding lower image $\mathcal{D}^*$. The Benson algorithm yields simultaneously a solution to (P) and gives the upper image $\mathcal{P} = \mathcal{Z}_t$. We know from Proposition 2.1 that if C contains no lines, then $\mathcal{D}^*$ can be identified with a section of the epigraph of the support function Z of $-\mathcal{P}$. Since $\mathcal{P} = \mathcal{Z}_t$, it follows that $Z = Z_t$ is the function from the dual construction in Sect. 3.3.

A complication arises when the solvency cone $\mathcal{K}_t$ contains some lines, which means that there are currencies which can be exchanged into one another without incurring any transaction costs. This is dealt with by taking P to be the matrix representing the so-called liquidation map, a linear map which amounts to liquidating all but one of the assets that can be exchanged into one another without transaction costs; see (4.1) in [17] for the precise definition of P. In this case $C = P[\mathcal{K}_t]$ contains no lines because there are no longer any assets that can be exchanged into one another without transaction costs. Then the upper image of the linear vector optimisation problem (P) is

$$\mathcal{P} = P[S] + C = P[\mathcal{W}_t + \mathcal{K}_t] = P[\mathcal{Z}_t].$$

Since C contains no lines, Benson's algorithm can also be applied in this case to compute a solution to the dual problem (D*) and hence the corresponding lower image $\mathcal{D}^*$. The Benson algorithm yields simultaneously a solution to (P) and gives the upper image $\mathcal{P} = P[\mathcal{Z}_t]$. This then gives $\mathcal{Z}_t = \{x \in \mathcal{L}_t : Px \in \mathcal{P}\}$ as the inverse image of $\mathcal{P}$ under P. Once again by Proposition 2.1, since C contains no lines, it follows that $\mathcal{D}^*$ can be identified with a section of the epigraph of the support function Z of $-\mathcal{P} = -P[\mathcal{Z}_t]$. This is related to Z_t, the support function of $-\mathcal{Z}_t$, by $Z(x) = Z_t(P^T x)$.

4 Example

In this section we present an example to illustrate the numerical procedures discussed in Sect. 3.3. Consider a model involving three assets, with time horizon $\tau = 1$ and with $T = 4$ time steps. Two of the assets are risky with correlated returns, and follow the two-asset recombinant [12] model with Cholesky decomposition. That is, there are $(t+1)^2$ possibilities for the stock prices $S_t = (S^1, S^2)$ at each time step $t = 0, \ldots, T$, indexed by pairs (j_1, j_2) where $1 \leq j_1, j_2 \leq t+1$, and each non-terminal node with stock price $S_t(j_1, j_2)$ has four successors, associated with the stock prices $S_{t+1}(j_1, j_2)$, $S_{t+1}(j_1+1, j_2)$, $S_{t+1}(j_1, j_2+1)$ and $S_{t+1}(j_1+1, j_2+1)$. With $\Delta = \frac{\tau}{T}$ defined for convenience, the stock prices are given by

$$
\begin{aligned}
S_t^1(j_1, j_2) &= S_0^1 e^{\left(r - \frac{1}{2}\sigma_1^2\right)t\Delta + (2j_1 - t - 2)\sigma_1\sqrt{\Delta}}, \\
S_t^2(j_1, j_2) &= S_0^2 e^{\left(r - \frac{1}{2}\sigma_2^2\right)t\Delta + \left((2j_1 - t - 2)\rho + (2j_2 - t - 2)\sqrt{1-\rho^2}\right)\sigma_2\sqrt{\Delta}}
\end{aligned}
$$

for $t = 0, \ldots, T$ and $j_1, j_2 = 1, \ldots, t+1$, where $S_0^1 = 45$ and $S_0^2 = 50$ are the initial stock prices, $\sigma_1 = 15$ and $\sigma_2 = 20\,\%$ are the volatilities of the returns and $\rho = 20\,\%$ is the correlation between the log returns on the two stocks. The third asset is a risk-free bond with nominal interest rate $r = 5\,\%$ and value process

$$B_t = (1 + r\Delta)^{-(T-t)} \text{ for } t = 0, \ldots, T.$$

Proportional transaction costs are introduced by allowing the asset prices to have constant (proportional) bid-ask spreads, i.e. the bid and ask prices are

$$
\begin{aligned}
S_t^{1b} &= (1 - k_1)S_t^1, & S_t^{1a} &= (1 + k_1)S_t^1, \\
S_t^{2b} &= (1 - k_2)S_t^2, & S_t^{2a} &= (1 + k_2)S_t^2, \\
B_t^b &= (1 - k_3)B_t, & B_t^a &= (1 + k_3)B_t
\end{aligned}
$$

for $t = 0, \ldots, T$, where $k_1 = 2$, $k_2 = 4$ and $k_3 = 1\,\%$. The matrix of exchange rates at each time step t is then

$$
\begin{pmatrix} \pi_t^{11} & \pi_t^{12} & \pi_t^{13} \\ \pi_t^{21} & \pi_t^{22} & \pi_t^{23} \\ \pi_t^{31} & \pi_t^{32} & \pi_t^{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{S_t^{2a}}{S_t^{1b}} & \frac{B_t^a}{S_t^{1b}} \\ \frac{S_t^{1a}}{S_t^{2b}} & 1 & \frac{B_t^a}{S_t^{2b}} \\ \frac{S_t^{1a}}{B_t^b} & \frac{S_t^{2a}}{B_t^b} & 1 \end{pmatrix},
$$

and the solvency cone is

$$
\mathcal{K}_t = \operatorname{cone}\left\{ \begin{pmatrix} S_t^{2a} \\ -S_t^{1b} \\ 0 \end{pmatrix}, \begin{pmatrix} B_t^a \\ 0 \\ -S_t^{1b} \end{pmatrix}, \begin{pmatrix} -S_t^{2b} \\ S_t^{1a} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_t^a \\ -S_t^{1b} \end{pmatrix}, \begin{pmatrix} -B_t^b \\ 0 \\ S_t^{1a} \end{pmatrix}, \begin{pmatrix} 0 \\ -B_t^b \\ S_t^{2a} \end{pmatrix} \right\}.
$$

This model was also considered by [17, Section 5.2]; note that the assets have been reordered in the present paper.

Consider an exchange option with physical delivery and payoff

$$\xi = (\mathbf{1}_{\{S_T^{1a} \geq S_T^{2a}\}}, -\mathbf{1}_{\{S_T^{1a} \geq S_T^{2a}\}}, 0)$$

that matures at time step T. [17, Example 5.3] reported

$$SHP_0 = \operatorname{conv}\left\{\begin{pmatrix} 0.584 \\ -0.260 \\ -7.760 \end{pmatrix}, \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}, \begin{pmatrix} 0.347 \\ -0.446 \\ 13.341 \end{pmatrix}\right\} + \mathcal{K}_0,$$

and gave the ask price of the exchange option in terms of the bond as

$$\pi_3^a(\xi) = 7.418.$$

The boundary of SHP_0 is depicted in Fig. 2. Application of the primal construction in Sect. 3.3 produces

$$\mathcal{Z}_0 = \operatorname{conv}\left\{\begin{pmatrix} 0.584 \\ -0.260 \\ -7.760 \end{pmatrix}, \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}, \begin{pmatrix} 0.399 \\ -0.406 \\ 8.714 \end{pmatrix}, \begin{pmatrix} 0.424 \\ -0.388 \\ 6.564 \end{pmatrix}\right\} + \mathcal{K}_0,$$

from which the ask price of the exchange option in terms of each of three assets can be computed as

$$\pi_1^a(\xi) = 0.152, \qquad \pi_2^a(\xi) = 0.146, \qquad \pi_3^a(\xi) = 7.418.$$

There is substantial agreement between SHP_0 and $\mathcal{Z}_0$, which can be confirmed visually (see Fig. 2), and in view of the agreement on the ask price $\pi_3^a(\xi)$, we ascribe the differences in the specifications of SHP_0 and $\mathcal{Z}_0$ to the error level chosen in

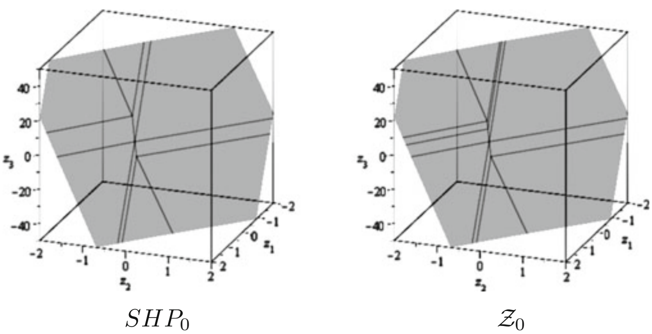


Fig. 2 Boundary of the set of superhedging endowments

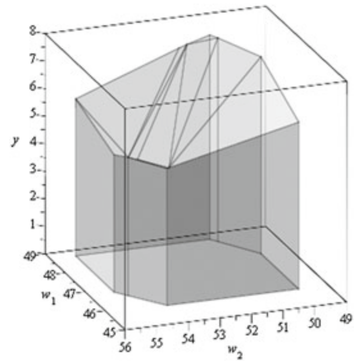


Fig. 3 Lower image $\mathcal{D}_0^*$ associated with Z_0

Benson's algorithm. Finally, application of the dual construction in Sect. 3.3 produces the support function Z_0 of $-\mathcal{Z}_0$. The set

$$\mathcal{D}_0^* = \{(w_1, w_2, y) : y \leq -Z_0(w_1, w_2, 1)\}$$

is the lower image of the dual problem (D*) with the choice $c = (0, 0, 1)^T$. It has 12 vertices

$$\begin{pmatrix} 48.726 \\ 51.930 \\ 7.081 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 51.681 \\ 7.178 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 54.050 \\ 4.981 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 55.201 \\ 5.702 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 49.946 \\ 6.048 \end{pmatrix}, \begin{pmatrix} 48.726 \\ 50.955 \\ 7.418 \end{pmatrix},$$
$$\begin{pmatrix} 48.573 \\ 50.796 \\ 7.395 \end{pmatrix}, \begin{pmatrix} 47.761 \\ 49.946 \\ 7.141 \end{pmatrix}, \begin{pmatrix} 46.565 \\ 54.907 \\ 5.012 \end{pmatrix}, \begin{pmatrix} 46.815 \\ 55.201 \\ 4.982 \end{pmatrix}, \begin{pmatrix} 46.405 \\ 54.718 \\ 5.018 \end{pmatrix}, \begin{pmatrix} 45.888 \\ 54.108 \\ 4.962 \end{pmatrix},$$

and is depicted in Fig. 3. The maximum of $\mathcal{D}_0^*$ in the y-direction is

$$\pi_3^a(\xi) = 7.418.$$

We conclude this numerical example by demonstrating the procedure of finding a superhedging strategy $y = (y_t)_{t=0}^T$ starting from the initial endowment

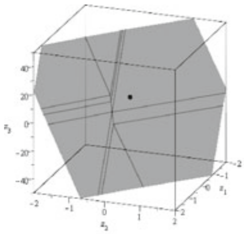
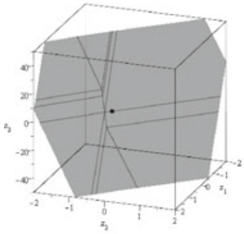
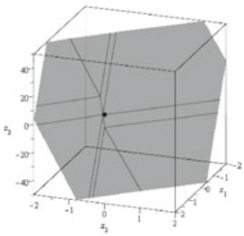
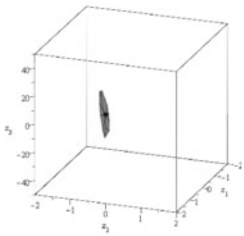
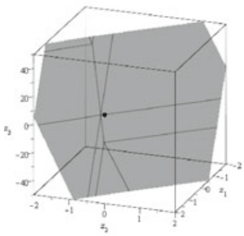
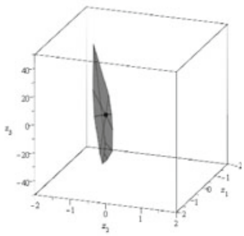
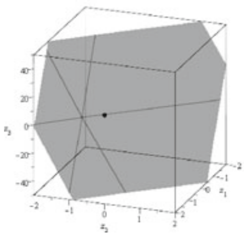
$$y_0 = (0, 0, \pi_3^a(\xi))^T \in \mathcal{Z}_0$$

along the price path in Table 1. At each time step t the portfolio y_t (indicated by a dot on the graph of the boundary of $\mathcal{Z}_t$ in Table 1) is rebalanced into a portfolio

$$y_{t+1} \in (y_t - \mathcal{K}_t) \cap \mathcal{W}_t \subseteq \mathcal{Z}_{t+1}.$$

As can be seen in Table 1, for this particular path the set $(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$ is a singleton at time steps $t = 0$ and $t = 1$, which means that there is only one choice for y_{t+1}. At

Table 1 Superhedging strategy along a path

t	(j_1, j_2)	y_t	$\mathcal{Z}_t$	$(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$
0	(1,1)	$\begin{pmatrix} 0.000 \\ 0.000 \\ 7.418 \end{pmatrix}$		$\left\{ \begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix} \right\}$
1	(2,1)	$\begin{pmatrix} 0.498 \\ -0.331 \\ 0.000 \end{pmatrix}$		$\left\{ \begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix} \right\}$
2	(2,1)	$\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$		
3	(3,2)	$\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$		
4	(3,2)	$\begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix}$		N/A

time steps $t = 2$ and $t = 3$ this set is a convex polytope, and the choice of y_{t+1} is no longer unique, which means that other considerations (e.g. a preference for holding one asset over another, or a preference not to trade) may be used to select y_{t+1} in $(y_t - \mathcal{K}_t) \cap \mathcal{W}_t$. In this demonstration we adopted a minimum-trading rule, that is, whenever possible we selected $y_{t+1} = y_t$. At the final time step $t = 4$ we have

$$y_4 - \xi = \begin{pmatrix} 0.641 \\ -0.491 \\ 0.000 \end{pmatrix} - \begin{pmatrix} 1.000 \\ -1.000 \\ 0.000 \end{pmatrix} = \begin{pmatrix} -0.359 \\ 0.509 \\ 0.000 \end{pmatrix} \in \mathcal{K}_4.$$

5 Representation of Superhedging Price

In this section we briefly present and compare the result of [17, 21] concerning the representation of the superhedging price of a European option in terms of risk-neutral expectations of the payoff ξ:

$$\pi_i^a(\xi) = \sup_{(\mathbb{Q},S)\in\mathcal{P}^i} \mathbb{E}_{\mathbb{Q}}((\xi^T S_T)), \tag{5.1}$$

where $\mathcal{P}^i$ is the set of pairs $(\mathbb{Q}, S)$ consisting of a probability measure $\mathbb{Q}$ and an $\mathbb{R}^d$-valued martingale S under $\mathbb{Q}$ satisfying the conditions of Theorem 3.1 and such that $S_t^i = 1$ for each $t = 0, \ldots, T$.

In Theorem 6.1 of [17] this result was proved under the so-called robust no-arbitrage condition of [22] and subject to the simplifying assumption that the solvency cone $\mathcal{K}_t$ contains no lines for any t (that is, the transaction costs are non-zero for any t). Their proof is based on the scalarisation procedure of [5] for the dual representation of the set SHP_0 of superhedging portfolios.

By comparison, the result in [21] is free of these restrictions: it works under the assumption that there is no arbitrage opportunity as defined in Sect. 3.1, which is weaker than the robust no-arbitrage condition, and without the need to assume that the solvency cone $\mathcal{K}_t$ contain no lines. It is also a much more general result that applies to American type derivatives, which reduces to (5.1) for European options. The proof is based on the dual construction from Sect. 3.3, which can in fact be used to produce a pair $(\mathbb{Q}, S)$ that realises the supremum in (5.1) (though in general such a pair does not lie in $\mathcal{P}^i$ as $\mathbb{Q}$ may be a degenerate measure, absolutely continuous with respect to but not necessarily equivalent to $\mathbb{P}$).

6 Conclusions

We have established a close link, indeed an equivalence between the three approaches: the above primal and dual constructions and the SHP-algorithm of [17]. The primal construction involves primal objects only. The dual construction deals exclusively with dual objects (support functions). Meanwhile, the SHP-algorithm switches back and forth between primal and dual objects (in this case the lower images of the dual problem (D*). By Proposition 2.1, these two types of dual objects are in one-to-one correspondence, which means that the apparent differences between the algorithms are merely superficial.

Moreover, all three approaches lend themselves well to numerical implementation: the primal and dual constructions utilise available software libraries for handling convex sets, whereas the SHP-algorithm makes an innovative use of Benson's procedure. In both approaches the procedure limiting computational efficiency is vertex enumeration. An advantage offered by Benson's algorithm is the ability to control the accuracy versus efficiency by choosing an error level. On the other hand, the *Maple* package *Convex* used by [21] employs exact arithmetic with rational numbers, hence there is no rounding beyond the conversion (as accurate as one needs it to be) of input data from real to rational numbers. While accurate rational arithmetic carries obvious computational overheads, the primal and dual algorithms are efficient enough so this does not become a problem in realistic multi-step and multi-asset examples that have been investigated, where the computation times were of the order of a couple of minutes on a standard PC machine.

One major difference as compared with the SHP-algorithm approach is that the primal and dual constructions have been developed in [21] for the much wider class of American type options, and can handle early exercise problems. In this context, European options are a particularly straightforward special case. It remains an open question whether or not the SHP-algorithm of [17] could be extended to American options, at least in the case of hedging and pricing the seller's position. It would be exciting to see this happen.

On the other hand, there are limits to what can be expected of the SHP-algorithm. American options present a particular obstacle that this approach is unlikely to be able to overcome. Namely, the case of hedging and pricing the buyer's (rather than the seller's) position in an American option leads to a non-convex optimisation problem, which is unlikely to yield to the power of linear vector optimisation methods and geometric duality. For the same reason, the dual construction collapses as there are no convex dual objects to work with in the first place. Nonetheless, the primal construction can still be adapted to handle this case; see Example 7.1 in [21] for details.

References

1. Bagnara, R., Hill, P.M., Zaffanella, E.: The parma polyhedra library: toward a complete set of numerical abstractions for the analysis and verification of hardware and software systems. Sci. Comput. Program. **72**(1–2), 3–21 (2008)
2. Benson, H.P.: An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. J. Global Optim. **13**(1), 1–24 (1998)
3. Franz, M.: Convex—a mdaple package for convex geometry. http://www.math.uwo.ca/~mfranz/convex/ (2009)
4. Hamel, A.H., Löhne, A. Rudloff, B.: Benson type algorithms for linear vector optimization and applications. arXiv:1302.2415v3 (2013)
5. Hamel, A., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**(1), 66–95 (2010)
6. Heyde, F.: Geometric duality for convex vector optimization problems. J. Convex Anal. **20**(3), 813–832 (2013)
7. Heyde, F., Löhne, A.: Geometric duality in multiple objective linear programming. SIAM J. Optim. **19**(2), 836–845 (2008)
8. IRISA.: Polylib—a library of polyhedral functions. http://www.irisa.fr/polylib/ (2001)
9. Kabanov, Y.M.: Hedging and liquidation under transaction costs in currency markets. Finance Stochast. **3**, 237–248 (1999)
10. Kabanov, Y.M., Rásonyi, M., Stricker, C.: No-arbitrage criteria for financial markets with efficient friction. Finance Stochast. **6**, 371–382 (2002)
11. Kabanov, Y.M., Stricker, C.: The Harrison-Pliska arbitrage pricing theorem under transaction costs. J. Math. Econ. **35**, 185–196 (2001)
12. Korn, R., Müller, S.: The decoupling approach to binomial pricing of multi-asset options. J. Comput. Finance **12**(3), 1–30 (2009)
13. Le Verge, H., : A Note on Chernikova's Algorithm. Publication interne 635, IRISA, Rennes (1992)
14. Loechner, V.: PolyLib: A library of polyhedral functions. http://icps.u-strasbg.fr/polylib/ (2010)
15. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
16. Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios and strategies in markets with transaction costs. arXiv:1107.5720v2 (2011)
17. Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. Int. J. Theor. Appl. Finance **17**(2), 1450012–1–1450012–33 (2014)
18. Luc, D.T.: On duality in multiple objective linear programming. Eur. J. Oper. Res. **210**(2), 158–168 (2011)
19. Rockafellar, R.T.: Convex analysis. Princeton Landmarks in Mathematics and Physics. Princeton University Press, Princeton (1996)
20. Roux, A., Tokarz, K., Zastawniak, T.: Options under proportional transaction costs: an algorithmic approach to pricing and hedging. Acta Applicandae Mathematicae **103**(2), 201–219 (2008)
21. Roux, A., Zastawniak, T.: American and Bermudan options in currency markets under proportional transaction costs, Acta Applicandae Mathematicae, published online (2015). doi:10.1007/s10440-015-0010-9, arXiv:1108.1910v3
22. Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Math. Finance **14**(1), 19–48 (2004)
23. Wilde, D.K.: A library for doing polyhedral operations, Rapport de recherche 2157. IRISA, Rennes (1993)

Part II
Special Topics

Conditional Analysis on $\mathbb{R}^d$

Patrick Cheridito, Michael Kupper and Nicolas Vogelpoth

Abstract This paper provides versions of classical results from linear algebra, real analysis and convex analysis in a free module of finite rank over the ring L^0 of measurable functions on a σ-finite measure space. We study the question whether a submodule is finitely generated and introduce the more general concepts of L^0-affine sets, L^0-convex sets, L^0-convex cones, L^0-hyperplanes and L^0-halfspaces. We investigate orthogonal complements, orthogonal decompositions and the existence of orthonormal bases. We also study L^0-linear, L^0-affine, L^0-convex and L^0-sublinear functions and introduce notions of continuity, differentiability, directional derivatives and subgradients. We use a conditional version of the Bolzano–Weierstrass theorem to show that conditional Cauchy sequences converge and give conditions under which conditional optimization problems have optimal solutions. We prove results on the separation of L^0-convex sets by L^0-hyperplanes and study L^0-convex conjugate functions. We provide a result on the existence of L^0-subgradients of L^0-convex functions, prove a conditional version of the Fenchel–Moreau theorem and study conditional inf-convolutions.

Keywords L^0-modules · Random sets · Conditional optimization · L^0-differentiability · L^0-convexity · Separating L^0-hyperplanes · L^0-convex conjugation · L^0-subgradients

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1 Introduction

Let L^0 be the set of all real-valued measurable functions on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$, where two of them are identified if they agree μ-almost everywhere. The purpose of this paper is to study the set $(L^0)^d$ of all d-dimensional vectors with components in L^0 and functions $f : (L^0)^d \to L^0$. Its main motivation are applications in the following two special cases:

- If μ is a probability measure, the elements of L^0 are random variables, and subsets $C \subseteq (L^0)^d$ can be understood as random sets in $\mathbb{R}^d$. A typical function $f : (L^0)^d \to L^0$ would, for example, be a mapping that conditionally on $\mathcal{F}$, assigns to every random point $X \in (L^0)^d$ its Euclidean distance to C.
- Let $(\Omega, \mathcal{G}, \mu)$ be the product of a σ-finite measure space $(\mathbb{T}, \mathcal{H}, \nu)$ and a probability space $(E, \mathcal{E}, P)$. If $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{G}$, the elements of L^0 are stochastic processes $(X_t)_{t \in \mathbb{T}}$ on $(E, \mathcal{E}, P)$. A subset $C \subseteq (L^0)^d$ could, for instance, describe the set of admissible strategies in a stochastic control problem, and an optimal strategy could be characterized as the conditional optimizer of an appropriate function $f : (L^0)^d \to L^0$ over C.

Unless Ω is the union of finitely many atoms, $(L^0)^d$ is an infinite-dimensional vector space over $\mathbb{R}$. But conditioned on $\mathcal{F}$, it is only d-dimensional. Or put differently, it is a free module of rank d over the ring L^0. This allows us to derive conditional analogs of classical results from linear algebra, real analysis and convex analysis that depend on the fact that $\mathbb{R}^d$ is a finite-dimensional vector space. L^0-modules have been studied before; see, for instance, Filipović et al. [4], Kupper and Vogelpoth [9], Guo [6], Guo [7] and the references in these papers. But since we consider free modules of finite rank, we are able to provide stronger results under weaker assumptions, and moreover, do not need Zorn's lemma or the axiom of choice. Our approach differs from standard measurable selection arguments in that we work modulo null-sets with respect to the measure μ and do not use ω-wise arguments. This has the advantage that one never leaves the world of measurable functions. But it only works in situations where a measure μ is given, and the quantities of interest do not depend on μ-null sets.

The results in this paper are theoretical. But they have already been applied several times: in Cheridito and Hu [1], they were used to describe stochastic constraints and characterize optimal strategies in a dynamic consumption and investment problem. In Cheridito and Stadje [3] they guaranteed the existence of a conditional subgradient. In Cheridito and Stadje [3] they were applied to show existence and uniqueness of economic equilibria in incomplete market models.

The structure of the paper is as follows: In Sect. 2 we investigate when an L^0-submodule of $(L^0)^d$ is finitely generated. Then we study conditional orthogonality and introduce L^0-affine sets, L^0-convex sets and L^0-convex cones. It turns out that the notion of σ-stability plays a crucial role. In Sect. 3 we investigate almost everywhere converging sequences in $(L^0)^d$ and the corresponding notion of closure. We define L^0-linear and L^0-affine functions $f : (L^0)^d \to (L^0)^k$ and show that they are continuous with respect to almost everywhere converging sequences. We also give a conditional version of the Bolzano–Weierstrass theorem and show that conditional Cauchy sequences converge. Moreover, we define L^0-bounded sets and give a condition for L^0-convex sets to be L^0-bounded. In Sect. 4 we study sequentially semicontinuous and L^0-convex functions $f : (L^0)^d \to L^0$ and prove a result which guarantees that a conditional optimization problem has an optimal solution. Section 5 is devoted to L^0-open sets, interiors and relative interiors. L^0-open sets form a topology, but they are not complements of sequentially closed sets. In Sect. 6 we give strong, weak and proper separation results of L^0-convex sets by L^0-hyperplanes. Section 7 studies L^0-convex functions and introduces conditional notions of differentiability, directional derivatives, subgradients and convex conjugation. We also provide results on the existence of conditional subgradients and give a conditional version of the Fenchel–Moreau theorem. In Sect. 8 we study conditional inf-convolutions.

Notation. We assume $\mu(\Omega) > 0$ and define $\mathcal{F}_+ := \{A \in \mathcal{F} : \mu[A] > 0\}$. By L we denote the set of all measurable functions $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, where two of them are identified if they agree a.e. (almost everywhere). In particular, for $X, Y \in L$, $X = Y$, $X > Y$ and $X \geq Y$ will be understood in the a.e. sense. Analogously, for sets $A, B \in \mathcal{F}$, we write $A = B$ if $\mu[A \triangle B] = 0$ and $A \subseteq B$ if $\mu[A \setminus B] = 0$. The set $L^0 := \{X \in L : |X| < \infty\}$ with the a.e. order is a lattice ordered ring, and every non-empty subset C of L has a least upper bound and a greatest lower bound in L with respect to the a.e. order. We follow the usual convention in measure theory and denote them by ess sup C and ess inf C, respectively. It is well-known (see for instance, [10]) that there exist sequences (X_n) and (Y_n) in C such that $\operatorname{ess\,sup} C = \sup_n X_n$ and $\operatorname{ess\,inf} C = \inf_n Y_n$. Moreover, if C is directed upwards, (X_n) can be chosen such that $X_{n+1} \geq X_n$, and if C is directed downwards, (Y_n) can be chosen so that $Y_{n+1} \leq Y_n$. For a set $A \in \mathcal{F}$, we denote by 1_A the characteristic function of A, that is, the function $1_A : \Omega \to \{0, 1\}$ which is 1 on A and 0 elsewhere. If $\mathcal{A}$ is a subset of $\mathcal{F}$, we set $\operatorname{ess\,sup} \mathcal{A} := \left\{\operatorname{ess\,sup}_{A \in \mathcal{A}} 1_A = 1\right\} \in \mathcal{F}$ and $\operatorname{ess\,inf} \mathcal{A} := \{\operatorname{ess\,inf}_{A \in \mathcal{A}} 1_A = 1\} \in \mathcal{F}$. Furthermore, we use the notation $L^0_+ := \{X \in L^0 : X \geq 0\}$, $L^0_{++} := \{X \in L^0 : X > 0\}$, $\overline{L} := \{X \in L : X > -\infty\}$, $\underline{L} := \{X \in L : X < +\infty\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. By $\mathbb{N}(\mathcal{F})$ we denote the set of all measurable functions $N : \Omega \to \mathbb{N}$.

2 Algebraic Structures and Generating Sets

We fix $d \in \mathbb{N}$ and consider the set $(L^0)^d := \left\{(X^1, \ldots, X^d) : X^i \in L^0\right\}$. On $(L^0)^d$ we define the conditional inner product and conditional 2-norm by

$$\langle X, Y \rangle := \sum_{i=1}^{d} X^i Y^i \quad \text{and} \quad ||X|| := \langle X, X \rangle^{1/2}.$$

For every $A \in \mathcal{F}$, $1_A L^0$ is a subring of L^0, and provided that $\mu[A] > 0$, $1_A (L^0)^d$ is a free $1_A L^0$-module of rank d generated by the base $1_A e_i$, $i = 1, \ldots, d$, where e_i is the ith unit vector in $\mathbb{R}^d \subseteq (L^0)^d$. In particular, $(L^0)^d$ is a free L^0-module of rank d.

Definition 2.1 We call a subset C of $(L^0)^d$

- stable if $1_A X + 1_{A^c} Y \in C$ for all $X, Y \in C$ and $A \in \mathcal{F}$;
- σ-stable if $\sum_{n\in\mathbb{N}} 1_{A_n} X_n \in C$ for every sequence $(X_n)_{n\in\mathbb{N}}$ in C and pairwise disjoint sets $A_n \in \mathcal{F}$ satisfying $\Omega = \bigcup_{n\in\mathbb{N}} A_n$;
- L^0-convex if $\lambda X + (1-\lambda)Y \in C$ for all $X, Y \in C$ and $\lambda \in L^0$ such that $0 \le \lambda \le 1$;
- an L^0-convex cone if it is L^0-convex and $\lambda X \in C$ for all $X \in C$ and $\lambda \in L^0_{++}$;
- L^0-affine if $\lambda X + (1-\lambda)Y \in C$ for all $X, Y \in C$ and $\lambda \in L^0$;
- L^0-linear (or an L^0-submodule) if $\lambda X + Y \in C$ for all $X, Y \in C$ and $\lambda \in L^0$.

For an arbitrary subset C of $(L^0)^d$ and $A \in \mathcal{F}$, we denote by $\mathrm{st}_A(C)$, $\mathrm{sst}_A(C)$, $\mathrm{conv}_A(C)$, $\mathrm{ccone}_A(C)$, $\mathrm{aff}_A(C)$, $\mathrm{lin}_A(C)$ the smallest subset of $1_A(L^0)^d$ containing $1_A C$ that is stable, σ-stable, L^0-convex, an L^0-convex cone, L^0-affine, or L^0-linear, respectively. If $A = \Omega$, we just write $\mathrm{st}(C)$, $\mathrm{sst}(C)$, $\mathrm{conv}(C)$, $\mathrm{ccone}(C)$, $\mathrm{aff}(C)$, $\mathrm{lin}(C)$ for these sets.

Remark 2.2 It can easily be checked that if C is a non-empty subset of $(L^0)^d$ and $A \in \mathcal{F}$, then

$$\mathrm{st}_A(C) = \left\{ \sum_{n=1}^k 1_{A_n} X_n : k \in \mathbb{N},\ X_n \in C,\ A_n \in \mathcal{F},\ \bigcup_{n=1}^k A_n = A,\ A_m \cap A_n = \emptyset \text{ for } m \neq n \right\};$$

$$\mathrm{sst}_A(C) = \left\{ \sum_{n\in\mathbb{N}} 1_{A_n} X_n : X_n \in C,\ A_n \in \mathcal{F},\ \bigcup_{n\in\mathbb{N}} A_n = A,\ A_m \cap A_n = \emptyset \text{ for } m \neq n \right\};$$

$$\mathrm{conv}_A(C) = \left\{ \sum_{n=1}^k \lambda_n X_n : k \in \mathbb{N},\ X_n \in C,\ \lambda_n \in 1_A L^0_+,\ \sum_{n=1}^k \lambda_n = 1_A \right\};$$

$$\mathrm{ccone}_A(C) = \left\{ \sum_{n=1}^k \lambda_n X_n : k \in \mathbb{N},\ X_n \in C,\ \lambda_n \in 1_A L^0_+,\ \sum_{n=1}^k \lambda_n \in 1_A L^0_{++} \right\};$$

$$\mathrm{aff}_A(C) = \left\{ \sum_{n=1}^k \lambda_n X_n : k \in \mathbb{N},\ X_n \in C,\ \lambda_n \in 1_A L^0,\ \sum_{n=1}^k \lambda_n = 1_A \right\};$$

$$\mathrm{lin}_A(C) = \left\{ \sum_{n=1}^k \lambda_n X_n : k \in \mathbb{N},\ X_n \in C,\ \lambda_n \in 1_A L^0 \right\}.$$

It follows that if $C = \{X_1, \dots, X_k\}$ for finitely many $X_1, \dots, X_k \in (L^0)^d$, then the sets $\mathrm{conv}_A(C)$, $\mathrm{ccone}_A(C)$, $\mathrm{aff}_A(C)$, $\mathrm{lin}_A(C)$ are all σ-stable.

Definition 2.3 Let $A \in \mathcal{F}_+$ and $k \in \mathbb{N}$. We call $X_1, \dots, X_k \in (L^0)^d$ linearly independent on A if $1_A X_1, \dots, 1_A X_k$ are linearly independent in the $1_A L^0$-module $1_A(L^0)^d$, that is, $(0, \dots, 0)$ is the only vector $(\lambda_1, \dots, \lambda_k) \in 1_A(L^0)^k$ satisfying

$$\lambda_1 X_1 + \cdots + \lambda_k X_k = 0.$$

We say that $X_1, \dots, X_k$ are orthogonal on A if $1_A \langle X_i, X_j \rangle = 0$ for $i \neq j$ and orthonormal on A if in addition, $1_A ||X_i|| = 1_A$, $1 \le i \le k$. If $X_1, \dots, X_k$ are linearly independent on A and $\mathrm{lin}_A\{X_1, \ldots, X_k\} = 1_A C$ for some subset C of $(L^0)^d$, we call them a basis of C on A. If in addition, $X_1, \ldots, X_k$ are orthogonal or orthonormal on A, we say $X_1, \ldots, X_k$ is an orthogonal or orthonormal basis of C on A, respectively.

Lemma 2.4 *Let $A \in \mathcal{F}$ and $X_1, \ldots, X_k, Y \in (L^0)^d$ for some $k \in \mathbb{N}$. Then there exists a largest subset $B \in \mathcal{F}$ of A such that $1_B Y \in \mathrm{lin}_B\{X_1, \ldots, X_k\}$.*

Proof The set

$$\mathcal{A} := \{B \in \mathcal{F} : B \subseteq A \text{ and } 1_B Y \in \mathrm{lin}_B\{X_1, \ldots, X_k\}\}$$

is directed upwards. So it contains an increasing sequence $(B_n)_{n\in\mathbb{N}}$ such that $B := \bigcup_n B_n = \mathrm{ess\,sup}\,\mathcal{A}$. B is the largest element of $\mathcal{A}$. □

Proposition 2.5 *Let $A \in \mathcal{F}_+$ and $k, l \in \mathbb{N}$. Assume $X_1, \ldots, X_k \in (L^0)^d$ are linearly independent on A and $\mathrm{lin}_A\{X_1, \ldots, X_k\} \subseteq \mathrm{lin}_A\{Y_1, \ldots, Y_l\}$ for some $Y_1, \ldots, Y_l \in (L^0)^d$. Then $k \le l$. Moreover, if $k = l$, then $Y_1, \ldots, Y_l$ are linearly independent on A and $\mathrm{lin}_A\{X_1, \ldots, X_k\} = \mathrm{lin}_A\{Y_1, \ldots, Y_l\}$.*

Proof One can write $1_A X_1 = \sum_{i=1}^l \lambda_i 1_A Y_i$ for some $\lambda_i \in L^0$. So there exists a $\sigma(1) \in \{1, \ldots, l\}$ such that $A_1 := A \cap \left\{\lambda_{\sigma(1)} \neq 0\right\} \in \mathcal{F}_+$, and one obtains

$$\mathrm{lin}_{A_1}\{X_1, \ldots, X_k\} \subseteq \mathrm{lin}_{A_1}\{Y_1, \ldots, Y_l\} = \mathrm{lin}_{A_1}(\{X_1, Y_1, \ldots, Y_l\} \setminus \left\{Y_{\sigma(1)}\right\}).$$

In particular, if $k \ge 2$, one must have $l \ge 2$, and it follows inductively that there exist $A_2, \ldots, A_d \in \mathcal{F}_+$ and an injection $\sigma : \{1, \ldots, k\} \to \{1, \ldots, l\}$ such that for all $i \in \{1, \ldots, k\}$,

$$\mathrm{lin}_{A_i}\{X_1, \ldots, X_k\} \subseteq \mathrm{lin}_{A_i}\{Y_1, \ldots, Y_l\} = \mathrm{lin}_{A_i}(\{X_1, \ldots, X_i, Y_1, \ldots, Y_l\} \setminus \left\{Y_{\sigma(1)}, \ldots, Y_{\sigma(i)}\right\}).$$

This shows that $k \le l$.

Now assume $k = l$ and $Y_1, \ldots, Y_l$ are not linearly independent on A. Then there exist $B \in \mathcal{F}_+$ and $j \in \{1, \ldots, k\}$ such that

$$\mathrm{lin}_B\{X_1, \ldots, X_k\} \subseteq \mathrm{lin}_B\{Y_1, \ldots, Y_k\} = \mathrm{lin}_B(\{Y_1, \ldots, Y_k\} \setminus \left\{Y_j\right\}),$$

a contradiction to the first part of the proposition. So if $k = l$, $Y_1, \ldots, Y_k$ must be linearly independent on A, and it remains to show that $\mathrm{lin}_A\{X_1, \ldots, X_k\} = \mathrm{lin}_A\{Y_1, \ldots, Y_k\}$. To do this, we assume that $\mathrm{lin}_A\{X_1, \ldots, X_k\} \subsetneq \mathrm{lin}_A\{Y_1, \ldots, Y_k\}$. Then $Y_j \notin \mathrm{lin}_A\{X_1, \ldots, X_k\}$ for at least one $j \in \{1, \ldots, k\}$. By Lemma 2.4, there exists a largest subset $B \in \mathcal{F}$ of A such that $1_B Y_j \in \mathrm{lin}_B\{X_1, \ldots, X_k\}$. The set $D := A \setminus B$ is in $\mathcal{F}_+$, and $X_1, \ldots, X_k, Y_j$ are linearly independent on D. But then

$$\mathrm{lin}_D\left\{X_1, \ldots, X_k, Y_j\right\} \subseteq \mathrm{lin}_D\{Y_1, \ldots, Y_k\},$$

again contradicts the first part of the proposition, and the proof is complete. □

Corollary 2.6 *Let $A \in \mathcal{F}_+$ and $k, l \in \mathbb{N}$. Assume $X_1, \ldots, X_k \in (L^0)^d$ are linearly independent on A and $\text{lin}_A\{X_1, \ldots, X_k\} = \text{lin}_A\{Y_1, \ldots, Y_l\}$ for some $Y_1, \ldots, Y_l \in (L^0)^d$ that are also linearly independent on A. Then $k = l \leq d$, and if $k = l = d$, one has $\text{lin}_A\{X_1, \ldots, X_k\} = \text{lin}_A\{Y_1, \ldots, Y_l\} = 1_A(L^0)^d$.*

Proof The corollary follows from Proposition 2.5 by noticing that

$$\text{lin}_A\{X_1, \ldots, X_k\} = \text{lin}_A\{Y_1, \ldots, Y_l\} \subseteq \text{lin}_A(e_1, \ldots, e_d) = 1_A(L^0)^d. \qquad \square$$

Lemma 2.7 *Let C be a non-empty σ-stable subset of $(L^0)^d$ and $X_1, \ldots, X_k \in (L^0)^d$ for some $k \in \mathbb{N}$. Then for given $A \in \mathcal{F}_+$, each of the collections*

$$\{B \in \mathcal{F}_+ : B \subseteq A \text{ and there exists a } Y \in C \text{ such that } ||Y|| > 0 \text{ on } B\} \tag{2.1}$$

and

$$\{B \in \mathcal{F}_+ : B \subseteq A \text{ and there exists } Y \in C \text{ such that } X_1, \ldots, X_k, Y \text{ are linearly independent on } B\} \tag{2.2}$$

is either empty or contains a largest set.

Proof Let us denote the collection (2.1) by $\mathcal{A}_1$ and (2.2) by $\mathcal{A}_2$. Both are directed upwards. So if either one of them is non-empty, it contains an increasing sequence of sets B_n with corresponding $Y_n \in C$, $n \in \mathbb{N}$, such that $B := \bigcup_n B_n = \text{ess sup}\, \mathcal{A}_i$. Since C is σ-stable,

$$Y := Y_1 1_{B_1 \cup B^c} + \sum_{n \geq 2} 1_{B_n \setminus B_{n-1}} Y_n$$

belongs to C. In the first case one has $||Y|| > 0$ on B, and in the second one, $X_1, \ldots, X_k, Y$ are linearly independent on B. This proves the lemma. $\square$

Theorem 2.8 *Let C be a σ-stable subset of $(L^0)^d$ containing an element $X \neq 0$. Then there exist a unique number $k \in \{1, \ldots, d\}$, unique pairwise disjoint sets $A_0, \ldots, A_k \in \mathcal{F}$ and $X_1, \ldots, X_k \in C$ such that the following hold:*

(i) $\bigcup_{i=0}^k A_i = \Omega$ *and* $\mu[A_k] > 0$;
(ii) $1_{A_0} C = \{0\}$;
(iii) For all $i \in \{1, \ldots, k\}$ *satisfying* $\mu[A_i] > 0$, $X_1, \ldots, X_i$ *is a basis of* $\text{lin}(C)$ *on* A_i.

Proof That k and the sets $A_0, \ldots, A_k$ are unique follows from Corollary 2.6. To show the existence of A_i and X_i satisfying (i)–(iii), we construct them inductively. Since C contains an element $X \neq 0$, it follows from Lemma 2.7 that there exists a largest set $B_1 \in \mathcal{F}_+$ such that $||Y|| > 0$ on B_1 for some $Y \in C$. Choose such a Y and call it X_1. One must have $1_{B_1^c} C = \{0\}$. If there exist no $B \in \mathcal{F}_+$ and $Y \in C$ such that X_1, Y are linearly independent on B, one obtains from Lemma 2.4 that $1_{B_1} Y \in \text{lin}_{B_1}\{X_1\}$ for all $Y \in C$, and therefore, $\text{lin}_{B_1}(C) = \text{lin}_{B_1}\{X_1\}$. So one can set $k = 1$, $A_0 = B_1^c$

and $A_1 = B_1$. On the other hand, if there exists a $B \in \mathcal{F}_+$ and $Y \in C$ such that X_1, Y are linearly independent on B, Lemma 2.7 yields a largest such set B_2 with a corresponding $X_2 \in C$. If there exists no $B \in \mathcal{F}_+$ and $Y \in C$ such that X_1, X_2, Y are linearly independent on B, then $\mathrm{lin}_{B_2}(C) = \mathrm{lin}_{B_2}\{X_1, X_2\}$ and one can set $k = 2$, $A_0 = B_c^1$, $A_1 = B_1 \setminus B_2$ and $A_2 = B_2$. Otherwise, one continues like this until there is no $B \in \mathcal{F}_+$ and $Y \in C$ such that $X_1, \dots, X_k, Y$ are linearly independent on B. Such a k must exist and $k \le d$. Otherwise one would have $X_1, \dots, X_{d+1} \in C$ that are linearly independent on some $B \in \mathcal{F}_+$, a contradiction to Corollary 2.6. One sets $A_0 = B_c^1$, $A_1 = B_1 \setminus B_2, \dots, A_{k-1} = B_{k-1} \setminus B_k$, $A_k = B_k$. □

Corollary 2.9 *Let C be a non-empty σ-stable subset of $(L^0)^d$ and $A \in \mathcal{F}$. Then $\mathrm{aff}_A(C)$ and $\mathrm{lin}_A(C)$ are again σ-stable.*

Proof If $1_A C = \{0\}$, then $\mathrm{aff}_A(C) = \mathrm{lin}_A(C) = \{0\}$, and the corollary is clear. Otherwise, one obtains from Theorem 2.8 that there exists a $k \in \{1, \dots, d\}$, disjoint sets $A_0, \dots, A_k \in \mathcal{F}$ and $X_1, \dots, X_k \in C$ such that $\bigcup_{i=0}^k A_i = A$, $1_{A_0} C = \{0\}$ and for all $i \in \{1, \dots, k\}$ satisfying $\mu[A_i] > 0$, $X_1, \dots, X_i$ is a basis of $\mathrm{lin}_A(C)$ on A_i. Now it can easily be verified that $\mathrm{lin}_A(C)$ is σ-stable. To see that $\mathrm{aff}_A(C)$ is σ-stable, one picks an $X \in 1_A C$. Then $\mathrm{aff}_A(C) - X = \mathrm{lin}_A(C - X)$ is σ-stable. So $\mathrm{aff}_A(C)$ is σ-stable too. □

Definition 2.10 The orthogonal complement of a non-empty subset C of $(L^0)^d$ is given by

$$C^\perp := \left\{X \in (L^0)^d : \langle X, Y \rangle = 0 \text{ for all } Y \in C\right\}.$$

It is clear that $C^\perp$ is an L^0-linear subset of $(L^0)^d$ satisfying

$$C \cap C^\perp \subseteq \{0\} \quad \text{and} \quad C \subseteq C^{\perp\perp}.$$

As a consequence of Theorem 2.8, one obtains the following corollary.

Corollary 2.11 *Let C be a non-empty σ-stable L^0-linear subset of $(L^0)^d$. Then there exist unique pairwise disjoint sets $A_0, \dots, A_d \in \mathcal{F}$ satisfying $\bigcup_{i=0}^d A_i = \Omega$ and an orthonormal basis $X_1, \dots, X_d$ of $(L^0)^d$ on Ω such that $1_{A_0} C = \{0\}$, $1_{A_d} C = 1_{A_d}(L^0)^d$ and*

$$1_{A_i} C = \mathrm{lin}_{A_i}\{X_1, \dots, X_i\}, \quad 1_{A_i} C^\perp = \mathrm{lin}_{A_i}\{X_{i+1}, \dots, X_d\} \quad \textit{for } 1 \le i \le d-1.$$

In particular, $C + C^\perp = (L^0)^d$, $C \cap C^\perp = \{0\}$ and $C = C^{\perp\perp}$.

Proof The uniqueness of the sets $A_1, \dots, A_d$ follows from Corollary 2.6, and in the special case $C = \{0\}$, one can choose $A_0 = \Omega$, $A_i = \emptyset$, $X_i = e_i$, $i = 1, \dots, d$.

If C is different from $\{0\}$, it follows from Theorem 2.8 that there exist a unique number $k \in \{1, \dots, d\}$, unique pairwise disjoint sets $A_0, \dots, A_k \in \mathcal{F}$ and $Y_1, \dots, Y_k \in C$ such that $\bigcup_{i=0}^{k} A_i = \Omega$, $\mu[A_k] > 0$, $1_{A_0} C = \{0\}$ and for all $i \in \{1, \dots, k\}$ satisfying $\mu[A_i] > 0$, $Y_1, \dots, Y_i$ is a basis of C on A_i. Let us define

$$U_1 := 1_{A_1 \cup \dots \cup A_k} \frac{Y_1}{||Y_1||} \in C$$

and

$$Z_i := Y_i - \sum_{j=1}^{i-1} \langle Y_i, U_j \rangle U_j, \quad U_i = 1_{A_i \cup \dots \cup A_k} \frac{Z_i}{||Z_i||} \quad \text{for } 2 \le i \le k.$$

Then for every $i \in \{1, \dots, k\}$ satisfying $\mu[A_i] > 0$, $U_1, \dots, U_i$ is an orthonormal basis of C on A_i. If $k = d$, one obtains from Corollary 2.6 that $1_{A_d} C = \text{lin}_{A_d} \{U_1, \dots, U_d\} = 1_{A_d} (L^0)^d$. If $k < d$, we set $A_{k+1} = \dots = A_d = \emptyset$, and $1_{A_d} C = 1_{A_d} (L^0)^d$ holds trivially. By Corollary 2.6 and Lemma 2.7, there exist $V_i \in C$, $i = 1, \dots, d$ such that

$$1_{A_0} (L^0)^d = \text{lin}_{A_0} \{V_1, \dots, V_d\}$$

and

$$1_{A_i} (L^0)^d = \text{lin}_{A_i} \{U_1, \dots, U_i, V_{i+1} \dots, V_d\} \quad \text{for all } i = 1, \dots, d-1.$$

Set

$$X_1 := 1_{A_1 \cup \dots \cup A_d} U_1 + 1_{A_0} \frac{V_1}{||V_1||}$$

and

$$W_i := V_i - \sum_{j=1}^{i-1} \langle V_i, X_j \rangle X_j, \quad X_i = 1_{A_i \cup \dots \cup A_d} U_i + 1_{A_0 \cup \dots \cup A_{i-1}} \frac{W_i}{||W_i||} \quad \text{for } 2 \le i \le d.$$

Then $X_1, \dots, X_d$ are orthonormal on Ω such that

$$1_{A_i} C = \text{lin}_{A_i} \{X_1, \dots, X_i\}, \quad 1_{A_i} C^{\perp} = \text{lin}_{A_i} \{X_{i+1}, \dots, X_d\} \quad \text{for } 1 \le i \le d-1.$$

It is clear that $C + C^{\perp} = (L^0)^d$, $C \cap C^{\perp} = \{0\}$ and $C = C^{\perp\perp}$. □

Corollary 2.12 *Let C be a non-empty σ-stable L^0-linear subset of $(L^0)^d$. Then every $X \in (L^0)^d$ has a unique decomposition $X = Y + Z$ for $Y \in C$, $Z \in C^{\perp}$, and $||Z|| \le ||X - V||$ for every $V \in C$.*

Proof That X has a unique decomposition $X = Y + Z$, $Y \in C$, $Z \in C^\perp$ is a consequence of Corollary 2.11. Moreover, if $V \in C$, then

$$||Z||^2 \leq ||Z||^2 + ||Y - V||^2 = ||Z + Y - V||^2 = ||X - V||^2. \qquad \square$$

3 Converging Sequences, Sequential Closures and Sequential Continuity

Definition 3.1 We call a subset C of $(L^0)^d$ sequentially closed if it contains every $X \in (L^0)^d$ that is an a.e. limit of a sequence $(X_n)_{n\in\mathbb{N}}$ in C. For an arbitrary subset C of $(L^0)^d$ and $A \in \mathcal{F}_+$, we denote by $\lim_A(C)$ the set consisting of all a.e. limits of sequences in $1_A C$ and by $\mathrm{cl}_A(C)$ the smallest sequentially closed subset of $1_A(L^0)^d$ containing $1_A C$. In the special case $A = \Omega$, we just write $\lim(C)$ and $\mathrm{cl}(C)$, respectively.

Proposition 3.2 *For all subsets C of $(L^0)^d$ and $A \in \mathcal{F}_+$ one has* $\lim_A(C) = \mathrm{cl}_A(C)$.

Proof It is clear that $\lim_A(C) \subseteq \mathrm{cl}_A(C)$. To show that the two sets are equal, it is enough to prove that $\lim_A(C)$ is sequentially closed. So let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\lim_A(C)$ that converges a.e. to some $X \in 1_A(L^0)^d$. Since $(\Omega, \mathcal{F}, \mu)$ is σ-finite, there exists an increasing sequence A_n, $n \in \mathbb{N}$, of measurable sets such that $\bigcup_n A_n = A$ and $\mu[A_n] < +\infty$. For every n there exists a sequence $(Y_m)_{m\in\mathbb{N}}$ in $1_A C$ converging a.e. to X_n. Therefore,

$$\mu[A_n \cap \{|Y_m - X_n| > 1/n\}] \to 0 \quad \text{for } m \to \infty,$$

and one can choose $m_n \in \mathbb{N}$ such that

$$\mu[B_n] \leq 2^{-n}, \quad \text{where } B_n = A_n \cap \left\{|Y_{m_n} - X_n| > 1/n\right\}.$$

It follows from the Borel–Cantelli lemma that $\mu\left[\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} B_n\right] = 0$, which implies $Y_{m_n} \to X$ a.e. for $n \to \infty$. So $X \in \lim_A(C)$, and the proof is complete. $\square$

Corollary 3.3 *If C is a stable subset of $(L^0)^d$ and $A \in \mathcal{F}_+$, then*

$$\lim_A(C) = 1_A \lim(C) = \mathrm{cl}_A(C) = 1_A \mathrm{cl}(C).$$

In particular, if C is stable and sequentially closed, then so is $1_A C$.

Proof $\lim_A(C) = 1_A \lim(C)$ is a consequence of the stability of C. Moreover, it follows from Proposition 3.2 that $\lim_A(C) = \mathrm{cl}_A(C)$ and $\lim(C) = \mathrm{cl}(C)$. This proves the corollary. $\square$

Corollary 3.4 *If C is a stable subset of $(L^0)^d$ and $A \in \mathcal{F}_+$, then $\mathrm{cl}_A(C)$ is σ-stable. Moreover, if C is L^0-convex, an L^0-convex cone, L^0-affine or L^0-linear, then so is $\mathrm{cl}_A(C)$.*

Proof By Proposition 3.2, $\mathrm{cl}_A(C)$ is equal to $\lim_A(C)$. So for all $X, Y \in \mathrm{cl}_A(C)$ there exist sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ in $1_A C$ such that $X_n \to X$ a.e. and $Y_n \to Y$ a.e. Since for all $B \in \mathcal{F}$, $1_B X_n + 1_{B^c} Y_n \in 1_A C$ and $1_B X_n + 1_{B^c} Y_n \to 1_B X + 1_{B^c} Y$ a.e., one obtains that $1_B X + 1_{B^c} Y$ belongs to $\lim_A(C) = \mathrm{cl}_A(C)$. This shows that $\mathrm{cl}_A(C)$ is stable. Since it is also sequentially closed, it must be σ-stable. The rest of the corollary follows similarly. □

Proposition 3.5 *Every σ-stable L^0-affine subset C of $(L^0)^d$ is sequentially closed.*

Proof If C is empty, the corollary is trivial. Otherwise, choose $X \in C$. Then $D = C - X$ is a σ-stable L^0-linear subset of $(L^0)^d$, and the corollary follows if we can show that D is sequentially closed. So let $(Y_n)_{n\in\mathbb{N}}$ be a sequence in D converging a.e. to some $Y \in (L^0)^d$. By Corollary 2.11, there exist unique pairwise disjoint sets $A_0, \ldots, A_d \in \mathcal{F}$ satisfying $\bigcup_{i=0}^d A_i = \Omega$ and an orthonormal basis $X_1, \ldots, X_d$ of $(L^0)^d$ on Ω such that $1_{A_0} D = \{0\}$ and $1_{A_i} D = \mathrm{lin}_{A_i}\{X_1, \ldots, X_i\}$ for $1 \le i \le d$. Define λ_n and λ in $(L^0)^d$ by $\lambda_n^j := \langle Y_n, X_j \rangle$ and $\lambda^j := \langle Y, X_j \rangle$. Since $Y_n \to Y$ a.e., one has $\lambda_n^j \to \lambda^j$ a.e. In particular, $\lambda^j = 0$ on A_i such that $i < j$. This shows that $Y = \sum_j \lambda^j X_j \in D$. □

The following example shows that L^0-affine subsets of $(L^0)^d$ that are not σ-stable need not be sequentially closed.

Example 3.6 Let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$ and μ the counting measure. Set $X_n = 1_{\{n\}} e_1$. Then

$$\mathrm{lin}(X_n : n \in \mathbb{N}) = \left\{ \sum_{n=1}^k \lambda_n X_n : k \in \mathbb{N},\ \lambda_1, \ldots, \lambda_k \in L^0 \right\}$$

is an L^0-linear subset of $(L^0)^d$ that is not σ-stable, and $Y_k = \sum_{n=1}^k X_n$ is a sequence in $\mathrm{lin}(X_n : n \in \mathbb{N})$ that converges a.e. to $\sum_{n\in\mathbb{N}} X_n \notin \mathrm{lin}(X_n : n \in \mathbb{N})$. Note that $\mathrm{lin}(X_n : n \in \mathbb{N})$ is an L^0-submodule of $(L^0)^d$ that is not finitely generated.

The next result is a conditional version of the Bolzano–Weierstrass theorem. It is already known (see for instance, Lemma 2 in Kabanov and Stricker [8] or Lemma 1.63 in Föllmer and Schied [5]). But since it is important to some of our later results, we give a short proof. To state the result we need the following definition.

Definition 3.7 We call a subset C of $(L^0)^d$ L^0-bounded if $\operatorname{ess\,sup}_{X\in C} ||X|| \in L^0$.

Note that if $(X_n)_{n\in\mathbb{N}}$ is a sequence in $(L^0)^d$ and $N \in \mathbb{N}(\mathcal{F})$, X_N can be written as

$$X_N = \sum_{n\in\mathbb{N}} 1_{\{N=n\}} X_n.$$

In particular, X_N is in $(L^0)^d$. Moreover, if all X_n belong to a σ-stable subset C of $(L^0)^d$, then X_N is again in C.

Theorem 3.8 (Conditional version of the Bolzano–Weierstrass theorem)
Let $(X_n)_{n\in\mathbb{N}}$ be an L^0-bounded sequence in $(L^0)^d$. Then there exists an $X \in (L^0)^d$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_{N_n} = X$ a.e.

Proof There exists a $Y \in L^0_+$ such that $||X_n|| \le Y$ for all $n \in \mathbb{N}$. Therefore, the a.e. limit $X^1 := \lim_{n\to\infty} \inf_{m\ge n} X^1_m$ exists and is in L^0. Define $N^1_0 := 0$ and

$$N^1_n(\omega) := \min\left\{m \in \mathbb{N} : m > N^1_{n-1}(\omega) \text{ and } X^1_m(\omega) \le X^1(\omega) + 1/n\right\} \in \mathbb{N}(\mathcal{F}), \quad n \in \mathbb{N}.$$

Then $N^1_{n+1} > N^1_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X^1_{N^1_n} = X^1$ a.e. Now set $Y^2_n = X^2_{N^1_n}$. Then there exists a sequence $(M^2_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $M^2_{n+1} > M^2_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} Y^2_{M^2_n} = X^2 := \lim_{n\to\infty} \inf_{m\ge n} Y^2_m$ a.e. $N^2_n := N^1_{M^2_n}$, $n \in \mathbb{N}$, defines a sequence in $\mathbb{N}(\mathcal{F})$ satisfying $N^2_{n+1} > N^2_n$ for all $n \in \mathbb{N}$, and one has $\lim_{n\to\infty} X^i_{N^2_n} = X^i$ a.e. for $i = 1, 2$. If one continues like this, one obtains $X^1, \dots, X^d \in L^0$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_{N_n} = X = (X^1, \dots, X^d)$ a.e. □

Corollary 3.9 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in a sequentially closed L^0-bounded stable subset C of $(L^0)^d$. Then there exists an $X \in C$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_{N_n} = X$ a.e.*

Proof Since $(X_n)_{n\in\mathbb{N}}$ is L^0-bounded, it follows from Theorem 3.8 that there exists $X \in (L^0)^d$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_{N_n} = X$ a.e. It remains to show that X belongs to C. By Corollary 3.4 the subset C is σ-stable. Hence, X_{N_n} belongs to C for all $n \in \mathbb{N}$, which implies that X is in C too. □

Corollary 3.10 *Let C and D be non-empty sequentially closed stable subsets of $(L^0)^d$ such that D is L^0-bounded. Then $C + D$ is sequentially closed and stable.*

Proof That $C + D$ is stable is clear. To show that $C + D$ is sequentially closed, choose a sequence $(X_n)_{n\in\mathbb{N}}$ in C and a sequence $(Y_n)_{n\in\mathbb{N}}$ in D such that $X_n + Y_n \to Z$ a.e. for some $Z \in (L^0)^d$. By Theorem 3.8, there exists $Y \in D$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} Y_{N_n} = Y$ a.e. It follows that $\lim_{n\to\infty} X_{N_n} = Z - Y$ a.e. Since C is and sequentially closed, $Z - Y$ belongs to C. Hence, Z is in $C + D$. □

Another consequence of Theorem 3.8 is that conditional Cauchy sequences converge if they are defined as follows:

Definition 3.11 We call a sequence $(X_n)_{n\in\mathbb{N}}$ in $(L^0)^d$ L^0-Cauchy if for every $\varepsilon \in L^0_{++}$ there exists an $N_0 \in \mathbb{N}(\mathcal{F})$ such that $||X_{N_1} - X_{N_2}|| \le \varepsilon$ for all $N_1, N_2 \in \mathbb{N}(\mathcal{F})$ satisfying $N_1, N_2 \ge N_0$.

Theorem 3.12 *Every L^0-Cauchy sequence $(X_n)_{n\in\mathbb{N}}$ in $(L^0)^d$ converges a.e. to some $X \in (L^0)^d$.*

Proof Choose $N_0 \in \mathbb{N}(\mathcal{F})$ such that $||X_{N_1} - X_{N_2}|| \leq 1$ for all $N_1, N_2 \in \mathbb{N}(\mathcal{F})$ satisfying $N_1, N_2 \geq N_0$. Then

$$||X_n|| \leq 1 + \sum_{m\in\mathbb{N}} 1_{\{m\leq N_0\}}||X_m|| \in L^0$$

for all $n \in \mathbb{N}$. So it follows from Theorem 3.8 that there exist $X \in (L^0)^d$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_{N_n} = X$ a.e. But since $(X_n)_{n\in\mathbb{N}}$ is L^0-Cauchy, one has $\lim_{n\to\infty} X_n = X$ a.e. □

The following result gives necessary and sufficient conditions for a sequentially closed L^0-convex subset of $(L^0)^d$ to be L^0-bounded.

Theorem 3.13 *Let C be a sequentially closed L^0-convex subset of $(L^0)^d$ containing* 0. *Then C is L^0-bounded if and only if for any $X \in C \setminus \{0\}$ there exists a $k \in \mathbb{N}$ such that $kX \notin C$.*

Proof Suppose that C is L^0-bounded. Then for every $0 \neq X \in C$, there exists a $k \in \mathbb{N}$ such that $\mu\left[\|kX\| > \operatorname{ess\,sup}_{Y\in C}\|Y\|\right] > 0$, and therefore $kX \notin C$.

Conversely, suppose that C is not L^0-bounded. The sequence

$$A_n := \operatorname{ess\,sup}\{B \in \mathcal{F} : \|X\| \geq n \text{ on } B \text{ for some } X \in C\}, \quad n \in \mathbb{N}\cup\{0\},$$

is decreasing with limit $A := \bigcap_n A_n$. One must have $\mu[A] > 0$, since otherwise, $\|X\| \leq \sum_{n\in\mathbb{N}} n 1_{\{A_n^c\setminus A_{n-1}^c\}} \in L^0$ for all $X \in C$. Since C is sequentially closed, L^0-convex and therefore stable, it is σ-stable. It follows that there exists a sequence $(X_n)_{n\in\mathbb{N}}$ in C such that $\|X_n\| \geq n$ on A. Since the sequence $Y_n = 1_A X_n/\|X_n\|$ is L^0-bounded, it follows from Theorem 3.8 that there exists $Y \in (L^0)^d$ and a sequence $(N_n)_{n\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ and $\lim_{n\to\infty} Y_{N_n} = Y$ a.e. Obviously, $1_A||Y|| = 1_A$, and in particular, $Y \neq 0$. Since C is L^0-convex, sequentially closed and contains 0, one has for all $n \geq k$,

$$kY_{N_n} = 1_A \frac{k}{\|X_{N_n}\|} X_{N_n} \in C.$$

But $\lim_{n\to\infty} kY_{N_n} = kY$. So $kY \in C$ for all $k \in \mathbb{N}$. □

Definition 3.14 Let C be a non-empty subset of $(L^0)^d$ and $k \in \mathbb{N}$. We call a function $f : C \to (L^0)^k$

- sequentially continuous at $X \in C$ if $f(X_n) \to f(X)$ a.e. for every sequence $(X_n)_{n\in\mathbb{N}}$ in C converging to X a.e.;
- sequentially continuous if it is sequentially continuous at every $X \in C$;

- L^0-affine if $f(\lambda X + (1-\lambda)Y) = \lambda f(X) + (1-\lambda) f(Y)$ for all $X, Y \in (L^0)^d$ and $\lambda \in L^0$ such that $\lambda X + (1-\lambda)Y \in C$;
- L^0-linear if $f(\lambda X + Y) = \lambda f(X) + f(Y)$ for all $X, Y \in (L^0)^d$ and $\lambda \in L^0$ such that $\lambda X + Y \in C$.
- We define the conditional norm of f by $||f|| := \operatorname{ess\,sup}_{X \in C,\, ||X|| \le 1} ||f(X)|| \in \overline{L}$.

Proposition 3.15 *Let C be a non-empty σ-stable L^0-linear subset of $(L^0)^d$. Then $||f|| \in L^0_+$ for every L^0-linear function $f : C \to (L^0)^k$, $k \in \mathbb{N}$.*

Proof By Corollary 2.11, there exist unique pairwise disjoint sets $A_0, \dots, A_d \in \mathcal{F}$ satisfying $\bigcup_{i=0}^d A_i = \Omega$ and an orthonormal basis $X_1, \dots, X_d$ of $(L^0)^d$ on Ω such that $1_{A_0} C = \{0\}$ and $1_{A_i} C = \operatorname{lin}_{A_i}\{X_1, \dots, X_i\}$ for $1 \le i \le d$. For every $X \in C$ there exists a unique $\lambda \in (L^0)^d$ such that $X = \sum_{j=1}^d \lambda_j X_j$. On the set A_0 one has $f(X) = X = 0$, and on A_i for $1 \le i \le d$, $||X|| = \left(\sum_{j=1}^i \lambda_j^2\right)^{1/2}$ as well as

$$||f(X)|| = ||\sum_{j=1}^i \lambda_j f(X_j)|| \le \sum_{j=1}^i |\lambda_j| ||f(X_j)|| \le \left(\sum_{j=1}^i \lambda_j^2\right)^{1/2} \left(\sum_{j=1}^i ||f(X_j)||^2\right)^{1/2}.$$

Therefore, $||f|| \le \sum_{i=1}^d 1_{A_i} \left(\sum_{j=1}^i ||f(X_j)||^2\right)^{1/2}$. □

Corollary 3.16 *Let C be a non-empty σ-stable L^0-affine subset of $(L^0)^d$. Then every L^0-affine function $f : C \to (L^0)^k$, $k \in \mathbb{N}$, is sequentially continuous.*

Proof Choose an $X_0 \in C$. Then $D = C - X_0$ is a non-empty σ-stable L^0-linear subset of $(L^0)^d$ and $g(X) = f(X + X_0) - f(X_0)$ is an L^0-linear function on D. By Proposition 3.15, one has $||g|| \in L^0_+$. Moreover, $||f(X) - f(Y)|| = ||g(X-Y)|| \le ||g||\, ||X - Y||$, and it follows that f is sequentially continuous. □

Corollary 3.17 *Let C be a non-empty sequentially closed subset of a non-empty σ-stable L^0-affine subset D of $(L^0)^d$. Then for every injective L^0-affine function $f : D \to (L^0)^k$, $k \in \mathbb{N}$, $f(C)$ is a sequentially closed subset of $(L^0)^k$.*

Proof Pick an $X_0 \in C$. The corollary follows if we can show that $f(C) - f(X_0)$ is sequentially closed. So by replacing C with $C - X_0$, D with $D - X_0$ and f with $f(X + X_0) - f(X_0)$, one can assume that $X_0 = 0$, D is a σ-stable L^0-linear subset of $(L^0)^d$ and f is injective L^0-linear. By Corollary 3.16, f is sequentially continuous. Therefore, $f(D)$ is a non-empty σ-stable L^0-linear subset of $(L^0)^k$, and it follows from Proposition 3.5 that it is sequentially closed. Since $f^{-1} : f(D) \to D$ is again L^0-linear, it is also sequentially continuous. So if $(Y_n)_{n \in \mathbb{N}}$ is a sequence in $f(C)$ converging a.e. to some $Y \in (L^0)^k$, then $Y \in f(D)$ and $f^{-1}(Y_n)$ is a sequence in C converging a.e. to $f^{-1}(Y) \in D$. It follows that $f^{-1}(Y) \in C$ and $Y = f(f^{-1}(Y)) \in f(C)$. □

Lemma 3.18 *Let C be a non-empty σ-stable L^0-linear subset of $(L^0)^d$ and $k \in \mathbb{N}$. Then every L^0-linear function $f : C \to (L^0)^k$ has an L^0-linear extension $F : (L^0)^d \to (L^0)^k$ such that $||f|| = ||F||$.*

Proof By Corollary 2.12, every $X \in (L^0)^d$ has a unique decomposition $X = Y + Z$ such that $Y \in C$ and $Z \in C^{\perp}$. $F(X) := f(Y)$ defines an L^0-linear extension of f to $(L^0)^d$ such that $||f|| = ||F||$. □

4 Conditional Optimization

Definition 4.1 Let C be a non-empty subset of $(L^0)^d$. We call a function $f : C \to L$

- sequentially lsc (lower semicontinuous) at $X \in C$ if $f(X) \leq \liminf_{n\to\infty} f(X_n)$ for every sequence $(X_n)_{n\in\mathbb{N}}$ in C with a.e. limit X;
- sequentially lsc if it is sequentially lsc at every $X \in C$;
- sequentially usc (upper semicontinuous) at $X \in C$ if $-f$ is sequentially lsc at X;
- sequentially usc if it is sequentially usc at every $X \in C$;
- sequentially continuous at $X \in C$ if it is sequentially lsc and usc at X;
- sequentially continuous if it is sequentially continuous at every $X \in C$.

In the following definition $+\infty - \infty$ is understood as $+\infty$ and $0 \cdot (\pm\infty)$ as 0.

Definition 4.2 Let $f : C \to L$ be a function on a non-empty subset C of $(L^0)^d$.

- If C is stable, we call f stable if
$$f(1_A X + 1_{A^c} Y) = 1_A f(X) + 1_{A^c} f(Y)$$
for all $X, Y \in C$ and $A \in \mathcal{F}_+$;
- If C is L^0-convex, we call f L^0-convex if
$$f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda) f(Y)$$
for all $X, Y \in C$ and $\lambda \in L^0$ such that $0 \leq \lambda \leq 1$;
- If C is L^0-convex, we call f strictly L^0-convex if
$$f(\lambda X + (1-\lambda)Y) < \lambda f(X) + (1-\lambda) f(Y) \quad \text{on the set } \{X \neq \lambda X + (1-\lambda)Y \neq Y\}$$
for all $X, Y \in C$ and $\lambda \in L^0$ such that $0 \leq \lambda \leq 1$.

Lemma 4.3 *Let $f : C \to L$ be an L^0-convex function on an L^0-convex subset C of $(L^0)^d$. Then f is also stable.*

Proof Let $X, Y \in C$ and $A \in \mathcal{F}_+$. Denote $Z = 1_A X + 1_{A^c} Y$. Then one has $1_A f(Z) \leq 1_A f(X)$ and $1_A f(X) = 1_A f(1_A Z + 1_{A^c} X) \leq 1_A f(Z)$. This shows that $1_A f(Z) = 1_A f(X)$. Analogously, one obtains $1_{A^c} f(Z) = 1_{A^c} f(Y)$ and therefore $f(Z) = 1_A f(X) + 1_{A^c} f(Y)$. □

Theorem 4.4 *Let C be a sequentially closed stable subset of $(L^0)^d$ and $f : C \to \overline{L}$ a sequentially lsc stable function. Assume there exists an $X_0 \in C$ such that the set*

$$\{X \in C : f(X) \leq f(X_0)\}$$

is L^0-bounded. Then there exists an $\hat{X} \in C$ such that

$$f(\hat{X}) = \operatorname*{ess\,inf}_{X \in C} f(X).$$

If C and f are L^0-convex, then the set

$$\left\{X \in C : f(X) = f(\hat{X})\right\}$$

is L^0-convex. If in addition, f is strictly L^0-convex, then

$$\left\{X \in C : f(X) = f(\hat{X})\right\} = \left\{\hat{X}\right\}.$$

Proof The set $D := \{X \in C : f(X) \leq f(X_0)\}$ is sequentially closed, stable and L^0-bounded. It follows that $\{f(X) : X \in D\}$ is directed downwards. Therefore, there exists a sequence $(X_n)_{n \in \mathbb{N}}$ in D such that $f(X_n)$ decreases a.e. to $I := \operatorname{ess\,inf}_{X \in D} f(X)$. By Corollary 3.9, there exists a sequence $(N_n)_{n \in \mathbb{N}}$ in $\mathbb{N}(\mathcal{F})$ such that $N_{n+1} > N_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} X_{N_n} = \hat{X}$ a.e. for some $\hat{X} \in D$. Since X_{N_n} belongs to D and

$$f(X_{N_n}) = \sum_{m \geq n} 1_{\{N_n = m\}} f(X_m) \leq f(X_n) \quad \text{for all } n,$$

one obtains from the L^0-lower semicontinuity of f that

$$f(\hat{X}) \leq \liminf_{n \to \infty} f(X_{N_n}) \leq \lim_{n \to \infty} f(X_n) = I.$$

This shows the first part of the theorem. That $\left\{X \in C : f(X) = f(\hat{X})\right\}$ is L^0-convex if C and f are L^0-convex, is clear. Finally, assume C is L^0-convex and f strictly L^0-convex. Then if there exists an X in C such that $f(X) = f(\hat{X})$, one has

$$f\left(\frac{X + \hat{X}}{2}\right) < \frac{f(X) + f(\hat{X})}{2}$$

on the set $\left\{X \neq \hat{X}\right\}$. It follows that $\mu[X \neq \hat{X}] = 0$. □

Corollary 4.5 *Let C and D be non-empty sequentially closed stable subsets of $L^0(\mathcal{F})^d$ such that D is L^0-bounded. Then there exist $\hat{X} \in C$ and $\hat{Y} \in D$ such that*

$$||\hat{X} - \hat{Y}|| = \operatorname*{ess\,inf}_{X \in C,\, Y \in D} ||X - Y||. \tag{4.1}$$

If in addition, C and D are L^0-convex, then $\hat{X} - \hat{Y}$ is unique.

Proof By Corollary 3.10, the set $E = C - D$ is sequentially closed and stable. Moreover, $Z \mapsto ||Z||$ is a sequentially continuous L^0-convex function from E to L^0, and for every $Z_0 \in E$, the set $\{Z \in E : ||Z|| \le ||Z_0||\}$ is L^0-bounded. So one obtains from Theorem 4.4 that there exists a $\hat{Z} \in E$ such that $||\hat{Z}|| = \operatorname{ess\,inf}_{Z \in E} ||Z||$. This shows that there exist $\hat{X} \in C$ and $\hat{Y} \in D$ satisfying (4.1). If C and D are L^0-convex, then so is E, and for every $Z \in E$ satisfying $||Z|| = ||\hat{Z}||$, one has $(Z + \hat{Z})/2 \in E$ and $||(Z + \hat{Z})/2|| < ||\hat{Z}||$ on the set $\left\{Z \neq \hat{Z}\right\}$. It follows that $\mu[Z \neq \hat{Z}] = 0$, and the proof is complete. □

5 Interior, Relative Interior and L^0-open Sets

Definition 5.1 Let C be a non-empty subset of $(L^0)^d$ and $A \in \mathcal{F}_+$.

- For $X \in (L^0)^d$ and $\varepsilon \in L^0_{++}$, we denote

$$B^\varepsilon_A(X) := \left\{Y \in 1_A (L^0)^d : 1_A ||Y - X|| \le \varepsilon\right\}.$$

- The interior $\text{int}_A(C)$ of C on A consists of elements $X \in 1_A C$ for which there exists an $\varepsilon \in L^0_{++}$ such that $B^\varepsilon_A(X) \subseteq 1_A C$. If $A = \Omega$, we just write $\text{int}(C)$ for $\text{int}_A(C)$.
- The relative interior $\text{ri}_A(C)$ of C on A consists of elements $X \in 1_A C$ for which there exists an $\varepsilon \in L^0_{++}$ such that $B^\varepsilon_A(X) \cap \text{aff}_A(C) \subseteq 1_A(C)$. If $A = \Omega$, we write $\text{ri}(C)$ instead of $\text{ri}_A(C)$.
- We say C is L^0-open on A if $1_A C = \text{int}_A(C)$. We call it L^0-open if it is L^0-open on Ω.

Note that one always has $1_A \text{int}(C) \subseteq \text{int}_A(C)$ but not necessarily the other way around. The collection of all L^0-open subsets of $(L^0)^d$ forms a topology. It is studied in Filipović et al. [4] and is related to (ε, λ)-topologies on random locally convex modules (see [6]). We point out that sequentially closed sets in $(L^0)^d$ are different from complements of L^0-open sets. But one has the following relation between the two:

Lemma 5.2 *Let C be a σ-stable subset of $(L^0)^d$. Then* $\text{cl}(C) \cap \text{int}(C^c) = \emptyset$.

Proof Assume $X \in \text{cl}(C) \cap \text{int}(C^c)$. By Proposition 3.2, there exists a sequence $(X_n)_{n \in \mathbb{N}}$ in C such that $X_n \to X$ a.e. On the other hand, there is an $\varepsilon \in L^0_{++}$ such that

$Y \in C^c$ for every $Y \in (L^0)^d$ satisfying $||X - Y|| \le \varepsilon$. $N(\omega) := \min\{n \in \mathbb{N} : ||X_n(\omega) - X(\omega)|| \le \varepsilon(\omega)\}$ is an element of $\mathbb{N}(\mathcal{F})$, and since C is σ-stable, X_N belongs to C. But at the same time one has $||X_N - X|| \le \varepsilon$, implying $X_N \in C^c$. This yields a contradiction. So $\mathrm{cl}(C) \cap \mathrm{int}(C^c) = \emptyset$. □

Lemma 5.3 *Let C be a non-empty L^0-convex subset of $(L^0)^d$, $A \in \mathcal{F}_+$ and $\lambda \in L^0$ such that $0 < \lambda \le 1$. Then*

$$\lambda X + (1-\lambda)Y \in \mathrm{int}_A(C) \quad \text{for all } X \in \mathrm{int}_A(C),\ Y \in 1_A C \tag{5.1}$$

and

$$\lambda X + (1-\lambda)Y \in \mathrm{ri}_A(C) \quad \text{for all } X \in \mathrm{ri}_A(C),\ Y \in 1_A C. \tag{5.2}$$

If in addition, C is σ-stable, then (5.1) and (5.2) also hold for $Y \in \mathrm{cl}_A(C)$.

Proof Let $X \in \mathrm{int}_A(C)$ and $Y \in 1_A C$. There exists an $\varepsilon \in L^0_{++}$ such that $B^\varepsilon_A(X)$ is contained in $1_A C$. So

$$\lambda X + (1-\lambda)Y + Z = \lambda(X + Z/\lambda) + (1-\lambda)Y \subseteq 1_A C$$

for all $Z \in B^{\varepsilon\lambda}_A(0)$. This shows (5.1).

To prove (5.2), we assume that $X \in \mathrm{ri}_A(C)$ and $Y \in 1_A C$. There exists an $\varepsilon \in L^0_{++}$ such that $B^\varepsilon_A(X) \cap \mathrm{aff}_A(C) \subseteq 1_A C$. Choose $Z \in B^{\varepsilon\lambda}_A(0)$ such that

$$\lambda X + (1-\lambda)Y + Z \in \mathrm{aff}_A(C).$$

Then $X + Z/\lambda \in \mathrm{aff}_A(C)$, and therefore $X + Z/\lambda \in 1_A C$. It follows that

$$\lambda X + (1-\lambda)Y + Z = \lambda(X + Z/\lambda) + (1-\lambda)Y \subseteq 1_A C.$$

This shows (5.2).

If C is σ-stable, $X \in \mathrm{int}_A(C)$ and $Y \in \mathrm{cl}_A(C)$, there exists an $\varepsilon \in L^0_{++}$ such that $B^{2\varepsilon}_A(X) \subseteq 1_A C$. From Proposition 3.2 we know that there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ in $1_A C$ converging a.e. to Y. $N(\omega) := \min\{n \in \mathbb{N} : (1-\lambda(\omega))||Y(\omega) - Y_n(\omega)|| \le \lambda(\omega)\varepsilon(\omega)\}$ belongs to $\mathbb{N}(\mathcal{F})$, and Y_N is an element of C satisfying $(1-\lambda)||Y - Y_N|| \le \lambda\varepsilon$. So for $Z \in B^{\lambda\varepsilon}_A(0)$, one has

$$\lambda X + (1-\lambda)Y + Z = \lambda\left(X + \frac{(1-\lambda)}{\lambda}(Y - Y_N) + \frac{1}{\lambda}Z\right) + (1-\lambda)Y_N \in 1_A C,$$

which shows that $\lambda X + (1-\lambda)Y \in \mathrm{int}_A(C)$.

If X is in $\mathrm{ri}_A(C)$ instead of $\mathrm{int}_A(C)$, there exists an $\varepsilon \in L^0_{++}$ such that $B^{2\varepsilon}_A(X) \cap \mathrm{aff}_A(C) \subseteq 1_A C$. Let $Z \in B^{\lambda\varepsilon}_A(0)$ such that

$$\lambda X + (1-\lambda)Y + Z \in \mathrm{aff}_A(C),$$

then

$$X + \frac{(1-\lambda)}{\lambda}(Y - Y_N) + \frac{1}{\lambda} Z \in \mathrm{aff}_A(C).$$

Hence

$$X + \frac{(1-\lambda)}{\lambda}(Y - Y_N) + \frac{1}{\lambda} Z \in 1_A C,$$

and it follows that

$$\lambda X + (1-\lambda)Y + Z = \lambda\left(X + \frac{(1-\lambda)}{\lambda}(Y - Y_N) + \frac{1}{\lambda} Z\right) + (1-\lambda)Y_N \in 1_A C.$$

So $\lambda X + (1-\lambda)Y \in \mathrm{ri}_A(C)$, and the proof is complete. □

Corollary 5.4 *Let C be an L^0-convex subset of $(L^0)^d$ and $A \in \mathcal{F}_+$. Then* $\mathrm{int}_A(C)$ *and* $\mathrm{ri}_A(C)$ *are again L^0-convex.*

Proof Since C is stable, it follows from Lemma 5.3 that for $X, Y \in \mathrm{int}_A(C)$ and $\lambda \in L^0$ satisfying $0 \le \lambda \le 1$, one has

$$\lambda X + (1-\lambda)Y = 1_{\{\lambda > 0\}}(\lambda X + (1-\lambda)Y) + 1_{\{\lambda = 0\}} Y \in \mathrm{int}_A(C).$$

This shows that $\mathrm{int}_A(C)$ is L^0-convex. The same argument shows that $\mathrm{ri}_A(C)$ is L^0-convex. □

Definition 5.5 Let $A \in \mathcal{F}_+$. We call a subset C of $(L^0)^d$

- an L^0-hyperplane on A if $1_A C = \left\{X \in 1_A (L^0)^d : \langle X, Z\rangle = V\right\}$
- an L^0-halfspace on A if $1_A C = \left\{X \in 1_A (L^0)^d : \langle X, Z\rangle \ge V\right\}$

for some $V \in 1_A L^0$ and $Z \in 1_A (L^0)^d$ such that $||Z|| > 0$ on A.

Lemma 5.6 *A subset C of $(L^0)^d$ is an L^0-hyperplane on $A \in \mathcal{F}_+$ if and only if there exist $X_0 \in 1_A (L^0)^d$ and an orthonormal basis $X_1, \dots, X_d$ of $(L^0)^d$ on A such that*

$$1_A C = \left\{X_0 + \sum_{i=1}^{d-1} \lambda_i X_i : \lambda_i \in 1_A L^0\right\}. \tag{5.3}$$

Similarly, C is an L^0-halfspace on $A \in \mathcal{F}_+$ if and only if there exist $X_0 \in 1_A(L^0)^d$ and an orthonormal basis $X_1, \ldots, X_d$ of $(L^0)^d$ on A such that

$$1_A C = \left\{ X_0 + \sum_{i=1}^{d} \lambda_i X_i : \lambda_i \in 1_A L^0,\ \lambda_d \geq 0 \right\}. \tag{5.4}$$

Proof If $1_A C$ is of the form (5.3), then $1_A C = \{X \in 1_A(L^0)^d : \langle X, X_d \rangle = \langle X_0, X_d \rangle\}$. Now assume that $1_A C = \left\{X \in 1_A(L^0)^d : \langle X, Z \rangle = V\right\}$ for some $V \in 1_A L^0$ and $Z \in 1_A(L^0)^d$ such that $||Z|| > 0$ on A. By Corollary 2.11, there exists an orthonormal basis $X_1, \ldots, X_d$ of $(L^0)^d$ on A such that $1_A Z^{\perp} = \mathrm{lin}_A \{X_1, \ldots, X_{d-1}\}$ and $X_d = 1_A Z/||Z||$. Choose $X_0 \in 1_A(L^0)^d$ such that $\langle X_0, Z \rangle = V$. Then $1_A C$ is of the form (5.3). That C is an L^0-halfspace on $A \in \mathcal{F}_+$ if and only if $1_A C$ is of the form (5.4) follows similarly. □

Lemma 5.7 *Let C be a σ-stable L^0-convex subset of $(L^0)^d$ and $A \in \mathcal{F}_+$. Then* $\mathrm{int}_A(C) \neq \emptyset$ *if and only if* $\mathrm{aff}_A(C) = 1_A(L^0)^d$.

Proof Let us first assume that $X_0 \in \mathrm{int}_A(C)$. Then $0 \in \mathrm{int}_A(C - X_0)$, and it follows that

$$\mathrm{aff}_A(C) = \mathrm{aff}_A(C - X_0) + X_0 = \mathrm{lin}_A(C - X_0) + X_0 = 1_A(L^0)^d + X_0 = 1_A(L^0)^d.$$

On the other hand, if $\mathrm{aff}_A(C) = 1_A(L^0)^d$, choose $X_0 \in 1_A C$. Then

$$\mathrm{lin}_A(C - X_0) = \mathrm{aff}_A(C - X_0) = \mathrm{aff}_A(C) - X_0 = 1_A(L^0)^d.$$

So it follows from Theorem 2.8 that there exist $X_1, \ldots, X_d$ in $1_A C$ such that $X_i - X_0$, $i = 1, \ldots, d$, form a basis of $(L^0)^d$ on A. Set

$$\hat{X} := \frac{1}{d+1} \sum_{i=0}^{d} X_i.$$

It follows from Corollary 2.11 and Lemma 5.6 that for every $i = 0, \ldots, d$, there exist $V_i \in L^0$ and $Z_i \in (L^0)^d$ such that for all $j \neq i$,

$$\left\langle \hat{X}, Z_i \right\rangle > V_i = \langle X_j, Z_i \rangle \text{ on } A.$$

This shows that $\hat{X} \in \mathrm{int}_A \left\{X \in 1_A(L^0)^d : \langle X, Z_i \rangle \geq V_i\right\}$ for all i, which implies $\hat{X} \in \mathrm{int}_A(C)$ since

$$\bigcap_{i=0}^{d} \left\{X \in 1_A(L^0)^d : \langle X, Z_i \rangle \geq V_i\right\} = \mathrm{conv}_A \{X_0, \ldots, X_d\} \subseteq 1_A C.$$

6 Separation by L^0-hyperplanes

In this section we prove results on the separation of two L^0-convex sets in $(L^0)^d$ by an L^0-hyperplane. As a corollary we obtain a version of the Hahn–Banach extension theorem. Hahn–Banach extension and separation results have been proved in more general modules; see e.g., Filipović et al. [4], Guo [6] and the references therein. However, due to the special form of $(L^0)^d$, we here are able to derive analogs of results that hold in $\mathbb{R}^d$ but not in infinite-dimensional vector spaces. Moreover, we do not need Zorn's lemma or the axiom of choice.

Theorem 6.1 (Strong separation) *Let C and D be non-empty L^0-convex subsets of $(L^0)^d$. Then there exists $Z \in (L^0)^d$ such that*

$$\operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle > \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z\rangle \tag{6.1}$$

if and only if $0 \notin \mathrm{cl}_A(C - D)$ *for all* $A \in \mathcal{F}_+$.

Proof Let us first assume that there exists an $A \in \mathcal{F}_+$ such that $0 \in \mathrm{cl}_A(C - D)$. From Proposition 3.2 we know that $\mathrm{cl}_A(C - D) = \lim_A(C - D)$. So there exists a sequence $(X_n)_{n\in\mathbb{N}}$ in $1_A(C - D)$ such that $X_n \to 0$ a.e. It follows that there can exist no $Z \in (L^0)^d$ satisfying (6.1).

Now assume $0 \notin \mathrm{cl}_A(C - D)$ for all $A \in \mathcal{F}_+$. It follows from Corollary 3.4 that $\mathrm{cl}(C - D)$ is L^0-convex. So one obtains from Corollary 4.5 that there exists a $Z \in \mathrm{cl}(C - D)$ such that

$$\|Z\|^2 \le \|(1-\lambda)Z + \lambda W\|^2 = \|Z\|^2 + 2\lambda\,\langle Z, W - Z\rangle + \lambda^2\|W - Z\|^2$$

for all $W \in \mathrm{cl}(C - D)$ and $\lambda \in L^0$ such that $0 < \lambda \le 1$. Division by 2λ and sending λ to 0 yields $\langle W, Z\rangle \ge \|Z\|^2$. In particular,

$$\langle W, Z\rangle \ge ||Z||^2 \quad \text{for all } W \in C - D,$$

and therefore,

$$\operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle \ge \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z\rangle + ||Z||^2.$$

It remains to show that $\|Z\| > 0$. But if this were not the case, the set $A = \{Z = 0\}$ would belong to $\mathcal{F}_+$ and $1_A Z = 0$. However, by assumption and Corollary 3.3, one has $0 \notin \mathrm{cl}_A(C - D) = 1_A\mathrm{cl}(C - D)$ for all $A \in \mathcal{F}_+$, a contradiction. □

Corollary 6.2 *Let C and D be non-empty sequentially closed L^0-convex subsets of $(L^0)^d$ such that D is L^0-bounded and $1_A C$ is disjoint from $1_A D$ for all $A \in \mathcal{F}_+$. Then there exists a $Z \in (L^0)^d$ such that*

$$\operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle > \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z\rangle\,.$$

Proof $C - D$ is a non-empty L^0-convex set, which by Corollary 3.10 is sequentially closed. It follows from the assumptions that $0 \notin 1_A(C - D)$ for all $A \in \mathcal{F}_+$, and we know from Corollary 3.3 that $1_A(C - D) = \mathrm{cl}_A(C - D)$. So the corollary is a consequence of Theorem 6.1. □

Lemma 6.3 *Let C be a non-empty σ-stable L^0-convex cone in $(L^0)^d$ such that $1_A C \neq 1_A(L^0)^d$ for all $A \in \mathcal{F}_+$. Then there exists a $Z \in (L^0)^d$ such that*

$$||Z|| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle \geq 0. \tag{6.2}$$

Proof If $C = \{0\}$, the lemma is clear. Otherwise one obtains from Theorem 2.8 that there exist $A \in \mathcal{F}$ and $X_1, \ldots, X_{d-1} \in C$ such that $\mathrm{lin}_A(C) = \mathrm{lin}_A(L^0)^d$ and $\mathrm{lin}_{A^c}(C) \subseteq \mathrm{lin}_{A^c}\{X_1, \ldots, X_{d-1}\}$. By Corollary 2.11, there exists $W \in \mathrm{lin}_{A^c}\{X_1, \ldots, X_{d-1}\}^{\perp}$ such that $||W|| > 0$ on A^c. If $\mu[A] = 0$, then $Z = W$ satisfies (6.2), and the proof is complete. If $\mu[A] > 0$, one notes that since C is an L^0-convex cone, one has $\mathrm{aff}_A(C) = \mathrm{lin}_A(C) = 1_A(L^0)^d$. It follows from Lemma 5.7 that there exists a $Y \in \mathrm{int}_A(C)$. Then $1_B Y \in \mathrm{int}_B(C)$ for every subset $B \in \mathcal{F}_+$ of A. But this implies that $-1_B Y$ cannot be in $\mathrm{cl}_B(C)$. Otherwise it would follow from Lemma 5.3 that 0 belongs to $\mathrm{int}_B(C)$, implying that $1_B C = 1_B(L^0)^d$ and contradicting the assumptions. So Theorem 6.1 applied to $1_A C$ and $\{-Y\}$ viewed as subsets of $1_A(L^0)^d$ yields a $V \in 1_A(L^0)^d$ such that

$$\operatorname*{ess\,inf}_{X \in 1_A C} \langle X, V\rangle > \langle -Y, V\rangle \quad \text{on } A.$$

Since C is an L^0-convex cone, $Z = 1_A V + 1_{A^c} W$ satisfies condition (6.2). □

Theorem 6.4 (Weak separation) *Let C and D be non-empty σ-stable L^0-convex subsets of $(L^0)^d$. Then there exists a $Z \in (L^0)^d$ such that*

$$||Z|| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle \geq \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z\rangle \tag{6.3}$$

if and only if $0 \notin \mathrm{int}_A(C - D)$ *for all* $A \in \mathcal{F}_+$.

Proof If there is an $A \in \mathcal{F}_+$ such that $0 \in \mathrm{int}_A(C - D)$, there can exist no $Z \in (L^0)^d$ such that (6.3) holds. Hence, (6.3) implies $0 \notin \mathrm{int}_A(C - D)$ for all $A \in \mathcal{F}_+$.

To show the converse implication, assume that $0 \notin \mathrm{int}_A(C - D)$ for all $A \in \mathcal{F}_+$. Clearly, $C - D$ is σ-stable and L^0-convex. Therefore, one has $\mathrm{ccone}(C - D) = \left\{\lambda X : \lambda \in L^0_{++}, X \in C - D\right\}$, from which it can be seen that $\mathrm{ccone}(C - D)$ is σ-stable and satisfies $1_A \mathrm{ccone}(C - D) \neq 1_A(L^0)^d$ for all $A \in \mathcal{F}_+$. So one obtains from Lemma 6.3 that there exists a $Z \in (L^0)^d$ such that

$$||Z|| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in E} \langle X, Z\rangle \geq 0.$$

This implies (6.3). □

Corollary 6.5 *Let C and D be two non-empty σ-stable L^0-convex subsets of $(L^0)^d$ such that $1_A C$ is disjoint from $1_A D$ for all $A \in \mathcal{F}_+$ and D is L^0-open. Then there exists a $Z \in (L^0)^d$ such that*

$$\operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle > \langle Y, Z\rangle \quad \textit{for all } Y \in D.$$

Proof It follows from Theorem 6.4 that there exists a $Z \in (L^0)^d$ such that

$$||Z|| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle \geq \operatorname*{ess\,sup}_{V \in D} \langle V, Z\rangle\,,$$

and since D is L^0-open, one has

$$\operatorname*{ess\,sup}_{V \in D} \langle V, Z\rangle > \langle Y, Z\rangle \quad \text{for all } Y \in D.$$

□

As another consequence of Theorem 6.4 we obtain a conditional version of the Hahn–Banach extension theorem.

Corollary 6.6 (Conditional version of the Hahn–Banach extension theorem) *Let $f : (L^0)^d \to L^0$ be an L^0-convex function such that $f(\lambda X) = \lambda f(X)$ for all $\lambda \in L^0_+$ and $g : E \to L^0$ an L^0-linear mapping on a σ-stable L^0-linear subset E of $(L^0)^d$ such that $g(X) \leq f(X)$ for all $X \in E$. Then there exists an L^0-linear extension $h : (L^0)^d \to L^0$ of g such that $h(X) \leq f(X)$ for all $X \in (L^0)^d$.*

Proof Note that

$$C := \left\{(X, V) \in (L^0)^d \times L^0 : f(X) \leq V\right\} \quad \text{and} \quad D := \{(Y, g(Y)) : Y \in E\}$$

are L^0-convex sets in $(L^0)^d \times L^0$. By Lemma 4.3, f and g are stable. It follows that C and D are σ-stable. Moreover, since $C - D$ is an L^0-convex cone and $1_A(0, -1) \notin 1_A(C - D)$ for all $A \in \mathcal{F}_+$, one has $(0, 0) \notin \operatorname{int}_A(C - D)$ for all $A \in \mathcal{F}_+$. So one obtains from Theorem 6.4 that there exists a pair $(Z, W) \in (L^0)^d \times L^0$ such that

$$||Z|| + |W| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{(X,V) \in C} \{\langle X, Z\rangle + VW\} \geq \operatorname*{ess\,sup}_{Y \in E} \{\langle Y, Z\rangle + g(Y)W\}\,. \tag{6.4}$$

It follows that $W > 0$. By multiplying (Z, W) with $1/W$, one can assume that $W = 1$. Since E and g are L^0-linear, the ess sup in (6.4) must be zero, and it follows that $g(Y) = \langle Y, -Z\rangle$ for all $Y \in E$. Moreover, $f(X) \geq \langle X, -Z\rangle$ for all $X \in (L^0)^d$. So $h(X) := \langle X, -Z\rangle$ is the desired extension of g to $(L^0)^d$. □

Theorem 6.7 (Proper separation) *Let C and D be two non-empty σ-stable L^0-convex subsets of $(L^0)^d$. Then there exists a $Z \in (L^0)^d$ such that*

$$\operatorname*{ess\,inf}_{X \in C} \langle X, Z\rangle \geq \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z\rangle \quad \textit{and} \quad \operatorname*{ess\,sup}_{X \in C} \langle X, Z\rangle > \operatorname*{ess\,inf}_{Y \in D} \langle Y, Z\rangle \tag{6.5}$$

if and only if $0 \notin \operatorname{ri}_A(C - D)$ for all $A \in \mathcal{F}_+$.

Proof Denote $E = \text{aff}(C - D)$. By Corollary 2.9, $1_A E$ is for all $A \in \mathcal{F}_+$ σ-stable, and therefore, by Proposition 3.5, sequentially closed.

If there exists an $A \in \mathcal{F}_+$ such that $0 \in \text{ri}_A(C - D)$, $1_A E$ is L^0-linear and there exists an $\varepsilon \in L^0_{++}$ such that $B^A_\varepsilon(0) \cap 1_A E \subseteq 1_A(C - D)$. Suppose there exists $Z \in (L^0)^d$ satisfying (6.5). Then

$$\langle X, Z \rangle \geq 0 \text{ for all } X \in \text{cl}_A(C - D) \tag{6.6}$$

and

$$\langle X, Z \rangle > 0 \text{ on } A \text{ for some } X \in 1_A(C - D). \tag{6.7}$$

One obtains from Corollary 2.12 that $Z = Z_1 + Z_2$ for some $Z_1 \in 1_A E$ and $Z_2 \in (1_A E)^\perp$. It follows from (6.6) that $Z_1 = 0$. But this contradicts (6.7). So (6.5) implies that $0 \notin \text{ri}_A(C - D)$ for all $A \in \mathcal{F}_+$.

Now assume $0 \notin \text{ri}_A(C - D)$ for all $A \in \mathcal{F}_+$. Since E is σ-stable, there exists a largest $B \in \mathcal{F}$ such that $0 \in 1_B E$. If $\mu[B] = 0$, one has $0 \notin 1_A E$ for all $A \in \mathcal{F}_+$, and it follows from Corollary 6.2 that there exists a $Z \in (L^0)^d$ such that $\text{ess inf}_{X \in E} \langle X, Z \rangle > 0$, which implies (6.5). If $\mu[B] > 0$, denote $A := \Omega \setminus B$. The same argument as before yields a $Z_0 \in 1_A(L^0)^d$ satisfying (6.6)–(6.7). On the other hand, $1_B E$ is L^0-linear. So it follows from Corollary 2.11 that there exist disjoint sets $B_1, \ldots, B_d \in \mathcal{F}$ satisfying $\bigcup_{i=1}^d B_i = B$ and an orthonormal basis $X_1, \ldots, X_d$ of $(L^0)^d$ on B such that $1_{B_i} E = \text{lin}_{B_i}\{X_1, \ldots, X_i\}$ for all $i = 1, \ldots, d$. For every $i \in \mathcal{I} := \left\{ j = 1, \ldots, d : \mu[B_j] > 0 \right\}$ one can apply Theorem 6.4 in the L^0-linear subset $1_{B_i} E$ to obtain a $Z_i \in 1_{B_i} E$ such that

$$||Z_i|| > 0 \text{ on } B_i \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in C} \langle X, Z_i \rangle \geq \operatorname*{ess\,sup}_{Y \in D} \langle Y, Z_i \rangle .$$

Since $0 \notin \text{ri}_A(C - D)$ for all $A \in \mathcal{F}_+$, one has

$$\operatorname*{ess\,sup}_{X \in C} \langle X, Z_i \rangle > \operatorname*{ess\,inf}_{Y \in D} \langle Y, Z_i \rangle \quad \text{on } B_i .$$

If one sets $Z = 1_A Z_0 + \bigcup_{i \in \mathcal{I}} 1_{B_i} Z_i$, one obtains (6.5), and the proof is complete. □

7 Properties of L^0-convex Functions

Definition 7.1 Consider a function $f : (L^0)^d \to L$ and an $X_0 \in (L^0)^d$.

- We call $Y \in (L^0)^d$ an L^0-subgradient of f at X_0 if

$$f(X_0) \in L^0 \quad \text{and} \quad f(X_0 + X) - f(X_0) \geq \langle X, Y \rangle \quad \text{for all } X \in (L^0)^d .$$

By $\partial f(X_0)$ we denote the set of all L^0-subgradients of f at X_0.

- If $f(X_0) \in L^0$ and for some $X \in (L^0)^d$ the limit

$$f'(X_0; X) := \lim_{n\to\infty} n\,[f(X_0 + X/n) - f(X_0)]$$

exists a.e. ($+\infty$ and $-\infty$ are allowed as limits), we call it L^0-directional derivative of f at X_0 in the direction X.

- We say f is L^0-differentiable at X_0 if $f(X_0) \in L^0$ and there exists a $Y \in (L^0)^d$ such that

$$\frac{f(X_0 + X_n) - f(X_0) - \langle X_n, Y\rangle}{||X_n||} \to 0 \text{ a.e.}$$

for every sequence $(X_n)_{n\in\mathbb{N}}$ in $(L^0)^d$ satisfying $X_n \to 0$ a.e. and $||X_n|| > 0$ for all $n \in \mathbb{N}$. If such a Y exists, we call it the L^0-derivative of f at X_0 and denote it by $\nabla f(X_0)$.

- The L^0-convex conjugate $f^* : (L^0)^d \to L$ is given by

$$f^*(Y) := \operatorname*{ess\,sup}_{X\in(L^0)^d} \{\langle X, Y\rangle - f(X)\}\,.$$

- If f is L^0-convex, we set

$$\operatorname{dom} f := \left\{X \in (L^0)^d : f(X) < +\infty\right\}.$$

- By $\operatorname{conv} f$ we denote the largest L^0-convex function below f and by $\underline{\operatorname{conv}}\, f$ the largest sequentially lsc L^0-convex function below f.
- If f is L^0-convex and satisfies $f(\lambda X) = \lambda f(X)$ for all $\lambda \in L^0_{++}$ and $X \in (L^0)^d$, we call f L^0-sublinear.
- For every pair $(Y, Z) \in (L^0)^d \times L^0$ we denote by $f^{Y,Z}$ the function from $(L^0)^d$ to L^0 given by $f^{Y,Z}(X) := \langle X, Y\rangle + Z$.

Theorem 7.2 *Let $f : (L^0)^d \to L$ be an L^0-convex function and $X_0 \in \operatorname{int}(\operatorname{dom} f)$ such that $f(X_0) \in L^0$. Then $f(X) \in \overline{L}$ for all $X \in (L^0)^d$ and f is sequentially continuous on* $\operatorname{int}(\operatorname{dom} f)$.

Proof Since $X_0 \in \operatorname{int}(\operatorname{dom} f)$, there exists an $\varepsilon \in L^0_{++}$ such that $V := \max_i f(X_0 \pm \varepsilon e_i) < +\infty$. By L^0-convexity, one has $f(X) \leq V$ for all $X \in X_0 + U$, where

$$U := \left\{X \in (L^0)^d : \sum_{i=1}^d |X^i| \leq \varepsilon\right\}.$$

Assume that there exist $X \in (L^0)^d$ and $A \in \mathcal{F}_+$ such that $f(X) = -\infty$ on A. Then one can choose a $Z \in X_0 + U$ and a $\lambda \in L^0$ such that $0 < \lambda \leq 1$ and $X_0 = \lambda X + (1-\lambda)Z$. It follows that $f(X_0) \leq \lambda f(X) + (1-\lambda) f(Z) = -\infty$ on A. But this contradicts the assumptions. So $f(X) \in \overline{L}$ for all $X \in (L^0)^d$.

Now pick an $X \in U$ and a $\lambda \in L^0$ such that $0 < \lambda \le 1$. Then

$$f(X_0 + \lambda X) = f(\lambda(X_0 + X) + (1-\lambda)X_0) \le \lambda f(X_0 + X) + (1-\lambda) f(X_0),$$

and therefore,

$$f(X_0 + \lambda X) - f(X_0) \le \lambda[f(X_0 + X) - f(X_0)] \le \lambda(V - f(X_0)).$$

On the other hand,

$$X_0 = \frac{1}{1+\lambda}(X_0 + \lambda X) + \frac{\lambda}{1+\lambda}(X_0 - X).$$

So

$$f(X_0) \le \frac{1}{1+\lambda} f(X_0 + \lambda X) + \frac{\lambda}{1+\lambda} f(X_0 - X),$$

which gives

$$f(X_0) - f(X_0 + \lambda X) \le \lambda[f(X_0 - X) - f(X_0)] \le \lambda(V - f(X_0)).$$

Hence, we have shown that

$$|f(X) - f(X_0)| \le \lambda(V - f(X_0)) \quad \text{for all } X \in X_0 + \lambda U.$$

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $(L^0)^d$ converging a.e. to X_0. For every $k \in \mathbb{N}$, the sets

$$A_m^k := \bigcap_{n \ge m} \{X_n - X_0 \in U/k\}$$

are increasing in m with $\bigcup_{m \ge 1} A_m^k = \Omega$. By Lemma 4.3, f is stable. Therefore,

$$|f(X_n) - f(X_0)| \le (V - f(X_0))/k \quad \text{for all } n \ge m \quad \text{on } A_m^k,$$

and one obtains

$$\mu\left[\bigcup_{k \ge 1} \bigcap_{m \ge 1} \bigcup_{n \ge m} \{|f(X_n) - f(X_0)| > (V - f(X_0))/k\}\right] = 0.$$

So $f(X_n) \to f(X_0)$ a.e., and the theorem follows. □

As an immediate consequence of Theorem 7.2 one obtains the following

Corollary 7.3 *An L^0-convex function $f : (L^0)^d \to \overline{L}$ is sequentially continuous on* $\operatorname{int}(\operatorname{dom} f)$.

Theorem 7.4 *Let $f : (L^0)^d \to \overline{L}$ be an L^0-convex function and $X_0 \in \mathrm{ri}(\mathrm{dom}\, f)$. Then $\partial f(X_0) \neq \emptyset$. In particular, if $f(X) \in L^0$ for all $X \in (L^0)^d$, then $\partial f(X_0) \neq \emptyset$ for all $X \in (L^0)^d$.*

Proof By Lemma 4.3, f is stable. Therefore,

$$C := \left\{(X, V) \in (L^0)^d \times L^0 : f(X) \leq V\right\}$$

is an L^0-convex, σ-stable subset of $(L^0)^d \times L^0$. Since $(X_0, f(X_0) + 1)$ is in C, one has $(0, 0) \notin \mathrm{ri}_A(C - (X_0, f(X_0)))$ for all $A \in \mathcal{F}_+$. So it follows from Theorem 6.7 that there exists $(Y, Z) \in (L^0)^d \times L^0$ such that

$$\operatorname*{ess\,inf}_{(X,V)\in C} \{\langle X, Y\rangle + VZ\} \geq \langle X_0, Y\rangle + f(X_0)Z \tag{7.1}$$

and

$$\operatorname*{ess\,sup}_{(X,V)\in C} \{\langle X, Y\rangle + VZ\} > \langle X_0, Y\rangle + f(X_0)Z. \tag{7.2}$$

Equation (7.1) implies that $Z \geq 0$. Now assume there exists an $A \in \mathcal{F}_+$ such that $1_A Z = 0$. Then since $X_0 \in \mathrm{ri}(\mathrm{dom}\, f)$, (7.2) contradicts (7.1). So one must have $Z > 0$, and by multiplying (Y, Z) with $1/Z$, one can assume $Z = 1$. It follows from (7.1) that

$$\operatorname*{ess\,inf}_{X\in \mathrm{dom}\, f} \{\langle X, Y\rangle + f(X)\} = \langle X_0, Y\rangle + f(X_0),$$

which shows that $-Y$ is an L^0-subgradient of f at X_0. □

Lemma 7.5 *Let $f, g : (L^0)^d \to L$ be functions such that $f \geq g$. Then the following hold:*

(i) f^* *is sequentially lsc and* L^0*-convex;*
(ii) $f^*(Y) \geq \langle X, Y\rangle - f(X)$ *for all* $X, Y \in (L^0)^d$*;*
(iii) $Y \in \partial f(X)$ *if and only if* $f(X) \in L^0$ *and* $f^*(Y) = \langle X, Y\rangle - f(X)$*;*
(iv) $f^* \leq g^*$ *and* $f^{**} \geq g^{**}$*;*
(v) $f \geq f^{**}$ *and* $f^* = f^{***}$*.*

Proof To prove (i) let $(Y_n)_{n\in\mathbb{N}}$ be a sequence in $(L^0)^d$ converging a.e. to some $Y \in (L^0)^d$. Then

$$\begin{aligned}
\liminf_{n\to\infty} f^*(Y_n) &= \sup_{m\geq 1} \inf_{n\geq m} \operatorname*{ess\,sup}_{X\in(L^0)^d} \{\langle X, Y_n\rangle - f(X)\} \\
&\geq \operatorname*{ess\,sup}_{X\in(L^0)^d} \sup_{m\geq 1} \inf_{n\geq m} \{\langle X, Y_n\rangle - f(X)\} \\
&= \operatorname*{ess\,sup}_{X\in(L^0)^d} \{\langle X, Y\rangle - f(X)\} = f^*(Y).
\end{aligned}$$

Hence, f^* is sequentially lsc. To show that it is L^0-convex, choose $Y, Z \in (L^0)^d$ and $\lambda \in L^0$ such that $0 \leq \lambda \leq 1$. Then, $\lambda f^*(Y) + (1 - \lambda) f^*(Z) \geq \langle X, \lambda Y + (1 -$

$\lambda)Z\rangle - f(X)$ for all $X \in (L^0)^d$ and therefore, $\lambda f^*(Y) + (1-\lambda)f^*(Z) \geq f^*(\lambda Y + (1-\lambda)Z)$. (ii) is immediate from the definition of f^*. Now assume that $f(X) \in L^0$. For any $X' \in (L^0)^d$, $f(X') - f(X) \geq \langle X' - X, Y\rangle$ is equivalent to $\langle X, Y\rangle - f(X) \geq \langle X', Y\rangle - f(X')$. This shows (iii). (iv) is clear. From (ii) one obtains that $f(X) \geq \langle X, Y\rangle - f^*(Y)$ for all $X, Y \in (L^0)^d$. So $f \geq f^{**}$. The same inequality applied to f^* gives $f^* \geq f^{***}$. On the other hand, we know from (iv) that $f^* \leq f^{***}$. This proves (v). □

Lemma 7.6 *Let $f : (L^0)^d \to \overline{L}$ be a sequentially lsc L^0-convex function. Then one has for all $X \in (L^0)^d$,*

$$f(X) = \operatorname{ess\,sup}\left\{f^{Y,Z}(X) : (Y, Z) \in (L^0)^d \times L^0,\ f \geq f^{Y,Z}\right\}.$$

Proof Note that the set

$$\mathcal{A} := \left\{A \in \mathcal{F} : \text{ there exists an } X \in (L^0)^d \text{ such that } 1_A f(X) \in L^0\right\}$$

is directed upwards. Therefore, there exists an increasing sequence A_n in $\mathcal{A}$ with corresponding X_n, $n \in \mathbb{N}$, such that $A_n \uparrow A := \operatorname{ess\,sup} \mathcal{A}$ a.e. Set

$$X_0 := 1_{A_1 \cup A^c} X_1 + \sum_{n \geq 2} 1_{A_n \setminus A_{n-1}} X_n.$$

By Lemma 4.3, f is stable. Hence, $f(X_0) < +\infty$ on A, and $f(X) = +\infty$ on A^c for all $X \in (L^0)^d$. The lemma can be proved on A and A^c separately, and on A^c it is obvious. Therefore, we can assume $A = \Omega$. Then $\operatorname{dom} f \neq \emptyset$, and it follows that

$$C := \left\{(X, V) \in \operatorname{dom} f \times L^0 : f(X) \leq V\right\}$$

is a non-empty sequentially closed L^0-convex subset of $(L^0)^d \times L^0$. Choose a pair $(U, W) \in (L^0)^d \times L^0$ such that $1_A(U, W) \notin 1_A C$ for all $A \in \mathcal{F}_+$. By Corollary 6.2, there exists $(Y, Z) \in (L^0)^d \times L^0$ such that

$$I := \inf_{(X,V) \in C} \{\langle X, Y\rangle + VZ\} > \langle U, Y\rangle + WZ.$$

It follows that $Z \geq 0$. On the set $B := \{Z > 0\}$ one can multiply (Y, Z) with $1/Z$ and assume $Z = 1$. Then one obtains that on B,

$$f(X) \geq f^{-Y,I}(X) \quad \text{for all } X \in (L^0)^d \quad \text{and} \quad f^{-Y,I}(U) > W.$$

On B^c one has $\lambda := I - \langle U, Y\rangle > 0$. Pick a $U' \in \operatorname{dom} f$. Since $1_A(U', f(U') - 1) \notin 1_A C$ for all $A \in \mathcal{F}_+$, one obtains from Corollary 6.2 that there exists a pair $(Y', Z') \in (L^0)^d \times L^0$ such that

$$I' := \inf_{(X,V)\in C} \left\{\left\langle X, Y'\right\rangle + VZ'\right\} > \left\langle U', Y'\right\rangle + (f(U') - 1)Z'.$$

Since $U' \in \operatorname{dom} f$, one must have $Z' > 0$. By multiplying with $1/Z'$, one can assume $Z' = 1$. Now choose a $\delta \in 1_{B^c} L^0_+$ such that

$$\delta > \frac{1}{\lambda}(W + \left\langle U, Y'\right\rangle - I')^+ \quad \text{on } B^c$$

and set $Y'' := \delta Y + Y'$. Then, on B^c,

$$I'' := \inf_{(X,V)\in C}(\left\langle X, Y''\right\rangle + V) \geq \delta I + I' = \delta\lambda + \delta \langle U, Y\rangle + I' > \left\langle U, Y''\right\rangle + W.$$

So on B^c, one has

$$f(X) \geq f^{-Y'',I''}(X) \quad \text{for all } X \in (L^0)^d \quad \text{and} \quad f^{-Y'',I''}(U) > W.$$

Now define $(\hat{Y}, \hat{I}) := 1_B(-Y, I) + 1_{B^c}(-Y'', I'')$. Then

$$f(X) \geq f^{\hat{Y},\hat{I}}(X) \quad \text{for all } X \in (L^0)^d \quad \text{and} \quad f^{\hat{Y},\hat{I}}(U) > W.$$

This proves the lemma. □

Theorem 7.7 (Conditional version of the Fenchel–Moreau theorem) *Let $f : (L^0)^d \to \overline{L}$ be a function such that $\underline{\operatorname{conv}} f$ takes values in $\overline{L}$. Then $\underline{\operatorname{conv}} f = f^{**}$. In particular, if f is sequentially lsc and L^0-convex, then $f = f^{**}$.*

Proof We know from Lemma 7.5 that f^{**} is a sequentially lsc L^0-convex minorant of f. So $\underline{\operatorname{conv}} f \geq f^{**}$. On the other hand, it follows from Lemma 7.6 that

$$\underline{\operatorname{conv}} f = \operatorname{ess\,sup}\left\{f^{Y,Z}(X) : (Y, Z) \in (L^0)^d \times L^0,\ \underline{\operatorname{conv}} f \geq f^{Y,Z}\right\},$$

and it can easily be checked that $(f^{Y,Z})^{**} = f^{Y,Z}$ for all $(Y, Z) \in (L^0)^d \times L^0$. So one obtains from Lemma 7.5 that $f^{**} \geq (f^{Y,Z})^{**} = f^{Y,Z}$ for every pair $(Y, Z) \in (L^0)^d \times L^0$ satisfying $f \geq f^{Y,Z}$. This shows that $f^{**} \geq \underline{\operatorname{conv}} f$. □

Lemma 7.8 *Let $f : (L^0)^d \to L$ be an L^0-convex function and $X_0 \in (L^0)^d$ such that $f(X_0) \in L^0$. Then $f'(X_0; X)$ exists for all $X \in (L^0)^d$, $f'(X_0, 0) = 0$ and $f'(X_0; .)$ is L^0-sublinear. Moreover, $\partial f(X_0) = \partial g(0)$, where $g(X) := f'(X_0; X)$.*

Proof It follows from L^0-convexity that for every $X \in (L^0)^d$, $n[f(X_0 + X/n) - f(X_0)]$ is decreasing in n. This implies that $f'(X_0; X)$ exists. $f'(X_0; 0) = 0$ is clear. That $f'(X_0; .)$ is L^0-sublinear and $\partial f(X_0) = \partial g(0)$ are straightforward to check. □

Lemma 7.9 *Let $f : (L^0)^d \to \overline{L}$ be a sequentially lsc L^0-sublinear function. If there exists an $X_0 \in (L^0)^d$ such that $f(X_0) \in L^0$, then $\partial f(0) \neq \emptyset$ and $f(X) = \operatorname{ess\,sup}_{Y \in \partial f(0)} \langle X, Y \rangle$ for all $X \in (L^0)^d$. In particular, $f(0) = 0$.*

Proof By Theorem 7.7, one has $f = f^{**}$. This implies that the set

$$C := \left\{ Y \in (L^0)^d : \langle X, Y \rangle \leq f(X) \text{ for all } X \in (L^0)^d \right\}$$

is non-empty and $f(X) = \operatorname{ess\,sup}_{Y \in C} \langle X, Y \rangle$. It follows that $f(0) = 0$ and $\partial f(0) = C$. This proves the lemma. □

Theorem 7.10 *Let $f : (L^0)^d \to \overline{L}$ be an L^0-convex function. Assume there exist $X_0 \in (L^0)^d$ and $V \in L^0_+$ such that $f(X_0) \in L^0$ and*

$$f(X_0 + X) \geq f(X_0) - V||X|| \quad \textit{for all } X \in (L^0)^d. \tag{7.3}$$

Then there exists a $Y \in \partial f(X_0)$ such that $||Y|| \leq V$.

Proof Denote $g(X) := f'(X_0; X)$. Then $h = \underline{\text{conv}}\, g$ is a sequentially lsc L^0-sublinear function which by (7.3), satisfies

$$h(X) \geq -V||X|| \quad \text{for all} \quad X \in (L^0)^d. \tag{7.4}$$

It follows that $h(0) = 0$ and $\partial h(0) \subseteq \partial g(0) = \partial f(X_0)$. Since $\partial h(0)$ and

$$B^V(0) := \left\{ Y \in (L^0)^d : ||Y|| \leq V \right\}$$

are L^0-convex and sequentially closed, they are both σ-stable. Therefore, there exists a largest set $A \in \mathcal{F}$ such that $1_A \partial h(0) \cap 1_A B^V(0)$ is non-empty. Assume that $A^c \in \mathcal{F}_+$. Then, if one restricts attention to A^c and assumes $\Omega = A^c$, the sets $\partial h(0)$ and $B^V(0)$ satisfy the assumptions of Corollary 6.2. So there exists a $Z \in (L^0)^d$ such that

$$-V||Z|| = \operatorname*{ess\,inf}_{Y \in B^V(0)} \langle Y, Z \rangle > \operatorname*{ess\,sup}_{Y \in \partial h(0)} \langle Y, Z \rangle .$$

But by Lemma 7.9, one has $h(Z) = \operatorname{ess\,sup}_{Y \in \partial h(0)} \langle Y, Z \rangle$, and one obtains a contradiction to (7.4). It follows that $A = \Omega$, which proves the theorem. □

Theorem 7.11 *Let $f : (L^0)^d \to \overline{L}$ be an L^0-convex function and X_0 in $(L^0)^d$ such that $f(X_0) \in L^0$. Assume that $\partial f(X_0) = \{Y\}$ for some $Y \in (L^0)^d$. Then f is L^0-differentiable at X_0 with $\nabla f(X_0) = Y$.*

Proof By Lemma 7.8, one has $\partial g(0) = \{Y\}$ for the L^0-sublinear function $g(X) := f'(X_0; X)$. It follows that

$$g^*(Z) = 1_{\{Z \neq Y\}}(+\infty) \quad \text{and} \quad g^{**}(X) = \langle X, Y \rangle\,. \tag{7.5}$$

Set

$$\mathcal{A} := \left\{ A \in \mathcal{F} : \text{ there exists an } X \in (L^0)^d \text{ such that } g(X) = +\infty \text{ on } A \right\}.$$

By Lemma 4.3, g is stable. Therefore, there exists a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ with corresponding X_n such that $A_n \uparrow A := \operatorname{ess\,sup} \mathcal{A}$. The element

$$X_0 := 1_{A_1 \cup A^c} X_1 + \sum_{n \geq 2} 1_{A_n \setminus A_{n-1}} X_n$$

satisfies $g(X_0) = +\infty$ on A. We want to show that $\mu[A] = 0$. So let us assume $\mu[A] > 0$. If one replaces Ω with A, one has $0 \notin 1_B(\operatorname{dom} g - X_0)$ for all $B \in \mathcal{F}_+$. By Theorem 6.4, there exists a $Z \in (L^0)^d$ such that

$$||Z|| > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{X \in \operatorname{dom} g} \langle X, Z \rangle \geq \langle X_0, Z \rangle\,.$$

Define the sequentially lsc L^0-convex function $h : (L^0)^d \to \overline{L}$ as follows:

$$h(X) := \langle X, Y \rangle\, 1_{\{\langle X, Z \rangle \geq \langle X_0, Z \rangle\}} + \infty 1_{\{\langle X, Z \rangle < \langle X_0, Z \rangle\}}.$$

Then $g \geq h$ and $h(X) = +\infty$ for all $X \in (L^0)^d$ satisfying $\langle X, Z \rangle < \langle X_0, Z \rangle$. It follows that $\underline{\operatorname{conv}} g(X) = +\infty$ for all $X \in (L^0)^d$ satisfying $\langle X, Z \rangle < \langle X_0, Z \rangle$. Moreover, since $Y \in \partial g(0)$, g fulfills the assumptions of Theorem 7.7, and one obtains $\underline{\operatorname{conv}} g = g^{**}$, contradicting (7.5). So one must have $\mu[A] = 0$, or in other words, $g(X) \in L^0$ for all $X \in (L^0)^d$. It follows from Theorem 7.2 that g is sequentially continuous, and therefore, $g(X) = g^{**}(X) = \langle X, Y \rangle$ for all $X \in (L^0)^d$.

Now let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $(L^0)^d$ such that $X_n \to 0$ a.e. and $||X_n|| > 0$ for all n. Denote $||X_n||_1 := \sum_{i=1}^d |X_n^i|$ and notice that there exists a constant $c > 0$ such that $||X_n||_1 \leq c||X_n||$ for all n. Since $g(X) = \langle X, Y \rangle$, one has for all $i = 1, \ldots, d$,

$$\frac{f(X_0 \pm ||X_n||_1 e_i) - f(X_0)}{||X_n||_1} \to \pm Y^i \quad \text{a.e.}$$

Therefore,

$$\frac{f(X_0+X_n)-f(X_0)-\langle X_n,Y\rangle}{||X_n||} \le c\frac{f(X_0+X_n)-f(X_0)-\langle X_n,Y\rangle}{||X_n||_1}$$
$$\le c\sum_{i=1}^{d}\frac{|X_n^i|}{||X_n||_1}\left\{\frac{f(X_0+||X_n||_1 \text{sign}(X_n^i)e_i)-f(X_0)}{||X_n||_1}-\text{sign}(X_n^i)Y^i\right\}\to 0 \quad \text{a.e.}$$

□

8 Inf-Convolution

Definition 8.1 We define the inf-convolution of finitely many functions $f_j : (L^0)^d \to \overline{L}$, $j = 1, \dots, n$, by

$$\square_{j=1}^n f_j(X) := \operatorname*{ess\,inf}_{X_1+\cdots+X_n=X} \sum_{j=1}^{n} f_j(X_j).$$

Lemma 8.2 *If f_j, $j = 1, \dots, n$, are L^0-convex functions from $(L^0)^d$ to $\overline{L}$, then $\square_{j=1}^n f_j$ is L^0-convex too.*

Proof Denote $f = \square_{j=1}^n f_j$. Choose $X, Y \in (L^0)^d$ and $V, W \in \overline{L}$ such that $f(X) \le V$ and $f(Y) \le W$. Let $\varepsilon \in L^0_{++}$ and $\lambda \in L^0$ such that $0 \le \lambda \le 1$. By Lemma 4.3, the functions f_j are stable. Therefore, the family $\left\{\sum_j f_j(X_j) : \sum_j X_j = X\right\}$ is directed downwards. So there exist sequences X_j^k, $k \in \mathbb{N}$, such that $\sum_j X_j^k = X$ and $\sum_j f_j(X_j^k)$ decreases to $f(X)$ a.e. It follows that the sets $A_k := \left\{\sum_j f_j(X_j^k) \le V + \varepsilon\right\}$ increase to Ω as $k \to \infty$. So for every $j = 1, \dots, n$,

$$X_j := \sum_{k\ge 1} 1_{A_k \setminus A_{k-1}} X_j^k, \quad \text{where } A_0 := \emptyset.$$

defines an element in $(L^0)^d$ such that $\sum_{j=1}^n X_j = X$ and $\sum_{j=1}^n f(X_j) \le V + \varepsilon$. Analogously, there exist $Y_j \in (L^0)^d$, $j = 1, \dots, n$, such that $\sum_{j=1}^n Y_j = Y$ and $\sum_{j=1}^n f(Y_j) \le W + \varepsilon$. Set $Z_j = \lambda X_j + (1-\lambda)Y_j$. Then $Z := \sum_{j=1}^n Z_j = \lambda X + (1-\lambda)Y$ and

$$f(Z) \le \sum_{j=1}^{n} f_j(Z_j) \le \sum_{j=1}^{n} \lambda f_j(X_j) + (1-\lambda)f(Y_j) \le \lambda V + (1-\lambda)W + \varepsilon.$$

It follows that $f(Z) \le \lambda f(X) + (1-\lambda)f(Y)$. □

Lemma 8.3 *Let $f_j : (L^0)^d \to \overline{L}$, $j = 1, \ldots, n$, be L^0-convex functions and denote $f = \square_{j=1}^n f_j$. Assume $f(X_0) = \sum_{j=1}^n f_j(X_j) < +\infty$ for some $X_j \in (L^0)^d$ summing up to X_0. If $X_1 \in \operatorname{int}(\operatorname{dom} f_1)$, then $f(X) \in \overline{L}$ for all $X \in (L^0)^d$, $X_0 \in \operatorname{int}(\operatorname{dom} f)$ and f is sequentially continuous on* $\operatorname{int}(\operatorname{dom} f)$.

Proof By definition of f, one has

$$f(X_0 + X) - f(X_0) \le f_1(X_1 + X) + \sum_{j=2}^n f_j(X_j) - \sum_{j=1}^n f_j(X_j) = f_1(X_1 + X) - f_1(X_1)$$

for all $X \in (L^0)^d$. This shows that $X_0 \in \operatorname{int}(\operatorname{dom} f)$. Since $f(X_0) = \sum_{j=1}^n f_j(X_j) \in L^0$, the rest of the lemma follows from Theorem 7.2. □

Lemma 8.4 *Consider functions $f_j : (L^0)^d \to \overline{L}$, $j = 1, \ldots, n$, and denote $f = \square_{j=1}^n f_j$. Assume $f(X_0) = \sum_{j=1}^n f_j(X_j) < +\infty$ for some $X_j \in (L^0)^d$ summing up to X_0. Then $\partial f(X_0) = \bigcap_{j=1}^n \partial f_j(X_j)$.*

Proof Assume $Y \in \partial f(X_0)$ and $X \in (L^0)^d$. Then

$$f_1(X_1 + X) - f_1(X_1) = f_1(X_1 + X) + \sum_{j=2}^n f_j(X_j) - \sum_{j=1}^n f_j(X_j) \ge f(X_0 + X) - f(X_0) \ge \langle X, Y \rangle .$$

Hence $Y \in \partial f_1(X_1)$, and by symmetry, $\partial f(X_0) \subseteq \bigcap_{j=1}^n \partial f_j(X_j)$. On the other hand, if $Y \in \bigcap_{j=1}^n \partial f_j(X_j)$ and $X \in (L^0)^d$, choose Z_j such that $\sum_{j=1}^n Z_j = X_0 + X$. Then

$$\sum_{j=1}^n f_j(Z_j) \ge \sum_{j=1}^n f_j(X_j) + \left\langle Z_j - X_j, Y \right\rangle = \sum_{j=1}^n f_j(X_j) + \langle X, Y \rangle .$$

So $f(X_0 + X) - f(X_0) \ge \langle X, Y \rangle$, and the lemma follows. □

Lemma 8.5 *Let $f_j : (L^0)^d \to \overline{L}$, $j = 1, \ldots, n$, be L^0-convex functions and denote $f = \square_{j=1}^n f_j$. Assume $f(X_0) = \sum_j f_j(X_j) < +\infty$ for some $X_j \in (L^0)^d$ summing up to X_0 and f_1 is L^0-differentiable at X_1. Then f is L^0-differentiable at X_0 with $\nabla f(X_0) = \nabla f_1(X_1)$.*

Proof One has

$$f(X_0 + X) - f(X_0) \le f_1(X_1 + X) + \sum_{j=2}^n f_j(X_j) - \sum_{j=1}^n f_j(X_j) = f_1(X_1 + X) - f_1(X_1)$$

for all $X \in (L^0)^d$. It follows that the L^0-directional derivative $g(X) := f'(X_0; X)$ satisfies

$$g(X) \le f_1'(X_1; X) = \langle X, \nabla f_1(X_1)\rangle$$

for all $X \in (L^0)^d$. But by Lemma 8.2, f is L^0-convex. It follows that g is L^0-sublinear, and therefore, $g(X) = \langle X, \nabla f_1(X_1)\rangle$. This implies that $\partial f(X_0) = \partial g(0) = \{\nabla f_1(X_1)\}$. Now the lemma follows from Theorem 7.11. □

Lemma 8.6 *Consider functions $f_j : (L^0)^d \to \overline{L}$, $j = 1, \dots, n$. Then $\left(\square_{j=1}^n f_j\right)^* = \sum_{j=1}^n f_j^*$, where the sum is understood to be $-\infty$ if at least one of the terms is $-\infty$.*

Proof

$$\begin{aligned}\left(\square_{j=1}^n f_j\right)^*(Y) &= \operatorname*{ess\,sup}_X \left\{\langle X, Y\rangle - \square_{j=1}^n f_j(X)\right\}\\ &= \operatorname*{ess\,sup}_{X_1,\dots,X_n} \sum_{j=1}^n \left\{\langle X_j, Y\rangle - f_j(X_j)\right\} = \sum_{j=1}^n f_j^*(Y).\end{aligned}$$ □

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References

1. Cheridito, P., Hu, Y.: Optimal consumption and investment in incomplete markets with general constraints. Stoch. Dyn. **11**(2), 283–299 (2011)
2. Cheridito, P., Horst, U., Kupper, M., Pirvu, T.: Equilibrium pricing in incomplete markets under translation invariant preferences. SSRN Preprint (2014)
3. Cheridito, P., Stadje, M.: BSΔEs and BSDEs with non-Lipschitz drivers: comparison, convergence and robustness (2012). Forthcoming in Bernoulli
4. Filipović, D., Kupper, M., Vogelpoth, N.: Separation and duality in locally L^0-convex modules. J. Funct. Anal. **256**, 3996–4029 (2009)
5. Föllmer, H., Schied, A.: Stochastic Finance, An Introduction in Discrete Time. de Gruyter Studies in Mathematics, vol. 27, 2nd edn. (2004)
6. Guo, T.: Relations between some basic results derived from two kinds of topologies for a random locally convex module. J. Funct. Anal. **258**, 3024–3047 (2010)
7. Guo, T.: Recent progress in random metric theory and its applications to conditional risk measures. Sci. China Math. **54**(4), 633–660 (2011)
8. Kabanov, Y., Stricker, Ch.: A teacher's note on no-arbitrage criteria. Séminaire de probabilités **35**, 149–152 (2001)
9. Kupper, M., Vogelpoth, N.: Complete L^0-normed modules and automatic continuity of monotone convex functions. Preprint (2009)
10. Neveu, J.: Discrete-Parameter Martingales. North-Holland Publishing Company Amsterdam, Oxford (1975)

Set Optimization Meets Variational Inequalities

Giovanni P. Crespi and Carola Schrage

Abstract We study necessary and sufficient conditions to attain solutions of set optimization problems in terms of variational inequalities of Stampacchia and Minty type. The notion of solution we deal with has been introduced by Heyde and Löhne in 2011. To define the set-valued variational inequality, we introduce a set-valued directional derivative that we relate to Dini derivatives of a family of scalar problems. Optimality conditions are given by Stampacchia and Minty type variational inequalities, defined both by set-valued directional derivatives and by Dini derivatives of the scalarizations. The main results allow to obtain known variational characterizations for vector optimization problems as special cases.

Keywords Set optimization · Variational inequalities · Dini derivatives · Scalarization · Vector optimization

1 Introduction

Since the seminal papers by Giannessi (see [9, 10]), variational inequalities have been applied to obtain necessary and sufficient optimality conditions in vector optimization. In [18] a new approach to study set-valued problems has been applied to have a *fresh look* to vector optimization. Indeed, it turns out that vector optimization can be treated as a special case of set optimization. The aim of this paper is to provide some variational characterization of (convex) set-valued optimization. Following the approach known as set optimization we introduce set-valued variational inequalities, both of Stampacchia and Minty type, by means of Dini-type derivatives (see e.g.

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[15]). Under suitable assumptions (e.g. lower semicontinuity type assumptions), we can prove equivalence between solutions of the variational inequalities and solutions of a (primitive) set optimization problem, as introduced in [18] and deepened in [21]. To prove the main results we need also to deal with scalarization problems. However, while in the vector case this might only be a technical need, we prove that eventually the set-valued variational inequalities and their scalar counterparts provide different insights into the problem. Some relevant information on the solution of the set optimization problem is provided only through the scalar version of the inequality. The special case of vector optimization is finally studied, to recover classical results stated in [4, 26].

The paper is organized as follows. Section 2 is devoted to preliminary results on set optimization that will be used throughout the paper. The concept of solution to a set optimization problem and the Dini type derivatives are presented and some properties are proved. Section 3 presents the main results. As the solution concept relies on two properties, we develop two different sets of relations between our variational inequalities and the set optimization. The first one provides a variational characterization of "infimizer", while the second one is devoted to characterize "minimizers". Finally, Sect. 4 applies the previous results to vector optimization. The relations proved for the convex case in this paper reproduce those already known for the vector case between optimization and variational inequalities. We leave as an open question whether convexity can be relaxed, as indeed can be done for vector-valued functions.

2 Preliminaries

2.1 Order and Operations with Sets

Throughout the paper, unless explicitly stated otherwise, we assume the setting and notation introduced in this section.

Let Z be a locally convex Hausdorff space with dual space Z^*. The set $\mathcal{U}$ is the set of all closed, convex and balanced 0-neighborhoods in Z, a 0-neighborhood base of Z. By $\operatorname{cl} A$, $\operatorname{co} A$ and $\operatorname{int} A$, we denote the closed or convex hull of a set $A \subseteq Z$ and the topological interior of A, respectively. The conical hull of a set A is $\operatorname{cone} A = \{ta \mid a \in A,\ 0 < t\}$.

The recession cone of a nonempty closed convex set $A \subseteq Z$ is given by

$$0^+A = \{z \in Z \mid A + \{z\} \subseteq A\}, \tag{2.1}$$

a closed convex cone, [27, p. 6]. By definition, $0^+\emptyset = \emptyset$.

Z is pre-ordered through a closed convex cone C with $0 \in C$ and nontrivial negative dual cone

$$C^- = \left\{z^* \in Z^* \mid \forall c \in C :\ z^*(c) \le 0\right\},$$

$C^- \setminus \{0\} \neq \emptyset$ by setting

$$z_1 \leq z_2 \quad \Leftrightarrow \quad \{z_2\} + C \subseteq \{z_1\} + C$$

for all $z_1, z_2 \in Z$. This relation is extended to $\mathcal{P}(Z)$, the power set of Z including $\emptyset$ and Z (compare [13] and the references therein) by setting

$$A_1 \preccurlyeq A_2 \quad \Leftrightarrow \quad A_2 + C \subseteq A_1 + C$$

for all $A_1, A_2 \subseteq Z$.

We introduce the subset

$$\mathcal{G}(Z, C) = \{A \subseteq Z \mid A = \operatorname{cl}\operatorname{co}(A + C)\}$$

which is an order complete lattice and $A_1 \preccurlyeq A_2$ is equivalent to $A_1 \supseteq A_2$ whenever $A_1, A_2 \in \mathcal{G}(Z, C)$. For any subset $\mathcal{A} \subseteq \mathcal{G}(Z, C)$, supremum and infimum of $\mathcal{A}$ in $\mathcal{G}(Z, C)$ are given by

$$\inf \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A; \quad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A \tag{2.2}$$

and for a net $\{A_i\}_{i \in I}$ in $\mathcal{G}(Z, C)$, limit inferior and limit superior are defined accordingly,

$$\liminf \mathcal{A} = \bigcap_{j \in I} \operatorname{cl}\operatorname{co} \bigcup_{i \geq j} A_i; \quad \limsup \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{j \in I} \bigcap_{i \geq j} A_i. \tag{2.3}$$

When $\mathcal{A} = \emptyset$, then we agree on $\inf \mathcal{A} = \emptyset$ and $\sup \mathcal{A} = Z$. Especially, $\mathcal{G}(Z, C)$ possesses a greatest and smallest element $\inf \mathcal{G}(Z, C) = Z$ and $\sup \mathcal{G}(Z, C) = \emptyset$.

The Minkowsky addition and multiplication with negative reals need to be slightly adjusted to provide operations on $\mathcal{G}(Z, C)$. We define

$$\forall A, B \in \mathcal{G}(Z, C): \quad A \oplus B = \operatorname{cl}\{a + b \in Z \mid a \in A,\ b \in B\}; \tag{2.4}$$

$$\forall A \in \mathcal{G}(Z, C),\ \forall 0 < t: \quad t \cdot A = \{ta \in Z \mid a \in A\}; \tag{2.5}$$

$$\forall A \in \mathcal{G}(Z, C): \quad 0 \cdot A = C. \tag{2.6}$$

Especially, $0 \cdot \emptyset = 0 \cdot Z = C$ and $\emptyset$ dominates the addition in the sense that $A \oplus \emptyset = \emptyset$ is true for all $A \in \mathcal{G}(Z, C)$. Moreover, $A \oplus C = A$ is satisfied for all $A \in \mathcal{G}(Z, C)$, thus C is the neutral element with respect to addition.

As a consequence,

$$\forall \mathcal{A} \subseteq \mathcal{G}(Z, C),\ \forall B \in \mathcal{G}(Z, C): \quad B \oplus \inf \mathcal{A} = \inf\{B \oplus A \mid A \in \mathcal{A}\}, \tag{2.7}$$

or, equivalently, the inf–residual

$$A \dot{-} B = \inf \{M \in \mathcal{G}(Z, C) \mid A \preccurlyeq B \oplus M\} \tag{2.8}$$

exists for all $A, B \in \mathcal{G}(Z, C)$. The following properties are well known in lattice theory, compare also [14, Theorem 2.1].

$$A \dot{-} B = \{z \in Z \mid B + \{z\} \subseteq A\}; \tag{2.9}$$
$$A \preccurlyeq B \oplus (A \dot{-} B) \tag{2.10}$$

Overall, the structure of $\mathcal{G}^\vartriangle = (\mathcal{G}(Z, C), \oplus, \cdot, C, \preccurlyeq)$ is that of an inf–residuated conlinear space, compare also [6–8, 12, 22].

Historically, it is interesting to note that Dedekind [5] introduced the residuation concept and used it in order to construct the real numbers. The construction above is in this line of ideas, but in a rather abstract setting.

Example 2.1 Let us consider $Z = \mathbb{R}$, $C = \mathbb{R}_+$. Then $\mathcal{G}(Z, C) = \{\, [r, +\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\emptyset\}$, and $\mathcal{G}^\vartriangle$ can be identified (with respect to the algebraic and order structures which turn $\mathcal{G}(\mathbb{R}, \mathbb{R}_+)$ into an ordered conlinear space and a complete lattice admitting an inf-residuation) with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ using the 'inf-addition' $\dot{+}$ (see [14, 23]). The inf-residuation on $\overline{\mathbb{R}}$ is given by

$$r \dot{-} s = \inf \left\{t \in \mathbb{R} \mid r \leq s \dot{+} t\right\}$$

for all $r, s \in \overline{\mathbb{R}}$, compare [14] for further details.

Each element of $\mathcal{G}^\vartriangle$ is closed and convex and $A = A + C$. Hence, by a separation argument we can prove

$$\forall A \in \mathcal{G}^\vartriangle : \quad A = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z \mid -\sigma(z^*|A) \leq -z^*(z)\right\}, \tag{2.11}$$

where $\sigma(z^*|A) = \sup \{z^*(z) \mid z \in A\}$ is the support function of A at z^*. Especially, $A = \emptyset$ if, and only if, there exists $z^* \in C^- \setminus \{0\}$ such that $-\sigma(z^*|A) = +\infty$.

Lemma 2.2 ([25, Proposition 3.5]) *Let $\mathcal{A} \subseteq \mathcal{G}^\vartriangle$ be a given subset, then*

$$\inf \mathcal{A} = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z \mid \inf \left\{-\sigma(z^*|A) \mid A \in \mathcal{A}\right\} \leq -z^*(z)\right\} \tag{2.12}$$
$$\forall z^* \in C^- \setminus \{0\} : \quad -\sigma(z^*| \inf \mathcal{A}) = \inf \left\{-\sigma(z^*|A) \mid A \in \mathcal{A}\right\}. \tag{2.13}$$

Lemma 2.3 ([14, Proposition 5.20]) *Let $A, B \in \mathcal{G}^\vartriangle$, then*

$$A \dot{-} B = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z \mid (-\sigma(z^*|A)) \dot{-} (-\sigma(z^*|B)) \leq -z^*(z)\right\}; \tag{2.14}$$

$$\forall z^* \in C^- \setminus \{0\}: \quad \left(-\sigma(z^*|A)\right) \dot{-} \left(-\sigma\left(z^*|B\right)\right) \leq -\sigma(z^*|A \dot{-} B). \tag{2.15}$$

In general, the difference of the scalarizations and the scalarization of the difference do not coincide, as the following example shows.

Example 2.4 Let $Z = \mathbb{R}^2$ and $C = \operatorname{cl}\operatorname{cone}(0, 1)^T$, $B = \left\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ 0 \leq y\right\}$ and $A = C$. Then $(-\sigma(z^*|A)) \dot{-} (-\sigma(z^*|B)) \in \mathbb{R}$ is satisfied for all $z^* \in C^- \setminus \{0\}$ and

$$A \dot{-} B = \left\{z \in Z \mid 1 \leq (-1, 0)^T z,\ 1 \leq (1, 0)^T z,\ 0 \leq (0, 1)^T z\right\} = \emptyset,$$

thus $-\sigma(A \dot{-} B) = +\infty$.

The following rules will be used frequently later on.

Lemma 2.5 *Let $A, B, D \in \mathcal{G}^\vartriangle$, $0 < s$ and $t \in (0, 1)$ be given, then*

(a) $s(A \dot{-} B) = sA \dot{-} sB$;
(b) $(tA \oplus (1-t)B) \dot{-} D \preccurlyeq t(A \dot{-} D) \oplus (1-t)(B \dot{-} D)$;
(c) $A \dot{-} D \preccurlyeq \left(A \dot{-} B\right) \oplus \left(B \dot{-} D\right)$;
(d) If $A \neq \emptyset$, then $0^+A = \left(A \dot{-} A\right)$.

Proof (a) It holds $z \in (A \dot{-} B)$ if, and only if, $B + \{z\} \subseteq A$ or equivalently $sA \preccurlyeq sB + \{sz\}$.

(b) As $D \in \mathcal{G}^\vartriangle$ is assumed, $tD \oplus (1-t)D = D$. Let $z_A \in A \dot{-} D$ and $z_B \in B \dot{-} D$ be given, then $(tA \oplus (1-t)B) \preccurlyeq D + (tz_A + (1-t)z_B)$ is satisfied.

(c) The inclusion is true if, and only if,

$$A \preccurlyeq \left(A \dot{-} B\right) \oplus \left(B \dot{-} D\right) \oplus D.$$

As we know that $B \preccurlyeq \left(B \dot{-} D\right) \oplus D$ and $A \preccurlyeq \left(A \dot{-} B\right) \oplus B$, this inclusion is true.

(d) This is immediate from the definition of 0^+A.

□

Lemma 2.5 (d) suggests that, if needed, we can use the recession cone of a set as generalized 0-element in certain inequalities. It is remarkable that for any $A \in \mathcal{G}^\vartriangle$, either $A = \emptyset$, or $0^+A \preccurlyeq C$. To implement these remarks in the sequel, we use the following properties of recession cones.

Proposition 2.6 *Let $A \in \mathcal{G}^\vartriangle \setminus \{\emptyset\}$, then*

$$0^+A = \left\{z \in Z \mid \forall z^* \in C^- \setminus \{0\}: \ -\sigma(z^*|A) = -\infty \ \vee \ 0 \leq -z^*(z)\right\}.$$

Especially, for all $A \in \mathcal{G}^\vartriangle$, either $A = \emptyset$, or

$$0^+A = \bigcap_{\substack{z^* \in C^- \setminus \{0\} \\ -\sigma(z^*|A) \in \mathbb{R}}} \left\{z \in Z \mid 0 \leq -z^*(z)\right\}. \tag{2.16}$$

Proof Assume $z \notin 0^+A$, then either $A = \emptyset$ or there exists a $z^* \in Z^*$ such that $\sigma(z^*|A) < z^*(a+z)$ is satisfied for some $a \in A$. As $z^*(a+z) \le \sigma(z^*|A) + z^*(z)$, this implies $-z^*(z) < 0$ and $-\sigma(z^*|A) \ne -\infty$ and therefore $z^* \in C^- \setminus \{0\}$. On the other hand, assume $z \in 0^+A$, then A is nonempty and $A + \{z\} \subseteq A$, hence for all $z^* \in Z^*$ it holds $\sigma(z^*|A + \{z\}) \le \sigma(z^*|A)$, hence $\sigma(z^*|A) + z^*(z) \le \sigma(z^*|A)$. This implies that either $-\sigma(z^*|A) = -\infty$ or $0 \le -z^*(z)$ is true for all $z^* \in Z^*$ and thus especially for $z^* \in C^- \setminus \{0\}$.

If $A = Z$, then $-\sigma(z^*|Z) = -\infty \notin \mathbb{R}$ is satisfied for all $z^* \in C^- \setminus \{0\}$, hence (2.16) is true with $0^+Z = Z$. Hence let $A \ne Z$ or $\emptyset$, then $-\sigma(z^*|A) \notin \mathbb{R}$ implies $-\sigma(z^*|A) = -\infty$ and the statement is proved. □

Lemma 2.7 *Let* $A \in \mathcal{G}^\vartriangle \setminus \{\emptyset\}$, *then*

$$\left\{z^* \in Z^* \mid -\sigma(z^*|A) \in \mathbb{R}\right\} \subseteq (0^+A)^- \subseteq C^-.$$

Proof Assume $-\sigma(z^*|A) \in \mathbb{R}$ and $A + \{z\} \subseteq A$. Then

$$-\sigma(z^*|A) \le -\sigma(z^*|A + \{z\}) = -\sigma(z^*|A) + (-z^*(z))$$

implies $0 \le -z^*(z)$, in other words $z^* \in (0^+A)^-$. The second inclusion is immediate, as $A \in \mathcal{G}^\vartriangle \setminus \{\emptyset\}$ implies $0^+A \supseteq C$. □

Lemma 2.8 *Let* $A, B \in \mathcal{G}^\vartriangle \setminus \{\emptyset\}$, *then*

$$0^+(A \oplus B) \preccurlyeq \operatorname{cl}\operatorname{co}\left(0^+A \cup 0^+B\right) = 0^+A \oplus 0^+B;$$
$$A \preccurlyeq B \quad \Rightarrow \quad 0^+A \preccurlyeq 0^+B.$$

Proof Assume $A + \{z_A\} \subseteq A$ and $B + \{z_B\} \subseteq B$, then for all $a \in A$ and all $b \in B$ it holds

$$a + b + (z_A + z_B) \in A \oplus B$$

and as both 0^+A and 0^+B are convex cones, for all $t \in [0, 1]$ it holds

$$ta + (1-t)b + (z_A + z_B) \in A \oplus B.$$

If $z \in A \oplus B$, then for all $U \in \mathcal{U}$ there exist $a \in A$, $b \in B$ and $t \in [0, 1]$ with $ta + (1-t)b \in \{z\} + U$, such that

$$ta + (1-t)b + (z_A + z_B) \in \{z + (z_A + z_B)\} + U,$$

and hence $z + (z_A + z_B) \in A \oplus B$, proving $0^+A + 0^+B \subseteq 0^+(A \oplus B)$. As $A \oplus B$ is a closed convex set, the recession cone is a closed convex cone, so

$$0^+A \oplus 0^+B = \operatorname{cl}\operatorname{co}(0^+A + 0^+B) \subseteq 0^+(A \oplus B).$$

Since $0 \in 0^+A \cap 0^+B$ implies $0^+A \cup 0^+B \subseteq 0^+A \oplus 0^+B$, also $\operatorname{cl}\operatorname{co}\left(0^+A \cup 0^+B\right) \subseteq 0^+A \oplus 0^+B$ holds true. On the other hand, if $z_A \in 0^+A$ and $z_B \in 0^+B$ are given, then $z_A + z_B \in \operatorname{co}\left(0^+A \cup 0^+B\right)$, hence $\operatorname{cl}\operatorname{co}\left(0^+A \cup 0^+B\right) \supseteq 0^+A \oplus 0^+B$ proves equality.

Finally, let $A \preccurlyeq B$ be satisfied, $B + \{z\} \subseteq B$ and $a + z \notin A$ for some $a \in A$. Then there exists a neighborhood $U \in \mathcal{U}$ such that $\{a + z\} + U \cap A = \emptyset$, as A is closed and thus there exists $t \in (0, 1)$ such that

$$t\left(b + \frac{1}{t}z\right) + (1 - t)a = a + z + t(b - a) \in \{a + z\} + U.$$

But since A is convex and 0^+B is a cone, this implies

$$t\left(b + \frac{1}{t}z\right) + (1 - t)a \in \operatorname{co}(B + A) \subseteq A,$$

a contradiction. □

Moreover, we can remark that for any $A \in \mathcal{G}^{\triangle}$ the following properties hold true

(i) $0^+A \oplus 0^+\emptyset = 0^+(A \oplus \emptyset) = \emptyset$;
(ii) $0^+A \preccurlyeq 0^+\emptyset = \emptyset$.

The inequality $0^+A \oplus 0^+\emptyset \preccurlyeq 0^+A \cup 0^+\emptyset$ is true only if only if $A = \emptyset$.

Lemma 2.9 *If $A \dot{-} B \neq \emptyset$, then*

$$0^+(A \dot{-} B) \preccurlyeq 0^+A \preccurlyeq 0^+B.$$

If additionally $B \neq \emptyset$, then we also get

$$0^+(A \dot{-} B) = 0^+A.$$

Proof Assume $A \dot{-} B \neq \emptyset$. If $B = \emptyset$, then $A \dot{-} B = Z$ and the first equation is immediate. Hence let $B \neq \emptyset$. Then $\emptyset \neq B \oplus (A \dot{-} B) \subseteq A$ and because A is closed and convex by assumption, we can apply Lemma 2.8 to prove

$$0^+B \cup 0^+(A \dot{-} B) \subseteq 0^+(B \oplus \left(A \dot{-} B\right)) \subseteq 0^+A.$$

On the other hand, if $B + \{z\} \subseteq A$, that is $z \in A \dot{-} B$, then for all $z_0 \in 0^+A$ it holds $B + \{z + z_0\} \subseteq A$, hence $0^+A \subseteq 0^+(A \dot{-} B)$. □

2.2 Set-valued Functions

Let X be a linear space. A function $f\colon X \to \mathcal{G}^{\triangle}$ is convex when

$$\forall x_1, x_2 \in X,\ \forall t \in (0,1): f(tx_1 + (1-t)x_2) \preccurlyeq tf(x_1) \oplus (1-t) f(x_2). \tag{2.17}$$

It is an easy exercise (see, for instance, [13]) to show that f is convex if, and only if, the set

$$\operatorname{graph} f = \{(x, z) \in X \times Z\colon z \in f(x)\}$$

is convex. A $\mathcal{G}^{\triangle}$-valued function f is called positively homogeneous when

$$\forall 0 < t, \forall x \in X\colon f(tx) \preccurlyeq tf(x),$$

and it is called sublinear if it is positively homogeneous and convex. It can be shown that f is sublinear if, and only if, graph f is a convex cone. Compare also [2, Definition 2.1.1.] on above definitions.

The (effective) domain of a function $f : X \to \mathcal{G}^{\triangle}$ is the set $\operatorname{dom} f = \{x \in X \mid f(x) \neq \emptyset\}$. Since $\emptyset$ is the supremum of $\mathcal{G}^{\triangle}$, the previous notion of domain of a set-valued function extends the scalar notion of effective domain. The image set of a subset $A \subseteq X$ through f is denoted by

$$f[A] = \left\{ f(x) \in \mathcal{G}^{\triangle} \mid x \in A \right\} \subseteq \mathcal{G}^{\triangle}.$$

We underline that $f[A]$ is a subset of $\mathcal{P}(Z)$ rather then a subset of Z, while $\inf f[A] = \operatorname{cl}\operatorname{co} \bigcup\limits_{a\in A} f(a)$ is an element of $\mathcal{P}(Z)$, hence a subset of Z.

Proposition 2.10 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. If $x \in \operatorname{dom} f$, then $t \mapsto 0^+(f(x + t(x_0 - x)))$ is constant on $(0, 1)$ and $0^+(f(x + t(x_0 - x))) \preccurlyeq 0^+ f(x) \cup 0^+ f(x_0)$ is satisfied for all $t \in (0, 1)$.*

Proof Let $t \in [0, 1]$ and denote $x_t = x + t(x_0 - x)$. By convexity of f, for any $z_0 \in 0^+ f(x_0)$ and $z \in 0^+ f(x)$, $z_t = tz + (1-t)z_0 \in 0^+ f(x_t)$ is satisfied. Since both recession cones contain 0, we have $z_0 + 0 \in 0^+ f(x_t)$ and $z + 0 \in 0^+ f(x_t)$. Therefore $0^+ f(x_t) \supseteq 0^+ f(x_0) \cup 0^+ f(x)$.

Moreover let $0 < s < t < 1$ be given. By replacing x with x_t in above argument we have

$$0^+ f(x_s) \supseteq 0^+ f(x_0) \cup 0^+ f(x_t) = 0^+ f(x_t)$$

and by replacing x_0 by x_s instead we have

$$0^+ f(x_t) \supseteq 0^+ f(x_s) \cup 0^+ f(x) = 0^+ f(x_s),$$

hence $0^+ f(x_s) = 0^+ f(x_t)$ is proven for all $s, t \in (0, 1)$. $\square$

Given a function $f\colon X \to \mathcal{G}^{\triangle}$, the family of extended real-valued functions $\varphi_{f,z^*}\colon X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{f,z^*}(x) = \inf\left\{-z^*(z) \mid z \in f(x)\right\},\ z^* \in C^- \setminus \{0\}$$

is the family of scalarizations of f. Some properties of f are inherited by its scalarizations and vice versa. For instance, f is convex if, and only if, φ_{f,z^*} is convex for each $z^* \in C^- \setminus \{0\}$. In turn, convexity of φ_{f,z^*} is equivalent to convexity of the function $f_{z^*}\colon X \to \mathcal{G}^{\triangle}$ given by

$$f_{z^*}(x) = \left\{z \in Z \mid \varphi_{f,z^*}(x) \le -z^*(z)\right\}.$$

Moreover, a standard separation argument shows that

$$\forall x \in X\colon\ f(x) = \bigcap_{z^* \in C^- \setminus \{0\}} f_{z^*}(x).$$

Remark 2.11 The function $f_{z^*}\colon X \to \mathcal{G}^{\triangle}$ maps x to the sublevel set $L^{\le}_{z^*}(-\varphi_{f,z^*}(x))$ of z^* at level $-\varphi_{f,z^*}(x)$. For all $z^* \in C^- \setminus \{0\}$ and all $x \in X$ it holds

$$f_{z^*}(x) = L^{\le}_{z^*}(\sigma(z^*|f(x))) = \left\{z \in Z \mid z^*(z) \le -\varphi_{f,z^*}(x)\right\}. \tag{2.18}$$

Therefore either $f_{z^*}(x) \in \{\emptyset, Z\}$, or it is a closed affine half space with a supporting point $z \in f_{z^*}(x)$ such that $\varphi_{f,z^*}(x) = -z^*(z)$. If $f(x) \ne \emptyset$, then either $f_{z^*}(x) = Z$, or $\varphi_{f,z^*}(x) \in \mathbb{R}$, thus

$$\forall x \in X:\quad f(x) = \emptyset\ \vee\ f(x) = \bigcap_{\substack{z^* \in C^- \setminus \{0\}:\\ \varphi_{f,z^*}(x) \in \mathbb{R}}} f_{z^*}(x).$$

Definition 2.12 (a) Let $\varphi : X \to \overline{\mathbb{R}}$ be a function, $x_0 \in X$. Then φ is said to be lower semicontinuous (l.s.c.) at x_0, when

$$\forall r \in \mathbb{R}:\quad r < \varphi(x_0)\ \Rightarrow\ \exists U \in \mathcal{U} : \forall u \in U : r < \varphi(x_0 + u).$$

(b) Let $f : X \to \mathcal{G}^{\triangle}$ be a function, $M^* \subseteq C^- \setminus \{0\}$. Then f is said M^*– lower semicontinuous (M^*–l.s.c.) at x_0, when φ_{f,z^*} is l.s.c. at x_0 for all $z^* \in M^*$.

(c) Let $f : X \to \mathcal{G}^{\triangle}$ be a function. If

$$f(x) \preccurlyeq \liminf_{u \to 0} f(x+u) = \bigcap_{U \in \mathcal{U}} \operatorname{cl} \operatorname{co} \bigcup_{u \in U} f(x+u)$$

is satisfied, then f is lattice lower semicontinuous (lattice l.s.c.) at x.

(d) A function $f : X \to \mathcal{G}^{\triangle}$ is lattice l.s.c. when it it is lattice l.s.c. everywhere.

In [19], it has been proven that if f is $\left(C^- \setminus \{0\}\right)$–l.s.c. at x, then it is also lattice l.s.c. at x. One can show that if f is convex, then f is lattice l.s.c. if, and only if, graph $f = \{(x, z) \mid z \in f(x)\} \subseteq X \times Z$ is a closed set with respect to the product topology, see [15].

In [19], a detailed study of continuity concepts for set valued functions is proposed. Indeed it is also shown that none of the concepts in Definition 2.12 coincides with those used in some literature (see e.g. [1, 2, 11]).

Remark 2.13 For notational simplicity the restriction of a set-valued function $f : X \to \mathcal{G}^\triangle$ to a segment with end points $x_0, x \in X$ is denoted by $f_{x_0,x} : \mathbb{R} \to \mathcal{G}^\triangle$ with

$$f_{x_0,x}(t) = \begin{cases} f(x_0 + t(x - x_0)), & \text{if } t \in [0, 1]\,; \\ \emptyset, & \text{elsewhere.} \end{cases}$$

Equivalently, the restriction of a scalar-valued function $\varphi : X \to \overline{\mathbb{R}}$ to the same segment is defined by

$$\varphi_{x_0,x}(t) = \begin{cases} \varphi(x_t), & \text{if } t \in [0, 1]\,; \\ +\infty, & \text{elsewhere.} \end{cases}$$

Setting $x_t = x_0 + t(x - x_0)$ for all $t \in \mathbb{R}$, the scalarization of the restricted function $f_{x_0,x}$ is equal to the restriction of the scalarization of f for all $z^* \in C^- \setminus \{0\}$.

If f is convex, $x_0, x_t \in \operatorname{dom} f$ for some $t \in (0, 1)$, then $\left(\varphi_{f,z^*}\right)_{x_0,x}$ is lower semi-continuous on $(0, t)$, hence $f_{x_0,x}$ is lattice l.s.c. on $(0, t)$.

The following notion, introduced in [15], is used in the sequel.

Definition 2.14 Let $f : X \to \mathcal{G}^\triangle$ be a function and $M \subseteq X$. We define the inf-translation of f by M to be the function $\hat{f}(\cdot; M) : X \to \mathcal{G}^\triangle$ given by

$$\hat{f}(x; M) = \inf f\,[M + \{x\}] = \operatorname{cl}\operatorname{co} \bigcup_{m \in M} f(m + x)\,. \tag{2.19}$$

The function $\hat{f}(\cdot; M)$ is nothing but the canonical extension of f at $M + \{x\}$ as defined in [18]. The following properties of the inf-translation are used in the proofs of the main results.

Lemma 2.15 ([15, Lemma 5.8(b)]) *Let $f : X \to \mathcal{G}^\triangle$ be convex, $M \subseteq X$, then $\hat{f}(\cdot; \operatorname{co} M) : X \to \mathcal{G}^\triangle$ is convex.*

Lemma 2.16 *Let $f : X \to \mathcal{G}^\triangle$, $z^* \in C^- \setminus \{0\}$ and $M \subseteq X$ be nonempty. Then*

$$\forall x \in X : \quad \inf \varphi_{f,z^*}\,[M + \{x\}] = \varphi_{\hat{f}(\cdot;M),z^*}(x).$$

Moreover, by defining $\hat{\varphi}_{f,z^*}(x; M) = \inf \varphi_{f,z^*}\,[M + \{x\}]$, it holds

$$\forall x \in X: \quad \hat{\varphi}_{f,z^*}(x; M) = \varphi_{\hat{f}(\cdot;M),z^*}(x),$$

that is the operations of taking the inf translation of a function and taking its scalarization commute.

Proof The statement is an easy consequence of Lemma 2.2. □

Lemma 2.17 *Let $f : X \to \mathcal{G}^\triangle$ and $M \subseteq X$ be nonempty, then the domain of $\hat{f}(\cdot; M) : X \to \mathcal{G}^\triangle$ is the set*

$$\operatorname{dom} \hat{f}(\cdot; M) = \bigcup_{m \in M} \operatorname{dom} f + \{-m\}. \tag{2.20}$$

Proof Since $x \in \operatorname{dom} \hat{f}(\cdot; M)$ if, and only if, $\inf f[M + \{x\}] \neq \emptyset$, there exists $m \in M$ such that $f(m + x) \neq \emptyset$. Therefore, $x \in \operatorname{dom} \hat{f}(\cdot; M)$ if, and only if, $m + x \in \operatorname{dom} f$ for some $m \in M$. In other words $x \in \bigcup\limits_{m \in M} \operatorname{dom} f + \{-m\}$. □

Lemma 2.18 *Let $f : X \to \mathcal{G}^\triangle$ be convex, $M \subseteq X$ a nonempty set and $z^* \in C^- \setminus \{0\}$, If any of the following conditions is satisfied, then the restriction of $\hat{f}(\cdot; \operatorname{co} M)$ to the segment $[0, x]$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0 for all $x \in X$.*

(a) $\hat{f}(0; M) = \inf f[X]$*;*
(b) $0 \in \operatorname{int} \bigcup\limits_{m \in \operatorname{co} M} (\operatorname{dom} f + \{-m\})$*;*
(c) $\left(\varphi_{f,z^*}\right)_{m,x} : X \to \overline{\mathbb{R}}$ *is continuous at* 0 *for all* $m \in \operatorname{co} M$*,* $x \in X$ *and all* $z^* \in C^- \setminus \{0\}$.

Proof (a) If $\hat{f}(0; M) = \inf f[X]$, then $\varphi_{\hat{f}(\cdot;\operatorname{co} M),z^*}(0) = \inf \varphi_{\hat{f}(\cdot;\operatorname{co} M),z^*}[X]$ is true for all $z^* \in C^- \setminus \{0\}$. Hence each scalarization $\varphi_{\hat{f}(\cdot;\operatorname{co} M),z^*}$ is l.s.c. at 0 and therefore $\hat{f}(\cdot; \operatorname{co} M)$ is $C^- \setminus \{0\}$–l.s.c at 0.

(b) By Lemma 2.17, $\bigcup\limits_{m \in \operatorname{co} M} (\operatorname{dom} f + \{-m\})$ is the domain of $\hat{f}(\cdot; \operatorname{co} M)$ and by Lemma 2.15, $\hat{f}(\cdot; \operatorname{co} M)$ is convex. This is true if, and only if, each scalarization of $\hat{f}(\cdot; \operatorname{co} M)$ i.e. $\left(\hat{\varphi}_{f,z^*}\right)(\cdot; \operatorname{co} M)$ is convex, compare Lemma 2.16. If $0 \in \operatorname{int} \bigcup\limits_{m \in \operatorname{co} M} (\operatorname{dom} f + \{-m\})$ is assumed, then the restriction of each scalarization $\varphi_{f,z^*}(\cdot; \operatorname{co} M)$ to $[x_0, x]$ is l.s.c. at 0, as $\operatorname{dom} \hat{f}(\cdot; \operatorname{co} M) = \operatorname{dom} \left(\hat{\varphi}_{f,z^*}\right)(\cdot; \operatorname{co} M)$.

(c) Let $\left(\varphi_{f,z^*}\right)_{m,x} : X \to \overline{\mathbb{R}}$ be continuous at 0 for all $m \in \operatorname{co} M$ and all $x \in X$. In this case,

$$\begin{aligned}
\limsup_{t \downarrow 0} (\varphi_{\hat{f}(\cdot;\operatorname{co} M),z^*})_{0,x}(t) &= \limsup_{t \downarrow 0} \inf_{m \in \operatorname{co} M} \left(\varphi_{f,z^*}\right)_{m,x}(t) \\
&\leq \inf_{m \in \operatorname{co} M} \limsup_{t \downarrow 0} \left(\varphi_{f,z^*}\right)_{m,x}(t) \\
&= \inf_{m \in \operatorname{co} M} \left(\varphi_{f,z^*}\right)_{m,x}(0) \\
&= \hat{\varphi}_{f,z^*}(0; \operatorname{co} M).
\end{aligned}$$

Hence for each $z^* \in C^- \setminus \{0\}$, the restriction of $\varphi_{f,z^*}(\cdot; \operatorname{co} M)$ to $[0, x]$ is convex and u.s.c. at 0, therefore l.s.c. at 0, too. □

In this framework, we are interested to study the problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in X \tag{P}$$

where f is a $\mathcal{G}^\triangle$-valued function. Following [18], to solve (P) means to look for both the infimum in $\mathcal{G}^\triangle$, as introduced in (2.2), and for subsets of X where the infimum is attained. This approach is different from most other ones in set optimization, see for example [20, Definition 14.2], [16, 17] and the references therein.

More formally, the solution concept based on Definitions 2.19 and 2.23 is stated in Definition 2.25.

Definition 2.19 Let $f : X \to \mathcal{G}^\triangle$. A subset $M \subseteq X$ is called an *infimizer* of f when

$$\inf \{f(m) \mid m \in M\} = \inf \{f(x) \mid x \in X\}.$$

According to the definition of $\hat{f}(\cdot; M) : X \to \mathcal{G}^\triangle$, it follows easily that

$$\forall M \neq \emptyset : \quad \inf \left\{\hat{f}(x; M) \mid x \in X\right\} = \inf \{f(x) \mid x \in X\}$$

and M is an infimizer of f when $\{0\}$ is an infimizer of $\hat{f}(\cdot; M) : X \to \mathcal{G}^\triangle$,

$$\hat{f}(0; M) = \inf \left\{\hat{f}(x; M) \mid x \in X\right\} \quad \Leftrightarrow \quad \inf \{f(m) \mid m \in M\} = \inf \{f(x) \mid x \in X\}.$$

Proposition 2.20 ([15, Proposition 5.9]) *Let $f : X \to \mathcal{G}^\triangle$ be convex and $M \subseteq X$, then the following are equivalent*

(a) M is an infimizer of f;
(b) $\{0\}$ is an infimizer of $\hat{f}(\cdot; M)$;
(c) $\{0\}$ is an infimizer of $\hat{f}(\cdot; \operatorname{co} M)$ and $\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$.

Proposition 2.21 *Let $f : X \to \mathcal{G}^\triangle$ and $x_0 \in \operatorname{dom} f$. Then the following are equivalent*

(a) $f(x_0) = \inf f[X]$;
(b) $\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x)$;
(c) $\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) \mathbin{-^{\!\cdot}} \varphi_{f,z^*}(x) \leq 0$;
(d) $\forall x \in X : 0 \in f(x_0) \mathbin{-^{\!\cdot}} f(x)$.
(e) $\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) = -\infty \ \vee\ 0 \leq \varphi_{f,z^*}(x) \mathbin{-^{\!\cdot}} \varphi_{f,z^*}(x_0)$.

Each of these conditions implies

(f) $\forall x \in X : 0^+ f(x_0) \preccurlyeq f(x) \dot{-} f(x_0)$.

Proof The equivalence between (a), (b), (c) and (e) is immediate. By Lemma 2.3 (c) and (d) are equivalent and by Proposition 2.6, (e) implies (f). □

Remark 2.22 For scalars $a, b \in \mathbb{R}$, $a \le b$ can be equivalently stated as $a - b \le 0$ or $0 \le b - a$. For $A,\ B \in \mathcal{G}^\triangle \setminus \{\emptyset\}$ we have a similar result for the equivalence between $A \preccurlyeq B$ and $A \dot{-} B \preccurlyeq C$ (and actually as '$A \dot{-} B \preccurlyeq 0^+ A$' or $0 \in A \dot{-} B$).

On the other hand, $A \preccurlyeq B$ only implies $0^+ A \preccurlyeq B - A$. Moreover $0^+ B$ is not necessarily equal to C, the neutral element in $\mathcal{G}^\triangle$, but $0^+ A \preccurlyeq C$, whenever $A \neq \emptyset$.

As dom f is always an infimizer of f, further requirements are usually assumed on the values $f(x)$, $x \in M$, for M to be a solution. e.g. $f(x)$ is minimal in some sense, compare [15, 18, 21].

Definition 2.23 Let $f : X \to \mathcal{G}^\triangle$ be given. An element $x_0 \in X$ is called a *minimizer* of f, when $f(x_0)$ is minimal in $f[X]$, i.e.

$$\forall x \in X : \quad f(x) \preccurlyeq f(x_0) \quad \Rightarrow \quad f(x) = f(x_0). \tag{Min}$$

The set of all minimal elements of $f[X]$ is denoted by Min $f[X]$.

If x_0 is a minimizer of a convex (set-valued) function f, then $f(x) = f(x_0)$ is satisfied if, and only if, f is constant on the set $\{x_t \in X \mid x_t = x_0 + t(x - x_0),\ t \in [0, 1]\}$.

Notice that if $M = \{x\}$ is an infimizer, then x automatically is a minimizer of f. On the other hand, a set of minimizers is not necessarily an infimizer. Let $\psi : S \subseteq X \to Z$ and its epigraphical extension $f = \psi^C : X \to \mathcal{G}^\triangle$, defined by

$$f(x) = \begin{cases} \{\psi(x)\} + C, & \text{if } x \in S; \\ \emptyset, & \text{elsewhere.} \end{cases} \tag{2.21}$$

Then $x_0 \in S$ is a minimizer of f if, and only if, it is an efficient element to ψ, i.e. $(\{\psi(x_0)\} + (-C)) \cap \bigcup_{x \in S} \psi(x) \subseteq \{\psi(x_0)\} + C$. A set $M \subseteq X$ is an infimizer if and only if the following domination property is satisfied

$$\bigcup_{x \in X} f(x) \subseteq \operatorname{cl} \operatorname{co} \bigcup_{m \in M} f(m).$$

The next result provides some characterizations of minimizers via scalarizations.

Proposition 2.24 *Let* $f : X \to \mathcal{G}^\triangle$ *and* $x_0 \in$ dom f. *Then the following are equivalent*

(a) $f(x_0) \in \operatorname{Min} f[X]$;
(b) $f(x) \neq f(x_0) \ \Rightarrow\ \exists z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) < \varphi_{f,z^*}(x)$;

(c) $f(x) \neq f(x_0) \;\Rightarrow\; \exists z^* \in C^- \setminus \{0\} : \; \varphi_{f,z^*}(x) \neq -\infty \;\wedge$
$\varphi_{f,z^*}(x_0) \dot{-} \varphi_{f,z^*}(x) < 0$;
(d) $f(x) \neq f(x_0) \;\Rightarrow\; \exists z^* \in C^- \setminus \{0\} : \; 0 < \varphi_{f,z^*}(x) \dot{-} \varphi_{f,z^*}(x_0)$;
(e) $f(x) \neq f(x_0) \;\Rightarrow\; 0 \notin f(x) \dot{-} f(x_0)$.

Proof Equivalences from (a) through (d) are immediate and by Lemma 2.3, (d) and (e) are equivalent. □

Definition 2.25 [18] Let $f : X \to \mathcal{G}^\triangle$. An infimizer of f consisting of only minimizers is called a *solution* of the optimization problem (P).

Example 2.26 Let $f : \mathbb{R} \to \mathcal{G}(\mathbb{R}^2, \mathbb{R}^2_+)$ be given as

$$f(x) = \{(-x, -x)\} \oplus R^2_+.$$

Then $\mathbb{N} \subseteq \mathbb{R}$ as well as any interval $(x, +\infty) \subseteq \mathbb{R}$ are infimizers of f. However, $\operatorname{Min} f\,[\mathbb{R}] = \emptyset$. Hence no solution of f exists.

In [15] the concept of z^*–minimizers was introduced, defining $x_0 \in X$ as a z^*–minimizer of $f : X \to \mathcal{G}^\triangle$ if, and only if, x_0 is a minimizer of $\varphi_{f,z^*} : X \to \overline{\mathbb{R}}$. In fact, this concept is independent from the one we are investigating. The following Example 2.27(a) due to F. Heyde proves that a solution in the sense of Definition 2.25 does not need to be a z^*–solution, while Example 2.27(b) provides a counterexample to the reverse implication.

Example 2.27 (a) Let $X = Z = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. The (closed and convex) function $f : X \to \mathcal{G}^\triangle$ is defined as follows

$$f(x) = \begin{cases} \{z \in -x_1 + x_2 \leq z_1, \; -x_1 - x_2 \leq z_2, \; x_1 \leq z_1 + z_2\}, & \text{if } 0 \leq x_1; \\ \emptyset, & \text{else.} \end{cases}$$

Then each $x_0 \in \operatorname{dom} f$ is minimal and $M = \{x \in X \mid 0 < x_1, x_2\}$ is a solution of (P), while no $x \in M$ is a z^*–solution for any $z^* \in C^- \setminus \{0\}$.

(b) Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. The (closed and convex) function $f : X \to \mathcal{G}^\triangle$ is defined as follows

$$f(x) = \begin{cases} \left\{z \in Z \mid \frac{1}{z_1} \leq z_2\right\}, & \text{if } 0 = x; \\ \{z \in Z \mid 0 \leq z_1, z_2\}, & \text{if } 1 = x; \\ xf(1) \oplus (1-x)f(0), & \text{if } 0 \leq x \leq 1; \\ \emptyset, & \text{else.} \end{cases}$$

Then each $x_0 \in \operatorname{dom} f$ is z^*–minimal with respect to $z^* \in \left\{(0,-1)^T, (-1,0)^T\right\}$, but the only minimizer of f is $x = 1$ and $M = \{1\}$ is the only solution of (P).

2.3 Directional Derivatives

The notions of variational inequalities related to an optimization problem involves the concept of directional derivatives.

We apply the following definition to convex functions $f : X \to \mathcal{G}^{\vartriangle}$ which extends the concept of (lower) Dini derivatives to functions mapping to any inf–residuated image space.

We stress that this approach allows to extend the classical Dini derivative for scalar-valued functions to extended real-valued functions (see e.g. [15, 24]), as discussed in Example 2.33 below.

Definition 2.28 Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex, $x, u \in X$, then the directional derivative of f at x along direction u is defined as

$$f'(x,u) = \liminf_{t \downarrow 0} \frac{1}{t}\left(f(x+tu) \mathbin{\dot{-}} f(x)\right) = \bigcap_{0<t_0} \operatorname{cl}\operatorname{co} \bigcup_{t \in (0,t_0)} \frac{1}{t}\left(f(x+tu) \mathbin{\dot{-}} f(x)\right).$$

For convex (set-valued) functions, the differential quotient is monotone.

Proposition 2.29 *Let* $f : X \to \mathcal{G}^{\vartriangle}$ *be convex,* $x_0 \in X$ *and* $g : (0, +\infty) \to \mathcal{G}^{\vartriangle}$ *be given by* $g(t) = \frac{1}{t}\left(f(x+tu) \mathbin{\dot{-}} f(x)\right)$. *Then for all* $0 < s \leq t$ *it holds* $g(s) \preccurlyeq g(t)$.

Proof Let $z_t \in g(t)$ and $0 < s < t$ be satisfied, then there exists an $r \in (0, 1)$ such that $s = rt$ and $f(x+su) \preccurlyeq (1-r)f(x) \oplus rf(x+tu)$. Thus,

$$f(x+su) \mathbin{\dot{-}} f(x) \preccurlyeq r(f(x+tu) \mathbin{\dot{-}} f(x)),$$

which in turn implies that

$$\frac{1}{s}\left(f(x+su) \mathbin{\dot{-}} f(x)\right) \preccurlyeq \frac{r}{rt}(f(x+tu) \mathbin{\dot{-}} f(x)),$$

as desired. □

The following result extends a well known property of Dini derivatives for convex single-valued functions.

Proposition 2.30 *Let* $f : X \to \mathcal{G}^{\vartriangle}$ *be convex,* $x \in \operatorname{dom} f$ *and* $u \in X$. *Then*

$$f'(x,u) = \inf_{0<t} \frac{1}{t}\left(f(x+tu) \mathbin{\dot{-}} f(x)\right),$$

$f'(x, 0) = 0^+ f(x)$ *and the function* $u \mapsto f'(x, u)$ *is sublinear as a function from* X *to* $\mathcal{G}(Z, 0^+ f(x))$.

Proof The first statement comes directly from Proposition 2.29.

For all $x \in X$, $f'(x,0) = \inf \frac{1}{t}\left(f(x) -^{\!\cdot} f(x)\right)$ and thus

$$f'(x,0) = \begin{cases} 0^+ f(x) & \text{, if } x \in \operatorname{dom} f; \\ Z & \text{, elsewhere.} \end{cases}$$

By definition, for all $0 < s, u \in X$ it holds

$$f'(x, su) = s \cdot \inf_{0<t} \frac{1}{st}\left(f(x + tsu) -^{\!\cdot} f(x)\right) = sf'(x,u).$$

Let $x, u_1, u_2 \in X$ and $s \in (0,1)$ be assumed. By Proposition 2.29 the differential quotient is decreasing, so for all $0 < t_0$ it holds

$$f'(x, su_1 + (1-s)u_2) = \inf_{0<t\le t_0} \frac{1}{t}\left(f(s(x+tu_1) + (1-s)(x+tu_2)) -^{\!\cdot} f(x)\right).$$

Convexity and Lemma 2.5 (b) imply

$$f'(x, su_1 + (1-s)u_2) \preccurlyeq \inf_{0<t\le t_0} \frac{1}{t}\left(s\left(f(x+tu_1) -^{\!\cdot} f(x)\right) \oplus (1-s)\left(f(x+tu_2) -^{\!\cdot} f(x)\right)\right).$$

Since $\mathcal{G}^{\vartriangle}$ is inf–residuated and by Proposition 2.29,

$$\begin{aligned} f'(x, su_1 + (1-s)u_2) &\preccurlyeq \frac{1}{t_0}\left(s\left(f(x+t_0u_1) -^{\!\cdot} f(x)\right)\right) \oplus (1-s) \inf_{0<t\le t_0} \frac{1}{t}\left(f(x+tu_2) -^{\!\cdot} f(x)\right) \\ &= s\frac{1}{t_0}\left(\left(f(x+t_0u_1) -^{\!\cdot} f(x)\right)\right) \oplus (1-s)f'(x,u_2). \end{aligned}$$

But, as this is true for all $0 < t_0$ and $\mathcal{G}^{\vartriangle}$ is inf–residuated,

$$f'(x, su_1 + (1-s)u_2) \preccurlyeq sf'(x,u_1) \oplus (1-s)f'(x,u_2)$$

is satisfied. □

Remark 2.31 Since the differential quotients $\frac{1}{t}\left(f(x+tu) -^{\!\cdot} f(x)\right)$ of a convex function $f : X \to \mathcal{G}^{\vartriangle}$ form a decreasing net of convex sets, their union is convex. Therefore in this case the following equation holds true.

$$f'(x,u) = \operatorname{cl}\operatorname{co} \bigcup_{t>0} \frac{1}{t}\left(f(x+tu) -^{\!\cdot} f(x)\right) = \operatorname{cl} \bigcup_{t>0} \frac{1}{t}\left(f(x+tu) -^{\!\cdot} f(x)\right)$$

Remark 2.32 Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex, $x_0 \in \operatorname{dom} f$ and $x \in X$.

If $f'(x_0, x - x_0) \neq \emptyset$, then $[0, t_0] \subseteq \operatorname{dom} f_{x_0,x}$ is true for some $t_0 \in (0,1)$ and for all $t \in (0, t_0)$ it holds

$$0^+ f'(x_0, x - x_0) \preccurlyeq 0^+ f(x_t) \preccurlyeq 0^+ f(x_0).$$

Indeed, as f is convex, $0^+ f(x_t)$ is constant on the set $(0, t_0)$ and $0^+ f(x_t) \preccurlyeq 0^+ f(x_0)$. Also,

$$f'(x_0, x - x_0) \preccurlyeq \frac{1}{t} \left(f(x_t) \dot{-} f(x_0) \right)$$

and both sets are convex, hence $0^+ f'(x_0, x - x_0) \preccurlyeq 0^+ f(x_t)$ by Lemma 2.9.

Example 2.33 Let $\varphi : X \to \overline{\mathrm{IR}}$ be convex, $f : X \to \mathcal{G}(\mathrm{IR}, \mathrm{IR}_+)$ its epigraphical extension as defined in (2.21). If $\varphi : X \to \overline{\mathrm{IR}}$ is proper, $x \in \operatorname{dom} \varphi$, then $f'(x, u)$ coincides with the upper Dedekind cut of the classic directional derivative of φ, while in general,

$$f'(x, u) = \left(\inf_{0<t} \frac{1}{t} \left(\varphi(x + tu) \dot{-} \varphi(x) \right) \right) + \mathrm{IR}_+.$$

Especially, if $\varphi(x) = +\infty$, then $f'(x, u) = \mathrm{IR}$ for all $u \in X$, while if $x \in \operatorname{dom} \varphi$ and $\varphi(x) = -\infty$, then a careful case study provides

$$f'(x, u) = \begin{cases} \mathrm{IR}, & \text{if } u \in \text{cone } (\operatorname{dom} \varphi - \{x\}) \\ \emptyset, & \text{else.} \end{cases}$$

Therefore

$$\varphi'(x, u) = \inf_{0<t} \frac{1}{t} \left(\varphi(x + tu) \dot{-} \varphi(x) \right)$$

for all $x, u \in X$ provides an extension of Dini derivatives to the case where φ is improper or $x \notin \operatorname{dom} \varphi$.

Remark 2.34 Let $f : X \to \mathcal{G}^\vartriangle$ be convex. It is easy to see that if $x \notin \operatorname{dom} f$, then $f'(x, u) = Z$ and $\varphi'_{f,z^*}(x, u) = -\infty$ are satisfied for all $u \in X$ and all $z^* \in C^- \setminus \{0\}$.

On the other hand, if $x \in \operatorname{dom} f$, then $\operatorname{dom} \varphi'_{f,z^*}(x, \cdot) = \text{cone } \{\operatorname{dom} f + \{-x\}\} \cup \{0\}$ is true for all $z^* \in C^- \setminus \{0\}$ and the derivative is sublinear. Hence, $\varphi'_{f,z^*}(x, u) = -\infty$ implies either $\varphi_{f,z^*}(x) = -\infty$, or $\varphi'_{f,z^*}(x, -u) = +\infty$.

Especially, $\operatorname{dom} f'(x, \cdot) \subseteq \operatorname{dom} \varphi'_{f,z^*}(x, \cdot)$ is always satisfied. Hence if $\varphi_{f,z^*}(x) \in \mathrm{IR}$, then either $x - tu \notin \operatorname{dom} f$ for all $0 < t$, or $-\infty < \varphi'_{f,z^*}(x, u) \leq \varphi_{f'(x,\cdot),z^*}(u)$.

If for some $z^* \in C^- \setminus \{0\}$ it holds $f(x) = f_{z^*}(x)$ for all $x \in X$ and f is convex, then the scalarization of the derivative is equal to the derivative of the scalarization, $\varphi_{f'_{z^*}(x,\cdot),z^*}(u) = \varphi'_{f,z^*}(x, u)$ for all $x, u \in X$. However, in general only the following inequality can be proven

$$\forall z^* \in C^- \setminus \{0\}, \ \forall x, u \in X : \quad \varphi'_{f,z^*}(x, u) \leq \varphi_{f'(x,\cdot),z^*}(u).$$

Example 2.35 Let $f : \mathbb{R} \to \mathcal{G}(\mathbb{R}, \{0\})$ be defined as $f(x) = \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right]$, whenever $x \in [-1, 1]$ and $f(x) = \emptyset$, else. Then $f(0) + \{z\} \not\subseteq f(t)$ for any $t \neq 0$, so $f'(0, u) = \emptyset$. On the other hand, $\varphi_{f,s}(x) = -|s| \cdot \sqrt{1-x^2}$ for all $s \neq 0$ and thus $\varphi'_{f,s}(x, u) = -|s| \cdot \frac{x}{\sqrt{1-x^2}} \cdot u$ for all $x \in (-1, 1)$, especially $\varphi'_{f,s}(0, u) = 0$ for all $s \neq 0$. Hence,

$$\emptyset = f'(0, u) \subsetneq \bigcap_{z^* \in (\{0\})^- \setminus \{0\}} f'_{z^*}(0, u) = \{0\}$$

Proposition 2.36 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex and $x, u \in X$. Then*

$$\bigcap_{z^* \in C^- \setminus \{0\}} f'_{z^*}(x, u) \preccurlyeq f'(x, u);$$
$$\forall z^* \in C^- \setminus \{0\} : \quad \varphi'_{f,z^*}(x, u) \leq \varphi_{f'(x,\cdot),z^*}(u).$$

Proof By definition and Lemmas 2.3 and 2.2,

$$\begin{aligned} f'(x, u) &= \operatorname{cl} \operatorname{co} \bigcup_{0<t} \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \mid \frac{1}{t} \left(\varphi_{f,z^*}(x + tu) \dot{-} \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\} \\ &\subseteq \bigcap_{z^* \in C^- \setminus \{0\}} \operatorname{cl} \operatorname{co} \bigcup_{0<t} \left\{ z \in Z \mid \frac{1}{t} \left(\varphi_{f,z^*}(x + tu) \dot{-} \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\} \\ &= \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \mid \inf_{0<t} \frac{1}{t} \left(\varphi_{f,z^*}(x + tu) \dot{-} \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\} \end{aligned}$$

hence the inclusion is proven, implying the inequality as well. □

In the sequel, some results require equality in at least one of the inequalities in Proposition 2.36. By *strong regularity*, we refer to condition

$$\forall z^* \in C^- \setminus \{0\} : \quad \varphi_{f'(x,\cdot),z^*}(u) = \varphi'_{f,z^*}(x, u) \tag{SR}$$

and by *weak regularity* to the following condition.

$$f'(x, u) = \bigcap_{z^* \in C^- \setminus \{0\}} f'_{z^*}(x, u) \tag{WR}$$

Clearly, (SR) implies (WR).

3 Main Results

As our solution concept involves both attainment of the infimum in a set and minimality of each element in this set, we need suitable inequalities for each of these properties. Beginning with the infimizer's part, we need to consider that the solution of a variational inequality is usually a singleton, while the infimizer of (P) is a set. However, Proposition 2.20 allows to characterize an infimizer M by proving $\hat{f}(0; M) = \inf f[X]$, or in other words $\{0\}$ is a single-valued infimizer of the optimization problem

$$\text{minimize} \quad \hat{f}(x; M) \quad \text{subject to} \quad x \in X.$$

Given a single-valued convex function $\varphi : X \to \overline{\mathbb{R}}$, a solution to a variational inequality of Stampacchia type is a point $x_0 \in X$ such that $0 \le \varphi'(x_0, x - x_0)$ for all $x \in X$. According to our setting, a natural extension of this property is given in the following definition.

Definition 3.1 Let $f : X \to \mathcal{G}^\triangle$ be convex and $x_0 \in \operatorname{dom} f$. Then x_0 solves the strict set-valued Stampacchia inequality when

$$\forall x \in X : \quad 0^+ f(x_0) \preccurlyeq f'(x_0, x - x_0). \qquad (SVI_I)$$

However, it turns out that, in the set–valued case, infimizers (and minimizers) are often characterized more adequately if a scalar type of variational inequalities is considered.

Definition 3.2 Let $f : X \to \mathcal{G}^\triangle$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the strict scalarized Stampacchia inequality when

$$\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \quad \varphi_{f,z^*}(x_0) = -\infty \ \vee\ 0 \le \varphi'_{f,z^*}(x_0, x - x_0). \qquad (svi_I)$$

Scalarized and set-valued variational inequalities are not equivalent without further assumptions.

Proposition 3.3 *Let $f : X \to \mathcal{G}^\triangle$ be convex, $x_0 \in \operatorname{dom} f$. If x_0 solves (svi_I), then it also solves (SVI_I). If additionally the strong regularity condition (SR) is satisfied, then the reverse implication is true as well.*

Proof By Proposition 2.36, (svi_I) implies

$$\bigcap_{\substack{z^* \in C^- \setminus \{0\} \\ \varphi_{(f,z^*)}(x_0) \ne -\infty}} \left\{ z \in Z \mid 0 \le -z^*(z) \right\} \preccurlyeq \bigcap_{z^* \in C^- \setminus \{0\}} f'_{z^*}(x_0, x - x_0) \preccurlyeq f'(x_0, x - x_0).$$

By (2.16) this implies (SVI_I) as $\operatorname{dom} f = \operatorname{dom} \varphi_{(f,z^*)}$ is true for all $z^* \in C^- \setminus \{0\}$. On the other hand, by Proposition 2.6, (SR) combined with (SVI_I) implies (svi_I). □

Theorem 3.4 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves (svi_I) if and only if $f(x_0) = \inf f[X]$.*

Proof By Proposition 2.21 (e), $f(x_0) = \inf f[X]$ is true if, and only if,

$$\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \quad \varphi_{f,z^*}(x_0) = -\infty \ \vee\ 0 \le \varphi_{f,z^*}(x) \dot{-} \varphi_{f,z^*}(x_0),$$

which immediately implies (svi_I). The opposite implication is true, as convexity of f implies φ_{f,z^*} is convex. □

Remark 3.5 According to Proposition 3.3 and Theorem 3.4, the set-valued variational inequality (SVI_I) is a necessary condition for $\{x_0\}$ to be an infimizer of f. Under the regularity condition (SR) it is also a sufficient condition.

Given a single-valued convex function $\varphi : X \to \overline{\mathbb{R}}$, a solution to a variational inequality of Minty type is a point $x_0 \in X$ such that $\varphi'(x, x_0 - x) \le 0$ for all $x \in X$.

Definition 3.6 Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the strict set-valued Minty inequality when

$$\forall x \in X : \quad f'(x, x_0 - x) \preccurlyeq 0^+ f(x_0). \qquad (MVI_I)$$

Equivalently, x_0 is a solution to the strict set-valued Minty inequality if, and only if,

$$\forall x \in X : \quad 0 \in f'(x, x_0 - x).$$

The previous definition can be related to the following family of a scalar Minty inequalities.

Definition 3.7 Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the strict scalarized Minty inequality when

$$\forall x \in X,\ \forall z^* \in C^- \setminus \{0\} : \quad \varphi'_{f,z^*}(x, x_0 - x) \le 0. \qquad (mvi_I)$$

Proposition 3.8 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. If x_0 solves (MVI_I), then it also solves (mvi_I). If additionally the regularity condition (WR) is satisfied, the reverse implication holds true.*

Proof If x_0 solves (MVI_I), then Proposition 2.36 implies (mvi_I). On the other hand, assuming (mvi_I) and the regularity condition (WR), then $0 \in f'(x, x_0 - x)$ is satisfied for all $x \in X$, in other words (MVI_I). □

Theorem 3.9 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then $f(x_0) = \inf f[X]$ if, and only if, x_0 solves (MVI_I) and for all $x \in X$ the function $f_{x_0,x} : [0, 1] \to \mathcal{G}^{\triangle}$ is lattice l.s.c. at 0.*

If x_0 solves (mvi_I) and for all $x \in X$ the function $f_{x_0,x}$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0, then $f(x_0) = \inf f[X]$.

Proof By Proposition 2.21 (d), $f(x_0) = \inf f[X]$ if, and only if, $0 \in f(x_0) \dot{-} f(x)$ for all $x \in X$, hence by the monotonicity of the differential quotient (see Proposition 2.29)

$$f'(x, x_0 - x) \preccurlyeq f(x_0) \dot{-} f(x) \preccurlyeq 0^+ f(x_0)$$

is satisfied, proving (MVI_I). When $f(x_0) \preccurlyeq f(x)$ is assumed,

$$f(x_0) \preccurlyeq \bigcap_{t_0 \in (0,1)} \operatorname{cl} \operatorname{co} \bigcup_{t \in (0, t_0)} f_{x_0,x}(t)$$

is satisfied and hence $f_{x_0,x}$ is lattice l.s.c. at 0 for all $x \in X$.

On the other hand, (MVI_I) combined with convexity of f implies

$$\forall x \in X, \forall s, t \in (0, 1]: \quad s < t \;\Rightarrow\; f(x_s) \preccurlyeq f(x_t).$$

Hence if $f_{x_0,x}$ is lattice l.s.c. at 0, then we obtain

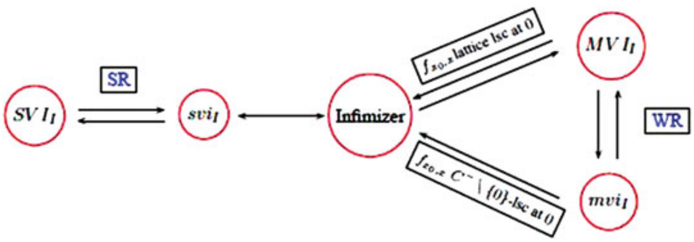
$$\forall x \in X: \quad f(x_0) = \inf f_{x_0,x}[0, 1] \preccurlyeq f(x)$$

and $f(x_0) = \inf f[X]$ is proven.

The proof of the last implication goes along the same lines. □

Recall that if $f_{x_0,x} : [0, 1] \to \mathcal{G}^{\vartriangle}$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0 for all $x \in X$, then each such function is also lattice l.s.c. at 0. In this case, (MVI_I) and (mvi_I) are equivalent.

Remark 3.10 The previous results are summarized in the following scheme of relations.



Applying the previous relations and the inf–translation we get a variational characterization of a set M to be an infimizer of f.

Corollary 3.11 *Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex, $M \subseteq X$ a set with $M \cap \operatorname{dom} f \neq \emptyset$ and*

$$\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M).$$

Then, M is an infimizer of f if, and only if, (svi_I) is satisfied at 0 for $\hat{f}(\cdot; \operatorname{co} M)$. In this case, $\hat{f}(\cdot; \operatorname{co} M)$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0 and (MVI_I) (and (mvi_I)) is satisfied at 0 for $\hat{f}(\cdot; \operatorname{co} M)$.

On the other hand, if (MVI_I) *(or* (mvi_I)*) is satisfied at* 0 *for* $\hat{f}(\cdot; \operatorname{co} M)$ *and one of the conditions* (b) *or* (c) *of Lemma 2.18 is satisfied, then* $\hat{f}(\cdot; \operatorname{co} M)$ *is* $\left(C^- \setminus \{0\}\right)$*–l.s.c. at* 0 *and* M *is an infimizer of* f*.*

In the remainder of this section, we deal with the relation between solutions of variational inequalities and minimizers. The variational inequalities of Stampacchia, as well as Minty type are presented both in a set-valued and a scalar(ized) form.

Definition 3.12 Let $f : X \to \mathcal{G}^\triangle$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the set-valued Stampacchia inequality when

$$f(x_0) = Z \ \vee \ \forall x \in \operatorname{dom} f : \ f(x) \neq f(x_0) \ \Rightarrow \ 0 \notin f'(x_0, x - x_0). \qquad (SVI_M)$$

Remark 3.13 In (SVI_M), the condition '$0 \notin f'(x_0, x - x_0)$' provides a set-valued version of the property '$\varphi'(x_0, x - x_0) \not\leq 0$' for scalar convex functions. The same inequality could be expressed also by the condition

$$f(x_0) = Z \ \vee \ \forall x \in \operatorname{dom} f : \ f(x) \neq f(x_0) \ \Rightarrow \ f'(x_0, x - x_0) \cap -0^+ f(x_0) = \emptyset. \tag{3.1}$$

However, since $\mathcal{G}^\triangle$ is not totally ordered, there is a notable difference between these and the condition $f'(x_0, x - x_0) \subset 0^+ f(x_0)$.

Definition 3.14 Let $f : X \to \mathcal{G}^\triangle$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the scalarized Stampacchia inequality, when

$$f(x_0) = Z \ \vee \ \forall x \in \operatorname{dom} f : \ f(x) \neq f(x_0) \ \Rightarrow \ \exists z^* \in C^- \setminus \{0\} : \ 0 < \varphi'_{f,z^*}(x_0, x - x_0). \qquad (svi_M)$$

Property (svi_M) also implies

$$\begin{cases} \forall x \in \operatorname{dom} f : \quad f(x_0) \neq f(x) \quad \Rightarrow \\ \exists z^* \in C^- \setminus \{0\} : \quad -\infty = \varphi_{f,z^*}(x_0) < \varphi_{f,z^*}(x) \ \vee \ 0 < \varphi'_{f,z^*}(x_0, x - x_0). \end{cases} \tag{3.2}$$

If additionally $f_{x_0,x} : \mathbb{R} \to \overline{\mathbb{R}}$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 1 for all $x \in X$, then (svi_M) and (3.2) are equivalent.

Proposition 3.15 *Let* $f : X \to \mathcal{G}^\triangle$ *be convex,* $x_0 \in \operatorname{dom} f$*. If* x_0 *solves* (svi_M)*, then it also solves* (SVI_M)*. If additionally the regularity condition* (WR) *is satisfied, then* x_0 *solves* (SVI_M) *if, and only if, it solves* (svi_M)*.*

Proof By Proposition 2.36, (svi_M) implies (SVI_M).

On the other hand, (SVI_M) combined with the regularity condition (WR) implies (svi_M). □

For the sake of completeness, we quote [15, Proposition 5.5], where it is proven that, if dom $f \neq \emptyset$, then

$$f_{z^*}(x) = Z \quad \vee \quad \forall x \in X: \quad 0 \leq (\varphi_{f,z^*})'(x_0, x - x_0)$$

is equivalent to $f_{z^*}(x_0) = \inf f_{z^*}[X]$. However, as it has already been shown in Example 2.27, this concept of optimality is not equivalent to the one investigated in this paper.

Theorem 3.16 *Let $f : X \to \mathcal{G}^{\triangle}$ be convex and $x_0 \in \operatorname{dom} f$. If x_0 solves (SVI_M) or (3.2), then $f(x_0) \in \operatorname{Min} f[X]$.*

Proof Let x_0 be a solution of (SVI_M), then

$$f(x) \neq f(x_0) \quad \Rightarrow \quad 0 \notin f(x) \dot{-} f(x_0)$$

is immediate, hence by Proposition 2.24 (e) x_0 is a minimizer of f. Assuming (3.2) is satisfied, then

$$f(x) \neq f(x_0) \quad \Rightarrow \quad \exists z^* \in C^- \setminus \{0\}: \ 0 < \varphi_{f,z^*}(x) \dot{-} \varphi_{f,z^*}(x_0)$$

is satisfied for all $x \in \operatorname{dom} f$, by Proposition 2.24 (d) implying $f(x_0) \in \operatorname{Min} f[X]$. □

The reverse implication of Theorem 3.16 is not true, as the following example illustrates.

Example 3.17 Let $\psi : \mathbb{R} \to \mathbb{R}$ be given as $\psi(x) = 1$ whenever $-1 \leq x \leq 1$ and $\psi(x) = |x|$, elsewhere, $f : X \to \mathcal{G}(\mathbb{R}, \mathbb{R}_+)$ its epigraphical extension. The negative dual cone of $C = \mathbb{R}_+$ is the set cone $(\{-1\}) \cup \{0\}$ and $\varphi_{f,z^*}(x) = -z^*\psi(x)$ for all $z^* \in C^- \setminus \{0\}$. Notably, it is sufficient to consider the single scalarization $\varphi_{f,-1} : \mathbb{R} \to \overline{\mathbb{R}}$ with $\varphi_{f,-1}(x) = \psi(x)$ for all $x \in \mathbb{R}$ and (SR) is satisfied. It holds $f(0) \in \operatorname{Min} f[X]$, but neither (SVI_M) nor (3.2) are satisfied, as $\psi'(0, -x) = 0$ and $f'(0, -x) = \mathbb{R}_+$ holds for all $x \in \mathbb{R}$.

In a similar way, we approach the Minty type inequalities.

Definition 3.18 Let $f : X \to \mathcal{G}^{\triangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the set-valued Minty inequality when

$$f(x) \neq f(x_0) \quad \Rightarrow \quad 0^+ f(x) \not\preccurlyeq f'(x, x_0 - x). \qquad (MVI_M)$$

Again, (MVI_M) can be interpreted as the set-valued version of the scalar Minty variational inequality, given by

$$\varphi(x) \neq \varphi(x_0) \quad \Rightarrow \quad 0 \not\leq \varphi'(x, x_0 - x),$$

but it is significantly different from the condition $0^+ f(x) \subset f'(x, x_0 - x)$, as $\mathcal{G}^{\vartriangle}$ is not totally ordered.

Definition 3.19 Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex, $x_0 \in \operatorname{dom} f$. Then x_0 solves the scalarized Minty inequality when

$$f(x) \neq f(x_0) \quad \Rightarrow \quad \exists z^* \in C^- \setminus \{0\} : \quad \varphi_{f,z^*}(x) \neq -\infty \wedge \varphi'_{f,z^*}(x, x_0 - x) < 0. \tag{mvi_M}$$

Proposition 3.20 *Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex, $x_0 \in \operatorname{dom} f$. If x_0 solves (MVI_M), then it also solves (mvi_M). If additionally the regularity condition (SR) is satisfied, then x_0 solves (MVI_M) if, and only if, it solves (mvi_M).*

Proof If x_0 solves (MVI_M), then Proposition 2.36 implies (mvi_I). On the other hand, assuming (mvi_M) and (SR), then for all $x \in X$ with $f(x) \neq f(x_0)$ there exists an element $z \in f'(x, x_0 - x) \setminus 0^+ f(x)$ (compare Proposition 2.6 and Remark 2.34), in other words (MVI_M) is satisfied. □

Proposition 3.21 *Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex and $x_0 \in \operatorname{dom} f$. Then x_0 solves (mvi_M) if, and only if, for all $x \in \operatorname{dom} f$*

$$f(x) \neq f(x_0) \quad \Rightarrow \quad \inf f_{x_0,x}(0, 1) \preccurlyeq f(x) \wedge \inf f_{x_0,x}(0, 1) \neq f(x). \tag{3.3}$$

Proof Let x_0 be a solution of (mvi_M). This is equivalent to state that for each $x \in \operatorname{dom} f$ with $f(x) \neq f(x_0)$ there exists an element $z^* \in C^- \setminus \{0\}$ and $t \in (0, 1)$ such that $\varphi_{f,z^*}(x_t) \dot{-} \varphi_{f,z^*}(x) < 0$ and $\varphi_{f,z^*}(x) \neq -\infty$, or equivalently $\varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x)$.

In this case, (3.3) is immediate, as

$$\inf f_{x_0,x}(0, 1) \preccurlyeq \bigcap_{t_0 \in (0,1)} \operatorname{cl} \bigcup_{t \in (t_0,1)} f_{x_0,x}(t) \preccurlyeq f(x)$$

by convexity and $\inf f_{x_0,x}(0, 1) \preccurlyeq f(x_t)$, hence strict inclusion is satisfied.

On the other hand, (3.3) implies that, if $f(x) \neq f(x_0)$, then there exists $t \in (0, 1)$ and $z^* \in C^- \setminus \{0\}$ such that $\varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x)$. Hence $\varphi_{f,z^*}(x) \neq -\infty$ and $\varphi'_{f,z^*}(x, x_0 - x) < 0$ are satisfied, as the scalarization $\varphi_{f,z^*} : X \to \overline{\mathrm{IR}}$ is convex. □

Theorem 3.22 *Let $f : X \to \mathcal{G}^{\vartriangle}$ be convex and $x_0 \in \operatorname{dom} f$. If $f(x_0) \in \operatorname{Min} f[X]$, then x_0 solves (mvi_M). If x_0 solves*

$$f(x) \neq f(x_0) \quad \Rightarrow \quad \exists z^* \in M^* : \quad \varphi_{f,z^*}(x) \neq -\infty \wedge \varphi'_{f,z^*}(x, x_0 - x) < 0 \tag{3.4}$$

where $M^* \subseteq C^- \setminus \{0\}$ is a finite set. If additionally $f_{x_0,x}$ is M^*–l.s.c. at 0, then $f(x_0) \in \operatorname{Min} f[X]$.

Proof Let $f(x_0) \in \operatorname{Min} f[X]$ be assumed, then by Proposition 2.24 (c)

$$f(x) \neq f(x_0) \quad \Rightarrow \quad \exists z^* \in C^- \setminus \{0\}: \ \varphi_{f,z^*}(x) \neq -\infty \ \wedge \ \varphi_{f,z^*}(x_0) \mathbin{-^{\!\cdot}} \varphi_{f,z^*}(x) < 0.$$

As the differential quotient is decreasing, this implies (mvi_M).

On the other hand, let (3.4) be satisfied and let $\left(\varphi_{f,z^*}\right)_{x,x_0} : [0,1] \to \overline{\mathbb{R}}$ be l.s.c. at 0 for all $z^* \in M^*$. Then $f(x) \neq f(x_0)$ and convexity and lower semicontinuity of the scalarizations imply that there exist $z^* \in M^*$ and $t \in [0,1)$ such that

$$\inf \left(\varphi_{f,z^*}\right)_{x_0,x} [0,1] = \varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x).$$

Now either $f(x_t) = f(x_0)$ and $f(x) \not\preccurlyeq f(x_0)$, or there exist $t_1 \in [0,t)$ and $z_1^* \in M^* \setminus \{z^*\}$ such that

$$\inf \left(\varphi_{f,z_1^*}\right)_{x_0,x} [0,1] = \varphi_{f,z_1^*}(x_{t_1}) < \varphi_{f,z_1^*}(x_t) \leq \varphi_{f,z_1^*}(x).$$

Especially,

$$\begin{aligned} &\varphi_{f,z^*}(x_t) = -\infty \ \vee \ 0 \leq \varphi'_{f,z^*}(x_t, x_0 - x) \\ &\varphi_{f,z_1^*}(x) \neq -\infty \ \wedge \ \varphi'_{f,z_1^*}(x, x_0 - x) < 0 \end{aligned}$$

are satisfied. As M^* is finite, there exists $t_0 \in [0,1)$ such that

$$\begin{aligned} \exists z_0^* \in M^* &: \ \inf \left(\varphi_{f,z_0^*}\right)_{x_0,x} [0,1] = \varphi_{f,z_0^*}(x_{t_0}) < \varphi_{f,z_0^*}(x); \\ \forall z^* \in M^* &: \ 0 \leq \varphi'_{f,z^*}(x_{t_0}, x_0 - x) \ \vee \ \varphi_{f,z^*}(x_{t_0}) = -\infty. \end{aligned}$$

Hence especially $f(x_{t_0}) = f(x_0)$ and $f(x) \not\preccurlyeq f(x_0)$. □

Property (3.4) implies (mvi_M), as the relevant set of directions M^* is a subset of $C^- \setminus \{0\}$. The reverse implication does not hold and the finiteness assumption in Theorem 3.22 cannot be relaxed, as the following example shows.

Example 3.23 Define $z_i^* = -\frac{1}{i+1}(1,i)^T \in (R^2_+)^- \setminus \{0\}$ for all $i \in \mathbb{N} = \{0,1,2,\ldots\}$. Let $f\colon \mathbb{R} \to \mathcal{G}(\mathbb{R}^2, \mathbb{R}^2_+)$ be defined by

$$\forall x \in \mathbb{R}: \quad f(x) = \bigcap_{i \in \mathbb{N}} \left\{ z \in Z \mid -\psi_{z_i^*}(x) \leq -z_i^*(z) \right\}$$

where

$$\psi_{z_i^*}(x) = \begin{cases} -(i+1)\min\{1-x, ix\} & : \text{if } x \in [0,1] \text{ and } i \in \mathbb{N}; \\ +\infty & : \text{ elsewhere.} \end{cases}$$

As $\psi_{z_i^*} : \mathbb{R} \to \overline{\mathbb{R}}$ is convex and l.s.c. for all $i \in \mathbb{N}$, f is $\left(C^- \setminus \{0\}\right)$–l.s.c. and convex, and it is easy to see that $f(0) = f(1) = \mathbb{R}^2_+$.

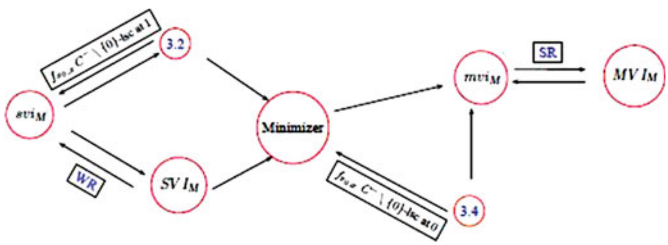
Defining $z_i(x) \in \mathbb{R}^2$ by

$$\forall i \in \mathbb{N} \setminus \{0\} : \; \{z_i(x)\} = \left\{z \in Z \mid z_{i-1}^*(z) = \varphi_{z_{i-1}^*}(x)\right\} \cap \left\{z \in Z \mid z_i^*(z) = \varphi_{z_i^*}(x)\right\}$$

then $f(x) = \text{co}\,\{z_i(x) \mid i \in \mathbb{N} \setminus \{0\}\} + C$ is true for all $x \in (0, 1)$. This implies that $\varphi_{f,z_i^*}(x) = \psi_{z_i^*}(x)$ is true for all $x \in [0, 1]$ and all $i \in \mathbb{N}$ and therefore $f(x) \supsetneq f(0)$ is satisfied for all $x \in (0, 1)$ so 0 is no minimizer of f.

On the other hand, for any given $x \in (0, 1)$, it exists an $i \in \mathbb{N} \setminus \{0\}$ such that $x \in \left(\frac{1}{i+1}, 1\right)$, hence $\varphi'_{f,z_i^*}(x, 0 - x) = -(i + 1) < 0$ and $-i \le \varphi_{f,z_i^*}(x) \ne -\infty$. Hence the assumptions of Theorem 3.22 are satisfied for $x_0 = 0$, replacing the finite set M^* by $C^- \setminus \{0\}$, although 0 is no minimizer of f.

Remark 3.24 The previous results can be summarized in the following scheme of relations.



4 Application to Vector Optimization

In this section, we consider a vector-valued function $\psi : S \subseteq X \to Z$ and its epigraphical extension as defined in (2.21). In the sequel, we refer only to dom $f = S$, which is the effective domain of ψ.

The function ψ is called C–convex, when for all $x_1, x_2 \in S$ and all $t \in (0, 1)$ it holds $(1 - t)\psi(x_1) + t\psi(x_2) \in \{\psi(x_1 + t(x_2 - x_1))\} + C$, or equivalently when graph $f = \text{epi}\,\psi = \{(x, z) \in X \times Z \mid z \in \{\psi(x)\} + C\}$ is a convex set, compare [20, Definition 14.6].

Lemma 4.1 *Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0, x \in S$. Then for all $t \in (0, 1)$ it holds*

$$\frac{1}{t}\left(f(x_0 + t(x - x_0)) \dot{-} f(x_0)\right) = \left\{\frac{1}{t}\left(\psi(x_0 + t(x - x_0)) - \psi(x_0)\right)\right\} + C.$$

Moreover $\frac{1}{t}(\psi(x_0 + t(x - x_0)) - \psi(x_0))$ *is decreasing as* t *converges to* 0 *and* (SR) *is satisfied.*

Proof By definition, $f(x_t) = \{\psi(x_t)\} + C$, as $x_0, x \in S$. Hence

$$\forall t \in (0,1): \quad \left(z \in \frac{1}{t}\left(f(x_t) \dot{-} f(x_0)\right) \quad \Leftrightarrow \quad \psi(x_0) + tz \in \{\psi(x_t)\} + C\right),$$

or equivalently $z \in \left\{\frac{1}{t}(\psi(x_0 + t(x - x_0)) - \psi(x_0))\right\} + C$. By Proposition 2.29, the differential quotient is decreasing as t converges to 0 and by Lemma 2.2

$$-\sigma(z^*|f'(x_0, x - x_0)) = \inf\left\{-\sigma(z^*|\frac{1}{t}\left(f(x_0 + t(x - x_0)) \dot{-} f(x_0)\right)) \mid 0 < t\right\}$$

for all $z^* \in C^- \setminus \{0\}$. But $\varphi_{f,z^*}(x) = -z^*\psi(x)$ is satisfied for all $z^* \in C^- \setminus \{0\}$ and all $x \in S$, hence

$$-\sigma(z^*|\frac{1}{t}\left(f(x_0 + t(x - x_0)) \dot{-} f(x_0)\right)) = -\frac{1}{t}\left(z^*\psi(x_0 + t(x - x_0)) - z^*\psi(x_0)\right),$$

for all $z^* \in C^- \setminus \{0\}$, proving the statement. □

Following the approach in [3] we introduce the set of infinite elements $Z_\infty = \{z_\infty \mid z \in Z\}$. An element z_∞ is the infinite element in direction z, in other words

$$z_\infty = \lim_{t \uparrow \infty} tz.$$

It holds $z_\infty = y_\infty$ if, and only if, $y = \lambda z$ for some $0 < \lambda$ and $0_\infty = 0 \in Z$. For any $z^* \in Z^*$ and $z \in Z$, we define $z^*(z_\infty) = \lim_{t \uparrow +\infty} z^*(tz)$. Especially, $z^*(z_\infty) \in \mathbb{R}$ is satisfied if, and only if, $z^*(z_\infty) = z^*(z) = 0$.

For a subset $S \subseteq Z$, S_∞ denotes the set of all $z_\infty \in Z_\infty$ with $z \in S \setminus \{0\}$.

The space $\tilde{Z} = Z \cup Z_\infty$ can be endowed with a topology defined by local bases of neighborhoods as follows. For any element $z \in Z$, the set $\mathcal{U}(z) = \mathcal{U} + \{z\}$ is a local base of neighborhoods in $\tilde{Z}$. For any element $z \in Z \setminus \{0\}$, the set

$$\mathcal{U}(z_\infty) = \{(\{tz\} + \text{cone}\,(U + \{z\})) \cup (U + \{z\})_\infty \mid 0 < t,\ U \in \mathcal{U}(z)\}$$

is a local base of neighborhoods of z_∞. Especially, if $K \subseteq Z$ is an open cone with $z \in K$ and $y \in Z$, then $(\{y\} + K) \cup K_\infty$ is a neighborhood of z_∞, for details, compare [3].

Lemma 4.2 *Let* $z \in Z$ *be given and define* $(z_\infty + C) = \liminf_{t \uparrow \infty} (\{tz\} + C)$, *then* $(z_\infty + C) = \limsup_{t \uparrow \infty} (\{tz\} + C)$ *is satisfied.*

If $z \notin -C$, *then* $(z_\infty + C) = \sup_{0<t} (\{tz\} + C) = \emptyset$. *Otherwise,* $(z_\infty + C) = \inf_{0<t} (\{tz\} + C)$ *holds true.*

Especially, $(z_\infty + C) = C$, *if* $z \in C \cap -C$ *and* $(z_\infty + C) = Z$, *if for all* $z^* \in C^- \setminus \{0\}$ *it holds* $-z^*(z) < 0$.

Proof By definition, $(z_\infty + C) = \bigcap\limits_{0<t_0} \operatorname{cl}\operatorname{co} \bigcup\limits_{t_0 \le t} (\{tz\} + C)$. Let $z \in -C$, then $\bigcap\limits_{t_0 \le t} (\{tz\} + C) = \{t_0 z\} + C$ and we claim $\left(\operatorname{cl}\operatorname{co} \bigcup\limits_{0<t_0} \bigcap\limits_{t_0 \le t} (\{tz\} + C)\right) = \operatorname{cl}\operatorname{co} \bigcup\limits_{0<t_0} (\{t_0 z\} + C)$, or equivalently

$$\limsup_{t\uparrow\infty} (\{tz\} + C) = \inf_{t>0} (\{tz\} + C).$$

Since $\inf\limits_{t>0} (\{tz\} + C) \preccurlyeq \liminf\limits_{t\uparrow\infty} (\{tz\} + C) \preccurlyeq \limsup\limits_{t\uparrow\infty} (\{tz\} + C)$ always holds true, this implies

$$(z_\infty + C) = \inf_{t>0} (\{tz\} + C) = \limsup_{t\uparrow\infty} (\{tz\} + C).$$

On the other hand, let $z \notin -C$ be assumed. Then $0 < -z^*(z)$ is satisfied for some $z^* \in C^- \setminus \{0\}$. Thus,

$$-\sigma(z^* | \operatorname{cl}\operatorname{co} \bigcup_{t_0 \le t} (\{tz\} + C)) = -z^*(t_0 z)$$

converges to $+\infty$ as t_0 converges to $+\infty$, hence $(z_\infty + C) = \emptyset$. But, since

$$\emptyset = \liminf_{t\uparrow\infty} (\{tz\} + C) \preccurlyeq \limsup_{t\uparrow\infty} (\{tz\} + C) \preccurlyeq \emptyset$$

it is proven that

$$(z_\infty + C) = \sup_{t>0} (\{tz\} + C) = \limsup_{t\uparrow\infty} (\{tz\} + C).$$

Finally, by Lemma 2.2 for $z \in -C$ it holds

$$(z_\infty + C) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ y \in Z \mid \inf_{0<t} -z^*(tz) \le -z^*(y) \right\}.$$

Hence if $z \in C \cap -C$, it is immediate that

$$(z_\infty + C) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ y \in Z \mid 0 \le -z^*(y) \right\} = C,$$

while if for all $z^* \in C^- \setminus \{0\}$ it is assumed that $-z^*(z) < 0$ holds true, then $(z_\infty + C) = Z$. □

In [3] infinite elements play a crucial role to define a Dini directional derivative of $\psi : S \subseteq X \to Z$ at $x_0 \in S$ in direction $(x - x_0)$ with $x \in S$. The proposed derivative is computed as

$$\psi'(x_0, x - x_0) = \operatorname*{Lim\,sup}_{t \downarrow 0} \left\{ \frac{1}{t} \left(\psi(x_0 + t(x - x_0)) - \psi(x_0) \right) \right\} \subseteq \tilde{Z}$$

where $\operatorname*{Lim\,sup}_{t\downarrow 0} \{z_t\} = \left\{ \tilde{z} \in \tilde{Z} \mid \exists \left\{z_{t_i}\right\}_{i \in \mathbb{N}} \subseteq \{z_t\}_{0<t}, z_{t_n} \to \tilde{z} \right\}$ is the outer Painlevé-Kuratowski limit in $\tilde{Z}$ of a net $\{z_t\}_{t\downarrow 0} \subseteq Z$.

We provide some comparison between the derivative defined in [3] and our set-valued derivative computed for ψ^C.

Lemma 4.3 *Let $\psi : S \subseteq X \to Z$ be C–convex, $f(x) = \psi^C(x)$ for all $x \in X$ and $x_0, x \in S$.*

(a) If $z \in \psi'(x_0, x - x_0) \cap Z$, then $\{z\} + C = f'(x_0, x - x_0)$ and for all $z^* \in C^- \setminus \{0\}$ it holds $\varphi'_{f,z^*}(x_0, x - x_0) = -z^*(z)$;
(b) If $z_\infty \in \psi'(x_0, x - x_0) \cap Z_\infty$, then $z \in -C$ and $(z_\infty + C) \subseteq 0^+ f'(x_0, x - x_0)$;
(c) If $\psi'(x_0, x - x_0) \cap Z \neq \emptyset$ and $z_\infty \in \psi'(x_0, x - x_0) \cap Z_\infty$, then $z \in C \cap -C$.

Proof (a) By definition, $z \in \psi'(x_0, x - x_0) \cap Z$ is satisfied if, and only if, there is a decreasing sequence $\{t_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $\frac{1}{t_i}(\psi(x_0 + t_i(x - x_0)) - \psi(x_0))$ converges to z as i converges to $+\infty$. But this implies

$$\forall z^* \in C^- \setminus \{0\} : \quad -z^*(z) \le \varphi'_{f,z^*}(x_0, x - x_0),$$

hence $\{z\} + C \supseteq f'(x_0, x - x_0)$. On the other hand,

$$z \in \operatorname{cl} \bigcup_{0<t} \left(\left\{ \frac{1}{t} \left(\psi(x_0 + t(x - x_0)) - \psi(x_0) \right) \right\} + C \right) = f'(x_0, x - x_0).$$

(b) Assume to the contrary that $z_\infty \in \psi'(x_0, x - x_0)$ and $z \notin -C$. Then there exists $U \in \mathcal{U}$ such that $\operatorname{cone}(U + \{z\}) \cap -C = \emptyset$ and a subsequence $z_i = \frac{1}{t_i}(\psi(x_0 + t_i(x - x_0)) - \psi(x_0))$ with $i \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists a $i_0 \in \mathbb{N}$ such that for all $i_0 \le i$ it holds $z_i \in \{nz\} + \operatorname{cone}(U + \{z\})$, especially $(\{z_1\} + (-C)) \cap (\{nz\} + \operatorname{cone}(U + \{z\})) \neq \emptyset$ for all $n \in \mathbb{N}$. However, choosing n sufficiently large, $nz - z_1 \in \operatorname{cone}(U + \{z\})$ is satisfied, implying $\emptyset \neq -C \cap (\{nz - z_1\} + \operatorname{cone}(U + \{z\})) \subseteq -C \cap \operatorname{cone}(U + \{z\}) = \emptyset$, a contradiction.

(c) Especially by (a), $y \in \psi'(x_0, x - x_0) \cap Z$ is a lower bound of the set

$$\left\{ \frac{1}{t} \left(\psi(x_0 + t(x - x_0)) - \psi(x_0) \right) \mid 0 < t \right\},$$

hence if $z_\infty \in \psi'(x_0, x - x_0)$, then $\forall z^* \in C^- \setminus \{0\}: \quad -z^*(y) \leq -z^*(z_\infty)$, hence by (b) $z \in C \cap -C$.

□

More generally, we remark that taking the limit over a net of singletons and adding the ordering cone does not commute.

Example 4.4 Let $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ be given, $\{z_t\}_{0<t} \subseteq Z$ a subset of Z with $z_t = (-t, -t^2)$. Then $\{z_t\}_{0<t}$ is decreasing as t converges to $+\infty$ and $\operatorname{Lim}\sup_{t\uparrow+\infty} \{z_t\} = (0, -1)_\infty$. However,

$$\operatorname{Lim}\sup_{t\uparrow+\infty} \{z_t\} + C = \{z = (z_1, z_2) \in Z \mid 0 \leq z_1\} \subsetneq \lim_{t\uparrow\infty} (\{z_t\} + C) = Z.$$

Proposition 4.5 ([3]) *If Z has finite dimension, then $\tilde{Z}$ is compact.*

By Proposition 4.5, if Z has finite dimension, then for a C–convex function $\psi : S \subseteq X \to Z$, $x_0, x \in S$ it holds

$$\emptyset \neq \psi'(x_0, x - x_0) \subseteq Z \cup (-C)_\infty,$$

so each element of $\psi'(x_0, x - x_0)$ is either finite (i.e. an element of Z), or an element of $(-C)_\infty$, (that is an infinite element of $\tilde{Z}$ which is "less or equal" than $0 \in Z$).

The set of all efficient elements of $\psi[X]$ is given by

$$\mathrm{Eff}\,\psi[\mathrm{X}] = \{\mathrm{z} \in \psi[\mathrm{X}] | \forall \mathrm{y} \in \psi[\mathrm{X}] : \mathrm{z} \in \{\mathrm{y}\} + \mathrm{C} \Rightarrow \mathrm{z} \in \{\mathrm{y}\} + (-\mathrm{C} \cap \mathrm{C})\}. \quad (\mathrm{Eff})$$

and $x_0 \in \operatorname{dom} f$ is an efficient solution if, and only if, $\psi(x_0) \in \mathrm{Eff}\,\psi[X]$. An element $x_0 \in \operatorname{dom} f$ is a minimizer of f if, and only if, it is an efficient solution to ψ. Moreover,

$$\bigcup_{f(x)\in \mathrm{Min}\, f[X]} f(x) = \mathrm{Eff}\,\psi[X] + C \tag{4.1}$$

and a solution to (P) exists if, and only if, $\operatorname{cl}\operatorname{co}(\mathrm{Eff}\,\psi[X] + C) = \operatorname{cl}\operatorname{co}(\psi[X] + C)$.

In the sequel, we only focus on the characterization of minimizers of $f = \psi^C$ or equivalently efficient solutions of ψ. In this setting, we do not get any new results about infimizer but those already obtained in Sect. 3, as the inf–translation $\left(\psi^C\right)(\cdot, M) : X \to \mathcal{G}^\triangle$ is in general not the epigraphical extension of a vector-valued function.

Corollary 4.6 *Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. Then (SVI_M), (svi_M) and (3.2) are equivalent. Especially, if for all $x \in S$ with $\psi(x) \neq \psi(x_0)$ there exists $z \in Z$ such that $z \in \psi'(x_0, x - x_0) \setminus -C$, then $\psi(x_0) \in \mathrm{Eff}\,\psi[X]$.*

Proof The first part of the statement is proven in Proposition 3.15, as by Lemma 4.1, (SR) and hence especially (WR) are satisfied. The existence of $z \in Z$ with $z \in \psi'(x_0, x - x_0) \setminus -C$ implies the existence of a $z^* \in C^- \setminus \{0\}$ with $0 < \varphi'_{f,z^*}(x_0, x - x_0)$, compare Lemma 4.3 (a). Thus (3.2) is satisfied, proving the statement. □

Corollary 4.7 *Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. Then x_0 solves (MVI_M) if, and only if, it solves (mvi_M). Moreover, (MVI_M) is equivalent to*

$$(x \in S,\ t \in (0,1)\,,\ \psi(x_t) \neq \psi(x_0)) \ \Rightarrow\ \exists z^* \in C^- \setminus \{0\} :\ (-z^*\psi)'(x_t, x_0 - x) < 0. \tag{4.2}$$

Proof The first part of the statement is true as (SR) is guaranteed by Lemma 4.1 (compare Proposition 3.8). As $f(x) = \psi^C(x)$ for all $x \in X$ is assumed, $\varphi_{f,z^*}(x) \neq -\infty$ is always true for all $z^* \in C^- \setminus \{0\}$. It is left to prove that (4.2) implies (mvi_M).

Let $x \in S$ and $\psi(x_t) \neq \psi(x_0)$ be assumed for some $t \in (0,1)$. By convexity of $\varphi_{f,z^*} : X \to \overline{\mathrm{I\!R}}$, $(-z^*\psi)'(x_t, x_0 - x) < 0$ implies $(-z^*\psi)'(x, x_0 - x) < 0$. On the other hand, if $\psi(x_t) = \psi(x_0)$ is satisfied for all $t \in (0,1)$, then by convexity of the scalarizations

$$-z^*\psi(x_0) = \liminf_{t \downarrow 0}(-z^*\psi(x_t)) \leq -z^*\psi(x)$$

is satisfied for all $z^* \in C^- \setminus \{0\}$. Especially, $\psi(x) \neq \psi(x_0)$ implies

$$\exists z^* \in C^- \setminus \{0\} : \quad -z^*(x_0) = -z^*\psi(x_t) < -z^*\psi(x),$$

hence $\varphi'_{f,z^*}(x, x_0 - x) = -\infty < 0$. □

Remark 4.8 As $(-z^*\psi)'(x, \cdot) : X \to \overline{\mathrm{I\!R}}$ is sublinear, if $\psi : S \subseteq X \to Z$ is C–convex, $x_0, x \in S$ implies $(-z^*\psi)'(x_t, x_0 - x) \in \mathrm{I\!R}$ for all $z^* \in C^- \setminus \{0\}$ and all $t \in (0,1)$. In this case, $z_\infty \in \psi'(x_t, x_0 - x)$ implies $z \in C \cap -C$.

Indeed, under the given assumptions, $-z^*\psi(x_t) \in \mathrm{I\!R}$ is true for all $t \in (0,1)$, hence

$$0 = (-z^*\psi)'(x_t, 0) \leq (-z^*\psi)'(x_t, x - x_0) \dot{+} (-z^*\psi)'(x_t, x_0 - x)$$

and $(-z^*\psi)'(x_t, x_0 - x) = -\infty$ implies $(-z^*\psi)'(x_t, x - x_0) = +\infty$. But as dom $(-z^*\psi)'(x_t, \cdot) = \text{cone }(S + \{-x_t\})$, this is a contradiction. By Lemma 4.3 (b), $z_\infty \in \psi'(x_t, x_0 - x)$ implies $z \in -C$. Assuming $z \notin C$ would imply the existence of a $z^* \in C^- \setminus \{0\}$ such that $\psi'(x_t, x_0 - x) = -\infty$, a contradiction.

Proposition 4.9 *Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. If $x \in S$ and $t \in (0,1)$ imply*

$$\psi(x_t) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_t, x_0 - x) \not\subseteq (C \cup C_\infty)\,,$$

then x_0 solves (MVI_M) and

$$\psi(x_t) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_t, x_0 - x) \subseteq (C \cap -C)_\infty \cup (Z \setminus C)\,.$$

Proof Under the given assumptions, let $\psi(x_t) \neq \psi(x_0)$. Then $\psi'(x_t, x_0 - x) \neq \emptyset$ and especially,

$$\psi'(x_t, x_0 - x) \cap (((-C)_\infty \setminus C_\infty) \cup (Z \setminus C)) \neq \emptyset.$$

Thus if $z \in \psi'(x_t, x_0 - x) \cap (Z \setminus C)$, then there exists an element $z^* \in C^- \setminus \{0\}$ satisfying $\varphi'_{f,z^*}(x_t, x_0 - x) < 0$. On the other hand, if $z_\infty \in \psi'(x_t, x_0 - x) \cap (((-C)_\infty \setminus C_\infty))$, then $\varphi'_{f,z^*}(x_t, x_0 - x) = -\infty$ is satisfied for some $z^* \in C^- \setminus \{0\}$, a contradiction. Hence

$$\emptyset \neq \psi'(x_t, x_0 - x) \subseteq ((-C)_\infty \cap C_\infty) \cup Z$$

and thus by assumption

$$\emptyset \neq \psi'(x_t, x_0 - x) \cap (Z \setminus C)\,.$$

But this implies

$$\forall z \in \psi'(x_t, x_0 - x) \cap (Z \setminus C): \quad \emptyset \neq \psi'(x_t, x_0 - x) \cap Z \subseteq \{z\} + (C \cap -C) \subseteq Z \setminus C,$$

implying the existence of a $z^* \in C^- \setminus \{0\}$ satisfying $\varphi'_{f,z^*}(x_t, x_0 - x) < 0$, hence (mvi_M) and therefore (MVI_M) is satisfied. □

We can prove that under certain assumptions the efficient solutions of a vector valued function are identical with the solutions to the set-valued Minty variational inequality of its epigraphical extension.

Theorem 4.10 *Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. If $f_{x_0,x}$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0 for all $x \in X$ and C is polyhedral, then x_0 solves (MVI_M) if, and only if, $\psi(x_0) \in \operatorname{Eff} \psi\,[X]$.*

Proof If C is polyhedral, then so is C^-, that is there exists a finite set $M^* = \{m_1, ..., m_n\} \in C^- \setminus \{0\}$ such that

$$C = \bigcap_{i=1}^{n} \left\{z \in Z \mid 0 \leq -m_i^*(z)\right\}.$$

Also, for all $z^* \in C^- \setminus \{0\}$, $z^* \in \operatorname{cone} \operatorname{co} M^*$ and for all $z \in Z$ and all $z^* \in C^- \setminus \{0\}$, if

$$z^* = \sum_{i=1}^{n} t_i m_i^*,\ 0 \leq t_1, ..., t_n,$$

then $-z^*(z) = -\sum_{i=1}^{n} t_i m_i^*(z)$. Let $(-z^*\psi)'(x, x_0 - x) < 0$ be satisfied for some $z^* = \sum_{i=1}^{n} t_i m_i^* \in C^- \setminus \{0\}$ and $x_0 \in S$. Then there exists $0 < \bar{s}$ such that (for all

$s \in (0, \bar{s})$)

$$-z^*(\frac{1}{s}(\psi(x + s(x_0 - x)) - \psi(x))) < 0,$$

hence there exists at least one $i \in \{1, ..., n\}$ such that

$$-m_i^*(\frac{1}{s}(\psi(x_t + s(x_0 - x)) - \psi(x_t))) < 0,$$

implying $(-m_i^*\psi)'(x, x_0 - x) < 0$. In this case, (mvi_M) implies (3.4), thus they are equivalent. Moreover, by Corollary 4.7, (mvi_M) and (MVI_M) are equivalent. As $\psi(x_0) \in \mathrm{Eff}\,\psi[X]$ is satisfied if, and only if, $f(x_0) \in \mathrm{Min}\, f[X]$, Theorem 3.22 proves the statement. □

Theorem 4.10 provides as special case the following Minty variational principle for vector-valued functions, which can be found in e.g. [4, 26].

Corollary 4.11 *Let $Z = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$. Let $\psi : S \subseteq X \to Z$ be C–convex, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. If $f_{x_0,x}$ is $\left(C^- \setminus \{0\}\right)$–l.s.c. at 0 for all $x \in X$, then $\psi(x_0) \in \mathrm{Eff}\,\psi[X]$ is satisfied if, and only if, $x \in S$ and $t \in (0, 1)$ imply*

$$\psi(x_t) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_t, x_0 - x) \subseteq Z \setminus C.$$

Especially in this case, $\psi'(x_t, x_0 - x) \subseteq Z$ is single-valued.

Proof By Proposition 4.5, $\psi'(x_t, x_0 - x) \neq \emptyset$ is satisfied under the given assumptions and C is polyhedral and pointed, i.e. $C \cap -C = \{0\}$. Thus $\emptyset \neq \psi'(x_t, x_0 - x) \subseteq Z$ holds true for all $x \in S$ and all $t \in (0, 1)$ and $\psi'(x_t, x_0 - x)$ is single-valued. Hence, $\psi'(x_t, x_0 - x) \subseteq Z \setminus C$ is equivalent to $\psi'(x_t, x_0 - x) \nsubseteq (C \cup C_\infty)$. Moreover, under the given assumptions (MVI_M) is satisfied (compare Proposition 4.9). By Theorem 4.10, (MVI_M) is equivalent to $\psi(x_0) \in \mathrm{Eff}\,\psi[X]$.

On the other hand, by Corollary 4.7, (MVI_M) is equivalent to (4.2), implying

$$t \in (0, 1)\,,\ \psi(x_t) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_t, x_0 - x) \setminus C \neq \emptyset,$$

which in turn implies

$$t \in (0, 1)\,,\ \psi(x_t) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_t, x_0 - x) \subseteq Z \setminus C,$$

as proposed. □

5 Conclusion

By means of conlinear spaces we developed a variational inequalities scheme to characterize solutions to set optimization problems. The results proved actually allow to recover results previously proved in vector optimization under convexity assumptions. It is an open question how far the convexity assumption can be relaxed for set-valued problems.

The graphics in the paper summarize the implications proved. Counterexamples are given for the equivalences that do not hold for the formulation presented in the paper.

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References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis: A Hitchhiker's Guide. Studies in Economic Theory, vol. 4. Springer, Berlin (2006)
2. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Systems & Control: Foundations & Applications, vol. 2. Birkhäuser Boston Inc., Boston, MA (1990)
3. Crespi, G.P., Ginchev, I., Rocca, M.: First-order optimality conditions in set-valued optimization. Math. Methods Oper. Res. **63**(1), 87–106 (2006)
4. Crespi, G.P., Ginchev, I., Rocca, M.: Some remarks on the minty vector variational principle. J. Math. Anal. Appl. **345**(1), 165–175 (2008)
5. Dedekind, R.: Stetigkeit und irrationale Zahlen. In: Fricke, R., Noether, E., Ore, Ö. (eds.) Richard Dedekind Gesammelte Mathematische Werke, 5th edn 1927 (1872). Druck und Verlag von Friedr. Vieweg & Sohn Akt.-Ges, Braunschweig (1932)
6. Fuchs, L.: Teilweise Geordnete Algebraische Strukturen. Vandenhoeck u. Ruprecht, Göttingen (1966)
7. Galatos, N.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Elsevier Science Ltd, Boston (2007)
8. Getan, J., Martinez-Legaz, J.E., Singer, I.: (*, s)-Dualities. J. Math. Sci. **115**(4), 2506–2541 (2003)
9. Giannessi, F.:Theorems of alternative, quadratic programs and complementarity problems, in Variational inequalities and complementarity problems. In: Proc. Internat. School, Erice, pp. 151–186. Wiley, Chichester, 1978 (1980)
10. Giannessi, F.: On Minty variational principle, in New Trends in Mathematical Programming. Series on Applied Optimization, 13:93–99. Kluwer Academic Publishers, Boston (1998)
11. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 17. Springer, New York (2003)
12. Hamel, A.H.: Variational Principles on Metric and Uniform Spaces. Habilitation thesis, MLU Halle-Wittenberg, Halle (2005)
13. Hamel, A.H.: A duality theory for set-valued functions i: fenchel conjugation theory. Set-Valued Var. Anal. **17**(2), 153–182 (2009)
14. Hamel, A.H., Schrage, C.: Notes on extended real- and set-valued functions. J. Convex Anal. **19**(2), 355–384 (2012)
15. Hamel, A.H., Schrage, C.: Directional derivative and subdifferentials of set–valued convex functions. Pac. J. Optim. **10**(4), 667–689 (2014)
16. Hernández, E., Rodríguez-Marín, L.: Duality in set optimization with set-valued maps. Pac. J. Optim. **3**(2), 245–255 (2007)
17. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**(6), 1726–1736 (2007)

18. Heyde, F., Löhne, A.: Solution concepts in vector optimization. a fresh look at an old story. Optim. **60**(12), 1421–1440 (2011)
19. Heyde, F., Schrage, C.: Continuity of set-valued maps and a fundamental duality formula for set-valued optimization. J. Math. Anal. Appl. **397**(2), 772–784 (2013)
20. Jahn, J.: Vector Optimization: Theory, Applications, and Extensions. Springer, Berlin (2004)
21. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
22. Martınez-Legaz, J.E., Singer, I.: Dualities associated to binary operations on $\bar{R}$. J. Convex Anal. **2**(1–2), 185–209 (1995)
23. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Springer, Berlin (2004)
24. Schrage, C.: Set-valued Convex Analyis. Ph.D. thesis, MLU Halle-Wittenberg, Halle (2009)
25. Schrage, C.: Scalar representation and conjugation of set-valued functions. Optim. **64**(2), 197–223 (2015)
26. Yang, X.M., Yang, X.Q., Teo, K.L.: Some remarks on the minty vector variational inequality. J. Optim. Theory Appl. **121**, 193–201 (2004)
27. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing Co., Inc., River Edge (2002)

Estimates of Error Bounds for Some Sets of Efficient Solutions of a Set-Valued Optimization Problem

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Abstract In this paper, we establish some estimates of the global/local error bounds for the sets $S_{\bar{y}}^{\text{Pareto}}$, $S_{\leq \bar{y}}^{\text{W}}$ and S^{W}, where $S_{\bar{y}}^{\text{Pareto}}$ is the set of efficient solutions of a unconstrained set-valued optimization problem ($\mathcal{SP}$) corresponding to an efficient value $\bar{y}$ of a unconstrained set-valued optimization problem ($\mathcal{SP}$), $S_{\leq \bar{y}}^{\text{W}}$ is the set of weakly efficient solutions of ($\mathcal{SP}$) corresponding to weakly efficient values smaller than a weakly efficient value $\bar{y}$ and S^{W} is the set of all weakly efficient solutions of ($\mathcal{SP}$). These estimates are expressed in terms of the approximate coderivative, the limiting Fréchet/basic coderivatives and the coderivative of convex analysis. Thus, we establish conditions ensuring the existence of weak sharp minima for ($\mathcal{SP}$). We also extend the concept of the good asymptotic behavior to a convex or cone-convex set-valued map.

Keywords Error bound · Set-valued map · Set optimization · Subdifferential · Coderivative · Marginal function

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1 Introduction

Error bounds for a given subset of a metric space is an inequality that bounds the distance from points of a test set to the given set in terms of a residual function (see [45]), or of a merit function (see [40]). Let S be a nonempty subset of a metric space X. A function $\theta : X \to \mathbb{R}_+$ is a merit function for S if

$$\theta(x) = 0 \text{ if and only if } x \in S.$$

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We recall the concepts of global/local error bounds for the distance to the set S. Let $d(.;S)$ be the distance function associated with S.

Definition 1.1 We say that the distance to the set S has a global error bound with a merit function θ if there exists a scalar $\tau > 0$ such that

$$\tau d(x;S) \leq \theta(x) \text{ for all } x \in X$$

and denote by $\sigma_g(S)$ the supremum of such τ's.

Definition 1.2 Let $\bar{x} \in S$. We say that the distance to the set S has a local error bound at $\bar{x}$ with a merit function θ if there exists a scalar $\tau > 0$ such that

$$\tau d(x;S) \leq \theta(x) \text{ for all } x \text{ near } \bar{x}$$

and denote by $\sigma_l(S)$ the supremum of such τ's.

For the sake of simplicity, we will use the expression "the set S has a global/local error bound" instead of "the distance to the set S has a global/local error bound".

The case when S is a lower level set $\{x \mid f(x) \leq \alpha\}$ of a lower semicontinuous (in brief l.s.c.) function $f : X \to \mathbb{R} \cup \{+\infty\}$ has attracted attention of many authors due to its applications in subdifferential calculus, optimality conditions, sensitivity analysis and convergence of numerical methods. There have been obtained a number of criteria for the global/local error bounds of the lower level set, which are expressed in terms of various derivative-like objects defined either in the primal space (directional derivatives, slopes, etc.) or in the dual space (different kinds of subdifferentials), see [4, 6, 11, 20, 53] for an overview.

Among concepts which are closely related to the error bounds, we would like to mention the concepts of sharp minimum, weak sharp minima and good asymptotic behavior. The notion of a *sharp minimum* was first introduced by Polyak [48] under the assumption that the optimization problem has only one solution. The terminology *weak sharp minima* was proposed by Ferris [21] to include the possibility of non-unique solution set. Recall that $S \subset \mathbb{R}^n$ is the set of weak sharp minima for a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ relative to the set $\Omega \subset \mathbb{R}^n$ where $S \subset \Omega$ if there is a constant $c > 0$ such that

$$f(x) \geq f(y) + cd(x,S), \text{ for all } \quad x \in \Omega \text{ and all } y \in S.$$

Clearly, S is the set of global minima of f over Ω and the above inequality says that the set S has a global error bound (on the set Ω) with the merit function $\theta(x) = f(x) - \min_\Omega f$. Weak sharp minima frequently occur in linear programming, linear complementarity, in convex and (under additional assumptions) nonconvex problems and play important roles in sensitivity analysis and convergence analysis of many optimization algorithms, see [13–16, 56, 57] and references therein. Recall that a convex l.s.c. function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has a *good asymptotic behavior* in the sense of Auslender and Crouzeix [2] if for any sequence $(x_i) \subset X$ we have

$$d(0; \partial f(x_i)) \to 0 \text{ implies } f(x_i) \to \inf_X f$$

(∂f denotes the subdifferential of convex analysis of the function f). As noted in [4], the existence of the global error bound at any level $\alpha > \inf_X f$ is necessary and sufficient for f to have a good asymptotic behavior.

Recently, error bounds and weak sharp minima have been considered for the case when a (single-valued) map under consideration has values in partially ordered spaces. Characterizations of error bounds in terms of various slopes and subdifferentials have been obtained for level sets corresponding to Pareto efficiency or Pareto weak efficiency in [10]. We would like to note that in the vector case, sets of weak sharp minima have been defined either as the set of Pareto efficient/weakly efficient solutions corresponding to *one* Pareto efficient/weakly efficient value of a vector optimization problem, see [8, 9, 22, 26, 47, 51] or as the set of *all* Pareto efficient/weakly efficient solutions of linear, piecewise linear or quadratic vector optimization problems, see [19, 39, 40, 52, 55, 58, 59] (see also [49]).

The subjects of our study are weak sharp minima of an unconstrained set-valued optimization problem ($\mathcal{SP}$) and a good asymptotic behavior of a convex/cone-convex set-valued map. In particular, we obtain coderivative estimates of error bounds for the sets $S_{\bar{y}}^{\text{Pareto}}$, $S_{\leq\bar{y}}^{\text{W}}$ and S^{W}, which are the set of efficient solutions of ($\mathcal{SP}$) corresponding to an efficient value $\bar{y}$, the set of weakly efficient solutions of ($\mathcal{SP}$) corresponding to weakly efficient values smaller than a weakly efficient value $\bar{y}$ and the set of all weakly efficient solutions of ($\mathcal{SP}$), respectively.

The paper is organized as follows. In Sect. 2, we recall some concepts of subdifferentials, coderivatives, some results about subdifferentials of a marginal function and error bounds of a lower level set of a scalar function. We also study properties of a merit function defined by means of the Hiriart-Urruty signed distance function. In Sect. 3, we obtain coderivative estimates of the global/local error bounds for a lower level set of a set-valued map and apply them to study error bounds of the sets $S_{\bar{y}}^{\text{Pareto}}$, $S_{\leq\bar{y}}^{\text{W}}$ and S^{W}. The last section is devoted to a good asymptotic behavior of a convex/cone-convex set-valued map.

2 Preliminaries

2.1 A Set-Valued Optimization Problem

In this subsection, we recall some concepts from vector optimization, see [34, 41]. Throughout the paper, unless otherwise stated, let X and Y be Banach spaces with the duals X^* and Y^*, $K \subset Y$ be a nonempty closed pointed convex cone with apex at zero (pointedness means $K \cap (-K) = \{0\}$). For a nonempty set A, by int A, cl A, cone$A$ and conv$A$ we mean the interior, the closure, the conic hull and the convex hull of A, respectively. A closed unit ball in any space, say Y, is denoted by $\mathbb{B}_Y$. Let $K^{+i} = \{y^* \in Y^* \mid y^*(k) > 0 \;\text{ for all } k \in K \setminus \{0\}\}$. In some cases, the cone K has a nonempty interior, for instance, when K is the nonnegative orthant in $\mathbb{R}^n$, $C_{[0,1]}$ [35] or K is a Bishop-Phelps cone in any Banach space Y, which is representable in the form $K = \{y \in Y \mid \phi(y) \geq t\|y\|\}$ for some functional $\phi \in Y^*$ with $\|\phi\| > 1$ and some scalar $t > 0$ [33].

In this paper we use the following order relations: For $y_1, y_2, v \in Y$ and $A \subset Y$, we define

$$y_1 \leq_K y_2 \text{ if } y_2 - y_1 \in K,$$

$$y_1 \ll_K y_2 \text{ if } y_2 - y_1 \in \operatorname{int} K$$

and

$$A \preceq_l v \text{ if } v \in A + K.$$

For the order relation $\preceq_l$ in its more general form, when some nonempty set $B \subset Y$ stays in the place of v, see the Kuroiwa's paper [37].

Let $A \subset Y$ be a nonempty set. Denote by $Min(A, K)$ and $WMin(A, K)$ the sets of efficient points and weakly efficient points of A (w.r.t. K), i.e.

$$Min(A, K) := \{a \in A \mid A \cap (a - K) = \{a\}\}$$
$$WMin(A, K) := \{a \in A \mid A \cap (a - \operatorname{int} K) = \emptyset\}$$

(we always assume that $\operatorname{int} K \neq \emptyset$ whenever we are concerning with the weak efficiency).

Throughout the paper, let F be a set-valued map between the spaces X and Y. Denote $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$, $F(X) := \cup_{x \in X} F(x)$ and $\operatorname{gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

Consider an unconstrained set-valued optimization problem $(\mathcal{SP})$

$$\text{Minimize } \ F(x) \ \text{ subject to } \ x \in X.$$

Recall that: (i) $\bar{x} \in X$ is said to be a (Pareto) efficient solution of $(\mathcal{SP})$ if there exists $\bar{y} \in F(\bar{x}) \cap Min(F(X), K)$, the vector $\bar{y}$ is then called an efficient value of $(\mathcal{SP})$; (ii) $\bar{x} \in X$ is said to be a weakly efficient solution of $(\mathcal{SP})$ if there exists $\bar{y} \in F(\bar{x}) \cap WMin(F(X), K)$, the vector $\bar{y}$ is then called a weakly efficient value of $(\mathcal{SP})$.

2.2 *Subdifferentials and Coderivatives*

Let a function $f : X \to \mathbb{R} \cup \{+\infty\}$ be given. Denote $\operatorname{dom} f := \{x \in X \mid f(x) \neq +\infty\}$ and $\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$. Let us recall some concepts of convex analysis. The subdifferential of a convex function f at $x \in \operatorname{dom} f$ is defined by $\partial f(x) := \{x^* \in X^* \mid \langle x^*, x' - x\rangle \leq f(x') - f(x), \forall x' \in X\}$. Recall that the normal cone $N(x; \Omega)$ of a nonempty convex set $\Omega \subset X$ at $x \in \Omega$ is the set

$N(x; \Omega) := \{x^* \in X^* \mid \langle x^*, x' - x\rangle \le 0, \forall x' \in \Omega\}$ and when a set-valued map F between Banach spaces X and Y has a convex graph, its convex coderivative $D^*F(x, y)$ at $(x, y) \in \text{gr } F$ is defined as follows [1]: for any $y^* \in Y^*$,

$$D^*F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((x, y); \text{gr } F)\}.$$

Next, we recall the concept of approximate subdifferential introduced by Ioffe [29] (see also [30]) and the concept of limiting subdifferential introduced by Mordukhovich [36, 42] (see also [43]) and provide estimates of subdifferentials of a marginal function.

Let $x \in \text{dom } f$. *The approximate subdifferential* $\partial_A f(x)$ of f at x in the case f being a locally Lipschitz function is the set

$$\partial_A f(x) := \bigcap_{L \in \mathcal{L}} \limsup_{(\varepsilon, y) \to (0^+, x)} \partial_\varepsilon^- f_{y+L}(y),$$

where $\mathcal{L}$ is the collection of all finite dimensional subspaces of X, $f_{y+L}(u) = f(u)$ if $u \in y + L$ and $f_{y+L}(u) = +\infty$ otherwise, for $\varepsilon \ge 0$, $\partial_\varepsilon^- f_{y+L}(y)$ is the ϵ-subdifferential

$$\partial_\varepsilon^- f_{y+L}(y) := \{x^* \in X^* \mid x^*(v) \le \varepsilon\|v\| + \liminf_{t \to 0^+} \frac{f_{y+L}(y + tv) - f_{y+L}(y)}{t}, \forall v \in X\}.$$

The approximate normal cone $N_A(x; \Omega)$ to a set Ω at $x \in \Omega$ is defined as the cone generated by the approximate subdifferential of the distance function of Ω:

$$N_A(x; \Omega) := \bigcup_{\lambda > 0} \lambda \partial_A d(x; \Omega).$$

The approximate subdifferential $\partial_A f(x)$ of f at x in the case f being an l.s.c. function is defined by means of the corresponding approximate normal cone as follows

$$\partial_A f(x) := \{x^* \in X^* \mid (x^*, -1) \in N_A((x, f(x)); \text{epi } f)\}.$$

The limiting subdifferential (also called basic subdifferential, Mordukhovich's subdifferential) $\partial_L f(x)$ of f at x in the case f being an l.s.c. function is defined by means of the corresponding limiting normal cone $N_L(x; \Omega)$ as follows

$$\partial_L f(x) := \{x^* \in X^* \mid (x^*, -1) \in N_L((x, f(x)); \text{epi } f)\},$$

where

$$N_L(x;\Omega) := \limsup_{x' \xrightarrow{\Omega} x,\ \epsilon \to 0^+} \hat{N}_\epsilon(x';\Omega),$$

$$\hat{N}_\epsilon(x;\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{x' \xrightarrow{\Omega} x} \frac{x^*(x'-x)}{\|x'-x\|} \le \epsilon \right\}$$

and the limit in the right-hand side of the definition of $N_L(x;\Omega)$ means the sequential Kuratowski-Painlevé upper limit with respect to the norm topology in X and the weak-star ω^* topology in X^*, $x' \xrightarrow{\Omega} x$ refers to all sequences converging to x which remain in Ω.

When $X = \mathbb{R}^n$, the approximate subdifferential and the limiting subdifferential coincide and when f is convex, they reduce to the subdifferential ∂f of convex analysis. Moreover, while the subdifferential of convex analysis satisfies the *exact sum rule* (one function is continuous and the other one—l.s.c.) see [46, Theorem 3.23] and [32, Theorem 0.3.3], the approximate subdifferential in any Banach space and the limiting subdifferential in any Asplund space satisfy the *semi-Lipschitzian sum rule* (one function is locally Lipschitz and the other one—l.s.c.) see [31, 44]. Recall that X is Asplund if every continuous convex function defined on it is Fréchet differentiable on a dense set of points and the Banach spaces $\mathbb{R}^n$, $L^p_{[0,1]}$ and l_p $(1 < p < \infty)$ are Asplund.

The approximate coderivative $D^*_A F(x,y)$ and the limiting coderivative $D^*_L F(x,y)$ of F at $(x,y) \in \text{gr } F$ are defined by means of the corresponding normal cone as follows: for any $y^* \in Y^*$,

$$D^*_A F(x,y)(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N_A((x,y);\text{gr } F)\}$$

and

$$D^*_L F(x,y)(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N_L((x,y);\text{gr } F)\}.$$

We make a *convention* that by ∂ and D^* we mean the approximate subdifferential and approximate coderivative in Banach space settings, the limiting subdifferential and the limiting coderivative in Asplund space settings and the subdifferential and coderivative of convex analysis when the considered map is assumed to be convex.

2.3 Properties of a Merit Function

Let us first study properties of a function which will play the role of a merit function. In the remaining of the paper, we associate with the set-valued map F and the vector $\bar{y} \in Y$ a scalarizing function g defined by

$$g(x) := \inf_{y \in F(x)} \Delta_{-K}(y - \bar{y}) \tag{1}$$

and a set-valued map V defined by

$$V(x) := \{y \in F(x) \mid \Delta_{-K}(y - \bar{y}) = g(x)\}, \tag{2}$$

where $\Delta_{-K}(y) := d(y; -K) - d(y; Y \setminus (-K))$. We make a convention that $g(x) = +\infty$ and $V(x) = \emptyset$ for $x \notin \text{dom}\, F$.

Recall that the Hiriart-Urruty signed distance function Δ_A associated with a nonempty set $A \subset Y$ [27] possesses nice properties, especially when A is solid (i.e. A has nonempty interior), and has been used for scalarization in the study of vector optimization problems [23, 25], for studying error bounds in convex programming [18] and error bounds for vector-valued maps [12, 40]. We list some known properties of this function for the special case $A = -K$ presented in [24] and refer the interested reader to [27] for the general case.

Proposition 2.1 *(a) The function Δ_{-K} is convex, Lipschitz with rank 1 on Y.*
(b) The function Δ_{-K} satisfies the triangle inequality, i.e., for any $y_1, y_2 \in Y$, one has $\Delta_{-K}(y_1 + y_2) \le \Delta_{-K}(y_1) + \Delta_{-K}(y_2)$.
(c) If K has a nonempty interior, then $\Delta_{-\text{int}\, K} = \Delta_{-K}$ and $\partial \Delta_{-K}(y) \subseteq (K^+ \cap \mathbb{B}_{Y^*}) \setminus \{0\}$ for any $y \in Y$.

Example 2.1 Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$. Then

$$\Delta_{-\mathbb{R}^2_+}(y, z) = \begin{cases} \sqrt{y^2 + z^2}\} & \text{if } y \ge 0, z \ge 0 \\ y & \text{if } y \le z, y < 0 \\ z & \text{if } y > z, z < 0 \end{cases}$$

and

$$\partial \Delta_{-\mathbb{R}^2_+}(y, z) = \begin{cases} \{(y/\sqrt{y^2 + z^2}, z/\sqrt{y^2 + z^2}\} & \text{if } y \ge 0, z \ge 0, y, z \ne 0 \\ \{(0, 1)\} & \text{if } y < z, y < 0 \\ \{(1, 0)\} & \text{if } y > z, z < 0 \\ \text{conv}\{(0, 1), (1, 0)\} & \text{if } y = z < 0 \\ \{(u, v) \in \mathbb{R}^2_+ \mid u^2 + v^2 \le 1 \le u + v\} & \text{if } y = z = 0 \end{cases}$$

We refer the interested reader to [18] for the formula of the subdifferential of the signed distance function Δ_U in the case U is an arbitrary solid convex set in $\mathbb{R}^n$.

Let us recall some concepts of set-valued analysis and vector optimization [1, 38, 41]. We say that

- A set $A \subset Y$ is K-compact if any open cover of the form $\{V_\alpha + K \mid \alpha \in I, V_\alpha \text{ are open}\}$ admits a finite subcover.
- F is upper semicontinuous, in brief u.s.c., (K-upper semicontinuous, in brief K-u.s.c.) at $\bar{x} \in X$ if for any open set V such that $F(\bar{x}) \subset V$, there exists an open

neighborhood U of $\bar{x}$ such that $F(x) \subset V$ (respectively, $F(x) \subset V + K$) for any $x \in U$.

- F is closed if its graph is closed and F is convex if its graph is convex.
- F is K-convex if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ one has $\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K$.

It is well-known that if F is u.s.c., compact-valued, then it is closed. One can check that if A is compact, then $A + K$ is K-compact and if F is convex, then it is is K-convex.

Some properties of the function g are listed in the following proposition.

Proposition 2.2 *(a) If $F(x)$ is K-compact, then $g(x) > -\infty$ and $V(x)$ is nonempty.*
(b) If F is K-u.s.c. at x, then g is l.s.c. at x.
(c) If F is K-convex, then g is convex.

Proof (a) Since $F(x)$ is K-compact, there exists a bounded set $T \subset Y$ such that $F(x) \subset T + K$ (see [41]). Then for any $y \in F(x)$ there exists $t \in T$ and $k \in K$ such that $y = t + k$. Using the triangle property of the function Δ_{-K} (see Proposition 2.1), we get

$$\Delta_{-K}(y - \bar{y}) = \Delta_{-K}(t - \bar{y} + k) \geq \Delta_{-K}(t - \bar{y}k) - \Delta_{-K}(-k) \geq \\ \Delta_{-K}(t - \bar{y}) \geq \|t - \bar{y}\|$$

and since T is a bounded set, it follows that $g(x) = \inf_{y \in F(x)} \Delta_{-K}(y - \bar{y}) > -\infty$. Further, we show that $V(x) \neq \emptyset$. Suppose to the contrary that $\Delta_{-K}(. - \bar{y})$ does not attain its infimum on $F(x)$. Then for any $y \in F(x)$ there exists a positive scalar $\epsilon(y)$ depending on y such that $\Delta_{-K}(y - \bar{y}) > g(x) + \epsilon(y)$. For each $y \in F(x)$, let $U_y := \{v \in Y \,|\Delta_{-K}(v - \bar{y}) > g(x) + \epsilon(y)\}$. One can check that U_y are nonempty open sets and $F(x) \subset \cup_{y \in F(x)} U_y$. Further, using the triangle property of the function Δ_{-K} one can verify that $U_y = U_y + K$. On the other hand, the K-compactness of $F(x)$ implies that there exist $y_1, \ldots, y_i$ such that $y_j \in F(x)$ for all $j = 1, \ldots, i$ and $F(x) \subset \cup_{j=1}^{i}(U_{y_j} + K) = \cup_{j=1}^{i} U_{y_j}$. We obtain that $g(x) = \inf_{y \in F(x)} \Delta_{-K}(y - \bar{y}) > g(x) + \inf\{\epsilon(y_j) \mid j = 1, \ldots, i\} > g(x)$, a contradiction.

(b) Since F is K-u.s.c. at x, for $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(u) \subset F(x) + \epsilon \mathbb{B}_Y + K \quad \text{for any } u \in X \text{ such that } d(u; x) \leq \delta.$$

Fix $u \in X$ such that $d(u; x) \leq \delta$. For any $v \in F(u)$ there exist $e \in \mathbb{B}_Y$, $y \in F(x)$ and $k \in K$ such that $v = y - \epsilon e + k$. Properties of the signed distance function (see Proposition 2.1) imply

$$\begin{aligned}\Delta_{-K}(v - \bar{y}) = \Delta_{-K}(y - \epsilon e + k - \bar{y}) &\geq \Delta_{-K}(y - \bar{y}) - \Delta_{-K}(\epsilon e - k) \geq \Delta_{-K}(y - \bar{y}) - \\ &- \Delta_{-K}(\epsilon e) - \Delta_{-K}(-k) \geq \Delta_{-K}(y - \bar{y}) - \Delta_{-K}(\epsilon e) \geq \Delta_{-K}(y - \bar{y}) - \epsilon e \\ &\geq \inf_{y' \in F(x)} \Delta_{-K}(y' - \bar{y}) - \epsilon = g(x) - \epsilon,\end{aligned}$$

which gives $g(u) - g(x) \geq -\epsilon$ for any $u \in X$ such that $d(u; x) \leq \delta$. Therefore, g is l.s.c. at x.

(c) By the definition, for any $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ there exist $y_\lambda \in F(x_\lambda)$ and $k \in K$ such that $\lambda y_1 + (1-\lambda)y_2 = y_\lambda + k$. The convexity and the triangle inequality of the function Δ_{-K} (see Proposition 2.1) imply

$$\begin{aligned} g(x_\lambda) &= \inf_{y' \in F(x_\lambda)} \Delta_{-K}(y' - \bar{y}) \le \Delta_{-K}(y_\lambda - \bar{y}) \\ &\le \Delta_{-K}(y_\lambda - \bar{y}) + d_{Y\setminus(-K)}(-k) = \Delta_{-K}(y_\lambda - \bar{y}) - \Delta_{-K}(-k) \\ &\le \Delta_{-K}(y_\lambda - \bar{y} + k) = \Delta_{-K}(\lambda y_1 + (1-\lambda)y_2 - \bar{y}) \\ &\le \lambda \Delta_{-K}(y_1 - \bar{y}) + (1-\lambda)\Delta_{-K}(y_2 - \bar{y}). \end{aligned}$$

Since $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ are arbitrarily chosen, we deduce that $g(\lambda x_1 + (1-\lambda)x_2) = g(x_\lambda) \le \lambda g(x_1) + (1-\lambda)g(x_2)$, which means that the function g is convex. □

Next, we provide estimates of the approximate subdifferential and the limiting subdifferential of g in terms of the corresponding coderivatives of F.

Proposition 2.3 *Assume that F is u.s.c. compact-valued and one of the following conditions is satisfied*

(a) V is topologically lower semicompact at $x \in$ dom g.
(b) X and Y are Asplund spaces, V is sequentially lower semicompact at $x \in$ dom g.

Then we have

$$\partial g(x) \subseteq \cup_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*),$$

where ∂ and D^* are the approximate subdifferential and the approximate coderivative in the case (a) and the limiting subdifferential and the limiting coderivative in the case (b).

Recall that V is topologically lower semicompact at x [31] if for any net (x_α) converging to x there is a subnet (x_{ν_α}) and a net (y_α) converging to a certain $y \in F(x)$ and such that $y_\alpha \in F(x_{\nu_\alpha})$ for all α and that V is sequential lower semicompact at $\bar{x}$ [44] if there exists a neighborhood U of $\bar{x}$ such that for any $x \in U$ and any sequence (x_i) converging to x as $i \to \infty$ there is a sequence (y_i) with $y_i \in V(x_i), i = 1, 2, \ldots$, which contains a subsequence convergent in the norm topology of Y. The lower semicompactness is a condition useful for dealing with the infinite-dimensional space case and it is often satisfied in finite-dimensional spaces.

Proof The assertion in the case (a) follows from [31, Proposition 3.3] applied to the function $(x, y) \to \Delta_{-K}(y - \bar{y}) + \tau_{\mathrm{gr}\, F}(x, y)$, where $\tau_{\mathrm{gr}\, F}(x, y)$ is the indicator function associated with gr F, i.e., $\tau_{\mathrm{gr}\, F}(x, y) = 0$ if $(x, y) \in$ gr F and $\tau_{\mathrm{gr}\, F}(x, y) = \infty$ otherwise. The assertion in the case (b) is a special Lipschitz case of [44, Theorem 6.1]. □

In the convex case, we have a stronger conclusion.

Proposition 2.4 *Assume that F is convex closed and either X is separable or F is u.s.c. Let $y_x \in V(x)$. Then*

$$\partial g(x) = \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*).$$

Proof This is a consequence of [25, Theorem 3.3]. □

To deal with the case F is K-convex, we need some auxiliary results. Let $\mathcal{F}$ be the epigraphical set-valued map associated with F defined by

$$\mathcal{F}(x) = F(x) + K.$$

Recall that the epigraph of the set-valued map F (w.r.t. the cone K) is the set

$$\text{epi}\, F =: \{(x, v) \in X \times Y \mid v \in F(x) + K\}.$$

It is clear that the graph of $\mathcal{F}$ coincides with the epigraph of F. Denote by $(\mathcal{SP}_K)$ the unconstrained set-valued optimization problem with the objective map $\mathcal{F}$. We associated to $\mathcal{F}$ the function $g_{\mathcal{F}}$ and the map $V_{\mathcal{F}}$ defined by (1) and (2) with $\mathcal{F}$ being in the place of F.

Proposition 2.5 *(a) If F is K-convex, then $\mathcal{F}$ is convex and if F is u.s.c. compact-valued, then $\mathcal{F}$ is closed.*

(b) One has $g(x) = g_{\mathcal{F}}(x)$ and $V(x) \subset V_{\mathcal{F}}(x)$.

(c) The sets of weakly efficient values of $(\mathcal{SP})$ and of $(\mathcal{SP}_K)$ coincide and the sets of weakly efficient values of $(\mathcal{SP})$ and of $(\mathcal{SP}_K)$ corresponding to one weakly efficient value coincide.

Proof (a) It is easy to check that if F is K-convex, then its epigraph (which coincides with the graph of $\mathcal{F}$) is convex and therefore, $\mathcal{F}$ is convex. Further, it is clear that if F is K-u.s.c. K-compact-valued, then so is $\mathcal{F}$. Suppose that there is a sequence (x_i, z_i) converging to (x, z), where $z_i \in \mathcal{F}(x_i)$ for $i = 1, 2, \ldots$ Recalling that $\mathcal{F}$ is K-u.s.c. and passing to a subsequence if necessary, we can assume that $\mathcal{F}(x_i) \subset \mathcal{F}(x) + 1/i\mathring{\mathbb{B}}_Y + K = \mathcal{F}(x) + 1/i\mathring{\mathbb{B}}_Y$ for $i = 1, 2, \ldots$ For each i let $\hat{z}_i \in \mathcal{F}(x)$ and $e_i \in \mathring{\mathbb{B}}_Y$ be such that $z_i = \hat{z}_i + 1/ie_i$. By the assumption, the sequence z_i converges to z and therefore, so does the sequence $\hat{z}_i$. Since $F(x)$ is K-compact, it is K-closed [41], i.e., $\mathcal{F}(x) = F(x) + K$ is closed. Therefore, $z \in \mathcal{F}(x)$ and thus, $\mathcal{F}$ is closed.

(b) Since $F(x) \subset \mathcal{F}(x)$, we get $g(x) \geq g_{\mathcal{F}}(x)$. On the other hand, using the triangle property of the function Δ_{-K}, see Proposition 2.1, we have $\Delta_{-K}(y + k - \bar{y}) \geq \Delta_{-K}(y - \bar{y}) \geq g(x)$ for any $y \in F(x)$ and $k \in K$ and therefore, $g_{\mathcal{F}}(x) \geq g(x)$. Hence, the equality $g(x) = g_{\mathcal{F}}(x)$ holds. This equality together with the inclusion $F(x) \subset \mathcal{F}(x)$ implies $V(x) \subset V_{\mathcal{F}}(x)$.

(c) It suffices to show that $WMin(A, K) = WMin(A + K, K)$ whenever $A \subset Y$ is a nonempty set. By the definition, $\bar{a} \in WMin(A, K)$ iff $(A - \bar{a}) \cap (-\text{int}\, K) =$

$\emptyset$. Hence, $WMin(A + K, K) \subset WMin(A, K)$. Next, let $\bar{a} \in WMin(A, K)$. If $\bar{a} \notin WMin(A + K, K)$, then one can find $a \in A$, $k \in K$ and $k_0 \in \text{int}K$ such that $a + k - \bar{a} = -k_0$ and hence, $a - \bar{a} = -k_0 - k \in -\text{int}K$, a contradiction to $\bar{a} \in WMin(A, K)$. □

2.4 Some Subdifferential Estimates of Error Bounds for a Lower Level Set of a l.s.c. Function

In this subsection, we recall some known subdifferential estimates of error bounds for a lower level set of a l.s.c. function f defined on the Banach space X. For a unified presentation of results, we will use the notions given in Definitions 1.1 and 1.2 and refer the interested reader to the papers [4, 6, 20] for the classical notions of error bounds.

Through the subsection, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function and S be the lower level set

$$S = [f \leq \alpha] := \{x \in X \mid f(x) \leq \alpha\}$$

with $\alpha \in \mathbb{R}$. For any $\alpha \in \mathbb{R}$, denote $\alpha^+ = \max\{\alpha, 0\}$.

Theorem 2.1 *Let $\sigma_g(S)$ be as in Definition 1.1 with the merit function $\theta(x) = [f(x) - \alpha]^+$.*

(a) If f is l.s.c., then $\sigma_g(S) \geq \inf_{x \in [\alpha < f]} d(0; \partial f(x))$.
(b) If, in addition, f is convex, then $\sigma_g(S) = \inf_{x \notin [f \leq \alpha]} d(0; \partial f(x))$.

The assertions (a) of Theorem 2.1 is a special case of Proposition 4.1, Corollary 4.1 and Remark 4.1(b) presented in the survey paper [6]. Remark that Ioffe proved a version of the estimate formulated in this assertion under the Lipschitz assumption in [28] and, following Ioffe's idea and the scheme of Ioffe's proof, other authors established corresponding results in some general cases. The global error bound in the convex case has been studied in [5, 50] and the assertion (b) of Theorem 2.1 is immediate from [6, Theorems 3.1 and 3.2].

Theorem 2.2 *Let $\alpha = f(\bar{x})$ and $\sigma_l(S)$ be as in Definition 1.2 with the merit function $\theta(x) = [f(x) - \alpha]^+$.*

(a) If f is l.s.c., then $\sigma_l(S) \geq \liminf_{(x, f(x)) \to (\bar{x}, f(\bar{x}))} d(0; \partial f(x))$.
(b) If, in addition, f is convex, then

$$\sigma_l(S) = \liminf_{x \to \bar{x}, f(x) \downarrow f(\bar{x})} d(0; \partial f(x)).$$

The inequality of Theorem 2.2 follows from [20, Proposition 1, Theorems 1 and 2] and the arguments used in the proof of [20, Proposition 5(ii)] and the equality is formulated in [20, Theorem 5(ii)].

3 Error Bounds for a Lower Level Set of a Set-Valued Map

Let F be as before a set-valued map between the spaces X and Y. Let an arbitrary vector $\bar{y} \in Y$ be given. We define a lower level set $[F \preceq \bar{y}]$ of F as follows:

$$[F \preceq \bar{y}] := \{x \in X \mid \bar{y} \in F(x) + K\} = \{x \in X \mid \exists y \in F(x) \text{ such that } y \leq_K \bar{y}\}.$$

3.1 Global/Local Error Bounds of the Set $[F \preceq_l \bar{y}]$

The following result plays an important role in our scalarization techniques.

Proposition 3.1 *If F is K-compact-valued then we have*

$$[F \preceq_l \bar{y}] = [g \leq 0].$$

Proof Let $x \in [F \preceq_l \bar{y}]$, i.e. $F(x) \preceq_l \bar{y}$. There exist $y \in F(x)$ such that $y \leq_K \bar{y}$ and we have $\Delta_{-K}(y - \bar{y}) \leq 0$. Hence, $g(x) \leq 0$. Next, suppose that $x \in [g \leq 0]$ but $F(x) \not\preceq_l \bar{y}$ or $\bar{y} \notin F(x) + K$. For all $y \in F(x)$ one has $y - \bar{y} \not\leq_K 0$ and $\Delta_{-K}(y - \bar{y}) > 0$. It follows from Proposition 2.2 that $\Delta_{-K}(. - \bar{y})$ attains its minimum on $F(x)$ and therefore, we obtain $g(x) > 0$, a contradiction. □

In this section, let $\sigma_g([F \preceq_l \bar{y}), \sigma_g([g \leq 0])$ and $\sigma_g([F \preceq_l \bar{y}), \sigma_l([g \leq 0])$ be as in Definitions 1.1 and 1.2 with the merit function $\theta = g^+$, where $g^+(x) := [g(x)]^+$. We will assume that $\bar{x} \in [F \preceq_l \bar{y}]$ such that $\bar{y} \in V(\bar{x})$ whenever the local error bound is under our consideration. Note that the relation $\bar{y} \in V(\bar{x})$ implies $g(\bar{x}) = 0$.

The first result is stated as follows.

Theorem 3.1 *Assume that F is u.s.c. compact-valued and one of the following conditions is satisfied*

(a) V is topologically lower semicompact on dom g.
(b) X and Y are Asplund spaces and V is sequentially lower semicompact on dom g.

Then the inequalities

$$\sigma_g(S) \geq \inf_{x \notin S} d(0; \cup_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^*F(x, y_x)(y^*)) \tag{3}$$

and

$$\sigma_l(S) > \liminf_{x \to \bar{x}, g(x) \to 0} d(0; \cup_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^*F(x, y_x)(y^*)), \tag{4}$$

hold, where D^* is the approximate coderivative in the case (a) and the limiting coderivative in the case (b).

Proof Applying Theorem 2.1(a) to the set $[g \leq 0]$ we obtain

$$\sigma_g([g \leq 0]) \geq \inf_{x \notin [g \leq 0]} d(0; \partial g(x))$$

and applying Proposition 2.3 we obtain

$$\sigma_g([g \leq 0]) \geq \inf_{x \notin [g \leq 0]} d(0; \cup_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)),$$

where ∂ and D^* stand for the approximate subdifferential and the approximate coderivative when (a) is satisfied and the limiting subdifferential and the limiting coderivative when (b) is satisfied. Since $[F \preceq_l \bar{y}] = [g \leq 0]$ by Proposition 3.1, the inequality (3) holds. The inequality (4) can be proved analogously. □

Let us now consider the case when F is convex/K-convex.

Theorem 3.2 *Suppose that one of the following conditions is satisfied.*

(a) The set-valued map F is u.s.c., convex and compact-valued.
(b) X is separable and the set-valued map F is closed, convex and K-compact-valued.
(c) X is separable and the set-valued map F is u.s.c. K-convex and compact-valued.

Then the equalities

$$\sigma_g(S) = \inf_{x \notin S} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)), \tag{5}$$

$$\sigma_l(S) = \liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \downarrow 0} d(0, \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)) \tag{6}$$

hold when F is convex and the equalities

$$\sigma_g(S) = \inf_{x \notin S} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* \mathcal{F}(x, y_x)(y^*)), \tag{7}$$

$$\sigma_l(S) = \liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \downarrow 0} d(0, \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* \mathcal{F}(x, y_x)(y^*)) \tag{8}$$

hold when F is K-convex, where $y_x \in V(x)$ can be arbitrarily chosen.

Proof Let us begin with the convex cases (a)–(b). By Proposition 2.2, the function g is l.s.c. and convex. Theorems 2.1(b) and 2.2(b) applied to the set $[g \leq 0]$ yield that the equalities

$$\sigma_g([g \leq 0]) = \inf_{x \notin [g \leq 0]} d(0; \partial g(x))$$

and

$$\sigma_l([g \leq 0]) = \liminf_{x \to \bar{x}, g(x) \downarrow 0} d(0; \partial g(x))$$

hold. Next, Proposition 2.4 gives

$$\partial g(x) = \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*),$$

where $y_x \in V(x)$ is arbitrarily chosen. By Proposition 3.1 we have $[F \preceq_l \bar{y}] = [g \leq 0]$. Hence, the equalities (5) and (6) hold.

Next, let us consider the case (c). Proposition 3.1 applied to the map $\mathcal{F}$ gives $[\mathcal{F} \preceq_l \bar{y}] = [g_{\mathcal{F}} \leq 0]$. Since $g = g_{\mathcal{F}}$ by Proposition 2.5 and $[F \preceq_l \bar{y}] = [g \leq 0]$, we get $[F \preceq_l \bar{y}] = [\mathcal{F} \preceq_l \bar{y}]$. Note that the map $\mathcal{F}$ satisfies all conditions of (b). Hence, replacing F by $\mathcal{F}$ in (5) and (6), we obtain (7) and (8) □

Remark 3.1 Proposition 2.1(c) implies that for any y^* in the right-hand side of (4) and (6) one has $y^* \in (K^* \setminus \{0\}) \cap \mathbb{B}_{Y^*}$.

3.2 Applications: Error Bounds for Some Sets of Efficient Solutions

Let S be one of the following sets

- The set $S_{\bar{y}}^{\text{Pareto}}$ of efficient solutions of $(\mathcal{SP})$ corresponding to an efficient value $\bar{y}$ of $(\mathcal{SP})$:
$$S_{\bar{y}}^{\text{Pareto}} := \{x \in X \mid \bar{y} \in F(x)\}.$$
(Here, we assume that $\bar{y} \in Min(F(X), K)$).
- The set $S_{\leq \bar{y}}^{\text{W}}$ of weakly efficient solutions of $(\mathcal{SP})$ corresponding to wealky efficient values which are smaller than some weakly efficient value $\bar{y}$ of $(\mathcal{SP})$:
$$S_{\leq \bar{y}}^{\text{W}} := \{x \in X \mid \exists y \in F(x) \cap WMin(F(X), K) \text{ such that } y \leq_K \bar{y}\}.$$
(Here, we assume that $\bar{y} \in WMin(F(X), K) \setminus Min(F(X), K)$).
- The set S^{W} of all weakly efficient solutions of $(\mathcal{SP})$
$$S^{\text{W}} := \{x \in X \mid \exists y \in F(x) \cap WMin(F(X), K)\}.$$

Proposition 3.2 *Suppose that there is a maximal weakly efficient value $\bar{y}$ of $(\mathcal{SP})$ (this means that $\bar{y} \in WMin(F(X), K)$ and $y \leq_K \bar{y}$ for any $y \in WMin(F(X), K)$). Then*

$$S^{\text{W}} = S_{\leq \bar{y}}^{\text{W}}.$$

Proof It is obvious that $S_{\leq \bar{y}}^{\text{W}} \subset S^{\text{W}}$. On the other hand, since is $\bar{y}$ is a maximal weakly efficient value of $(\mathcal{SP})$, we get $S^{\text{W}} \subset S_{\leq \bar{y}}^{\text{W}}$. The desired equality follows. □

Remark 3.2 Note that since the cone K is pointed, there exists at most one maximal weakly efficient value. When such a value $\bar{y}$ exists, it follows from Proposition 3.2 that any statement which holds true for $S^{\mathrm{W}}_{\leq \bar{y}}$ also holds true for S^{W}.

We provide an illustrating example.

Example 3.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$.

(i) Let F be a set-valued map defined by

$$F(x) = \begin{cases} \{(u, v) \mid u^2 + (v-1)^2 \leq 1\} & \text{if } x = 0 \\ \{(u, v) \mid u = |x| \text{ and } v = 0\} & \text{if } 0 < |x| < 2 \\ \{(2, 2)\} & \text{if } 2 \leq |x| \end{cases}$$

Then we have $MIN(F(X), \mathbb{R}^2_+) = \{(u, v) \mid u^2 + (v-1)^2 = 1, -1 \leq u \leq 0, v \leq 1\}$, $WMIN(F(X), \mathbb{R}^2_+) = MIN(F(X), \mathbb{R}^2_+) \cup [0, 2[\times\{0\}$, $S^{\mathrm{W}}_{\leq \bar{y}} = [-1, 1]$ for $\bar{y} = (1, 0)$ and $S^{\mathrm{W}} =]-2, 2[$. Note that in this case there does not exist any maximal weakly efficient value of $(\mathcal{SP})$.

(ii) Let F be a set-valued map defined by

$$F(x) = \begin{cases} \{(u, v) \mid u^2 + (v-1)^2 \leq 1\} & \text{if } x = 0 \\ \{(u, v) \mid u = x \text{ and } v = 0\} & \text{if } 0 < |x| \leq 2 \\ \{(2, 2)\} & \text{if } 2 < |x| \end{cases}$$

Then we have $MIN(F(X), \mathbb{R}^2_+) = \{(-2, 0)\}$, $WMIN(F(X), \mathbb{R}^2_+) = [-2, 2] \times \{0\}$. Since $\bar{y} = (2, 0)$ is a maximal weakly efficient value of $(\mathcal{SP})$, we have $S^{\mathrm{W}} = S^{\mathrm{W}}_{\leq \bar{y}} = [-2, 2]$.

Proposition 3.3 *Let S be any of the sets $S^{\text{Pareto}}_{\bar{y}}$ of $S^{\mathrm{W}}_{\leq \bar{y}}$. Then we have $S = [F \preceq_l \bar{y}]$, and $g(x) = \inf_{y \in F(x)} d(y - \bar{y}; -K) \geq 0$ for all $x \in X$.*

Proof We will consider only the case $S = S^{\mathrm{W}}_{\leq \bar{y}}$ because the case $S = S^{\text{Pareto}}_{\bar{y}}$ can be checked with similar arguments.

Let $x \in S^{\mathrm{W}}_{\leq \bar{y}}$. By the definition, there exists $y \in F(x)$ such that $y \in WMin(F(X), K)$ and $y \leq_K \bar{y}$. This means that $x \in [F \preceq_l \bar{y}]$. Next, let $x \in [F \preceq_l \bar{y}]$. Then there exists $y \in F(x)$ such that $y \leq_K \bar{y}$. We show that $y \in WMin(F(X), K)$. Indeed, if $y \notin WMin(F(X), K)$ then there is $v \in F(X)$ such that $v \ll_K y$. Since $y \leq_K \bar{y}$, we get $v \ll_K \bar{y}$, a contradiction to $\bar{y} \in WMin(F(X), K)$. Thus, $S^{\mathrm{W}}_{\leq \bar{y}} = [F \preceq_l \bar{y}]$.

Further, since $\bar{y} \in WMin(F(X), K)$, for any $y \in F(X)$ we have $y - \bar{y} \notin -\text{int}\, K$ and $\Delta_{-\text{int}\, K}(y - \bar{y}) = d(y - \bar{y}; -\text{int}\, K) \geq 0$. Therefore, $\Delta_{-K}(y - \bar{y}) = \Delta_{-\text{int}\, K}(y - \bar{y}) = d(y - \bar{y}; -\text{int}\, K) = d(y - \bar{y}; -K) \geq 0$ for all $y \in F(X)$. It follows that $g(x) = \inf_{y \in F(x)} d(y - \bar{y}; -K) \geq 0$ for all $x \in X$. □

Proposition 3.3 shows that results about coderivative estimates obtained for the set $[F \preceq_l \bar{y}]$ in the previous subsection can be applied to obtain similar results for the sets $S^{\text{Pareto}}_{\bar{y}}$, $S^{\text{W}}_{\leq\bar{y}}$ and (when a maximal weakly efficient value exists) S^{W}. Moreover, the merit function in these cases has the form

$$\theta(x) = [g(x)]^+ = \inf_{y \in F(x)} d(y - \bar{y}; -K).$$

Below are some examples which have illustrating character to show possible applications of Theorems 3.1 and 3.2 to the set $S^{\text{W}}_{\leq\bar{y}}$.

Example 3.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$.

(a) Consider a set-valued map

$$F(x) = \begin{cases} \{(u, v) \mid x \leq u \leq x + 2, -1 \leq v \leq 1\} & \text{if } x \geq 0 \\ \{(u, v) \mid 0 \leq u \leq 2, -1 - x \leq v \leq 1 - x\} & \text{if } x < 0 \end{cases}$$

The map F is u.s.c. and compact-valued. The graph of F is

$$\begin{aligned} \text{gr } F = &\{(x, u, v) \in \mathbb{R}^3 \mid -x \leq 0, x - u \leq 0, u - x - 2 \leq 0, -1 - v \leq 0, v - 1 \leq 0\} \\ &\cup \{(x, u, v) \in \mathbb{R}^3 \mid x < 0, -u \leq 0, u - 2 \leq 0, -1 - x - v \leq 0, -1 + x + v \leq 0\} \end{aligned}$$

and therefore, it is not convex but is locally convex at any (x, u, v) with $x \neq 0$. Further, one can check that

$$\begin{aligned} F(X) &= [0, +\infty[\times[-1, 1] \cup [0, 2] \times [1, +\infty[, \\ Min(F(X), \mathbb{R}^2_+) &= \{(0, -1)\}, \\ WMin(F(X), \mathbb{R}^2_+) &= [0, +\infty[\times\{-1\}. \end{aligned}$$

and its graph is locally convex but not convex. We will apply the estimate (3) to the case $\bar{y} = (2, -1)$. It is easy to see that $\bar{y} \in WMin(F(X), \mathbb{R}^2_+) \setminus Min(F(X), \mathbb{R}^2_+)$ and $S^{\text{W}}_{\leq\bar{y}} = [0, 2]$. Let be given $x \notin S^{\text{W}}_{\leq\bar{y}}$. Suppose that $x > 2$. Then $V(x) = \{y_x\} = \{(x, -1)\}$ and it follows from Example 2.1 that $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(1, 0)\}$. The normal cone of convex analysis to the graph of F at (x, y_x) is given by

$$N((x, (x, -1)); \text{gr } F) = \{r(t, -t, -1 + t) \mid t \in [0, 1], r \in \mathbb{R}_+\}.$$

The local convexity of the graph of F at (x, y_x) implies that we can consider the coderivative $D^*F(x, y_x)(y^*)$ of convex analysis. By the definition of the coderivative, $x^* \in D^*F(x, y_x)(y^*)$ with $y^* = (1, 0)$ iff $(x^*, (-1, 0)) \in N((x, y_x); \text{gr } F)$. It follows that $x^* = 1$. Hence,

$$\cup_{y^* \in \partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y})} D^*F(x, y_x)(y^*) = \{1\}.$$

Next, suppose that $x < 0$. Then $V(x) = \{y_x \mid y_x = (s, -1 - x), s \in [0, 2]\}$, and it follows from Example 2.1 that $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(0, 1)\}$. The normal cone of convex analysis to the graph of F at (x, y_x) with $s = 0$ is

$$N((x,(0,-1-x)); \operatorname{gr} F) = \{r(t-1, t, t-1) \mid t \in [0,1], r \in \mathbb{R}_+\} \tag{9}$$

and at (x, y_x) with $s \in]0, 2[$ is

$$N((x,(s,-1-x)); \operatorname{gr} F) = \{r(-1, 0, -1) \mid r \in \mathbb{R}_+\}. \tag{10}$$

It follows from (9) and (10) that in case $x < 0$ one has $\cup_{y^* \in \partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y})} D^*F(x, y_x)(y^*) = \{-1\}$. Thus, for all $x \notin S^{\mathrm{W}}_{\leq \bar{y}}$ we have $d(0; \cup_{y_x \in V(x)} \cup_{y^* \in \partial\Delta_{-K}(y_x - \bar{y})} D^*F(x, y_x)(y^*)) = 1$ and the estimate (3) of Theorem 3.1 implies that $\sigma_g(S^{\mathrm{W}}_{\leq \bar{y}}) \geq 1$.

(b) Consider a set-valued map

$$F(x) = \begin{cases} \{(u,v) \mid x \leq u \leq x+2, -1 \leq v \leq 1\} & \text{if } x \geq 0 \\ \{(u,v) \mid -x \leq u \leq 2+x, -1-x \leq v \leq 1+x\} & \text{if } -1 \leq x < 0 \\ \emptyset & \text{if } x < -1 \end{cases}$$

The map F is u.s.c. and compact-valued and its graph

$$\begin{aligned} \operatorname{gr} F = \{&(x,u,v) \in \mathbb{R}^3 \mid -x \leq 0, x-u \leq 0, u-x-2 \leq 0, -1-v \leq 0, v-1 \leq 0\} \\ &\cup \{(x,u,v) \in \mathbb{R}^3 \mid x < 0, -x-1 \leq 0, -x-u \leq 0, -x+u-2 \leq 0, \\ &-1-x-v \leq 0, -1-x+v \leq 0\} \end{aligned}$$

is convex. Further, one can check that

$$\begin{aligned} F(X) &= [0, +\infty[\times[-1, 1], \\ Min(F(X), \mathbb{R}^2_+) &= \{(0, -1)\}, \\ WMin(F(X), \mathbb{R}^2_+) &= [0, +\infty[\times\{-1\}. \end{aligned}$$

Let $\bar{y} = (2, -1)$. It is easy to see that $\bar{y} \in WMin(F(X), \mathbb{R}^2_+) \setminus Min(F(X), \mathbb{R}^2_+)$ and $S^{\mathrm{W}}_{\leq \bar{y}} = [0, 2]$. Let be given $x \notin S^{\mathrm{W}}_{\leq \bar{y}}$. Suppose that $x > 2$. It follows from the example presented in the part (a) that

$$\cup_{y^* \in \partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y})} D^*F(x, y_x)(y^*) = \{1\}.$$

Next, suppose that $-1 \leq x < 0$. One can check that $V(x) = \{y_x \mid y_x = (s, -1-x), s \in [-x, 2+x]\}$, $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(0, 1)\}$. One also has

$$N((x, y_x); \operatorname{gr} F) = N((x,(-x,-1-x)); \operatorname{gr} F) = \{r(-1, -t, t-1) \mid t \in [0,1], r \in \mathbb{R}_+\} \tag{11}$$

in case $x > -1$, $y_x = (-x, -1)$, and

$$N((x, y_x); \operatorname{gr} F) = N((-1,(1,0)); \operatorname{gr} F) = \{(-1, 0, -1)\} \tag{12}$$

in case $x = -1$ $y_x = (-1, -1)$. It follows from (11) and (12) that in case $-1 \leq x < 0$ one has

$$\cup_{y^*\in\partial\Delta_{-\mathbb{R}^2_+}(y_x-\bar{y})} D^*F(x, y_x)(y^*) = \{-1\}.$$

Thus, for all $x \notin S^{\mathrm{W}}_{\leq\bar{y}}$ we have $d(0; \cup_{y^*\in\partial\Delta_{-K}(y_x-\bar{y})} D^*F(x, y_x)(y^*)) = 1$ and the estimate (5) of Theorem 3.2 implies that $\sigma_g(S^{\mathrm{W}}_{\leq\bar{y}}) = 1$.

(c) Consider a set-valued map

$$F(x) = \begin{cases} \{(u, v) \mid 0 \leq u \leq 2, -2l - 1 + x \leq v \leq -2l + 1 + x\} & \text{if } 2l \leq x < 2l + 1, l = 0, 1, \ldots \\ \{(u, v) \mid 0 \leq u \leq 2, 2l - 1 - x \leq v \leq 2l + 1 - x\} & \text{if } 2l - 1 \leq x < 2l, l = 1, 2, \ldots \\ \{(u, v) \mid 0 \leq u \leq 2, -1 - x \leq v \leq 1 - x\} & \text{if } x < 0 \end{cases}$$

The map F is u.s.c. and compact-valued and its graph is not convex. It is easy to see that $Min(F(X), \mathbb{R}^2_+) = \{(0, -1)\}$, $WMin(F(X), \mathbb{R}^2_+) = \{(u, -1) \mid 0 \leq u \leq 2\}$ and $S^{\mathrm{W}} = \{0, 2, 4, \ldots\}$. Let $\bar{y} = (2, -1)$ and $\bar{x} = 0$. It is clear that $\bar{y} \in WMin(F(X), \mathbb{R}^2_+) \setminus Min(F(X), \mathbb{R}^2_+)$, $\bar{y} \in F(\bar{x})$, $\bar{y}$ is a maximal weakly efficient value and $\bar{x} \in S^{\mathrm{W}}$.

We will consider points x near $\bar{x}$ so we can assume that $x \in [-1/2, 1/2]$. Firstly, we consider the case $x \in]0, 1/2]$. We have $V(x) = \{y_x = (u, -1 + x) \mid 0 \leq u \leq 2\}\}$. Let $y_x = (0, -1 + x)$. It follows from Example 2.1 that $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(0, 1)\}$. The local convexity of the graph of F at (x, y_x) implies the normal cone of convex analysis to the graph of F at (x, y_x) is given by

$$N((x, (0, -1 + x)); \text{gr } F) = \{r(1 - t, -t, -1 + t) \mid t \in [0, 1], r \in \mathbb{R}_+\}$$

and that we can consider the coderivative $D^*F(x, y_x)(y^*)$ of convex analysis. Since, $x^* \in D^*F(x, y_x)(y^*)$ with $y^* = (0, 1)$ iff $(x^*, (0, -1)) \in N((x, y_x); \text{gr } F)$, it follows that $x^* = 1$. Similarly, in case $y_x = (u, -1 + x)$ $(0 < u < 2)$ we have $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(0, 1)\}$, $N((x, (u, -1 + x)); \text{gr } F) = \{r(1, 0, -1) \mid r \in \mathbb{R}_+\}$ and $x^* \in D^*F(x, y_x)(y^*)$ with $y^* = (0, 1)$ iff $x^* = 1$ and in case $y_x = (2, -1 + x)$ we have $\partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y}) = \{(0, 1)\}$, $N((x, (2, -1 + x)); \text{gr } F) = \{r(1 - t, t, t - 1) \mid r \in \mathbb{R}_+\}$ and $x^* \in D^*F(x, y_x)(y^*)$ with $y^* = (0, 1)$ iff $x^* = 1$. Thus, for $x \in]0, 1/2]$ we have that $D^*F(x, y_x)(y^*) = \{1\}$ for any $y_x \in V(x)$ and $y^* \in \partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y})$.

By a similar argument, one can prove that for $x \in [-1/2, 0[$ one has that $D^*F(x, y_x)(y^*) = \{-1\}$ for any $y_x \in V(x)$ and any $y^* \in \partial\Delta_{-\mathbb{R}^2_+}(y_x - \bar{y})$.

Therefore,

$$\liminf_{x\to\bar{x}, g(x)\to 0} d(0; \cup_{y_x\in V(x)} \cup_{y^*\in\partial\Delta_{-K}(y_x-\bar{y})} D^*F(x, y_x)(y^*)) = 1$$

and the inequality (7) in Theorem 3.2 implies that $\sigma_l(S^{\mathrm{W}}) \geq 1$.

Remark 3.3 Theorems 3.1 and 3.2 may not be applicable when $\bar{x} \in S^{\mathrm{W}}_{\leq\bar{y}}$ is not an isolated weakly efficient solution. In such a situation, a weakly efficient solution, say x, exists in any neighborhood of $\bar{x}$ and for $y \in F(x) \cap WMin(F(X), K)$ it may happen that

$$0 \in D_L^* F(x, y)(y^*) \quad \text{or} \quad 0 \in D^* F(x, y)(y^*)$$

for some $y^* \in K^+ \setminus \{0\}$, due to the fact that the Fermat rule holds for weakly efficient solutions [7, 54].

4 A Good Asymptotic Behavior of a Set-Valued Map

In this section, we present the concepts of an almost good/good asymptotical behavior for the set-valued map F. We will assume that

- $\bar{y}$ is an efficient, ideally efficient or weakly efficient point of cl $F(X)$.

Recall that $\bar{y}$ is an ideally efficient point of cl $F(X)$ if $\bar{y} \leq_K y$ for all $y \in \text{cl}\, F(X)$.

Definition 4.1 Assume that F is convex. We say that

(i) F has an almost good asymptotic behavior at $\bar{y}$ if for any sequence $(x_i) \subset X$ we have

$$d(0, \cup_{y^* \in \partial \Delta_{-K}(y_i - \bar{y})} D^* F(x_i, y_i)(y^*)) \to 0 \text{ for } y_i \in V(x_i) \text{ implies } d(y_i - \bar{y}; -K) \to 0. \tag{13}$$

(ii) F has a good asymptotic behavior at $\bar{y}$ if for any sequence $(x_i) \subset X$ we have

$$d(0, \cup_{y^* \in \partial \Delta_{-K}(y_i - \bar{y})} D^* F(x_i, y_i)(y^*)) \to 0 \text{ for } y_i \in V(x_i) \text{ implies } y_n \to \bar{y}. \tag{14}$$

Clearly, if F has a good asymptotic behavior at $\bar{y}$, then it has an almost good asymptotic behavior at that point. It turns out that the converse holds true under some additional conditions. To show this, we need the concept of a normal cone. Recall that the cone K is normal if there exists a scalar $\mathcal{N} > 0$ such that for any pair $k_1, k_2 \in K$ satisfying $k_1 \leq_K k_2$ one has $\|k_1\| \leq \mathcal{N} \|k_2\|$. Note that the nonnegative orthants in $\mathbb{R}^n$ and $C_{[0,1]}$ are normal and they also have nonempty interior [35].

Proposition 4.1 *Suppose that the cone K is normal. If $\bar{y}$ is an ideally efficient point of* cl $F(X)$*, then the concepts of good asymptotical behavior and almost good asymptotical behavior coincide.*

Proof It suffices to show that under the assumptions made, $d(y_i - \bar{y}; -K) \to 0$ implies $\|y_i - \bar{y}\| \to 0$ whenever $y_i \in F(X)$. Since $d(y_i - \bar{y}; -K) \to 0$, there exists a sequence $(k_i) \subset K$ such that $\|y_i - \bar{y} + k_i\| \to 0$ as $i \to \infty$. As $\bar{y}$ is an ideally efficient point of cl $F(X)$, we have $y_i - \bar{y} \in K$ and, therefore, $0 \leq_K y_i - \bar{y} \leq_K y_i - \bar{y} + k_i$. Let $\mathcal{N} > 0$ be the scalar stated in the definition of the normal cone. Then we have $\|y_i - \bar{y}\| \leq \mathcal{N} \|y_i - \bar{y} + k_i\|$ and since $\|y_i - \bar{y} + k_i\| \to 0$, we get $\|y_i - \bar{y}\| \to 0$ as it was to be shown. □

We will show that, under some conditions, the almost good asymptotic behavior of F equivalent to the good asymptotic behavior in the sense of Auslender and Crouzeix [2] of the convex function g, which means that for any sequence $(x_i) \subset X$ we have

$$d(0, \partial g(x_i)) \to 0 \text{ implies } g(x_i) \to \inf_X g. \tag{15}$$

Proposition 4.2 *Suppose that one of the following conditions is satisfied.*

(a) The set-valued map F is u.s.c., convex and compact-valued.
(b) X is separable and F is closed, convex, u.s.c. and K-compact-valued.

Then F has an almost good asymptotic behavior at $\bar{y}$ if and only if g has a good asymptotic behavior.

Proof Firstly, observe that Proposition 2.4 gives

$$\partial g(x_i) = \cup_{y^* \in \partial \Delta_{-K}(y_i - \bar{y})} D^* F(x_i, y_i)(y^*) \tag{16}$$

for any $y_i \in V(x_i)$. By the definition we have $g(x_i) = \Delta_{-K}(y_i - \bar{y})$. Let $x \in X$ and $y_x \in V(x)$. Since $\bar{y}$ is a Pareto (ideally, weakly) efficient point of cl $F(X)$, we have either $y_x - \bar{y} \notin -K$ or $y_x - \bar{y} \notin -\text{int}\, K$. Then we have $g(x) = \Delta_{-K}(y_x - \bar{y}) = d(y_x - \bar{y}; -K) \geq 0$ for all $x \in X$. In particular, for $i = 1, 2, \ldots$ we have

$$g(x_i) = \Delta_{-K}(y_i - \bar{y}) = d(y_i - \bar{y}; -K). \tag{17}$$

Further, we show that

$$\inf_X g = 0. \tag{18}$$

Since $\bar{y} \in \text{cl}\, F(X)$, there exists a sequence $((u_i, v_i)) \subset \text{gr}\, F$ such that $v_i \to \bar{y}$. Recall that the function $\Delta_{-K}(. - \bar{y})$ is Lipschitz, by Proposition 2.1(a), we obtain $\Delta_{-K}(v_i - \bar{y}) \to 0$. From the inequalities $0 \leq g(u_i) \leq \Delta_{-K}(v_i - \bar{y})$ we deduce that $g(u_i) \to 0$ and (18) holds.

Finally, it is easy to see that the relations (16)–(18) yield the equivalence between (15) and (13). □

The main result of this section is the following.

Theorem 4.1 *Suppose that X and Y are Banach spaces and one of the following conditions is satisfied.*

(i) The set-valued map F is u.s.c., convex and compact-valued.
(ii) X is separable and F is closed, u.s.c., convex and K-compact-valued.

Then

(a) Let $\bar{y} \in WMin(\text{cl}\, F(X), K)$. For F to have an almost good asymptotic behavior at $\bar{y}$

Necessary condition: For every $\hat{y}$ *satisfying* $\bar{y} \ll \hat{y}$ *one has*

$$\inf\{\tau > 0 \mid \tau d(x; [F \preceq_l \hat{y}]) \le [g(x) - d(\hat{y} - \bar{y}; Y \setminus K)]^+ \text{ for all } x \in X\} > 0. \tag{19}$$

Sufficient condition: For every $\hat{y}$ *satisfying* $\bar{y} \ll \hat{y}$ *one has*

$$\inf\{\tau > 0 \mid \tau d(x; [F \preceq_l \hat{y}]) \le [g(x) - d(\hat{y} - \bar{y}; -K)]^+ \text{ for all } x \in X\} > 0. \tag{20}$$

(b) If the cone K *is normal and* $\bar{y}$ *is an ideally efficient point of* cl $F(X)$ *then the above conditions also are necessary and sufficient, respectively, for* F *to have a good asymptotic behavior at* $\bar{y}$*.*

To prove Theorem 4.1, we need the following auxiliary fact.

Lemma 4.1 *Suppose that* F *is* K*-compact-valued and* $\bar{y} \ll \hat{y}$*. Then*

$$[g \le d(\hat{y} - \bar{y}; Y \setminus K)] \subset [F \preceq_l \hat{y}] \subset [g \le d(\hat{y} - \bar{y}; -K)].$$

Proof Let us begin with the first inclusion. Suppose that $x \in [g \le d(\hat{y} - \bar{y}; Y \setminus K)]$. Assume to the contrary that $x \notin [F \preceq_l \hat{y}]$. Then $y - \hat{y} \notin -K$ for all $y \in F(x)$. Hence, we get $\Delta_{-K}(y - \hat{y}) = d(y - \hat{y}; -K) > 0$ and

$$\begin{aligned}\Delta_{-K}(y - \bar{y}) &\ge \Delta_{-K}(y - \hat{y}) - \Delta_{-K}(\bar{y} - \hat{y}) = d(y - \hat{y}; -K) + d(\bar{y} - \hat{y}; Y \setminus (-K)) \\ &> d(\bar{y} - \hat{y}; Y \setminus (-K)) = d(\hat{y} - \bar{y}; Y \setminus K)\end{aligned}$$

for all $y \in F(x)$. Since F(x) is K-compact, $g(x)$ attained its infimum at some $y \in F(x)$, see Proposition 2.2. Hence, we get $g(x) > d(\hat{y} - \bar{y}; Y \setminus K)$, a contradiction.

Next, we prove the second inclusion. Suppose that $x \in [F \preceq_l \hat{y}]$. Then there exists $y \in F(x)$ such that $y - \hat{y} \in -K$ and $\Delta_{-K}(y - \hat{y}) = -d(y - \hat{y}; Y \setminus (-K))$. Hence, we obtain

$$\begin{aligned}\Delta_{-K}(y - \bar{y}) \le \Delta_{-K}(y - \hat{y}) + \Delta_{-K}(\hat{y} - \bar{y}) = -d(y - \hat{y};\\ Y \setminus (-K)) + d(\hat{y} - \bar{y}; -K) \le d(\hat{y} - \bar{y}; -K),\end{aligned}$$

which implies $g(x) \le d(\hat{y} - \bar{y}; -K)$. □

Remark that the assertion of Lemma 4.1 hods true for an arbitrary $\bar{y}$ not necessarily a Pareto (ideally, weakly) efficient point of cl $F(X)$.

Let us return to the proof of Theorem 4.1.

Proof Our proof is based on the fact that the convex l.s.c. function g has a good asymptotical behavior if and only if $\sigma_g([g \le \alpha]) > 0$ for any $\alpha > \inf_X g$, see [3] and [17, Theorem 5.2].

(a) To prove the necessary condition, suppose that F has an almost good asymptotic behavior at $\bar{y}$ and $\hat{y} \in Y$ such that $\hat{y} - \bar{y} \in \text{int}\, K$. Recall that from the proof of Proposition 4.2, we have $\inf_X g = 0$. By Proposition 4.2, the function g has a good asymptotical behavior and since $d(\hat{y}-\bar{y}, Y \setminus K) > 0 = \inf_X g$, we get $\sigma_{glob}([g \le d(\hat{y}-\bar{y}, Y \setminus K)]) > 0$ or

$$\inf\{\tau > 0 \mid \tau d(x; [g \le d(\hat{y}-\bar{y}, Y \setminus K)]) \le [g(x) - d(\hat{y}-\bar{y}; Y \setminus K)]^+ \text{ for all } x \in X\} > 0. \tag{21}$$

On the other hand, by Lemma 4.1, we have

$$[g \le d(\hat{y}-\bar{y}, Y \setminus K)] \subset [F \preceq \hat{y}].$$

It follows that $d(x; [F \preceq \hat{y}]) \le d(x; [g \le d(\hat{y}-\bar{y}, Y \setminus K)])$. This and the relation (21) yield

$$\inf\{\tau > 0 \mid \tau d(x; [F \preceq_l \hat{y}]) \le [g(x) - d(\hat{y}-\bar{y}; Y \setminus K)]^+ \text{ for all } x \in X\} > 0$$

and (19) holds.

Next, we prove the sufficient condition. We need only to check that (20) implies $\sigma_g([g \le \alpha]) > 0$ for any $\alpha > 0 = \inf_X g$. Indeed, this means that g has a good asymptotic behavior and therefore, the set-valued map F has an almost good asymptotic behavior at $\bar{y}$, by Proposition 4.2. Let $\alpha > 0$ be an arbitrary scalar. Let $k_0 \in \operatorname{int} K$ such that $d(k_0; -K) = \alpha$ and let $\hat{y} = \bar{y} + k_0$. Then $d(\hat{y}-\bar{y}; -K) = d(k_0; -K) = \alpha$ and the relation (20) applied to $\hat{y}$ yields

$$\inf\{\tau > 0 \mid \tau d(x; [F \preceq_l \hat{y}]) \le [g(x) - \alpha]^+ \text{ for all } x \in X\} > 0. \tag{22}$$

On the other hand, by Lemma 4.1, we have

$$[F \preceq \hat{y}] \subset [g \le d(\hat{y}-\bar{y}, -K)] = [g \le \alpha].$$

It follows that $d(x; [g \le \alpha]) \le d(x; [F \preceq \hat{y}])$. This and the relation (22) yield $\sigma_g([g \le \alpha]) > 0$, as it was to be shown.

(b) The assertion follows from the assertion(a) and Proposition 4.1. □

We conclude the section with the remark that replacing F in (13) and (14) by $\mathcal{F}$, one can define analogous concepts for the case when F is K-convex. It is easy to see that when $Y = \mathbb{R}$, $K = \mathbb{R}_+$ and F is single-valued, the concepts of almost good/good asymptotic behavior defined for F being K-convex reduce to the classical one introduced by Auslender and Crouzeix for a convex function.

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References

1. Aubin, J.-P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1983)
2. Auslender, A., Crouzeix, J.-P.: Well behaved asymptotical convex functions. Ann. Inst. Henri Poincare Anal. Non Lineaire **6**, 101–121 (1989)
3. Auslender, A., Cominetti, R., Crouzeix, J.-P.: Convex functions with unbounded level sets. SIAM J. Optim. **3**, 669–687 (1993)
4. Azé, D.: A survey on error bounds for lower semicontinuous functions. ESAIM Proc. **13**, 1–17 (2003)
5. Azé, D., Corvellec, J.-N.: On the sensitivity analysis of Hoffman constants for systems of linear inequalities. SIAM J. Optim. **12**, 913–927 (2002)
6. Azé, D., Corvellec, J.-N.: Characterizations of error bounds for lower semicontinuous functions on metric spaces. ESAIM Control Optim. Calc. Var. **10**, 409–425 (2004)
7. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. Math. Program. Ser. A **122**, 301–347 (2010)
8. Bernarczuk, E.M.: Weak sharp efficiency and growth condition for vector-valued functions with applications. Optimization **53**, 455–474 (2004)
9. Bernarczuk, E.M.: On weak sharp minima in vector optimization with applications to parametric problems. Control Cybern. **36**, 563–570 (2007)
10. Bernarczuk, E.M., Kruger, A.Y.: Error bounds for vector-valued functions: necessary and sufficient conditions. Nonlinear Anal. **75**, 1124–1140 (2012)
11. Bosch, P., Jourani, A., Henrion, R.: Sufficient conditions for error bounds and applications. Appl. Math. Optim. **50**, 161–181 (2004)
12. Bot, R.I., Csetnek, E.R.: Error bound results for convex inequality systems via conjugate duality. TOP **20**(2), 296–309 (2012)
13. Burke, J.V., Ferris, M.C.: Weak sharp minima in mathematical programming. SIAM J. Control Optim. **31**, 1340–1359 (1993)
14. Burke, J.V., Deng, S.: Weak sharp minima revisited. I. Basic theory. Control Cybern. **31**, 439–469 (2002)
15. Burke, J.V., Deng, S.: Weak sharp minima revisited, part II: application to linear regularity and error bounds. Math. Program. Ser. B **104**, 235–261 (2005)
16. Burke, J.V., Deng, S.: Weak sharp minima revisited, part III: Error bounds for differentiable convex inclusions. Math. Program. Ser. B **116**, 37–56 (2009)
17. Cornejo, O., Jourani, A., Zalinescu, C.: Conditioning and upper-Lipschitz inverse subdifferentials in nonsmooth optimization problems. J. Optim. Theory Appl. **95**, 127–148 (1997)
18. Coulibaly, A., Crouzeix, J.-P.: Condition numbers and error bounds in convex programming. Math. Program. Ser. B **116**, 79–113 (2009)
19. Deng, S., Yang, X.Q.: Weak sharp minima in multicriteria programming. SIAM J. Optim. **15**, 456–460 (2004)
20. Fabian, M.J., Henrion, R., Kruger, A.Y., Outrata, J.V.: Error bounds: necessary and sufficient conditions. Set-Valued Var. Anal. **18**, 121–149 (2010)
21. Ferris, M.C.: Weak sharp minima and penalty functions in mathematical programming. Ph.D. thesis, University of Cambridge, Cambridge, 1988
22. Flores-Bazán, F., Jiménez, B.: Strict efficient in set-valued optimization. SIAM J. Control Optim. **48**, 881–908 (2009)
23. Gorokhovich, V.V.: Convex and Nonsmooth Problems of Vector Optimization. Nauka i Tekhnika, Minsk (1990)
24. Ha, T.X.D.: The Ekeland variational principle for set-valued maps involving coderivatives. J. Math. Anal. Appl. **286**, 509–523 (2003)
25. Ha, T.X.D.: Lagrange multipliers for set-valued optimization problems associated with coderivatives. J. Math. Anal. Appl. **311**, 647–663 (2005)
26. Ha, T.X.D.: Some criteria for error bounds in set optimization. Institute of Mathematics, Hanoi (2012) (preprint)

27. Hiriart-Urruty, J.B.: New concepts in nondifferentiable programming. Bull. Soc. Math. Fr. **60**, 57–85 (1979)
28. Ioffe, A.D.: Regular points of Lipschitz functions. Trans. Am. Math. Soc. **251**, 61–69 (1979)
29. Ioffe, A.D.: Nonsmooth analysis: differential calculus of nondifferentiable mappings. Trans. Am. Math. Soc. **266**, 1–56 (1981)
30. Ioffe, A.D.: Metric regularity and subdifferential calculus. Russ. Math. Surv. **55**, 501–558 (2000)
31. Ioffe, A.D., Penot, J.-P.: Subdifferentials of performance functions and calculus of coderivatives of set-valued mappings. Well-posedness and stability of variational problems. Serdica Math. J. **22**(3), 257–282 (1996)
32. Ioffe, A.D., Tikhomirov, V.M.: Theory of Extremal Problems. North-Holland Publishing Company, Amsterdam (1979)
33. Jahn, J.: Bishop-Phelps cones in optimization. Int. J. Optim. Theory Methods Appl. **1**, 123–139 (2009)
34. Jahn, J.: Vector Optimization—Theory, Applications, and Extensions, 2nd edn. Springer, Berlin (2011)
35. Krasnoselski, M.A.: Positive Solutions of Operator Equations. Nordhoff, Groningen (1964)
36. Kruger, A.Y., Mordukhovich, B.S.: Extremal points and the Euler equations in nonsmooth optimization. Dokl. Akad Nauk BSSR **24**, 684–687 (1980)
37. Kuroiwa, D.: The natural criteria in set-valued optimization. RIMS Kokyuroku **1998**, 85–90 (1031)
38. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 3, Athens (1996)
39. Li, J., Liu, Z., Pan, D.: A note on weak sharp minima in multicriteria linear programming. Appl. Math. Lett. **25**(7), 1071–1076 (2012)
40. Liu, C.G., Ng, K.F., Yang, W.H.: Merit functions in vector optimization. Math. Program. Ser. A **119**, 215–237 (2009)
41. Luc, D.T.: Theory of Vector Optimization. Springer, Berlin (1989)
42. Mordukhovich, B.S.: Maximum principle in problems of time optimal control with nonsmooth contraints. J. Appl. Math. Mech. **40**, 960–969 (1976)
43. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory, Grundlehren Series (Fundamental Principles of Mathematical Sciences), vol. 330. Springer, Berlin (2006)
44. Mordukhovich, B.S., Shao, Y.: Nonsmooth sequential analysis in Asplund spaces. Trans. Am. Math. Soc. **348**, 1235–1280 (1996)
45. Pang, J.S.: Error bounds in mathematical programming. Math. Program. **79**, 299–332 (1997)
46. Phelp, R.R.: Convex Functions, Monotone Operators and Differentiability. Springer, Berlin (1988)
47. Peng, Z.Y., Xu, S., Long, X.J.: Optimality conditions for weak ψ-sharp minima in vector optimization problems. Positivity **17**, 475–482 (2013)
48. Polyak, B.T.: Introduction to Optimization. Optimization Software, New York (1987)
49. Studniarski, M.: Weak sharp minima in multiobjective optimization. Control Cybern. **36**, 925–937 (2007)
50. Wu, Z., Ye, J.: On error bounds for lower semicontinuous functions. Math. Program. **92**, 301–314 (2002)
51. Xu, S., Li, S.J.: Weak ψ-sharp minima in vector optimization problems. Fixed Point Theory Appl. **2010**(Article Id 154598), 10 pp
52. Yang, X.Q., Yen, N.D.: Structure and weak sharp minimum of the Pareto solution set for piecewise linear multiobjective optimization. J. Optim. Theory Appl. **147**(1), 113–124 (2010)
53. Zheng, X.Y., Ng, K.F.: Error bounds for lower semicontinuous functions in normed spaces. SIAM J. Optim. **12**, 1–17 (2001)
54. Zheng, X.Y., Ng, K.F.: Fermat rule for multifunctions in Banach spaces. Math. Program. **104**, 69–90 (2005)

55. Zheng, X.Y., Ng, K.F.: Metric regularity of piecewise linear multifunction and applications to piecewise linear multiobjective optimization. SIAM J. Optim. **24**(1), 154–174 (2014)
56. Zheng, X.Y., Yang, X.Q.: Weak sharp minima for semi-infinite optimization problems with applications. SIAM J. Optim. **18**, 573–588 (2007)
57. Zheng, X.Y., Yang, X.Q.: Global weak sharp minima for convex (semi-)infinite optimization problems. J. Math. Anal. Appl. **348**, 1021–1028 (2008)
58. Zheng, X.Y., Yang, X.Q.: Weak sharp minima for piecewise linear multiobjective optimization in normed spaces. Nonlinear Anal. **68**(12), 3771–3779 (2008)
59. Zheng, X.Y., Yang, X.Q.: Conic positive definiteness and sharp minima of fractional orders in vector optimization problems. J. Math. Anal. Appl. **391**, 619–629 (2012)

On Supremal and Maximal Sets with Respect to Random Partial Orders

Yuri Kabanov and Emmanuel Lepinette

Abstract The paper deals with definition of supremal sets in a rather general framework where deterministic and random preference relations (preorders) and partial orders are defined by continuous multi-utility representations. It gives a short survey of the approach developed in (J. Math. Econ. 14(4–5):554–563, 2011 [4]), (J. Math. Econ. 49(6):478–487, 2013 [5]) with some new results on maximal sets.

Keywords Preference relation · Partial order · Random cones · Transaction costs · European/american options · Hedging

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1 Introduction

The classical notion of *essential supremum* plays an important role in the theory of frictionless markets serving to define the generalized Snell envelope of the pay-off process, an important tool to characterize the set of super-replicating prices of European and American contingent claims, see [6]. Contrarily to the frictionless financial markets, in models with proportional transaction costs, a portfolio process is vector-valued and its dynamic depends on a set-valued adapted process $(G_t)_{t\in[0,T]}$ in $\mathbf{R}^d$, $d \geq 1$, whose values are the solvency cones in physical units. In

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the discrete-time setting, a self-financing portfolio process $(\hat{V}_t)_{t=0,\cdots,T}$ has the increments $\hat{V}_t - \hat{V}_{t-1} \in -G_t$, $t \geq 1$, i.e. $\hat{V}_{t-1} \geq_{G_t} \hat{V}_t$ where $\geq_{G_t}$ denotes the random preorder (preference relation) generated by the random cone G_t. In this context, it is reasonable to characterize the minimal vector-valued prices and minimal portfolio processes super-hedging contingent claims in the sense of random preorders defined by the solvency cones. Mathematically, it is rather natural to study questions on existence and properties of suitably defined supremal sets in a much more general framework of random preorders or partial orders. This study was initiated in our papers [4, 5] where we use systematically the description of preorders and partial orders in terms of continuous multi-utility representation. Though our main interest is in the development of the partial ordering in the space of random vectors, it seems that our approach is new even in the deterministic case. Specific and (pleasant) features is that we do not require that the partial order generates a structure (i.e. any two elements admit maximum and minimum). The reader should be aware about the terminology. We are working with random partial orders in the sense of probability theory, i.e. with partial orders depending on ω (in analogy with random variables) while wording "stochastic orders" in the literature is usually attributed to deterministic orders in the space of probability distributions.

In the present article we give a short survey of the approach developed in [4, 5] with some new results on maximal sets. We discuss the relations with the previous works [7–10] where the notions of supset and optimal sets were studied in the case where the partial order is generated by a proper cone of a vector space. Moreover, we provide an application to models of financial markets with transaction costs.

2 Supremum and Maximum with Respect to a Preorder in a Deterministic Setting

2.1 Vocabulary

Let $\succeq$ be a *preorder* (=*preference relation*) in X, i.e. a binary relation between certain elements of a set X which is reflexive ($x \succeq x$) and transitive (if $x \succeq y$ and $y \succeq z$ then $x \succeq z$). The elements x and y are *equivalent* if $x \succeq y$ and $y \succeq x$; we write $x \sim y$ in this case. The preorder is called *partial order* if it is antisymmetric (if $x \succeq y$ and $y \succeq x$ then $x = y$). For the partial order, the classes of equivalence are singletons.

The word "binary relation" means simply that we are given an indicator function $I_D : X \times X \to \{0, 1\}$ and the notation $x \succeq y$ is equivalent to the equality $I_D(x, y) = 1$, i.e. (x, y) belongs to the set D. For the preorder, the diagonal of the product space should be a subset of D and if the points (x, y), (y, z) are in D, then $(x, z) \in D$. For the partial order, (x, y), (y, x) are in D if and only if they belong to the diagonal. Let a set $D \subseteq X \times X$ define a preorder. Then its subset D' not containing the diagonal and such that for any points (x, y), (y, z) in D' the point $(x, z) \in D'$ defines a non-reflexive transitive relation denoted $\succ$.

We define the order intervals $[x, y] := \{z \in X : y \succeq z \succeq x\}$,

$$]-\infty, x] := \{z \in X : x \succeq z\}, \qquad [x, \infty[:= \{z \in X : z \succeq x\}$$

(the latter objects are called sometimes *lower* and *upper contour sets*). If Γ is a subset of X, the notation $\Gamma \succeq x$ means that $y \succeq x$ for all $y \in \Gamma$ while $\Gamma_1 \succeq \Gamma$ means that $x \succeq y$ for all $x \in \Gamma_1$ and $y \in \Gamma$;

$$[\Gamma, \infty[:= \bigcap_{x \in \Gamma} \{z \in X : z \succeq x\}$$

is the set of upper bounds of Γ, and so on. We shall use the notation $x \preceq z$ synonymous with $z \succeq x$.

A preorder $\succeq$ in a topological space X is *continuous* if its graph, i.e. the set $\{(x, y) : x \succeq y\}$, is closed.

2.2 Maximal and Supremal Sets

In the ample literature on preference theory and vector optimization one can find a number of definitions of maximal and supremal sets under various hypotheses on X. On the informal level, the maximal set of Γ is defined from the primary object, i.e. from the set Γ itself, while the supremal set is defined from the object, dual to Γ in an appropriate sense, namely, from the set $[\Gamma, \infty[$ of upper bounds of Γ. The difference between the two approaches is noticeable already in the case of the real line $\mathbf{R}$ with its usual total (linear) order. Indeed, any set $\Gamma \subset \mathbf{R}$ bounded from above has a supremum but may not have a maximum in the usual sense. But, the set being "improved" by passing to its closure, will have one, coinciding with the supremum.

With partial orders the situation is more complicated. Let us consider the partial order in $\mathbf{R}^2$ where $x \succeq y$ means that $x^i \geq y^i$ for $i = 1, 2$ (that is the partial order generated by $\mathbf{R}^2_+$). Let $\Gamma = \{(0, 0), (1, 0), (0, 1)\}$ and let $\mathbf{1} = (1, 1)$. Then $[\Gamma, \infty[= \mathbf{1} + \mathbf{R}^2_+$. This set has a minimal element with respect to the partial order, namely, $\mathbf{1}$, which is a good candidate to be considered as the supremum of Γ. The unpleasant feature is that it lays far from Γ. On the other hand, the set $\{(1, 0), (0, 1)\}$ looks, intuitively, as a good candidate for the maximum: it is a subset of Γ and for any its element x the intersection of Γ and $[x, \infty[$ is the singleton $\{x\}$.

Generalizing the above examples we arrive to the following notions.

Definition 2.1 Let Γ be a non-empty subset of X and let $\succeq$ be a preorder. We denote by $\operatorname{Sup} \Gamma$ the **largest** subset $\hat{\Gamma}$ of X such that the following conditions hold:

(a_0) $\hat{\Gamma} \subseteq [\Gamma, \infty[$;
(b_0) if $x \in [\Gamma, \infty[$, then there is $\hat{x} \in \hat{\Gamma}$ such that $\hat{x} \preceq x$;
(c_0) if two elements $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$ are comparable, then they are equivalent.

In the case of partial order the word “largest” in the above definition can be omitted: then the comparable elements of $\hat{\Gamma}$ coincide and it is not difficult to prove that the set with the listed properties is unique, see Lemma 3.3 in [4].

Note that the definition does not involve any additional structure on X. In contrast to this, to define maximal sets we assume that X is a topological space.

Definition 2.2 Let Γ be a non-empty subset of a topological space X and let $\succeq$ be a preorder. We put

$$\mathrm{Max}\,\Gamma = \{x \in \overline{\Gamma} :\ \overline{\Gamma} \cap [x, \infty[= [x, x]\}.$$

Definition 2.3 Let Γ be a non-empty subset of a topological space X and let $\succeq$ be a preorder. We denote by $\mathrm{Max}_1\Gamma$ the largest subset $\hat{\Gamma} \subseteq \overline{\Gamma}$ (possibly empty) such that the following conditions hold:

(α) if $x \in \overline{\Gamma}$, then there is $\hat{x} \in \hat{\Gamma}$ such that $\hat{x} \succeq x$;
(β) if two elements $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$ are comparable, then they are equivalent.

It is easy to understand that in the case of $\mathbf{R}$ our definition of supremum coincides with the classical one: $\mathrm{Sup}\,\Gamma = \{\sup \Gamma\}$ when a (non-empty) set Γ is bounded from above. Moreover, if Γ is closed and bounded from above, then $\mathrm{Max}\,\Gamma = \mathrm{Max}_1\Gamma = \{\max \Gamma\} = \mathrm{Sup}\,\Gamma$. For non-closed bounded $\Gamma \neq \emptyset$ the value $\max \Gamma$ in the classical sense may not exist while the sets $\mathrm{Max}\,\Gamma$ and $\mathrm{Max}_1\Gamma$ are well-defined and non-empty. In the case of our introductory example above where $\Gamma = \{(0, 0),\ (1, 0),\ (0, 1)\}$, we have $\sup \Gamma = \mathbf{1}$ and $\max \Gamma = \mathrm{Max}_1\Gamma = \{(1, 0),\ (0, 1)\}$.

Remark 2.4 For a closed set Γ, the set $\mathrm{Max}\,\Gamma$ is just the set of maximal points of Γ and it plays an important role in multicriteria optimization. In the latter theory, the simplest standard problem is the following: given a compact set Γ and a continuous function $\mathbf{u} : \mathbf{R}^n \to \mathbf{R}^n$, find $\mathrm{Max}\,\Gamma$ with respect to the preorder defined by the multi-utility representation $\{u^i, 1 \le i \le d\}$, formed by the component of the vector function $\mathbf{u}$. Recall that, in this theory, the set $\mathrm{Max}\,\mathbf{u}(\Gamma) \subset \mathbf{R}^n$ defined by the “natural” partial order in $\mathbf{R}^n$, i.e. generated by the cone $\mathbf{R}^n_+$, is called the Pareto frontier.

Though the definitions of $\mathrm{Max}\,\Gamma$ and $\mathrm{Max}_1\Gamma$ look quite different, in the case where $\mathrm{Max}_1\Gamma \neq \emptyset$, both sets coincide.

Lemma 2.5 $\mathrm{Max}_1\Gamma \subseteq \mathrm{Max}\,\Gamma$.

Proof Assume that $\mathrm{Max}_1\Gamma \neq \emptyset$ (otherwise the claim is trivial). Consider $\hat{x}_1 \in \mathrm{Max}_1\Gamma$ and $x \in \bar{\Gamma}$ such that $\hat{x}_1 \preceq x$. By (α), there exists $\hat{x}_2 \in \mathrm{Max}_1\Gamma$ such that $x \preceq \hat{x}_2$. Hence, by (β), $\hat{x}_1 \sim \hat{x}_2 \sim x$, i.e. $\hat{x}_1 \in \mathrm{Max}\,\Gamma$. □

Lemma 2.6 *Let* $\mathrm{Max}_1\Gamma \neq \emptyset$. *Then* $\mathrm{Max}_1\Gamma = \mathrm{Max}\,\Gamma$.

Proof By Lemma 2.5 above it is sufficient to check that the set $\mathrm{Max}\,\Gamma$ satisfies the properties (α) and (β) in Definition 2.3. Since $\mathrm{Max}_1\Gamma \neq \emptyset$ and $\mathrm{Max}_1\Gamma \subseteq \mathrm{Max}\,\Gamma$ the condition (α) is satisfied and it remains to observe that (β) automatically holds by definition of $\mathrm{Max}\,\Gamma$. □

2.3 Existence Results

The existence theorem below is a synthesis of several, more general, results from [4, 5].

Theorem 2.7 *Let $\succeq$ be a continuous partial order in the Euclidean space $\mathbf{R}^d$ such that all order intervals $[x, y]$, $y \succeq x$, are compact. Let Γ be a non-empty subset bounded from above, i.e. the order interval $[\Gamma, \infty[\neq \emptyset$. Then $\mathrm{Sup}\,\Gamma \neq \emptyset$ and $\mathrm{Max}\,\Gamma = \mathrm{Max}_1\Gamma \neq \emptyset$.*

We recall some important facts on multi-utility representations of preorders on topological vector space X.

We say that a set $\mathcal{U}$ of real-valued functions defined on X is a multi-utility representation of the preorder $\succeq$ if for any $x, y \in X$,

$$x \succeq y \;\Leftrightarrow\; u(x) \geq u(y), \quad \forall u \in \mathcal{U}.$$

It is easy to see that any preorder admits a multi-utility representation given by the family of indicator functions $u(x) = I_{[x,\infty[}$, $x \in X$. Note that the terminology, taken from [3], does not coincide with the standard one: properties usually associated with the utility functions are not required from the elements of $\mathcal{U}$.

The interest in multi-utility representations lays in the possibility to formulate, in terms of comprehensive properties of functions, assumptions on preorders and partial orders.

For example, if a preorder admits a continuous multi-utility representation (i.e. given by continuous functions), then, of course, this preorder is continuous. Under a suitable assumption the converse is also true: if X is a locally compact and σ-compact Hausdorff space, then a continuous preorder admits a continuous multi-utility representation, see [3].

As a corollary, we get that any continuous preorder on $\mathbf{R}^d$ (or, more generally, on any locally compact and σ-compact Hausdorff space) admits a countable continuous multi-utility representation, see [5].

Proposition 2.8 *Let X be a σ-compact metric space. Suppose that a family $\mathcal{U}$ of continuous functions defines a preorder on X. Then, this preorder can be defined by a countable subfamily of $\mathcal{U}$.*

It is obvious that a preorder on $\mathbf{R}^d$ defined by a closed (convex) cone G ($x \succeq y$ means that $x - y \in G$) is continuous. Such a preorder is a partial order if and only if G is a proper cone, i.e. $G^0 := G \cap (-G) = \{0\}$.

Lemma 2.9 *Let $\succeq$ be a partial order defined by a closed proper cone G in $\mathbf{R}^d$. Then, the order intervals $[x, y]$ are compact.*

Proof Suppose that an element $z^n \in [x, y]$ and its Euclidean norm $|z^n| \to \infty$. Put $\tilde{z}^n = z^n/(1 + |z^n|)$, $\tilde{x}^n = x/(1 + |z^n|)$, and $\tilde{y}^n = y/(1 + |z^n|)$. Then $\tilde{x}^n \preceq \tilde{z}^n \preceq \tilde{y}^n$.

Using the compactness of the unit ball in $\mathbf{R}^d$, we may assume that $\tilde{z}^n \to \tilde{z}$ such that $|\tilde{z}| = 1$. On the other hand, $\tilde{x}^n,\ \tilde{y}^n \to 0$. Hence, $0 \preceq \tilde{z} \preceq 0$, i.e. $\tilde{z} \in G \cap (-G) = \{0\}$ contrary to the assumption. □

Thus, Theorem 2.7 implies as corollary the following result which seems to be well adapted to the needs of the theory of financial markets with proportional transaction costs:

Theorem 2.10 *Let $\succeq$ be a continuous partial order generated by a proper closed convex cone $G \subset \mathbf{R}^d$. Let Γ be a non-empty subset bounded from above. Then* $\mathrm{Sup}\,\Gamma \neq \emptyset$ *and* $\mathrm{Max}\,\Gamma = \mathrm{Max}_1\Gamma \neq \emptyset$.

It is rather natural to place the question on the existence of non-empty supremal sets in a more general context of a preorder on a topological space. Several results in this direction can be found in the paper [5] where the principal assumptions are: the preorder admits a countable multi-utility representation with lower semicontinuous functions and the order intervals in the quotient space $\tilde{X} = X/\sim$ (generated by the equivalence $x \sim y$) are compact. Of course, the quotient mapping q induces the relation between classes of equivalence which is a partial order and the quotient space is equipped with the weakest topology such that q is continuous. But even for the preorder in $\mathbf{R}^d$, the quotient space is, in general, an abstract topological space and formulations of the corresponding results are too technical except the case when the preorder is generated by a cone. Also the topological assumptions we need to use our techniques (requiring the existence of countable representing family) are considered in the literature as too restrictive. In view of this, in the present survey, we concentrate ourselves on partial orders.

Theorem 2.11 *Let $\succeq$ be a partial order in a topological space X represented by a countable family $\mathcal{U}$ of lower semicontinuous functions and such that all order intervals $[x, y]$, $y \succeq x$, are compact. If a subset Γ is bounded from above, then* $\mathrm{Sup}\,\Gamma \neq \emptyset$ *and* $\mathrm{Max}\,\Gamma = \mathrm{Max}_1\Gamma \neq \emptyset$.

Recall that a function $u : X \to \mathbf{R}$ is called lower semicontinuous (l.s.c.) if for any point $x \in X$

$$\liminf_{x_\alpha \to x} u(x_\alpha) \geq u(x).$$

Equivalently,[1] u is l.s.c. if all lower level sets $\{x \in X :\ u(x) \leq c\}$ are closed, see [1]. A function $g : \tilde{X} \to \mathbf{R}$ is l.s.c. if and only if $g \circ q : X \to \mathbf{R}$ is l.s.c. If a function $f : X \to \mathbf{R}$ is l.s.c. and constant on the classes of equivalences $[x]$, then the function $g : \tilde{X} \to \mathbf{R}$ with $g([x]) = f(x)$ is l.s.c.

We complete this section by an example showing that $\mathrm{Sup}\,\Gamma$ might not be closed even if Γ is closed.

[1] Recall that $(x_\alpha)_{\alpha\in I}$ designates a net, i.e. a sequence of elements in X indexed by an upward directed set I, such that for all open set $\mathcal{O}$ containing x, $(x_\alpha)_{\alpha\in I}$ eventually belongs to $\mathcal{O}$.

Example Let $G \subseteq \mathbf{R}^2_+$ be defined as

$$G = \{(0,0)\} \cup \{(x,y) \in \mathbf{R}^2_+ : \ x + y \geq 2\}.$$

As $G \cap (-G) = \{0\}$ and $G + G \subseteq G$, the relation $x \succeq y$ if $x - y \in G$ defines a partial order. Let us consider the set $\Gamma := \{(0,0), (4,-1)\}$. Then,

$$[\Gamma, \infty) = \{(x,y) \in \mathbf{R}_+ : \ x + y \geq 5 \text{ and } x \geq 4\}.$$

Indeed, $(x,y) \succeq \Gamma$ if and only if $(x,y) \in G$, i.e. $(x,y) \in \mathbf{R}^2_+$ and $x + y \geq 2$, and $(x-4, y+1) \in G$, i.e. $x \geq 4$ and $x - 4 + y + 1 \geq 2$. We deduce that

$$\operatorname{Sup} \Gamma = \{(x,y) \in \mathbf{R}_+ : \ 7 > x + y \geq 5 \text{ and } x \geq 4\}. \tag{2.1}$$

Indeed, it suffices to observe that

$$(4,1) + G = \{(4,1)\} \cup \{(x,y) \in \mathbf{R}_+ : \ x + y \geq 7; \ y \geq 1\},$$
$$(5,0) + G = \{(5,0)\} \cup \{(x,y) \in \mathbf{R}_+ : \ x + y \geq 7; \ x \geq 5\},$$

so that the points in $\mathbf{R}_+$ above the line $x + y \geq 7$ are greater that $(4,1)$ or $(5,0)$ with respect to G. If two points belong to the set given by (2.1), then they are not comparable. Observe that this set is not closed though G and Γ are closed.

Remark 2.12 For unbounded Γ it may happen that $\operatorname{Max} \Gamma \neq \operatorname{Max}_1 \Gamma = \emptyset$. Indeed, let us consider in $\mathbf{R}^2$ the partial order generated by the closed cone $G = \mathbf{R}_+ e_1$. For the set $\Gamma = \{e_2\} \cup G$ we have $\operatorname{Max} \Gamma = \{e_2\}$ while $\operatorname{Max}_1 \Gamma = \emptyset$.

2.4 Relations with Other Concepts

In this subsection we discuss some concepts existing in the literature and very close to those introduced above. We start with the notion of supset Γ.

Definition 2.13 Let Γ be a non-empty subset of X and let $\succeq$ be a partial order. Put

$$\operatorname{supset} \Gamma := \{z \in [\Gamma, \infty[: \ [\Gamma, \infty[\cap] - \infty, z]\} = \{z\}\}.$$

Since $\operatorname{Sup} \Gamma$ does not contain comparable elements, $\operatorname{Sup} \Gamma \subseteq \operatorname{supset} \Gamma$. The definition of the supset was introduced and studied in papers [7–10] under the assumption that X is a vector space and the partial order is given by a proper cone G such that $G - G = X$. In the paper [10], in the same setting, the optimal set of Γ is defined as the set of maximal elements of Γ. In the case where Γ is closed this definition coincides with that of $\operatorname{Max} \Gamma$.

In the abovementioned papers the authors introduced the property (called Condition (A)) of the space X requiring that for every subset Γ and every $a \in [\Gamma, \infty[$ there is a minimal element $x \in [\Gamma, \infty[$ such that $x \succeq a$, i.e. an element of supset Γ. Thus, supset Γ satisfies all properties of $\operatorname{Sup}\Gamma$ and, by the uniqueness of the latter, both sets coincide. Condition (A) is satisfied when the space X is finite-dimensional.

The proposition below asserts that supset Γ and $\operatorname{Sup}\Gamma$ coincide without any assumption on the partial ordered space provided that $\operatorname{Sup}\Gamma$ is non-empty and the counterexample shows that it may happen that supset $\Gamma \neq \emptyset$ though $\operatorname{Sup}\Gamma$ is empty.

Proposition 2.14 *Let* Γ *be a subset of partially ordered space* X. *Then* $\operatorname{Sup}\Gamma =$ supset Γ *when* $\operatorname{Sup}\Gamma \neq \emptyset$.

Proof It remains to prove the inclusion supset $\Gamma \subseteq \operatorname{Sup}\Gamma$. Since $\operatorname{Sup}\Gamma \neq \emptyset$, for all $z \in$ supset Γ, there exists $\hat{z} \in \operatorname{Sup}\Gamma$ such that $z \succeq \hat{z}$. By definition of supset Γ, we have $z = \hat{z}$ and finally $z \in \operatorname{Sup}\Gamma$, i.e. supset $\Gamma \subseteq \operatorname{Sup}\Gamma$. □

Counterexample. Let $X = \mathbf{R}_+$ and let u be an arbitrary real-valued function on X. Let $x \succeq y$ when either $x = y$ or, simultaneously, $u(x) > u(y)$ and $x > y$. Obviously, this is a partial order.

Let us consider the function u with $u(x) = 2x$ when $0 \leq x < 1/2$ and $1 < x \leq 3/2$, $u(x) = 1$ when $1/2 \leq x \leq 1$, and u(x) strictly decreasing from its value $u(3/2) = 3$ to unit when $x > 3/2$. This function is continuous at all points X except $x = 1$ where it is left continuous and $u(x) > 1$ for all $x > 1$. Its graph is given on Fig. 1.

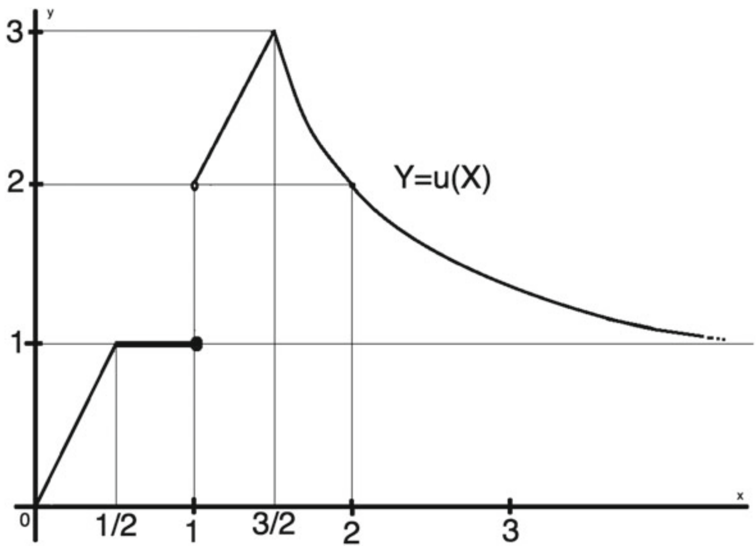

Fig. 1 Graphic representation of $x \mapsto u(x)$

Let $\Gamma := \{z : 0 \le z \le 1\}$. Note that $[\Gamma, \infty) = X \setminus \Gamma$. Indeed, the inclusion $\supseteq$ is obvious. On the other hand, if $x \in \Gamma$ and dominates Γ, then $x > y$ for any other element of Γ, i.e. $x = 1$. But $u(1/2) = u(1)$, i.e. we cannot have the relation $1 \succeq 1/2$.

Let us check that supset $\Gamma = \{x : x \ge 2\}$. Indeed, if $x \ge 2$ and $x \succeq y \succeq \Gamma$, then $x = y$: the inequalities $u(x) > u(y)$ and $1 \le y < x$ are not compatible. on the other hand, if $1 < x < 2$, there exists $y \succeq \Gamma$ such that $y < x$ and $u(y) < u(x)$, see Fig. 1, i.e. such that $1 \preceq y \preceq x$ and $y \ne x$. It follows that $x \notin$ supset Γ.

At last, suppose that $\sup \Gamma \ne \emptyset$. Since $3/2 \succeq \Gamma$, by virtue of (b_0), there exists $\hat{x} \in \sup \Gamma$ such that $3/2 \succeq \hat{x}$. By Proposition 2.14, $\hat{x} \in$ supset Γ. Hence, $\hat{x} \ge 2$. So, $3/2 \ne \hat{x}$ implying that $3/2 > 2$. A contradiction. □

3 Essential Supremum in $L^0(X, \mathcal{F})$

In this section, we discuss the concept of the Essential Supremum for sets of random variables. In the scalar case, the traditional definition is obtained by lifting the linear order of the real line (i.e. the order generated by the cone (ray) $\mathbf{R}_+$) to the space L^0 of classes of equivalent random variables. The straightforward analog for the vector-valued random variables could be a procedure consisting in lifting the preorder or partial order in $\mathbf{R}^d$ given by a fixed cone. A slightly more sophisticated generalization is related with random partial orders given by random cones in $\mathbf{R}^d$, the situation, typical in models of financial markets with transaction costs. In view of the previous section, it is natural to study the notion Esssup for the case when the preorder in $L^0(\mathbf{R}^d)$ is given by a countable random multi-utility representation. We consider the setting where the supremal set consists of $\mathcal{H}$-measurable random vectors, where $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{F}$. This additional feature seems to be new even in the scalar case where, usually, either $\mathcal{H} = \mathcal{F}$, or $\mathcal{H} = \{\emptyset, \Omega\}$, but we do not insist on this.

3.1 Essential Supremum in a General Setting

Let $(X, \mathcal{B}_X)$ be a separable metric space with its Borel σ-algebra and let $(\Omega, \mathcal{F}, P)$ be a probability space. As usual, $E\xi$ is the expectation of a real-valued random variable ξ. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. On the space $L^0(X, \mathcal{F})$ (of classes of equivalence) of X-valued random variables a preference relation is defined by a countable family $\mathcal{U} = \{u_j : \ j = 1, 2, ...\}$ of Carathéodory functions $u_j : \Omega \times X \to \mathbf{R}$, i.e. functions with the following properties:

(i) $u_j(., x) \in L^0(X, \mathcal{F})$ for every $x \in X$;
(ii) $u_j(\omega, .)$ is continuous for almost all $\omega \in \Omega$.

We recall that the important property of a Carathéodory function u on a separable metric space is that it is $\mathcal{F} \otimes \mathcal{B}_X$-measurable. Note that an order generated by a random cone can be generated by a countable family of linear Carathéodory functions, see the next subsection.

If $\gamma_1, \gamma_2 \in L^0(X, \mathcal{F})$, the relation $\gamma_2 \succeq \gamma_1$ means that $u_j(\gamma_2) \succeq u_j(\gamma_1)$ a.s. for all j. The just mentioned property ensures that the superpositions $u_j(\gamma_1)$, $u_2(\gamma_1)$ are random variables. The equivalence relation $\gamma_2 \sim \gamma_1$ has an obvious meaning.

We associate with an order interval $[\gamma_1, \gamma_2]$ in $L^0(X, \mathcal{F})$ its ω-sections, that is the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ in X corresponding to the partial orders represented by the families $\mathcal{U}(\omega) = \{u_j(\omega) : \ j = 1, 2, ...\}$.

Definition 3.1 Let Γ be a subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-Esssup Γ the maximal subset $\hat{\Gamma}$ of $L^0(X, \mathcal{H})$ such that the following conditions hold:

(a) $\hat{\Gamma} \succeq \Gamma$;
(b) if $\gamma \in L^0(X, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\gamma \succeq \hat{\gamma}$;
(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

An inspection of the proof of Theorem 3.7 in [4] (which deals with a partial order in $\mathbf{R}^d$) leads to the following statement [5]:

Theorem 3.2 *Let $\succeq$ be a preference relation in $L^0(X, \mathcal{F})$ represented by a countable family of Carathéodory functions. Let $\Gamma \neq \emptyset$ be such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(X, \mathcal{H})$. Suppose that for every $\gamma \in L^0([\Gamma, \infty[, \mathcal{H})$*

$$\Lambda(\gamma) := \text{argmin}_{\zeta \in L^0([\Gamma, \gamma], \mathcal{H})} Eu(\zeta) \neq \emptyset, \tag{3.1}$$

where $u(\omega, z) = \sum_{j=1}^{\infty} 2^{-j} \arctan u_j(\omega, z)$. Then

$$\mathcal{H}\text{−Esssup}\, \Gamma = \cup_{\gamma \in L^0([\Gamma, \infty], \mathcal{H})} \Lambda(\gamma) \neq \emptyset. \tag{3.2}$$

Solving (3.1), we "minimize" the $\mathcal{H}$-measurable random variables ζ dominating Γ with respect to every "direction" u_j. It is easy to check that under (3.1) the set defined by the right-hand side of (3.2) satisfies the properties required of $\mathcal{H}$−Esssup Γ. The verification of the condition (3.1) is far from being trivial. At the moment we are able to do this only in the case where the ω-sections of the order intervals $[\gamma_1, \gamma_2]$ are compact, see Theorem 3.7 in [4]. It is not clear how to extend this theorem, for a general $\mathcal{H}$ to the case of preorders, even under the assumption of compactness of the order intervals in the quotient space (the difficulty is that the quotient mapping is only $\mathcal{F}$-measurable).

3.2 Essential Supremum in $L^0(X)$ with Respect to a Random Cone

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let X be a separable Hilbert space. Let $\omega \mapsto G(\omega) \subseteq X$ be a measurable set-valued mapping whose values are closed convex cones. The measurability means that

$$\text{graph } G := \{(\omega, x) \in \Omega \times X : \ x \in G(\omega)\} \in \mathcal{F} \otimes \mathcal{B}_X.$$

The positive dual $G^*(\omega)$ of $G(\omega)$ is defined as the set

$$G^*(\omega) := \{x \in X : \ xy \geq 0, \ \forall y \in G(\omega)\},$$

where xy is the scalar product generating the norm $||.||$ in X. Recall that a measurable mapping whose values are closed subsets admits a Castaing representation, i.e. there is a countable set of measurable selectors ξ_i of G such that $G(\omega) = \overline{\{\xi_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$. Thus,

$$\text{graph } G^* = \{(\omega, y) \in \Omega \times X : \ y\xi_i(\omega) \geq 0, \ \forall i \in \mathbf{N}\} \in \mathcal{F} \otimes \mathcal{B}_X,$$

hence, G^* is a measurable mapping and it admits a Castaing representation, i.e. there exists a countable set of $\mathcal{G}$-measurable selectors η_i of G^* such that $G^*(\omega) = \overline{\{\eta_i(\omega) : \ i \in \mathbf{N}\}}$ for all $\omega \in \Omega$.

Since $G = (G^*)^*$,

$$G(\omega) = \{(\omega, x) \in \Omega \times X : \ \eta_i(\omega)x \geq 0, \ \forall i \in \mathbf{N}\}. \tag{3.3}$$

Therefore, the relation $\gamma_2 - \gamma_1 \in G$ (a.s.) defines a preference relation $\gamma_2 \succeq \gamma_1$ in $L^0(X, \mathcal{F})$ and possesses a countable multi-utility representation given by the random linear functions $u_j(\omega, x) = \eta_j(\omega)x$ where η_j is a Castaing representation of G^*.

Notation. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$ and let $\Gamma \subseteq L^0(X, \mathcal{F})$. We shall use the notation $(\mathcal{H}, G)$-Esssup Γ instead of $\mathcal{H}$-Esssup Γ to indicate that the partial order is generated by the random cone G. In the following, we use the notation $G^0 := (-G) \cap G$.

We get the following [4]:

Theorem 3.3 *Let X be a separable Hilbert space and let $\succeq$ be a preference relation in $L^0(X, \mathcal{F})$ defined by a random cone G. Suppose that the subspaces $(G^0(\omega))^\perp$ are finite-dimensional a.s. Let $\Gamma \neq \emptyset$ be such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(X, \mathcal{F})$. Then $\mathcal{F}-$Esssup $\Gamma \neq \emptyset$.*

Remark 3.4 If we suppose that the ω-sections of $G \subseteq \mathbf{R}^d$ are proper, i.e. $G^0 = \{0\}$, then the order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$ are compact. Therefore, the set $(\mathcal{H}, G)$-Esssup Γ exists if Γ is bounded from above (i.e. if there exists $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ such that $\bar{\gamma} - \Gamma \in G$.)

4 Essential Maximum in $L^0(X, \mathcal{F})$

4.1 Decomposability-Based Approach

Let us recall that the fastest proof of the existence of the element $\operatorname{esssup}\Gamma$ in the scalar case is assuming from the very beginning that the elements of Γ take values in the interval [0, 1]. It is sufficient to notice that one can replace Γ by the upward completion Γ^{up}, the smallest set containing Γ and closed with respect to operation $\vee$. Consider a sequence ξ_n on which $\sup_{\xi\in\Gamma}$ is attained, replace it by the monotone sequence $\xi^{(n)} = \xi_1 \vee ... \vee \xi_n$, and check that the limit of the latter is the required random variable. In the case where Γ^{up} is closed in L^0, it belongs to this set. It happens that this strategy of proof, appropriately modified, may work in the vector case and leads to a satisfactory definition of the maximal set. The approach presented in this section is developed in [4] and based on the notion of decomposability.

We start from some minor generalization of classical concepts, see, e.g. [6, 12].

Definition 4.1 The set $\Gamma \subseteq L^0(X, \mathcal{F})$ is $\mathcal{H}$-*decomposable* if for any its elements γ_1, γ_2 and $A \in \mathcal{H}$ the random variable $\gamma_1 I_A + \gamma_2 I_{A^c}$ belongs to Γ.

Definition 4.2 We denote by $\operatorname{env}_{\mathcal{H}}\Gamma$ the smallest $\mathcal{H}$-decomposable subset of $L^0(X, \mathcal{F})$ containing Γ and by $\operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$ its closure in $L^0(X, \mathcal{F})$.

The "interior" description of the $\mathcal{H}$-envelope of Γ is as follows:

Lemma 4.3 *The set* $\operatorname{env}_{\mathcal{H}}\Gamma$ *is formed by all random variables* $\sum \gamma_i I_{A_i}$ *where* $\gamma_i \in \Gamma$ *and* $\{A_i\}$ *is an arbitrary finite partition of* Ω *into* $\mathcal{H}$*-measurable subsets. Moreover,* $\mathcal{H}$-$\operatorname{cl}\operatorname{env}\Gamma$ *is* $\mathcal{H}$*-decomposable.*

We recall the two notions of Essential Maximum introduced in [5]:

Definition 4.4 Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We put

$$\mathcal{H}{-}\operatorname{Essmax}\Gamma = \{\gamma \in \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma :\ \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma \cap [\gamma, \infty[= [\gamma, \gamma]\}.$$

Definition 4.5 Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-$\operatorname{Essmax}_1\Gamma$ the largest subset $\hat{\Gamma} \subseteq \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$ such that the following conditions hold:

(i) if $\gamma \in \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{\gamma} \succeq \gamma$;
(ii) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

In the same way we define similar notions of $\mathcal{H}$-Essmin and $\mathcal{H}$-Essmin_1.

The proofs of the following two lemmata is exactly the same as of Lemmas 2.5 and 2.6.

Lemma 4.6 $\mathcal{H}$-$\operatorname{Essmax}_1\Gamma \subseteq \mathcal{H}$-$\operatorname{Essmax}\Gamma$.

Lemma 4.7 *Let* $\mathcal{H}$-$\operatorname{Essmax}_1\Gamma \neq \emptyset$. *Then* $\mathcal{H}$-$\operatorname{Essmax}_1\Gamma = \mathcal{H}$-$\operatorname{Essmax}\Gamma$.

In [5], it is shown that $\mathcal{H}$-Essmax$_1\Gamma = \mathcal{H}$-Essmax $\Gamma \neq \emptyset$ as soon as the preference relation is a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. The same result is also claimed for preference relations when Γ is a set of measurable selectors of a closed random set.

Corollary 4.8 *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d)$ such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. Let $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ be such that there is $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$ such that $\bar{\gamma} \succeq \Gamma$. Then*

$$\mathcal{H}\text{−Esssup}\,\Gamma = \text{Essmin}\, L^0([\Gamma, \infty), \mathcal{H}) = \text{Essmin}_1 L^0([\Gamma, \infty), \mathcal{H}) \neq \emptyset.$$

Proof Observe that $L^0([\Gamma, \infty), \mathcal{H}) \succeq \tilde{\gamma}$ where $\tilde{\gamma} \in \Gamma$ is arbitrary. Moreover, $L^0([\Gamma, \infty), \mathcal{H})$ is closed and $\mathcal{H}$-decomposable. Therefore, by [5], the set Essmin $L^0([\Gamma, \infty), \mathcal{H}) = \text{Essmin}_1 L^0([\Gamma, \infty), \mathcal{H}) \neq \emptyset$ is nonempty and satisfies the required properties to be the unique set $\mathcal{H}$−EsssupΓ. □

4.2 Convexity-Based Approach

In this subsection we suggest a new notion of the maximal set for the case where X is a separable normed space. The most interesting case: $\mathbf{R}^d$ with a partial order defined by a closed cone. Of course, for the scalar case and a closed set Γ, this maximal set is reduced to the singleton, containing the maximal point of Γ.

We denote by conv Γ the smallest convex subset of $L^0(X, \mathcal{F})$ containing Γ and by cl conv Γ its closure in $L^0(X, \mathcal{F})$.

Definition 4.9 Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We put

$$\text{Essmax}^c\,\Gamma = \{\gamma \in \text{cl conv}\,\Gamma : \ \text{cl conv}\,\Gamma \cap [\gamma, \infty[= [\gamma, \gamma]\}.$$

Remark 4.10 Suppose that $\Gamma \subseteq L^0(X, \mathcal{F})$ is both $\mathcal{H}$-decomposable, convex and closed where $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra. Then $\Gamma = \text{cl conv}\,\Gamma = \text{cl env}_{\mathcal{H}}\Gamma$ and, therefore, Essmax$^c\,\Gamma = \mathcal{H}$−Essmax Γ.

Definition 4.11 Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$.

We denote by Essmax$^c_1\Gamma$ the largest subset $\hat{\Gamma} \subseteq \text{cl conv}\,\Gamma$ such that the following conditions hold:

(i) if $\gamma \in \text{cl conv}\,\Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{\gamma} \succeq \gamma$;
(ii) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

As in the last section we have the following:

Lemma 4.12 $\mathcal{H}$-Essmax$^c_1\Gamma \subseteq \mathcal{H}$-Essmax$^c\Gamma$.

Lemma 4.13 *Let* $\mathcal{H}$-Essmax$^c_1\Gamma \neq \emptyset$. *Then* $\mathcal{H}$-Essmax$^c_1\Gamma = \mathcal{H}$-Essmax$^c\Gamma$.

Proposition 4.14 *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family of linear functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. Let Γ be a non-empty subset of $L^0(\mathbf{R}^d, \mathcal{F})$. Suppose that there exists $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F})$ such that $\bar{\gamma} \succeq \Gamma$. Then* $\mathrm{Essmax}^c\,\Gamma = \mathrm{Essmax}_1^c\Gamma \neq \emptyset$.

Proof Note that the set $\mathrm{Essmax}^c\,\Gamma$ obviously satisfies (ii) and it remains only to check (i) and that $\mathrm{Essmax}^c\,\Gamma \neq \emptyset$. For $\gamma \in \mathrm{cl\,conv}\,\Gamma$, we define random variables

$$\alpha_j(\gamma) := \alpha_j(\omega, \gamma) := 2^{-j}/(1 + |u_j(\gamma(\omega))| + |u_j(\bar{\gamma}(\omega))|).$$

Put

$$u(x, \gamma) := u(x, \omega, \gamma) := \sum_j \alpha_j(\gamma) u_j(\omega, x).$$

Then the mapping $\xi \mapsto u(\xi, \gamma)$ is well-defined for $\xi \in [\gamma, \bar{\gamma}]$ and for such an argument $u(\xi, \gamma)$ is a random variable with values in the interval $[-1, 1]$. Let

$$c := \sup_{\tilde{\gamma} \in \mathrm{cl\,conv}\,\Gamma \cap L^0([\gamma, \infty), \mathcal{F})} Eu(\tilde{\gamma}).$$

Let $(\tilde{\gamma}_n)$ be a sequence on which the supremum in the above definition is attained. As the set $\mathrm{cl\,conv}\,\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{H})$ is convex and $[\gamma, \bar{\gamma}]$ is compact a.s., we may assume without loss of generality (by applying Theorem 5.2.3 [6] on convergent subsequences) that the sequence of $\tilde{\gamma}_n$ converges a.s. to some $\tilde{\gamma}_\infty \in \mathrm{cl\,conv}\,\Gamma \cap L^0([\gamma, \infty), \mathcal{F})$ such that $c := Eu(\tilde{\gamma}_\infty)$.

By definition of c, it is straightforward that $\tilde{\gamma}_\infty \in \mathrm{Essmax}\,\Gamma$ and the conclusion follows. □

4.3 Comment on Essential Maximum of Processes

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in \mathbf{R}_+}, P)$ be a stochastic basis satisfying the usual assumptions and let $X = (X_t)_{t \in \mathbf{R}_+}$ and $Y = (Y_t)_{t \in \mathbf{R}_+}$ be two real-valued measurable processes. Following Dellacherie [2], we say that the process Y is *essential majorant* of X if the set $\{X > Y\} = \{(\omega, t) : \; X_t(\omega) > Y_t(\omega)\}$ is negligeable, i.e. its projection on Ω has zero probability (the projection is measurable because the σ-algebra is complete). Let Γ be an arbitrary set of measurable processes. The measurable process Y is the *essential supremum* of Γ (notation: $Y = \mathrm{ess.sup}\,\Gamma$) if Y is an essential majorant for every process from Γ and any other process Y' with the same property is an essential majorant of Y. Of course, in this definition the word "measurable" can be replaced by the words "optional", "predictable", etc. To get results one needs to impose certain assumptions on the regularity of trajectories. In view of financial applications, it is interesting to study the problem for the vector-valued processes. This is a problem for further studies.

5 Application in Finance: Hedging of European Options in a Discrete-Time Model with Transaction Costs

In this section, we recall an application in finance developed in [4, 5] for a market model with proportional transaction costs. Let us consider a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,\dots,T}, P)$ and a d-dimensional adapted price process $S = (S_t)$ with strictly positive components. Let $K \subset \mathbf{R}^d$ be a closed proper convex cone whose interior contains $\mathbf{R}^d_+ \setminus \{0\}$. We interpret K as the set of all solvent portfolio positions (expressed in some numéraire) that can be liquidated without any debt. Define the random diagonal operators

$$\phi_t : (x^1, \dots, x^d) \mapsto (x^1/S^1_t, \dots, x^d/S^d_t), \qquad t = 0, \dots, T,$$

and relate with them the random cones $\widehat{K}_t := \phi_t K$. We consider the set $\widehat{\mathcal{V}}$ of $\mathbf{R}^d$-valued adapted processes $\widehat{V}$ such that $\Delta \widehat{V}_t := \widehat{V}_t - \widehat{V}_{t-1} \in -\widehat{K}_t$ for all t and the set $\mathcal{V}$ whose elements are the processes V with $V_t = \phi_t^{-1} \widehat{V}_t$, $\widehat{V} \in \widehat{\mathcal{V}}$.

In the context of the theory of markets with proportional transaction costs, K is the solvency cone in a model with efficient friction corresponding to the description in terms of a numéraire, $\mathcal{V}$ is the set of value processes of self-financing portfolios. The notations with hat correspond to the description of the model in terms of "physical" units where the portfolio dynamics is much simpler because it does not depend on price movements. A typical example is the model of currency market defined via the matrix of transaction costs coefficients $\Lambda = (\lambda^{ij})$ with non-negative entries and $\lambda^{ii} = 0$. In this case

$$K = \text{cone}\,\{(1 + \lambda^{ij})e_i - e_j,\ e_i, 1 \le i, j \le d\}.$$

Another example is the commodity market where all transactions are payed from the money account. In this case

$$K = \text{cone}\,\{\gamma^{ij} e_1 + e_i,\ (1 + \gamma^{1i})e_1 - e_i,\ (-1 + \gamma^{j1})e_1 + e_j,\ e_i,\ 1 \le i, j \le d\}.$$

We assumed for simplicity that K is constant. In general, $K = (K_t)$ is an adapted random process whose values are convex closed proper cones, e.g., given by an adapted matrix-valued process $\Lambda = (\Lambda_t)$. But even in the constant case, $\widehat{K} = (\widehat{K}_t)$ is a random cone-valued process. Note that one can use modeling involving only $\widehat{K}$ defined, e.g., by the bid-ask (adapted matrix-valued) process but this is just a different parametrization leading to the same geometric structure.

In this model, the contingent claim is a d-dimensional random vector. We shall use the notation Y_T when the contingent claim is expressed in units of the numéraire and $\widehat{Y}_T$ when it is expressed in physical units. The relation is obvious: $\widehat{Y}_T = \phi_T Y_T$.

We shall work under the assumption that $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t), t \le T-1$, i.e. the absence of arbitrage opportunities of the second kind (**NA2**), see [6], Theorem

3.2.20. Note that it is always fulfilled when the price process S admits an equivalent martingale measure.

In the following, we use the convention that $\widehat{K}_{T+1} = \widehat{K}_T$. The value process $\widehat{V} \in \widehat{\mathcal{V}}$ is called *cheap* if $\widehat{V}_T = \widehat{Y}_T$ and any process $\widehat{W} \in \widehat{\mathcal{V}}$ such that $\widehat{W}_T = \widehat{Y}_T$ and $\widehat{W}_t \preceq_{\widehat{K}_{t+1}} \widehat{V}_t$ for all $t \le T$ coincides with $\widehat{V}$. Observe that under NA2 Condition, the inequality $\widehat{W}_t \preceq_{\widehat{K}_{t+1}} \widehat{V}_t$ is equivalent to $\widehat{W}_t \preceq_{\widehat{K}_{t+1} \cap \widehat{K}_t} \widehat{V}_t$, i.e. $\widehat{W}_t$ is cheaper than $\widehat{V}_t$ at times t and $t+1$ in the sense that we may turn the position $\widehat{V}_t$ into $\widehat{W}_t$ either at time t or $t+1$ by paying transaction costs. The questions of interest are whether cheap portfolios do exist and how they can be found. We denote $\widehat{\mathcal{V}}_{cheap}$ the set of all cheap processes.

Proposition 5.1 *Suppose that* $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \le T-1$, *and suppose there exits at least one* $\widehat{V} \in \widehat{\mathcal{V}}$ *such that* $\widehat{V}_T \ge_{\hat{K}_T} \hat{Y}_T$. *Then* $\widehat{\mathcal{V}}^E_{cheap} \neq \emptyset$ *and* $\widehat{\mathcal{V}}^E_{cheap}$ *coincides with the set of solutions of backward inclusions*

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_{t+1})\text{−Esssup}\{\widehat{V}_{t+1}\}, \quad t \le T-1, \quad \widehat{V}_T = \widehat{Y}_T. \tag{5.4}$$

Moreover, any $\widehat{W} \in \mathcal{V}$ *with* $\widehat{W}_T \succeq \widehat{Y}_T$ *is such that* $\widehat{W}_t \succeq_{\widehat{K}_t} \widehat{V}_t$, $t = 0, \cdots, T$, *for some* $\widehat{V} \in \mathcal{V}^E_{cheap}$.

Proof Let $\widehat{W} \in \widehat{\mathcal{V}}$ be such that $\widehat{W}_T \succeq_{\widehat{K}_T} \widehat{Y}_T$. Since $\Delta \widehat{W}_T \in -\widehat{K}_T$, we have $\widehat{W}_{T-1} \succeq_{\widehat{K}_T} \widehat{W}_T \succeq_{\widehat{K}_T} \widehat{Y}_T$. By definition of $(\mathcal{F}_{T-1}, \widehat{K}_T)$−Esssup and Theorem 3.2, we get that $\widehat{W}_{T-1} \succeq_{\widehat{K}_T} \widehat{V}_{T-1}$ where $\widehat{V}_{T-1} \in (\mathcal{F}_{T-1}, \widehat{K}_T)$−Esssup $\{\widehat{Y}_T\} \neq \emptyset$. Therefore, by the hypothesis, $\widehat{W}_{T-1} \succeq_{\widehat{K}_{T-1}} \widehat{V}_{T-1}$. Continuing the backward induction, we obtain that $\widehat{W}_t \succeq_{\widehat{K}_t} \widehat{V}_t$ where $\widehat{V}_t$ satisfies (5.4). We deduce that any portfolio $\widehat{W} \in \widehat{\mathcal{V}}_{cheap}$ satisfy (5.4). The same backward induction allows us to conclude that any $\widehat{V} \in \widehat{\mathcal{V}}$ which satisfies (5.4) is cheap. Indeed, let $\widehat{V}$ be such a portfolio process such that $\widehat{V}_t \succeq_{\widehat{K}_{t+1}} \widehat{W}_t$ for all t where $W \in \widehat{\mathcal{V}}$ satisfies $\widehat{W}_T \succeq_{\widehat{K}_T} \widehat{Y}_T$. Since $\widehat{V}_T = \widehat{Y}_T$, we get that $\widehat{V}_T = \widehat{W}_T$. By the first step, $\widehat{W}_{T-1} \succeq_{\widehat{K}_T} \widehat{U}_{T-1}$ where $\widehat{U}_{T-1} \in (\mathcal{F}_{T-1}, \widehat{K}_T)$−Esssup $\{\widehat{Y}_T\} \ni \widehat{V}_{T-1}$. Therefore, $\widehat{V}_{T-1} \succeq_{\widehat{K}_T} \widehat{U}_{T-1}$ which implies that $\widehat{V}_{T-1} = \widehat{U}_{T-1}$. We pursue the reasoning and finally get that $U = V$. □

The minimal portfolio processes $\widehat{\mathcal{V}}^E_{cheap}$ are obtained by solving expected utility minimization problems. Therefore, we have a constructive approach for superhedging prices for the European claim $\widehat{Y_T}$:

Corollary 5.2 *Suppose that* $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \le T-1$, *and suppose there exits at least one* $\widehat{V} \in \widehat{\mathcal{V}}$ *such that* $\widehat{V}_T \ge_{\widehat{K}_T} \widehat{Y}_T$. *The set of all superhedging prices for the contigent claim* $\widehat{Y_T}$ *is obtained by adding an arbitrary element of* G_0 *to an initial value of a portfolio process* $\widehat{V} \in \widehat{\mathcal{V}}^E_{cheap}$ *satisfying (5.4).*

An algorithm to compute the hedging sets for market models with transaction costs can be found in [11].

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References

1. Aliprantis, C., Border, K.: Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer, Berlin (2006)
2. Dellacherie, C.: Sur l'existence de certains essinf et essup de familles de processus mesurables. In: Seminaire de Probabilitś, XII, Lectures Notes in Mathematics, vol. 816, pp. 512-513. Springer, Berlin (1981)
3. Evren, O., Ok, E.A.: On the multi-utility representation of preference relations. J. Math. Econ. **14**(4–5), 554–563 (2011)
4. Kabanov, Y., Lepinette, E.: Essential supremum with respect to a random partial order. J. Math. Econ. **49**(6), 478–487 (2013)
5. Kabanov, Y., Lepinette, E.: Essential supremum and essential maximum with respect to preference relations. J. Math. Econ. **49**(6), 488–495 (2013)
6. Kabanov, Yu., Safarian, M.: Markets with Transaction Costs. Springer, Mathematical Theory (2009)
7. Komuro, N.: The set of upper bounds in ordered linear spaces. In: Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publisher, Tokyo, pp. 197–207 (2003)
8. Komuro, N., Koshi, S.: Supsets on partially ordered topological linear spaces. Taiwanese J. Math. **4**(2), 275–284 (2000)
9. Komuro, N.: Properties of the set of upper bounds in ordered linear spaces. Departmental Bulletin Paper, vol. 1298, pp. 12-17 (2002), . http://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/42677/1/1298_03.pdf?origin=publication_detail
10. Koshi, S.: Optimal set in ordered linear space. Results Math. **37**(3–4), 274–282 (2000)
11. Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. Int. J. Theor. Appl. Finance **17**, 2 (2014)
12. Molchanov, I.: Theory of Random Sets. Springer, Berlin (2005)

Generalized Minimality in Set Optimization

Daishi Kuroiwa

Abstract In this paper, we propose a generalized minimality in set optimization. At first, we introduce parametrized embedding functions, which includes the embedding function in the previous literatures. By using the embedding functions, we generalize notions of minimal solutions for set optimization, and give existence results of the generalized minimal solutions. Also we introduce parametrized scalarizing functions which are generalizations of scalarizing functions defined in the previous literatures, and we characterize the generalized minimal solutions by using the scalarizing functions.

Keywords Set optimization · Embedding approach · Unification and generalization of minimal solution · Existence of minimal solution · Generalized scalarizing function

1 Introduction

We study the following optimization problem (SP):

$$\text{(SP)}\quad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in X, \end{array}$$

where X is a nonempty set, F is a set-valued map from X to an ordered vector space E. Notions of minimal solutions of (SP) are defined in accordance with *set relations*, which are binary relations on the power set of E, e.g., see [12]. Such optimization problem (SP) is called *set optimization*.

For every set relation, notions of minimal solutions of (SP) can be defined. For example, l-minimal and u-minimal solutions are given by using set relations $\preceq_K^l$ and

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$\preceq^u_K$, respectively. We studied various notions and properties in each set relation, that is, notions of weak and proper minimal solutions of (SP), conditions for the existence for such minimal notions, duality results for such minimal notions, notions of convex functions for set-valued maps, and notions of derivatives for set-valued maps in each set relation. Therefore, it sometimes takes much time to observe them.

In this paper, we propose a unified approach to study set optimization which covers the study with respect to set relations $\preceq^l_K$ and $\preceq^u_K$, and we define a notion of minimality which is a generalization of l and u-minimality but also s-minimality, see [6]. In Sects. 2 and 3, we give preliminaries about vector and set optimization. In Sect. 4, we introduce parametrized embedding functions by observing behavior of a singleton, which is a generalization of the previous embedding function defined by the author, and we study properties of the parametrized embedding functions. By using the parametrized embedding functions, we define generalized minimal solutions for set optimization, and show existence theorems of the generalized minimal solutions. In Sect. 5, we introduce parametrized scalarizing functions which are generalizations of scalarizing functions defined in the previous literatures. By using the scalarizing functions, we characterize the generalized minimal solutions.

2 Preliminaries—Vector Optimization

Let C be a closed convex cone of a topological vector space E over $\mathbb{R}$ satisfying $C \cap (-C) = \{0\}$ and $\mathrm{int}C \neq \emptyset$, where 0 is the null vector and $\mathrm{int}C$ is the set of all interior points of C. The partial order $\leq_C$ is given by

$$x \leq_C y \text{ if and only if } y - x \in C,$$

and binary relation $<_C$ by

$$x <_C y \text{ if and only if } y - x \in \mathrm{int}C.$$

For any subset A of E, the set of all minimal elements of A with respect to C is written by

$$\begin{aligned}\mathrm{Min}(A \mid C) &= \{a \in A \mid (a - C) \cap A = \{a\}\}\\ &= \{a \in A \mid a' \in A, a' \leq_C a \Rightarrow a \leq_C a'\},\end{aligned}$$

and the set of all weak minimal elements of A with respect to C is written by

$$\begin{aligned}w\mathrm{Min}(A \mid C) &= \{a \in A \mid (a - \mathrm{int}C) \cap A = \emptyset\}\\ &= \{a \in A \mid \nexists a' \in A \text{ such that } a' <_C a\}.\end{aligned}$$

The positive polar cone of C is given by

$$C^+ = \{c^* \in E^* \mid \langle c^*, c\rangle \geq 0, \forall c \in C\},$$

where E^* is the continuous dual space of E, and it is well known that C^{++}, the second positive polar cone of C, which is given by

$$C^{++} = \{c \in E \mid \langle c^*, c\rangle \geq 0, \forall c^* \in C^*\},$$

coincides with C.

For a nonempty convex subset A of E, $x_0 \in w\text{Min}(A \mid C)$, that is, $A \cap (x_0 - \text{int}C) = \emptyset$ if and only if there exists $c^* \in C^+$ such that $\langle c^*, x_0\rangle = \min_{x \in A} \langle c^*, x\rangle$ by using a separation theorem.

In the nonconvex case, nonlinear scalarization is a well-known tool to study minimal and weak minimal elements. Such scalarizing functions are given as follows:

$$z(x) = \inf\{t \in \mathbb{R} \mid x \in te - C\},$$

or

$$f(x) = \inf\{t \in \mathbb{R} \mid x \in te + a - \text{int}C\},$$

for fixed $e \in C$ and $a \in E$, see [2, 14]. These two scalarizing functions, which are essentially the same because $z(x - a) = f(x)$ under the assumptions of this section, play very important roles to study vector optimization.

3 Preliminaries—Set Optimization

In the rest of the paper, let E be a normed vector space, and $\mathcal{C} := \mathcal{C}(E)$ be the family of all nonempty compact convex subsets of E. For each $A, B \in \mathcal{C}$ and $\lambda \in \mathbb{R}$,

$$A + B = \{x + y \mid x \in A, y \in B\} \text{ and } \lambda A = \{\lambda x \mid x \in A\},$$

and also $A - B = A + (-B)$. It is clear that $\mathcal{C}$ is not a vector space under these operators, because there does not exist $C \in \mathcal{C}$ satisfying $A + C = \{0\}$ for given $A \in \mathcal{C}$ which has at least two points.

Set relations are binary relations on $\mathcal{C}$ based on an ordering cone and these are the most important notions to consider set optimization problems. Throughout the paper, let K be a closed convex cone of E satisfying $K \cap (-K) = \{0\}$ and $\text{int}K \neq \emptyset$. We introduce set relations $\preceq^l_K$ and $\preceq^u_K$ on $\mathcal{C}$: for each $A, B \in \mathcal{C}$,

$$A \preceq^l_K B \text{ if and only if } A + K \supset B,$$

$$A \preceq^u_K B \text{ if and only if } A \subset B - K,$$

and weak set relations $\prec_K^l$ and $\prec_K^u$ on $\mathcal{C}$: for each $A, B \in \mathcal{C}$,

$$A \prec_K^l B \text{ if and only if } A + \operatorname{int} K \supset B,$$

$$A \prec_K^u B \text{ if and only if } A \subset B - \operatorname{int} K.$$

Also we define $A \sim_K^l B$ if $A \preceq_K^l B$ and $B \preceq_K^l A$. In the previous literatures [12], above set relations are called type (iii) or type (v), and then, these are written by $A \preceq_K^{(\mathrm{iii})} B$, $A \preceq_K^{(\mathrm{v})} B$, $A \prec_K^{(\mathrm{iii})} B$, and $A \prec_K^{(\mathrm{v})} B$, respectively.

Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$. By using these set relations, notions of minimality of $\mathcal{A}$ with respect to K are defined as follows: a set $A \in \mathcal{A}$ is said to be an l-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\ B \preceq_K^l A \Rightarrow A \preceq_K^l B,$$

and a set $A \in \mathcal{A}$ is said to be a weak l-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\ B \prec_K^l A \Rightarrow A \prec_K^l B,$$

or equivalently,

$$\nexists B \in \mathcal{A} \text{ such that } B \prec_K^l A.$$

Replacing l by u, notions of u-minimality and weak u-minimality of $\mathcal{A}$ are given.

Consider the following set-valued optimization problem:

$$\begin{array}{lll} \text{(SP)} & \text{Minimize} & F(x) \\ & \text{subject to} & x \in X, \end{array}$$

where X is a nonempty set and $F : X \to \mathcal{C}$. By using the notions of minimality defined above, we define notions of solutions of (SP) with respect to K. An element $x_0 \in X$ is said to be an l-minimal solution of (SP) if and only if $F(x_0)$ is an l-minimal element of $\{F(x) \mid x \in X\}$, and is said to be a weak l-minimal solution of (SP) if and only if $F(x_0)$ is a weak l-minimal element of $\{F(x) \mid x \in X\}$. In similar way, u-minimal solutions and weak u-minimal solutions are defined.

To study set-valued optimization problem (SP), many researchers have proposed several generalizations of scalarizing function which is given in the last section, see [1, 4, 5, 13]. In these literature, such scalarizing functions are classified broadly into the following four types:

$$\begin{aligned} I_e^l(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^l te + B\}, \\ I_e^u(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^u te + B\}, \\ S_e^l(A; B) &= \sup\{t \in \mathbb{R} \mid A \preceq_K^l te + B\}, \text{ and} \\ S_e^u(A; B) &= \sup\{t \in \mathbb{R} \mid A \preceq_K^u te + B\}. \end{aligned}$$

In this paper, we propose an idea of a unification of the above minimalities, and a unification of the above scalarizing functions. For this purpose, we introduce a partially ordered normed vector space in which the family $\mathcal{C}$ is embedded in the next section.

4 An Embedding Space and an Embedding Function

In this study, we provide a vector space in which the class $\mathcal{C}$ is embedded, in order to reformulate set optimization problem (SP) as a vector optimization problem. There are several literatures with respect to the construction of a vector space in which a family of convex sets is embedded, for example, see [15, 16]. In this section, we introduce a specialized embedding vector space $\mathcal{C}^2/\equiv$ and an embedding function ψ to observe l-minimal solutions of (SP). All definitions and results are based on the previous literatures, see [10, 11].

Let $\equiv$ be a binary relation on $\mathcal{C}^2$ defined by

$$(A, B) \equiv (C, D) \text{ if and only if } A + D + K = B + C + K,$$

then $\equiv$ is an equivalence relation on $\mathcal{C}^2$. To show this, the following cancellation law is used: for each $A, B, C \in \mathcal{C}$,

$$A + C + K = B + C + K \Rightarrow A + K = B + K.$$

Denote the equivalence class of $(A, B) \in \mathcal{C}$ as $[A, B] = \{(C, D) \in \mathcal{C}^2 \mid (A, B) \equiv (C, D)\}$, and the quotient space of $\mathcal{C}^2$ by $\equiv$ as $\mathcal{C}^2/\equiv \, = \{[A, B] \mid (A, B) \in \mathcal{C}^2\}$. On the quotient space, we define addition and scalar multiplication as follows:

$$[A, B] + [C, D] = [A + C, B + D],$$
$$\lambda \cdot [A, B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{C}^2/\equiv, +, \cdot)$ becomes a vector space over $\mathbb{R}$ with the null vector $[\{0\}, \{0\}](=: \theta)$. Clearly, $[A, A] = \theta$ for each $A \in \mathcal{C}$ by using the cancellation law. Next we can define a norm on $\mathcal{C}^2/\equiv$ for a given bounded base W of K^+, that is $\cup_{\lambda \geq 0} \lambda W = K^+$, whose closure does not contain 0. The existence of such W is guaranteed by $\text{int}K \neq \emptyset$, for example, see [7]. Define

$$\|[A, B]\| = \sup_{w \in W} |\inf \langle w, A\rangle - \inf \langle w, B\rangle| ,$$

for every $[A, B] \in \mathcal{C}^2/\equiv$, then $\| \cdot \|$ is a norm on $\mathcal{C}^2/\equiv$, and we equip the vector space $\mathcal{C}^2/\equiv$ with the topology which is induced by the norm. Let $\mathcal{K}$ be defined as

$$\mathcal{K} = \{[A, B] \in \mathcal{C}^2/\equiv \mid B \preceq^l_K A\}.$$

Then $\mathcal{K}$ is a closed convex cone with nonempty interior, $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$ and moreover

$$\text{int}\mathcal{K} = \{[A, B] \in \mathcal{C}^2/\!\equiv \mid B \prec_K A\}.$$

From this, we can define the following partial order $\leq_{\mathcal{K}}$ and binary relation $<_{\mathcal{K}}$ on $\mathcal{C}^2/\!\equiv$ in the same manner to vector optimization: for each $[A, B], [C, D] \in \mathcal{C}^2/\!\equiv$,

$$[A, B] \leq_{\mathcal{K}} [C, D] \text{ if and only if } [C, D] - [A, B] \in \mathcal{K},$$

and

$$[A, B] <_{\mathcal{K}} [C, D] \text{ if and only if } [C, D] - [A, B] \in \text{int}\mathcal{K}.$$

Let $(\mathcal{C}^2/\!\equiv)^*$ be the continuous dual space of $\mathcal{C}^2/\!\equiv$. The positive polar cone of $\mathcal{K}$ is given by

$$\mathcal{K}^+ = \{T \in (\mathcal{C}^2/\!\equiv)^* \mid \langle T, [A, B]\rangle \geq 0, \forall [A, B] \in \mathcal{K}\},$$

and the second positive polar cone of $\mathcal{K}$ is given by

$$\mathcal{K}^{++} = \{[A, B] \in \mathcal{C}^2/\!\equiv \mid \langle T, [A, B]\rangle \geq 0, \forall T \in \mathcal{K}^+\}.$$

Also we have $\mathcal{K} = \mathcal{K}^{++}$ from the closedness of convex cone $\mathcal{K}$.

Define an embedding function $\psi : \mathcal{C} \to \mathcal{C}^2/\!\equiv$ by

$$\psi(A) = [A, \{0\}]$$

for all $A \in \mathcal{C}$. Because $\mathcal{C}^2/\!\equiv$ is an ordered normed vector space with convex cone $\mathcal{K}$, we reconsider notions of minimality with respect to $\preceq^l_K$ by using the embedding function. For a subfamily $\mathcal{A}$ of $\mathcal{C}$, $A \in \mathcal{A}$ is l-minimal of $\mathcal{A}$ with respect to K

$$\begin{aligned}
&\Longleftrightarrow B \in \mathcal{A}, B \preceq^l_K A \Rightarrow A \preceq^l_K B \\
&\Longleftrightarrow B \in \mathcal{A}, [A, B] \in \mathcal{K} \Rightarrow [B, A] \in \mathcal{K} \\
&\Longleftrightarrow B \in \mathcal{A}, [A, \{0\}] - [B, \{0\}] \in \mathcal{K} \Rightarrow [B, \{0\}] - [A, \{0\}] \in \mathcal{K} \\
&\Longleftrightarrow B \in \mathcal{A}, \psi(A) - \psi(B) \in \mathcal{K} \Rightarrow \psi(B) - \psi(A) \in \mathcal{K} \\
&\Longleftrightarrow B \in \mathcal{A}, \psi(A) - \psi(B) \in \mathcal{K} \Rightarrow \psi(B) - \psi(A) = \theta \\
&\Longleftrightarrow B \in \mathcal{A}, \psi(B) \in \psi(A) - \mathcal{K} \Rightarrow \psi(B) = \psi(A) \\
&\Longleftrightarrow \psi(\mathcal{A}) \cap (\psi(A) - \mathcal{K}) \subset \{\psi(A)\} \\
&\Longleftrightarrow \psi(A) \in \text{Min}(\psi(\mathcal{A}) \mid \mathcal{K}) \\
&\Longleftrightarrow \psi(A) \text{ is a minimal element of } \psi(\mathcal{A}) \text{ with respect to } \mathcal{K}.
\end{aligned}$$

Therefore l-minimality is represented by minimality of vector optimization. Also, $A \in \mathcal{A}$ is weak l-minimal of $\mathcal{A}$ with respect to K if and only if $\psi(A) \in w\text{Min}(\psi(\mathcal{A}) \mid \mathcal{K})$, that is, $\psi(A)$ is a weak minimal element of $\psi(\mathcal{A})$ with respect to $\mathcal{K}$. In the same way, set optimization

$$\text{(SP)}\ \ \text{Minimize}\ \ F(x)\\ \text{subject to}\ \ x \in X,$$

can be regarded as the following vector optimization:

$$\text{(VP)}\ \ \text{Minimize}\ \ \psi \circ F(x)\\ \text{subject to}\ \ x \in X.$$

An element $x_0 \in X$ is an l-minimal solution of (SP) in the last section if and only if $\psi \circ F(x_0) \in \text{Min}(\psi \circ F(X) \mid \mathcal{K})$ and $x_0 \in X$ is a weak l-minimal solution of (SP) in the last section if and only if $\psi \circ F(x_0) \in w\text{Min}(\psi \circ F(X) \mid \mathcal{K})$, where $\psi \circ F(X) = \{\psi(F(x)) \mid x \in X\}$.

The embedding space $\mathcal{C}^2/\equiv$ and the embedding function ψ play very important role to study l-minimal solutions and weak l-minimal solutions of set optimization problems. In the rest of this paper, we propose parameterized embedding functions ψ_λ, which include the previous embedding function ψ. By using the parametrized embedding functions, we define notions of generalized minimal solutions, and we characterize such solutions by using given parametrized scalarizing functions.

5 Parameterized Embedding Functions

At first, we give an important observation of a singleton $\{a\} \subset E$ as follows:

$$[\{a\}, \{0\}] = [\{0\}, -\{a\}] = [(1-\lambda)\{a\}, -\lambda\{a\}],$$

for each $\lambda \in \mathbb{R}$. Indeed, the first equality follows from $\{a\} + (-\{a\}) = \{0\} + \{0\}$ and the second equality follows from $\{0\} - \lambda\{a\} = -\{a\} + (1-\lambda)\{a\}$. From the observation, we define new embedding functions $\psi_\lambda : \mathcal{C} \to \mathcal{C}^2/\equiv$ as follows:

$$\psi_\lambda(A) = [(1-\lambda)A, -\lambda A]$$

for each $A \in \mathcal{C}$. Clearly ψ_0 is the same to ψ, which was given previously. By using the embedding function, we have the following remarkable proposition:

Proposition 1 *For each $A, B \in \mathcal{C}$, the following are satisfied:*

(i) $\psi_0(A) \leq_{\mathcal{K}} \psi_0(B)$ *if and only if* $A \preceq^l_K B$,
(ii) $\psi_0(A) <_{\mathcal{K}} \psi_0(B)$ *if and only if* $A \prec^l_K B$,
(iii) $\psi_0(A) = \psi_0(B)$ *if and only if* $A \sim^l_K B$,
(iv) $\psi_1(A) \leq_{\mathcal{K}} \psi_1(B)$ *if and only if* $A \preceq^u_K B$,
(v) $\psi_1(A) <_{\mathcal{K}} \psi_1(B)$ *if and only if* $A \prec^u_K B$, *and*
(vi) $\psi_1(A) = \psi_1(B)$ *if and only if* $A \sim^u_K B$.

Proof Proof of (i) is shown as follows:

$$\psi_0(A) \leq_{\mathcal{K}} \psi_0(B) \iff [A, \{0\}] \leq_{\mathcal{K}} [B, \{0\}] \iff \theta \leq_{\mathcal{K}} [B, A] \iff A \preceq^l_K B.$$

From this and $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$, (iii) is given immediately. Proof of (iv) is given in similar way:

$$\begin{aligned} \psi_1(A) \leq_{\mathcal{K}} \psi_1(B) &\iff [\{0\}, -A] \leq_{\mathcal{K}} [\{0\}, -B] \iff \theta \leq_{\mathcal{K}} [-A, -B] \\ &\iff -A \preceq^l_K -B \iff B \preceq^u_K A. \end{aligned}$$

Proofs of (ii), (v) and (vi) are similar and omitted. □

Motivated by Proposition 1, we give the following notations:

$$\begin{aligned} &A \preceq^\lambda_K B \textit{ if and only if } \psi_\lambda(A) \leq_{\mathcal{K}} \psi_\lambda(B), \\ &A \prec^\lambda_K B \textit{ if and only if } \psi_\lambda(A) <_{\mathcal{K}} \psi_\lambda(B), \quad \text{and} \\ &A \sim^\lambda_K B \textit{ if and only if } \psi_\lambda(A) = \psi_\lambda(B). \end{aligned}$$

Clearly these include binary relations $\preceq^l_K$, $\preceq^u_K$, $\prec^l_K$, $\prec^u_K$, $\sim^l_K$ and $\sim^u_K$.

Now we observe properties of the parametrized embedding functions.

Proposition 2 *For each $A \in \mathcal{C}$, the following are satisfied:*

(i) for each $\alpha, \beta \in [0, \infty)$, $\alpha A + \beta A = (\alpha + \beta)A$;
(ii) for each $\alpha, \beta \in [0, \infty)$, $[\alpha A, \beta A] = (\alpha - \beta)[A, \{0\}]$;
(iii) if $\lambda \leq 0$ then $\psi_\lambda(A) = \psi_0(A)$;
(iv) if $1 \leq \lambda$ then $\psi_\lambda(A) = \psi_1(A)$.

Proof Let $A \in \mathcal{C}$ and $\alpha, \beta \in [0, \infty)$. (i) is shown from the convexity of A. Indeed, we may assume that $\alpha + \beta > 0$. Then

$$\alpha A + \beta A = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}A + \frac{\beta}{\alpha + \beta}A\right) = (\alpha + \beta)A.$$

Next we show (ii). When $\alpha > \beta$, since $\alpha = (\alpha - \beta) + \beta$ and $\alpha - \beta, \beta \geq 0$, we have

$$\begin{aligned} [\alpha A, \beta A] &= [(\alpha - \beta)A + \beta A, \beta A] = [(\alpha - \beta)A, \{0\}] + [\beta A, \beta A] \\ &= [(\alpha - \beta)A, \{0\}] = (\alpha - \beta)[A, \{0\}]. \end{aligned}$$

The first equality is shown from (i). In similar way, when $\alpha \leq \beta$, since $\beta = \alpha + (\beta - \alpha)$ and $\alpha, \beta - \alpha \geq 0$, we have

$$\begin{aligned} [\alpha A, \beta A] &= [\alpha A, \alpha A + (\beta - \alpha)A] = [\alpha A, \alpha A] + [\{0\}, (\beta - \alpha)A] \\ &= [\{0\}, (\beta - \alpha)A] = (\beta - \alpha)[\{0\}, A] = (\alpha - \beta)[A, \{0\}]. \end{aligned}$$

Next we show (iii). Let $\lambda \leq 0$. Since $0 \leq -\lambda \leq 1 - \lambda$, by using (ii), we have

$$\psi_\lambda(A) = [(1-\lambda)A, -\lambda A] = ((1-\lambda) - (-\lambda))[A, \{0\}] = [A, \{0\}] = \psi_0(A).$$

The proof of (iv) is similar to (iii) and omitted. □

The next proposition is about monotonicity of the embedding functions with respect to variable λ for a given $A \in C$.

Proposition 3 *Let $A \in \mathcal{C}$ and $0 \leq \lambda_0 \leq \lambda_1 \leq 1$. Then the following are satisfied:*

(i) $\psi_{\lambda_1}(A) - \psi_{\lambda_0}(A) = (\lambda_1 - \lambda_0)[\{0\}, A - A]$*;*
(ii) $\psi_{(1-t)\lambda_0 + t\lambda_1}(A) = (1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A)$ *for each* $t \in [0, 1]$*;*
(iii) $\psi_{\lambda_0}(A) \leq_{\mathcal{K}} \psi_{\lambda_1}(A)$*;*
(iv) $\lambda_0 < \lambda_1$ *and* $A - A \prec^l_K \{0\}$ *if and only if* $\psi_{\lambda_0}(A) <_{\mathcal{K}} \psi_{\lambda_1}(A)$*;*
(v) if $\lambda_0 < \lambda_1$ *and A is not a singleton, then* $\psi_{\lambda_0}(A) \neq \psi_{\lambda_1}(A)$.

Proof Let $A \in \mathcal{C}$ and $0 \leq \lambda_0 \leq \lambda_1 \leq 1$. The proof of (i) is as follows:

$$\begin{aligned}
\psi_{\lambda_1}(A) - \psi_{\lambda_0}(A) &= [(1-\lambda_1)A, -\lambda_1 A] - [(1-\lambda_0)A, -\lambda_0 A] \\
&= [(1-\lambda_1)A - \lambda_0 A, (1-\lambda_0)A - \lambda_1 A] \\
&= [(1-\lambda_1)A, (1-\lambda_0)A] + [\lambda_0(-A), \lambda_1(-A)] \\
&= (\lambda_0 - \lambda_1)[A, \{0\}] + (\lambda_0 - \lambda_1)[-A, \{0\}] \\
&= (\lambda_0 - \lambda_1)[A - A, \{0\}] \\
&= (\lambda_1 - \lambda_0)[\{0\}, A - A].
\end{aligned}$$

The fourth equality is shown by using Proposition 2 (ii). Next we show (ii). For each $t \in [0, 1]$, by using (i),

$$\begin{aligned}
&\psi_{(1-t)\lambda_0 + t\lambda_1}(A) - \psi_{\lambda_0}(A) = t(\lambda_1 - \lambda_0)[\{0\}, A - A], \quad \text{and} \\
&\psi_{\lambda_1}(A) - \psi_{(1-t)\lambda_0 + t\lambda_1}(A) = (1-t)(\lambda_1 - \lambda_0)[\{0\}, A - A],
\end{aligned}$$

and then, we have the following equality, which is equivalent to (ii):

$$(1-t)(\psi_{(1-t)\lambda_0 + t\lambda_1}(A) - \psi_{\lambda_0}(A)) = t(\psi_{\lambda_1}(A) - \psi_{(1-t)\lambda_0 + t\lambda_1}(A)).$$

We show (iii). Since $A - A \ni 0$, it is clear that $A - A + K \supset \{0\}$, that is, $A - A \preceq^l_K \{0\}$, or equivalently $[\{0\}, A - A] \in \mathcal{K}$, and then we have $(\lambda_1 - \lambda_0)[\{0\}, A - A] \in \mathcal{K}$ because $\mathcal{K}$ is a cone and $\lambda_1 - \lambda_0 \geq 0$. The proof of (iv) is similar to (iii). Finally we show (v). Assume that $\lambda_0 < \lambda_1$, A is not a singleton, and $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$. From (i) and $\lambda_1 - \lambda_0 > 0$, we have $[\{0\}, A - A] = \theta$, or equivalently, $A - A + K = K$. Since A is not a singleton, there exist different two elements $a, a' \in A$. Since $A - A \subset K$, $a - a' \in K$ and $a' - a \in K$, therefore $a - a' \in K \cap (-K) = \{0\}$. This is a contradiction. □

We have observed that if A is a singleton then all embedding functions ψ_λ have the same image at the beginning of this section. The inverse implication holds from the following proposition:

Proposition 4 *For each $A \in \mathcal{C}$, the following are equivalent:*

(i) $A = \{a\}$ *for some* $a \in E$*;*
(ii) *there exist different* $\lambda_0, \lambda_1 \in [0, 1]$ *such that* $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$*;*
(iii) *for each* $\lambda_0, \lambda_1 \in [0, 1]$, $\psi_{\lambda_0}(A) = \psi_{\lambda_1}(A)$.

Proof It is clear that (i) implies (iii), which is the first observation of this section, and (iii) implies (ii). Also (ii) implies (i) from (v) of Proposition 3. □

The following property is essential to define generalized minimality of (SP):

Proposition 5 *Let* $A, B \in \mathcal{C}$ *and* $0 \le \lambda_0 < \lambda_1 \le 1$. *The following are satisfied:*

(i) *both* $A \preceq_K^{\lambda_0} B$ *and* $A \preceq_K^{\lambda_1} B$ *if and only if* $A \preceq_K^{\lambda} B$ *for every* $\lambda \in (\lambda_0, \lambda_1)$*;*
(ii) *both* $A \prec_K^{\lambda_0} B$ *and* $A \prec_K^{\lambda_1} B$ *if and only if* $A \prec_K^{\lambda} B$ *for every* $\lambda \in [\lambda_0, \lambda_1]$*;*
(iii) $\{\lambda \in [0, 1] \mid A \preceq_K^{\lambda} B\}$ *is a closed interval, a singleton or empty;*
(iv) $\{\lambda \in [0, 1] \mid A \prec_K^{\lambda} B\}$ *is an interval which is open in* $[0, 1]$ *or empty.*

Proof Let $A, B \in \mathcal{C}$ and $0 \le \lambda_0 < \lambda_1 \le 1$. We show (i). Assume that $A \preceq_K^{\lambda_0} B$ and $A \preceq_K^{\lambda_1} B$, that is, both $\psi_{\lambda_0}(A) \le_{\mathcal{K}} \psi_{\lambda_0}(B)$ and $\psi_{\lambda_1}(A) \le_{\mathcal{K}} \psi_{\lambda_1}(B)$. For any $\lambda \in (\lambda_0, \lambda_1)$, $\lambda = (1-t)\lambda_0 + t\lambda_1$ for some $t \in (0, 1)$. From (ii) of Proposition 3,

$$\psi_\lambda(A) = (1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A) \text{ and } \psi_\lambda(B) = (1-t)\psi_{\lambda_0}(B) + t\psi_{\lambda_1}(B).$$

This implies $\psi_\lambda(A) \le_{\mathcal{K}} \psi_\lambda(B)$, that is, $A \preceq_K^{\lambda} B$. Conversely, assume that $A \preceq_K^{\lambda} B$, that is, $\psi_\lambda(A) \le_{\mathcal{K}} \psi_\lambda(B)$ for every $\lambda \in (\lambda_0, \lambda_1)$. This is equivalent to

$$(1-t)\psi_{\lambda_0}(A) + t\psi_{\lambda_1}(A) \le_{\mathcal{K}} (1-t)\psi_{\lambda_0}(B) + t\psi_{\lambda_1}(B)$$

for every $t \in (0, 1)$ by using (ii) of Proposition 3. From the closedness of $\mathcal{K}$, we have $\psi_{\lambda_0}(A) \le_{\mathcal{K}} \psi_{\lambda_0}(B)$ and $\psi_{\lambda_1}(A) \le_{\mathcal{K}} \psi_{\lambda_1}(B)$ by considering the cases $t \searrow 0$ and $t \nearrow 1$. The proof of (ii) is similar to (i) and omitted. We show (iii). Put $\Lambda = \{\lambda \in [0, 1] \mid A \preceq_K^{\lambda} B\}$. We may assume that $|\Lambda| > 1$. For any $\lambda_0, \lambda_1 \in \Lambda$ such that $\lambda_0 < \lambda_1$, we have $(\lambda_0, \lambda_1) \subset \Lambda$ from (i). This shows that Λ is an interval in $[0, 1]$. To prove that Λ is closed, choose a sequence $\{\lambda_n\} \subset \Lambda$ converges to λ_0. We will show that $A \preceq_K^{\lambda_0} B$, that is,

$$(1-\lambda_0)A - \lambda_0 B + K \supset -\lambda_0 A + (1-\lambda_0)B.$$

For any $a \in A$ and $b \in B$, since

$$(1-\lambda_n)A - \lambda_n B + K \supset -\lambda_n A + (1-\lambda_n)B$$

for every $n \in \mathbb{N}$, there exist $\{a_n\} \subset A$, $\{b_n\} \subset B$ and $\{k_n\} \subset K$ such that

$$(1-\lambda_n)a_n - \lambda_n b_n + k_n = -\lambda_n a + (1-\lambda_n)b$$

for every $n \in \mathbb{N}$. From the compactness of A and B, we can choose a subsequence $\{n'\}$ of $\{n\}$ such that $\{a_{n'}\}$ converges to some $a_0 \in A$ and $\{b_{n'}\}$ converges to some $b_0 \in B$. Therefore $\{k_{n'}\}$ converges to $k_0 = (1-\lambda_0)(b-a_0) + \lambda_0(b_0 - a)$, which is an element of K because K is closed, and

$$(1-\lambda_0)A - \lambda_0 B + K \ni (1-\lambda_0)a_0 - \lambda_0 b_0 + k_0 = -\lambda_0 a + (1-\lambda_0)b.$$

Finally we show (iv). Put $\Lambda = \{\lambda \in [0,1] \mid A \prec_K^{\lambda} B\}$. In similar way to (iii), Λ is an interval. We show Λ is open in $[0,1]$. Let $\lambda_0 \in \Lambda$. Since $(1-\lambda_0)A - \lambda_0 B + \mathrm{int}K \supset -\lambda_0 A + (1-\lambda_0)B$, there exists $r > 0$ such that

$$(1-\lambda_0)A - \lambda_0 B + K \supset -\lambda_0 A + (1-\lambda_0)B + 3rU,$$

where U is the unit closed ball of E. Put $\varepsilon = r \inf \|W\| / \max\{\|[-A, B]\|, \|[A, -B]\|\}$. We will show that $\{\lambda \in [0,1] \mid |\lambda - \lambda_0| \leq \varepsilon\} \subset \Lambda$. For any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \varepsilon$,

$$|\lambda - \lambda_0| \|[-A, B]\| \leq r \inf \|W\| \;\text{ and }\; |\lambda - \lambda_0| \|[A, -B]\| \leq r \inf \|W\|,$$

then for any $w \in W$,

$$\begin{aligned}
&(\lambda_0 - \lambda)\left(\inf \langle w, -A\rangle - \inf \langle w, B\rangle\right) \leq r\|w\|, \quad \text{and} \\
&(\lambda - \lambda_0)\left(\inf \langle w, A\rangle - \inf \langle w, -B\rangle\right) \leq r\|w\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&\inf \langle w, -\lambda A + (1-\lambda)B\rangle \geq \inf \langle w, -\lambda_0 A + (1-\lambda_0)B\rangle - r\|w\|, \quad \text{and} \\
&\inf \langle w, (1-\lambda_0)A - \lambda_0 B\rangle + r\|w\| \geq \inf \langle w, (1-\lambda)A - \lambda B\rangle.
\end{aligned}$$

Therefore, for any $w \in W$,

$$\begin{aligned}
\inf \langle w, -\lambda A + (1-\lambda)B + rU\rangle &= \inf \langle w, -\lambda A + (1-\lambda)B\rangle - r\|w\| \\
&\geq \inf \langle w, -\lambda_0 A + (1-\lambda_0)B\rangle - 2r\|w\| \\
&= \inf \langle w, -\lambda_0 A + (1-\lambda_0)B + 3rU\rangle + r\|w\| \\
&\geq \inf \langle w, (1-\lambda_0)A - \lambda_0 B\rangle + r\|w\| \\
&\geq \inf \langle w, (1-\lambda)A - \lambda B\rangle.
\end{aligned}$$

This shows

$$(1-\lambda)A - \lambda B + K \supset -\lambda A + (1-\lambda)B + rU,$$

that is, $A \prec_K^\lambda B$. This completes the proof. □

Motivated by Proposition 5, we define Λ-minimality as follows:

Definition 1 Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$, $A \in \mathcal{A}$, and Λ be a nonempty subset of $[0, 1]$. The set A is said to be a Λ-minimal element of $\mathcal{A}$ with respect to $\mathcal{K}$ if and only if

$$B \in \mathcal{A},\ B \preceq_K^\lambda A \text{ for any } \lambda \in \Lambda \Rightarrow A \preceq_K^\lambda B \text{ for any } \lambda \in \Lambda,$$

or equivalently,

$$\nexists B \in \mathcal{A} \text{ s.t. } \forall\lambda \in \Lambda,\ B \preceq_K^\lambda A \text{ and } \exists\lambda_0 \in \Lambda \text{ s.t. } A \not\preceq_K^{\lambda_0} B,$$

and A is said to be a weak Λ-minimal element of $\mathcal{A}$ with respect to $\mathcal{K}$ if and only if

$$\nexists B \in \mathcal{A} \text{ s.t. } \forall\lambda \in \Lambda,\ B \prec_K^\lambda A \text{ and } \exists\lambda_0 \in \Lambda \text{ s.t. } A \not\prec_K^{\lambda_0} B,$$

When $\Lambda = \{\lambda\}$, λ-minimality and weak λ-minimality mean Λ-minimality and weak Λ-minimality respectively.

Clearly, $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if

$$\psi_\lambda(A) \in \mathrm{Min}(\psi_\lambda(\mathcal{A}) \mid \mathcal{K})$$

and $A \in \mathcal{A}$ is a weak λ-minimal element of $\mathcal{A}$ if and only if

$$\psi_\lambda(A) \in w\mathrm{Min}(\psi_\lambda(\mathcal{A}) \mid \mathcal{K}).$$

The notion of Λ-minimality includes not only the notions of l and u-minimality, but also the notion of s-minimality, which was introduced in [6]. Indeed, 0-minimality, weak 0-minimality, 1-minimality, and weak 1-minimality are equivalent to l-minimality, weak l-minimality, u-minimality, and weak u-minimality, respectively. For a given family $\mathcal{A} \subset \mathcal{C}$, remember that $A \in \mathcal{A}$ is said to be an s-minimal element of $\mathcal{A}$ if and only if

$$B \in \mathcal{A},\ B \preceq_K^s A \Rightarrow A \preceq_K^s B,$$

where set relation $A \preceq_K^s B$ is defined by $A \preceq_K^l B$ and $A \preceq_K^u B$. From Proposition 5,

$$A \preceq_K^s B \iff A \preceq_K^0 B \text{ and } A \preceq_K^1 B \iff A \preceq_K^\lambda B \text{ for all } \lambda \in [0, 1],$$

and this shows the equivalence of s-minimality and $[0, 1]$-minimality. More generally, Λ-minimality is equivalent to $(\text{co}\,\Lambda)$-minimality, where $\text{co}\,\Lambda$ is the convex hull of Λ.

Proposition 6 *Let $\mathcal{A}$ be a subfamily of $\mathcal{C}$, $A \in \mathcal{A}$, and Λ, Λ' be nonempty subsets of* $[0, 1]$. *The following are satisfied:*

(i) A is a Λ-minimal element of $\mathcal{A}$ if and only if A is a $\text{co}(\Lambda)$*-minimal element of $\mathcal{A}$;*
(ii) A is a weak Λ-minimal element of $\mathcal{A}$ if and only if A is a weak $\text{co}(\Lambda)$*-minimal element of $\mathcal{A}$;*
(iii) if A is a Λ-minimal element of $\mathcal{A}$ and A is a Λ'-minimal element of $\mathcal{A}$, then A is a $\Lambda \cup \Lambda'$-minimal element of $\mathcal{A}$;
(iv) if A is a weak Λ-minimal element of $\mathcal{A}$ and A is a weak Λ'-minimal element of $\mathcal{A}$, then A is a weak $\Lambda \cup \Lambda'$-minimal element of $\mathcal{A}$.

Proof We show (i). Assume that A is a Λ-minimal element of $\mathcal{A}$, $B \in \mathcal{A}$ and $B \preceq_K^{\lambda} A$ for all $\lambda \in \text{co}(\Lambda)$. Since $\Lambda \subset \text{co}(\Lambda)$ and A is a Λ-minimal element of $\mathcal{A}$, $A \preceq_K^{\lambda} B$ for all $\lambda \in \Lambda$. For any $\lambda \in \text{co}(\Lambda) \setminus \Lambda$, there exist $\lambda_0, \lambda_1 \in \Lambda$ such that $\lambda \in (\lambda_0, \lambda_1)$. Since $A \preceq_K^{\lambda_0} B$ and $A \preceq_K^{\lambda_1} B$, $A \preceq_K^{\lambda} B$ holds by using (i) of Proposition 5. This shows A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$. Conversely, Assume that A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$, $B \in \mathcal{A}$ and $B \preceq_K^{\lambda} A$ for all $\lambda \in \Lambda$. By using (i) of Proposition 5, we have $B \preceq_K^{\lambda} A$ for all $\lambda \in \text{co}(\Lambda)$. Since A is a $\text{co}(\Lambda)$-minimal element of $\mathcal{A}$, $A \preceq_K^{\lambda} B$ hold for all $\lambda \in \text{co}(\Lambda)$, and from $\Lambda \subset \text{co}(\Lambda)$, we have A is a Λ-minimal element of $\mathcal{A}$. To prove (ii), we use (ii) of Proposition 5. The proof is similar to (i) and left to the reader. Proofs of (iii) and (iv) are easy and omitted. □

We define notions of Λ-minimal solutions of (SP) with respect to K by using the notions of Λ-minimality defined above. Remember

$$\text{(SP)} \quad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in X, \end{array}$$

where X is a nonempty set, and $F : X \to \mathcal{C}$. An element $x_0 \in X$ is said to be a Λ-minimal solution of (SP) if and only if $F(x_0)$ is a Λ-minimal element of $\{F(x) \mid x \in X\}$, and is said to be a weak Λ-minimal solution of (SP) if and only if $F(x_0)$ is a weak Λ-minimal element of $\{F(x) \mid x \in X\}$. Next we give examples of Λ-minimality.

Example 1 Let $A = \{(0, 0)\}$, $B = \text{co}\{(1, 1), (-1, -1), (0, -2), (2, 0)\}$, $\mathcal{A} = \{A, B\}$ and $K = \{(x, y) \mid x, y \geq 0\}$. For any $\lambda \in [0, 1]$,

$$A \preceq_K^{\lambda} B \iff -\lambda B + K \supset (1 - \lambda)B \iff \frac{2}{3} \leq \lambda, \quad \text{and}$$
$$B \preceq_K^{\lambda} A \iff (1 - \lambda)B + K \supset -\lambda B \iff \lambda \leq \frac{1}{3}.$$

So, A is a $[\frac{2}{3}, 1]$-minimal element of $\mathcal{A}$, and B is a $[0, \frac{1}{3}]$-minimal element of $\mathcal{A}$. Clearly, A and B are u and l-minimal elements of $\mathcal{A}$ respectively. The notion of Λ-minimality shows attributes characteristic of A and B in $\mathcal{A}$.

Example 2 Let $F : [0, 2] \to 2^{\mathbb{R}}$ be a set-valued map defined by $F(x) = [f(x), g(x)]$, where $f(x) = x - 2$ and $g(x) = \frac{1}{2}|x - 1| - x + \frac{3}{2}$, for each $x \in [0, 2]$. Consider

$$\text{(SP)} \quad \text{Minimize} \quad F(x) \\ \text{subject to} \quad x \in [0, 2],$$

with $K = [0, +\infty)$. For fixed $\lambda \in [0, 1]$,

$$\begin{aligned} & F(x) \preceq_K^{\lambda} F(y) \\ & \iff [(1-\lambda)[f(x), g(x)], -\lambda[f(x), g(x)]] \leq_{\mathcal{K}} [(1-\lambda)[f(y), g(y)], -\lambda[f(y), g(y)]] \\ & \iff [(1-\lambda)[f(y), g(y)] - \lambda[f(x), g(x)], (1-\lambda)[f(x), g(x)] - \lambda[f(y), g(y)]] \in \mathcal{K} \\ & \iff (1-\lambda)[f(x), g(x)] - \lambda[f(y), g(y)] + K \supset (1-\lambda)[f(y), g(y)] - \lambda[f(x), g(x)] \\ & \iff (1-\lambda)f(x) - \lambda g(y) \leq (1-\lambda)f(y) - \lambda g(x) \\ & \iff (1-\lambda)f(x) + \lambda g(x) \leq (1-\lambda)f(y) + \lambda g(y). \end{aligned}$$

Then $\bar{x} \in [0, 2]$ is a Λ-minimal solution of (SP) if and only if $(1-\lambda)f(\bar{x}) + \lambda g(\bar{x}) \leq (1-\lambda)f(x) + \lambda g(x)$ for any $x \in [0, 2]$ and $\lambda \in \Lambda$. Therefore 0 is a $[0, \frac{2}{5}]$-minimal solution, each element of $(0, 1)$ is a $\frac{2}{5}$-minimal solution, 1 is a $[\frac{2}{5}, \frac{2}{3}]$-minimal solution, each element of $(1, 2)$ is a $\frac{2}{3}$-minimal solution, and 2 is a $[\frac{2}{3}, 1]$-minimal solution.

At the end of this section, we study the existence of λ-minimal solutions of set optimization problem (SP) because λ_0 and λ_1-minimality implies $[\lambda_0, \lambda_1]$-minimality from Proposition 6. We give proofs of the existence theorems in similar ways to the previous existence theorems of l-minimal solutions of (SP) in [8, 9].

Theorem 1 *Let F be a function from a compact topological space X to $\mathcal{C}$. Assume that the following property: if $\{x_\alpha\}_{\alpha \in T}$ is a totally ordered λ-decreasing net in X, that is, T is totally ordered, and $\alpha < \alpha'$ implies $F(x_{\alpha'}) \preceq_{\mathcal{K}}^{\lambda} F(x_\alpha)$, and if $\{x_\alpha\}_{\alpha \in T}$ converges x_0, then $\psi_\lambda(F(x_0)) \in \bigcap_{\alpha \in T}(\psi_\lambda(F(x_\alpha)) - \mathcal{K})$. Then there exists a λ-minimal solution of (SP).*

Proof Let $\{\psi_\lambda(F(x))\}_{x \in T}$ be a totally ordered set of $\{\psi_\lambda(F(x))\}_{x \in X}$. Define a reflexive and transitive binary relation $<$ on T by $x < x'$ if $\psi_\lambda(F(x')) \leq_{\mathcal{K}} \psi_\lambda(F(x))$ for each $x, x' \in T$, then $(T, <)$ is a directed set. Since X is compact set, we can choose a subnet T' of T such that T' converges to some element x_0 of X. From the assumption of the theorem, $\psi_\lambda(F(x_0)) \in \bigcap_{x \in T'}(\psi_\lambda(F(x)) - \mathcal{K})$.

Now, we will show that $\psi_\lambda(F(x_0)) \leq_{\mathcal{K}} \psi_\lambda(F(x))$ for each $x \in T$. If not, there exists $\hat{x} \in T$ such that $\psi_\lambda(F(x_0)) \not\leq_{\mathcal{K}} \psi_\lambda(F(\hat{x}))$. For each $x \in T'$ satisfying $\hat{x} < x$, $\psi_\lambda(F(x)) \leq_{\mathcal{K}} \psi_\lambda(F(\hat{x}))$, therefore $\bigcap_{x \in T', \hat{x} < x}(\psi_\lambda(F(x)) - \mathcal{K}) \subset \psi_\lambda(F(\hat{x})) - \mathcal{K}$.

Clearly $\bigcap_{x\in T'}(\psi_\lambda(F(x))-\mathcal{K})\subset\bigcap_{x\in T',\hat{x}<x}(\psi_\lambda(F(x))-\mathcal{K})$, we have $\psi_\lambda(F(x_0))\in\psi_\lambda(F(\hat{x}))-\mathcal{K}$, or equivalently $\psi_\lambda(F(x_0))\leq_{\mathcal{K}}\psi_\lambda(F(\hat{x}))$. This is a contradiction. Hence, we have that $\psi_\lambda(F(x_0))$ is a lower bound of $\{\psi_\lambda(F(x))\}_{x\in T}$ with respect to $\leq_{\mathcal{K}}$. From Zorn's lemma, there exists a minimal element of $\{\psi_\lambda(F(x))\}_{x\in X}$, that is, there exists a λ-minimal solution of (SP). □

When $\lambda = 0$, the condition of F in Theorem 1 is weaker than the notion of Hausdorff cone-upper continuity; F is Hausdorff K-upper continuous at x_0 if for any neighborhood V of the origin in E, there is a neighborhood U of x_0 in X such that $F(x)\subset F(x_0)+V+K$ for all $x\in U\cap X$, for example, see [3]. From this fact and Theorem 1, the following result is shown, the proof is left to the reader:

Corollary 1 *Let F be a function from a compact topological space X to $\mathcal{C}$. If F is Hausdorff K-upper continuous at every $x\in X$, then there exists an l-minimal solution of (SP). If F is Hausdorff $(-K)$-lower continuous at every $x\in X$, that is, for every $x\in X$ and for any neighborhood V of the origin in E, there is a neighborhood U of x in X such that $F(x)\subset F(x')+V-K$ for all $x'\in U\cap X$, then there exists an u-minimal solution of (SP).*

Define λ-level sets of F by

$$\mathrm{Lev}_\lambda(A)=\{x\in X\mid F(x)\leq^\lambda_{\mathcal{K}}A\},$$

where $A\in\mathcal{C}$.

Theorem 2 *If (X,d) is a complete metric space,* $\mathrm{Lev}_\lambda(F(x))$ *is closed for each $x\in X$, and the following condition is satisfied:*

> *there exists a function $l: X\to[0,+\infty)$ such that for each $x_1,x_2\in X$, $F(x_1)\preceq^\lambda_K F(x_2)$ implies $d(x_2,x_1)\leq l(x_2)-l(x_1)$.*

Then, there exists a λ-minimal solution of (SP).

Proof Let $x_0\in X$. We construct a sequence $\{x_k\}\subset X$ by induction as follows:

(i) if $\mathrm{Lev}_\lambda(F(x_k))\neq\{x_k\}$, since $\psi_\lambda(F(x'))\leq_{\mathcal{K}}\psi_\lambda(F(x_k))$ for some $x'\neq x_k$,

$$0<d(x_k,x')\leq l(x_k)-l(x')\leq l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x).$$

Since $l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)>0$, we can choose $x_{k+1}\in\mathrm{Lev}_\lambda(F(x_k))$ such that

$$l(x_{k+1})\leq\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)+\frac{1}{2}\left\{l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)\right\}.$$

(ii) if $\mathrm{Lev}_\lambda(F(x_k))=\{x_k\}$, let $x_{k+1}:=x_k$.

In each case, we can check easily that $\mathrm{Lev}_\lambda(F(x_{k+1}))\subset\mathrm{Lev}_\lambda(F(x_k))$ and

$$l(x_{k+1})-\inf_{x\in\mathrm{Lev}_\lambda(F(x_{k+1}))}l(x)\leq\frac{1}{2}\left\{l(x_k)-\inf_{x\in\mathrm{Lev}_\lambda(F(x_k))}l(x)\right\}.$$

Now we show that $\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \to 0$ as $k \to +\infty$. Indeed, let $u \in \mathrm{Lev}_\lambda(F(x_k))$. From the assumption and $\psi_\lambda(F(u)) \leq_{\mathcal{K}} \psi_\lambda(F(x_k))$, we have $d(x_k, u) \leq l(x_k) - l(u)$. Hence

$$\begin{aligned} d(x_k, u) &\leq l(x_k) - l(u) \\ &\leq l(x_k) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_k))} l(x) \\ &\leq \frac{1}{2}\left\{ l(x_{k-1}) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_{k-1}))} l(x) \right\} \\ &\leq \cdots \leq \frac{1}{2^k}\left\{ l(x_0) - \inf_{x \in \mathrm{Lev}_\lambda(F(x_0))} l(x) \right\} \leq \cdots \leq \frac{1}{2^k} l(x_0). \end{aligned}$$

This shows us

$$\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \leq \frac{1}{2^{k-1}} l(x_0),$$

therefore, we have $\mathrm{diam}(\mathrm{Lev}_\lambda(F(x_k))) \to 0$ as $k \to +\infty$. Since $\mathrm{Lev}_\lambda(F(x_k))$ is non-empty closed, $\mathrm{Lev}_\lambda(F(x_{k+1})) \subset \mathrm{Lev}_\lambda(F(x_k))$, and (X, d) is complete, we conclude $\bigcap_{k \in \mathbb{N}} \mathrm{Lev}_\lambda(F(x_k)) = \{\hat{x}\}$ for some $\hat{x} \in X$. This implies $\mathrm{Lev}_\lambda(F(\hat{x})) = \{\hat{x}\}$ and, it follows that $\hat{x}$ is a λ-minimal solution of (SP). □

6 A Generalized Scalarizing Function on $\mathcal{C}$

Since $\mathcal{C}^2/\!\equiv$ is an ordered vector space with convex cone $\mathcal{K}$, the scalarizing function from $\mathcal{C}^2/\!\equiv$ to $\mathbb{R}$ is given in this way:

$$\varphi_{[P,Q]}([A, B]) = \inf\{t \in \mathbb{R} \mid [A, B] \in t[P, Q] - \mathcal{K}\},$$

for fixed $[P, Q] \in \mathcal{C}^2/\!\equiv$. From the definition, it is clear that $\varphi_{[P,Q]}([A, B] + r[P, Q]) = \varphi_{[P,Q]}([A, B]) + r$. When $[P, Q] \in \mathrm{int}\mathcal{K}$, this function $\varphi_{[P,Q]}$ has the following properties: it is a special case of vector-valued version in [3], and the proof of the following theorem is omitted.

Theorem 3 *If* $[P, Q] \in \mathrm{int}\mathcal{K}$, *then* $\varphi_{[P,Q]} : \mathcal{C}^2/\!\equiv \to \mathbb{R}$ *is a well-defined continuous function, and for each* $[A, B], [C, D] \in \mathcal{C}^2/\!\equiv$, *we have*

(i) $\varphi_{[P,Q]}([A, B]) \leq r$ *if and only if* $[A, B] \in r[P, Q] - \mathcal{K}$;
(ii) $\varphi_{[P,Q]}([A, B]) < r$ *if and only if* $[A, B] \in r[P, Q] - \mathrm{int}\mathcal{K}$;
(iii) $\varphi_{[P,Q]}([A, B]) > r$ *if and only if* $[A, B] \notin r[P, Q] - \mathcal{K}$;
(iv) $\varphi_{[P,Q]}([A, B]) \geq r$ *if and only if* $[A, B] \notin r[P, Q] - \mathrm{int}\mathcal{K}$;
(v) $[A, B] \leq_{\mathcal{K}} [C, D]$ *implies* $\varphi_{[P,Q]}([A, B]) \leq \varphi_{[P,Q]}([C, D])$;
(vi) $[A, B] <_{\mathcal{K}} [C, D]$ *implies* $\varphi_{[P,Q]}([A, B]) < \varphi_{[P,Q]}([C, D])$.

Now we characterize solutions of (SP) by using the scalarizing function. At first we observe λ-minimal elements of a subfamily $\mathcal{A} \subset \mathcal{C}$ with respect to K:

Theorem 4 *Let $[P, Q] \in \mathrm{int}\mathcal{K}$ and $\mathcal{A}$ be a subfamily of $\mathcal{C}$. The set $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) > 0$ or $B \sim_K^\lambda A$. The set $A \in \mathcal{A}$ is a weak λ-minimal element of $\mathcal{A}$ if and only if for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) \geq 0$.*

Proof The set $A \in \mathcal{A}$ is a λ-minimal element of $\mathcal{A}$ if and only if $B \in \mathcal{A}$, $\psi_\lambda(B) \leq_\mathcal{K} \psi_\lambda(A)$ implies $\psi_\lambda(A) \leq_\mathcal{K} \psi_\lambda(B)$, that is, for each $B \in \mathcal{A}$, $\psi_\lambda(B) \not\leq_\mathcal{K} \psi_\lambda(A)$ or else $\psi_\lambda(B) = \psi_\lambda(A)$. By using Theorem 3, this is equivalent to for each $B \in \mathcal{A}$, $\varphi_{[P,Q]}(\psi_\lambda(B) - \psi_\lambda(A)) > 0$ or $B \sim_K^\lambda A$. The latter is shown in the similar way. □

From this theorem, we may choose any $[P, Q] \in \mathrm{int}\mathcal{K}$ to observe λ-minimal elements and weak λ-minimal elements. When $e \in \mathrm{int}K$, we can check that $[\{e\}, \{0\}] \in \mathrm{int}\mathcal{K}$, and embedding function $\psi_{[P,Q]}$ is a generalization of $I_e^l(A; B)$ and $I_e^u(A; B)$, indeed,

$$\begin{aligned} I_e^l(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^l te + B\} \\ &= \inf\{t \in \mathbb{R} \mid [A, \{0\}] \leq_\mathcal{K} t[\{e\}, \{0\}] + [B, \{0\}]\} \\ &= \varphi_{\psi_0(\{e\})}(\psi_0(A) - \psi_0(B)), \text{ and} \end{aligned}$$

$$\begin{aligned} I_e^u(A; B) &= \inf\{t \in \mathbb{R} \mid A \preceq_K^u te + B\} \\ &= \inf\{t \in \mathbb{R} \mid -B \preceq_K^l te - A\} \\ &= \inf\{t \in \mathbb{R} \mid [-B, \{0\}] \leq_\mathcal{K} t[\{e\}, \{0\}] + [-A, \{0\}]\} \\ &= \varphi_{\psi_1(\{e\})}(\psi_1(A) - \psi_1(B)). \end{aligned}$$

Also $S_e^l(A; B)$ and $S_e^u(A; B)$ can be written by using φ because $S_e^l(A; B) = -I_{-e}^l(A; B)$ and $S_e^u(A; B) = -I_{-e}^u(A; B)$. Motivated by the observation, we give the following simple notation $\varphi_e^\lambda(A, B)$ as follows: for each $\lambda \in [0, 1]$,

$$\varphi_e^\lambda(A, B) = \varphi_{\psi_\lambda(\{e\})}(\psi_\lambda(A) - \psi_\lambda(B)).$$

Clearly we have

$$\begin{aligned} &\varphi_e^0(A, B) = I_e^l(A; B), \quad \varphi_e^1(A, B) = I_e^u(A; B), \\ &\varphi_e^0(A, B) = -S_{-e}^l(A; B), \text{ and } \varphi_e^1(A, B) = -S_{-e}^u(A; B), \end{aligned}$$

and we can characterize solutions of (SP) by using the function:

Corollary 2 *Let X be a nonempty set, $F : X \to \mathcal{C}$, and $e \in \mathrm{int}K$. The element $x_0 \in X$ is a λ-minimal solution of (SP) if and only if for each $x \in X$, $\varphi_e^\lambda(F(x), F(x_0)) > 0$ or $F(x) \sim_K^\lambda F(x_0)$. The element $x_0 \in X$ is a weak λ-minimal solution of (SP) if and only if for each $x \in X$, $\varphi_e^\lambda(F(x), F(x_0)) \geq 0$ or $F(x) \sim_K^\lambda F(x_0)$.*

The above characterizations are generalizations of the previous ones of set optimization problems. Finally, we observe the following example:

Example 3 Let $F : X \to 2^{\mathbb{R}^n}$ be a set-valued map defined by

$$F(x) = (f_0(x) + K) \cap (f_1(x) - K)$$

where functions $f_0, f_1 : X \to \mathbb{R}^n$ satisfy $f_0(x) \leq_K f_1(x)$ for each $x \in X$, and consider a set optimization problem

$$\begin{aligned} \text{(SP)} \;\; &\text{Minimize} \;\; F(x) \\ &\text{subject to} \;\; x \in X. \end{aligned}$$

For given $e \in \text{int}K$ and for any $\lambda \in [0, 1]$, we can check that

$$\varphi_e^\lambda(F(x), F(y)) = \inf\{t \in \mathbb{R} \mid f_\lambda(x) \leq_K f_\lambda(y) + te\},$$

in the similar way to Example 2, where $f_\lambda = (1 - \lambda) f_0 + \lambda f_1$. The right-hand side of the above equality can be regarded as a convolution of f_λ and the scalarizing function in Sect. 2. Then the λ-minimal solutions of (SP) is characterized by the K-minimal solutions of the following vector optimization (VP):

$$\begin{aligned} \text{(VP)} \;\; &\text{Minimize} \;\; f_\lambda(x) \\ &\text{subject to} \;\; x \in X. \end{aligned}$$

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References

1. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**, 3821–3835 (2012)
2. Gerth, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
3. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational methods in partially ordered spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 17. Springer, New York (2003)
4. Hamel, A., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)
5. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
6. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
7. Jameson, G.: Ordered linear spaces. Lecture Notes in Mathematics, vol. 141. Springer, Berlin (1970)
8. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**, 73–84 (2003)

9. Kuroiwa, D.: Existence of efficient points of set optimization with weighted criteria. J. Nonlinear Convex Anal. **4**, 117–123 (2003)
10. Kuroiwa, D.: On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. **10**, 41–50 (2009)
11. Kuroiwa, D., Nuriya, T.: A generalized embedding vector space in set optimization. Nonlinear analysis and convex analysis, pp. 297–303. Yokohama Publisher, Yokohama (2007)
12. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Anal. **30**, 1487–1496 (1997)
13. Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets and set-valued Ky Fan minimax inequality. J. Nonlinear Convex Anal. **11**, 513–525 (2010)
14. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
15. Pallaschke, D., Urbański, R.: Pairs of Compact Convex Sets, Fractional Arithmetic with Convex Sets, Mathematics and its Applications, vol. 548. Kluwer Academic Publishers, Dordrecht (2002)
16. Rådström, H.: An embedding theorem for spaces of convex sets. Proc. Am. Math. Soc. **3**, 165–169 (1952)

On Characterization of Nash Equilibrium Strategy in Bi-Matrix Games with Set Payoffs

Takashi Maeda

Abstract In this paper, we consider set-valued payoff bi-matrix games where each player's payoffs are given by non-empty sets in n-dimensional Euclidean spaces $\mathbb{R}^n$. First, we define several types of set-orderings on the set of all non-empty subsets in $\mathbb{R}^n$. Second, by using these orderings, we define four kinds of concepts of Nash equilibrium strategies to the games and investigate their properties. Finally, we give sufficient conditions for which there exists these types of Nash equilibrium strategy.

Keywords Set-ordering · Maximal element · Set-valued map · Nonlinear scalarization · Set payoff games · Nash equilibrium strategy · Maximal Nash equilibrium strategy · Pareto Nash equilibrium strategy · Incomplete information game · Weak Pareto Nash equilibrium strategy · Fixed point theorem

AMC: 90C29 · 90C46

1 Introduction

Since seminal works by Neumann and Morgenstern [26] and Nash [24, 25], Game theory has played an important role in the fields of decision making theory such as economics, management, and operations research, etc.

When we apply the game theory to model some practical problems with which we encounter in real situations, we have to know (1) who are players, (2) what are strategies for each player, and (3) values of payoffs for each player to receive. However it is difficult for us to know the exact values of payoffs and could only know the values of payoffs approximately, or with some imprecise degree in general. In order to model such a situation with game theory, a great number of efforts have been devoted to the developments of game theory from the theoretical and practical points of views.

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For the games where the payoffs are given by random variables, Harsanyi [6] has defined Bayesian games, which is games with incomplete information on players' payoffs, and the concept of Bayesian Nash equilibrium to the games, and investigated the properties.

Campos [4] has considered fuzzy matrix games where the payoffs are given by fuzzy numbers, and proposed some methods to solve fuzzy matrix games based on linear programming, but has not defined explicit concepts of equilibrium strategies.

For fuzzy bi-matrix games with fuzzy payoffs, Maeda [18] has defined three types of Nash equilibrium strategies based on possibility and necessity measures and investigated their properties; Maeda [19] has defined three types of Nash equilibrium strategies by using fuzzy max ordering, and proved that fuzzy bi-matrix games are equivalent to the games with vector payoffs games. While, for fuzzy matrix games, Maeda [20] has defined fuzzy minimax equilibrium strategies based on fuzzy max order and investigated their properties.

Aghassi and Bertsimas [1] have considered matrix games where payoffs are uncertain and players have no information about the probability distributions, and investigated their properties based on robust optimization methods in mathematical programming. Liu and Kao [17], and Li [16] have considered matrix games where the payoffs are compact intervals in $\mathbb{R}$ and proposed some methods to solve the matrix games based on linear programming approaches. However, Liu and Kao [17] and Li [16] have not defined explicit concepts of equilibrium strategies.

In this paper, we consider the bi-matrix games where each player's payoffs are given by non-empty sets in n-dimensional Euclidean spaces $\mathbb{R}^n$, including interval-valued payoffs, which means both players don't know exact values of payoffs but they know their ranges. Namely, we consider the games that payoffs are deterministic uncertainty (See Leitmann [15]). Based on set-valued maps optimization methods (Maeda [21–23]), for set payoff game, we define four kinds of concepts of Nash equilibrium and give sufficient conditions under which there exist these Nash equilibrium strategies.

For those purposes, this paper is organized as follows. In Sect. 2, we introduce several types of set orderings on the set of all non-empty subsets in n-dimensional Euclidean space $\mathbb{R}^n$ and investigate their properties. In Sect. 3, we introduce two types of extended real-valued set functions defined on the set of all non-empty subsets of $\mathbb{R}^n$, which are extensions of the non-convex separation functions (Gerth and Weidner [5], Hamel [7] and Hernández and Rodríguez-Marín [8]) and investigate their properties. In particular, we show that these set functions are monotone and positively homogeneous with respect to the set orderings given in Sect. 2. In Sect. 4, we consider set payoffs bi-matrix games, where the payoffs for each player are compact convex sets in $\mathbb{R}^\ell$. First, we define the concepts of Nash equilibrium strategy to the game; then, associated with the set payoff bi-matrix games, we define the two-person games with scalar payoffs; bi-matrix games; we investigate relationships between set payoff bi-matrix games and scalar payoff two-person games. In Sect. 5, we give sufficient conditions under which there exists at least one Nash equilibrium strategy to set payoff bi-matrix games.

2 Orderings on Sets in $\mathbb{R}^n$ and Set-Valued Maps

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant, respectively. By $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}^n)$, we denote the sets of all non-empty subsets of $\mathbb{R}^n$ and the set of all non-empty compact subsets of $\mathbb{R}^n$, respectively. For any elements $A,\ B \in \mathcal{P}(\mathbb{R}^n)$ and any real number $\lambda \in \mathbb{R}$, we write $A + B := \{z \in \mathbb{R}^n \mid z = x + y,\ x \in A,\ y \in B\}$ and $\lambda A := \{z \in \mathbb{R}^n \mid z = \lambda x,\ x \in A\}$. Whenever $A \in \mathcal{P}(\mathbb{R}^n)$ is a singleton, say $A = \{a\}$, we abuse notations and write a instead of $\{a\}$.

Definition 2.1 For $A,\ B \in \mathcal{P}(\mathbb{R}^n)$, we write

$$A \leqq_L B \quad \text{iff} \quad B \subseteq A + \mathbb{R}^n_+, \tag{1}$$
$$A \leqq_U B \quad \text{iff} \quad A \subseteq B - \mathbb{R}^n_+, \tag{2}$$
$$A \leqq B \quad \text{iff} \quad A \leqq_L B \ \text{ and } \ A \leqq_U B. \tag{3}$$
$$A \prec_L B \quad \text{iff} \quad \mathrm{cl}\, B \subseteq \mathrm{cl}\, A + \mathrm{int}\, \mathbb{R}^n_+, \tag{4}$$
$$A \prec_U B \quad \text{iff} \quad \mathrm{cl}\, A \subseteq \mathrm{cl}\, B - \mathrm{int}\, \mathbb{R}^n_+, \tag{5}$$
$$A \prec B \quad \text{iff} \quad A \prec_L B \ \text{ and } \ A \prec_U B, \tag{6}$$
$$A \preceq B \quad \text{iff} \quad A \prec_L B \ \text{ and } \ A \leqq_U B, \ \text{ or } \ A \leqq_L B \ \text{ and } \ A \prec_U B. \tag{7}$$

where cl A denotes the closure of the set A.

It is easy to see that the binary relations $\leqq_L$, $\leqq_U$, and $\leqq$ are reflexive and transitive, but not antisymmetric. In fact, for any $A, B \in \mathcal{P}(\mathbb{R}^n)$, $A \leqq B$ and $B \leqq A$ implies that $A + \mathbb{R}^n_+ = B + \mathbb{R}^n_+$, in general. Therefore, the binary relations $\leqq_L$, $\leqq_U$, and $\leqq$ are quasi orderings on $\mathcal{P}(\mathbb{R}^n)$. On the other hand, binary relations $\prec_L$, $\prec_U$, $\prec$, and $\preceq$ are strict partial orderings.

The set-orderings $\leqq$ and $\prec$ are introduced by Young [27]. Kuroiwa [12, 13] use the set-orderings $\leqq_L$, $\leqq_U$, $\prec_L$ and $\prec_U$ to study set optimization problems where the objective map is given by set-valued map. By using the set-ordering $\leqq$, Maeda [21] gave the condition that fuzzy mathematical problems are equivalent to set-valued optimization problems. For the relationships among these set orderings and other set orderings, see Jahn and Ha [10].

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^n)$ be any non-empty subset and $A \in \mathcal{A}$ be any set. Then, the set $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set-ordering $\leqq$ iff $A' \in \mathcal{A},\ \ A \leqq A'$ imply $A' \leqq A$. While, $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set-ordering $\preceq$ iff there is no $\bar{A} \in \mathcal{A}$ such that $\mathcal{A} \preceq \bar{A}$, and $A \in \mathcal{A}$ is said to be a maximal element in $\mathcal{A}$ with respect to the set ordering $\prec$ iff there is no $\bar{A} \in \mathcal{A}$ such that $A \prec \bar{A}$. Similarly, we could define various types of maximal element in $\mathcal{A}$ with respect to other set-orderings given in Definition 2.1

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ be any set-valued map. By $\mathrm{Dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\mathrm{Gr}(F) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell \mid y \in F(x)\}$, we denote the effective domain and the graph of F, respectively. Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ be any set-valued map and $S \subseteq \mathrm{Dom}(F)$ be any non-empty convex set. Then F is said to be $\leqq$-concave on S if $(1 - \lambda)F(x) + \lambda F(y) \leqq F((1 - \lambda)x + \lambda y)$ holds for $\forall x,\ y \in S$ and $\forall \lambda \in [0, 1]$;

$F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be $\leqq$-convex on S if $F((1-\lambda)x + \lambda y) \leqq (1-\lambda)F(x) + \lambda F(y)$ holds for $\forall x,\ y \in S$ and $\forall \lambda \in [0, 1]$ (See Maeda [23]). While $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be convex-valued if $F(x)$ is convex for $\forall x \in \mathrm{Dom}(F)$; $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be compact-valued if $F(x)$ is compact for $\forall x \in \mathrm{Dom}(F)$.

A set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be upper semi-continuous at $x^o \in \mathrm{Dom}(F)$ if, for any sequences $\{(x^\nu, y^\nu)\}_{\nu=1}^\infty \subseteq \mathrm{Gr}(F)$ converging to $(x^o, y^o) \in \mathbb{R}^n \times \mathbb{R}^\ell$, we have $y^o \in F(x^o)$. While, $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be lower semi-continuous at $x^o \in \mathrm{Dom}(F)$ if, for any $(x^o, y^o) \in \mathrm{Gr}(F)$, and any sequence $\{x^\nu\}_{\nu=1}^\infty \subseteq \mathrm{Dom}(F)$ such that $\{x^\nu\}_{\nu=1}^\infty$ converging to x^o, there exists a subsequence $\{(x^{\nu'}, y^{\nu'})\}_{\nu'=1}^\infty \subseteq \mathrm{Gr}(F)$ such that the sequence $\{y^{\nu'}\}_{\nu'=1}^\infty$ converges to y^o; $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ is said to be continuous at $x^o \in \mathbb{R}^n$ if F is upper semi-continuous and lower semi-continuous at x^o. We say F is continuous on $\mathrm{Dom}(F)$ if, for any $x^o \in \mathrm{Dom}(F)$, F is continuous at x^o (see Aubin [3]).

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be any set-valued map, $S \subseteq \mathrm{Dom}(F)$ be any non-empty set and let $x^o \in S$ be any point. Then F is said to be uniformly compact near $x^o \in S$ if there exists a neighborhood $N(x^o)$ of x^o such that $\mathrm{cl} \cup_{x \in N(x^o)} F(x)$ is compact, where cl denotes the closure of the set $\cup_{x \in N(x^o)} F(x)$. F is said to be uniformly compact on S if F is uniformly compact near x for all $x \in S$.

3 Scalarizaion Methods of Set-Valued Maps in $\mathbb{R}^n$

In this section, we define two types of extended real-valued functions defined on $\mathcal{C}(\mathbb{R}^n)$, which are extensions of Gerstewitz's functions and investigate their properties (see [5, 7, 8], Maeda [23] and Araya [2]) .

Let $A \in \mathcal{C}(\mathbb{R}^n)$ be any non-empty compact set and let $k^o \in \mathrm{int}\ \mathbb{R}^n_+$ be any point. We define the real-valued set functions $\phi^i(\cdot; k^o) : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$ by $\phi^i(A; k^o) := \sup\{t \in \mathbb{R} \mid tk^o \leqq_i A\}$, $i = L, U$. Note that, for each $a \in A$, $\phi^L(a; k^o) = \phi^U(a; k^o) = \min\{a_i / k_i \mid i = 1, 2, \cdots, n\}$ and $\phi^L(\cdot; k^o), \phi^U(\cdot; k^o) : \mathbb{R}^n \to \mathbb{R}$ are continuous on $\mathbb{R}^n$ as functions defined on $\mathbb{R}^n$. Then we have the following lemma (see Hamel [7]).

Lemma 3.1 *Let $A \in \mathcal{C}(\mathbb{R}^n)$ be any compact set and let $k^o \in \mathrm{int}\mathbb{R}^n_+$ be any point. Then we have*

$$\phi^L(A; k^o) = \min\{\phi^L(a; k^o) \mid a \in A\}, \tag{8}$$

$$\phi^U(A; k^o) = \max\{\phi^U(a; k^o) \mid a \in A\}, \tag{9}$$

$$A \subseteq \phi^L(A; k^o)k^o + \mathbb{R}^n_+, \tag{10}$$

$$\phi^U(A; k^o)k^o \subseteq A - \mathbb{R}^n_+. \tag{11}$$

The following theorem shows that the set functions $\phi^L(\cdot; k^o)$ and $\phi^U(\cdot; k^o)$ are superadditive and positively homogeneous on $\mathcal{C}(\mathbb{R}^n)$, namely $\phi^i(\cdot, k^o)$, $i = L, U$ are concave set functions.

Theorem 3.1 *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets, $k^o \in \operatorname{int} \mathbb{R}^n_+$ be any point and let $\lambda \in \mathbb{R}_+$ be any real number. Then it holds that*

$$\phi^i(A; k^o) + \phi^i(B; k^o) \leqq \phi^i(A + B; k^o), \quad i = L, U, \tag{12}$$

$$\phi^i(\lambda A; k^o) = \lambda \phi^i(A; k^o) \quad i = L, U. \tag{13}$$

Proof First we show that (12) and (13) hold for $\phi^L(\cdot, k^o)$. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets. From (8), there exist points $\bar{a} \in A$ and $\bar{b} \in B$ such that

$$\begin{aligned}
\phi^L(A + B; k^o) &= \phi^L(\bar{a} + \bar{b}; k^o) \\
&= \min\{\frac{\bar{a}_i + \bar{b}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} \\
&\geqq \min\{\frac{\bar{a}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} + \min\{\frac{\bar{a}_i}{k^o_i} \mid i = 1, 2, \cdots, n\} \\
&= \phi^L(\bar{a}; k^o) + \phi^L(\bar{b}; k^o) \\
&\geqq \phi^L(A; k^o) + \phi^L(B; k^o)
\end{aligned}$$

Next we show that $\phi^L(\cdot; k^o)$ is positively homogeneous. Let $\lambda > 0$ be any real number,

$$\begin{aligned}
\phi^L(\lambda A; k^o) &= \sup\{t \in \mathbb{R} \mid \lambda A \subseteq t k^o + \mathbb{R}^n_+\} \\
&= \sup\{t \in \mathbb{R} \mid A \subseteq (t/\lambda) k^o + \mathbb{R}^n_+\} \\
&= \sup\{\lambda t' \in \mathbb{R} \mid A \subseteq t' k^o + \mathbb{R}^n_+\} \\
&= \lambda \phi^L(A; k^o).
\end{aligned}$$

For $\lambda = 0$, it obvious that (13) holds. By a similar way, we could show that (12) and (13) hold for $\phi^U(\cdot, k^o)$. □

Corollary 3.1 *Let $A, B \in \mathcal{C}(\mathbb{R})$ be any intervals and let $k^o \in \operatorname{int} \mathbb{R}_+$ be any positive real number. Then it holds that*

$$\phi^i(A + B; k^o) = \phi^i(A; k^o) + \phi^i(B; k^o), \quad i = L, U.$$

Proof We omit the proof. □

The following theorem shows that the set functions $\phi^L(\cdot; k^o)$ and $\phi^U(\cdot; k^o)$ are monotone increasing with respect to the set orderings $\leqq_i$, $\prec_i$, $i = L, U$ for any given $k^o \in \operatorname{int} \mathbb{R}^n_+$.

Theorem 3.2 *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any compact sets, and let $k^o \in \operatorname{int} \mathbb{R}^n_+$ be any vector and $\lambda \in \mathbb{R}_+$ be any positive number. Then it holds that*

$$\phi^L(A;k^o) \leqq \phi^L(B;k^o) \text{ if } A \leqq_L B, \tag{14}$$
$$\phi^U(A;k^o) \leqq \phi^U(B;k^o) \text{ if } A \leqq_U B, \tag{15}$$
$$\phi^L(A;k^o) < \phi^L(B;k^o) \text{ if } A \prec_L B, \tag{16}$$
$$\phi^U(A;k^o) < \phi^U(B;k^o) \text{ if } A \prec_U B. \tag{17}$$

Proof First we show that (14) holds. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any elements such that $A \leqq_L B$ holds. Since the set orderings $\leqq_L$ is a quasi ordering, from Lemma 3.1, we have $\phi^L(A;k^o)k^o \leqq_L A \leqq_L B$, which implies that $\phi^L(A;k^o) \leqq \phi^L(B;k^o)$. Second we show that (15) holds. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$ be any elements such that $A \leqq_U B$ holds. Since the set ordering $\leqq_U$ is quasi ordering, from Lemma 3.1, we have $\phi^U(A;k^o)k^o \leqq_U A \leqq_U B$, which implies that $\phi^U(A;k^o) \leqq \phi^U(B;k^o)$.

Third, we show that (16) holds. Note that there exists a vector $\bar{b} \in B$ such that $\phi^L(B;k^o) = \phi^L(\bar{b};k^o)$. Since $A \prec_L B$, there exists a vector $\bar{a} \in A$ and a real number $\varepsilon > 0$ such that $\bar{b} = \bar{a} + \varepsilon k^o$ holds. Therefore, we have $\phi^L(B;k^o) = \phi^L(\bar{b};k^o) = \phi^L(\bar{a};k^o) + \varepsilon > \phi^L(A;k^o)$.

Finally, we show (17) holds. Let $\bar{a} \in A$ be any vector such that $\phi^U(A;k^o) = \phi^U(\bar{a};k^o)$. Then there exist a vector $\bar{b} \in B$ and a real number $\varepsilon > 0$ such that $\bar{a} = \bar{b} - \varepsilon k^o$ holds. Hence we have $\phi^U(A;k^o) = \phi^U(\bar{b};k^o) - \varepsilon < \phi^U(\bar{b};k^o) \leqq \phi^U(B;k^o)$. □

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be any set-valued map with compact image, $k^o \in \operatorname{int} \mathbb{R}^m_+$ be any point, and let $S \subseteq \mathrm{Dom}(F)$ be any non-empty set. We define real-valued functions $h^i(\cdot;k^o) : S \to \mathbb{R}$ by $h^i(x;k^o) := \phi^i(F(x);k^o)$, $i = L, U$. Then we have the following theorem.

Theorem 3.3 *Suppose that the set-valued map $F : S \rightsquigarrow \mathbb{R}^m$ is convex-valued and compact-valued and S is a convex set. If F is $\leqq$-concave on S, then real-valued functions $h^i(\cdot;k^o) : S \to \mathbb{R}$, $i = L, U$ are concave on S.*

Proof Let $x, y \in S$ be any elements and $\lambda \in [0, 1]$ be any real number. By assumptions, since the set S is convex and F is $\leqq$-concave, from Theorem 3.1 and 3.2, we have

$$\begin{aligned} h^i((1-\lambda)x + \lambda y;k^o) &= \phi^i(F((1-\lambda)x + \lambda y);k^o) \\ &\geqq \phi^i((1-\lambda)F(x) + \lambda F(y);k^o) \\ &\geqq (1-\lambda)\phi^i(F(x);k^o) + \lambda\phi^i(F(y);k^o) \\ &= (1-\lambda)h^i(x;k^o) + \lambda h^i(y;k^o), \end{aligned}$$

which implies that $h^i(\cdot;k^o)$ is concave on S, $i = L, U$. □

Theorem 3.4 *Suppose that S is compact, the set-valued map $F : S \rightsquigarrow \mathbb{R}^m$ is compact-valued and uniformly compact on S. If the set-valued map F is continuous on S, then functions $h^i(\cdot;k^o)$, $i = L, U$ are continuous on S.*

Proof From Lemma 3.1, it holds that $h^L(x; k^o) = \min\{\phi^L(z; k^o) \mid z \in F(x)\}$ and $h^U(x; k^o) = \max\{\phi^U(z; k^o) \mid z \in F(x)\}$ for $\forall x \in S$. By assumptions, since F is continuous on S and uniformly compact on S, $\phi^i(\cdot; k^o),\ i = L, U$ are continuous as functions defined on $\mathbb{R}^\ell$. Hence, $h^i(\cdot, k^o),\ i = L, U$ are continuous on S (See Hogan [9], Theorems 5 and 6). □

4 Bi-Matrix Game with Set Payoffs and Its Equilibrium Strategy

Let $I,\ J$ denote players and let $M := \{1, 2, \ldots, m\}$ and $N := \{1, 2, \ldots, n\}$ be the sets of all pure strategies available for players I and J, respectively. We denote the sets of all mixed strategies available for players I and J by $S_I := \{x := (x_1, x_2, \ldots, x_m) \in R^m_+ \mid x_i \geqq 0,\ i = 1, 2, \ldots, m,\ \sum_{i=1}^m x_i = 1\}$, $S_J := \{y := (y_1, y_2, \ldots, y_n) \in R^n_+ \mid y_j \geqq 0,\ j = 1, 2, \ldots, n,\ \sum_{j=1}^n y_j = 1\}$.

By $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell)$, we denote the payoffs that player I receives and J receives when player I plays the pure strategy i and player J plays the pure strategy j, respectively. We set $\mathcal{A} := (A_{ij})$ and $\mathcal{B} := (B_{ij})$, where $\mathcal{A}$ and $\mathcal{B}$ are $m \times n$ matrices whose i, j th elements are A_{ij} and B_{ij}, respectively.

Now we define bi-matrix game with set payoffs by $\Gamma := \langle\{I, J\}, S_I \times S_J, \{\mathcal{A}, \mathcal{B}\}\rangle$ or

		Player J			
		1	2	$\cdots$	n
Player I	1	(A_{11}, B_{11})	(A_{12}, B_{12})	$\cdots$	(A_{1n}, B_{1n})
	2	(A_{21}, B_{21})	(A_{22}, B_{22})	$\cdots$	(A_{2n}, B_{2n})
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	m	(A_{m1}, B_{m1})	(B_{m2}, B_{m2})	$\cdots$	(A_{mn}, B_{mn})

Let $x \in S_I$ and $y \in S_J$ be any mixed strategies. For each player I and J, we define the set-valued payoff maps $F, G : S_I \times S_J \rightsquigarrow \mathbb{R}^\ell$ by $F(x, y) := \sum_{i=1}^m \sum_{j=1}^n x_i A_{ij} y_j$ and $G(x, y) := \sum_{i=1}^m \sum_{j=1}^n x_i B_{ij} y_j$, which are called expected payoffs.

Player I is said to be L-type, U-type, and LU-type if he maximizes his expected payoff $F(x, y)$ with respect to the set-orderings $\leqq_L,\ \leqq_U$, and $\leqq$ for given $y \in S_J$ and player J is said to be L-type, U-type, and LU-type if he maximizes his expected payoff $G(x, y)$ with respect to the set-orderings $\leqq_L, \leqq_U$, and $\leqq$ for given $x \in S_I$. By $\Gamma(\leqq_L, \leqq_U)$, we denote the game where player I is L-type and J is U-type. Then, set-payoff game Γ is classified into the following five set payoff games $\Gamma(\leqq_L, \leqq_L)$, $\Gamma(\leqq_L, \leqq_U)$, $\Gamma(\leqq_U, \leqq_L)$, $\Gamma(\leqq_U, \leqq_U)$, and $\Gamma(\leqq, \leqq)$.

In the above games, each player knows his/her own type, but does not know the other player's type. Therefore, set-payoff games are considered to be incomplete information games and Nash equilibrium strategy is characterized by Bayesian Nash equilibrium strategy (see [6]).

While, from Theorem 3.2, noting that for any $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \leqq B$ implies that $\phi^L(A; k^o) \leqq \phi^L(B; k^o)$ and $\phi^U(A; k^o) \leqq \phi^U(B; k^o)$ hold, we may assume that both players use the set-ordering $\leqq$ to maximize their expected payoffs and this is common knowledge between players. Moreover, from practical point of views, our approach is useful to study bi-matrix with fuzzy vector payoffs. Hence, in the following, we assume that both players I and J are LU type and this is a common knowledge for the players.

For each $(x, y) \in S_I \times S_J$, we set $\mathcal{F}(y) :=\{F(x, y) \mid x \in S_I\}$ and $\mathcal{G}(x) := \{G(x, y) \mid y \in S_J\}$ and denote the sets of all maximal elements in $\mathcal{F}(y)$ with respect to the set orderings $\leqq, \preceq$ and $\prec$ by $\mathcal{F}(y)^{\leqq}$, $\mathcal{F}(y)^{\preceq}$, and $\mathcal{F}(y)^{\prec}$, respectively. Similarly, we denote the sets of all maximal elements in $\mathcal{G}(x)$ with respect to the set orderings $\leqq, \preceq$ and $\prec$ by $\mathcal{G}(x)^{\leqq}$, $\mathcal{G}(x)^{\preceq}$, and $\mathcal{G}(x)^{\prec}$. By definitions, for each $(x, y) \in S_I \times S_J$, it holds that $\mathcal{F}(y)^{\leqq} \subseteq \mathcal{F}(y)^{\preceq} \subseteq \mathcal{F}(y)^{\prec}$ and $\mathcal{G}(x)^{\leqq} \subseteq \mathcal{G}(x)^{\preceq} \subseteq \mathcal{G}(x)^{\prec}$.

Now we define the concept of Nash equilibrium strategies to game Γ.

Definition 4.1 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Nash equilibrium strategy to game Γ if it holds that

(i) $F(x, y^*) \leqq F(x^*, y^*), \quad \forall x \in S_I,$
(ii) $G(x^*, y) \leqq G(x^*, x^*), \quad \forall y \in S_J.$

The pair of sets $(F(x^*, y^*), G(x^*, y^*))$ is said to be the value of game Γ.

We define set-valued maps $\mathcal{B}_I : S_J \rightsquigarrow S_I, \mathcal{B}_J : S_I \rightsquigarrow S_J$ and $\mathcal{B} : S_I \times S_J \rightsquigarrow S_I \times S_J$ by $\mathcal{B}_I(y) := \{x \in S_I \mid F(u, y) \leqq F(x, y), \quad \forall u \in S_I\}$, $\mathcal{B}_J(x) := \{y \in S_J \mid G(x, v) \leqq G(x, y), \ \forall v \in S_J\}$ and $\mathcal{B}(x, y) := \mathcal{B}_I(y) \times \mathcal{B}_J(x)$. Then, it is obvious that the pair of strategies $(x, y) \in S_I \times S_J$ is a Nash equilibrium if and only if $(x, y) \in \mathcal{B}(x, y)$ holds.

Example 4.1 We consider the following bi-matrix game with interval-valued payoffs. In Game 1 (Fig. 1), there is no pair of pure strategies such that the pair is a Nash equilibrium. We show that there exists a unique mixed Nash equilibrium in Game 1. Let $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$ be any strategies. Then, by simple calculations, we have

$$F(x, y) = [(1 - 2y_1)x_1 + 3y_1 + 1, (1 - 2y_1)x_1 + 3y_1 + 3], \tag{18}$$

$$G(x, y) = [(2x_1 - 1)y_1 + x_1 + 3, (2x_1 - 1)y_1 + x_1 + 5]. \tag{19}$$

From (18) and (19), we have the the following best response maps:

$$\mathcal{B}_I(y_1, y_2) = \begin{cases} \{(1,0)\} & \text{if } \ y_1 \in [0, 1/2), \\ S_I & \text{if } \ y_1 = 1/2, \\ \{(0,1)\} & \text{if } \ y_1 \in (1/2, 1], \end{cases} \quad \text{and } \mathcal{B}_J(x_1, x_2) = \begin{cases} \{(0,1)\} & \text{if } \ x_1 \in [0, 1/2), \\ S_J & \text{if } \ x_1 = 1/2, \\ \{(1,0)\} & \text{if } \ x_1 \in (1/2, 1]. \end{cases}$$

Then, we have $((0.5, 0.5), (0.5, 0.5)) \in \mathcal{B}_I((0.5, 0.5)) \times \mathcal{B}_J((0.5, 0.5))$, which implies that the pair of strategies $\{(0.5, 0.5), (0.5, 0.5)\}$ is a unique Nash equilibrium strategy in Game 1.

		Player J	
		C	D
Player I	C	$([3,5],[5,7])$	$([2,4],[4,6])$
	D	$([4,6],[2,4])$	$([1,3],[3,5])$

Fig. 1 Game 1

The following example shows that there is no Nash equilibrium strategy in set-payoff games in general.

Example 4.2 We consider the following bi-matrix game with interval-valued payoffs (Fig. 2). For any $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, we have

$$F(x, y) = [(3 - 2y_1)x_1 - 2y_1 + 4, (y_1 - 2)x_1 - 4y_1 + 10].$$

Then, it holds that $\mathcal{B}_I(y) = \emptyset, \ \forall y \in S_J$. Therefore, there is no Nash equilibrium in Game 2.

Based on the above example, we introduce three types of concepts of Nash equilibrium strategies.

Definition 4.2 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a maximal Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\leqq} \times \mathcal{G}(x^*)^{\leqq}$.

Definition 4.3 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Pareto Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\preceq} \times \mathcal{G}(x^*)^{\preceq}$.

		Player J	
		C	D
Player I	C	$([4,5],[5,7])$	$([7,8],[4,6])$
	D	$([2,6],[2,4])$	$([4,10],[3,5])$

Fig. 2 Game 2

Definition 4.4 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a weak Pareto Nash equilibrium to game Γ if it holds that $(F(x^*, y^*), G(x^*, y^*)) \in \mathcal{F}(y^*)^{\prec} \times \mathcal{G}(x^*)^{\prec}$.

It is easy to see that pairs of pure strategies $\{(C, C)\}$ and $\{(D, D)\}$ are maximal Nash equilibriums in Game 2 (Fig. 2).

Example 4.3 Consider the following bi-matrix game with interval-valued payoffs (Fig. 3).

It is easy to see that the pairs of the pure strategies $\{(C, C)\}$ and $\{(D, D)\}$ are Nash equilibrium strategies in Game 3. For each $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, the set-valued payoff maps for players I and J are given by

$$F(x, y) = [2(5y_1 - 1)x_1 - y_1 + 3, 2x_1y_1 + 5y_1 + 4], \tag{20}$$
$$G(x, y) = [2(5x_1 - 1)y_1 - x_1 + 3, 2x_1y_1 + 5x_1 + 4]. \tag{21}$$

The pair of strategy $\{(1/6, 5/6), (1/6, 5/6)\}$ is a maximal Nash equilibrium strategy in Game 3. But there are infinite number of maximal Nash equilibrium strategies in Game 3, and the set of all maximal Nash equilibrium strategies is given by $\{(x, 1 - x), (y, 1 - y) \in S_I \times S_J \mid 0 < x < 1/5,\ 0 < y < 1/5\} \cup \{(C, C)\} \cup \{(D, D)\}$.

Let $k^o \in \text{int } \mathbb{R}^{\ell}_{+}$ be any point and $\lambda_i, \mu_i \in \mathbb{R}_+$, $i = L, U$ be any real numbers such that $\lambda_L + \lambda_U = \mu_L + \mu_U = 1$. Now we define real-valued functions $f, g : S_I \times S_J \to \mathbb{R}$ by

$$f(x, y; k^o, \lambda_L, \lambda_U) := \lambda_L \phi^L(F(x, y); k^o) + \lambda_U \phi^U(F(x, y); k^o),$$
$$g(x, y; k^o, \mu_L, \mu_U) := \mu_L \phi^L(G(x, y); k^o) + \mu_U \phi^U(G(x, y); k^o).$$

Associated with game Γ, we define the following two person non-cooperative game with scalar payoffs $\Gamma(k^o, \lambda_L, \lambda_U\ \mu_L, \mu_U)$ by

$$\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) := \langle \{I, J\}, S_I \times S_J, \{f(\cdot, \cdot; k^o, \lambda_L, \lambda_U), g(\cdot, \cdot; k^o, \mu_L, \mu_U)\} \rangle.$$

		Player J	
		C	D
Player I	C	([10, 11], [10, 11])	([1, 4], [2, 9])
	D	([2, 9], [1, 4])	([3, 4], [3, 4])

Fig. 3 Game 3

We assume that λ_i and μ_i, $i = L, U$ are common knowledge in $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Definition 4.5 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ if it holds that

(i) $f(x, y^*; k^o, \lambda_L, \lambda_U) \leqq f(x^*, y^*; k^o, \lambda_L, \lambda_U) \quad \forall x \in S_I$,
(ii) $g(x^*, y; k^o, \mu_L, \mu_U) \leqq g(x^*, y^*; k^o, \mu_L, \mu_U) \quad \forall y \in S_J$.

Definition 4.6 A pair of strategies $(x^*, y^*) \in S_I \times S_J$ is said to be a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ if it holds that

(i) $f(x, y^*; k^o, \lambda_L, \lambda_U) < f(x^*, y^*; k^o, \lambda_L, \lambda_U) \quad \forall x \in S_I,\ x \neq x^*$,
(ii) $g(x^*, y; k^o, \mu_L, \mu_U) < g(x^*, y^*; k^o, \mu_L, \mu_U) \quad \forall y \in S_J,\ y \neq y^*$.

The following theorem holds between game Γ and game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Theorem 4.1 *Let $(x^*, y^*) \in S_I \times S_J$ be any pair of strategies to game Γ. Then, if the pair of strategies $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, then it is a weak Pareto Nash equilibrium to game Γ.*

Proof Suppose that there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \prec F(\bar{x}, y^*)$ holds. Then, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L) < f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$, which contradicts that (x^*, y^*) is Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \prec G(x^*, \bar{y})$ holds. Then, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) < g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. □

Theorem 4.2 *Let $(x^*, y^*) \in S_I \times S_J$ be any pair of strategies and suppose that $\lambda_i, \mu_i \in \text{int}\,\mathbb{R}_+$, $i = L, U$ are positive numbers in game $\Gamma(k^o, \lambda_L, \lambda_U\, \mu_L, \mu_U)$. Then we have the following:*

(i) *If the pair of strategies (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it is a Pareto Nash equilibrium to game Γ.*
(ii) *If the pair of strategies (x^*, y^*) is a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it is a maximal Nash equilibrium to game Γ.*

Proof First, we show that (i) holds. On the contrary, we suppose that the pair of strategies (x^*, y^*) is not a Pareto Nash equilibrium to game Γ. Then there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \preceq F(\bar{x}, y^*)$ holds. Since $\lambda_i > 0$, $i = L, U$, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L) < f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \preceq G(x^*, \bar{y})$ holds. Since $\mu_i > 0$, $i = L, U$, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) < g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$, which contradicts that (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$.

Next we show that (ii) holds. Suppose that there exists a strategy $\bar{x} \in S_I$ such that $F(x^*, y^*) \leqq F(\bar{x}, y^*)$ holds. Since $\lambda_i > 0$, $i = L, U$, and (x^*, y^*) is a strict

Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, from Theorem 3.2, we have $f(x^*, y^*; k^o, \lambda_L, \lambda_U) = f(\bar{x}, y^*; k^o, \lambda_L, \lambda_U)$ and $x^* = \bar{x}$. Therefore, we have $F(\bar{x}, y^*) \leqq F(x^*, y^*)$.

Next we suppose that there exists a strategy $\bar{y} \in S_J$ such that $G(x^*, y^*) \leqq G(x^*, \bar{y})$ holds. Then, since $\mu_i > 0$, $i = L, U$ and (x^*, y^*) is a strict Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, from Theorem 3.2, we have $g(x^*, y^*; k^o, \mu_L, \mu_U) = g(x^*, \bar{y}; k^o, \mu_L, \mu_U)$ and $y^* = \bar{y}$. Therefore, we have $G(\bar{x}, y^*) \leqq G(x^*, y^*)$. □

From Theorems 4.1 and 4.2, by varying parameters λ_i, μ_i, $i = L, U$, we could obtain another maximal Nash, Pareto Nash and weak Pareto Nash equilibrium strategies to game Γ.

We consider Game 3 given in Example 4.3 again. Let $k^o = 1 \in \mathbb{R}$ and λ_i, $\mu_i \in \operatorname{int} \mathbb{R}_+$, $i = L, U$ be any positive numbers such that $\lambda_L + \lambda_U = \mu_L + \mu_U = 1$. Then, for each $x := (x_1, x_2) \in S_I$ and $y := (y_1, y_2) \in S_J$, real-valued payoff functions $f, g : S_I \times S_J \to \mathbb{R}$ for each player I and J are given by

$$f(x, y; k^o, \lambda_L, \lambda_U) := 2\{(5\lambda_L + \lambda_U)y_1 - 2\lambda_L\}x_1 - (\lambda_L - 5\lambda_U)y_1 + (3\lambda_L + 4\lambda_U),$$
$$g(x, y; k^o, \mu_L, \mu_U) := 2\{(5\mu_L + \mu_U)x_1 - 2\mu_L\}y_1 - (\mu_L - 5\mu_U)x_1 + (3\mu_L + 4\mu_U).$$

We set $x_1^* := \mu_L/(5\mu_L + \mu_U) \in (0, 1/5)$ and $y_1^*; = \lambda_L/(5\lambda_L + \lambda_U) \in (0, 1/5)$. Then, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a Nash equilibrium in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. From Theorem 4.2, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a Pareto Nash equilibrium in game Γ. We show that the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a maximal Nash equilibrium in game Γ. Suppose that there exists a strategy $(x_1, x_2) \in S_I$ such that $F((x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)) \leqq F((x_1, x_2), (y_1^*, 1 - y_2^*))$. Then, by simple calculations, we have $(x_1, x_2) = (x_1^*, 1 - x_1^*)$. Similarly, we could show that $G((x_1^*, 1 - x^*), (y_1^*, 1 - y_1^*)) \leqq G((x_1^*, 1 - x_1^*), (y_1, y_2))$ implies $(y_1, y_2) = (y_1^*, 1 - y_1^*)$. Therefore, the pair of strategies $\{(x_1^*, 1 - x_1^*), (y_1^*, 1 - y_1^*)\}$ is a maximal Nash equilibrium in game Γ.

Note that, the scalar-payoff game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ induced from set-payoff game Γ, is the game with incomplete informations. Because in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, player I does not know μ_L and μ_U, while player J does not know the value of λ_L and λ_U which are necessary for each player to find best response strategies. Moreover, each player may choose different k^o in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Therefore, the Nash equilibrium strategy in the scalar-payoff game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is a Bayesian Nash equilibrium (See Harsanyi [6]).

5 Existence of Nash Equilibrium Strategy to Game Γ

In the previous section, for any given set payoff bi-matrix games, we define two person games with scalar-valued payoff functions, and investigate relationships between these games. In this section, we shall give some conditions under which there exists at least one maximal Nash, Pareto Nash and weak Pareto Nash equilibrium strategies to game Γ.

Lemma 5.1 *In game* Γ*, suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets. Then, it holds that*

(i) $(1-\lambda)F(x^1, y) + \lambda F(x^2, y) = F((1-\lambda)x^1 + \lambda x^2, y)\ \ \forall x^1, x^2 \in S_I,\ \forall y \in S_J,\ \ \forall \lambda \in [0, 1]$,
(ii) *The set-valued map* $F : S_I \times S_J \rightsquigarrow \mathbb{R}^\ell$ *is continuous on* $S_I \times S_J$*,*
(iii) F *is uniformly compact on* $S_I \times S_J$*,*
(iv) $(1-\lambda)G(x, y^1) + \lambda G(x, y^2) = G(x, (1-\lambda)y^1 + \lambda y^2)\ \ \forall x \in S_I,\ \forall y^1, y^2 \in S_J,\ \ \forall \lambda \in [0, 1]$,
(v) *The set-valued map* $G : \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^\ell$ *is continuous on* $S_I \times S_J$*,*
(vi) G *is uniformly compact on* $S_I \times S_J$*.*

Proof We shall show that (i), (ii) and (iii) hold. Let $(x^1, y), (x^2, y) \in S_I \times S_J$ be any strategies and $\lambda \in [0, 1]$ be any real number. Then by simple calculations, we have

$$\begin{aligned}
(1-\lambda)F(x^1, y) + \lambda F(x^2, y) &= (1-\lambda)\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^1 A_{ij} y_j + \lambda \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^2 A_{ij} y_j \\
&= (1-\lambda)\sum_{i=1}^{m} x_i^1 \sum_{j=1}^{n} A_{ij} y_j + \lambda \sum_{i=1}^{m} x_i^2 \sum_{j=1}^{n} A_{ij} y_j \\
&= \sum_{i=1}^{m}\{(1-\lambda)x_i^1 + \lambda x_i^2\} \sum_{j=1}^{n} A_{ij} y_j \\
&= F((1-\lambda)x^1 + \lambda x^2, y).
\end{aligned}$$

Next we shall show that (ii) holds. First, we shall show that $F(\cdot, \cdot)$ is upper semi-continuous on $S_I \times S_J$. Let $\{(x^\nu, y^\nu, z^\nu)\}_{\nu=1}^\infty \subseteq \mathrm{Gr}(F)$ be any sequence converging to $(x^o, y^o, z^o) \in S_I \times S_J \times \mathbb{R}^\ell$. By Definition, for each ν, there exits an $a_{ij}^\nu \in A_{ij},\ i \in M,\ j \in N$ such that $z^\nu = \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^\nu y_j^\nu$. Since A_{ij} is compact, without loss of any generality we assume that $\{a_{ij}^\nu\}_{\nu=1}^\infty$ converges to some point $a_{ij}^o \in A_{ij}$. Therefore, we have $z^o \in F(x^o, y^o)$.

Second we shall show that $F(\cdot, \cdot)$ is lower semi-continuous on $S_I \times S_J$. Let $\{(x^\nu, y^\nu)\}_{\nu=1}^\infty \subseteq S_I \times S_J$ be any sequence converging to $(x^o, y^o) \in S_I \times S_J$ and $z^o \in F(x^o, y^o)$ be any point. Then there exists $a_{ij}^o \in A_{ij}$ such that $z^o = \sum_{i=1}^m \sum_{j=1}^n x_i^o a_{ij}^o y_j^o$. By setting $z^\nu := \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^o y_j^\nu$, we have $z^\nu \in F(x^\nu, y^\nu)$ and $z^\nu \to z^o$, which implies that $F(\cdot, \cdot)$ is lower semi-continuous on $S_I \times S_J$. From the above, $F(\cdot, \cdot)$ is continuous on $S_I \times S_J$.

Finally we show that (iii) holds. In order to show that F is uniformly compact on $S_I \times S_J$, it suffices to show that $F(S_I, S_J)$ is compact. Let $\{z^\nu\}_{\nu=1}^\infty \subseteq F(S_I, S_J)$ be any sequence. By definition, for each ν, there exist points $a_{ij}^\nu \in A_{ij}, x_i^\nu \in S_I$ and $y_j^\nu \in S_J,\ i \in M,\ j \in N$ such that $z^\nu = \sum_{i=1}^m \sum_{j=1}^n x_i^\nu a_{ij}^\nu y_j^\nu$. By assumptions, $A_{ij},\ S_I$ and S_J are compact, without loss of any generality, we may assume that $x_i^\nu \to x_i^o,\ y_j^\nu \to y_j^o$, and $a_{ij}^\nu \to a_{ij}^o,\ i \in M,\ j \in N$, which implies that $z^\nu \to z^o \in$

$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^o a_{ij}^o y_j^o \in F(S_I, S_J)$. By a similar way, we could show that (iv), (v), and (vi) hold. □

From Theorems 3.3, 3.4 and Lemma 5.1, we have the following lemma.

Lemma 5.2 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets in game* Γ. *Then, it holds that*

(i) $f(\cdot, y; k^o, \lambda_L, \lambda_U) : S_I \to \mathbb{R}$ *is concave and* $f(\cdot, \cdot; k^o, \lambda_L, \lambda_U)$ *is continuous on* $S_I \times S_J$,
(ii) $g(x, \cdot; k^o, \mu_L, \mu_U) : S_J \to \mathbb{R}$ *is concave and* $g(\cdot, \cdot; k^o, \mu_L, \mu_U)$ *is continuous on* $S_I \times S_J$.

Let $(x, y) \in S_I \times S_J$ be any pair of strategies in game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. We define the set-valued maps $B_I(\cdot; k^o, \lambda_L, \lambda_U) : S_J \rightsquigarrow S_I$ and $B_J(\cdot; k^o, \mu_L, \mu_U) : S_I \rightsquigarrow S_J$ by $B_I(y; k^o; \lambda_L, \lambda_U) := \{u \in S_I \mid f(u, y; k^o, \lambda_L, \lambda_U) \geqq f(x, y; k^o, \lambda_L, \lambda_U)\ \forall x \in S_I\}$ and $B_J(x; k^o, \mu_L, \mu_U) := \{v \in S_J \mid g(x, v; k^o, \mu_L, \mu_U) \geqq g(x, y; k^o, \mu_L, \mu_U)\ \forall y \in S_J\}$, which are called the best response maps for players I and J, respectively.

From Lemma 5.2, we have the following lemmas.

Lemma 5.3 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets in game* Γ. *Then, it holds that*

(i) $B_I(y; k^o, \lambda_L, \lambda_U)$ *and* $B_J(x; k^o, \mu_L, \mu_U)$ *are non-empty, compact and convex set for each* $(x, y) \in S_I \times S_J$.
(ii) $B_I(\cdot; k^o, \lambda_L, \lambda_U) : S_J \rightsquigarrow S_I$ *and* $B_J(\cdot; k^o, \mu_L, \mu_U) : S_I \rightsquigarrow S_J$ *are upper semi-continuous on* S_J *and* S_I *respectively.*

Proof First we show that (i) holds. From Lemma 5.2, for each $y \in S_J$, $f(\cdot, y; k^o, \lambda_L, \lambda_U)$ is concave and continuous on S_I. Since S_I is compact and convex, it holds that $B_I(y; k^o, \lambda_L, \lambda_U)$ is non-empty, compact and convex for all $y \in S_J$. Similarly, we could show that (i) holds for $B_J(x; k^o, \mu_L, \mu_U)$.

Next, we prove that (ii) holds for $B_I(\cdot; k^o, \lambda_L, \lambda_U)$. Let $\{(x^\nu, y^\nu)\} \subseteq \mathrm{Gr}(B_I(\cdot; k^o, \lambda_L, \lambda_U))$ be any sequence converging to $(x^o, y^o) \in S_I \times S_J$. Then it holds that $f(x^\nu, y^\nu; k^o, \lambda_L, \lambda_U) \geqq f(u, y^\nu; k^o, \lambda_L, \lambda_U)$ for $\forall u \in S_I$. From Lemma 5.2, since $f(\cdot, \cdot; k^o, \lambda_L, \lambda_U)$ is continuous on $S_I \times S_J$, we have $f(x^o, y^o; k^o, \lambda_L, \lambda_U) \geqq f(u, y^o; k^o, \lambda_L, \lambda_U)$ for $\forall u \in S_I$, which implies that $x^o \in B_I(y^o; k^o, \lambda_L, \lambda_U)$. Similarly, we could show that (ii) holds for $B_J(\cdot; k^o, \mu_L, \mu_U)$. □

Now we define the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) : S_I \times S_J \rightsquigarrow S_I \times S_J$ by $B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) := B_I(y; k^o, \lambda_L, \lambda_U) \times B_J(x; k^o, \mu_L, \mu_U)$. Then from Lemma 5.3, we have the following lemma.

Lemma 5.4 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets in game* Γ. *Then, it holds that*

(i) $B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ *is non-empty, compact and convex for each* $(x, y) \in S_I \times S_J$.

(ii) *The set-valued map* $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) : S_I \times S_J \rightsquigarrow S_I \times S_J$ *is upper semi-continuous on* $S_I \times S_J$.

Lemma 5.5 *The pair of strategies* $(x^*, y^*) \in S_I \times S_J$ *is a Nash equilibrium to game* $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ *if and only if* $(x^*, y^*) \in B(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ *holds (See Kakutani [11]).*

Lemma 5.5 shows that a pair of strategies (x^*, y^*) is a Nash equilibrium if and only if it is a fixed point of the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. From the above lemmas, we have the following theorem.

Theorem 5.1 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell), i \in M,\ j \in N$ *are compact convex sets in game* Γ. *Then, there exists at least one Pareto Nash equilibrium strategy in game* Γ.

Proof Let $k^o \in \operatorname{int} \mathbb{R}^\ell_+$ be any point and $\lambda_i,\ \mu_i \in \mathbb{R}_+,\ i = L, U$ be any positive numbers. Then, from Lemma 5.3, $B(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is non-empty, compact and convex for each $(x, y) \in S_I \times S_J$ and the set-valued map $B(\cdot, \cdot; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is upper semi-continuous on $S_I \times S_J$. Therefore, from Kakutani's fixed point theorem [11], there exists at least one point $(x^*, y^*) \in S_I \times S_J$ such that $(x^*, y^*) \in B(x^*, y^*; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Therefore, from Lemma 5.5 and Theorem 4.2, the point (x^*, y^*) is a Pareto Nash equilibrium strategy in game Γ. □

Theorem 5.2 *Suppose that* $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}), i \in M,\ j \in N$ *are compact convex sets in game* Γ. *Then there exists at least one maximal Nash equilibrium strategy in game* Γ.

Proof Without loss of any generality, we assume that $k^o = 1,\ \lambda_i = \mu_i = 1, i = L, U$ and $A_{ij} = [a^L_{ij}, a^U_{ij}]$ and $B_{ij} = [b^L_{ij}, b^U_{ij}],\ i \in M,\ j \in N$. Then, from Corollary 3.1,

$$f(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (\phi^L(A_{ij}; k^o) + \phi^U(A_{ij}; k^o)) y_j,$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (a^L_i + a^U_{ij}) y_j, \tag{22}$$
$$g(x, y; k^o, \lambda_L, \lambda_U, \mu_L, \mu_U) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (\phi^L(B_{ij}; k^o) + \phi^U(B_{ij}; k^o)) y_j$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (b^L_{ij} + b^U_{ij}) y_j, \tag{23}$$

Namely, game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$ is a bi-matrix game with scalar valued payoffs. Let $(x^*, y^*) \in S_I \times S_J$ be any Nash equilibrium strategy to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$. Suppose that there exists a strategy $\bar{x} \in S_I$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*[a_{ij}^L, a_{ij}^U]y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i[a_{ij}^L, a_{ij}^U]y_j^*. \tag{24}$$

Then by Definition 2.1, it holds

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^L y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^L y_j^*, \tag{25}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^U y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^U y_j^*. \tag{26}$$

Since (x^*, y^*) is a Nash equilibrium to game $\Gamma(k^o, \lambda_L, \lambda_U, \mu_L, \mu_U)$, it must hold that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*(a_{ij}^L + a_{ij}^U)y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i(a_{ij}^L + a_{ijj}^U)y_j^*. \tag{27}$$

From (25), (26) and (27), it follows that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^L y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^L y_j^*$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij}^U y_j^* = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i a_{ij}^U y_j^*,$$

which implies that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i[a_{ij}^L, a_{ij}^U]y_j^* \leqq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*[a_{ij}^L, a_{ij}^U]y_j^*.$$

Namely, $F(x^*, y^*)$ is a maximal element in $\mathcal{F}(y^*)^{\leqq}$. By a similar way, we could show that $G(x^*, y^*)$ is a maximal element in $\mathcal{G}(x^*)^{\leqq}$. Hence (x^*, y^*) is a maximal Nash equilibrium strategy in game Γ. □

Theorem 5.3 *Suppose that $A_{ij}, B_{ij} \in \mathcal{C}(\mathbb{R}^\ell)$, $i \in M$, $j \in N$ are compact convex sets and that $\mathcal{B}(x, y) \neq \emptyset$ for each $(x, y) \in S_I \times S_J$ holds in game Γ. Then, there exists a Nash equilibrium in game Γ.*

Proof From Kakutani's fixed point theorem, it suffices to show that the set-valued map $\mathcal{B}$ is convex-valued and upper semi-continuous on $S_I \times S_J$.

First we show that set-valued map $\mathcal{B}_I$ is convex-valued. Let $y \in S_J$ be any element. Then from Lemma 5.1, for each $x^1, x^2 \in \mathcal{B}_I(y)$ and any $\lambda \in [0, 1]$, it holds that

$$(1-\lambda)F(x^1,y)+\lambda F(x^2,y)=F((1-\lambda)x^1+\lambda x^2,y)\subseteq F(u,y)+\mathbb{R}^\ell_+ \quad \forall u\in S_I,$$
$$F(u,y)\subseteq (1-\lambda)F(x^1,y)+\lambda F(x^2,y)=F((1-\lambda)x^1+\lambda x^2,y)-\mathbb{R}^\ell_+ \quad \forall u\in S_I,$$

which implies $(1-\lambda)x^1+\lambda x^2\in\mathcal{B}_I(y)$. Namely, the set-valued map $\mathcal{B}_I$ is convex-valued. Similarly, we could prove that the set -valued map $\mathcal{B}_J$ is convex-valued.

Next, we show that set-valued map $\mathcal{B}_I$ is upper semi-continuous on S_J. Let $\{(y^\nu,x^\nu)\}\subseteq S_I\times S_J$ be any sequence converging to (y^o,x^o) such that $x^\nu\in\mathcal{B}_I(y^\nu)$ for $\forall\nu$. It suffices to show that $F(u,y^o)\leqq F(x^o,y^o)$ holds for $\forall u\in S_I$. Let $z^o\in F(x^o,y^o)\subseteq F(u,y^o)+\mathbb{R}^\ell_+$ be any element. From Lemma 5.1, since F is continuous on $S_I\times S_J$, there exists a sequence $\{z^\nu\}$ converging to z^o such that $z^\nu\in F(x^\nu,y^\nu)\subseteq F(u,y^\nu)+\mathbb{R}^\ell_+$. By Definition, it holds that $F(u,y^\nu)\leqq_L F(x^\nu,y^\nu)$ for $\forall u\in S_I$ and for $\forall\nu$. Again, from Lemma 5.1, $F(u,y^\nu)$ is compact, we have $z^o\in F(u,y^o)+\mathbb{R}^\ell_+$, $\forall u\in S_I$, which implies that $F(u,y^o)\leqq_L F(x^o,y^o)$ holds for $\forall u\in S_I$.

Finally we show that $F(u,y^o)\leqq_U F(x^o,y^o)$ holds for $\forall u\in S_I$. Let $u\in S_I$ be any element. From the continuity of F, for any $z^o\in F(u,y^o)$, there exists $z^\nu\in F(u,y^\nu)\subseteq F(x^\nu,y^\nu)-\mathbb{R}^\ell_+$ such that $z^o\in F(u,y^o)$. Since F is compact-valued and continuous on $S_I\times S_J$, we have $z^o\in F(x^o,y^o)-\mathbb{R}^\ell$, which implies that $F(u,y^o)\leqq_U F(x^o,y^o)$ holds for $\forall u\in S_I$. By a similar way we could show that $\mathcal{B}_J$ is upper semi-continuous on S_I. Hence $\mathcal{B}$ is upper semi-continuous on $S_I\times S_J$. □

6 Conclusion

In this paper, we considered set payoff bi-matrix games where payoffs for each player are given by compact convex sets in $\mathbb{R}^\ell$, namely, players don't know the values of payoffs but the ranges of the payoffs. We call this environment deterministic uncertainty. This type of game may encompass interval payoff games, fuzzy payoff games and robust games. First, we define several types of set orderings on the set of all non-empty subsets in n-dimensional Euclidean space $\mathbb{R}^n$. Second, by using these orderings, we define four kinds of concepts of Nash equilibrium strategies, that is, Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium strategies to the games and investigate their properties. In particular, we investigate the relationships between set-payoff games and incomplete information games. Finally, we give sufficient conditions under which there exists these Nash equilibrium strategies in bi-matrix games with set-valued payoffs and necessary condition under which there exists Nash equilibrium strategies in bi-matrix games with interval-valued payoffs.

In this paper, we use the set-orderings $\leqq$, $\preceq$ and $\prec$ to define the concepts of Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium to the games with set payoff. However, it is easy to use other types of set-orderings, say $\leqq_i$, $i=L,U$ etc. to define the concepts of Nash, maximal Nash, Pareto Nash, and weak Pareto Nash equilibrium to the games with set payoff and we could derive similar results to the games with set payoffs. Moreover, we could define the incomplete information game with set payoffs, where each player chooses a set-orderings among set-ordering given in Definition 2.1, after that, each players plays the game with set payoff, without knowing the set-ordering which the other player chooses each other. This means that there are deep relationships between set-payoff games and incomplete information games.

References

1. Aghassi, M., Bertsimas, D.: Robust game theory. Math. Program. **107**, 231–273 (2006)
2. Araya, Y.: Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. **75**, 3821–3835 (2012)
3. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
4. Campos, L.: Fuzzy linear programming models to solve fuzzy matrix games. Fuzzy Sets Syst. **32**, 275–289 (1989)
5. Gerth, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
6. Harsanyi, J.: Games with incomplete information played by Bayesian players. Parts I, II, III Manage. Sci. **14**, 159–182, 320–334, 486–502 (1967–1968)
7. Hamel, A., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)
8. Hernández, H., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization. J. Math. Anal. Appl. **325**, 1–18 (2007)
9. Hogan, W.W.: Point-to-set maps in mathematical programming. SIAM Rev. **15**(13), 59–603 (1973)
10. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appli. **148**, 209–236 (2011)
11. Kakutani, S.: A generalization of Brouwer's fixed point theorem. Duke Math. J. **8**, 457–458 (1941)
12. Kuroiwa, D., Tanaka, T., Truong, X.D.H.: Cone convexity of set-valued maps. Nonlinear Anal. **30**(3), 1487–1496 (1997)
13. Kuroiwa, D.: The natural criteria in set-valued optimization. RIMS Kokyuroku **1031**, 85–90 (1998)
14. Kuroiwa, D.: On set-valued optimization. Nonlinear Anal. **47**, 1395–1400 (2001)
15. Leitmann, G.: Deterministic control of uncertain systems via a constructive use of Lyapunov stability theory. Syst. Model. Optim. **143**, 38–55 (1990)
16. Li, D.-F.: Linear programming approach to solve interval-valued matrix games. Omega **39**, 655–666 (2011)
17. Liu, S.H., Kao, C.: Matrix games with interval data. Comput. Ind. Eng. **56**, 1697–1700 (2009)
18. Maeda, T.: Characterization of the equilibrium strategy of bi-matrix game with fuzzy payoff. J. Math. Anal. Appl. **251**, 885–896 (2000)
19. Maeda, T.: On characterization of Nash equilibrium strategy of bi-matrix game with fuzzy payoff. RIMS kokyuroku **1297**, 154–162 (2002)
20. Maeda, T.: On characterization of equilibrium strategy of two person zero-sum games with fuzzy payoffs. Fuzzy Sets Syst. **139**(2), 283–296 (2003)
21. Maeda, T.: On characterization of fuzzy vectors and its applications to fuzzy mathematical programming problems. Fuzzy Sets Syst. **159**, 3336–3346 (2008)
22. Maeda, T.: On optimization problems with set-valued objective maps. Appl. Math. Comp. **217**, 1150–1157 (2010)

23. Maeda, T.: On optimization problems with set-valued objective maps, existence and optimality. J. Optim. Theory Appl. **153**, 263–279 (2012)
24. Nash, J.F.: Equilibrium points in n-person games. Proc. Natl. Acad. Sci. **36**, 48–49 (1950)
25. Nash, J.F.: Noncooperative games. Annals Math. **54**, 286–295 (1951)
26. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1944)
27. Young, R.C.: The algebra of many-valued quantities. Mathematische Annalen **104**, 260–290 (1931)