An Abstract Algebraic Logic View on Judgment Aggregation⋆

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Abstract. In the present paper, we propose Abstract Algebraic Logic (AAL) as a general logical framework for Judgment Aggregation. Our main contribution is a generalization of Herzberg's algebraic approach to characterization results on judgment aggregation and propositional-attitude aggregation, characterizing certain Arrovian classes of aggregators as Boolean algebra and MV-algebra homomorphisms, respectively. The characterization result of the present paper applies to agendas of formulas of an arbitrary *selfextensional* logic. This notion comes from AAL, and encompasses a vast class of logics, of which classical, intuitionistic, modal, many-valued and relevance logics are special cases. To each selfextensional logic $\mathcal{S}$, a unique class of algebras $\mathbb{A}\mathrm{lg}\mathcal{S}$ is canonically associated by the general theory of AAL. We show that for any selfextensional logic $\mathcal{S}$ such that $\mathbb{A}\mathrm{lg}\mathcal{S}$ is closed under direct products, any algebra in $\mathbb{A}\mathrm{lg}\mathcal{S}$ can be taken as the set of truth values on which an aggregation problem can be formulated. In this way, judgment aggregation on agendas formalized in classical, intuitionistic, modal, many-valued and relevance logic can be uniformly captured as special cases. This paves the way to the systematic study of a wide array of "realistic agendas" made up of complex formulas, the propositional connectives of which are interpreted in ways which depart from their classical interpretation. This is particularly interesting given that, as observed by Dietrich, nonclassical (subjunctive) interpretation of logical connectives can provide a strategy for escaping impossibility results.

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1 Introduction

Social Choice and Judgment Aggregation. The theory of *social choice* is the formal study of mechanisms for collective decision making, and investigates issues of philosophical, economic, and political significance, stemming from the classical Arrovian problem of how the preferences of the members of a group can be "fairly" aggregated into one outcome.

In the last decades, many results appeared generalizing the original Arrovian problem, which gave rise to a research area called *judgment aggregation* (JA) [25]. While the original work of Arrow [1] focuses on preference aggregation, this can be recognized as a special instance of the aggregation of consistent judgments, expressed by each member of a group of individuals over a given set of logically interconnected propositions (the *agenda*): each proposition in the agenda is either accepted or rejected by each group member, so as to satisfy certain requirements of logical consistency. Within the JA framework, the Arrovian-type *impossibility results* (axiomatically providing sufficient conditions for aggregator functions to turn into degenerate rules, such as dictatorship) are obtained as consequences of *characterization theorems* [26], which provide necessary and sufficient conditions for agendas to have aggregator functions on them satisfying given axiomatic conditions.

In the same logical vein, in [24], *attitude aggregation theory* was introduced; this direction has been further pursued in [19], where a characterization theorem has been given for certain many-valued propositional-attitude aggregators as MV-algebra homomorphisms.

The Ultrafilter Argument and its Generalizations. Methodologically, the *ultrafilter argument* is the tool underlying the generalizations and unifications mentioned above. It can be sketched as follows: to prove impossibility theorems for finite electorates, one shows that the axiomatic conditions on the aggregation function force the set of all decisive coalitions to be an (ultra)filter on the powerset of the electorate. If the electorate is finite, this implies that all the decisive coalitions must contain one and the same (singleton) coalition: the oligarchs (the dictator). Employed in [11] and [23] for a proof of Arrow's theorem alternative to the original one[1], this argument was applied to obtain elegant and concise proofs of impossibility theorems also in judgment aggregation [7]. More recently, it gave rise to characterization theorems, e.g. establishing a bijective correspondence between Arrovian aggregation rules and ultrafilters on the set of individuals [20]. Moreover, the ultrafilter argument has been generalized by Herzberg and Eckert [20] to obtain a generalized Kirman-Sondermann correspondence as a consequence of which Arrow-rational aggregators can be identified with those arising as ultraproducts of profiles (see also [2], in which the results in [20] have been generalized to a setting accounting for vote abstention), and—using the well-known correspondence between ultrafilters and Boolean homomorphisms—similar cor-

[1] See also [16] for further information about the genesis and application of the technique.

respondences have been established between Arrovian judgment aggregators and Boolean algebra homomorphisms [18].

Escaping Impossibility via Nonclassical Logics. While much research in this area explored the limits of the applicability of Arrow-type results, at the same time the question of how to 'escape impossibility' started attracting increasing interest. In [5], Dietrich provides a unified model of judgment aggregation which applies to predicate logic as well as to modal logic and fuzzy logics. In [6], Dietrich argues that impossibility results do not apply to a wide class of realistic agendas once propositions of the form 'if a then b' are modelled as *subjunctive* implications rather than material implications. Besides their theoretical value, these results are of practical interest, given that subjunctive implication models the meaning of if-then statements in natural language more accurately than material implication. In [27] and [28], Porello discusses judgment aggregation in the setting of intuitionistic, linear and substructural logics. In particular, in [28], it is shown that linear logic is a viable way to circumvent impossibility theorems in judgment aggregation.

Aim. A natural question arising in the light of these results is how to uniformly account for the role played by the different logics (understood both as formal language and deductive machinery) underlying the given agenda in characterization theorems for JA.

The present paper focuses on *Abstract Algebraic Logic* as a natural theoretical setting for Herzberg's results [17, 19], and the theory of *(fully) selfextensional logics* as the appropriate logical framework for a nonclassical interpretation of logical connectives, in line with the approach of [6].

Abstract Algebraic Logic and Selfextensional Logics. Abstract Algebraic Logic (AAL) [14] is a forty-year old research field in mathematical logic. It was conceived as the framework for an algebraic approach to the investigation of classes of logics. Its main goal was establishing a notion of *canonical algebraic semantics* uniformly holding for classes of logics, and using it to systematically investigate (metalogical) properties of logics in connection with properties of their algebraic counterparts.

Selfextensionality is the metalogical property holding of those logical systems whose associated relation of logical equivalence on formulas is a congruence of the term algebra. Wójcicki [29] characterized selfextensional logics as the logics which admit a so-called *referential semantics* (which is a general version of the well known possible-world semantics of modal and intuitionistic logics), and in [22], a characterization was given of the particularly well behaved subclass of the fully selfextensional logics in general duality-theoretic terms. This subclass includes many well-known logics, such as classical, intuitionistic, modal, many-valued and relevance logic. These and other results in this line of research (cf. e.g. [8, 9, 15, 21]) establish a systematic connection between possible world semantics and the logical account of intensionality.

Contributions. In the present paper, we generalize and refine Herzberg's characterization result in [19] from the MV-algebra setting to any class of algebras canonically associated with some selfextensional logic. This generalization simultaneously accounts for agendas expressed in the language of such logics as modal, intuitionistic, relevance, substructural and many-valued logics. Besides having introduced the connection between AAL and Judgment Aggregation, the added value of this approach is that it is parametric in the logical system $\mathcal{S}$. In particular, the properties of agendas are formulated independently of a specific logical signature and are slightly different than those in Herzberg's setting. In contrast with Herzberg's characterization result, which consisted of two slightly asymmetric parts, the two propositions which yield the characterization result in the present paper (cf. Propositions 1 and 2) are symmetric. Aggregation of propositional attitudes modeled in classical, intuitionistic, modal, Łukasiewicz and relevance logic can be uniformly captured as special cases of the present result. This makes it possible to fine-tune the expressive and deductive power of the formal language of the agenda, so as to capture e.g. intensional or vague statements.

Structure of the Paper. In Section 2, relevant preliminaries are collected on Abstract Algebraic Logic. In Section 3, Herzberg's algebraic framework for aggregation theory is generalized from MV-algebras to $\mathcal{S}$-algebras, where $\mathcal{S}$ is an arbitrary selfextensional logic. In Section 4, the main characterization result is stated. In Section 5, the impossibility theorem for judgment aggregation is deduced as a corollary of the main result, and one well known setting accounting for the subjunctive reading of implication is discussed.

2 Preliminaries on Abstract Algebraic Logic

The present section collects the basic concepts of Abstract Algebraic Logic that we will use in the paper. For a general view of AAL the reader is addressed to [13] and the references therein.

2.1 General Approach.

As mentioned in the introduction, in AAL, logics are not studied in isolation, and in particular, investigation focuses on classes of logics and their identifying *metalogical properties*. Moreover, the notion of *consequence* rather than the notion of *theoremhood* is taken as basic: consequently, *sentential logics*, the primitive objects studied in AAL, are defined as tuples $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\mathbf{Fm}$ is the algebra of formulas of type $\mathcal{L}_{\mathcal{S}}$ over a denumerable set of propositional variables Var, and $\vdash_{\mathcal{S}}$ is a *consequence relation* on (the carrier of) $\mathbf{Fm}$ (cf. Subsection 2.3).

This notion encompasses logics that are defined by any sort of proof-theoretic calculus (Gentzen-style, Hilbert-style, tableaux, etc.), as well as logics arising from some classes of (set-theoretic, order-theoretic, topological, algebraic, etc.) semantic structures, and in fact it allows to treat logics independently of the way in which they have been originally introduced. Another perhaps more common approach in logic takes the notion of theoremhood as basic and consequently sees logics as sets of formulas (possibly closed under some rules of inference).

This approach is easily recaptured by the notion of sentential logic adopted in AAL: Every sentential logic $\mathcal{S}$ is uniquely associated with the set $Thm(\mathcal{S}) = \{\varphi \in Fm \mid \emptyset \vdash_{\mathcal{S}} \varphi\}$ of its *theorems*.

2.2 Consequence Operations

For any set A, a *consequence operation* (or closure operator) on A is a map $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$: (1) $X \subseteq C(X)$, (2) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$ and (3) $C(C(X)) = C(X)$. The closure operator C is *finitary* if in addition satisfies (4) $C(X) = \bigcup\{C(Z) : Z \subseteq X, Z \text{ finite}\}$. For any consequence operation C on A, a set $X \subseteq A$ is C-*closed* if $C(X) = X$. Let $\mathcal{C}_C$ be the collection of C-closed subsets of A.

For any set A, a *closure system* on A is a collection $\mathcal{C} \subseteq \mathcal{P}(A)$ such that $A \in \mathcal{C}$, and $\mathcal{C}$ is closed under intersections of arbitrary non-empty families. A closure system is *algebraic* if it is closed under unions of up-directed[2] families.

For any closure operator C on A, the collection $\mathcal{C}_C$ of the C-closed subsets of A is a closure system on A. If C is finitary, then $\mathcal{C}_C$ is algebraic. Any closure system $\mathcal{C}$ on A defines a consequence operation $C_{\mathcal{C}}$ on A by setting $C_{\mathcal{C}}(X) = \bigcap\{Y \in \mathcal{C} : X \subseteq Y\}$ for every $X \subseteq A$. The $C_{\mathcal{C}}$-closed sets are exactly the elements of $\mathcal{C}$. Moreover, $\mathcal{C}$ is algebraic if and only if $C_{\mathcal{C}}$ is finitary.

2.3 Logics

Let $\mathcal{L}$ be a propositional language type (i.e. a set of connectives and their arities, which we will also regard as a set of function symbols) and let $\mathbf{Fm}_{\mathcal{L}}$ denote the algebra of formulas (or term algebra) of $\mathcal{L}$ over a denumerable set V of propositional variables. Let $Fm_{\mathcal{L}}$ be the carrier of the algebra $\mathbf{Fm}_{\mathcal{L}}$. A *logic* (or deductive system) of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}} \rangle$ such that $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ such that the operator $C_{\vdash_{\mathcal{S}}} : \mathcal{P}(Fm_{\mathcal{L}}) \to \mathcal{P}(Fm_{\mathcal{L}})$ defined by

$$\varphi \in C_{\vdash_{\mathcal{S}}}(\Gamma) \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi$$

is a consequence operation with the property of *invariance under substitutions*; this means that for every substitution σ (i.e. for every $\mathcal{L}$-homomorphism $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$) and for every $\Gamma \subseteq Fm_{\mathcal{L}}$,

$$\sigma[C_{\vdash_{\mathcal{S}}}(\Gamma)] \subseteq C_{\vdash_{\mathcal{S}}}(\sigma[\Gamma]).$$

For every $\mathcal{S}$, the relation $\vdash_{\mathcal{S}}$ is the *consequence* or *entailment* relation of $\mathcal{S}$. A logic is *finitary* if the consequence operation $C_{\vdash_{\mathcal{S}}}$ is finitary. Sometimes we will use the symbol $\mathcal{L}_{\mathcal{S}}$ to refer to the propositional language of a logic $\mathcal{S}$.

The *interderivability relation* of a logic $\mathcal{S}$ is the relation $\equiv_{\mathcal{S}}$ defined by

$$\varphi \equiv_{\mathcal{S}} \psi \quad \text{iff} \quad \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi.$$

$\mathcal{S}$ satisfies the *congruence property* if $\equiv_{\mathcal{S}}$ is a congruence of $\mathbf{Fm}_{\mathcal{L}}$.

[2] For $\langle P, \leq \rangle$ a poset, $U \subseteq P$ is *up-directed* when for any $a, b \in U$ there exists $c \in U$ such that $a, b \leq c$.

2.4 Logical Filters

Let $\mathcal{S}$ be a logic of type $\mathcal{L}$ and let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra (from now on, we will drop reference to the type $\mathcal{L}$, and when we refer to an algebra or class of algebras in relation with $\mathcal{S}$, we will always assume that the algebra and the algebras in the class are of type $\mathcal{L}$).

A subset $F \subseteq A$ is an $\mathcal{S}$*-filter* of $\boldsymbol{A}$ if for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and every $h \in \mathrm{Hom}(\mathbf{Fm}_{\mathcal{L}}, \boldsymbol{A})$,

$$\text{if } \Gamma \vdash_{\mathcal{S}} \varphi \text{ and } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

The collection $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ of the $\mathcal{S}$-filters of $\boldsymbol{A}$ is a closure system. Moreover, $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ is an algebraic closure system if $\mathcal{S}$ is finitary. The consequence operation associated with $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ is denoted by $C^{\boldsymbol{A}}_{\mathcal{S}}$. For every $X \subseteq A$, the closed set $C^{\boldsymbol{A}}_{\mathcal{S}}(X)$ is the $\mathcal{S}$-filter of $\boldsymbol{A}$ generated by X. If $\mathcal{S}$ is finitary, then $C^{\boldsymbol{A}}_{\mathcal{S}}$ is finitary for every algebra $\boldsymbol{A}$.

On the algebra of formulas $\mathbf{Fm}$, the closure operator $C^{\mathbf{Fm}}_{\mathcal{S}}$ coincides with $C_{\vdash_{\mathcal{S}}}$ and the $C^{\mathbf{Fm}}_{\mathcal{S}}$-closed sets are exactly the $\mathcal{S}$*-theories*; that is, the sets of formulas which are closed under the relation $\vdash_{\mathcal{S}}$.

2.5 $\mathcal{S}$-algebras and Selfextensional Logics

One of the basic topics of AAL is how to associate in a uniform way a class of algebras with an arbitrary logic $\mathcal{S}$. According to contemporary AAL [13], the canonical algebraic counterpart of $\mathcal{S}$ is the class $\mathbb{A}\mathrm{lg}\mathcal{S}$, whose elements are called $\mathcal{S}$*-algebras*. This class can be defined via the notion of Tarski congruence.

For any algebra $\mathbf{A}$ (of the same type as $\mathcal{S}$) and any closure system $\mathcal{C}$ on $\mathbf{A}$, the *Tarski congruence of* $\mathcal{C}$ *relative to* $\mathbf{A}$, denoted by $\tilde{\mathbf{\Omega}}_{\mathbf{A}}(\mathcal{C})$, is the greatest congruence which is compatible with all $F \in \mathcal{C}$, that is, which does not relate elements of F with elements which do not belong to F. The Tarski congruence of the closure system consisting of all $\mathcal{S}$-theories relative to $\mathbf{Fm}$ is denoted by $\tilde{\mathbf{\Omega}}(\mathcal{S})$. The quotient algebra $\mathbf{Fm}/\tilde{\mathbf{\Omega}}(\mathcal{S})$ is called the *Lindenbaum-Tarski algebra* of $\mathcal{S}$.

For any algebra $\mathbf{A}$, we say that $\mathbf{A}$ is an $\mathcal{S}$*-algebra* (cf. [13, Definition 2.16]) if the Tarski congruence of $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ relative to $\mathbf{A}$ is the identity. It is well-known (cf. [13, Theorem 2.23] and ensuing discussion) that $\mathbb{A}\mathrm{lg}\mathcal{S}$ is closed under direct products. Moreover, for any logic $\mathcal{S}$, the Lindenbaum-Tarski algebra is an $\mathcal{S}$-algebra (see page 36 in [13]).

A logic $\mathcal{S}$ is *selfextensional* (cf. [29]) when the relation of *logical equivalence* between formulas

$$\varphi \equiv_{\mathcal{S}} \psi \qquad \text{iff} \qquad \varphi \vdash_{\mathcal{S}} \psi \quad \text{and} \quad \psi \vdash_{\mathcal{S}} \varphi$$

is a congruence relation of the formula algebra $\mathbf{Fm}$. An equivalent definition of selfextensionality (see page 48 in [13]) is given as follows: $\mathcal{S}$ is selfextensional iff the Tarski congruence $\tilde{\mathbf{\Omega}}(\mathcal{S})$ and the relation of logical equivalence $\equiv_{\mathcal{S}}$ coincide. In such case the Lindenbaum-Tarski algebra reduces to $\mathbf{Fm}/\equiv_{\mathcal{S}}$. Examples of selfextensional logics besides classical propositional logic are intuitionistic logic, positive modal logic [4], the $\{\wedge, \vee\}$-fragment of classical propositional logic, Belnap's four-valued logic [3], the local consequence relation on modal formulas arising from Kripke frames, the (order-induced) consequence relation associated with MV-algebras and defined by "preserving degrees of truth" (cf. [12]), and the order-induced consequence relation of linear logic. Examples of non-selfextensional logics include the $\top$-induced consequence relation of linear logic, the ($\top$-induced) consequence relation associated with MV-algebras and defined by "preserving absolute truth" (cf. [12]), and the global consequence relation on modal formulas arising from Kripke frames.

From now on we assume that $\mathcal{S}$ is a selfextensional logic and $\mathbf{B} \in \mathbb{A}\mathrm{lg}\mathcal{S}$. For any formula $\varphi \in Fm$, we say that φ is *provably equivalent* to a propositional variable iff there exist a propositional variable x such that $\varphi \equiv x$.

3 Formal Framework

In the present section, we generalize Herzberg's algebraic framework for aggregation theory from MV-propositional attitudes to $\mathcal{S}$-propositional attitudes, where $\mathcal{S}$ is an arbitrary selfextensional logic. Our conventional notation is similar to [19]. Let $\mathcal{L}$ be a logical language which contains countably many connectives, each of which has arity at most n, and let Fm be the collection of $\mathcal{L}$-formulas.

3.1 The Agenda

The *agenda* will be given by a set of formulas $X \subseteq \mathbf{Fm}$. Let $\bar{X}$ denote the closure of X under the connectives of the language, i.e. the smallest set containing all formulas in X and the 0-ary connectives in $\mathcal{L}$, and closed under the connectives in the language. Notice that for any constant $c \in \mathcal{L}$, we have $c \in \bar{X}$.

We want the agenda to contain a sufficiently rich collection of formulas. In the classical case, it is customary to assume that the agenda contains at least two propositional variables. In our general framework, this translates in the requirement that the agenda contains at least n formulas that 'behave' like propositional variables, in the sense that their interpretation is not constrained by the interpretation of any other formula in the agenda.

We could just assume that the agenda contains at least n different propositional variables, but we will deal with a slightly more general situation, namely, we assume that the agenda is n-pseudo-rich:

Definition 1. *An agenda is n-*pseudo-rich *if it contains at least n formulas $\{\delta_1, \dots, \delta_n\}$ such that each δ_i is provably equivalent to x_i for some set $\{x_1, \dots, x_n\}$ of pairwise different propositional variables.*

3.2 Attitude Functions, Profiles and Attitude Aggregators

An *attitude function* is a function $A \in \mathbf{B}^X$ which assigns an element of the algebra $\mathbf{B}$ to each formula in the agenda.

The *electorate* will be given by some (finite or infinite) set N. Each $i \in N$ is called an *individual.*

An *attitude profile* is an N-sequence of attitude functions, i.e. $\boldsymbol{A} \in (\mathbf{B}^X)^N$. For each $\varphi \in X$, we denote the N-sequence $\{A_i(\varphi)\}_{i\in N} \in \mathbf{B}^N$ by $\boldsymbol{A}(\varphi)$.

An attitude aggregator is a function which maps each profile of individual attitude functions in some domain to a collective attitude function, interpreted as the set of preferences of the electorate as a whole. Formally, an *attitude aggregator* is a partial map $F : (\mathbf{B}^X)^N \nrightarrow \mathbf{B}^X$.

3.3 Rationality

Let the agenda contain formulas $\varphi_1, \ldots, \varphi_m, g(\varphi_1, \ldots, \varphi_m) \in X$, where $g \in \mathcal{L}$ is an m-ary connective of the language and $m \leq n$. Among all attitude functions $A \in \mathbf{B}^X$, those for which it holds that $A(g(\varphi_1, \ldots, \varphi_n)) = g^{\mathbf{B}}(A(\varphi_1), \ldots, A(\varphi_n))$ are of special interest. In general, we will focus on attitude functions which are 'consistent' with the logic $\mathcal{S}$ in the following sense.

We say that an attitude function $A \in \mathbf{B}^X$ is *rational* if it can be extended to a homomorphism $\bar{A} : \mathbf{Fm}_{/\equiv} \longrightarrow \mathbf{B}$ of $\mathcal{S}$-algebras. In particular, if A is rational, then it can be uniquely extended to $\bar{X}$, and we will implicitly use this fact in what follows.

We say that a profile $\boldsymbol{A} \in (\mathbf{B}^X)^N$ is *rational* if A_i is a rational attitude function for each $i \in N$.

We say that an attitude aggregator $F : (\mathbf{B}^X)^N \nrightarrow \mathbf{B}^X$ is *rational* if for all rational profiles $\boldsymbol{A} \in dom(F)$ in its domain, $F(\boldsymbol{A})$ is a rational attitude function. Moreover, we say that F is *universal* if $\boldsymbol{A} \in dom(F)$ for any rational profile $\boldsymbol{A}$. In other words, an aggregator is universal whenever its domain contains all rational profiles, and it is rational whenever it gives a rational output provided a rational input.

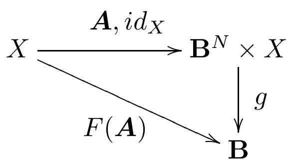
3.4 Decision Criteria and Systematicity

A *decision criterion* for F is a partial map $f : \mathbf{B}^N \nrightarrow \mathbf{B}$ such that for all $\boldsymbol{A} \in dom(F)$ and all $\varphi \in X$,

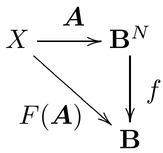
$$F(\boldsymbol{A})(\varphi) = f(\boldsymbol{A}(\varphi)). \tag{3.1}$$

As observed by Herzberg [19], an aggregator is independent if the aggregate attitude towards any proposition φ does not depend on the individuals attitudes towards propositions other than φ:

An aggregator F is *independent* if there exists some map $g : \mathbf{B}^N \times X \nrightarrow \mathbf{B}$ such that for all $\boldsymbol{A} \in dom(F)$, the following diagram commutes (whenever the partial maps are defined):

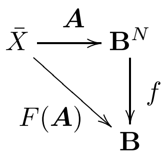


An aggregator F is *systematic* if there exists some *decision criterion* f for F, i.e. there exists some map $f : \mathbf{B}^N \nrightarrow \mathbf{B}$ such that for all $\boldsymbol{A} \in dom(F)$, the following diagram commutes (whenever the partial maps are defined):



Systematic aggregation is a special case of independent aggregation, in which the output of g does not depend on the input in the second coordinate. Thus, g is reduced to a decision criterion $f : \mathbf{B}^N \nrightarrow \mathbf{B}$.

An aggregator F is *strongly systematic* if there exists some decision criterion f for F, such that for all $\boldsymbol{A} \in dom(F)$, the following diagram commutes (whenever the partial maps are defined):



Notice that the diagram above differs from the previous one in that the agenda X is now replaced by its closure $\bar{X}$ under the connectives of the language. If X is closed under the operations in $\mathcal{L}_{\mathcal{S}}$, then systematicity and strong systematicity coincide.

A formula $\varphi \in \mathbf{Fm}$ is *strictly contingent* if for all $a \in \mathbf{B}$ there exists some homomorphism $v : \mathbf{Fm} \to \mathbf{B}$ such that $v(\varphi) = a$. Notice that for any $n \geq 1$, any n-pseudo rich agenda (cf. Definition 1) always contains a strictly contingent formula. Moreover, if the agenda contains some strictly contingent formula φ, then any universal systematic attitude aggregator F has a unique decision criterion (cf. [19, Remark 3.5]).

Before moving on to the main section, we mention four definitions which appear in Herzberg's paper, namely that of *Paretian* attitude aggregator (cf. [19, Definition 3.7]), *complex* and *rich* agendas (cf. [19, Definition 3.8]), and *strongly systematizable* aggregators (cf. [19, Definition 3.9]). Unlike the previous ones, these definitions rely on the specific MV-signature, and thus do not have a natural counterpart in the present, vastly more general setting. However, as we will see, our main result can be formulated independently of these definitions. Moreover, a generalization of the Pareto condition follows from the assumptions of F being universal, rational and strongly systematic, as then it holds that for any constant $c \in \mathcal{L}_{\mathcal{S}}$, and any $\varphi \in \mathbf{Fm}$, if $A_i(\varphi) = c$ for all $i \in N$, then $F(\boldsymbol{A})(\varphi) = c$.

4 Characterization Results

In the present section, the main results of the paper are presented. In what follows, we fix a language type $\mathcal{L}$ and a selfextensional logic $\mathcal{S}$. Recall that $\equiv$ indicates the interderivability relation associated with $\mathcal{S}$ and $\mathbf{B}$ is an arbitrary algebra in $\mathbb{A}\mathrm{lg}\mathcal{S}$.

Lemma 1. *Let X be an n-pseudo-rich agenda, $m \leq n$, $g \in \mathcal{L}$ be an m-ary connective and $a_1, \ldots, a_m \in \mathbf{B}$. Then there exist formulas $\delta_1, \ldots, \delta_m \in X$ in the agenda and a rational attitude function $A : X \longrightarrow \mathbf{B}$ such that $A(\delta_j) = a_j$ for each $j \in \{1, \ldots, m\}$.*

Proof. As the agenda is n-pseudo-rich, there are formulas $\delta_1, \ldots, \delta_m \in X$ each of which is provably equivalent to a different propositional variable x_i. Notice that this implies that the formulas $\delta_1, \ldots, \delta_m$ are not pairwise interderivable. So the $\equiv$-equivalence cells $[\delta_1], \ldots, [\delta_m]$ are pairwise different, and moreover there exists a valuation $v : \mathbf{Fm}/{\equiv} \longrightarrow \mathbf{B}$ such that $v(\delta_i) = a_i$ for all $i \in \{1, \ldots, m\}$. Let $A := v \circ \pi_{\restriction X}$, where $\pi_{\restriction X} : X \to \mathbf{Fm}/{\equiv}$ is the restriction of the canonical projection $\pi : \mathbf{Fm} \to \mathbf{Fm}/{\equiv}$ to X. Then clearly $A : X \to \mathbf{B}$ is the required rational attitude function.

Lemma 2. *Let X be an n-pseudo-rich agenda, $m \leq n$, $g \in \mathcal{L}$ be an m-ary connective and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbf{B}^N$. Then there exist formulas $\delta_1, \ldots, \delta_m \in X$ in the agenda and a rational attitude profile $\boldsymbol{A} : X \longrightarrow \mathbf{B}^N$ such that $\boldsymbol{A}(\delta_j) = \boldsymbol{a}_j$ for each $j \in \{1, \ldots, m\}$.*

Proof. As the agenda is n-pseudo-rich, there are formulas $\delta_1, \ldots, \delta_m \in X$ each of which is provably equivalent to a different propositional variable x_i. By the previous lemma, for each $i \in N$, there exists a rational attitude function $A_i : X \longrightarrow \mathbf{B}$ such that $A_i(\delta_j) = \boldsymbol{a}_j(i)$ for each $j \in \{1, \ldots, m\}$. Thus it is easy to check that the sequence of attitudes $\boldsymbol{A} := \{A_i\}_{i \in N}$ is a rational profile such that $\boldsymbol{A}(\delta_j) = \boldsymbol{a}_j$ for each $j \in \{1, \ldots, m\}$.

Recall that given that X is n-pseudo rich, there exists a unique decision criterion for any strongly systematic attitude aggregator F (cf. page 85). We omit the proofs of the following propositions, which can be found in an extended version of the present paper (cf. [10]):

Proposition 1. *Let F be a rational, universal and strongly systematic attitude aggregator. Then the decision criterion of F is a homomorphism of $\mathcal{S}$-algebras.*

Proposition 2. *Let $f : \mathbf{B}^N \twoheadrightarrow \mathbf{B}$ be a homomorphism of $\mathcal{S}$-algebras. Then the function $F : (\mathbf{B}^X)^N \twoheadrightarrow \mathbf{B}^X$, defined for any rational profile $\boldsymbol{A}$ and any $\varphi \in X$ by the following assignment:*

$$F(\boldsymbol{A})(\varphi) = f(\boldsymbol{A}(\varphi)),$$

is a rational, universal and strongly systematic attitude aggregator.

Finally, the conclusion of the following corollary expresses a property which is a generalization of the Pareto condition (cf. [19, Definition 3.7]).

Corollary 1. *If F is universal, rational and strongly systematic, then for any constant $c \in \mathcal{L}_{\mathcal{S}}$ and $\varphi \in \mathbf{Fm}$, if $A_i(\varphi) = c^{\mathbf{B}}$ for all $i \in N$, then $F(\boldsymbol{A})(\varphi) = c^{\mathbf{B}}$.*

Proof. Let $c \in \mathcal{L}_{\mathcal{S}}$ and $\varphi \in \mathbf{Fm}$. Notice that by definition of the product algebra, the sequence $\{c^{\mathbf{B}}\}_{i \in N}$ is precisely $c^{\mathbf{B}^N}$. If $A_i(\varphi) = c^{\mathbf{B}}$ for all $i \in N$, i.e. $\boldsymbol{A}(\varphi) = c^{\mathbf{B}^N}$, then by Proposition 1, $F(\boldsymbol{A})(\varphi) = f(\boldsymbol{A}(\varphi)) = f(c^{\mathbf{B}^N}) = c^{\mathbf{B}}$, as required.

5 Applications

In the present section, we show how the setting in the present paper relates to existing settings in the literature.

5.1 Arrow-Type Impossibility Theorem for Judgment Aggregation

Let $\mathcal{S}$ be the classical propositional logic. Its algebraic counterpart $\mathbb{A}\mathrm{lg}\mathcal{S} = \mathbb{BA}$ is the variety of Boolean algebras. Let $\mathcal{L} = \{\neg, \vee\}$ be its language (the connectives $\wedge, \rightarrow, \leftrightarrow$ are definable from the primitive ones). Let $\mathbf{B} = \mathbf{2}$ be the two-element Boolean algebra. Let $X \subseteq Fm_{\mathcal{L}}$ be a 2-pseudo-rich agenda.

By Propositions 1 and 2, for every electorate N, there exists a bijection between rational, universal and strongly systematic attitude aggregators $F : (\mathbf{2}^X)^N \longrightarrow \mathbf{2}^X$ [3] and Boolean homomorphisms $f : \mathbf{2}^N \longrightarrow \mathbf{2}$.

Recall that there is a bijective correspondence between Boolean homomorphisms $f : \mathbf{2}^N \longrightarrow \mathbf{2}$ and ultrafilters of $\mathbf{2}^N$. Moreover, if N is finite, every ultrafilter of $\mathbf{2}^N$ is principal. In this case, a decision criterion corresponds to an ultrafilter exactly when it is dictatorial.

5.2 A Mathematical Environment for the Subjunctive Interpretation of 'if – then' Statements

In [6], Dietrich argues that, in order to reflect the meaning of connection rules (i.e. formulas of the form $p \rightarrow q$ or $p \leftrightarrow q$ such that p and q are conjunctions of atomic propositions or negated atomic propositions) as they are understood and used in natural language, the connective $\rightarrow$ should be interpreted subjunctively. That is, the formula $p \rightarrow q$ should not be understood as a statement about the actual world, but about whether q holds in hypothetical world(s) where p holds, depends on q's truth value in possibly non-actual worlds. Dietrich proposes that, in the context of connection rules, any such implication should satisfy the following conditions:

(a) for any atomic propositions p and q, $p \rightarrow q$ is inconsistent with $\{p, \neg q\}$ but consistent with each of $\{p, q\}$ $\{\neg p, q\}$ $\{\neg p, \neg q\}$;

[3] Note that in this case an alternative presentation of F is $F : \mathcal{P}(X)^N \longrightarrow \mathcal{P}(X)$, which is the standard one.

(b) for any atomic propositions p and q, $\neg(p \to q)$ is consistent with each of $\{p, \neg q\}$, $\{p, q\}$, $\{\neg p, q\}$ and $\{\neg p, \neg q\}$.

Clearly, the classical interpretation of $p \to q$ as $\neg p \vee q$ satisfies only condition (a) but not (b). The subjunctive interpretation of $\to$ has been formalised in various settings based on possible-worlds semantics. One such setting, which is different from the one adopted by Dietrich's, is given by Boolean algebras with operators (BAOs). These are Boolean algebras endowed with an additional unary operation $\Box$ satisfying the identities $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$. Let us further restrict ourselves to the class of BAOs such that the inequality $\Box x \leq x$ is valid. This class coincides with $\mathbb{A}\mathrm{lg}\mathcal{S}$, where $\mathcal{S}$ is the normal modal logic **T** with the so-called local consequence relation. It is well known that **T** is selfextensional and is complete w.r.t. the class of reflexive Kripke frames. In this setting, let us stipulate that $p \to q$ is interpreted as $\Box(\neg p \vee q)$.

It is easy to see that this interpretation satisfies both conditions (a) and (b). To show that $p \to q$ is inconsistent with $\{p, \neg q\}$, observe that $\Box(\neg p \vee q) \wedge p \wedge \neg q \leq (\neg p \vee q) \wedge p \wedge \neg q = (\neg p \wedge (p \wedge \neg q)) \vee (q \wedge (p \wedge \neg q)) = \bot \vee \bot = \bot$.

To show that $p \to q$ is consistent with $\{p, q\}$,consider the two-element BAO s.t. $\Box 1 = 1$ and $\Box 0 = 0$. The assignment mapping p and q to 1 witnesses the required consistency statement. The remaining part of the proof is similar and hence is omitted.

Clearly, the characterization theorem given by Propositions 1 and 2 applies also to this setting. However, the main interest of this setting is given by the possibility theorems. It would be a worthwile future research direction to explore the interplay and the scope of these results.

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