Chapter 5
Spatial Simple Closed Chains

Subject of this chapter is the kinematics of 1-d.o.f. spatial simple closed chains with axial joints. The simple closed chain is explained in Fig. 4.1b. Axial joints are the cylindrical joint (C), the revolute joint (R) and the prismatic joint (P). Overconstrained mechanisms are excluded from consideration. Then it is known from Theorem 4.1 that a 1-d.o.f. spatial simple closed chain has seven joint variables. The variable of either one revolute joint or one prismatic joint is declared independent. The problem to be solved is to determine the six dependent variables in terms of the independent variable and of constant mechanism parameters. The solution to this problem is of great engineering importance because 1-d.o.f. spatial simple closed chains are basic elements of machine mechanisms.

A simple closed chain is specified by the ordered sequence of letters C, R and P of its joints. Neither cyclic permutation of the sequence of letters nor reversal of the sequence changes the mechanism. Thus, the sequences RCPRC, CPRCR and CRPCR represent one and the same mechanism whereas the sequence CCRRP represents a different mechanism with the same set of joints. The sequence of letters does not tell which joint variable is considered as independent variable.

Let n_{C}, n_{R} and n_{P} be the numbers of cylindrical, of revolute and of prismatic joints, respectively. Important characteristic numbers are the total number n_φ of angular variables, the total number n_{t} of translatory variables and the total number n of joints and of bodies. These numbers are

$$n_\varphi = n_{\mathrm{C}} + n_{\mathrm{R}}\ , \qquad n_{\mathrm{t}} = n_{\mathrm{C}} + n_{\mathrm{P}} = 7 - n_\varphi\ , \qquad n = n_{\mathrm{C}} + n_{\mathrm{R}} + n_{\mathrm{P}}\ . \quad (5.1)$$

The numbers n_{C}, n_{R} and n_{P} are subject to the constraint $n_\varphi + n_{\mathrm{t}} = 2n_{\mathrm{C}} + n_{\mathrm{R}} + n_{\mathrm{P}} = 7$ and also to the constraint $n_\varphi = n_{\mathrm{C}} + n_{\mathrm{R}} > 3$ because with three or fewer rotation axes no change of angular positions is possible. An equivalent formulation is $n_{\mathrm{t}} = n_{\mathrm{C}} + n_{\mathrm{P}} \le 3$. It expresses the fact that three is the maximum number of independent translations. Under these constraints

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Table 5.1 Numbers n_C, n_R, n_P, n_φ, n and associated mechanisms. Symmetrical forms printed boldface. N_φ and N_t are the numbers of configurations for a given value of the independent variable in a revolute joint and in a prismatic joint, respectively

	n_C	n_R	n_P	n_φ	n	mechanisms	N_φ	N_t
1	3	1	0	4	4	RCCC	2	–
2	2	2	1	4	5	RCPRC, CCPRR, **RCPCR**	2	4,8,8
3	1	3	2	4	6	RRRPPC, RRRPCP, RPRPCR, RPRCRP, RPRCPR	2	8
4	0	4	3	4	7	**RRPPPRR**, **RPRPRPR**, **PRRPRRP**, RRRPRPP	2	8
5	2	3	0	5	5	**RCRCR**, **CRRRC**	$4^{*)}$, 8	–
6	1	4	1	5	6	RRCRPR, RRCPRR, RRCRRP	8	16
7	0	5	2	5	7	**PRRRRRP**, **RPRRRPR**, **RRPRPRR**	8	16
8	1	5	0	6	6	5R-C	16	–
9	0	6	1	6	7	**6R-P**	16	16
10	0	7	0	7	7	**7R**	16	–

$^{*)}$ $N_\varphi = 4$ configurations exist if (i) the mechanism is R̲CRCR̲ and (ii) the independent angle is in one of the underscored revolutes. $N_\varphi = 8$ otherwise

altogether ten combinations of numbers n_C, n_R and n_P are possible. They are listed in Table 5.1 in the order of increasing numbers n_φ. For each combination the complete list of mechanisms with the respective combination is given. Whenever possible the letter sequence is shown in a form symmetrical to the central letter. Symmetrical letter sequences are printed boldface. The numbers N_φ and N_t in the last columns are results of the kinematics analysis to come. N_φ is the number of configurations which a mechanism has for a given value of the independent angular variable in a revolute joint, and N_t is the number of configurations which a mechanism has for a given value of the independent translatory variable in a prismatic joint. The planar four-bar mechanism is known to have two configurations for a given value of the input crank angle. The mechanisms in Table 5.1 have two or four or eight or sixteen configurations.

About the relationship between N_φ and N_t the following can be said without any analysis. Let the angular and the translatory variable in a cylindrical joint λ be called φ_λ and h_λ, respectively. If joint λ is a prismatic joint, φ_λ is a constant φ_λ^* and h_λ is variable. If joint λ is a revolute joint, φ_λ is variable and h_λ is a constant h_λ^*. Imagine now two mechanisms I and II with the only difference that joint λ is a prismatic joint in mechanism I and a revolute joint in mechanism II. Furthermore, it is assumed that the independent variables are chosen such that in both mechanisms $h_\lambda = h_\lambda^*$ and $\varphi_\lambda = \varphi_\lambda^*$. Then both mechanisms are identical in this position of their joint λ. Consequently, both mechanisms have the same sets of solutions for the six dependent variables. This means that N_t for mechanism I equals N_φ for mechanism II. This equality is shown in Table 5.1. By replacing in a mechanism of row 2, 3, 4, 6, 7 or 9 a prismatic joint by a revolute joint a mechanism of row 5, 6, 7, 8, 9 or 10, respectively, is produced. The number N_t for the former equals the number N_φ for the latter.

Before starting the kinematics analysis joint variables and mechanism parameters must be defined. This is the subject of the following Sect. 5.1. Section 5.2 is devoted to coordinate transformations of relevant vectors. In Sect. 5.3 on closure conditions basic equations are formulated for the kinematics analysis. The application of these equations to the mechanisms of Table 5.1 is demonstrated in Sect. 5.4.

5.1 Joint Variables. Denavit-Hartenberg Parameters

A single-loop mechanism with n bodies has n joints $(4 \leq n \leq 7)$. Bodies as well as joints are labeled from 1 to n in such a way that the joint axes i and $i+1$ are located on body i $(i = 1, \ldots, n$ cyclic). Figure 5.1 shows the bodies $i-1$, i and $i+1$ together with their joint axes. The most general case is assumed that the two joint axes of each body are skew. Then the two joint axes of each body i have a common normal which is fixed on the respective body i. The joint axis i is, in turn, the common normal of the thus defined common normals on bodies $i-1$ and i. On the joint axis i the dual unit line vector $\hat{\mathbf{n}}_i$ is defined, and on the common normal of the joint axes i and $i+1$, i.e., also fixed on body i, the dual unit line vector $\hat{\mathbf{a}}_i$ is defined $(i = 1, \ldots, n)$.

The unit line vector $\hat{\mathbf{n}}_{i+1}$ is produced from $\hat{\mathbf{n}}_i$ by a screw displacement with $\hat{\mathbf{a}}_i$ being the screw axis. As shown in Fig. 5.1 the rotation angle about the screw axis is called α_i and the translation along the screw axis is called ℓ_i. These two constants (positive or negative) are the only kinematical parameters of body i. Together they define the constant dual screw angle

$$\hat{\alpha}_i = \alpha_i + \varepsilon \ell_i \,. \tag{5.2}$$

In the same way the unit line vector $\hat{\mathbf{a}}_i$ is produced from $\hat{\mathbf{a}}_{i-1}$ by a screw displacement with the screw axis $\hat{\mathbf{n}}_i$ and with a dual screw angle

$$\hat{\varphi}_i = \varphi_i + \varepsilon h_i \,. \tag{5.3}$$

Figure 5.1 shows also φ_i and h_i. In a cylindrical joint φ_i and h_i are joint variables. In a revolute joint h_i is a constant parameter and only φ_i is variable. In a prismatic joint φ_i is a constant parameter and h_i is variable. The constant h_i in a revolute joint is referred to as *offset*. The $4n$ quantities α_i, ℓ_i, φ_i, h_i $(i = 1, \ldots, n)$ are the so-called *Denavit-Hartenberg parameters* of the mechanism (see Denavit/Hartenberg [3]). Seven among them are variables, and $4n - 7$ are constant system parameters. The number of con-

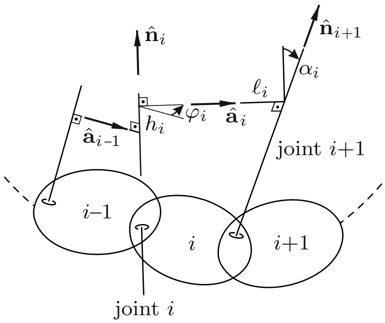

Fig. 5.1 Bodies $i{-}1$, i and $i{+}1$ with joint axes. Dual unit line vectors, body parameters and joint variables

stant system parameters ranges between nine for the mechanism RCCC and twenty-one for all mechanisms with $n = 7$.

The n line vectors $h_i\hat{\mathbf{n}}_i$ and the n line vectors $\ell_i\hat{\mathbf{a}}_i$ $(i = 1, \ldots, n)$ create a mobile spatial polygon with right angles at every corner. Analyzing the mechanism means analyzing this polygon.

The assumption that consecutive joint axes i and $i+1$ are skew can now be dropped. If they intersect, then $\ell_i = 0$, $\hat{\alpha}_i = \alpha_i$, and the unit line vector $\hat{\mathbf{a}}_i$ is along the common normal of the joint axes through their point of intersection. If the axes of three consecutive revolutes $i-1$, i and $i+1$ intersect at a single point, these axes are equivalent to a spherical joint S located at this point. Example: The mechanism RRSRR is a special case of the mechanism 7R .

If two consecutive joint axes i and $i+1$ are parallel, then $\alpha_i = 0$. The common normal to the joint axes can be placed such that either $h_i = 0$ or $h_{i+1} = 0$.

5.2 Screw Displacements. Coordinate Transformations

This section is devoted to screw displacements in the mobile spatial polygon spanned be the line vectors $h_i\hat{\mathbf{n}}_i$ and $\ell_i\hat{\mathbf{a}}_i$ $(i = 1, \ldots, n)$. The screw displacements relating $\hat{\mathbf{n}}_i$ to $\hat{\mathbf{n}}_{i+1}$ and $\hat{\mathbf{a}}_{i+1}$ to $\hat{\mathbf{a}}_i$ are described by the equations

$$\hat{\mathbf{n}}_{i+1} = \hat{\mathbf{n}}_i \cos\hat{\alpha}_i + \hat{\mathbf{a}}_i \times \hat{\mathbf{n}}_i \sin\hat{\alpha}_i \ , \qquad \hat{\mathbf{a}}_i = \hat{\mathbf{a}}_{i-1} \cos\hat{\varphi}_i + \hat{\mathbf{n}}_i \times \hat{\mathbf{a}}_{i-1} \sin\hat{\varphi}_i \ . \quad (5.4)$$

This is the dualized form of (1.125) which describes the rotation shown in Fig. 1.5 . For a more compact formulation the following abbreviations are introduced:

$$\hat{C}_i = \cos\hat{\alpha}_i\ , \qquad \hat{S}_i = \sin\hat{\alpha}_i\ , \qquad \hat{c}_i = \cos\hat{\varphi}_i\ , \qquad \hat{s}_i = \sin\hat{\varphi}_i \tag{5.5}$$

$(i = 1, \ldots, n$ cyclic) . The equations then read (in the second equation all indices are increased by one):

$$\hat{\mathbf{n}}_{i+1} = \hat{C}_i\hat{\mathbf{n}}_i + \hat{S}_i\hat{\mathbf{a}}_i \times \hat{\mathbf{n}}_i\ , \qquad \hat{\mathbf{a}}_{i+1} = \hat{c}_{i+1}\hat{\mathbf{a}}_i + \hat{s}_{i+1}\hat{\mathbf{n}}_{i+1} \times \hat{\mathbf{a}}_i \tag{5.6}$$

$(i = 1, \ldots, n$ cyclic) . On an *indeterminate* body k of the mechanism a dual cartesian basis is defined with unit basis vectors

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{n}}_k\ , \qquad \hat{\mathbf{e}}_2 = \hat{\mathbf{a}}_k\ , \qquad \hat{\mathbf{e}}_3 = \hat{\mathbf{n}}_k \times \hat{\mathbf{a}}_k\ . \tag{5.7}$$

Equations (5.6) constitute two-step recursion formulas for the coordinate decomposition of all vectors $\hat{\mathbf{n}}_{k+j}$ and $\hat{\mathbf{a}}_{k+j}$ $(j = 1, 2, \ldots)$ in this basis. From $\hat{\mathbf{a}}_k$ and $\hat{\mathbf{n}}_{k+1}$

$$\begin{aligned} \hat{\mathbf{a}}_{k+1} &= \hat{c}_{k+1}\hat{\mathbf{a}}_k + \hat{s}_{k+1}(\hat{C}_k\hat{\mathbf{n}}_k + \hat{S}_k\hat{\mathbf{a}}_k \times \hat{\mathbf{n}}_k) \times \hat{\mathbf{a}}_k \\ &= \hat{s}_{k+1}\hat{S}_k\hat{\mathbf{n}}_k + \hat{c}_{k+1}\hat{\mathbf{a}}_k + \hat{s}_{k+1}\hat{C}_k\hat{\mathbf{n}}_k \times \hat{\mathbf{a}}_k\ . \end{aligned} \tag{5.8}$$

From $\hat{\mathbf{a}}_{k+1}$ and $\hat{\mathbf{n}}_{k+1}$

$$\begin{aligned} \hat{\mathbf{n}}_{k+2} &= \hat{C}_{k+1}\hat{\mathbf{n}}_{k+1} + \hat{S}_{k+1}\hat{\mathbf{a}}_{k+1} \times \hat{\mathbf{n}}_{k+1} \\ &= (\hat{C}_{k+1}\hat{C}_k - \hat{S}_{k+1}\hat{S}_k\hat{c}_{k+1})\hat{\mathbf{n}}_k + \hat{S}_{k+1}\hat{s}_{k+1}\hat{\mathbf{a}}_k \\ &\quad -(\hat{C}_{k+1}\hat{S}_k + \hat{S}_{k+1}\hat{C}_k\hat{c}_{k+1})\hat{\mathbf{n}}_k \times \hat{\mathbf{a}}_k\ . \end{aligned} \tag{5.9}$$

In the next step $\hat{\mathbf{a}}_{k+2}$ and $\hat{\mathbf{n}}_{k+3}$ are calculated by increasing in (5.8) and (5.9) all indices by one and by substituting for $\hat{\mathbf{a}}_{k+1}$ and $\hat{\mathbf{n}}_{k+2}$ the previous formulas. Formulas for $\hat{\mathbf{n}}_{k+4}$ and $\hat{\mathbf{a}}_{k+4}$ are obtained with the least effort if in the expressions for $\hat{\mathbf{n}}_{k+2}$ and $\hat{\mathbf{a}}_{k+2}$ all indices are increased by two. Example:

$$\begin{aligned} \hat{\mathbf{n}}_{k+4} = &\quad (\hat{C}_{k+3}\hat{C}_{k+2} - \hat{S}_{k+3}\hat{S}_{k+2}\hat{c}_{k+3})\hat{\mathbf{n}}_{k+2} + \hat{S}_{k+3}\hat{s}_{k+3}\hat{\mathbf{a}}_{k+2} \\ &-(\hat{C}_{k+3}\hat{S}_{k+2} + \hat{S}_{k+3}\hat{C}_{k+2}\hat{c}_{k+3})\hat{\mathbf{n}}_{k+2} \times \hat{\mathbf{a}}_{k+2}\ . \end{aligned} \tag{5.10}$$

Into this expression the previously obtained expressions for $\hat{\mathbf{n}}_{k+2}$ and $\hat{\mathbf{a}}_{k+2}$ are substituted. Every expression thus obtained is linear with respect to every sine and to every cosine involved. Products sine × cosine of one and the same angle do not appear.

From the coordinates of the vectors $\hat{\mathbf{n}}_{k+j}$ and $\hat{\mathbf{a}}_{k+j}$ (j arbitrary) in the basis (5.7) the coordinates of $\hat{\mathbf{n}}_{k-j}$ and $\hat{\mathbf{a}}_{k-j}$ are obtained by a change of symbols. This is shown as follows. Inversion of the relationships (5.4) is achieved by reversing the sign of the dual angle:

$$\hat{\mathbf{n}}_i = \hat{\mathbf{n}}_{i+1}\cos\hat{\alpha}_i - \hat{\mathbf{a}}_i \times \hat{\mathbf{n}}_{i+1}\sin\hat{\alpha}_i\ , \qquad \hat{\mathbf{a}}_{i-1} = \hat{\mathbf{a}}_i\cos\hat{\varphi}_i - \hat{\mathbf{n}}_i \times \hat{\mathbf{a}}_i\sin\hat{\varphi}_i\ . \tag{5.11}$$

With (5.5) these are the equations (in the first equation all indices are decreased by one)

$$\hat{\mathbf{n}}_{i-1} = \hat{C}_{i-1}\hat{\mathbf{n}}_i - \hat{S}_{i-1}\hat{\mathbf{a}}_{i-1} \times \hat{\mathbf{n}}_i \ , \qquad \hat{\mathbf{a}}_{i-1} = \hat{c}_i\hat{\mathbf{a}}_i - \hat{s}_i\hat{\mathbf{n}}_i \times \hat{\mathbf{a}}_i \ . \tag{5.12}$$

Comparison with (5.6) reveals: For arbitrary j the coordinates of $\hat{\mathbf{n}}_{k+j}$ in the basis (5.7) become those of $\hat{\mathbf{a}}_{k-j}$ and vice versa if (i) $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are interchanged, (ii) j is replaced by $-j$ and (iii) $\hat{\alpha}$ and $\hat{\varphi}$ are interchanged.

Table 5.2 contains the coordinates of all vectors $\hat{\mathbf{n}}_{k\pm j}$ and $\hat{\mathbf{a}}_{k\pm j}$ (k arbitrary; $j = 0, 1, 2, 3$) and of $\hat{\mathbf{n}}_{k+4}$ and $\hat{\mathbf{a}}_{k-4}$. The table is used for calculating sums and products of vectors. It greatly simplifies the kinematics analysis of simple closed chains in this chapter. In Chap. 6 it is applied to overconstrained closed chains and in Chap. 7 to serial chains. When the symbol ˆ is deleted everywhere, the table gives the coordinates of first Plücker vectors $\mathbf{n}_{k\pm j}$ and $\mathbf{a}_{k\pm j}$.

Table 5.2 Vector coordinates in the dual basis with unit vectors $\hat{\mathbf{e}}_1 = \hat{\mathbf{n}}_k$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{a}}_k$, $\hat{\mathbf{e}}_3 = \hat{\mathbf{n}}_k \times \hat{\mathbf{a}}_k$. The three coordinates of each vector are separated by semicolons. $\hat{C}_i = \cos\hat{\alpha}_i$, $\hat{S}_i = \sin\hat{\alpha}_i$, $\hat{c}_i = \cos\hat{\varphi}_i$, $\hat{s}_i = \sin\hat{\varphi}_i$

$\hat{\mathbf{n}}_k$	$[1 \ ; \ 0 \ ; \ 0]$
$\hat{\mathbf{a}}_k$	$[0 \ ; \ 1 \ ; \ 0]$
$\hat{\mathbf{n}}_{k+1}$	$[\hat{C}_k \ ; \ 0 \ ; \ -\hat{S}_k]$
$\hat{\mathbf{a}}_{k-1}$	$[0 \ ; \ \hat{c}_k \ ; \ -\hat{s}_k]$
$\hat{\mathbf{a}}_{k+1}$	$[\hat{s}_{k+1}\hat{S}_k \ ; \ \hat{c}_{k+1} \ ; \ \hat{s}_{k+1}\hat{C}_k]$
$\hat{\mathbf{n}}_{k-1}$	$[\hat{C}_{k-1} \ ; \ \hat{S}_{k-1}\hat{s}_k \ ; \ \hat{S}_{k-1}\hat{c}_k]$
$\hat{\mathbf{n}}_{k+2}$	$[\hat{C}_{k+1}\hat{C}_k - \hat{S}_{k+1}\hat{S}_k\hat{c}_{k+1} \ ; \ \hat{S}_{k+1}\hat{s}_{k+1} \ ; \ -(\hat{C}_{k+1}\hat{S}_k + \hat{S}_{k+1}\hat{C}_k\hat{c}_{k+1})]$
$\hat{\mathbf{a}}_{k-2}$	$[\hat{s}_{k-1}\hat{S}_{k-1} \ ; \ \hat{c}_{k-1}\hat{c}_k - \hat{s}_{k-1}\hat{s}_k\hat{C}_{k-1} \ ; \ -(\hat{c}_{k-1}\hat{s}_k + \hat{s}_{k-1}\hat{c}_k\hat{C}_{k-1})]$
$\hat{\mathbf{a}}_{k+2}$	$[\hat{c}_{k+2}\hat{s}_{k+1}\hat{S}_k + \hat{s}_{k+2}(\hat{S}_{k+1}\hat{C}_k + \hat{C}_{k+1}\hat{S}_k\hat{c}_{k+1}) \ ; \ \hat{c}_{k+2}\hat{c}_{k+1} - \hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1} \ ;$ $\hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_k - \hat{s}_{k+2}(\hat{S}_{k+1}\hat{S}_k - \hat{C}_{k+1}\hat{C}_k\hat{c}_{k+1})]$
$\hat{\mathbf{n}}_{k-2}$	$[\hat{C}_{k-2}\hat{C}_{k-1} - \hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1} \ ; \ \hat{C}_{k-2}\hat{S}_{k-1}\hat{s}_k + \hat{S}_{k-2}(\hat{s}_{k-1}\hat{c}_k + \hat{c}_{k-1}\hat{s}_k\hat{C}_{k-1}) \ ;$ $\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_k - \hat{S}_{k-2}(\hat{s}_{k-1}\hat{s}_k - \hat{c}_{k-1}\hat{c}_k\hat{C}_{k-1})]$
$\hat{\mathbf{n}}_{k+3}$	$\Big[\ \hat{C}_{k+2}(\hat{C}_{k+1}\hat{C}_k - \hat{S}_{k+1}\hat{S}_k\hat{c}_{k+1}) + \hat{S}_{k+2}[\hat{s}_{k+2}\hat{s}_{k+1}\hat{S}_k$ $-\hat{c}_{k+2}(\hat{S}_{k+1}\hat{C}_k + \hat{C}_{k+1}\hat{S}_k\hat{c}_{k+1})] \ ;$ $\hat{C}_{k+2}\hat{S}_{k+1}\hat{s}_{k+1} + \hat{S}_{k+2}(\hat{s}_{k+2}\hat{c}_{k+1} + \hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1}) \ ;$ $-\hat{C}_{k+2}(\hat{C}_{k+1}\hat{S}_k + \hat{S}_{k+1}\hat{C}_k\hat{c}_{k+1}) + \hat{S}_{k+2}[\hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_k$ $+\hat{c}_{k+2}(\hat{S}_{k+1}\hat{S}_k - \hat{C}_{k+1}\hat{C}_k\hat{c}_{k+1})]\Big]$

Table 5.2 continued

$$
\begin{array}{ll}
\hat{\mathbf{a}}_{k-3} & \Big[\ \hat{c}_{k-2}\hat{s}_{k-1}\hat{S}_{k-1}+\hat{s}_{k-2}(\hat{S}_{k-2}\hat{C}_{k-1}+\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1}) \quad ; \\
 & \ \hat{c}_{k-2}(\hat{c}_{k-1}\hat{c}_{k}-\hat{s}_{k-1}\hat{s}_{k}\hat{C}_{k-1}) \\
 & +\hat{s}_{k-2}[\hat{S}_{k-2}\hat{S}_{k-1}\hat{s}_{k}-\hat{C}_{k-2}(\hat{s}_{k-1}\hat{c}_{k}+\hat{c}_{k-1}\hat{s}_{k}\hat{C}_{k-1})] \quad ; \\
 & -\hat{c}_{k-2}(\hat{c}_{k-1}\hat{s}_{k}+\hat{s}_{k-1}\hat{c}_{k}\hat{C}_{k-1}) \\
 & +\hat{s}_{k-2}[\hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k}+\hat{C}_{k-2}(\hat{s}_{k-1}\hat{s}_{k}-\hat{c}_{k-1}\hat{c}_{k}\hat{C}_{k-1})]\Big]
\end{array}
$$

$$
\begin{array}{ll}
\hat{\mathbf{a}}_{k+3} & \Big\{\ \hat{c}_{k+3}[\hat{c}_{k+2}\hat{s}_{k+1}\hat{S}_{k}+\hat{s}_{k+2}(\hat{S}_{k+1}\hat{C}_{k}+\hat{C}_{k+1}\hat{S}_{k}\hat{c}_{k+1})] \\
 & -\hat{s}_{k+3}\Big[\hat{C}_{k+2}[\hat{s}_{k+2}\hat{s}_{k+1}\hat{S}_{k}-\hat{c}_{k+2}(\hat{S}_{k+1}\hat{C}_{k}+\hat{C}_{k+1}\hat{S}_{k}\hat{c}_{k+1})] \\
 & -\hat{S}_{k+2}(\hat{C}_{k+1}\hat{C}_{k}-\hat{S}_{k+1}\hat{S}_{k}\hat{c}_{k+1})\Big] \ ; \\
 & \ \hat{s}_{k+3}[\hat{S}_{k+2}\hat{S}_{k+1}\hat{s}_{k+1}-\hat{C}_{k+2}(\hat{s}_{k+2}\hat{c}_{k+1}+\hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1})] \\
 & +\hat{c}_{k+3}(\hat{c}_{k+2}\hat{c}_{k+1}-\hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1}) \quad ; \quad \hat{c}_{k+3}[\hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_{k} \\
 & -\hat{s}_{k+2}(\hat{S}_{k+1}\hat{S}_{k}-\hat{C}_{k+1}\hat{C}_{k}\hat{c}_{k+1})]-\hat{s}_{k+3}\Big[\hat{C}_{k+2}[\hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_{k} \\
 & +\hat{c}_{k+2}(\hat{S}_{k+1}\hat{S}_{k}-\hat{C}_{k+1}\hat{C}_{k}\hat{c}_{k+1})]+\hat{S}_{k+2}(\hat{C}_{k+1}\hat{S}_{k}+\hat{S}_{k+1}\hat{C}_{k}\hat{c}_{k+1})\Big]\Big\}
\end{array}
$$

$$
\begin{array}{ll}
\hat{\mathbf{n}}_{k-3} & \Big\{\ \hat{S}_{k-3}[\hat{s}_{k-2}\hat{s}_{k-1}\hat{S}_{k-1}-\hat{c}_{k-2}(\hat{S}_{k-2}\hat{C}_{k-1}+\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1})] \\
 & +\hat{C}_{k-3}(\hat{C}_{k-2}\hat{C}_{k-1}-\hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1}) \quad ; \quad \hat{C}_{k-3}[\hat{C}_{k-2}\hat{S}_{k-1}\hat{s}_{k} \\
 & +\hat{S}_{k-2}(\hat{s}_{k-1}\hat{c}_{k}+\hat{c}_{k-1}\hat{s}_{k}\hat{C}_{k-1})]-\hat{S}_{k-3}\Big[\hat{c}_{k-2}[\hat{S}_{k-2}\hat{S}_{k-1}\hat{s}_{k} \\
 & -\hat{C}_{k-2}(\hat{s}_{k-1}\hat{c}_{k}+\hat{c}_{k-1}\hat{s}_{k}\hat{C}_{k-1})]-\hat{s}_{k-2}(\hat{c}_{k-1}\hat{c}_{k}-\hat{s}_{k-1}\hat{s}_{k}\hat{C}_{k-1})\Big] \ ; \\
 & \ \hat{C}_{k-3}[\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_{k}-\hat{S}_{k-2}(\hat{s}_{k-1}\hat{s}_{k}-\hat{c}_{k-1}\hat{c}_{k}\hat{C}_{k-1})] \\
 & -\hat{S}_{k-3}\Big[\hat{c}_{k-2}[\hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k}+\hat{C}_{k-2}(\hat{s}_{k-1}\hat{s}_{k}-\hat{c}_{k-1}\hat{c}_{k}\hat{C}_{k-1})] \\
 & +\hat{s}_{k-2}(\hat{c}_{k-1}\hat{s}_{k}+\hat{s}_{k-1}\hat{c}_{k}\hat{C}_{k-1})\Big]\Big\}
\end{array}
$$

$$
\begin{array}{ll}
\hat{\mathbf{n}}_{k+4} & \Big\{(\hat{C}_{k+3}\hat{C}_{k+2}-\hat{S}_{k+3}\hat{S}_{k+2}\hat{c}_{k+3})(\hat{C}_{k+1}\hat{C}_{k}-\hat{S}_{k+1}\hat{S}_{k}\hat{c}_{k+1}) \\
 & +\hat{S}_{k+3}\hat{s}_{k+3}[\hat{c}_{k+2}\hat{s}_{k+1}\hat{S}_{k}+\hat{s}_{k+2}(\hat{S}_{k+1}\hat{C}_{k}+\hat{C}_{k+1}\hat{S}_{k}\hat{c}_{k+1})] \\
 & +(\hat{C}_{k+3}\hat{S}_{k+2}+\hat{S}_{k+3}\hat{C}_{k+2}\hat{c}_{k+3})[\hat{s}_{k+2}\hat{s}_{k+1}\hat{S}_{k} \\
 & -\hat{c}_{k+2}(\hat{S}_{k+1}\hat{C}_{k}+\hat{C}_{k+1}\hat{S}_{k}\hat{c}_{k+1})] \quad ; \\
 & \ (\hat{C}_{k+3}\hat{C}_{k+2}-\hat{S}_{k+3}\hat{S}_{k+2}\hat{c}_{k+3})\hat{S}_{k+1}\hat{s}_{k+1} \\
 & +\hat{S}_{k+3}\hat{s}_{k+3}(\hat{c}_{k+2}\hat{c}_{k+1}-\hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1}) \\
 & +(\hat{C}_{k+3}\hat{S}_{k+2}+\hat{S}_{k+3}\hat{C}_{k+2}\hat{c}_{k+3})(\hat{s}_{k+2}\hat{c}_{k+1}+\hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_{k+1}) \quad ; \\
 & -(\hat{C}_{k+3}\hat{C}_{k+2}-\hat{S}_{k+3}\hat{S}_{k+2}\hat{c}_{k+3})(\hat{C}_{k+1}\hat{S}_{k}+\hat{S}_{k+1}\hat{C}_{k}\hat{c}_{k+1}) \\
 & +\hat{S}_{k+3}\hat{s}_{k+3}[\hat{c}_{k+2}\hat{s}_{k+1}\hat{C}_{k}-\hat{s}_{k+2}(\hat{S}_{k+1}\hat{S}_{k}-\hat{C}_{k+1}\hat{C}_{k}\hat{c}_{k+1})] \\
 & +(\hat{C}_{k+3}\hat{S}_{k+2}+\hat{S}_{k+3}\hat{C}_{k+2}\hat{c}_{k+3})[\hat{s}_{k+2}\hat{s}_{k+1}\hat{C}_{k} \\
 & +\hat{c}_{k+2}(\hat{S}_{k+1}\hat{S}_{k}-\hat{C}_{k+1}\hat{C}_{k}\hat{c}_{k+1})]\Big\}
\end{array}
$$

$$
\begin{array}{ll}
\hat{\mathbf{a}}_{k-4} & \Big\{\ (\hat{c}_{k-3}\hat{c}_{k-2}-\hat{s}_{k-3}\hat{s}_{k-2}\hat{C}_{k-3})\hat{s}_{k-1}\hat{S}_{k-1} \\
 & +\hat{s}_{k-3}\hat{S}_{k-3}(\hat{C}_{k-2}\hat{C}_{k-1}-\hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1}) \\
 & +(\hat{c}_{k-3}\hat{s}_{k-2}+\hat{s}_{k-3}\hat{c}_{k-2}\hat{C}_{k-3})(\hat{S}_{k-2}\hat{C}_{k-1}+\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_{k-1}) \quad ; \\
 & \ (\hat{c}_{k-3}\hat{c}_{k-2}-\hat{s}_{k-3}\hat{s}_{k-2}\hat{C}_{k-3})(\hat{c}_{k-1}\hat{c}_{k}-\hat{s}_{k-1}\hat{s}_{k}\hat{C}_{k-1}) \\
 & +\hat{s}_{k-3}\hat{S}_{k-3}[\hat{C}_{k-2}\hat{S}_{k-1}\hat{s}_{k}+\hat{S}_{k-2}(\hat{s}_{k-1}\hat{c}_{k}+\hat{c}_{k-1}\hat{s}_{k}\hat{C}_{k-1})] \\
 & +(\hat{c}_{k-3}\hat{s}_{k-2}+\hat{s}_{k-3}\hat{c}_{k-2}\hat{C}_{k-3})[\hat{S}_{k-2}\hat{S}_{k-1}\hat{s}_{k} \\
 & -\hat{C}_{k-2}(\hat{s}_{k-1}\hat{c}_{k}+\hat{c}_{k-1}\hat{s}_{k}\hat{C}_{k-1})] \quad ; \\
 & -(\hat{c}_{k-3}\hat{c}_{k-2}-\hat{s}_{k-3}\hat{s}_{k-2}\hat{C}_{k-3})(\hat{c}_{k-1}\hat{s}_{k}+\hat{s}_{k-1}\hat{c}_{k}\hat{C}_{k-1}) \\
 & +\hat{s}_{k-3}\hat{S}_{k-3}[\hat{C}_{k-2}\hat{S}_{k-1}\hat{c}_{k}-\hat{S}_{k-2}(\hat{s}_{k-1}\hat{s}_{k}-\hat{c}_{k-1}\hat{c}_{k}\hat{C}_{k-1})] \\
 & +(\hat{c}_{k-3}\hat{s}_{k-2}+\hat{s}_{k-3}\hat{c}_{k-2}\hat{C}_{k-3})[\hat{S}_{k-2}\hat{S}_{k-1}\hat{c}_{k} \\
 & +\hat{C}_{k-2}(\hat{s}_{k-1}\hat{s}_{k}-\hat{c}_{k-1}\hat{c}_{k}\hat{C}_{k-1})]\Big\}
\end{array}
$$

5.3 Closure Conditions

Having defined variables and parameters and having established basic relationships in the mobile spatial polygon of vectors $h_i\hat{\mathbf{n}}_i + \ell_i\hat{\mathbf{a}}_i$ $(i = 1, \ldots, n)$ we can finally turn to the kinematics problem of this chapter: Determine six dependent variables of a single-loop mechanism in terms of a single independent variable. For this purpose six scalar equations relating the altogether seven variables are required. Any such equation is called closure condition because it expresses the closure of the kinematical loop. Closure conditions are highly nonlinear. Therefore, it is not useful to formulate six fully coupled equations. It is essential to formulate a set of $m < 6$ equations for m unknowns with m being as small as possible. Once these equations are solved for the m unknowns it is simple to express the remaining $6 - m$ unknowns one by one in terms of previously determined unknowns. In the literature various methods for formulating closure conditions are found:
Yang [40] – [42], Duffy [4] – [8], Duffy/Crane [9], Yuan [44], Yuan/Freudenstein/Woo [45], [46], Dukkipati/Soni [13], Dukkipati [14], Soni/Pamidi [35], Soni [36], Lee [20, 27], Lee/Liang [21] – [24], Liang/Lee/Liao [28], Woernle [39], Lee/Woernle/Hiller [26], Raghavan/Roth [32], Lee/Roth [25], Nielsen/Roth [30], Crane/Duffy [2] and others. Many more references are found in [15]. An historical overview is found in Peisach [31].

In what follows, methods developed by Woernle [39] and Lee [27] are used. They lead to a minimal set of either $m = 1$ or $m = 2$ or $m = 4$ coupled equations depending on the type of mechanism. The formulations presented appeared in Wittenburg [47].

5.3.1 Woernle-Lee Equations

Figure 5.2 shows schematically the polygon of dual-vectors of a single-loop mechanism. Actually, only the vectors associated with two specific joints labeled a and b are shown. The rest of the polygon is indicated by dashed lines. How to choose the joints a and b is explained later. These joints divide the mechanism into a *left segment* and a *right segment*. The ordinary vector $\mathbf{r}$ shown in the figure joins the axes a and b. This vector has two representations. One as a sum of vectors fixed on bodies of the left segment and one as a sum of vectors fixed on bodies of the right segment. The vectors are the primary parts of dual vectors. The two representations are

$$\mathbf{r} = \begin{cases} -(h_b\mathbf{n}_b + \ell_b\mathbf{a}_b + h_{b+1}\mathbf{n}_{b+1} + \ldots + \ell_{a-1}\mathbf{a}_{a-1}) & \text{(left segment)} \\ \quad h_a\mathbf{n}_a + \ell_a\mathbf{a}_a + h_{a+1}\mathbf{n}_{a+1} + \ldots + \ell_{b-1}\mathbf{a}_{b-1} & \text{(right segment)}\ . \end{cases} \tag{5.13}$$

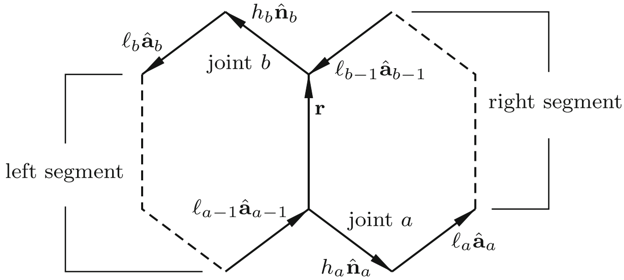

Fig. 5.2 Segments of a mechanism defined by two joints a and b

In the particular case $b = a+1$ the joints a and b are direct neighbors. In this case, the vector $\mathbf{r}$ in the right segment is $\mathbf{r} = h_a\mathbf{n}_a + \ell_a\mathbf{a}_a$. If the axes a and $a+1$ are nonparallel, then $\mathbf{a}_a = (1/S_a)(\mathbf{n}_a \times \mathbf{n}_{a+1})$ (see Fig. 5.2). In this case, (5.13) has the special form

$$\mathbf{r} = \begin{cases} -(h_{a+1}\mathbf{n}_{a+1} + \ell_{a+1}\mathbf{a}_{a+1} + \ldots + \ell_{a-1}\mathbf{a}_{a-1}) & \text{(left segment)} \\ h_a\mathbf{n}_a + \dfrac{\ell_a}{S_a}(\mathbf{n}_a \times \mathbf{n}_{a+1}) & \text{(right segment)}\ . \end{cases} \tag{5.14}$$

In terms of $\mathbf{n}_a$, $\mathbf{n}_b$ and $\mathbf{r}$ seven scalar quantities $F_1, \ldots, F_7$ are defined as follows:

$$\left.\begin{array}{ll} F_1 = \mathbf{n}_a \cdot \mathbf{n}_b\ , & F_2 = \mathbf{r} \cdot \mathbf{n}_a \times \mathbf{n}_b\ , \\ F_3 = \mathbf{n}_a \cdot \mathbf{r}\ , & F_4 = \mathbf{n}_b \cdot \mathbf{r}\ , \\ F_5 = \mathbf{r} \cdot \mathbf{n}_p \times \mathbf{n}_q & (\mathbf{r}\ \text{from (5.14)};\ p \neq q\ \text{arbitrary})\ , \\ F_6 = \mathbf{r}^2\ , & F_7 = \frac{1}{2}(\mathbf{n}_a \cdot \mathbf{n}_b)\mathbf{r}^2 - (\mathbf{n}_a \cdot \mathbf{r})(\mathbf{n}_b \cdot \mathbf{r}) \\ & \quad\ = \frac{1}{2}F_1F_6 - F_3F_4\ . \end{array}\right\} \tag{5.15}$$

The scalars F_1 and F_2 represent the primary part and the dual part, respectively, of the dual scalar product $\hat{\mathbf{n}}_a \cdot \hat{\mathbf{n}}_b$. The primary part is calculated with coordinates from Table 5.2 (without the symbol $\hat{}$). The dual part is calculated not as product $\mathbf{r} \cdot \mathbf{n}_a \times \mathbf{n}_b$, but as dual derivative of F_1 (see (3.50)). The MAPLE software tool developed by Sinigersky [34] has special routines for switching back and forth between the ordinary notation $\cos\alpha$, $\sin\alpha$, $\cos\varphi$, $\sin\varphi$ and the short-hand notation C, S, c, s. Dual differentiation is carried out automatically. By combining Table 5.2 with this tool kinematics equations for mechanisms can be formulated semi-automatically.

Each of the scalar quantities $F_1, \ldots, F_7$ is expressed in the left segment as function of variables and in the right segment as another function of variables. These functions are called F_k^ℓ and F_k^r $(k = 1, \ldots, 7)$. The equality of both scalars establishes the seven Woernle-Lee equations

$$F_k^\ell = F_k^r \qquad (k = 1, \ldots, 7)\ . \tag{5.16}$$

These equations have properties which make them useful as closure conditions. The most important properties are the following:
- the parameters φ_a and φ_b do not appear explicitly in any of the equations $F_k^\ell = F_k^r \ \ (k = 1, \ldots, 7)$. This is a consequence of the fact that each of the vectors $\mathbf{n}_a$ and $\mathbf{n}_b$ is fixed on a body of the left segment and also fixed on a body of the right segment
- the parameters h_a and h_b do not appear explicitly in the equation $F_2^\ell = F_2^r$. This is a consequence of the fact that this equation is the dual derivative of the equation $F_1^\ell = F_1^r$.

From these properties the following criteria for choosing the joints a and b are derived. The joint combination ab is
- CC in mechanisms with two or three cylindrical joints (rows 1, 2, 5 of Table 5.1)
- CR in mechanisms with one cylindrical joint (rows 3, 6, 8)
- RR in all other mechanisms
- the angles φ_a and φ_b must be dependent variables.

This choice of joints has the consequence that the maximum possible number of dependent variables is eliminated from the equations. The properties of the equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ can now be stated in more detail as follows. The equation $F_1^\ell = F_1^r$ is an equation in terms of *angular* Denavit-Hartenberg parameters only. Every sine and every cosine appears in linear form only. Products sine × cosine of one and the same angle do not appear.

The independent variable is either the angle in a specific revolute joint or the translatory variable in a specific prismatic joint. Until further below *it is assumed to be an angle*. The mechanisms in Table 5.1 have a total number n_φ of angular variables in the range $4 \le n_\varphi \le 7$. With one of them being independent and with φ_a and φ_b being eliminated the equation $F_1^\ell = F_1^r$ is an equation for $1 \le n_\varphi - 3 \le 4$ unknown angular variables.

Unknowns in the equation $F_2^\ell = F_2^r$ are the same $n_\varphi - 3$ angular variables and, in addition, ν translatory variables. The number ν is the difference between the total number $n_\mathrm{t} = n_\mathrm{C} + n_\mathrm{P}$ of translatory variables and the number of cylindrical joints among the joints a and b. This formula yields $\nu = 1$ for the mechanism RCCC and $\nu = n_\mathrm{P}$ for all other mechanisms. Table 5.1 shows that ν is in the range $0 \le \nu \le 3$. The translatory variables and the sines and cosines of angular variables appear in linear form only. The mechanisms in rows 5, 8 and 10 of Table 5.1 have angular variables only $(\nu = 0)$. For these mechanisms the equation $F_2^\ell = F_2^r$ has, with other coefficients, the same form as the equation $F_1^\ell = F_1^r$.

Let b be the larger of the indices a and b so that $1 \le a < b \le n$ and $b = a + d$ with $d > 0$. Because of the cyclic repetition of the indices $1, \ldots, n$ the identity holds: $b \equiv b - n = a + d - n$. With this identity the equation $F_1^\ell = F_1^r$ is written in the form

$$\mathbf{n}_k \cdot \mathbf{n}_{k+d} = \mathbf{n}_k \cdot \mathbf{n}_{k+d-n} \qquad (k = a) \,. \tag{5.17}$$

Example: In the case $a = 3$, $b = 6$, $n = 7$, the equation is $\mathbf{n}_k \cdot \mathbf{n}_{k+3} = \mathbf{n}_k \cdot \mathbf{n}_{k-4}$ with $k = 3$. The product $\mathbf{n}_k \cdot \mathbf{n}_{k+3}$ is found in Table 5.2 as first coordinate of the vector $\mathbf{n}_{k+3}$ with $k = 3$. It is a function of φ_4 and φ_5. For the product $\mathbf{n}_k \cdot \mathbf{n}_{k-4}$ the vector $\mathbf{n}_{k-4}$ is not found in the table. The table suffices, however, because the product has the alternative form $\mathbf{n}_{k-2} \cdot \mathbf{n}_{k+2}$ with $k = 1$. It is a function of $\varphi_{k-1} = \varphi_0 \equiv \varphi_7$, $\varphi_k = \varphi_1$ and $\varphi_{k+1} = \varphi_2$. For details see the right-hand side of (5.82). End of example.

Mechanisms with $n_\varphi = 4$: Let φ_1 and φ_λ be the independent and the dependent angular variable, respectively. The equation $F_1^\ell = F_1^r$ is

$$Ac_\lambda + Bs_\lambda = R\,. \tag{5.18}$$

The coefficients A, B and R are functions of φ_1. For every value of φ_1 the equation has two (not necessarily real) solutions φ_λ. This is an important result. It means that the mechanisms in rows 1 to 4 of Table 5.1 have two different configurations for every value of the independent angular variable. This is the number N_φ shown in Table 5.1.

Dual differentiation of (5.18) produces the equation $F_2^\ell = F_2^r$:

$$(A' + h_\lambda B)c_\lambda + (B' - h_\lambda A)s_\lambda = R'\,, \tag{5.19}$$

where A', B', R' are the dual derivatives of A, of B and of R, respectively. Since φ_λ is known from (5.18), this equation is a linear equation for $\nu = 1$ or 2 or 3 translatory variables. The mechanisms in rows 1 and 2 of Table 5.1 with $\nu = 1$ are the simplest mechanisms. Equation (5.19) determines the single translatory variable. For the mechanism RCCC this is shown in detail in Sect. 5.4.1.

Mechanisms with $n_\varphi = 5$: These are the mechanisms in rows 5, 6 and 7 of Table 5.1. Let φ_λ and φ_μ be the two dependent angular variables. The equation $F_1^\ell = F_1^r$ has either the form

$$A_2 c_\lambda + B_2 s_\lambda = A_1 c_\mu + B_1 s_\mu + R_1 \tag{5.20}$$

or the form

$$\underline{A}\,[\,c_\lambda c_\mu \quad c_\lambda s_\mu \quad c_\lambda \quad s_\lambda c_\mu \quad s_\lambda s_\mu \quad s_\lambda \quad c_\mu \quad s_\mu \quad 1\,]^T = 0\,. \tag{5.21}$$

The coefficients A_1, B_1, R_1, A_2, B_2 and the row matrix $\underline{A}$ are functions of the independent variable. Equation (5.20) occurs if the joints λ and μ belong to different segments created by the cylindrical joints, and (5.21) occurs if they belong to one and the same segment.

Unknowns in the equation $F_2^\ell = F_2^r$ are the same angles φ_λ and φ_μ and, in addition, $\nu = 0$ or 1 or 2 translatory variables. For the mechanisms RCRCR and CRRRC with $\nu = 0$ the equations $F_2^\ell = F_2^r$ and $F_1^\ell = F_1^r$

have identical forms, i.e., either the form (5.20) or the form (5.21). Both equations together determine φ_λ and φ_μ. Details see in Sect. 5.4.2.

The mechanisms in rows 6 and 7 of Table 5.1: It will be seen that the equation $F_5^\ell = F_5^r$ has the form (5.21) for the same unknowns φ_λ and φ_μ. Thus, all mechanisms with $n_\varphi = 5$ are governed either by two Eqs.(5.20) or by two Eqs.(5.21). Once the two unknowns φ_λ and φ_μ are determined the equation $F_2^\ell = F_2^r$ represents a linear equation for $\nu = 1$ or $\nu = 2$ unknown translatory variables.

Mechanisms with $n_\varphi = 6$: The only mechanisms of this type are the mechanisms 5R-C and 6R-P. The joints a and b (a cylindrical and a revolute joint or two revolute joints) are chosen such that two dependent angular variables are in one segment and the third dependent angular variable together with the independent variable is in the other. Let φ_λ, φ_μ and φ_ν be the dependent variables. The matrix form of the equation $F_1^\ell = F_1^r$ is

$$\underline{A}\,[\,c_\nu \quad s_\nu\,]^T = \underline{B}\,[\,c_\lambda c_\mu \quad c_\lambda s_\mu \quad c_\lambda \quad s_\lambda c_\mu \quad s_\lambda s_\mu \quad s_\lambda \quad c_\mu \quad s_\mu \quad 1\,]^T\,. \tag{5.22}$$

The row matrices $\underline{A}$ and $\underline{B}$ are either constant or functions of the independent variable. The equation $F_2^\ell = F_2^r$ for the mechanism 5R-C has the same form with other coefficient matrices $\underline{A}$ and $\underline{B}$.

In the equation $F_2^\ell = F_2^r$ for a mechanism 6R-P the unknown translatory variable of the prismatic joint appears in addition to the three unknown angular variables. If this translatory variable is called h_κ, the equation has the form

$$\begin{aligned}&\underline{A}\,[\,c_\nu \quad s_\nu \quad h_\kappa c_\nu \quad h_\kappa s_\nu \quad h_\kappa\,]^T\\ &= \underline{B}\,[\,c_\lambda c_\mu \quad c_\lambda s_\mu \quad c_\lambda \quad s_\lambda c_\mu \quad s_\lambda s_\mu \quad s_\lambda \quad c_\mu \quad s_\mu \quad 1\,]^T\,.\end{aligned} \tag{5.23}$$

Mechanisms with $n_\varphi = 7$: The only mechanism of this type is the mechanism 7R. The joints a and b are chosen such that two dependent angular variables are in each segment. Let φ_7 be the independent variable. Then $a = 3$, $b = 6$ is a possible choice. The four dependent variables are φ_1, φ_2, φ_4 and φ_5. The equation $F_1^\ell = F_1^r$ is known already from the example given for Eq.(5.17). Its matrix form is

$$\begin{aligned}&\underline{A}\,[\,c_4c_5 \quad c_4s_5 \quad c_4 \quad s_4c_5 \quad s_4s_5 \quad s_4 \quad c_5 \quad s_5\,]^T\\ &= \underline{B}\,[\,c_1c_2 \quad c_1s_2 \quad c_1 \quad s_1c_2 \quad s_1s_2 \quad s_1 \quad c_2 \quad s_2 \quad 1\,]^T\,.\end{aligned} \tag{5.24}$$

The row matrix $\underline{A}$ is constant, and $\underline{B}$ is a function of φ_7. The equation $F_2^\ell = F_2^r$ has the same form with other coefficient matrices $\underline{A}$ and $\underline{B}$. Details see in Sect. 5.4.7.

The equations $F_3^\ell = F_3^r$ *and* $F_4^\ell = F_4^r$

Unknowns in these equations are the same $1 \le n_\varphi - 3 \le 4$ angular variables which appear in the equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ and, in addition, all translatory variables including h_a and h_b (constant parameter or variable). In the equations for the mechanisms in rows 3, 4, 8 and 9 the total number of variables is four, so that these two equations together with the equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ represent a system of four equations for four unknowns.

The mechanisms in rows 3 and 4: The equation $F_1^\ell = F_1^r$ is Eq.(5.18). With each of its two solutions the other three equations are coupled linear equations for three unknown translatory variables.

The mechanisms 5R-C and 6R-P: Unknowns are three angular and one translatory variable. The equations $F_3^\ell = F_3^r$ and $F_4^\ell = F_4^r$ have the form (5.23), and the equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ have the form (5.22).

The mechanism 7R: The four unknowns are angular variables. The four equations $F_k^\ell = F_k^r$ $(k = 1, 2, 3, 4)$ have the form (5.24).

The equation $F_5^\ell = F_5^r$

This equation is formulated only for the mechanisms in rows 6 and 7 of Table 5.1. These mechanisms have $n_\varphi = 5$ angular variables and either $n = 6$ joints with one cylindrical and one prismatic joint or $n = 7$ joints with two prismatic joints. These joints are the only ones with translatory variables. They are chosen as joints p and q in the expression $F_5 = \mathbf{r} \cdot \mathbf{n}_p \times \mathbf{n}_q$. With (5.14) for $\mathbf{r}$ the equation $F_5^\ell = F_5^r$ is

$$\begin{aligned} &- (h_{a+1}\mathbf{n}_{a+1} + \ell_{a+1}\mathbf{a}_{a+1} + \ldots + \ell_{a-1}\mathbf{a}_{a-1} + h_a\mathbf{n}_a) \cdot (\mathbf{n}_p \times \mathbf{n}_q) \\ &= \frac{\ell_a}{S_a} (\mathbf{n}_a \times \mathbf{n}_{a+1}) \cdot (\mathbf{n}_p \times \mathbf{n}_q) \\ &= \frac{\ell_a}{S_a} [(\mathbf{n}_a \times \mathbf{n}_p) \cdot (\mathbf{n}_{a+1} \times \mathbf{n}_q) - (\mathbf{n}_a \times \mathbf{n}_q) \cdot (\mathbf{n}_{a+1} \times \mathbf{n}_p)] \,. \end{aligned} \tag{5.25}$$

The vectors $h_p\mathbf{n}_p$ and $h_q\mathbf{n}_q$ are among the vectors indicated by dots. Multiplication with $(\mathbf{n}_p \times \mathbf{n}_q)$ eliminates the variables h_p and h_q. The angular variables in the joints a and $a+1$ are eliminated as well. This means that only two dependent angular variables appear explicitly. Let them be denoted φ_λ and φ_μ. Then the equation has the form

$$\underline{A} [\, c_\lambda c_\mu \quad c_\lambda s_\mu \quad c_\lambda \quad s_\lambda c_\mu \quad s_\lambda s_\mu \quad s_\lambda \quad c_\mu \quad s_\mu \quad 1 \,]^T = 0 \,. \tag{5.26}$$

This is, with a different coefficient matrix $\underline{A}$, the form of the equation $F_1^\ell = F_1^r$ (see (5.21)). In both equations the unknowns φ_λ and φ_μ are the same if the joints a and $b = a + 1$ are the same.

It remains to be shown how to generate the coefficient matrix $\underline{A}$ in (5.26) with the help of Table 5.2. First, the left-hand side of (5.25) is considered. It is a linear combination of products $\mathbf{n}_\ell \cdot \mathbf{n}_p \times \mathbf{n}_q$ ($\ell \neq p, q$ arbitrary) and $\mathbf{a}_\ell \cdot \mathbf{n}_p \times \mathbf{n}_q$ ($\ell \neq a$ arbitrary). Every product is evaluated as (3×3)-determinant of vector coordinates copied from Table 5.2. The goal is to formulate the determinant such that the angles φ_a and φ_{a+1} do not appear explicitly, and that s_λ, c_λ, s_μ, c_μ appear in linear form only. This goal is achieved if the determinant is evaluated in three steps as follows.

Step 1: To those of the indices ℓ, p and q which are smaller than or equal to a the number n is added ($n = 6$ or $n = 7$). The new indices are called ℓ', p' and q'.

Step 2: The new indices ℓ', p' and q' are brought into a monotonically increasing order. One of them is the central index. To this index the name k is given. The other two indices are $k - d_1$ and $k + d_2$ with $d_1, d_2 \geq 0$.

Step 3: The rows of the (3×3)-determinant are the coordinates of the vectors with indices k, $k-d_1$ and $k+d_2$. These coordinates are copied from Table 5.2. The vector with the index k is either $\mathbf{n}_k$ with the coordinates $[1 \quad 0 \quad 0]$ or $\mathbf{a}_k$ with the coordinates $[0 \quad 1 \quad 0]$. With either one of these forms the determinant is reduced to a (2×2)-determinant. The coordinates of the vectors with indices $k - d_1$ and $k + d_2$ are linear with respect to c_λ, s_λ, c_μ and s_μ. The variables φ_a and φ_{a+1} do not appear.

Example: For a mechanism with $n = 7$ joints the product $\mathbf{a}_1 \cdot \mathbf{n}_5 \times \mathbf{n}_6$ is to be expressed such that the variables φ_a and φ_{a+1} with $a = 4$ do not appear explicitly.

Solution: The given indices are $\ell = 1 < a$, $p = 5 > a$ and $q = 6 > a$. Hence $\ell' = \ell + n = 8$, $p' = 5$, $q' = 6$. The desired form of the product is

$$\underbrace{\mathbf{a}_{k+2} \cdot \mathbf{n}_{k-1} \times \mathbf{n}_k}_{k=6}$$
$$= S_5 c_6 (c_1 c_7 - s_1 s_7 C_7) - S_5 s_6 [c_1 s_7 C_6 - s_1 (S_7 S_6 - C_7 C_6 c_7)] \,. \quad (5.27)$$

End of example.

For the left-hand side expression of (5.25) considered so far the desired form (free of φ_a and φ_{a+1} and linear in the sines and cosines of the remaining three angles) is possible no matter which angle is chosen as φ_a. For the right-hand side expression this choice is not arbitrary. For both sides the angle dictated by the right-hand side is chosen. The rule for choosing joint a is explained, first, for the mechanisms in row 6 with one prismatic joint and one cylindrical joint. Let the cylindrical joint be joint q. One of the joints a and $a+1$ must be the cylindrical joint q, and the other must be a revolute joint with a dependent angular variable. This allocation is possible no matter in which revolute joint the independent variable is located. In the case $a = q$, the right-hand side of (5.25) is

$$\frac{\ell_a}{S_a} \left(C_a \mathbf{n}_p \cdot \mathbf{n}_q - \mathbf{n}_p \cdot \mathbf{n}_{a+1} \right) . \quad (5.28)$$

In the case $a + 1 = q$, the right-hand side of (5.25) is

$$-\frac{\ell_a}{S_a}\left(C_a\mathbf{n}_p\cdot\mathbf{n}_q-\mathbf{n}_a\cdot\mathbf{n}_p\right). \tag{5.29}$$

In either case Table 5.2 yields an expression with the desired linearity properties.

Next, the mechanisms with two prismatic joints p and q are considered. The joints a and $a+1$ must be revolute joints with dependent angular variables, and one of them must be neighbor of a prismatic joint (p or q). This allocation is possible no matter in which revolute joint the independent variable is located. Since each of the two prismatic joints can be joint p, it suffices to distinguish whether joint a or joint $a+1$ is neighbor of joint p. If joint a is neighbor of joint p, then $a=p+1$ and $\mathbf{n}_a\times\mathbf{n}_p=-S_p\mathbf{a}_p$ and $\mathbf{n}_a\cdot\mathbf{n}_p=C_p$. The right-hand side of (5.25) is

$$\frac{\ell_a}{S_a}\left(-S_p\mathbf{a}_p\cdot\mathbf{n}_{a+1}\times\mathbf{n}_q-C_a\mathbf{n}_p\cdot\mathbf{n}_q+C_p\mathbf{n}_q\cdot\mathbf{n}_{a+1}\right). \tag{5.30}$$

If joint $a+1$ is neighbor of joint p, then $p=a+2$ and $\mathbf{n}_{a+1}\times\mathbf{n}_p=-S_{a+1}\mathbf{a}_{a+1}$ and $\mathbf{n}_p\cdot\mathbf{n}_{a+1}=C_{a+1}$. The right-hand side of (5.25) is

$$\frac{\ell_a}{S_a}\left(C_a\mathbf{n}_p\cdot\mathbf{n}_q-C_{a+1}\mathbf{n}_a\cdot\mathbf{n}_q+\mathbf{n}_a\times\mathbf{n}_q\cdot S_{a+1}\mathbf{a}_{a+1}\right). \tag{5.31}$$

In either case Table 5.2 yields an expression with the desired linearity properties. The products $\mathbf{a}_p\cdot\mathbf{n}_{a+1}\times\mathbf{n}_q$ and $\mathbf{n}_a\times\mathbf{n}_q\cdot\mathbf{a}_{a+1}$ are evaluated as determinants.

The equations $F_6^\ell=F_6^r$ *and* $F_7^\ell=F_7^r$

These equations are formulated only for the mechanism 7R which has angular variables only. In every scalar product appearing in the expressions for $\mathbf{r}^2$ every sine and every cosine appears in linear form only. Products sine $\times$ cosine of one and the same angle do not appear. If as joints a and b the joints 3 and 6 are chosen again, the equation $F_6^\ell=F_6^r$ has the form (5.24). Only the coefficient matrices are different.

Surprisingly, also the equation $F_7^\ell=F_7^r$ has the form (5.24) (with other coefficient matrices). In spite of the definition $F_7=\frac{1}{2}F_1F_6-F_3F_4$ the functions F_7^ℓ and F_7^r are both linear with respect to the sines and cosines of angles. For the function F_7^r this is proved as follows. With (5.13) for $\mathbf{r}$ the function F_7^r is a linear combination of products h_ih_j, $h_i\ell_j$ and $\ell_i\ell_j$ with $a\le i,j\le b$. As an example the coefficient of h_ih_j with $i\le j$ is considered. This coefficient denoted H_{ij} is

$$H_{ij}=(\mathbf{n}_a\cdot\mathbf{n}_b)(\mathbf{n}_i\cdot\mathbf{n}_j)-(\mathbf{n}_a\cdot\mathbf{n}_i)(\mathbf{n}_b\cdot\mathbf{n}_j)-(\mathbf{n}_a\cdot\mathbf{n}_j)(\mathbf{n}_b\cdot\mathbf{n}_i) \tag{5.32}$$

$$=(\mathbf{n}_a\times\mathbf{n}_i)\cdot(\mathbf{n}_b\times\mathbf{n}_j)-(\mathbf{n}_a\cdot\mathbf{n}_i)(\mathbf{n}_b\cdot\mathbf{n}_j)\qquad(a\le i\le j\le b). \tag{5.33}$$

The products $(\mathbf{n}_a\times\mathbf{n}_i)$ and $(\mathbf{n}_a\cdot\mathbf{n}_i)$ are linear with respect to the sines and cosines of angles between bodies a and i, and the products $(\mathbf{n}_b\times\mathbf{n}_j)$ and

$(\mathbf{n}_b \cdot \mathbf{n}_j)$ are linear with respect to the sines and cosines of angles between bodies b and j. From this follows the linearity of H_{ij} with respect to the sines and cosines of all angles. For other terms of F_7^r and for F_7^ℓ similar arguments hold. End of proof.

The discussion of Woernle-Lee equations is summarized as follows.

Mechanisms in rows 1 and 2 of Table 5.1: The equations with F_1 and F_2 are formulated. They have the forms (5.18) and (5.19). The first equation determines two solutions for a single unknown angle φ_λ. With each solution the second equation is a linear equation for a single translatory variable.

Mechanisms in rows 3 and 4: The equations with F_1, F_2, F_3 and F_4 are formulated. The first equation is Eq.(5.18). It determines two solutions for a single unknown angle φ_λ. With each solution the remaining three equations are linear equations for three unknown translatory variables.

Mechanisms in rows 5, 6, 7: Two equations are formulated. These are the equations with F_1 and F_2 for the mechanisms in row 5 and the equations with F_1 and F_5 for the mechanisms in rows 6 and 7. Both equations have either the form (5.20) or the form (5.21). These equations are easily decoupled. Methods of solution see in Sect. 5.4.3.

Mechanisms 5R-C, 6R-P and 7R: The four equations with F_1, F_2, F_3 and F_4 determine four unknowns. Without additional equations of another mathematical form it is impossible to decouple these equations. The necessary additional equations are the half-angle equations introduced further below.

Up to this point the independent variable was the angle of an arbitrarily chosen revolute joint. In what follows, it is the translatory variable in an arbitrarily chosen prismatic joint. This change has the effect that in each of the seven Woernle-Lee equations the number of unknown angular variables increases by one whereas the number of unknown translatory variables decreases by one. The two numbers of unknowns are $2 \le n_\varphi - 2 \le 4$ and $0 \le n_{\rm P} - 1 \le 2$. For the mechanisms of Table 5.1 this change has the following effect.

For the mechanisms in rows 2, 3 and 4 the same equations are formulated which are formulated for the mechanisms in rows 5, 6, 7 with an independent angular variable. These are the equations with F_1 and F_2 for the mechanisms in row 2 and the equations with F_1 and F_5 for the mechanisms in rows 3 and 4. Both equations have either the form (5.20) or the form (5.21).

For the mechanisms in rows 6, 7 and 9 the four equations with F_1, F_2, F_3 and F_4 determine four unknowns. For the mechanisms in row 6 they are the same equations which govern the mechanism 5R-C with an independent angular variable. For the mechanisms in row 7 they are the same equations which govern the mechanism 6R-P with an independent angular variable.

For the mechanism 6R-P they are the same equations which govern the mechanism 7R with an independent angular variable. These relationships were predicted when Table 5.1 was introduced.

5.3.2 Half-Angle Equations

These equations were first formulated by Lee [27]. Again, Fig. 5.2 is considered. The ordinary unit vectors $\mathbf{n}_b$, $\mathbf{a}_b$ und $\mathbf{n}_b \times \mathbf{a}_b$ form an orthogonal cartesian basis fixed on body b, and the unit vectors $\mathbf{n}_b$, $\mathbf{a}_{b-1}$, $\mathbf{n}_b \times \mathbf{a}_{b-1}$ form another basis fixed on body $b-1$. Temporarily, the abbreviations are used:

$$\mathbf{d}_b = \mathbf{n}_b \times \mathbf{a}_b\ , \qquad \mathbf{d}^*_{b-1} = \mathbf{n}_b \times \mathbf{a}_{b-1}\ . \tag{5.34}$$

The two bases are rotated against each other through the angle φ_b about the common axis $\mathbf{n}_b$. Let $\mathbf{v}$ be an arbitrary vector. Its coordinates in the two bases are related through the matrix equation

$$\begin{bmatrix} \mathbf{v}\cdot\mathbf{n}_b \\ \mathbf{v}\cdot\mathbf{a}_b \\ \mathbf{v}\cdot\mathbf{d}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_b & \sin\varphi_b \\ 0 & -\sin\varphi_b & \cos\varphi_b \end{bmatrix} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n}_b \\ \mathbf{v}\cdot\mathbf{a}_{b-1} \\ \mathbf{v}\cdot\mathbf{d}^*_{b-1} \end{bmatrix}\ . \tag{5.35}$$

The new variable $x_b = \tan\varphi_b/2$ is defined. The expressions $\cos\varphi_b = (1-x_b^2)/(1+x_b^2)$ and $\sin\varphi_b = 2x_b/(1+x_b^2)$ are substituted into (5.35). In order to avoid the quadratic term x_b^2 the equation is premultiplied by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x_b & 1 \\ 0 & 1 & -x_b \end{bmatrix}$. Following this, the identities $\sin\varphi_b - x_b\cos\varphi_b = x_b$ and $x_b\sin\varphi_b + \cos\varphi_b = 1$ are used. This results in the following equation which is linear with respect to x_b:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x_b & 1 \\ 0 & 1 & -x_b \end{bmatrix} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n}_b \\ \mathbf{v}\cdot\mathbf{a}_b \\ \mathbf{v}\cdot\mathbf{d}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -x_b & 1 \\ 0 & 1 & x_b \end{bmatrix} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n}_b \\ \mathbf{v}\cdot\mathbf{a}_{b-1} \\ \mathbf{v}\cdot\mathbf{d}^*_{b-1} \end{bmatrix}\ . \tag{5.36}$$

The first equation is the identity. The other two are, in terms of the original vectors in (5.34),

$$\left.\begin{aligned} x_b\,\mathbf{v}\cdot\mathbf{a}_b + \mathbf{v}\cdot\mathbf{n}_b\times\mathbf{a}_b &= -x_b\,\mathbf{v}\cdot\mathbf{a}_{b-1} + \mathbf{v}\cdot\mathbf{n}_b\times\mathbf{a}_{b-1}\ , \\ \mathbf{v}\cdot\mathbf{a}_b - x_b\,\mathbf{v}\cdot\mathbf{n}_b\times\mathbf{a}_b &= \mathbf{v}\cdot\mathbf{a}_{b-1} + x_b\,\mathbf{v}\cdot\mathbf{n}_b\times\mathbf{a}_{b-1}\ . \end{aligned}\right\} \tag{5.37}$$

This pair of equations is formulated with the vectors

$$\mathbf{v}_1 = \mathbf{n}_a \ , \qquad \mathbf{v}_2 = \mathbf{r} \ , \qquad \mathbf{v}_3 = \mathbf{n}_a \times \mathbf{r} \ , \qquad \mathbf{v}_4 = \frac{1}{2}\,\mathbf{r}^2\mathbf{n}_a - (\mathbf{n}_a \cdot \mathbf{r})\mathbf{r} \ . \quad (5.38)$$

The vectors $\mathbf{n}_a$ and $\mathbf{r}$ are the ones shown in Fig. 5.2 and in (5.13). In what follows, it is important that $-h_b\mathbf{n}_b$ is part of $\mathbf{r}$ in the left segment, and that $h_a\mathbf{n}_a$ is part of $\mathbf{r}$ in the right segment.

The scalar products $\mathbf{v}_i \cdot \mathbf{a}_b$ and $\mathbf{v}_i \cdot \mathbf{n}_b \times \mathbf{a}_b$ on the left-hand side of the equations are evaluated in the left segment, and the scalar products $\mathbf{v}_i \cdot \mathbf{a}_{b-1}$ and $\mathbf{v}_i \cdot \mathbf{n}_b \times \mathbf{a}_{b-1}$ on the right-hand side of the equations are evaluated in the right segment. With every vector $\mathbf{v}_i$ $(i = 1, 2, 3, 4)$ the vector products in (5.37) are linear with respect to the sines and cosines of all angles involved.

The expressions for $\mathbf{r}$ in (5.13) allow the following conclusions (note that the vector $\mathbf{n}_b \times \mathbf{a}_b$ is fixed on body b, and that the vector $\mathbf{n}_b \times \mathbf{a}_{b-1}$ is fixed on body $b-1$):

1. With all vectors $\mathbf{v}_i$ $(i = 1, 2, 3, 4)$ all scalar products are independent of both φ_a and φ_b.

2. The angle φ_b appears in every equation, however, only in the form $x_b = \tan\varphi_b/2$.

3. With $\mathbf{v}_i = \mathbf{v}_1$ only angular variables appear.

4. With $\mathbf{v}_i = \mathbf{v}_2$ all scalar products are independent of h_b. In the products on the right-hand side h_a occurs in linear form.

5. With $\mathbf{v}_i = \mathbf{v}_3$ all scalar products are independent of h_a. In the products on the left-hand side h_b occurs in linear form.

6. With $\mathbf{v}_i = \mathbf{v}_4$ both h_a and h_b occur in first and second-order terms. These equations are formulated only for the mechanism 7R in which the parameters $h_1, \ldots, h_7$ are constant.

For the evaluation of products in (5.37) Table 5.2 is used. The following facts are helpful. The vector $\mathbf{n}_b \times \mathbf{a}_b$ fixed on body b has the coordinates $[0\ 0\ 1]^T$ in the basis of body b. The vector $\mathbf{n}_b \times \mathbf{a}_{b-1}$ fixed on body $b-1$ has the coordinates $[C_{b-1}\ \ 0\ \ -S_{b-1}]^T$ in the basis of body $b-1$.

The product $\mathbf{v}_3 \cdot \mathbf{a}_b$ is formulated with $\mathbf{r} = \mathbf{r} + h_b\mathbf{n}_b - h_b\mathbf{n}_b$:

$$\mathbf{v}_3 \cdot \mathbf{a}_b = -[(\mathbf{r} + h_b\mathbf{n}_b) \times \mathbf{n}_a \cdot \mathbf{a}_b + h_b\mathbf{n}_a \cdot \mathbf{n}_b \times \mathbf{a}_b] \ . \qquad (5.39)$$

In the second term the previously given coordinates of $\mathbf{n}_b \times \mathbf{a}_b$ are used. In the first term the vector $\mathbf{r} + h_b\mathbf{n}_b$ joins the dual vectors $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{a}}_b$ (see Fig. 5.2). From this it follows that the first term represents the dual part of $\hat{\mathbf{n}}_a \cdot \hat{\mathbf{a}}_b$. It is calculated as dual derivative of $\mathbf{n}_a \cdot \mathbf{a}_b$. Similarly,

$$\mathbf{v}_3 \cdot \mathbf{a}_{b-1} = -\mathbf{r} \times \mathbf{n}_a \cdot \mathbf{a}_{b-1} = -(\text{dual derivative of } \mathbf{n}_a \cdot \mathbf{a}_{b-1}) \ . \qquad (5.40)$$

5.4 Systematic Analysis of Mechanisms

The previous section provided equations which are sufficient for the analysis of all mechanisms of Table 5.1 . In the present section methods for decoupling these equations are presented. For each type of mechanism a polynomial equation of minimal order is developed for a single unknown variable. This minimal order is the maximum number of configurations the mechanism under consideration can have for a given value of the independent variable. In these analyses all constant Denavit-Hartenberg parameters are assumed to be nonzero and arbitrary. As special cases a 7R-mechanism with three parallel joint axes and a 7R-mechanism with a spherical joint are analyzed.

5.4.1 RCCC

This is the simplest of all mechanisms listed in Table 5.1 . It has the smallest number of joints $(n = 4)$ and the smallest number of constant parameters $(4n - 7 = 9)$. Figure 5.3 shows schematically the mobile polygon with unit vectors $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, 2, 3, 4)$. The vectors are uniformly directed counterclockwise. Joint 1 represents the revolute joint. The variable φ_1 in this joint is the independent variable. The six dependent variables are the quantities φ_i , h_i in the cylindrical joints $i = 2, 3, 4$. The nine constant parameters are h_1 in joint 1 and the quantities α_i , ℓ_i of bodies $i = 1, 2, 3, 4$.
General remarks on vector polygons of mechanisms: A mechanism with n joints has a polygon with $2n$ sides. It can be drawn schematically as a regular $2n$-gon with all unit vectors $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, \ldots, n)$ pointing counterclockwise. A regular $2n$-gon has $2n$ axes of symmetry. Any closure condition formulated for a specific mechanism remains valid if the labels of its constant and variable Denavit-Hartenberg parameters are changed according to the

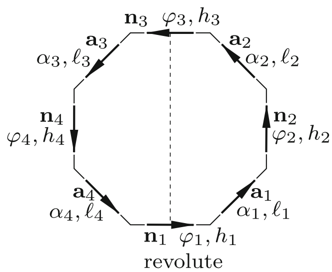

Fig. 5.3 Mechanism RCCC . Polygon with unit vectors $\mathbf{n}_i$, $\mathbf{a}_i$ and Denavit-Hartenberg parameters α_i , ℓ_i , φ_i , h_i $(i = 1, 2, 3, 4)$. Joint 1 is the revolute with the independent variable φ_1 . The dashed line of symmetry

symmetry. Symmetry is particularly useful if symmetrically located joints are of equal type, i.e., CC or RR or PP. In Fig. 5.3 this is the case for the dashed line of symmetry. Closure conditions remain valid if the quantities $(\alpha_1\,,\,\ell_1\,,\,\varphi_2\,,\,h_2\,,\,\alpha_2\,,\,\ell_2)$ are replaced by $(\alpha_4\,,\,\ell_4\,,\,\varphi_4\,,\,h_4\,,\,\alpha_3\,,\,\ell_3)$ and vice versa. Not every mechanism has such an axis of symmetry, for example not the mechanism RCPRC.

Also the following is true. Any closure condition formulated for a specific mechanism remains valid if the labels of its constant and variable Denavit-Hartenberg parameters are cyclicly increased by an arbitrary integer λ $(\pm 1, \pm 2, \ldots)$. This means that the two joints a and b on which the closure condition is founded are replaced by the joints $a+\lambda$ and $b+\lambda$, respectively. A cyclic rotation of labels by an integer λ is equivalent to a symmetry change of labels if the joints $a+\lambda$ and $b+\lambda$ are located symmetrically to the joints b and a, respectively. Cyclic rotation of labels and symmetry changes of labels are much simpler than re-formulations of closure conditions for new joints a and b.

Now back to the mechanism RCCC. Among the three cylindrical joints 2, 3 and 4 any two can be chosen as joints a and b. For each pair Eqs.(5.18) and (5.19) are formulated. The unknowns in these equations are the two variables of the third cylindrical joint. The first equation is the Woernle-Lee Eq.(5.17). It is given for only two joint combinations because any other combination is produced by a cyclic change of indices.

$$\left.\begin{array}{ll}\text{joint combination } a=2\,,\,b=3: & \mathbf{n}_k\cdot\mathbf{n}_{k-3}=\mathbf{n}_k\cdot\mathbf{n}_{k+1}\\ \text{joint combination } a=2\,,\,b=4: & \mathbf{n}_k\cdot\mathbf{n}_{k-2}=\mathbf{n}_k\cdot\mathbf{n}_{k+2}\end{array}\right\}\quad k=a\,. \tag{5.41}$$

For the joint combination $a=2\,,\,b=3$ Table 5.2 yields the equation

$$S_3[s_4s_1S_1-c_4(S_4C_1+C_4S_1c_1)]+C_3(C_4C_1-S_4S_1c_1)=C_2\,. \tag{5.42}$$

This is Eq.(5.18) for φ_4:

$$Ac_4+Bs_4=R\,. \tag{5.43}$$

The coefficients depend on φ_1:

$$\left.\begin{array}{ll}A=-S_3(S_4C_1+C_4S_1c_1)\,, & B=S_1S_3s_1\,,\\ R=C_2-C_3(C_4C_1-S_4S_1c_1)\,. & \end{array}\right\} \tag{5.44}$$

Dual differentiation of (5.43) yields Eq.(5.19) for h_4:

$$h_4(Bc_4-As_4)=Dc_4+Es_4+F\,. \tag{5.45}$$

The coefficients D, E and F are the dual derivatives of $-A$, $-B$ and R, respectively:

$$\left.\begin{aligned} D = & -\ell_4 S_3(S_1 S_4 c_1 - C_1 C_4) + \ell_1 S_3(C_1 C_4 c_1 - S_1 S_4) \\ & +\ell_3 C_3(S_1 C_4 c_1 + C_1 S_4) - h_1 S_1 S_3 C_4 s_1 \ , \\ F = & \ \ \ell_4 C_3(S_1 C_4 c_1 + C_1 S_4) + \ell_1 C_3(C_1 S_4 c_1 + S_1 C_4) \\ & -\ell_3 S_3(S_1 S_4 c_1 - C_1 C_4) - h_1 S_1 C_3 S_4 s_1 - \ell_2 S_2 \ , \\ E = & -s_1(\ell_1 C_1 S_3 + \ell_3 S_1 C_3) - h_1 S_1 S_3 c_1 \ . \end{aligned}\right\} \tag{5.46}$$

For every value of φ_1 Eq.(5.43) has two (not necessarily real) solutions φ_4. The associated solutions h_4 are determined by (5.45).

Next, equations for φ_2 and h_2 are produced. This is done by increasing all indices in (5.43) and (5.44) cyclicly by one. Equation (5.43) is replaced by the equation

$$S_4[s_1 s_2 S_2 - c_1(S_1 C_2 + C_1 S_2 c_2)] + C_4(C_1 C_2 - S_1 S_2 c_2) = C_3 \ . \tag{5.47}$$

This is the equation for φ_2:

$$A^* c_2 + B^* s_2 = R^* \ , \tag{5.48}$$

$$\left.\begin{aligned} A^* &= -S_2(S_1 C_4 + C_1 S_4 c_1) \ , \qquad B^* = S_4 S_2 s_1 \ , \\ R^* &= C_3 - C_2(C_4 C_1 - S_4 S_1 c_1) \ . \end{aligned}\right\} \tag{5.49}$$

The cyclic change of indices in (5.45) and (5.46) is left to the reader. Next, φ_3 is determined from (5.41) for the joint combination $a = 2$, $b = 4$. Table 5.2 yields the equation

$$C_3 C_2 - S_3 S_2 c_3 = C_1 C_4 - S_1 S_4 c_1 \ . \tag{5.50}$$

It displays the symmetry of Fig. 5.3. It determines c_3 as a linear function of c_1. Every value of φ_1 yields two solutions $\pm\varphi_3$. Dual differentiation of (5.50) produces for h_3 the formula

$$\begin{aligned} h_3 S_2 S_3 s_3 = h_1 S_1 S_4 s_1 &+ c_3(\ell_2 C_2 S_3 + \ell_3 S_2 C_3) + (\ell_2 S_2 C_3 + \ell_3 C_2 S_3) \\ &- c_1(\ell_1 C_1 S_4 + \ell_4 S_1 C_4) - (\ell_1 S_1 C_4 + \ell_4 C_1 S_4) \ . \end{aligned} \tag{5.51}$$

If h_3 is the solution associated with $+\varphi_3$, $-h_3$ is the solution associated with $-\varphi_3$.

The analysis is now complete except for one open problem. For every value of φ_1 there exist two solutions (φ_2, h_2), two solutions (φ_3, h_3) and two solutions (φ_4, h_4). What remains to be shown is which of the solutions φ_2, φ_4 occur together with $+\varphi_3$ and which with $-\varphi_3$. For solving this problem two closure conditions are needed relating φ_3 to φ_4 and φ_3 to φ_2, respectively. They are obtained by increasing the indices in (5.43) by two and by three, respectively. This produces the equations

$$S_1[s_2 s_3 S_3 - c_2(S_2 C_3 + C_3 S_3 c_3)] + C_1(C_2 C_3 - S_2 S_3 c_3) = C_4 \ , \tag{5.52}$$

$$S_2[s_3 s_4 S_4 - c_3(S_3 C_4 + C_3 S_4 c_4)] + C_2(C_3 C_4 - S_3 S_4 c_4) = C_1 \,. \tag{5.53}$$

The relationship between φ_4 and φ_2 is obtained more easily from (5.50) by a cyclic increase of indices by one:

$$C_4 C_3 - S_4 S_3 c_4 = C_2 C_1 - S_2 S_1 c_2 \,. \tag{5.54}$$

Any two of these equations determine whether numerically calculated angles φ_4 and φ_2 belong to $+\varphi_3$ or to $-\varphi_3$.

Special Cases: Bennett Mechanism and Spherical Four-Bar

Under certain conditions on the nine constant Denavit-Hartenberg parameters $\ell_1, \ldots, \ell_4$, $\alpha_1, \ldots, \alpha_4$ and h_1 of the mechanism RCCC the translatory variables h_2, h_3, h_4 are constant (actually identically zero) independent of the angle φ_1. This means that the mechanism has four revolute joints and, yet, a single degree of freedom. It is an overconstrained mechanism. There are two such special mechanisms. One of them is called Bennett mechanism. It is the subject of Sect. 6.2. The other is the spherical four-bar with four revolute joints the axes of which intersect at a single point. Intersection means that the Denavit-Hartenberg parameters $\ell_1, \ldots, \ell_4$ and h_1 are zero. Equations (5.46) yield $D = E = F \equiv 0$ and with this, (5.45) yields $h_4 \equiv 0$. Dual differentiation of (5.47) and (5.51) yields $h_2 \equiv 0$ and $h_3 \equiv 0$. The equations relating the angular variables $\varphi_1, \ldots, \varphi_4$ do not change (see (5.43), (5.44), (5.48), (5.49), (5.50), (5.54)). These results confirm that the spherical four-bar has a single degree of freedom.

In Fig. 5.4 a spherical four-bar is shown as quadrilateral A_0ABB_0 the links 1, 2, 3, 4 of which are arcs of great circles on the unit sphere about the intersection point 0 of the joint axes. The unit vectors $\mathbf{n}_1, \ldots, \mathbf{n}_4$ along the axes are pointing away from 0. The unit vector $\mathbf{a}_i$ normal to both $\mathbf{n}_i$ and $\mathbf{n}_{i+1}$ has the direction of $\mathbf{n}_i \times \mathbf{n}_{i+1}$ (here and in what follows, $i = 1, \ldots, 4$ cyclic). The angle α_i is the angle about $\mathbf{a}_i$ from $\mathbf{n}_i$ to $\mathbf{n}_{i+1}$, and φ_i is the angle about $\mathbf{n}_i$ from $\mathbf{a}_{i-1}$ to $\mathbf{a}_i$ (see Fig. 5.1). The angle α_i equals the arc of link i on the unit sphere. Link i is said to have the length α_i. At this

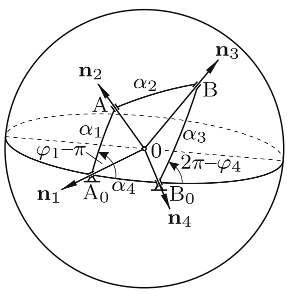

Fig. 5.4 Spherical four-bar

point the kinematics analysis is stopped. It is resumed in Chap. 18.

Final remark: The expressions in (5.42) – (5.46) are moderately complicated. For mechanisms with up to twenty-one instead of nine constant parameters much more complicated expressions are generated. In every case the generation requires only two operations. One is copying terms from Table 5.2 and the other is calculating dual derivatives. Both operations can be executed by the MAPLE software tool [34] already mentioned.

5.4.2 RCRCR and CRRRC

These mechanisms have $n = 5$ joints, $4n - 7 = 13$ constant parameters, two cylindrical joints and $n_\varphi = 5$ angular variables. Figures 5.5a and b show the mobile polygons with unit vectors $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, \ldots, 5)$. Each figure is structurally symmetric with respect to the dashed line.

With the cylindrical joints as joints a and b the equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ are formulated. From Sect. 5.3.1 it is known that these equations relate the angles in the three revolute joints. Two of them are unknown dependent angles. The equations have the form (5.20) if the unknowns are located in different segments created by the cylindrical joints, and they have the form (5.21) if they are located in one and the same segment. The case of location in different segments occurs if (i) the mechanism is R̲CRCR̲ and (ii) the independent angle is in one of the underscored revolutes. This simpler case is treated first. It is the case of Fig. 5.5a with φ_1 as independent variable. The cylindrical joints are the joints 3 and 5. The equation $F_1^\ell = F_1^r$ is the equation $\mathbf{n}_k \cdot \mathbf{n}_{k+2} = \mathbf{n}_k \cdot \mathbf{n}_{k-3}$ with $k = 3$. Table 5.2 yields the explicit form

$$\begin{aligned} &C_4 C_3 - S_4 S_3 c_4 \\ &= S_5[s_1 s_2 S_2 - c_1(S_1 C_2 + C_1 S_2 c_2)] + C_5(C_1 C_2 - S_1 S_2 c_2)\,. \end{aligned} \qquad (5.55)$$

This is a special form of (5.20):

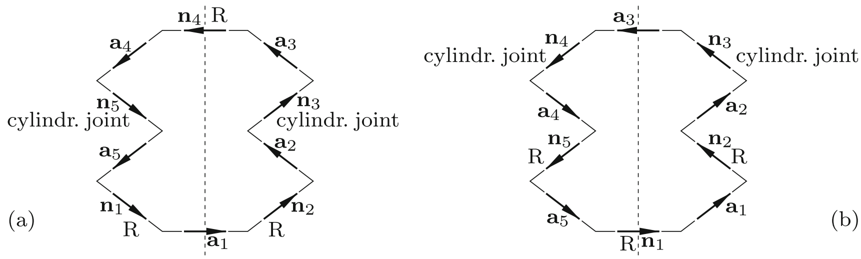

Fig. 5.5 Polygons with unit vectors for the mechanisms RCRCR **(a)** and CRRRC **(b)**

$$S_3S_4c_4 = A_1c_2 + B_1s_2 + R_1 \ . \tag{5.56}$$

The coefficients A_1, B_1 and R_1 depend on φ_1:

$$\left.\begin{array}{ll} A_1 = S_2(C_5S_1 + C_1S_5c_1) \ , & B_1 = -S_2S_5s_1 \ , \\ R_1 = C_3C_4 - C_2(C_1C_5 - S_1S_5c_1) \ . & \end{array}\right\} \tag{5.57}$$

Dual differentiation yields (5.21):

$$A_4c_4 - h_4S_3S_4s_4 = A_2c_2 + B_2s_2 + R_2 \ . \tag{5.58}$$

The coefficients are

$$A_4 = \ell_3C_3S_4 + \ell_4S_3C_4 \ , \quad A_2 = A_1' + h_2B_1 \ , \quad B_2 = B_1' - h_2A_1 \ , \quad R_2 = R_1' \tag{5.59}$$

with A_1', B_1', R_1' denoting the dual derivatives of A_1, of B_1 and of R_1, respectively.

Equations (5.56) and (5.58) are linear equations for c_4 and s_4. Substituting the solutions into the equation $c_4^2 + s_4^2 = 1$ results in an equation for φ_2 of the form

$$A\sin^2\varphi_2 + B\sin\varphi_2\cos\varphi_2 + C\sin\varphi_2 + D\cos^2\varphi_2 + E\cos\varphi_2 + F = 0 \tag{5.60}$$

with new coefficients which are functions of φ_1. The substitution $x = \tan\varphi_2/2$, $\cos\varphi_2 = (1-x^2)/(1+x^2)$, $\sin\varphi_2 = 2x/(1+x^2)$ produces for x the 4th-order equation

$$x^4(D-E+F) + 2x^3(C-B) + 2x^2(2A-D+F) + 2x(C+B) + D+E+F = 0 \ . \tag{5.61}$$

It has four (not necessarily real) solutions φ_2 for every value of φ_1. This is the number $N_\varphi = 4$ in Table 5.1. With every real solution φ_2 the associated values c_4 and s_4 and, thus, φ_4 are determined from (5.56) and (5.58).

Note: The substitution $x = \tan\varphi_2/2$ makes sense only if $\varphi_2 \neq \pi$. $\varphi_2 = \pi$ is a solution if $D - E + F = 0$ and a double solution if, in addition, also $C - B = 0$. These are the coefficients of x^4 and x^3 in the polynomial[1]. In the case $B = C = 0$, (5.60) is quadratic in $\cos\varphi_2$, and in the case $B = E = 0$, it is quadratic in $\sin\varphi_2$. In either case the substitution $x = \tan\varphi_2/2$ is unnecessary.

The variables φ_3, h_3 and φ_5, h_5 in the cylindrical joints are still unknown. The angles φ_3 and φ_5 are determined first as follows. One of the two half-angle equations (5.37) is formulated with $a = 5$, $b = 3$ and $\mathbf{v} = \mathbf{n}_5$. This is a linear equation for $x_3 = \tan\varphi_3/2$ as the only unknown. The same equation with $a = 3$, $b = 5$ and $\mathbf{v} = \mathbf{n}_3$ determines $x_5 = \tan\varphi_5/2$. Next,

[1] Example: The equation $4\sin^2\varphi + 3\sin\varphi\cos\varphi + 3\sin\varphi + 2\cos^2\varphi + \cos\varphi - 1 = 0$ has the solutions $\varphi_1 = \varphi_2 = \pi$. Equation (5.61) is $5x^2 + 6x - 1 = 0$. It has the solutions $\varphi_3 = -\pi/2$ and $\varphi_4 = -2\tan^{-1}(1/5)$

the unknowns h_3 and h_5 are expressed in terms of all other variables including the previously determined angles φ_3 and φ_5. A single linear equation for h_3 is obtained by formulating one of the half-angle equations (5.37) with $a = 5$, $b = 3$ and $\mathbf{v} = \mathbf{r}$. The same equation with $a = 3$, $b = 5$ and $\mathbf{v} = \mathbf{r}$ determines h_5. Note that the vector $\mathbf{r}$ depends upon a and b according to Fig. 5.2. This concludes the analysis of the mechanism RCRCR with φ_1 as independent variable.

Next, φ_4 is assumed to be the independent variable. In this case, both Eq.(5.55) and its dual derivative have the form (5.21). With new names for coefficients they are written in the form

$$c_1 \underbrace{(a_{i1}c_2 + a_{i2}s_2 + a_{i3})}_{A_i} + s_1 \underbrace{(b_{i1}c_2 + b_{i2}s_2 + b_{i3})}_{B_i} + \underbrace{(r_{i1}c_2 + r_{i2}s_2 + r_{i3})}_{R_i} = 0 \tag{5.62}$$

$(i = 1, 2)$. Equation (5.55) yields, for example, $a_{11} = S_5C_1S_2$, $a_{12} = 0$, $b_{11} = 0$, $b_{12} = -S_5S_2$, $r_{11} = C_5S_1S_2$, $r_{12} = 0$, $r_{13} = C_4C_3 - S_4S_3c_4 - C_5C_1C_2$. The two Eqs.(5.62) are solved as linear equations for c_1 and s_1:

$$c_1 = \frac{B_1R_2 - B_2R_1}{A_1B_2 - A_2B_1}\,, \qquad s_1 = -\frac{A_1R_2 - A_2R_1}{A_1B_2 - A_2B_1}\,. \tag{5.63}$$

The common denominator and the two numerator expressions contain zero, first and second-order terms of c_2 and s_2. Substitution into the constraint equation $c_1^2 + s_1^2 = 1$ eliminates φ_1. The resulting equation relates φ_2 to the independent variable φ_4. This is the equation $(A_1R_2 - A_2R_1)^2 + (B_1R_2 - B_2R_1)^2 - (A_1B_2 - A_2B_1)^2 = 0$. It contains zero, first, second, third and fourth-order terms of c_2 and s_2 with coefficients which are functions of φ_4:

$$A\cos^4\varphi_2 + B\cos^3\varphi_2\sin\varphi_2 + \ldots = 0\,. \tag{5.64}$$

The substitution $x = \tan\varphi_2/2$, $\cos\varphi_2 = (1-x^2)/(1+x^2)$, $\sin\varphi_2 = 2x/(1+x^2)$ produces for x an 8th-order polynomial equation[2]. For a given value of the independent variable φ_4 it has eight (not necessarily real) solutions φ_2. This is the number $N_\varphi = 8$ in Table 5.1. For every solution φ_2 the corresponding solution φ_1 is calculated from (5.63).

For the mechanism CRRRC in Fig. 5.5b the same set of Eqs.(5.62) is obtained. Only the indices of the unknown variables and the coefficients are different. The details are left to the reader. This concludes the analysis of the mechanisms RCRCR and CRRRC.

Equations (5.63) require a comment. It may happen that the common denominator and the two numerator expressions are linear functions of c_2 and s_2. This requires that in all three expressions c_2s_2 has the factor zero,

[2] $\varphi_2 = \pi$ is a root of multiplicity n if the highest-order term in the polynomial is x^{8-n}

and that c_2^2 and s_2^2 have identical factors. For the factors of c_2s_2 to be zero the three conditions on the left-hand side below must be satisfied. For the factors of c_2^2 and of s_2^2 to be identical the three conditions on the right-hand side must be satisfied.

$$\left.\begin{array}{ll} a_{11}r_{22}+a_{12}r_{21}=a_{21}r_{12}+a_{22}r_{11}\,, & a_{11}r_{21}-a_{21}r_{11}=a_{12}r_{22}-a_{22}r_{12}\,,\\ b_{11}r_{22}+b_{12}r_{21}=b_{21}r_{12}+b_{22}r_{11}\,, & b_{11}r_{21}-b_{21}r_{11}=b_{12}r_{22}-b_{22}r_{12}\,,\\ a_{11}b_{22}+a_{12}b_{21}=a_{21}b_{12}+a_{22}b_{11}\,, & a_{11}b_{21}-a_{21}b_{11}=a_{12}b_{22}-a_{22}b_{12}\,. \end{array}\right\} \tag{5.65}$$

Under these conditions the equation $(A_1R_2-A_2R_1)^2+(B_1R_2-B_2R_1)^2-(A_1B_2-A_2B_1)^2=0$ contains only zero, first and second-order terms of c_2 and s_2. Hence it does not have the form (5.64), but the form (5.60). This equation has four solutions for every value of the independent variable. The mechanism RRSRR analyzed in Sect. 5.5.2 is governed by a set of equations satisfying the conditions (5.65).

5.4.3 RCPRC, CCPRR and RCPCR. Independent Variable in the Prismatic Joint

These mechanisms are obtained from the previously investigated mechanisms RCRCR and CRRRC when one revolute joint is replaced by a cylindrical joint. The letter sequences are written such that the prismatic joint is joint 3. The translatory variable h_3 in this joint is the independent variable. As before, the two cylindrical joints are the joints a and b. As before, the closure conditions $F_1^\ell=F_1^r$ and $F_2^\ell=F_2^r$ are formulated and, as before, the angles φ_λ and φ_μ of two revolute joints are the only unknowns in these equations. In the mechanism RCPRC the two angles are located in different segments created by the cylindrical joints, and in the mechanisms CCPRR and RCPCR they are located in one and the same segment. For the first mechanism both equations have the form (5.20), and for the other two mechanisms they have the form (5.21). The numbers of solution are four for the former and eight for the latter. These are the numbers N_t given in Table 5.1. This concludes the analysis.

5.4.4 Mechanisms in Rows 6 and 7 of Table 5.1. Independent Variable is an Angle

It was shown that the closure condition $F_1^\ell=F_1^r$ is Eq.(5.21) with two unknowns φ_λ and φ_μ, and that the closure condition $F_5^\ell=F_5^r$ has the same form (see (5.26)). The unknowns φ_λ and φ_μ are the same in both equations if the joints a and b are the same for both equations. The choice of these joints is dictated by the method for formulating the right-hand side of (5.26). Both equations are written in the form (5.62). For a given value of the independent variable the number of solutions is eight since the conditions (5.65) are not satisfied. This concludes the analysis.

5.4.5 5R-C

This mechanism has $n = 6$ joints, $4n-7 = 17$ constant parameters, $n_\varphi = 6$ angular variables and a single translatory variable in the cylindrical joint. Figure 5.6 shows schematically the mobile polygon with unit vectors $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, \ldots, 6)$. Joint 3 is the cylindrical joint. The dashed line is an axis of structural symmetry. The vector $\mathbf{r}$ shown in the figure is

$$\mathbf{r} = \begin{cases} -(h_3\mathbf{n}_3 + \ell_3\mathbf{a}_3 + h_4\mathbf{n}_4 + \ell_4\mathbf{a}_4 + h_5\mathbf{n}_5 + \ell_5\mathbf{a}_5) & \text{(left segment)} \\ \quad h_6\mathbf{n}_6 + \ell_6\mathbf{a}_6 + h_1\mathbf{n}_1 + \ell_1\mathbf{a}_1 + h_2\mathbf{n}_2 + \ell_2\mathbf{a}_2 & \text{(right segment)}\ . \end{cases} \tag{5.66}$$

The revolute joint 6 and the cylindrical joint 3 are chosen as joints a and b, respectively. This has the effect that every closure condition displays the symmetry.

Let φ_5 be the independent variable. Then two unknown variables φ_1 and φ_2 appear in the right segment and the single unknown variable φ_4 in the left segment. First, the closure conditions $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ are formulated. The former is the equation $\mathbf{n}_k \cdot \mathbf{n}_{k+3} = \mathbf{n}_k \cdot \mathbf{n}_{k-3}$ with $k = 3$. The scalar products are copied from Table 5.2:

$$\begin{aligned} &C_5(C_4C_3 - S_4S_3c_4) + S_5[s_5s_4S_3 - c_5(S_4C_3 + C_4S_3c_4)] \\ &= S_6[s_1s_2S_2 - c_1(S_1C_2 + C_1S_2c_2)] + C_6(C_1C_2 - S_1S_2c_2)\ . \end{aligned} \tag{5.67}$$

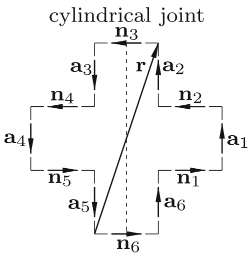

Fig. 5.6 Mechanism 5R-C. Polygon with unit vectors

This is written in the form (5.22) with $\varphi_\lambda = \varphi_1$, $\varphi_\mu = \varphi_2$ and $\varphi_\nu = \varphi_4$. The matrix $\underline{A}$ of this equation is

$$\underline{A} = [-S_3(S_4C_5 + C_4S_5c_5) \qquad S_3S_5s_5\ ]^T \ . \tag{5.68}$$

The equation $F_2^\ell = F_2^r$ is the dual derivative of (5.67). It has the same form. Both equations are combined in the matrix form

$$\underline{A}_1\underline{u}_\ell = \underline{B}_1 \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \ . \tag{5.69}$$

The column matrices $\underline{u}_\ell$ and $\underline{u}_r$ are

$$\underline{u}_\ell = [c_4 \quad s_4\ ]^T \ , \quad \underline{u}_r = [c_1c_2 \quad c_1s_2 \quad c_1 \quad s_1c_2 \quad s_1s_2 \quad s_1 \quad c_2 \quad s_2\ ]^T \ . \tag{5.70}$$

The coefficient matrices $\underline{A}_1$ and $\underline{B}_1$ are of size (2×2) and (2×9) , respectively. They are functions of the independent variable φ_5 .

No other closure condition yields a third equation for the same three unknowns. The closure conditions $F_3^\ell = F_3^r$ and $F_4^\ell = F_4^r$ with $F_3 = \mathbf{n}_6 \cdot \mathbf{r}$ and $F_4 = \mathbf{n}_3 \cdot \mathbf{r}$ have the form (5.23) with $h_\kappa = h_3$. One of these equations is used later for the calculation of h_3 once φ_1 , φ_2 and φ_4 are known. For the determination of these angles the two pairs of half-angle equations (5.37) with $\mathbf{v} = \mathbf{n}_6$ and with $\mathbf{v} = \mathbf{r}$ are formulated. In the four Eqs.(5.37) the same unknown elements of $\underline{u}_\ell$ are located on the left-hand sides and the elements of $\underline{u}_r$ on the right-hand sides. All these elements occur once without and once with the factor $x_6 = \tan\varphi_6/2$. The four equations are combined in the matrix form

$$(\underline{A}_2 + x_6\underline{A}_3)\underline{u}_\ell = (\underline{B}_2 + x_6\underline{B}_3) \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \ . \tag{5.71}$$

The coefficient matrices $\underline{A}_2$ and $\underline{A}_3$ are of size (4×4) , and $\underline{B}_2$ and $\underline{B}_3$ are of size (4×9) . They are functions of φ_5 .

The goal is now to deduce from the six Eqs.(5.69) and (5.71) a polynomial equation for a single unknown variable. Two additional equations are produced by multiplying (5.69) with x_6 . These two equations together with the two Eqs.(5.69) and the four Eqs.(5.71) represent a system of eight equations. It is written in the form

$$\underbrace{\begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_1 \\ \underline{A}_2 & \underline{A}_3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} \underline{u}_\ell \\ x_6\underline{u}_\ell \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} \underline{B}_1 & \underline{0} \\ \underline{0} & \underline{B}_1 \\ \underline{B}_2 & \underline{B}_3 \end{bmatrix}}_{\underline{B}} \underbrace{\begin{bmatrix} \underline{u}_r \\ 1 \\ x_6\underline{u}_r \\ x_6 \end{bmatrix}}_{\underline{z}} \qquad \text{or} \qquad \underline{A}\,\underline{y} = \underline{B}\,\underline{z} \ . \tag{5.72}$$

The coefficient matrices $\underline{A}$ and $\underline{B}$ are of size (8×4) and (8×18) , respectively. Four out of these eight equations are solved for $\underline{y}$ in terms of $\underline{z}$. The resulting expression is substituted into the last four equations. These four equations are then of the form

$$\underline{P}\,\underline{z} = \underline{0} \tag{5.73}$$

with a (4×18)-matrix $\underline{P}$. Next, two new variables $x_i = \tan\varphi_i/2 \ \ (i = 1,2)$ are defined. Substituting $c_i = (1 - x_i^2)/(1 + x_i^2)$ and $s_i = 2x_i/(1 + x_i^2)$ $(i = 1,2)$ into the submatrix $\underline{u}_r$ of $\underline{z}$ and re-arranging terms the four Eqs.(5.73) are given the forms

$$x_6(a_i x_2^2 + b_i x_2 + d_i) + (p_i x_2^2 + q_i x_2 + r_i) = 0 \qquad (i = 1,2,3,4)\ . \tag{5.74}$$

The coefficients a_i, b_i, d_i, p_i, q_i, r_i themselves are second-order functions of x_1 with coefficients depending on φ_5. The four Eqs.(5.74) are multiplied with x_2. These new equations and the four Eqs.(5.74) are combined in matrix form:

$$\begin{bmatrix} a_1 & p_1 & b_1 & q_1 & d_1 & r_1 & 0 & 0 \\ a_2 & p_2 & b_2 & q_2 & d_2 & r_2 & 0 & 0 \\ a_3 & p_3 & b_3 & q_3 & d_3 & r_3 & 0 & 0 \\ a_4 & p_4 & b_4 & q_4 & d_4 & r_4 & 0 & 0 \\ 0 & 0 & a_1 & p_1 & b_1 & q_1 & d_1 & r_1 \\ 0 & 0 & a_2 & p_2 & b_2 & q_2 & d_2 & r_2 \\ 0 & 0 & a_3 & p_3 & b_3 & q_3 & d_3 & r_3 \\ 0 & 0 & a_4 & p_4 & b_4 & q_4 & d_4 & r_4 \end{bmatrix} \begin{bmatrix} x_2^3 x_6 \\ x_2^3 \\ x_2^2 x_6 \\ x_2^2 \\ x_2 x_6 \\ x_2 \\ x_6 \\ 1 \end{bmatrix} = \underline{0}\ . \tag{5.75}$$

The (8×8) coefficient matrix must satisfy two conditions.
1. For a nontrivial solution to exist the determinant Δ of the coefficient matrix must be zero.
2. The nontrivial solution must have a nonzero 8th component. This latter condition is formulated as follows. Let $\underline{K}_1$ be the (8×7)-matrix formed by the first seven columns of the coefficient matrix, and let $\underline{K}_2$ be the eighth column. Furthermore, let $\underline{\xi}$ be the column matrix

$$\underline{\xi} = [\, x_2^3 x_6 \quad x_2^3 \quad x_2^2 x_6 \quad x_2^2 \quad x_2 x_6 \quad x_2 \quad x_6 \,]^T\ . \tag{5.76}$$

Equation (5.75) has the form $\underline{K}_1\underline{\xi} = -\underline{K}_2$. Premultiplication by $\underline{K}_1^T$ produces the equation

$$\underline{K}_1^T \underline{K}_1 \underline{\xi} = -\underline{K}_1^T \underline{K}_2\ . \tag{5.77}$$

This equation must have a unique solution. This is condition 2. The matrices $\underline{K}_1$ and $\underline{K}_2$ depend on x_1 which is calculated from the first condition $\Delta = 0$. This is a 16th-order equation[3] for x_1. With this equation it is proved that the mechanism 5R-C has at most sixteen different configurations for a

[3] The 16th-order polynomial is computed as interpolation-polynomial connecting sixteen numerically calculated points (x_{1_i}, Δ_i) $(i = 1, \ldots, 16)$. The general problem of *Numerical polynomial algebra* is the title of Stetter [38]

given value of the independent variable φ_5 . This is the number N_φ given in Table 5.1 . Lee [27] gave a numerical example with sixteen different real configurations.

Every solution x_1 determines $\varphi_1 = 2\tan^{-1} x_1$. The associated solutions x_2 and x_6 , i.e., φ_2 and φ_6 are determined by (5.77). With the solutions for φ_1 and φ_2 Eqs.(5.69) are two linear equations for c_4 and s_4 . They determine φ_4 . The last two unknowns are the variables φ_3 and h_3 in the cylindrical joint. The equation $F_3^\ell = F_3^r$ has the form (5.23) with $h_\kappa = h_3$. It determines h_3 . The angle φ_3 or rather $x_3 = \tan\varphi_3/2$, can be calculated from one half-angle equation (5.37) with $a = 6$, $b = 3$ and $\mathbf{v} = \mathbf{n}_6$. An alternative method is to formulate and to solve two linear equations for c_3 and s_3 , for example, the Woernle-Lee equations $F_1^\ell = F_1^r$ with $F_1 = \mathbf{n}_4 \cdot \mathbf{n}_1$ and with $F_1 = \mathbf{n}_5 \cdot \mathbf{n}_2$. They are obtained from (5.67) by a cyclic increase of all indices by one and by two, respectively. This concludes the analysis of the mechanism 5R-C .

5.4.6 RRCRPR , RRCPRR , RRCRRP . Independent Variable in the Prismatic Joint

Each of these mechanisms can be produced from the mechanism 5R-C by replacing one revolute joint by a cylindrical joint. To be specific, the mechanism RRCRPR is investigated. Its vector polygon has the form of Fig. 5.6 . The only difference as compared with the mechanism 5R-C is that joint 5 is replaced by a prismatic joint. In the previous analysis the angle φ_5 was the independent variable. Now, φ_5 is constant whereas h_5 , previously constant, is the independent variable. With the joints $a = 3$ and $b = 6$ the same equations $F_1^\ell = F_1^r$ and $F_2^\ell = F_2^r$ are formulated. These are Eq.(5.67) and its dual derivative. The matrix form of these two equations is, again, Eq.(5.69). The only difference is, that now the coefficient matrix $\underline{A}_1$ is a function not of φ_5 , but of h_5 . Also the rest of the analysis is the same as for the mechanism 5R-C . Two pairs of half-angle equations (5.37) with $\mathbf{v} = \mathbf{n}_6$ and with $\mathbf{v} = \mathbf{r}$ result in Eqs.(5.71). Via Eqs.(5.73) - (5.75) the existence of sixteen solutions $x_1 = \tan\varphi_1/2$ for a given value of the independent variable h_5 is proved. This concludes the analysis.

5.4.7 Mechanism 7R

This mechanism has $n = 7$ joints and $4n - 7 = 21$ constant parameters. The only variables are the angles $\varphi_1, \ldots, \varphi_7$ in the revolute joints. Figure 5.7 shows schematically the polygon with unit vectors $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, \ldots, 7)$.

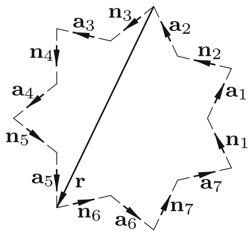

Fig. 5.7 Mechanism 7R. Polygon with unit vectors

Let φ_7 be the independent variable. Joints 3 and 6 are chosen as joints a and b for the formulation of closure conditions. The vector $\mathbf{r}$ shown in the figure is

$$\mathbf{r} = \begin{cases} h_3\mathbf{n}_3 + \ell_3\mathbf{a}_3 + h_4\mathbf{n}_4 + \ell_4\mathbf{a}_4 + h_5\mathbf{n}_5 + \ell_5\mathbf{a}_5 & \text{(left s.)} \\ -(h_6\mathbf{n}_6 + \ell_6\mathbf{a}_6 + h_7\mathbf{n}_7 + \ell_7\mathbf{a}_7 + h_1\mathbf{n}_1 + \ell_1\mathbf{a}_1 + h_2\mathbf{n}_2 + \ell_2\mathbf{a}_2) & \text{(right s.)}\ . \end{cases} \tag{5.78}$$

The unknown dependent variables are φ_1, φ_2 in the right segment and φ_4, φ_5 in the left segment. The four closure conditions $F_k^\ell = F_k^r$ $(k = 1, \ldots 4)$ with $F_1 = \mathbf{n}_3 \cdot \mathbf{n}_6$, $F_2 = \mathbf{r} \cdot \mathbf{n}_3 \times \mathbf{n}_6$, $F_3 = \mathbf{n}_3 \cdot \mathbf{r}$ and $F_4 = \mathbf{n}_6 \cdot \mathbf{r}$ are formulated. It was shown that all four equations have, with different coefficient matrices, the form (5.24). They are written in the matrix form

$$\underline{A}_i^* \underline{u}_\ell = \underline{B}_i^* \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \qquad (i = 1, \ldots, 4) \tag{5.79}$$

with the column matrices

$$\left.\begin{aligned} \underline{u}_\ell &= [\, c_4c_5 \quad c_4s_5 \quad c_4 \quad s_4c_5 \quad s_4s_5 \quad s_4 \quad c_5 \quad s_5 \,]^T\,, \\ \underline{u}_r &= [\, c_1c_2 \quad c_1s_2 \quad c_1 \quad s_1c_2 \quad s_1s_2 \quad s_1 \quad c_2 \quad s_2 \,]^T\,. \end{aligned}\right\} \tag{5.80}$$

In order to demonstrate the usefulness of Table 5.2 the four equations are developed in detail. The equation with F_1 is written in the form (see the example following (5.17))

$$\underbrace{\mathbf{n}_k \cdot \mathbf{n}_{k+3}}_{k=3} = \underbrace{\mathbf{n}_{k-2} \cdot \mathbf{n}_{k+2}}_{k=1}\ . \tag{5.81}$$

With Table 5.2 this is the equation

$$\begin{aligned} &C_5\ (C_4C_3 - S_4S_3c_4) + S_5[s_5s_4S_3 - c_5(S_4C_3 + C_4S_3c_4)] \\ &= (C_6C_7 - S_6S_7c_7)(C_2C_1 - S_2S_1c_2) + [C_6S_7s_1 + S_6(s_7c_1 + c_7s_1C_7)]S_2s_2 \\ &\quad -[C_6S_7c_1 - S_6(s_7s_1 - c_7c_1C_7)](C_2S_1 + S_2C_1c_2)\ . \end{aligned} \tag{5.82}$$

This is the first Eq.(5.79). The coefficient matrices are

$$\left.\begin{aligned}\underline{A}_1^* &= [\,-S_3C_4S_5 \quad 0 \quad -S_3S_4C_5 \quad 0 \quad S_3S_5 \quad 0 \quad -C_3S_4S_5 \quad 0\,]\,,\\ \underline{B}_1^* &= [\,-C_1S_2(C_6S_7+S_6C_7c_7) \quad S_2S_6s_7 \quad -S_1C_2(C_6S_7+S_6C_7c_7) \quad C_1S_2S_6s_7\\ &\qquad S_2(C_6S_7+S_6C_7c_7) \quad S_1C_2S_6s_7 \quad -S_1S_2(C_6C_7-S_6S_7c_7) \quad 0\\ &\qquad C_1C_2(C_6C_7-S_6S_7c_7)-C_3C_4C_5\,]\,.\end{aligned}\right\} \tag{5.83}$$

The second closure condition with F_2 is the dual derivative of the first equation. It is more complicated. Only the left-hand side expression is given:

$$\begin{aligned}&-\ell_5S_5(C_4C_3-S_4S_3c_4)+C_5[-\ell_4S_4C_3-\ell_3C_4S_3-(\ell_4C_4S_3+\ell_3S_4C_3)c_4+h_4S_4S_3s_4]\\&+\ell_5C_5[s_5s_4S_3-c_5(S_4C_3+C_4S_3c_4)]+S_5\{h_5c_5s_4S_3+h_4s_5c_4S_3+\ell_3s_5s_4C_3\\&+h_5s_5(S_4C_3+C_4S_3c_4)-c_5[\ell_4C_4C_3-\ell_3S_4S_3+(-\ell_4S_4S_3+\ell_3C_4C_3)c_4-h_4C_4S_3s_4]\}\\&=\dots\quad \text{(dual derivative of the right-hand side expression of (5.82))}\,.\end{aligned} \tag{5.84}$$

The two closure conditions with $F_3=\mathbf{n}_3\cdot\mathbf{r}$ and $F_4=\mathbf{n}_6\cdot\mathbf{r}$ read

$$\begin{aligned}&h_3+\underbrace{\mathbf{n}_k\cdot(h_4\mathbf{n}_{k+1}+\ell_4\mathbf{a}_{k+1}+h_5\mathbf{n}_{k+2}+\ell_5\mathbf{a}_{k+2})}_{k=3}\\&=-h_6\underbrace{\mathbf{n}_{k-2}\cdot\mathbf{n}_{k+2}}_{k=1}-\underbrace{\mathbf{n}_k\cdot(\ell_6\mathbf{a}_{k-4}+h_7\mathbf{n}_{k-3}+\ell_7\mathbf{a}_{k-3}+h_1\mathbf{n}_{k-2}+\ell_1\mathbf{a}_{k-2}+h_2\mathbf{n}_{k-1})}_{k=3}\,,\end{aligned} \tag{5.85}$$

$$\begin{aligned}&\underbrace{\mathbf{n}_k\cdot(h_3\mathbf{n}_{k-3}+\ell_3\mathbf{a}_{k-3}+h_4\mathbf{n}_{k-2}+\ell_4\mathbf{a}_{k-2}+h_5\mathbf{n}_{k-1})}_{k=6}\\&=-h_6-\underbrace{\mathbf{n}_k\cdot(h_7\mathbf{n}_{k+1}+\ell_7\mathbf{a}_{k+1}+h_1\mathbf{n}_{k+2}+\ell_1\mathbf{a}_{k+2}+h_2\mathbf{n}_{k+3}+\ell_2\mathbf{a}_{k+3})}_{k=6}\,.\end{aligned} \tag{5.86}$$

The scalar products are copied from Table 5.2 . Simple re-arrangements result in the following equations

$$\begin{aligned}&h_3+C_3(\ell_5s_5S_4+h_5C_4+h_4)+S_3s_4(\ell_5c_5+\ell_4)+S_3c_4(\ell_5s_5C_4-h_5S_4)\\&=-h_2C_2-(C_2C_1-S_2S_1c_2)(\ell_6s_7S_7+h_7C_7+h_1)\\&\quad -s_2S_2[\ell_6(c_7c_1-s_7s_1C_7)+h_7S_7s_1+\ell_7c_1+\ell_1]\\&\quad +(C_2S_1+S_2C_1c_2)[-\ell_6(c_7s_1+s_7c_1C_7)+h_7S_7c_1-\ell_7s_1]\\&\quad -h_6\times\ \text{right-hand side expression of (5.82)}\,,\end{aligned} \tag{5.87}$$

$$\begin{aligned}&h_5C_5+(C_5C_4-S_5S_4c_5)h_4+S_5s_5(\ell_4+\ell_3c_4)+(C_5S_4+S_5C_4c_5)\ell_3s_4\\&+h_3\times\ \text{left-hand side expression of (5.82)}\\&=-h_6-h_7C_6-\ell_7s_7S_6-(C_6C_7-S_6S_7c_7)(h_1+h_2C_1+\ell_2s_2S_1)\\&\quad -[C_6S_7s_1+S_6(s_7c_1+c_7s_1C_7)](\ell_1+\ell_2c_2)\\&\quad -[C_6S_7c_1-S_6(s_7s_1-c_7c_1C_7)](-h_2S_1+\ell_2s_2C_1)\,.\end{aligned} \tag{5.88}$$

The four Eqs.(5.82), (5.84), (5.87) and (5.88) for the unknowns φ_1, φ_2, φ_4, φ_5 are Eqs.(5.79) written in detail. Following Lee [27] the reduction to a 16th-order polynomial equation for a single unknown is achieved as follows. First, the two Woernle-Lee Eqs.(5.16) with $F_6 = \mathbf{r}^2$ and with $F_7 = \frac{1}{2}(\mathbf{n}_3 \cdot \mathbf{n}_6)\mathbf{r}^2 - (\mathbf{n}_3 \cdot \mathbf{r})(\mathbf{n}_6 \cdot \mathbf{r})$ are formulated. These equations have, with new coefficient matrices, the form (5.79). The altogether six equations are combined in the matrix form

$$\underline{A}_1 \underline{u}_\ell = \underline{B}_1 \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \tag{5.89}$$

with coefficient matrices $\underline{A}_1$ and $\underline{B}_1$ of size (6×8) and (6×9), respectively. They are functions of the independent variable φ_7.

Next, four pairs of half-angle equations (5.37) are formulated with the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ shown in (5.38) and with $a = 3$ and $b = 6$:

$$\left.\begin{aligned} x_6\, \mathbf{v}_i \cdot \mathbf{a}_6 + \quad \mathbf{v}_i \cdot \mathbf{n}_6 \times \mathbf{a}_6 &= -x_b\, \mathbf{v}_i \cdot \mathbf{a}_5 + \quad \mathbf{v}_i \cdot \mathbf{n}_6 \times \mathbf{a}_5 \;, \\ \mathbf{v}_i \cdot \mathbf{a}_6 - x_6\, \mathbf{v}_i \cdot \mathbf{n}_6 \times \mathbf{a}_6 &= \quad \mathbf{v}_i \cdot \mathbf{a}_5 + x_6\, \mathbf{v}_i \cdot \mathbf{n}_6 \times \mathbf{a}_5 \end{aligned}\right\} (i = 1, 2, 3, 4) \;, \tag{5.90}$$

$$\mathbf{v}_1 = \mathbf{n}_3 \;, \qquad \mathbf{v}_2 = \mathbf{r} \;, \qquad \mathbf{v}_3 = \mathbf{n}_3 \times \mathbf{r} \;, \qquad \mathbf{v}_4 = \frac{1}{2}\,\mathbf{r}^2 \mathbf{n}_3 - (\mathbf{n}_3 \cdot \mathbf{r})\mathbf{r} \;. \tag{5.91}$$

In each of these eight equations the elements of $\underline{u}_\ell$ appear on the left-hand side and the elements of $\underline{u}_r$ on the right-hand side. All these elements occur once without and once with the factor $x_6 = \tan\varphi_6/2$. The eight equations are combined in the matrix form

$$(\underline{A}_2 + x_6 \underline{A}_3)\underline{u}_\ell = (\underline{B}_2 + x_6 \underline{B}_3) \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \tag{5.92}$$

with coefficient matrices $\underline{A}_2$ and $\underline{A}_3$ of size (8×8) and $\underline{B}_2$ and $\underline{B}_3$ of size (8×9). They are functions of φ_7. Another six equations are produced by multiplying (5.89) with x_6. These six equations together with the six Eqs.(5.89) and the eight Eqs.(5.92) represent a system of twenty equations. It is written in the form

$$\underbrace{\begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_1 \\ \underline{A}_2 & \underline{A}_3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} \underline{u}_\ell \\ x_6 \underline{u}_\ell \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} \underline{B}_1 & \underline{0} \\ \underline{0} & \underline{B}_1 \\ \underline{B}_2 & \underline{B}_3 \end{bmatrix}}_{\underline{B}} \underbrace{\begin{bmatrix} \underline{u}_r \\ 1 \\ x_6 \underline{u}_r \\ x_6 \end{bmatrix}}_{\underline{z}} \qquad \text{or} \qquad \underline{A}\,\underline{y} = \underline{B}\,\underline{z}\;. \tag{5.93}$$

The coefficient matrices $\underline{A}$ and $\underline{B}$ are of size (20×16) and (20×18), respectively. Sixteen out of these twenty equations are solved for $\underline{y}$ in terms of $\underline{z}$. The resulting expression is substituted into the last four equations. These four equations are then of the form

$$\underline{P}\,\underline{z} = \underline{0} \tag{5.94}$$

with a (4×18)-matrix $\underline{P}$. They are formally identical with (5.73). Also (5.74) and (5.75) and the conditions on the coefficient matrix in (5.75) are valid again (the only difference being that now φ_7 is the independent variable instead of φ_5 and that a larger number of constant Denavit-Hartenberg parameters is involved). The condition that the determinant Δ must be zero results in a 16th-order equation for $x_1 = \tan\varphi_1/2$ with coefficients depending on φ_7. Thus, it is proved that the mechanism 7R has at most sixteen configurations for a given value of the independent variable φ_7. The unknown angles φ_1, φ_2 and φ_6 are calculated as was shown following (5.77). With φ_1, φ_2 and φ_6 the column matrix $\underline{u}_r$ in the four Eqs.(5.79) is known. These equations are solved for $\underline{u}_\ell$. This solution determines the angles φ_1 and φ_2. The last unknown φ_3 is found by the method explained in the section on the mechanism 5R-C. This concludes the analysis of the mechanism 7R.

5.4.8 4R-3P. Independent Variable is an Angle

These are the mechanisms in row 4 of Table 5.1. At the end of Sect. 5.3.1 on Woernle-Lee equations it was said that the closure conditions $F_k^\ell = F_k^r$ $(k = 1, 2, 3, 4)$ are formulated not only for the mechanism 7R, but also for the mechanisms 4R-3P. These mechanisms have seven joints and, consequently, the vector polygon shown in Fig. 5.7. The mechanism 7R is converted into a mechanism 4R-3P by replacing three out of the four revolute joints 1, 2, 4 and 5 by prismatic joints. This has the effect that in the four Eqs.(5.82), (5.84), (5.87) and (5.88) three out of the variables φ_1, φ_2, φ_4 and φ_5 are (arbitrary) constants. Only one of them, say φ_j, is still a variable. For this variable (5.82) becomes an equation of the form $Ac_j + Bs_j = R$. It has two solutions for every value of the independent variable φ_7. With each solution the remaining three Eqs.(5.84), (5.87) and (5.88) become linear equations with known coefficients for the translatory variables in the three prismatic joints. This concludes the analysis of the mechanisms 4R-3P.

5.4.9 6R-P. Independent Variable is an Angle

The mechanism 6R-P has the same vector polygon the mechanism 7R has (Fig. 5.7). The mechanism 7R is converted into the mechanism 6R-P by replacing a single revolute joint by a prismatic joint. Let this be joint 5. Again, the four Eqs.(5.82), (5.84), (5.87) and (5.88) are used. As before, φ_7 is the independent variable. But now, φ_5 is a constant and h_5 is a variable. The only places where this variable appears explicitly, are the left-hand sides of (5.84), (5.87) and (5.88). These three equations have the form (5.23) with

$\lambda = 1$, $\mu = 2$, $\nu = 4$ and $\kappa = 5$. These three equations and (5.82) are combined in the matrix form

$$\underline{A}_1 \begin{bmatrix} \underline{u}_\ell \\ h_5 \underline{u}_\ell \\ h_5 \end{bmatrix} = \underline{B}_1 \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} \tag{5.95}$$

with the column matrices

$$\underline{u}_\ell = [c_4 \quad s_4\,]^T, \quad \underline{u}_r = [c_1 c_2 \quad c_1 s_2 \quad c_1 \quad s_1 c_2 \quad s_1 s_2 \quad s_1 \quad c_2 \quad s_2\,]^T . \tag{5.96}$$

The coefficient matrices $\underline{A}_1$ and $\underline{B}_1$ are of size (4×5) and (4×9), respectively. These four equations are supplemented by three pairs of half-angle equations (5.37) formulated with the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ shown in (5.38). In each of these altogether six equations the column matrices of (5.95) appear once without and once with the factor $x_6 = \tan \varphi_6/2$. The six equations are combined in the matrix form

$$(\underline{A}_2 + x_6 \underline{A}_3) \begin{bmatrix} \underline{u}_\ell \\ h_5 \underline{u}_\ell \\ h_5 \end{bmatrix} = (\underline{B}_2 + x_6 \underline{B}_3) \begin{bmatrix} \underline{u}_r \\ 1 \end{bmatrix} . \tag{5.97}$$

The coefficient matrices $\underline{A}_2$ and $\underline{A}_3$ are of size (6×5) and $\underline{B}_2$ and $\underline{B}_3$ are of size (6×9). Another four equations are produced by multiplying (5.95) with x_6. These four equations together with the four Eqs.(5.95) and the six Eqs.(5.97) represent a system of fourteen equations. It is written in the form

$$\underbrace{\begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_1 \\ \underline{A}_2 & \underline{A}_3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} \underline{u}_\ell \\ h_5 \underline{u}_\ell \\ h_5 \\ x_6 \underline{u}_\ell \\ x_6 h_5 \underline{u}_\ell \\ x_6 h_5 \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} \underline{B}_1 & \underline{0} \\ \underline{0} & \underline{B}_1 \\ \underline{B}_2 & \underline{B}_3 \end{bmatrix}}_{\underline{B}} \underbrace{\begin{bmatrix} \underline{u}_r \\ 1 \\ x_6 \underline{u}_r \\ x_6 \end{bmatrix}}_{\underline{z}} \quad \text{or} \quad \underline{A}\,\underline{y} = \underline{B}\,\underline{z} . \tag{5.98}$$

The coefficient matrices $\underline{A}$ and $\underline{B}$ are of size (14×10) and (14×18), respectively. Ten out of these fourteen equations are solved for $\underline{y}$ in terms of $\underline{z}$. The resulting expression is substituted into the last four equations. These four equations are then of the form

$$\underline{P}\,\underline{z} = \underline{0} \tag{5.99}$$

with a (4×18)-matrix $\underline{P}$. They are formally identical with (5.73). The reduction to a 16th-order equation proceeds as before. Thus, it is proved that the mechanism 6R-P has at most sixteen configurations for a given value of the independent variable φ_7. The unknown angles φ_1, φ_2 and φ_6 are calculated as was shown following (5.77). With these results the column matrix $\underline{z}$ in the fourteen Eqs.(5.98) is known. The ten equations already mentioned yield $\underline{y}$ and with it the variables φ_4 and h_5. The last two unknowns φ_3 and φ_6 can be determined either from half-angle equations or from Woernle-Lee equations $F_1^\ell = F_1^r$. See the sections on the mechanisms RCRCR and 5R-C. This concludes the analysis of the mechanism 6R-P when the independent variable is an angle.

5.4.10 6R-P. Independent Variable in the Prismatic Joint

Again, the vector polygon shown in Fig. 5.7 is used. This time, joint 7 is the prismatic joint. As in the analysis of the mechanism 7R the same four Eqs.(5.82), (5.84), (5.87), (5.88), the same two Woernle-Lee equations with F_6 and F_7 and the same four pairs of half-angle equations are formulated. The only difference is that in these equations φ_7 is now constant whereas h_7 is the independent variable. The only unknowns are, as before, the angles φ_1, φ_2, φ_4 and φ_5. The final result of the analysis is, again, a 16th-order polynomial equation for $x_1 = \tan\varphi_1/2$. This explains the number $N_t = 16$ in Table 5.1 for this mechanism.

5.5 Mechanisms with Special Parameter Values

In the analysis of every mechanism up to now it was assumed that all constant Denavit-Hartenberg parameters are nonzero and arbitrary. The large number of parameters (between nine and twenty-one) allows for an enormous number of special cases. The Bennett mechanism and the spherical four-bar introduced in Sect. 5.4.1 are special cases of the mechanism RCCC. Table 5.2 shows that parallelity or orthogonality of the joint axes on a single body k results in substantial simplifications ($S_k = 0$, $C_k = 1$ in the former case and $S_k = 1$, $C_k = 0$ in the latter). In the following sections two special mechanisms are investigated.

5.5.1 7R with Three Parallel Joint Axes in Series

Subject of investigation is a mechanism 7R with three parallel successive joint axes, say $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$. The parallelity has the effect that the bodies 1, 2, 3 and 4 are in planar motion relative to each other. For the Denavit-

Hartenberg parameters this parallelity means $\alpha_2 = \alpha_3 = 0$ and, therefore, $S_2 = S_3 = 0$, $C_2 = C_3 = 1$. When this is substituted into (5.82), the unknowns φ_2 and φ_4 disappear. For the two remaining unknowns φ_1 and φ_5 the equation has the form

$$S_4 S_5 c_5 = A_1 c_1 + B_1 s_1 + R_1 \tag{5.100}$$

with coefficients

$$\left.\begin{aligned} A_1 &= S_1(C_6 S_7 + S_6 C_7 c_7)\,, \qquad B_1 = -S_1 S_6 s_7\,, \\ R_1 &= C_4 C_5 - C_1(C_6 C_7 - S_6 S_7 c_7)\,. \end{aligned}\right\} \tag{5.101}$$

Except for changes of indices (5.100) is identical with (5.56). Equation (5.84) is replaced by the dual derivative of (5.100). This equation is Eq.(5.58) with the same changes of indices. It was shown that these equations have four solutions (φ_1, φ_5) for every value of the independent variable φ_7. This ends the analysis of the special case.

5.5.2 RRSRR

The mechanism RRSRR is another special case of a mechanism 7R. The letter S stands for the spherical joint which was explained in Sect. 5.1. The polygon of vectors is shown in Fig. 5.8. Bodies 1, 2, 3, 4, 5 are coupled by revolute joints 1, 2, 3, 5 and by the spherical joint. Unit vectors $\mathbf{n}_i$ $(i = 1, 2, 3, 5)$ and $\mathbf{a}_i$ $(i = 1, 2, 5)$ are located on joint axes and on normals common to pairs of joint axes, respectively. The vectors $\ell_3 \mathbf{a}_3$ and $\ell_4 \mathbf{a}_4$ denote the normals from the spherical joint onto the joint axes 3 and 5, respectively. The mechanism has twelve constant parameters, namely, h_i $(i = 1, 2, 3, 5)$ in revolute joints and ℓ_i $(i = 1, \ldots, 5)$, α_i $(i = 1, 2, 5)$ on bodies. Variables are the angles φ_1, φ_2, φ_3, φ_5 in revolute joints and, in addition, three angles associated with the spherical joint. These latter ones are not considered.

The symmetry allows the same conclusions which were drawn from the symmetry of Fig. 5.3. The independent variable is either φ_5 or φ_1. Because of the symmetry only these two cases need be considered. As joints a and b in the sense of Fig. 5.2 the spherical joint and joint 3 are chosen. One of the two segments between the joints a and b has no joint variables. The other segment has the joint variables φ_1, φ_2 and φ_5. Two of these are unknowns. These two are determined from the two Woernle-Lee Eqs.(5.16) with $F_3 = \mathbf{n}_3 \cdot \mathbf{r}$ and $F_6 = \mathbf{r}^2$. The vector $\mathbf{r}$ is shown in Fig. 5.8:

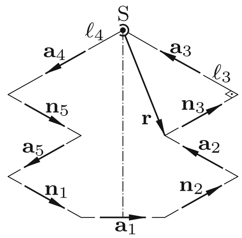

Fig. 5.8 Mechanism RRSRR with vector polygon

$$\mathbf{r} = \begin{cases} \ell_4\mathbf{a}_4 + h_5\mathbf{n}_5 + \ell_5\mathbf{a}_5 + h_1\mathbf{n}_1 + \ell_1\mathbf{a}_1 + h_2\mathbf{n}_2 + \ell_2\mathbf{a}_2 & \text{(left segment)} \\ -(h_3\mathbf{n}_3 + \ell_3\mathbf{a}_3) & \text{(right segment)}\ . \end{cases} \tag{5.102}$$

Taking into account the orthogonality $\mathbf{n}_3 \cdot \mathbf{a}_3 = 0$ the closure conditions are

$$\mathbf{n}_3 \cdot (\ell_4\mathbf{a}_4 + h_5\mathbf{n}_5 + \ell_5\mathbf{a}_5 + h_1\mathbf{n}_1 + \ell_1\mathbf{a}_1 + h_2\mathbf{n}_2 + \ell_2\mathbf{a}_2) = -h_3\ , \tag{5.103}$$

$$(\ell_4\mathbf{a}_4 + h_5\mathbf{n}_5 + \ell_5\mathbf{a}_5 + h_1\mathbf{n}_1 + \ell_1\mathbf{a}_1 + h_2\mathbf{n}_2 + \ell_2\mathbf{a}_2)^2 = \ell_3^2 + h_3^2\ . \tag{5.104}$$

In (5.104) the seven quadratic terms yield the constant $(\ell_4^2 + h_5^2 + \ldots + \ell_2^2)$. Each vector is orthogonal to its right-hand neighbor. This determines the products $\mathbf{a}_4 \cdot \mathbf{n}_5 = \mathbf{n}_5 \cdot \mathbf{a}_5 = \mathbf{a}_5 \cdot \mathbf{n}_1 = \mathbf{n}_1 \cdot \mathbf{a}_1 = \mathbf{a}_1 \cdot \mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{a}_2 = 0$. Between each vector and its second neighbor to the right either a joint angle or a constant angle is located. This determines the five products $\mathbf{a}_4 \cdot \mathbf{a}_5 = c_5$, $\mathbf{n}_5 \cdot \mathbf{n}_1 = C_5$, $\mathbf{a}_5 \cdot \mathbf{a}_1 = c_1$, $\mathbf{n}_1 \cdot \mathbf{n}_2 = C_1$, $\mathbf{a}_1 \cdot \mathbf{a}_2 = c_2$. For calculating the remaining products the vectors are decomposed in a basis fixed on body 1. With $k = 1$ the vectors are $\mathbf{a}_4 = \mathbf{a}_{k-2}$, $\mathbf{n}_5 = \mathbf{n}_{k-1}$, $\mathbf{a}_5 = \mathbf{a}_{k-1}$, $\mathbf{n}_1 = \mathbf{n}_k$, $\mathbf{a}_1 = \mathbf{a}_k$, $\mathbf{n}_2 = \mathbf{n}_{k+1}$, $\mathbf{a}_2 = \mathbf{a}_{k+1}$, $\mathbf{n}_3 = \mathbf{n}_{k+2}$. The coordinates are copied[4] from Table 5.2 into the following Table 5.3.

Table 5.3 Coordinates of vectors in (5.103) and (5.104)

$\mathbf{a}_4$	$\mathbf{n}_5$	$\mathbf{a}_5$	$\mathbf{n}_1$	$\mathbf{a}_1$	$\mathbf{n}_2$	$\mathbf{a}_2$	$\mathbf{n}_3$
s_5S_5	C_5	0	1	0	C_1	s_2S_1	$C_2C_1 - S_2S_1c_2$
$c_5c_1 - s_5s_1C_5$	S_5s_1	c_1	0	1	0	c_2	S_2s_2
$-(c_5s_1 + s_5c_1C_5)$	S_5c_1	$-s_1$	0	0	$-S_1$	s_2C_1	$-(C_2S_1 + S_2C_1c_2)$

With these coordinates (5.103) and (5.104) take the forms

$$c_\lambda(a_{i1}c_2 + a_{i2}s_2 + a_{i3}) + s_\lambda(b_{i1}c_2 + b_{i2}s_2 + b_{i3}) + (r_{i1}c_2 + r_{i2}s_2 + r_{i3}) = 0 \tag{5.105}$$

[4] Instead of using Table 5.3 the scalar products can be obtained directly from Table 5.2. Example: $\mathbf{n}_5 \cdot \mathbf{n}_2 = \mathbf{n}_k \cdot \mathbf{n}_{k+2}$ with $k = 5$ yields $C_1C_5 - S_1S_5c_1$

$(i=1,2)$. The index λ equals 5 if φ_1 is the independent variable and it equals 1 if φ_5 is the independent variable. The coefficients $a_{11},\ldots,r_{23}$ are functions of the independent variable (φ_1 or φ_5). They are given further below. The equations are identical with (5.62). The number of solutions for a given value of the independent variable is eight in the general case and four if the coefficients of c_2 and s_2 satisfy the six conditions (5.65). In the present case, these conditions are satisfied. This is shown as follows. If φ_1 is the independent variable, the auxiliary variables are defined:

$$\left.\begin{aligned} q_1 &= C_1C_5c_1 - S_1S_5 \ , \\ q_2 &= h_5S_5s_1 + \ell_5c_1 + \ell_1 \ , \\ q_3 &= h_5(C_1S_5c_1 + S_1C_5) - \ell_5C_1s_1 + h_1S_1 \ . \end{aligned}\right\} \tag{5.106}$$

In terms of these variables the coefficients of c_2 and s_2 are

$$\left.\begin{aligned} a_{11} &= \ell_2\ell_4c_1 \ , & a_{12} &= -\ell_2\ell_4C_1s_1 \ , \\ a_{21} &= S_2\ell_4C_1s_1 \ , & a_{22} &= S_2\ell_4c_1 \ , \\ b_{11} &= -\ell_2\ell_4C_5s_1 \ , & b_{12} &= -\ell_2\ell_4q_1 \ , \\ b_{21} &= S_2\ell_4q_1 \ , & b_{22} &= -S_2\ell_4C_5s_1 \ , \\ r_{11} &= \ell_2q_2 \ , & r_{12} &= \ell_2q_3 \ , \\ r_{21} &= -S_2q_3 \ , & r_{22} &= S_2q_2 \ . \end{aligned}\right\} \tag{5.107}$$

If φ_5 is the independent variable, new auxiliary variables are defined as follows:

$$q_1 = \ell_4c_5 + \ell_5 \ , \quad q_2 = \ell_4C_5s_5 - h_5S_5 \ , \quad q_3 = \ell_4S_5s_5 + h_5C_5 + h_1 \ . \tag{5.108}$$

In terms of these variables the new coefficients of c_2 and s_2 are

$$\left.\begin{aligned} a_{11} &= \ell_2q_1 \ , & a_{12} &= -\ell_2C_1q_2 \ , \\ a_{21} &= S_2C_1q_2 \ , & a_{22} &= S_2q_1 \ , \\ b_{11} &= -\ell_2q_2 \ , & b_{12} &= -\ell_2C_1q_1 \ , \\ b_{21} &= S_2C_1q_1 \ , & b_{22} &= -S_2q_2 \ , \\ r_{11} &= \ell_2\ell_1 \ , & r_{12} &= \ell_2S_1q_3 \ , \\ r_{21} &= -S_2S_1q_3 \ , & r_{22} &= S_2\ell_1 \ . \end{aligned}\right\} \tag{5.109}$$

In either case the conditions (5.65) are satisfied. The proof by inspection is elementary. This concludes the analysis of the mechanism RRSRR.

5.6 Generalized Velocities. Generalized Accelerations

The time derivatives of the seven variables φ_i and h_j are called generalized velocities $\dot{\varphi}_i$ and $\dot{h}_j$, respectively. One of them is independent and the other six are dependent. The time derivatives of six suitably chosen closure conditions constitute a system of six homogeneous linear equations for the seven generalized velocities. The solution of this system expresses each of the six dependent generalized velocities as some multiple of the independent generalized velocity. Suitable closure conditions are Woernle-Lee equations which are used also for the determination of the dependent variables. In what follows, details are shown for the mechanisms RCCC and 7R.

5.6.1 RCCC

The mechanism RCCC with the labeling of joints shown in Fig. 5.3 has the seven generalized velocities $\dot\varphi_1$, $\dot\varphi_2$, $\dot\varphi_3$, $\dot\varphi_4$, $\dot h_2$, $\dot h_3$, $\dot h_4$. For expressing six of them as multiples of the independent velocity $\dot\varphi_1$ six linear equations are required. The simplest equations are the total time derivatives of (5.43), (5.45), (5.50) and (5.51) and of (5.50), (5.51) with all indices increased by one. The time derivatives of (5.43) and (5.45) are

$$\left.\begin{aligned} \dot\varphi_4(-As_4+Bc_4) &= \dot\varphi_1(-A'c_4-B's_4+R')\ ,\\ \dot h_4(Bc_4-As_4)+\dot\varphi_4[h_4(-Bs_4-Ac_4)+Ds_4-Ec_4]&\\ =\dot\varphi_1[-h_4(B'c_4-A's_4)+D'c_4+E's_4+F']\ .& \end{aligned}\right\} \tag{5.110}$$

The scalars A', B', R', D', E', F' are the partial derivatives of A, B, R, D, E, F from (5.44) and (5.46) with respect to φ_1:

$$\left.\begin{aligned} &A'=S_3C_4S_1s_1\ ,\qquad B'=S_1S_3c_1\ ,\qquad R'=-C_3S_4S_1s_1\ ,\\ &D'=(\ \ell_4S_3S_1S_4-\ell_1S_3C_1C_4-\ell_3C_3S_1C_4)s_1-h_1S_1S_3C_4c_1\ ,\\ &F'=(-\ell_4C_3S_1C_4-\ell_1C_3C_1S_4+\ell_3S_3S_1S_4)s_1-h_1S_1C_3S_4c_1\ ,\\ &E'=-(\ell_1C_1S_3+\ell_3S_1C_3)c_1+h_1S_1S_3s_1\ . \end{aligned}\right\} \tag{5.111}$$

The time derivatives of (5.50) and (5.51) are

$$\left.\begin{aligned} \dot\varphi_3S_3S_2s_3 &= \dot\varphi_1S_1S_4s_1\ ,\\ \dot h_3S_2S_3s_3+\dot\varphi_3[h_3S_2S_3c_3+s_3(\ell_2C_2S_3+\ell_3S_2C_3)]&\\ =\dot\varphi_1[h_1S_1S_4c_1+s_1(\ell_1C_1S_4+\ell_4S_1C_4)]\ .& \end{aligned}\right\} \tag{5.112}$$

5.6.2 Mechanism 7R

The mechanism 7R has seven generalized velocities $\dot\varphi_1,\ldots,\dot\varphi_7$. The simplest closure condition is Eq.(5.82). Its matrix form is $\underline{A}_1^*\underline{u}_\ell = \underline{B}_1^*[\underline{u}_r^T \quad 1]^T$. The matrices $\underline{A}_1^*$, $\underline{B}_1^*$, $\underline{u}_\ell$ and $\underline{u}_r$ are given in (5.83) and (5.80). The time derivative of the equation is

$$\begin{aligned}
&\dot\varphi_4\underline{A}_1^*\,[\,-s_4c_5 \quad -s_4s_5 \quad -s_4 \quad c_4c_5 \quad c_4s_5 \quad c_4 \quad 0 \quad 0\,]^T \\
&+\dot\varphi_5\underline{A}_1^*\,[\,-c_4s_5 \quad c_4c_5 \quad 0 \quad -s_4s_5 \quad s_4c_5 \quad 0 \quad -s_5 \quad c_5\,]^T \\
&=\dot\varphi_1\underline{B}_1^*\,[\,-s_1c_2 \quad -s_1s_2 \quad -s_1 \quad c_1c_2 \quad c_1s_2 \quad c_1 \quad 0 \quad 0 \quad 0\,]^T \\
&+\dot\varphi_2\underline{B}_1^*\,[\,-c_1s_2 \quad c_1c_2 \quad 0 \quad -s_1s_2 \quad s_1c_2 \quad 0 \quad -s_2 \quad c_2 \quad 0\,]^T \\
&+\dot\varphi_7\underline{B}_1^{*\prime}[\,c_1c_2 \quad c_1s_2 \quad c_1 \quad s_1c_2 \quad s_1s_2 \quad s_1 \quad c_2 \quad s_2 \quad 1\,]^T\,.
\end{aligned} \tag{5.113}$$

The matrix $\underline{B}_1^{*\prime}$ is the partial derivative of $\underline{B}_1^*$ with respect to φ_7 :

$$\begin{aligned}
\underline{B}_1^{*\prime} = S_6[\,&C_1S_2C_7s_7 \quad S_2c_7 \quad S_1C_2C_7s_7 \quad C_1S_2c_7 \\
&-S_2C_7s_7 \quad S_1C_2c_7 \quad -S_1S_6S_7s_7 \quad 0 \quad C_1C_2S_7s_7\,]\,.
\end{aligned} \tag{5.114}$$

The other five equations are produced from this equation by a cyclic increase of all indices by one, by two, ..., by five (not the index 1 of $\underline{A}_1^*$ and $\underline{B}_1^*$, but all indices in $\underline{A}_1^*$ and $\underline{B}_1^*$).

It is a simple task to formulate the second time derivative. The result is a system of six linear equations for generalized accelerations $\ddot\varphi_1,\ldots,\ddot\varphi_7$. These equations contain additional terms with $\dot\varphi_i\dot\varphi_j$ $(i,j=1,2,4,5,7)$.

5.7 Spatial Serial Robots

A spatial serial robot consists of a stationary base, a robot hand and a serial kinematical chain connecting hand and base. The serial chain is the arm of the robot. With a suitable combination of cylindrical, revolute and prismatic joints in the arm with altogether six joint variables the hand has relative to the base three rotational and three translatory degrees of freedom. The minimal number of joints is three with the joint combination 3C and the maximal number is six with the joint combination 6R . The bodies are labeled $1,\ldots,n$ with the base being body 1 and the hand being body n . The joints are labeled $1,\ldots,n-1$ beginning at the base. The problem to be solved is the following. The pose, i.e., the position and the angular orientation of the hand relative to the base, is prescribed in terms of six variables of unspecified nature, for example, by three coordinates of a single point plus three angular variables. Determine all sets of six joint variables producing this prescribed pose. The solution is found as follows. In a preparatory step the six prescribed variables are converted into another set of six variables which are defined as follows. In the prescribed pose bodies n and 1 are imagined to be connected by a revolute joint labeled n with an axis of arbitrarily chosen location and direction and with locked joint variable φ_n . The six new variables are the six Denavit-Hartenberg parameters defined by this joint, namely, α_n , ℓ_n on body n , φ_n and h_n in joint n and α_1 , ℓ_1 on body 1 . Together with the real joints of the robot this fictitious joint creates a spatial single-loop mechanism with given constant Denavit-Hartenberg parameters and with a given value of the joint variable φ_n. Thus, the problem to be solved is the following. Determine six dependent joint variables of a spatial single-loop mechanism for a given value of a single independent variable φ_n. The complete solution is known from the previous sections. Table 5.4 is deduced from Table 5.1. Each of the ten rows shows the joint combination of the respective row in Table 5.1 with one revolute joint deleted. The deleted joint is the fictitious joint. The ten joint combinations represent all possible robot arms giving

Table 5.4 Serial robots with three rotational and three translatory degrees of freedom of the hand

	joint combin.	ν_r	N_φ	
1	3C	1	2	
2	2C-R-P	12	2	independent of joint sequence
3	C-2R-2P	30	2	independent of joint sequence
4	3R-3P	20	2	independent of joint sequence
5	2C-2R	6	4 8	(sequences CRCR, RCRC) (sequences CCRR, RRCC, RCCR, CRRC)
6	C-3R-P	20	8	independent of joint sequence
7	4R-2P	15	8	independent of joint sequence
8	C-4R	5	16	independent of joint sequence
9	5R-P	6	16	independent of joint sequence
10	6R	1	16	

the hand six degrees of freedom. The joints of a given joint combination can be ordered along the robot arm from base to hand in many different ways. Different sequences of letters represent different robots. Examples are the sequences (from base to hand) CRRPP, CRPRP, RRPPC etc. with the joint combination C-2R-2P. Let ν_r be the number of different robots that can be built with a given joint combination. It is calculated as follows. Let ν_C, ν_R and ν_P be the numbers of cylindrical, of revolute and of prismatic joints, respectively. The total number of joints in the arm is $\nu = \nu_\mathrm{C} + \nu_\mathrm{R} + \nu_\mathrm{P}$. With these numbers the number ν_r is

$$\nu_\mathrm{r} = \binom{\nu}{\nu_\mathrm{C}} \binom{\nu - \nu_\mathrm{C}}{\nu_\mathrm{R}} . \tag{5.115}$$

The pair of numbers ν_C, ν_R in this formula can be replaced by the pair ν_C, ν_P and also by the pair ν_R, ν_P.

Example: The joint combination C-2R-2P yields $\nu_\mathrm{r} = \binom{5}{1}\binom{4}{2} = 30$. In Table 5.4 the number ν_r is given for every joint combination.

The number N_φ is copied from Table 5.1 . It represents the number of (real or complex) six-tuples of joint variables for a given pose of the robot hand. Every real six-tuple determines an arm configuration producing the given pose. From Table 5.1 it is known that in row 5 the number N_φ is either four or eight depending on which revolute joint carries the independent angular variable. In Table 5.4 the independent variable is φ_5 in the fictitious joint 5 . This explains the correspondence between the numbers $N_\varphi = 4$ and 8 and the various joint sequences. For all other joint combinations the number N_φ is independent of the sequence of joints.

Lee [27] investigated the following problem. The hand of a 6R-robot is in pure translation with a point P of the hand moving along a given straight line. Let z be the coordinate of P along this line. The six Denavit-Hartenberg parameters α_n , ℓ_n , φ_n , h_n , α_1 , ℓ_1 of the fictitious joint are functions of z . The number of real solutions for the six joint variables, i.e., of arm configurations is a function of z , too. This function divides the z-axis into intervals with different numbers of arm configurations. With the parameter values chosen by Lee z-intervals with the maximum number $N_\varphi = 16$ were found.

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