Chapter 3
Finite Screw Displacement

Subject of this chapter are relationships between two positions of a rigid body without a fixed point. These two positions, referred to as initial and final position, respectively, are assumed to be arbitrary subject only to the restriction that the final position cannot be produced from the initial position by pure translation. Motions leading from the initial to the final position are not investigated.

3.1 (4 × 4) Transformation Matrix

In Fig. 3.1 the most general displacement of a rigid body is shown. The body is represented by a body-fixed basis $\underline{\mathbf{e}}^2$. In the initial position $\underline{\mathbf{e}}^2$ coincides with a reference basis $\underline{\mathbf{e}}^1$ with origin 0_1 . The displacement to the final position of $\underline{\mathbf{e}}^2$ with origin 0_2 is the result of a rotation $(\mathbf{n},\ \varphi)$ about 0_1 followed by the translatory displacement $\mathbf{r} = \overrightarrow{0_1 0_2}$. The position of $\underline{\mathbf{e}}^2$ after the rotation and prior to translation is referred to as intermediate position (shown in dotted lines). In the final position a body-fixed point Q with position vector $\boldsymbol{\varrho}_2$ in $\underline{\mathbf{e}}^2$ has in $\underline{\mathbf{e}}^1$ the position vector

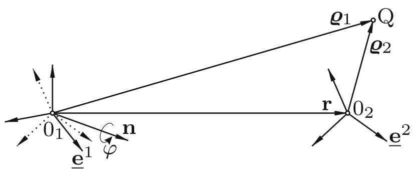

Fig. 3.1 Initial, intermediate and final positions of basis $\underline{\mathbf{e}}^2$. Position vectors of a body-fixed point Q . Rotation $(\mathbf{n},\ \varphi)$ about 0_1 and translatory displacement $\mathbf{r}$

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$$\boldsymbol{\varrho}_1 = \boldsymbol{\varrho}_2 + \mathbf{r}\ . \tag{3.1}$$

Decomposition of this equation in $\underline{\mathbf{e}}^1$ yields the equation

$$\underline{\varrho}_1^1 = \underline{A}^{12}\underline{\varrho}_2^2 + \underline{r}^1\ . \tag{3.2}$$

The matrix $\underline{A}^{12}$ is determined by the rotation $(\mathbf{n},\ \varphi)$ about 0_1. The inverse equation is (premultiply by $\underline{A}^{21} = \underline{A}^{12^T}$ and write $\underline{r}^2 = \underline{A}^{21}\underline{r}^1$)

$$\underline{\varrho}_2^2 = \underline{A}^{21}\underline{\varrho}_1^1 - \underline{r}^2\ . \tag{3.3}$$

It is convenient to write these equations in product form. This is achieved by adding an identity equation:

$$\begin{bmatrix}\underline{\varrho}_1^1\\ 1\end{bmatrix} = \begin{bmatrix}\underline{A}^{12} & \underline{r}^1\\ \underline{0}^T & 1\end{bmatrix}\begin{bmatrix}\underline{\varrho}_2^2\\ 1\end{bmatrix}\ ,\quad \begin{bmatrix}\underline{\varrho}_2^2\\ 1\end{bmatrix} = \begin{bmatrix}\underline{A}^{21} & -\underline{r}^2\\ \underline{0}^T & 1\end{bmatrix}\begin{bmatrix}\underline{\varrho}_1^1\\ 1\end{bmatrix}\ ,\quad \underline{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}\ . \tag{3.4}$$

The (4×4) matrices are transformation matrices. Inversion is carried out not by transposition, but by the rule shown in Eqs.(3.4).

Example: In Fig. 3.2 a serial robot with six revolute joints is shown. Starting at the base the bodies and joints are labeled from 1 to 7 and from 1 to 6, respectively. The locations of the joint axes on the bodies are specified by body-fixed vectors $\mathbf{r}_2, \ldots, \mathbf{r}_6$ pointing from one axis to the next and by body-fixed unit vectors $\mathbf{n}_1, \ldots, \mathbf{n}_6$ along joint axes. The variable angle of rotation in joint i is called φ_i. It is the angle of body $i+1$ relative to body i. The vector $\mathbf{r}_7$ locates a specified point P on the hand of the robot. An arbitrarily chosen position of the robot is declared as null position. In this position the angles are $\varphi_1 = \varphi_2 = \ldots = \varphi_6 = 0$. On body 1 a reference basis $\underline{\mathbf{e}}^1$ is fixed with its origin 0 on the joint axis 1. On each of the bodies $i = 2, \ldots, 7$ a basis $\underline{\mathbf{e}}^i$ is fixed in such a way that in the null position all bases are oriented parallel to basis $\underline{\mathbf{e}}^1$. The given data are
- the column matrices $\underline{r}_i^i$ of the coordinates of $\mathbf{r}_i$ in $\underline{\mathbf{e}}^i$ $(i = 2, \ldots, 7)$
- the coordinates of $\mathbf{n}_i$ in $\underline{\mathbf{e}}^i$ (identical with the coordinates in $\underline{\mathbf{e}}^{i+1}$) $(i = 1, \ldots, 6)$
- the angles φ_i $(i = 1, \ldots, 6)$.

The position of the robot hand in the reference basis $\underline{\mathbf{e}}^1$ is determined by the matrix $\underline{A}^{17}$ in the equation $\underline{\mathbf{e}}^1 = \underline{A}^{17}\underline{\mathbf{e}}^7$ and by the column matrix $\underline{r}_{\mathrm{P}}^1$ of the coordinates of the position vector $\mathbf{r}_{\mathrm{P}}$ in $\underline{\mathbf{e}}^1$. To be determined are $\underline{A}^{17}$ and $\underline{r}_{\mathrm{P}}^1$ as functions of $\varphi_1, \ldots, \varphi_6$.
Solution: With the coordinates n_{i1}, n_{i2}, n_{i3} of $\mathbf{n}_i$ in $\underline{\mathbf{e}}^i$ and with φ_i Eq.(1.49) determines the matrix $\underline{A}^{i-1,i}$ in the relationship $\underline{\mathbf{e}}^{i-1} = \underline{A}^{i-1,i}\underline{\mathbf{e}}^i$. The desired matrix $\underline{A}^{17}$ is the product $\underline{A}^{12}\underline{A}^{23}\underline{A}^{34}\underline{A}^{45}\underline{A}^{56}\underline{A}^{67}$. The position vector of P is $\mathbf{r}_{\mathrm{P}} = \mathbf{r}_2 + \mathbf{r}_3 + \cdots + \mathbf{r}_7$. The desired column matrix of its coordinates in $\underline{\mathbf{e}}^1$ is the expression (to be read from right to left)

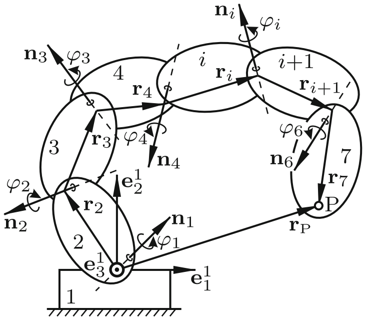

Fig. 3.2 Serial robot with six revolute joints

$$\underline{r}^1_\mathrm{P} = \underline{A}^{12}\Big(\underline{r}^2_2 + \underline{A}^{23}\Big[\underline{r}^3_3 + \underline{A}^{34}\{\underline{r}^4_4 + \underline{A}^{45}[\underline{r}^5_5 + \underline{A}^{56}(\underline{r}^6_6 + \underline{A}^{67}\underline{r}^7_7)]\}\Big]\Big) . \qquad (3.5)$$

In terms of (4×4) matrices this equation reads

$$\begin{bmatrix} \underline{r}^1_\mathrm{P} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{A}^{12} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{A}^{23} & \underline{r}^2_2 \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{A}^{34} & \underline{r}^3_3 \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{A}^{45} & \underline{r}^4_4 \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{A}^{56} & \underline{r}^5_5 \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{A}^{67} & \underline{r}^6_6 \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{r}^7_7 \\ 1 \end{bmatrix} . \qquad (3.6)$$

End of example.

3.2 Chasles' Theorem

In Fig. 3.3 the same general displacement of a rigid body is shown which was the subject of Fig. 3.1. The body is displaced from an initial position 1 to a final position 2. Let $\underline{\mathbf{e}}^2$ be a body-fixed basis which in position 1 coincides with a reference basis $\underline{\mathbf{e}}^1$ with origin 0_1 (arbitrary). In position 2 the origin of $\underline{\mathbf{e}}^2$ is at 0_2. The vector pointing from 0_1 to 0_2 is called $\mathbf{r}$. Dashed lines indicate an intermediate position $2'$ arrived at from position 1 by pure translation $\mathbf{r}$, and dotted lines indicate another intermediate position $1'$ arrived at from position 2 by pure translation $-\mathbf{r}$. The displacement from position 1 to position $1'$ is a rotation $(\mathbf{n}, \varphi)$ about 0_1, and the displacement from position $2'$ to position 2 is the same rotation $(\mathbf{n}, \varphi)$ about 0_2. Hence the conclusion: The displacement of the body from position 1 to position 2 can be interpreted in two ways, either as resultant of the rotation $(\mathbf{n}, \varphi)$ about 0_1 followed by the translation $\mathbf{r}$ or as resultant of the same translation $\mathbf{r}$ followed by the same rotation $(\mathbf{n}, \varphi)$ about 0_2.

For another origin $0'_1$ of the basis $\underline{\mathbf{e}}^1$ the rotation $(\mathbf{n}, \varphi)$ is the same because both $\underline{\mathbf{e}}^1$ and $\underline{\mathbf{e}}^2$ are oriented as before, but the translatory displacement $\mathbf{r}'$ from $0'_1$ to $0'_2$ is different. If $\boldsymbol{\varrho}$ is the vector pointing from 0_1 to

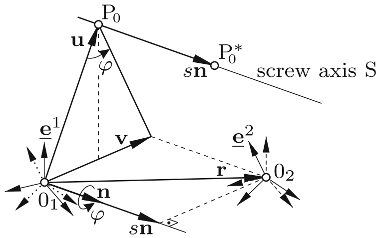

Fig. 3.3 Initial, final and two intermediate positions of a body-fixed basis $\underline{\mathbf{e}}^2$ specified by a rotation $(\mathbf{n},\,\varphi)$ and a translatory displacement $\mathbf{r}$. Screw axis S and displacement $s\mathbf{n}$

$0_1'$ and $\boldsymbol{\varrho}^*$ the vector pointing from 0_2 to $0_2'$, the relationship between $\mathbf{r}'$ and $\mathbf{r}$ is $\mathbf{r}' = \mathbf{r} + \boldsymbol{\varrho}^* - \boldsymbol{\varrho}$.

Proposition: For arbitrary $\mathbf{n}$ and $\varphi \neq 0$ there exists a uniquely determined body-fixed line having the direction of $\mathbf{n}$ all points of which have identical displacements $s\mathbf{n}$ ($s = \text{const}$) in the direction along this line. More precisely, this is the statement made by

Theorem 3.1. (Chasles [8])[1] *Any not purely translatory displacement of a rigid body from an initial to a final position can be represented in a unique way as screw displacement. This screw displacement is the resultant of a rotation* $(\mathbf{n}\,,\varphi)$ *about a body-fixed screw axis S and a translation* $s\mathbf{n}$ *along this axis. The screw displacement is the same regardless whether the rotation or the translation is carried out first.*

Proof: Starting from $(\mathbf{n}\,,\varphi)$ and from the displacement $\mathbf{r}$ the screw axis and the displacement $s\mathbf{n}$ are determined as follows. Let $\boldsymbol{\varrho}$ be the body-fixed vector pointing from 0_1 to another body-fixed point P (arbitrary). After the rotation $(\mathbf{n}\,,\varphi)$ about 0_1 P has the position vector (see (1.37))

$$\boldsymbol{\varrho}^* = \boldsymbol{\varrho} + (1 - \cos\varphi)\,\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\varrho}) + \sin\varphi\,\mathbf{n} \times \boldsymbol{\varrho}\,. \tag{3.7}$$

After the subsequent translation $\mathbf{r}$ the point has the position vector $\mathbf{r} + \boldsymbol{\varrho}^*$, so that the total displacement of P is $\mathbf{r} + \boldsymbol{\varrho}^* - \boldsymbol{\varrho}$. Points on the body-fixed screw axis, if it exists, have, prior to displacement, position vectors $\boldsymbol{\varrho} = \mathbf{u} + \lambda\mathbf{n}$ (λ arbitrary) with $\mathbf{u}\cdot\mathbf{n} = 0$. Thus, $\mathbf{u}$ is the perpendicular from 0_1 onto the screw axis. For proving the theorem it has to be shown that with these vectors $\boldsymbol{\varrho}$ the equation $\mathbf{r} + \boldsymbol{\varrho}^* - \boldsymbol{\varrho} \equiv s\mathbf{n}$ holds true independent of λ, and that, furthermore, the equation determines s and $\mathbf{u}$ uniquely. The first condition is satisfied, because λ is eliminated by the product $\mathbf{n} \times \boldsymbol{\varrho}$. The equation reads

$$\mathbf{r} - (1 - \cos\varphi)\,\mathbf{u} + \sin\varphi\,\mathbf{n} \times \mathbf{u} = s\mathbf{n}\,. \tag{3.8}$$

Scalar multiplication by $\mathbf{n}$ determines

[1] This theorem was known already to Mozzi (1765; see Giorgini [16])

$$s = \mathbf{n} \cdot \mathbf{r} \,, \tag{3.9}$$

and cross-multiplication by $\mathbf{n}$ produces the equation

$$\mathbf{n} \times \mathbf{r} - (1 - \cos\varphi)\, \mathbf{n} \times \mathbf{u} - \sin\varphi\, \mathbf{u} = \mathbf{0} \,. \tag{3.10}$$

This equation and (3.8) with $s\mathbf{n} = \mathbf{nn}\cdot\mathbf{r}$ are two linear equations for $\mathbf{u}$ and for the second Plücker vector $\mathbf{u} \times \mathbf{n}$ of the screw axis:

$$\left.\begin{aligned} \sin\varphi\, \mathbf{u} - (1 - \cos\varphi)\, \mathbf{u} \times \mathbf{n} &= \mathbf{n} \times \mathbf{r} \,, \\ (1 - \cos\varphi)\, \mathbf{u} + \sin\varphi\, \mathbf{u} \times \mathbf{n} &= \mathbf{r} - \mathbf{nn} \cdot \mathbf{r} = (\mathbf{n} \times \mathbf{r}) \times \mathbf{n} \,. \end{aligned}\right\} \tag{3.11}$$

With the formula $\sin\varphi/(1 - \cos\varphi) = \cot\varphi/2$ the solutions are

$$\mathbf{u} = \frac{1}{2}\Big[(\mathbf{n} \times \mathbf{r}) \times \mathbf{n} + \mathbf{n} \times \mathbf{r} \cot\frac{\varphi}{2}\Big] \,, \tag{3.12}$$

$$\mathbf{u} \times \mathbf{n} = \frac{1}{2}\Big[(\mathbf{n} \times \mathbf{r}) \times \mathbf{n} \cot\frac{\varphi}{2} - \mathbf{n} \times \mathbf{r}\Big] \,. \tag{3.13}$$

This concludes the proof. The geometrical interpretation of the formula for $\mathbf{u}$ is given in Fig. 3.3. The vector $\mathbf{r}$ is decomposed into its components $\mathbf{v}$ orthogonal to $\mathbf{n}$ and $s\mathbf{n}$ along $\mathbf{n}$, so that $s = \mathbf{n} \cdot \mathbf{r}$ in accordance with (3.9). Hence $\mathbf{r} = \mathbf{v} + s\mathbf{n}$. The screw axis is called S . It is passing through the apex P_0 of the isosceles triangle in the plane normal to $\mathbf{n}$ having $\mathbf{v}$ as base and φ as apex angle. The vector $\mathbf{u}$ is pointing from 0_1 to P_0 . The first term in the expression for $\mathbf{u}$ represents the vector $\mathbf{v}/2$, and the second is the altitude of the triangle above the base. Until further below (see Sect. 3.9) the screw displacement is denoted (S, $\mathbf{n}, \varphi, s$).

In the general formula for the displacement $\mathbf{r} + \boldsymbol{\varrho}^* - \boldsymbol{\varrho}$ of arbitrary points of the body the last two terms satisfy (1.44) and (1.77):

$$(\boldsymbol{\varrho}^* - \boldsymbol{\varrho}) \cdot \mathbf{n} = 0 \,, \qquad \boldsymbol{\varrho}^* - \boldsymbol{\varrho} = \mathbf{n} \tan\frac{\varphi}{2} \times (\boldsymbol{\varrho}^* + \boldsymbol{\varrho}) \,. \tag{3.14}$$

With the first equation it is verified that the component of the displacement along the screw axis is $\mathbf{nn} \cdot \mathbf{r} = s\mathbf{n}$ for all points of the body. With the second equation it is verified that the same screw axis is obtained if instead of 0_1 another point $0_1'$ is used as starting point. Let this point $0_1'$ be the point at the tip of a vector $\boldsymbol{\varrho}$ (arbitrary) from 0_1 . Furthermore, let $\mathbf{u}'$ be the perpendicular from $0_1'$ onto the screw axis. It is given by (3.12) if $\mathbf{r}$ is replaced by $\mathbf{r} + \boldsymbol{\varrho}^* - \boldsymbol{\varrho}$. It has to be verified that the second Plücker vector $(\mathbf{u}' + \boldsymbol{\varrho}) \times \mathbf{n}$ of the screw axis is identical with $\mathbf{u} \times \mathbf{n}$. Because of (3.14) this is, indeed, the case:

$$\begin{aligned}(\mathbf{u}' + \boldsymbol{\varrho}) \times \mathbf{n} &= \frac{1}{2}\Big[(\mathbf{n} \times \mathbf{r}) \times \mathbf{n} \cot\frac{\varphi}{2} - \mathbf{n} \times \mathbf{r}\Big] \\ &\quad + \frac{1}{2}\Big[[\mathbf{n} \times (\varrho^* - \boldsymbol{\varrho})] \times \mathbf{n} \cot\frac{\varphi}{2} - \mathbf{n} \times (\varrho^* - \boldsymbol{\varrho})\Big] + \boldsymbol{\varrho} \times \mathbf{n} \\ &= \mathbf{u} \times \mathbf{n} + \frac{1}{2}\Big[(\varrho^* - \boldsymbol{\varrho}) \cot\frac{\varphi}{2} - \mathbf{n} \times (\varrho^* + \boldsymbol{\varrho})\Big] = \mathbf{u} \times \mathbf{n}\,. \end{aligned} \quad (3.15)$$

Example: Given $\mathbf{n}$, φ, s and the position vector $\mathbf{r}_{\rm A}$ of a point A on the screw axis, determine the relationship between the position vectors $\mathbf{r}$ and $\mathbf{r}^*$ of an arbitrary body-fixed point before and after the screw displacement. Solution: From Fig. 1.3 it follows that the rotation is governed by (1.38) if $\mathbf{r}^*$ and $\mathbf{r}$ are replaced by $\mathbf{r}^* - \mathbf{r}_{\rm A}$ and $\mathbf{r} - \mathbf{r}_{\rm A}$, respectively. Hence the solution:

$$\mathbf{r}^* = \mathbf{r}_{\rm A} + \cos\varphi\,(\mathbf{r} - \mathbf{r}_{\rm A}) + (1 - \cos\varphi)\mathbf{n}\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_{\rm A}) + \sin\varphi\,\mathbf{n} \times (\mathbf{r} - \mathbf{r}_{\rm A}) + s\,\mathbf{n}\,. \quad (3.16)$$

In the special case $\varphi = \pi$, $s = 0$, the point is reflected in the screw axis:

$$\mathbf{r}^* = 2\mathbf{r}_{\rm A} - \mathbf{r} + 2\mathbf{n}\,\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_{\rm A})\,. \quad (3.17)$$

This formula is known from (2.11). End of example.

3.3 Scalar Measures of a Screw Displacement

For a screw displacement with infinitesimal quantities φ and s the quotient $p = s/\varphi$ is called pitch as is done for a machine screw. In the theory of finite screw displacements the quotient s/φ does not occur. As scalar measure of a finite screw displacement Dimentberg [12] defines the quantity

$$p_{\rm D} = \frac{s}{\sin\varphi}\,. \quad (3.18)$$

Parkin [34] defines the quantity[2]

$$p_{\rm P} = \frac{\dfrac{s}{2}}{\tan\dfrac{\varphi}{2}}\,. \quad (3.19)$$

These two measures are related through the equation

$$p_{\rm P} = p_{\rm D}\cos^2\frac{\varphi}{2}\,. \quad (3.20)$$

[2] This measure was already used by Schönflies [32] p.1014

In the special case of an infinitesimal screw displacement both measures are identical with the pitch $p = s/\varphi$. Both measures find applications (see Huang [20] and Eqs.(3.207), (6.3), (6.23), (6.88)).

3.4 Roth' Theorem

Starting point of this section is the trivial statement: If a body is rotated about a fixed point P, every point of the body has the property that its distance from P is the same before and after the rotation. Roth [43] proved

Theorem 3.2. *Given a point* P *and two positions of a body not resulting from each other by a rotation about* P*, there exists a body-fixed plane every point of which has the property that its distance from* P *is the same in both positions.*

In what follows, not only a proof of the theorem is given. The body-fixed plane is determined as well. Let $\mathbf{r}_i$ and $\mathbf{r}'_i$ be the vectors from P to an arbitrary body-fixed point in the initial and in the final position, respectively. The condition that the distances from P be the same in both positions reads

$$\mathbf{r}'^2_i = \mathbf{r}^2_i \,. \tag{3.21}$$

It suffices to prove that this condition is satisfied by three noncollinear body-fixed points. Then it is satisfied by every body-fixed point in the plane spanned by these points. Four noncoplanar body-fixed points satisfying (3.21) cannot exist since otherwise the displacement of the body would be a rotation about P contrary to the assumption.

In the general case, the displacement of the body is a screw displacement. Define $(\mathbf{n}, \varphi)$ to be the rotation, $R \geq 0$ the distance of P from the screw axis, $\mathbf{e}$ a unit vector through P normal to the screw axis (in the case $R > 0$, the vector $R\mathbf{e}$ is the perpendicular from P onto the screw axis). Finally, let s be the translation along the screw axis (see Fig. 3.4). The special cases of pure translation ($s \neq 0$, $\varphi = 0$) and of pure rotation ($s = 0$, $\varphi \neq 0$) are not excluded. Three distinguished points of the unknown body-fixed plane

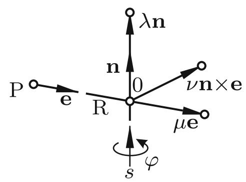

Fig. 3.4 Axes and quantities R, λ, μ, ν in the context of Roth' theorem

are its intersections with the line P$\mathbf{e}$, with the screw axis and with the line perpendicular to both screw axis and line P$\mathbf{e}$. The vectors $\mathbf{r}_i$ $(i = 1, 2, 3)$ from P to these points in the initial position have, with unknown quantities λ, μ and ν of dimension length, the forms

$$\mathbf{r}_1 = R\mathbf{e} + \lambda\mathbf{n}\,, \qquad \mathbf{r}_2 = (R + \mu)\mathbf{e}\,, \qquad \mathbf{r}_3 = R\mathbf{e} + \nu\mathbf{n} \times \mathbf{e}\,. \tag{3.22}$$

In the final position after the screw displacement the position vectors are found by simple inspection from Fig. 3.4:

$$\left.\begin{aligned} \mathbf{r}_1' = R\mathbf{e} + (s + \lambda)\mathbf{n}\,, \quad \mathbf{r}_2' &= s\mathbf{n} + (R + \mu\cos\varphi)\mathbf{e} + \mu\sin\varphi\,\mathbf{n} \times \mathbf{e}\,, \\ \mathbf{r}_3' &= s\mathbf{n} + (R - \nu\sin\varphi)\mathbf{e} + \nu\cos\varphi\,\mathbf{n} \times \mathbf{e}\,. \end{aligned}\right\} \tag{3.23}$$

Substitution into (3.21) yields for the unknowns the expressions

$$\lambda = -\frac{s}{2}\,, \qquad \mu = \frac{s^2}{2R(1 - \cos\varphi)}\,, \qquad \nu = \frac{s^2}{2R\sin\varphi}\,. \tag{3.24}$$

Except for a single case these formulas define three points and, hence, a plane. The single case is a pure rotation about an axis not passing through P. It is characterized by $s = 0$ and $R, \varphi \neq 0$. The corresponding solutions $\lambda = \mu = \nu = 0$ define only the point 0. However, even in this case, a body-fixed plane with the required property exists. Without calculation it is obvious that the plane contains the rotation axis. In the initial position it is rotated against the line $\overline{\mathrm{P0}}$ through $-\varphi/2$ and in the final position through $+\varphi/2$. In Fig. 3.5a the points P and 0 and the two positions of the plane are shown in the projection along the rotation axis $\mathbf{n}$. Thus, it is proved that Roth' Theorem is valid without any exception. In what follows, three more special cases are considered in which the body-fixed plane is predictable without the above analysis.

1. The special case $R = 0$, $\varphi, s \neq 0$ (screw displacement with a screw axis passing through P): Without calculation it is obvious that the plane is normal to the screw axis. The perpendicular from P onto the plane is $-(s/2)\mathbf{n}$ in the initial position and $+(s/2)\mathbf{n}$ in the final position of the body (Fig. 3.5b). The plane is defined by Eqs.(3.24) which, in this case, yield $\lambda = -s/2$ and $\mu, \nu \to \infty$. A point A in this plane is displaced to A$'$.

2. The special case $\varphi = 0$, $s \neq 0$, R unspecified (pure translation $s\mathbf{n}$): Without calculation it is obvious that Fig. 3.5b applies also to this case. As before, Eqs.(3.24) yield $\lambda = -s/2$ and $\mu, \nu \to \infty$.

3. The special case $\varphi = \pi$, R, $s \neq 0$ (screw displacement with 180°-turn): Equations (3.24) yield $\lambda = -s/2$, $\mu = s^2/(4R)$ and $\nu \to \infty$. These results indicate that the line $\mathbf{n} \times \mathbf{e}$ is parallel to the plane. In Fig. 3.5c the two positions of the plane are shown in the projection along $\mathbf{n} \times \mathbf{e}$. The points with position vectors $\mathbf{r}_1$, $\mathbf{r}_1'$ and $\mathbf{r}_2$, $\mathbf{r}_2'$ in Eqs.(3.22) and (3.23) are marked A , A$'$ and B, B$'$, respectively. In this case, the solution is less

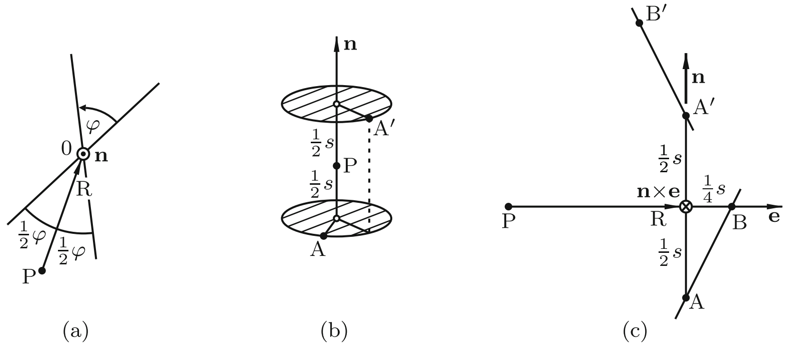

Fig. 3.5 Three special cases of Roth' theorem

obvious than in the previous cases, but it is still predictable without analysis. In all other cases λ, μ and ν are finite and different from zero.

3.5 Screw Displacement Determined from Displacements of Three Body Points

Problem: For three noncollinear body-fixed points P_1, P_2, P_3 the position vectors $\mathbf{z}_i$ before and $\mathbf{z}_i'$ after a displacement, respectively, are given $(i = 1, 2, 3)$. The three displacements $\mathbf{z}_i' - \mathbf{z}_i$ $(i = 1, 2, 3)$ are not identical. Therefore, the displacement of the body is not a translation, but a screw displacement. To be determined are the rotation $(\mathbf{n}, \varphi)$, a vector from 0 to some point on the screw axis and the translation s along the screw axis.

Solution: The rotation $(\mathbf{n}, \varphi)$ does not change if the displacement is superimposed by an arbitrary translation. Arbitrarily, the translation $-(\mathbf{z}_3' - \mathbf{z}_3)$ is superimposed. Then the resulting displacement is the rotation $(\mathbf{n}, \varphi)$ about the fixed point P_3. The position vectors from P_3 to the body-fixed points P_1 and P_2 before and after the rotation are given by

$$\mathbf{r}_i = \mathbf{z}_i - \mathbf{z}_3 \quad \text{and} \quad \mathbf{r}_i^* = \mathbf{z}_i' - \mathbf{z}_3' \qquad (i = 1, 2)\,, \tag{3.25}$$

respectively. These vectors determine the rotation $(\mathbf{n}, \varphi)$. Its Rodrigues vector $\mathbf{n}\tan\varphi/2$ is calculated from (1.210) – (1.217) in Sect. 1.15.7. The rotation is superimposed again by the translation $\mathbf{r} = \mathbf{z}_3' - \mathbf{z}_3$. From the now known quantities $\mathbf{n}$, φ and $\mathbf{r}$ the translation s along the screw axis and the perpendicular $\mathbf{u}$ from P_3 onto the screw axis are calculated from (3.9) and (3.12), respectively. The desired vector from 0 to the screw axis is $\mathbf{z}_3 + \mathbf{u}$. Note: In (1.210) – (1.217) $\mathbf{u}$ is the Rodrigues vector.

3.6 Halphen's Theorem

In 1882 Halphen [17] published

Theorem 3.3. *A screw displacement* (S, $\mathbf{n}, \varphi, s$) *can be represented as resultant of two successive reflections in lines* g_1 (*first reflection*) *and* g_2 *. The lines* g_1 *and* g_2 *intersect the screw axis* S *orthogonally. Line* g_2 *results from* g_1 *by the screw displacement* $(\mathrm{S}, \mathbf{n}, \varphi/2\,,\ s/2)$. *One of the two lines may be chosen arbitrarily.*

Proof: Figure 3.6 shows in two projections the screw axis S together with two lines g_1 and g_2 having the required properties. Let Q_1 be an arbitrary point of the body in the initial position prior to the first reflection. Its location relative to g_1 is specified by the quantities d and α explained in the figure. After the first reflection the body-fixed point is located at Q' and after the second reflection in g_2 it is located at Q_2 . The effect of the first reflection is a rotation of the body-fixed perpendicular from Q_1 onto S through the angle 2α about S and a displacement of Q_1 by $2d$ in the direction $\mathbf{n}$. The second reflection in g_2 increases the rotation angle by $2(\varphi/2 - \alpha)$ and the displacement along S by $2(s/2 - d)$. Hence the total rotation angle is φ , and the total displacement along S is s . This proves the theorem.

In the special case $s = 0$, Halphen's theorem reduces to the statement known from Sects. 1.15.2 and 1.16 that a rotation $(\mathbf{n}, \varphi)$ can be represented as resultant of two reflections in lines which intersect $\mathbf{n}$ orthogonally and which enclose the angle $\varphi/2$. A reflection in a line and a $180°$-rotation about this line result in one and the same displacement.

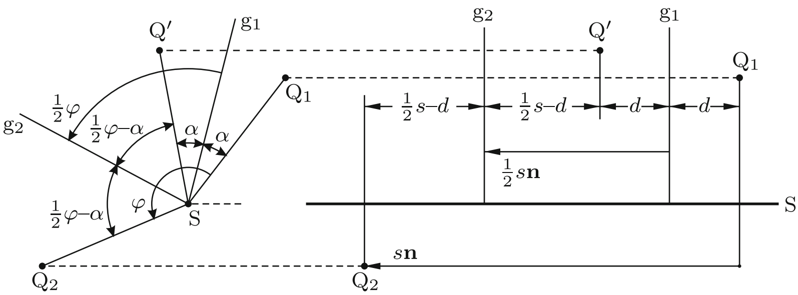

Fig. 3.6 Halphen's theorem

3.7 Resultant of two Screw Displacements. Screw Triangle

Consider now the displacement of a body which is the result of two successive screw displacements $(\mathrm{S}_1, \mathbf{n}_1, \varphi_1, s_1)$ (first screw displacement) and $(\mathrm{S}_2, \mathbf{n}_2, \varphi_2, s_2)$. According to Chasles' theorem the resultant displacement is itself a screw displacement called the resultant screw displacement $(\mathrm{S}_{\mathrm{res}}, \mathbf{n}_{\mathrm{res}}, \varphi_{\mathrm{res}}, s_{\mathrm{res}})$. This resultant is geometrically constructed by applying Halphen's theorem several times. First, the general case is considered in which the screw axes S_1 and S_2 are skew. Let g_1 be the common perpendicular of S_1 and S_2 (see Fig. 3.7). Each of the two screw displacements 1 and 2 is represented as resultant of two reflections. For the line of the second reflection of screw displacement 1 and also for the line of the first reflection of screw displacement 2 the common perpendicular g_1 is chosen. These two reflections cancel each other. Hence the resultant screw displacement is the resultant of the first reflection of screw displacement 1 and of the second reflection of screw displacement 2. The lines of these two reflections are called g_3 and g_2. According to Halphen's theorem, they are obtained by subjecting g_1 to the screw displacements $(\mathrm{S}_1, \mathbf{n}_1, -\varphi_1/2, -s_1/2)$ and $(\mathrm{S}_2, \mathbf{n}_2, \varphi_2/2, s_2/2)$, respectively. Again, according to Halphen's theorem, the resultant screw axis S_{res} is the common perpendicular of g_2 and g_3. Furthermore, $s_{\mathrm{res}}\mathbf{n}_{\mathrm{res}}/2$ is the vector along this common perpendicular shown in the figure, and $\varphi_{\mathrm{res}}/2$ is the projected angle between g_2 and g_3.

Up to now the screw axes S_1 and S_2 were assumed to be skew. Suppose now that they intersect at a point P. In this case, the common perpendicular g_1 is uniquely defined as normal through P of the plane spanned by S_1 and S_2. The length of the common perpendicular is zero.

The inverse of the resultant is the screw displacement $(\mathrm{S}_{\mathrm{res}}, \mathbf{n}_{\mathrm{res}}, -\varphi_{\mathrm{res}}, -s_{\mathrm{res}})$. It carries the body back to its initial position. The lines S_1, g_1, S_2,

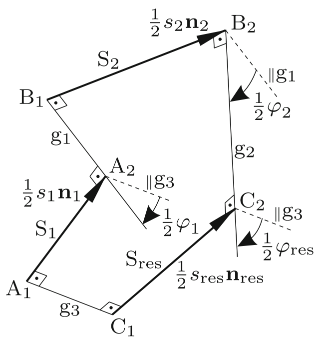

Fig. 3.7 Resultant of two screw displacements in the screw triangle

g_2 , S_3 , g_3 form a spatial hexagon with right angles at every corner. Three arbitrary pairwise skew lines S_1 , S_2 , S_3 have uniquely defined common perpendiculars g_1 , g_2 , g_3 . Hence the three lines determine uniquely a system of three screw displacements about these lines S_1 , S_2 , S_3 which carry a body from an initial position via two intermediate positions back into its initial position[3]. In an arbitrarily chosen reference frame the three lines are defined by their Plücker vectors. From this it follows that the position and the shape of the hexagon are determined by twelve independent parameters. For reasons explained later the hexagon is called *spatial triangle* or *screw triangle* (Yang [51], Roth [44]). In Sects. 3.11 and 3.12 analytical relationships are developed for the screw triangle.

When in the given screw displacements 1 and 2 s_1 and s_2 are changed (all other parameters held fixed), then the lines g_2 and g_3 undergo lateral displacements. This has no effect on $\varphi_{\rm res}$ whereas all other parameters of the resultant screw displacement are effected. In Fig. 3.8 the special case $s_1 = s_2 = 0$ is shown, i.e., the resultant of two pure rotations about skew axes (S_1 , $\mathbf{n}_1$, φ_1 , S_2 , $\mathbf{n}_2$, φ_2 and g_1 are the same as in Fig. 3.7). The points A_1 and A_2 coalesce in a single point A , and B_1 and B_2 coalesce in a single point B .

Remark: In 1848 Cayley [7],v.1 gave analytical solutions for the resultant of two successive screw displacements as well as for the inverse problem of decomposing a given screw displacement into two screw displacements with prescribed characteristics. He did not consider the special case of screw displacements with 180° rotation angles which was the subject of Halphen's paper [17] almost half a century later. Among the problems solved by Cayley are the determination of the resultant of two successive pure rotations about skew axes and the decomposition of a given screw displacement into two pure

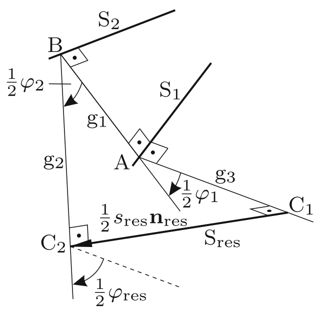

Fig. 3.8 Resultant of two rotations about skew axes

[3] The lines S_1 , S_2 , S_3 and their perpendiculars g_1 , g_2 , g_3 can change roles. Thus, the same hexagon determines three screw displacements about g_1 , g_2 , g_3 which carry the body from its initial position via two intermediate positions back into the initial position

rotations with prescribed characteristics. The decomposition is the subject of Cayley's

Theorem 3.4. *A given screw displacement can be represented as resultant of two subsequent pure rotations. The axis of one rotation may be prescribed arbitrarily (but not parallel to the axis of the resultant screw displacement). Then the axis of the other rotation as well as the two rotation angles are determined.*

The geometrical solution is explained in Fig. 3.8 . Let it be assumed that the resultant screw displacement and the axis S_2 are prescribed as shown. The axes S_2 and $\mathrm{S}_{\mathrm{res}}$ determine the common perpendicular g_2 and its endpoints B and C_2 . Point C_1 is determined by s_{res} , and g_3 is determined by φ_{res} . Point A is the point of intersection of g_3 with the plane through B and perpendicular to S_2 . The axis S_1 of the first rotation is the common perpendicular of g_3 and $\mathrm{g}_1 = \overline{\mathrm{AB}}$. Finally, $\varphi_1/2$ and $\varphi_2/2$ are the angles between g_1 and g_3 and between g_1 and g_2 , respectively. An analytical solution of the problem is given in Sect. 3.11 .

3.8 Dual Numbers

Let x and y be real numbers. The number $x+\varepsilon y$ is complex if $\varepsilon^2=-1$. Clifford [10] was the first to consider the case $\varepsilon^2=0$. In this case, $x+\varepsilon y$ is called a *dual number* with x being its *primary part* and y its *dual part*. It must be understood that $\varepsilon^2=0$ does not mean that also $\varepsilon=0$. The quantity ε is, just as $\mathrm{i}=\sqrt{-1}$, a unit, namely, the unit of the dual part. The sum and the product of two dual numbers $x_1+\varepsilon y_1$ and $x_2+\varepsilon y_2$ are defined by the formulas

$$\left.\begin{aligned}(x_1+\varepsilon y_1)+(x_2+\varepsilon y_2)&=x_1+x_2+\varepsilon(y_1+y_2)\;,\\(x_1+\varepsilon y_1)(x_2+\varepsilon y_2)&=x_1x_2+\varepsilon(x_1y_2+y_1x_2)\;.\end{aligned}\right\}\tag{3.26}$$

According to these definitions, for addition as well as for multiplication the laws of commutativity, associativity and distributivity are valid. As with real numbers expressions are multiplied out term by term always keeping in mind the rule $\varepsilon^2=0$. Together with $\varepsilon^2=0$ also all higher-order terms of ε are zero: $\varepsilon^3=\varepsilon\cdot\varepsilon^2=0$ etc. The difference of two dual numbers is defined uniquely via the sum. The zero dual number is the number $(0+\varepsilon\cdot 0)$ since only this number has the property that addition to $x+\varepsilon y$ with arbitrary x, y results in $x+\varepsilon y$.

Equation (3.26) shows that the product of two dual numbers is zero not only if at least one factor is the number $(0+\varepsilon\cdot 0)$, but also in the case $x_1=x_2=0$ with arbitrary y_1 , y_2 . From this it follows that division by $(x+\varepsilon y)$ is not defined if $x=0$. Indeed, multiplying both the numerator and the denominator of $1/(x+\varepsilon y)$ by $x-\varepsilon y$ results in the expression

$$\frac{1}{x+\varepsilon y}=\frac{x-\varepsilon y}{x^2}=\frac{1}{x}+\varepsilon\,\frac{-y}{x^2} \tag{3.27}$$

which is defined only in the case $x\neq 0$.

Dual numbers $x+\varepsilon y$ and (2×2)-matrices of the form $\begin{bmatrix} x & 0\\ y & x\end{bmatrix}$ have the same algebra as is shown by the formulas

$$\begin{bmatrix} x_1 & 0\\ y_1 & x_1\end{bmatrix}\begin{bmatrix} x_2 & 0\\ y_2 & x_2\end{bmatrix}=\begin{bmatrix} x_1x_2 & 0\\ x_1y_2+x_2y_1 & x_1x_2\end{bmatrix}, \quad \begin{bmatrix} x & 0\\ y & x\end{bmatrix}^{-1}=\begin{bmatrix} 1/x & 0\\ -y/x^2 & 1/x\end{bmatrix}. \tag{3.28}$$

Let $f(x+\varepsilon y)$ be a once differentiable function depending on the dual variable $x+\varepsilon y$ and possibly on additional parameters. The Taylor series expansion about the point x consists, because of $\varepsilon^2=0$, of two terms only:

$$f(x+\varepsilon y)=f(x)+\varepsilon y\,\frac{\partial f}{\partial x}\Big|_{y=0}. \tag{3.29}$$

Hence the function $f(x+\varepsilon y)$ is a dual number. Its primary part is the function of the primary part of its argument. Its dual part is the derivative of the primary part with respect to the primary part x of its argument multiplied by the dual part y of $x+\varepsilon y$. This dual part is referred to as *dual derivative*, and the process of calculating it is referred to as dual differentiation. Examples:

$$\cos(x+\varepsilon y)=\cos x-\varepsilon y\sin x\,,\qquad \sin(x+\varepsilon y)=\sin x+\varepsilon y\cos x\,, \tag{3.30}$$

$$\tan(x+\varepsilon y)=\tan x+\varepsilon\,\frac{y}{\cos^2 x}\,,\qquad \cot(x+\varepsilon y)=\cot x-\varepsilon\,\frac{y}{\sin^2 x}\,. \tag{3.31}$$

The product of two functions $f(x_1+\varepsilon y_1)\,g(x_2+\varepsilon y_2)$ is decomposed into primary and dual part as follows:

$$f(x_1+\varepsilon y_1)\,g(x_2+\varepsilon y_2)=\Big(f(x_1)+\varepsilon y_1\,\frac{\partial f}{\partial x_1}\Big|_{y_1=0}\Big)\Big(g(x_2)+\varepsilon y_2\,\frac{\partial g}{\partial x_2}\Big|_{y_2=0}\Big)$$

$$=f(x_1)g(x_2)+\varepsilon\Big(f(x_1)y_2\,\frac{\partial g}{\partial x_2}\Big|_{y_2=0}+y_1\,\frac{\partial f}{\partial x_1}\Big|_{y_1=0}g(x_2)\Big). \tag{3.32}$$

Example:

$$\sin(x_1+\varepsilon y_1)\cos(x_2+\varepsilon y_2)=\sin x_1\cos x_2+\varepsilon(y_1\cos x_1\cos x_2-y_2\sin x_1\sin x_2)\,. \tag{3.33}$$

Thus, the primary part is the product of the functions of the primary parts of their variables. The rule for calculating the dual part is the product rule of dual differentiation. The dual part is linear with respect to the dual parts y_1, y_2 of the arguments of the factors f and g, respectively. With these few rules all mathematical expressions encountered in later chapters can be decomposed into their primary and dual parts.

A MAPLE software tool developed by Sinigersky [45] has subroutines for the symbolic manipulation of dual numbers, dual vectors and dual quaternions and also for the dual differentiation of arbitrarily complex mathematical expressions.

3.9 Dual Vectors. Dual Angles

Figure 3.9 shows a *line vector* $\hat{\mathbf{v}}$ of given magnitude. This is a vector which is confined to its line. In contrast to a free vector $\mathbf{v}$ a line vector can slide along its line, but it cannot move lateral to it. A force is an example of a line vector. Its line is called line of action. Let $\mathbf{v}$ be the free vector having direction, sense of direction and magnitude in common with $\hat{\mathbf{v}}$. The line vector $\hat{\mathbf{v}}$ is uniquely determined if $\mathbf{v}$ is given and, in addition, the vector $\mathbf{r}$ from a reference point 0 to an arbitrary point of the line of $\hat{\mathbf{v}}$. The vectors $\mathbf{r}$ and $\mathbf{v}$ together define the *moment* of $\hat{\mathbf{v}}$ with respect to 0. It is abbreviated $\mathbf{w}$:

$$\mathbf{w} = \mathbf{r} \times \mathbf{v} \qquad \text{(equal for all points of the line of } \hat{\mathbf{v}}\text{)} . \tag{3.34}$$

The vectors $\mathbf{v}$ and $\mathbf{w} = \mathbf{r} \times \mathbf{v}$ represent the first and the second Plücker vectors of the line (see Sect. 2.2). They determine the line. With a free parameter λ it is given by the vector equation

$$\mathbf{r}^*(\lambda) = \lambda \mathbf{v} + \frac{\mathbf{v} \times \mathbf{w}}{\mathbf{v}^2} . \tag{3.35}$$

The Plücker vectors satisfy the conditions

$$\mathbf{v}^2 = \text{const} , \qquad \mathbf{v} \cdot \mathbf{w} = 0 . \tag{3.36}$$

Definition: The line vector $\hat{\mathbf{v}}$ is the dual vector

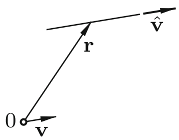

Fig. 3.9 Line vector $\hat{\mathbf{v}}$

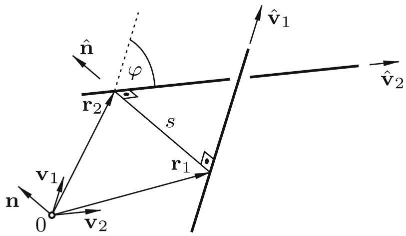

Fig. 3.10 Screw angle $\hat{\varphi} = \varphi + \varepsilon s$

$$\hat{\mathbf{v}} = \mathbf{v} + \varepsilon \mathbf{w} \; . \tag{3.37}$$

Because of (3.36) the scalar product of $\hat{\mathbf{v}}$ with itself is

$$\hat{\mathbf{v}}^2 = (\mathbf{v} + \varepsilon \mathbf{w})^2 = \mathbf{v}^2 + 2\varepsilon \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^2 \; . \tag{3.38}$$

In the case $\mathbf{v}^2 = 1$, $\hat{\mathbf{v}}$ is a unit dual vector (unit line vector).

In Fig. 3.10 two unit line vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are shown the lines of which are skew. By assumption, $\mathbf{v}_1^2 = \mathbf{v}_2^2 = 1$. As vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ the vectors to the feet of the common perpendicular of the lines are chosen. Then, by Eq.(3.37),

$$\hat{\mathbf{v}}_1 = \mathbf{v}_1 + \varepsilon \mathbf{w}_1 = \mathbf{v}_1 + \varepsilon \mathbf{r}_1 \times \mathbf{v}_1 \; , \qquad \hat{\mathbf{v}}_2 = \mathbf{v}_2 + \varepsilon \mathbf{w}_2 = \mathbf{v}_2 + \varepsilon \mathbf{r}_2 \times \mathbf{v}_2 \; . \tag{3.39}$$

These expressions are valid also when the lines are parallel. In this case, $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of an arbitrary common perpendicular.
Let $\mathbf{n}$ be the unit vector in the direction of $\mathbf{r}_2 - \mathbf{r}_1$:

$$\mathbf{n} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|} \; . \tag{3.40}$$

In the case of parallel lines only the first expression is useful and in the case of intersecting (not identical) lines only the second expression. Furthermore, the line vector is defined

$$\hat{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{r}_1 \times \mathbf{n} \; . \tag{3.41}$$

It has the direction of $\mathbf{n}$, and its line is the common perpendicular.

The line vector $\hat{\mathbf{v}}_2$ can be produced from $\hat{\mathbf{v}}_1$ by a screw displacement about the screw axis $\hat{\mathbf{n}}$. The rotation angle φ and the translation s of this screw displacement (both positive, zero or negative) are determined by the equations

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{n} \sin\varphi \; , \qquad \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{n}\, s \; . \tag{3.42}$$

The absolute values $|s|$ and $|\varphi|$ are the distance and the projected angle, respectively, between the two lines. The special cases of parallel or intersecting lines are characterized by $\varphi = 0$ or $s = 0$, respectively.
Between the various quantities just defined the following relationships exist:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \cos\varphi \; , \qquad (\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = s \sin\varphi \; . \tag{3.43}$$

Definition: The dual angle

$$\hat{\varphi} = \varphi + \varepsilon s \tag{3.44}$$

is called the *screw angle* of the screw displacement carrying $\hat{\mathbf{v}}_1$ into $\hat{\mathbf{v}}_2$. The screw displacement itself is denoted $(\hat{\mathbf{n}}, \hat{\varphi})$. This replaces the earlier notation $(\mathrm{S}, \mathbf{n}, \varphi, s)$. The pair $\mathrm{S}, \mathbf{n}$ is replaced by $\hat{\mathbf{n}}$, and the pair φ, s is replaced by $\hat{\varphi}$. The functions $\cos\hat{\varphi}$ and $\sin\hat{\varphi}$ are given in (3.30).

The definitions given for dual vectors and for dual screw angles are very useful. This is shown by calculating the dot product and the cross product of the dual unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ of Fig. 3.10 . With (3.39) and (3.43) the dot product is

$$\begin{aligned}
\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 &= \mathbf{v}_1 \cdot \mathbf{v}_2 + \varepsilon(\mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{v}_2 \cdot \mathbf{w}_1) && (3.45)\\
&= \mathbf{v}_1 \cdot \mathbf{v}_2 + \varepsilon(\mathbf{v}_1 \cdot \mathbf{r}_2 \times \mathbf{v}_2 + \mathbf{r}_1 \times \mathbf{v}_1 \cdot \mathbf{v}_2)\\
&= \mathbf{v}_1 \cdot \mathbf{v}_2 + \varepsilon(-\mathbf{r}_2 \cdot \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{r}_1 \cdot \mathbf{v}_1 \times \mathbf{v}_2)\\
&= \mathbf{v}_1 \cdot \mathbf{v}_2 - \varepsilon(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)\ . && (3.46)\\
&= \cos\varphi - \varepsilon s \sin\varphi && (3.47)
\end{aligned}$$

and with (3.30)

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 = \cos\hat{\varphi}\ . \qquad (3.48)$$

Thus, the rule for calculating the dot product of two ordinary unit vectors is transferred to dual unit vectors.

Comparison of (3.45) and (3.47) yields $s\sin\varphi = -(\mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{v}_2 \cdot \mathbf{w}_1)$ and the condition for two lines to intersect:

$$\mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{v}_2 \cdot \mathbf{w}_1 = 0\ . \qquad (3.49)$$

These equations repeat what is known from (2.17) and (2.18). Another important relationship is deduced from (3.46):

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \quad \text{is the dual derivative of} \quad \mathbf{v}_1 \cdot \mathbf{v}_2\ . \qquad (3.50)$$

The usefulness of this equation is demonstrated in Sects. 5.3.1, 6.3 and 6.4.4.

Next, the cross product of the dual unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ is calculated. With (3.39) it is

$$\begin{aligned}
&\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2 = (\mathbf{v}_1 + \varepsilon\mathbf{r}_1 \times \mathbf{v}_1) \times (\mathbf{v}_2 + \varepsilon\mathbf{r}_2 \times \mathbf{v}_2)\\
&= \mathbf{v}_1 \times \mathbf{v}_2 + \varepsilon[\mathbf{v}_1 \times (\mathbf{r}_2 \times \mathbf{v}_2) + (\mathbf{r}_1 \times \mathbf{v}_1) \times \mathbf{v}_2]\\
&= \mathbf{v}_1 \times \mathbf{v}_2 + \varepsilon[\mathbf{v}_1 \cdot \mathbf{v}_2\mathbf{r}_2 - \mathbf{v}_1 \cdot \mathbf{r}_2\mathbf{v}_2 + \mathbf{r}_1 \cdot \mathbf{v}_2\mathbf{v}_1 - \mathbf{v}_1 \cdot \mathbf{v}_2\mathbf{r}_1]\\
&= \mathbf{v}_1 \times \mathbf{v}_2 + \varepsilon[\mathbf{v}_1 \cdot \mathbf{v}_2(\mathbf{r}_2 - \mathbf{r}_1) + \mathbf{r}_1 \times (\mathbf{v}_1 \times \mathbf{v}_2)] \quad \text{(note } \mathbf{v}_1 \cdot \mathbf{r}_2 = \mathbf{v}_1 \cdot \mathbf{r}_1)\\
&= \mathbf{n}\sin\varphi + \varepsilon(\cos\varphi\,\mathbf{n}s + \mathbf{r}_1 \times \mathbf{n}\sin\varphi) \quad \text{(because of (3.43) and (3.42))}\\
&= (\mathbf{n} + \varepsilon\mathbf{r}_1 \times \mathbf{n})(\sin\varphi + \varepsilon s\cos\varphi)\ . \qquad (3.51)
\end{aligned}$$

The correctness of the last expression is verified by multiplying out again. With (3.37) and (3.30) the final result is

$$\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2 = \hat{\mathbf{n}}\sin\hat{\varphi}\ . \qquad (3.52)$$

Thus, also the rule for calculating the cross product of two ordinary unit vectors is transferred to dual unit vectors.

3.10 Principle of Transference

In Chap. 1 relationships were established between positions of a body before and after a rotation $(\mathbf{n}, \varphi)$ about a fixed point. Equations (3.48) and (3.52) represent the basis of the *principle of transference* first formulated by Kotelnikov [26] in 1886 and by Study [47] in 1903. It says: Given an equation relating positions of a body before and after a rotation $(\mathbf{n}, \varphi)$ about a fixed point. Replace the unit vector $\mathbf{n}$ along the axis by the dual unit vector $\hat{\mathbf{n}}$ and the rotation angle φ by the dual angle $\hat{\varphi}$. The equation thus obtained relates positions of a body before and after the screw displacement $(\hat{\mathbf{n}}, \hat{\varphi})$.

In Sects. 3.10.1 – 3.10.5 the notions of cartesian basis, direction cosine matrix, Euler angle, rotation tensor and quaternion of a rotation are *dualized*, i.e., transferred into respective dual quantities. Dualized equations relating such quantities must subsequently be split into their primary and dual parts. For this procedure it suffices to apply basic rules of (vector) algebra in combination with the rule of dual differentiation (see (3.29) – (3.33), in particular formulas (3.30) and (3.31) for trigonometric functions). These rules and formulas reveal the following facts.

1. The primary parts of dual equations contain neither second Plücker vectors $\mathbf{w}$ of screw axes nor translatory displacements s along screw axes. Thus, primary parts of equations describe a (usually nonlinear) problem of rotation about a fixed point.

2. The quantities $\mathbf{w}$ and s of screw displacements appear in the dual parts only and, moreover, in linear form only. The solution of these equations is an elementary problem. Note: First Plücker vectors $\mathbf{n}$ of screw axes and rotation angles appear in the dual parts as well. However, these quantities are known from solving the primary parts.

Due to these facts the principle of transference is a powerful tool for solving problems of very diverse nature. This is demonstrated in subsequent chapters of this book (Sects. 3.11 , 3.12 , 3.14 and Chaps. 5 , 7 , 8 , 9 and 13).
Literature: Löbell [29] (applications in kinematics, statics and differential geometry), Dimentberg [12] – [14], Keler [21] – [25], Yang [51, 52, 53], Yang/Freudenstein [54], Adams [1], Roth [44], Yuan/Freudenstein/Woo [55, 56], Veldkamp [49], Hsia/Yang [19], Castelain/Flamme/Gorla/Renaud [6], Pennock/Yang [37], Martinez/Duffy [30], Chevallier [9], Pennestri/Stefanelli [36] and the article by Pennock/Schaaf in Erdman [15].

3.10.1 Dual Basis. Dual Direction Cosine Matrix

Using Fig. 3.11 the notion of a (right-handed, orthogonal) dual basis is introduced. Point 0 is the origin of ordinary bases $\underline{\mathbf{e}}^1$ and $\underline{\mathbf{e}}^2$. These bases are related through the direction cosine matrix (see (1.6)):

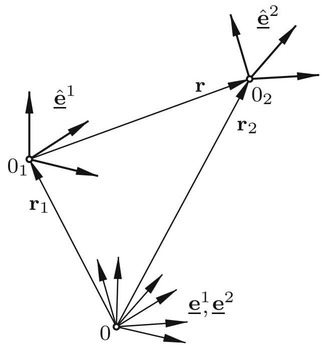

Fig. 3.11 Dual bases $\underline{\hat{\mathbf{e}}}^1$ and $\underline{\hat{\mathbf{e}}}^2$

$$\underline{\mathbf{e}}^1 = \underline{A}^{12}\underline{\mathbf{e}}^2 , \qquad \underline{A}^{12} = \underline{\mathbf{e}}^1 \cdot \underline{\mathbf{e}}^{2^T} . \tag{3.53}$$

Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ locate the origins 0_1 and 0_2, respectively, of two dual bases. These dual bases are formed by the dual unit vectors $\hat{\mathbf{e}}_i^1$ and $\hat{\mathbf{e}}_i^2$ $(i = 1, 2, 3)$ which are parallel to the basis vectors of $\underline{\mathbf{e}}^1$ and $\underline{\mathbf{e}}^2$, respectively. The dual basis vectors are

$$\hat{\mathbf{e}}_i^1 = \mathbf{e}_i^1 + \varepsilon\,\mathbf{r}_1 \times \mathbf{e}_i^1 \qquad \text{and} \qquad \hat{\mathbf{e}}_i^2 = \mathbf{e}_i^2 + \varepsilon\,\mathbf{r}_2 \times \mathbf{e}_i^2 \qquad (i = 1, 2, 3) . \tag{3.54}$$

Let $\underline{\hat{\mathbf{e}}}^1$ and $\underline{\hat{\mathbf{e}}}^2$ denote the two dual bases as well as the column matrices of their dual basis vectors. The two sets of equations are written in the matrix forms

$$\underline{\hat{\mathbf{e}}}^1 = \underline{\mathbf{e}}^1 + \varepsilon\mathbf{r}_1 \times \underline{\mathbf{e}}^1 , \qquad \underline{\hat{\mathbf{e}}}^2 = \underline{\mathbf{e}}^2 + \varepsilon\mathbf{r}_2 \times \underline{\mathbf{e}}^2 . \tag{3.55}$$

Every dual basis vector of $\underline{\hat{\mathbf{e}}}^1$ is a linear combination of the dual basis vectors of $\underline{\hat{\mathbf{e}}}^2$. This is written in the form

$$\hat{\mathbf{e}}_i^1 = \sum_{k=1}^{3} \hat{a}_{ik}^{12}\,\hat{\mathbf{e}}_k^2 \qquad (i = 1, 2, 3) . \tag{3.56}$$

By scalar multiplication of this equation by $\hat{\mathbf{e}}_j^2$ and by applying (3.48) and (3.49) it is shown that the coordinates represent dual direction cosines:

$$\hat{a}_{ij}^{12} = \hat{\mathbf{e}}_i^1 \cdot \hat{\mathbf{e}}_j^2 \qquad (i, j = 1, 2, 3) . \tag{3.57}$$

Since the dual basis vectors intersect orthogonally, the direction cosines satisfy the conditions

$$\sum_{k=1}^{3} \hat{a}_{ik}^{12}\,\hat{a}_{jk}^{12} = \delta_{ij} \qquad (i, j = 1, 2, 3) . \tag{3.58}$$

Let $\underline{\hat{A}}^{12}$ be the matrix of the dual direction cosines. Dualization of (3.53) yields

$$\underline{\hat{\mathbf{e}}}^1 = \underline{\hat{A}}^{12}\underline{\hat{\mathbf{e}}}^2\,, \qquad \underline{\hat{A}}^{12} = \underline{\hat{\mathbf{e}}}^1 \cdot \underline{\hat{\mathbf{e}}}^{2^T}\,. \tag{3.59}$$

Into the second equation the expressions (3.55) are substituted:

$$\begin{aligned}\underline{\hat{A}}^{12} &= (\underline{\mathbf{e}}^1 + \varepsilon\,\mathbf{r}_1 \times \underline{\mathbf{e}}^1) \cdot (\underline{\mathbf{e}}^2 + \varepsilon\,\mathbf{r}_2 \times \underline{\mathbf{e}}^2)^T \\ &= \underline{\mathbf{e}}^1 \cdot \underline{\mathbf{e}}^{2^T} - \varepsilon(\mathbf{r}_2 - \mathbf{r}_1) \cdot \underline{\mathbf{e}}^1 \times \underline{\mathbf{e}}^{2^T} \\ &= \underline{A}^{12} - \varepsilon\,(\mathbf{r}_2 - \mathbf{r}_1) \cdot \underline{\mathbf{e}}^1 \times \underline{\mathbf{e}}^{1^T}\underline{A}^{12}\,. \end{aligned} \tag{3.60}$$

Define $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \overrightarrow{0_1 0_2}$ and let $r_1\,,\ r_2\,,\ r_3$ be the coordinates of $\mathbf{r}$ in $\underline{\mathbf{e}}^1$. Then

$$\underline{\hat{A}}^{12} = (\underline{I} + \varepsilon\,\underline{\tilde{r}})\underline{A}^{12}\,, \qquad \underline{\tilde{r}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}\,. \tag{3.61}$$

From this it follows that the product $\underline{\hat{A}}^{12}\underline{\hat{A}}^{12^T}$ is the unit matrix. Hence its inverse is also its transpose:

$$(\underline{\hat{A}}^{12})^{-1} = \underline{\hat{A}}^{12^T}\,. \tag{3.62}$$

The determinant of $\underline{\hat{A}}^{12}$ equals 1 (the determinant of its dual part is zero).

Let $\underline{\hat{v}}^1$ and $\underline{\hat{v}}^2$ be the dual coordinate matrices of a dual vector of the form (3.37), $\hat{\mathbf{v}} = \mathbf{v} + \varepsilon\mathbf{w}$, in the dual bases $\underline{\hat{\mathbf{e}}}^1$ and $\underline{\hat{\mathbf{e}}}^2$, respectively. The vector need not be a unit vector. The first Eq.(3.59) is proof of the transformation rule

$$\underline{\hat{v}}^1 = \underline{\hat{A}}^{12}\underline{\hat{v}}^2\,. \tag{3.63}$$

This is the dualized form of the transformation rule $\underline{v}^1 = \underline{A}^{12}\underline{v}^2$ for ordinary vector coordinates.

3.10.2 Screw Axis, Screw Angle and Translation Determined from Dual Direction Cosines

From Chap. 1 on rotations about a fixed point Euler's Theorem 1.1 is known. It makes the following statements. The transformation matrix $\underline{A}^{12}$ has the eigenvalue one. In the case $\underline{A}^{12} \neq \underline{I}$, the matrix $\underline{A}^{12}$ determines uniquely a rotation $(\mathbf{n}, \varphi)$ carrying a body-fixed basis from the position $\underline{\mathbf{e}}^1$ into the position $\underline{\mathbf{e}}^2 = \underline{A}^{12^T}\underline{\mathbf{e}}^1$. The coordinate matrix $\underline{n}$ of $\mathbf{n}$, identical in both bases, is the solution of the equation

$$(\underline{A}^{12} - \underline{I})\underline{n} = \underline{0}\,. \tag{3.64}$$

If $\underline{A}^{12}$ is unsymmetric, φ and $\underline{n}$ are determined by (1.51) and (1.52):

$$\cos\varphi = \frac{1}{2}(\mathrm{tr}\,\underline{A}^{12}-1)\,, \quad 2n_i\sin\varphi = a^{12}_{kj}-a^{12}_{jk}\ \ (i,j,k=1,2,3\ \text{ cyclic})\,. \tag{3.65}$$

If $\underline{A}^{12}$ is symmetric, then $\varphi = \pm\pi$.

Transference of this theorem into dual form produces Chasles' Theorem 3.1 and the following statements. The dual transformation matrix $\underline{\hat{A}}^{12} = (\underline{I}+\varepsilon\,\underline{\tilde{r}})\underline{A}^{12}$ has the eigenvalue one. A formal proof is given further below. In the case $\underline{A}^{12} \neq \underline{I}$, the matrix $\underline{\hat{A}}^{12}$ determines uniquely a screw displacement $(\hat{\mathbf{n}}, \hat{\varphi})$ carrying a body-fixed dual basis from the position $\underline{\hat{\mathbf{e}}}^1$ into the position $\underline{\hat{\mathbf{e}}}^2 = \underline{\hat{A}}^{12T}\underline{\hat{\mathbf{e}}}^1$. As before, the dual angle $\hat{\varphi}$ and the dual unit line vector $\hat{\mathbf{n}}$ along the screw axis are written in the forms

$$\hat{\varphi} = \varphi + \varepsilon s\,, \qquad \hat{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{w} \qquad (\mathbf{n}^2 = 1\,,\ \mathbf{n}\cdot\mathbf{w} = 0)\,. \tag{3.66}$$

In what follows, it is shown how to determine the unknown scalars φ and s and the unknown Plücker vectors $\mathbf{n}$ and $\mathbf{w}$ if the matrix $\underline{\hat{A}}^{12}$ is given. The unknowns are determined from the equation $(\underline{\hat{A}}^{12} - \underline{I})\underline{\hat{n}} = 0$ or in detail

$$[(\underline{I} + \varepsilon\,\underline{\tilde{r}})\underline{A}^{12} - \underline{I}](\underline{n} + \varepsilon\underline{w}) = \underline{0}\,. \tag{3.67}$$

If $\underline{A}^{12}$ is unsymmetric, Eqs.(3.65) in dualized form are valid. Only the first equation is needed:

$$\cos\hat{\varphi} = \frac{1}{2}\Big\{\mathrm{tr}\,[(\underline{I} + \varepsilon\,\underline{\tilde{r}})\underline{A}^{12}] - 1\Big\}\,. \tag{3.68}$$

Equations (3.67) and (3.68) are split into their primary and dual parts. The primary parts are the original Eqs.(3.64) and (3.65) for $\underline{n}$ and φ. With the solutions for $\underline{n}$ and φ the dual parts of the equations determine $\underline{w}$ and s. The dual part of (3.67) is the equation $(\underline{A}^{12} - \underline{I})\underline{w} + \underline{\tilde{r}}\,\underline{A}^{12}\underline{n} = \underline{0}$ or, since $\underline{A}^{12}\underline{n} = \underline{n}$,

$$(\underline{A}^{12} - \underline{I})\underline{w} = -\underline{\tilde{r}}\,\underline{n}\,. \tag{3.69}$$

Since the matrix $\underline{A}^{12} - \underline{I}$ has rank two, the equation has a solution $\underline{w}$ only if the complete coefficient matrix including the right-hand side terms has rank two. This is, indeed, the case. Proof: The equation has the form $\underline{B}\,\underline{w} = -\underline{\tilde{r}}\,\underline{n}$. The homogeneous equation $\underline{B}\,\underline{w} = \underline{0}$ has the solution $\underline{w} = \mu\underline{n}$ (μ arbitrary). Because of the orthogonality of $\underline{A}^{12}$ also the equation $\underline{B}^T\,\underline{w} = \underline{0}$ has this solution. From this it follows that the rows of $\underline{B}^T$, i.e., the columns of $\underline{B}$ are in the plane orthogonal to $\underline{n}$. In this plane also the column matrix $-\underline{\tilde{r}}\,\underline{n}$ is located since it is the coordinate matrix of the vector $\mathbf{n}\times\mathbf{r}$. End of proof. Hence the inhomogeneous equation has a solution $\underline{w}_{\mathrm{p}}$. The complete solution is $\underline{w} = \mu\underline{n} + \underline{w}_{\mathrm{p}}$. From the conditions $\mathbf{n}^2 = 1$ and $\mathbf{n}\cdot\mathbf{w} = 0$ valid for Plücker vectors it follows that $\mu = -\underline{n}^T \underline{w}_\mathrm{p}$. Thus, the final solution for $\underline{w}$ is

$$\underline{w} = (\underline{I} - \underline{n}\underline{n}^T)\underline{w}_\mathrm{p} \ . \tag{3.70}$$

The perpendicular from 0_1 onto the screw axis is the vector $\mathbf{u} = \mathbf{n} \times \mathbf{w} = \mathbf{n} \times \mathbf{w}_\mathrm{p}$. It has the coordinate matrix

$$\underline{u} = \underline{\tilde{n}}\,\underline{w}_\mathrm{p} \ . \tag{3.71}$$

These results cover also the special case when the right-hand side of (3.69) equals zero which means $\mathbf{n} \times \mathbf{r} = \mathbf{0}$. Then $\underline{w}_\mathrm{p} = \underline{0}$ and $\underline{w} = \underline{0}$. This means that the screw axis is the line connecting the origins 0_1 and 0_2 . This is obvious without any analysis.

The dual part of (3.68) determines s . This equation reads (omitting the upper indices in the elements of $\underline{A}^{12}$) :

$$\begin{aligned} -s\sin\varphi &= \frac{1}{2}\mathrm{tr}\,(\underline{\tilde{r}}\,\underline{A}^{12}) \\ &= \frac{1}{2}\Big[(a_{23} - a_{32})r_1 + (a_{31} - a_{13})r_2 + (a_{12} - a_{21})r_3\Big] \ . \end{aligned} \tag{3.72}$$

The three differences of matrix elements are expressed with the help of the second Eq.(3.65). This yields for s the explicit expression

$$s = n_1 r_1 + n_2 r_2 + n_3 r_3 = \underline{n}^T \underline{r}^1 \ . \tag{3.73}$$

This is identical with (3.9). For the perpendicular $\mathbf{u}$ of a screw displacement with given quantities $\mathbf{n}$, φ and $\mathbf{r}$ vector methods led to (3.12). Comparison with (3.71), $\mathbf{u} = \mathbf{n} \times \mathbf{w}_\mathrm{p}$, allows an interpretation of $\mathbf{w}_\mathrm{p}$.

In what follows, it is proved that the dual matrix $\underline{\hat{A}}^{12}$ has the eigenvalue one. The characteristic equation is $\det[(\underline{I} + \varepsilon\,\underline{\tilde{r}})\underline{A}^{12} - \lambda\underline{I}] = 0$. The term free of λ consists of 24 expressions which cancel each other pairwise. The remaining terms are (the upper indices in the elements of $\underline{A}^{12}$ are omitted)

$$\det(\underline{A}^{21} - \lambda\underline{I}) + \varepsilon\,\lambda\sum_{i=1}^{3} r_i\Big[\lambda(a_{jk} - a_{kj}) - (a_{ij}a_{ki} - a_{ii}a_{kj}) + (a_{ji}a_{ik} - a_{ii}a_{jk})\Big] = 0 \tag{3.74}$$

$(i, j, k = 1, 2, 3$ cyclic). From (1.10) it follows that this equation is solved with $\lambda = 1$. End of proof.

The expression (3.61) for the dual transformation matrix was obtained by applying the transference principle to the matrix $\underline{A}^{12}$ of a rotation expressed in terms of direction cosines. In Chap. 1 the matrix $\underline{A}^{12}$ has been expressed in various ways by the unit vector $\mathbf{n}$ along the axis of a rotation and by the angle φ. The most useful expressions are those in (1.49) in terms of n_1 , n_2 , n_3 , $\sin\varphi$, $\cos\varphi$, in (1.79) in terms of Euler-Rodrigues parameters and in (1.170) in terms of the coordinates of the Rodrigues vector. All these expressions can be transferred into dual form. Transference of (1.49) yields the expression

$$\underline{\hat{A}}^{12} = \begin{bmatrix} \hat{n}_1^2 + (1-\hat{n}_1^2)\cos\hat{\varphi} & \hat{n}_1\hat{n}_2(1-\cos\hat{\varphi}) - \hat{n}_3\sin\hat{\varphi} & \hat{n}_1\hat{n}_3(1-\cos\hat{\varphi}) + \hat{n}_2\sin\hat{\varphi} \\ \hat{n}_1\hat{n}_2(1-\cos\hat{\varphi}) + \hat{n}_3\sin\hat{\varphi} & \hat{n}_2^2 + (1-\hat{n}_2^2)\cos\hat{\varphi} & \hat{n}_2\hat{n}_3(1-\cos\hat{\varphi}) - \hat{n}_1\sin\hat{\varphi} \\ \hat{n}_1\hat{n}_3(1-\cos\hat{\varphi}) - \hat{n}_2\sin\hat{\varphi} & \hat{n}_2\hat{n}_3(1-\cos\hat{\varphi}) + \hat{n}_1\sin\hat{\varphi} & \hat{n}_3^2 + (1-\hat{n}_3^2)\cos\hat{\varphi} \end{bmatrix} . \quad (3.75)$$

Example: Determine φ, $\underline{\hat{n}} = \underline{n} + \varepsilon\underline{w}$, $\underline{u} = \underline{\tilde{n}}\,\underline{w}$, $s = \underline{n}^T\underline{r}$ and the matrix $\underline{\hat{A}}^{12}$ from the given quantities

$$\underline{A}^{12} = \begin{bmatrix} \frac{2}{3} & -\frac{11}{15} & -\frac{2}{15} \\ \frac{1}{3} & \frac{2}{15} & \frac{14}{15} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad \underline{r} = \frac{2}{15}\begin{bmatrix} 50 \\ 30 \\ 11 \end{bmatrix} . \quad (3.76)$$

Solution: Equation (3.65) yields $\cos\varphi = 1/15$, $\sin\varphi = 4\sqrt{14}/15$ (arbitrarily positive), $\underline{n} = (1/\sqrt{14})\,[\,-3\;\;1\;\;2\,]^T$. Equation (3.69) reads

$$\begin{bmatrix} -5 & -11 & -2 \\ 5 & -13 & 14 \\ -10 & -10 & -10 \end{bmatrix} \underline{w} = \sqrt{14}\begin{bmatrix} -7 \\ 19 \\ -20 \end{bmatrix} . \quad (3.77)$$

It has the solution $\underline{w}_{\mathrm{p}} = \sqrt{14}\,[\,1\;\;0\;\;1\,]^T$. In (3.70) $\underline{n}^T\underline{w}_{\mathrm{p}} = -1$. Furthermore,

$$\underline{\hat{n}} = \frac{1}{\sqrt{14}}\begin{bmatrix} -3+11\varepsilon \\ 1+\varepsilon \\ 2+16\varepsilon \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \quad s = -\frac{14\sqrt{14}}{15} . \quad (3.78)$$

For $\underline{u}$ the same result is obtained from (3.13). For (3.75) the quantities are calculated:

$$\left.\begin{aligned} \cos\hat{\varphi} &= \cos\varphi - \varepsilon s\,\sin\varphi = \tfrac{1}{15^2}(15+28^2\varepsilon) , \\ \sin\hat{\varphi} &= \sin\varphi + \varepsilon s\,\cos\varphi = \tfrac{2\sqrt{14}}{15^2}(30-7\varepsilon) , \end{aligned}\right\} \quad (3.79)$$

$$\left.\begin{aligned} \hat{n}_1^2 = \tfrac{1}{14}(9-66\varepsilon) , \quad \hat{n}_1\hat{n}_2 &= \tfrac{1}{14}(-3+8\varepsilon) , \quad \hat{n}_1\hat{n}_3 = \tfrac{1}{14}(-6-26\varepsilon), \\ \hat{n}_2^2 &= \tfrac{1}{14}(1+2\varepsilon) , \quad \hat{n}_2\hat{n}_3 = \tfrac{1}{14}(2+18\varepsilon) , \\ \hat{n}_3^2 &= \tfrac{1}{14}(4+64\varepsilon) . \end{aligned}\right\} \quad (3.80)$$

The desired matrix is

$$\underline{\hat{A}}^{12} = \begin{bmatrix} \frac{2}{3} & -\frac{11}{15} & -\frac{2}{15} \\ \frac{1}{3} & \frac{2}{15} & \frac{14}{15} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} + \varepsilon \begin{bmatrix} -\frac{142}{45} & -\frac{644}{225} & -\frac{8}{225} \\ \frac{244}{45} & \frac{758}{225} & -\frac{544}{225} \\ -\frac{4}{9} & \frac{173}{45} & \frac{304}{45} \end{bmatrix} . \quad (3.81)$$

The primary part is the given matrix (3.76). End of Example.

3.10.3 Dual Euler Angles. Dual Bryan Angles

In Fig. 3.12 the dual basis $\underline{\hat{\mathbf{e}}}^2$ is produced from $\underline{\hat{\mathbf{e}}}^1$ by a screw displacement about the axis $\hat{\mathbf{e}}_1^1$ with the rotation angle ϕ_1 and the translation u_1. With $\hat{\phi}_1 = \phi_1 + \varepsilon u_1$ the dual direction cosine matrix is

$$\underline{\hat{A}}_1^{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\hat{\phi}_1 & -\sin\hat{\phi}_1 \\ 0 & \sin\hat{\phi}_1 & \cos\hat{\phi}_1 \end{bmatrix} . \tag{3.82}$$

Formulas for $\cos\hat{\phi}$ and $\sin\hat{\phi}$ are given in (3.30). With the abbreviations $c_1 = \cos\phi_1$ and $s_1 = \sin\phi_1$ these formulas are written in the forms $\cos\hat{\phi}_1 = c_1 - \varepsilon u_1 s_1$, $\sin\hat{\phi}_1 = s_1 + \varepsilon u_1 c_1$. Hence

$$\underline{\hat{A}}_1^{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} + \varepsilon u_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s_1 & -c_1 \\ 0 & c_1 & -s_1 \end{bmatrix} . \tag{3.83}$$

The same result is obtained from (3.61) with $r_1 = u_1$, $r_2 = r_3 = 0$. The dual part is calculated from the primary part by the rule of dual differentiation.

In what follows, the same idea is applied to the direction cosine matrix $\underline{A}^{12}$ expressed as function of Euler angles ψ, θ and ϕ (see (1.28)). Euler angles are defined in Fig. 1.1a. They are angles of subsequent rotations about the axes $\mathbf{e}_3^1$, $\mathbf{e}_1^{2''}$ and $\mathbf{e}_3^{2'} = \mathbf{e}_3^2$. Each of these three rotations is replaced by a screw displacement about the respective axis. The three dual screw angles are denoted $\hat{\psi} = \psi + \varepsilon u_\psi$, $\hat{\theta} = \theta + \varepsilon u_\theta$ and $\hat{\phi} = \phi + \varepsilon u_\phi$, respectively. They are dual Euler angles. The dual direction cosine matrix $\underline{\hat{A}}^{12}$ is obtained from the direction cosine matrix $\underline{A}^{12}$ by the rule of dual differentiation:

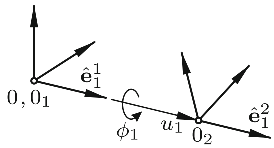

Fig. 3.12 Screw displacement $(\hat{\mathbf{e}}_1^1, \hat{\phi}_1)$ with $\hat{\phi}_1 = \phi_1 + \varepsilon u_1$

$$
\begin{aligned}
\underline{\hat{A}}^{12} = &\begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & -c_\psi s_\phi - s_\psi c_\theta c_\phi & s_\psi s_\theta \\ s_\psi c_\phi + c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & -c_\psi s_\theta \\ s_\theta s_\phi & s_\theta c_\phi & c_\theta \end{bmatrix} \\
&+\varepsilon \left(u_\psi \begin{bmatrix} -s_\psi c_\phi - c_\psi c_\theta s_\phi & s_\psi s_\phi - c_\psi c_\theta c_\phi & c_\psi s_\theta \\ c_\psi c_\phi - s_\psi c_\theta s_\phi & -c_\psi s_\phi - s_\psi c_\theta c_\phi & s_\psi s_\theta \\ 0 & 0 & 0 \end{bmatrix} \right. \\
&\quad +u_\theta \begin{bmatrix} s_\psi s_\theta s_\phi & s_\psi s_\theta c_\phi & s_\psi c_\theta \\ -c_\psi s_\theta s_\phi & -c_\psi s_\theta c_\phi & -c_\psi c_\theta \\ c_\theta s_\phi & c_\theta c_\phi & -s_\theta \end{bmatrix} \\
&\quad \left. +u_\phi \begin{bmatrix} -c_\psi s_\phi - s_\psi c_\theta c_\phi & -c_\psi c_\phi + s_\psi c_\theta s_\phi & 0 \\ -s_\psi s_\phi + c_\psi c_\theta c_\phi & -s_\psi c_\phi - c_\psi c_\theta s_\phi & 0 \\ s_\theta c_\phi & -s_\theta s_\phi & 0 \end{bmatrix} \right) . \qquad (3.84)
\end{aligned}
$$

The primary part is the matrix $\underline{A}^{12}$ of Eq.(1.28). Dual differentiation of the element (1,3), i.e., of the product $s_\psi s_\theta$, yields the expression $u_\psi c_\psi s_\theta + u_\theta\, s_\psi c_\theta$. These two terms are the elements (1,3) of the matrices associated with u_ψ and u_θ.

The same procedure is applied to the direction cosine matrix expressed as function of Bryan angles ϕ_1, ϕ_2 and ϕ_3 (see Fig. 1.2a and Eq.(1.32)). Each of the three subsequent rotations about the axes $\mathbf{e}_1^1$, $\mathbf{e}_2^{2''}$ and $\mathbf{e}_3^{2'} = \mathbf{e}_3^2$ is replaced by a screw displacement about the respective axis. The dual screw angles are denoted $\hat{\phi}_i = \phi_i + \varepsilon u_i \;\; (i = 1, 2, 3)$. They are dual Bryan angles. The associated dual direction cosine matrix is

$$
\begin{aligned}
\underline{\hat{A}}^{12} = &\begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ c_1 s_3 + s_1 s_2 c_3 & c_1 c_3 - s_1 s_2 s_3 & -s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 & s_1 c_3 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix} \\
&+\varepsilon \left(u_1 \begin{bmatrix} 0 & 0 & 0 \\ -s_1 s_3 + c_1 s_2 c_3 & -s_1 c_3 - c_1 s_2 s_3 & -c_1 c_2 \\ c_1 s_3 + s_1 s_2 c_3 & c_1 c_3 - s_1 s_2 s_3 & -s_1 c_2 \end{bmatrix} \right. \\
&\quad +u_2 \begin{bmatrix} -s_2 c_3 & s_2 s_3 & c_2 \\ s_1 c_2 c_3 & -s_1 c_2 s_3 & -s_1 s_2 \\ -c_1 c_2 c_3 & -c_1 c_2 s_3 & c_1 s_2 \end{bmatrix} \\
&\quad \left. +u_3 \begin{bmatrix} -c_2 s_3 & -c_2 c_3 & 0 \\ c_1 c_3 - s_1 s_2 s_3 & -c_1 s_3 - s_1 s_2 c_3 - c_1 s_3 & 0 \\ s_1 c_3 + c_1 s_2 s_3 & -s_1 s_3 + c_1 s_2 c_3 & 0 \end{bmatrix} \right) \qquad (3.85)
\end{aligned}
$$

The primary part is the matrix of Eq.(1.32). Linearization in the case of small angles yields

$$\underline{\hat{A}}^{12} \approx \begin{bmatrix} 1 & -\phi_3 & \phi_2 \\ \phi_3 & 1 & -\phi_1 \\ -\phi_2 & \phi_1 & 1 \end{bmatrix}$$
$$+ \varepsilon \left(u_1 \begin{bmatrix} 0 & 0 & 0 \\ \phi_2 & -\phi_1 & -1 \\ \phi_3 & 1 & -\phi_1 \end{bmatrix} + u_2 \begin{bmatrix} -\phi_2 & 0 & 1 \\ \phi_1 & 0 & 0 \\ -1 & -\phi_3 & \phi_2 \end{bmatrix} + u_3 \begin{bmatrix} -\phi_3 & -1 & 0 \\ 1 & -\phi_3 & 0 \\ \phi_1 & \phi_2 & 0 \end{bmatrix} \right) . \tag{3.86}$$

3.10.4 Dual Rodrigues Vector

The Rodrigues vector of a rotation $(\mathbf{n}, \varphi)$ is the vector $\mathbf{u} = \mathbf{n} \tan \varphi/2$. The dual Rodrigues vector of a screw displacement $(\hat{\mathbf{n}}, \hat{\varphi})$ with $\hat{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{w}$ and $\hat{\varphi} = \varphi + \varepsilon s$ is

$$\begin{aligned} \hat{\mathbf{u}} = \hat{\mathbf{n}} \tan \frac{\hat{\varphi}}{2} &= (\mathbf{n} + \varepsilon \mathbf{w}) \Big(\tan \frac{\varphi}{2} + \varepsilon \frac{s/2}{\cos^2 \frac{\varphi}{2}} \Big) \\ &= \mathbf{n} \tan \frac{\varphi}{2} + \varepsilon \Big[\mathbf{n} \frac{s}{2} \Big(1 + \tan^2 \frac{\varphi}{2} \Big) + \mathbf{w} \tan \frac{\varphi}{2} \Big] . \end{aligned} \tag{3.87}$$

3.10.5 Dual Euler-Rodrigues Parameters. Dual Quaternions

From (1.67) the Euler-Rodrigues parameters of a rotation $(\mathbf{n}, \varphi)$ are known:

$$q_0 = \cos \frac{\varphi}{2} , \qquad \mathbf{q} = \mathbf{n} \sin \frac{\varphi}{2} . \tag{3.88}$$

They satisfy the constraint equation

$$q_0^2 + \mathbf{q}^2 = 1 . \tag{3.89}$$

According to Fig. 1.3 and to (1.70) the parameters establish between the position vectors $\boldsymbol{\varrho}$ and $\boldsymbol{\varrho}^*$ of a body-fixed point before and after the rotation the relationship

$$\boldsymbol{\varrho}^* = \boldsymbol{\varrho} + 2[\mathbf{q} \times (\mathbf{q} \times \boldsymbol{\varrho}) + q_0 \mathbf{q} \times \boldsymbol{\varrho}] . \tag{3.90}$$

The quaternion of the rotation is

$$D = (q_0 \, , \, \mathbf{q}) . \tag{3.91}$$

Corresponding dual quantities are defined for a screw displacement $(\hat{\mathbf{n}}, \hat{\varphi})$. With the quantities shown in Fig. 3.3 the dual screw angle $\hat{\varphi}$ and the dual unit vector $\hat{\mathbf{n}}$ along the screw axis are

$$\hat{\varphi} = \varphi + \varepsilon s \,, \qquad \hat{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{u} \times \mathbf{n} \,, \qquad (\mathbf{u} \cdot \mathbf{n} = 0\,) \,. \tag{3.92}$$

For the translatory displacement $\mathbf{r} = \overrightarrow{0_1 0_2}$ in Fig. 3.3 Eq.(3.8) provides the expression

$$\mathbf{r} = s\mathbf{n} + \mathbf{u}(1 - \cos\varphi) + \mathbf{u} \times \mathbf{n} \sin\varphi \,. \tag{3.93}$$

The square of this vector is

$$\mathbf{r}^2 = s^2 + 2\mathbf{u}^2(1 - \cos\varphi) \,. \tag{3.94}$$

By definition, the dual Euler-Rodrigues parameters of the screw displacement are

$$\left.\begin{aligned} \hat{q}_0 &= \cos\frac{\hat{\varphi}}{2} &&= \cos\frac{\varphi}{2} - \varepsilon\frac{s}{2}\sin\frac{\varphi}{2} \,, \\ \hat{\mathbf{q}} &= \hat{\mathbf{n}}\sin\frac{\hat{\varphi}}{2} &&= (\mathbf{n} + \varepsilon\mathbf{u} \times \mathbf{n})\Big(\sin\frac{\hat{\varphi}}{2} + \varepsilon\frac{s}{2}\cos\frac{\varphi}{2}\Big) \\ & &&= \mathbf{n}\sin\frac{\varphi}{2} + \varepsilon\Big(\frac{s}{2}\mathbf{n}\cos\frac{\varphi}{2} + \mathbf{u} \times \mathbf{n}\sin\frac{\varphi}{2}\Big) \,. \end{aligned}\right\} \tag{3.95}$$

The dual parts are abbreviated

$$q_0' = -\frac{s}{2}\sin\frac{\varphi}{2} \,, \qquad \mathbf{q}' = \frac{s}{2}\mathbf{n}\cos\frac{\varphi}{2} + \mathbf{u} \times \mathbf{n}\sin\frac{\varphi}{2} \,. \tag{3.96}$$

With this notation

$$\hat{q}_0 = q_0 + \varepsilon q_0' \,, \qquad \hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}' \,. \tag{3.97}$$

The dual quaternion of the screw displacement is

$$\hat{D} = (\hat{q}_0 \,, \hat{\mathbf{q}}) = D + \varepsilon D' \tag{3.98}$$

with

$$D = (q_0 \,, \mathbf{q}) \,, \qquad D' = (q_0' \,, \mathbf{q}') \,. \tag{3.99}$$

The square of the norm of D' is

$$q_0'^{\,2} + \mathbf{q}'^2 = \frac{s^2}{4} + \mathbf{u}^2\sin^2\frac{\varphi}{2} = \frac{1}{4}[s^2 + 2\mathbf{u}^2(1 - \cos\varphi)] \,. \tag{3.100}$$

Comparison with (3.94) shows that

$$\mathbf{r}^2 = 4(q_0'^{\,2} + \mathbf{q}'^2) \,. \tag{3.101}$$

Next, the quaternion product $D\,\tilde{D}' = (q_0 \,, \mathbf{q})(q_0' \,, -\mathbf{q}')$ is calculated by the multiplication rule (1.98). The scalar part is

$$q_0 q_0' + \mathbf{q} \cdot \mathbf{q}' = 0 \,. \tag{3.102}$$

This equation is referred to as Study-quadric. It expresses the orthogonality of the primary and the dual part of the Euler-Rodrigues parameters. The vector part of $D\,\tilde{D}'$ is

$$\begin{aligned}-q_0\mathbf{q}' + q_0'\mathbf{q} - \mathbf{q}\times\mathbf{q}' &= -\Big[\cos\frac{\varphi}{2}\Big(\frac{s}{2}\mathbf{n}\cos\frac{\varphi}{2} + \mathbf{u}\times\mathbf{n}\sin\frac{\varphi}{2}\Big) \\ &\quad + \frac{s}{2}\sin\frac{\varphi}{2}\,\mathbf{n}\sin\frac{\varphi}{2} + \mathbf{n}\sin\frac{\varphi}{2}\times(\mathbf{u}\times\mathbf{n})\sin\frac{\varphi}{2}\Big] \\ &= -\frac{1}{2}[s\mathbf{n} + \mathbf{u}\times\mathbf{n}\sin\varphi + \mathbf{u}(1-\cos\varphi)]\ . \end{aligned} \tag{3.103}$$

Comparison with (3.93) reveals the equation

$$\mathbf{r} = 2(q_0\mathbf{q}' - q_0'\mathbf{q} + \mathbf{q}\times\mathbf{q}')\ . \tag{3.104}$$

Let $\boldsymbol{\varrho}$ and $\boldsymbol{\varrho}^*$ be the vectors from the reference point 0_1 to a body-fixed point before and after the screw displacement. According to (3.90) and (3.104) the relationship between these vectors is

$$\boldsymbol{\varrho}^* = \boldsymbol{\varrho} + 2[\mathbf{q}\times(\mathbf{q}\times\boldsymbol{\varrho}) + q_0\mathbf{q}\times\boldsymbol{\varrho}] + 2(q_0\mathbf{q}' - q_0'\mathbf{q} + \mathbf{q}\times\mathbf{q}')\ . \tag{3.105}$$

The dualized form of Theorem 1.4 is

Theorem 3.5. *The dual quaternion $\hat{D}_{\text{res}}$ of the resultant of two subsequent screw displacements with dual quaternions $\hat{D}_1$ (first screw displacement) and $\hat{D}_2$ is the product*

$$\hat{D}_{\text{res}} = \hat{D}_2\hat{D}_1\ . \tag{3.106}$$

Applications of the above equations see in Sect. 3.11 and in Chap. 8 . Additional material see in Ravani/Roth [41].

3.11 Resultant of two Screw Displacements. Dual-Quaternion Formulation

Halphen's geometrical construction of the resultant of two screw displacements resulted in the spatial hexagon shown in Fig. 3.7 . Extracting analytical expressions for the unknowns φ_{res} , s_{res} and S_{res} from this figure is difficult. Explicit solutions are most easily obtained on the basis of Theorem 3.5. The quaternion equation $D_{\text{res}} = D_2D_1$ for the resultant $(\mathbf{n}_{\text{res}}, \varphi_{\text{res}})$ of two successive rotations $(\mathbf{n}_1, \varphi_1)$ (first rotation) and $(\mathbf{n}_2, \varphi_2)$ resulted in the explicit coordinate-free Eqs.(1.118) and (1.119). Decomposition of vectors in the basis shown in Fig. 1.4 led to Eqs.(1.120) – (1.122):

$$\mathbf{n}_{1,2} = \mathbf{e}_1\cos\frac{\alpha}{2} \mp \mathbf{e}_2\sin\frac{\alpha}{2}\ , \tag{3.107}$$

$$\cos\frac{\varphi_{\text{res}}}{2} = \cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} - \sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\,\cos\alpha\,, \tag{3.108}$$

$$\begin{aligned}\mathbf{n}_{\text{res}}\sin\frac{\varphi_{\text{res}}}{2} &= \mathbf{e}_1\sin\frac{\varphi_1+\varphi_2}{2}\,\cos\frac{\alpha}{2} - \mathbf{e}_2\sin\frac{\varphi_1-\varphi_2}{2}\,\sin\frac{\alpha}{2}\\ &\quad -\mathbf{e}_3\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\,\sin\alpha\,.\end{aligned} \tag{3.109}$$

Theorem 3.5 states that the same equations are valid when vectors $\mathbf{n}$ of rotation axes are replaced by dual vectors $\hat{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{w}$ of screw axes and rotation angles φ by dual screw angles $\hat{\varphi} = \varphi + \varepsilon s$. The angle α between intersecting rotation axes is replaced by the dual angle $\hat{\alpha} = \alpha + \varepsilon\ell$ of the screw displacement which carries $\hat{\mathbf{n}}_1$ into $\hat{\mathbf{n}}_2$ (this means that ℓ, positive or negative, is the length of the common perpendicular of the two screw axes). For making (3.107) with $\hat{\mathbf{n}}_{1,2}$ and $\hat{\alpha}$ valid the origin 0 of the basis $\mathbf{e}_{1,2,3}$ of Fig. 1.4 must be the midpoint of the common perpendicular (see Fig. 3.13). The primary parts of the dualized equations are Eqs.(3.108) and (3.109). The dual parts are

$$\begin{aligned}s_{\text{res}}\sin\frac{\varphi_{\text{res}}}{2} = {}& s_1\Big(\sin\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} + \cos\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\cos\alpha\Big)\\ &+s_2\Big(\cos\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2} + \sin\frac{\varphi_1}{2}\,\cos\frac{\varphi_2}{2}\,\cos\alpha\Big)\\ &-2\ell\sin\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\sin\alpha\,,\end{aligned} \tag{3.110}$$

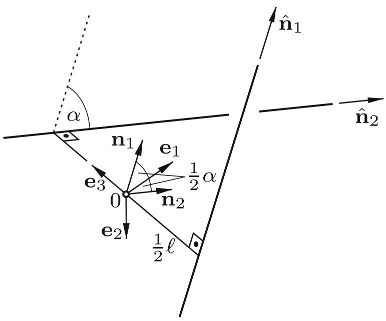

Fig. 3.13 Screw axes with reference basis $\mathbf{e}_{1,2,3}$ on the common perpendicular

$$\begin{aligned}
&\mathbf{n}_{\rm res}s_{\rm res}\cos\frac{\varphi_{\rm res}}{2}+2\mathbf{w}_{\rm res}\sin\frac{\varphi_{\rm res}}{2}\\
&\qquad=\quad \mathbf{e}_1\Big[(s_1+s_2)\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\alpha}{2}-\ell\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\alpha}{2}\Big]\\
&\qquad\quad -\mathbf{e}_2\Big[(s_1-s_2)\cos\frac{\varphi_1-\varphi_2}{2}\sin\frac{\alpha}{2}+\ell\sin\frac{\varphi_1-\varphi_2}{2}\cos\frac{\alpha}{2}\Big]\\
&\qquad\quad -\mathbf{e}_3\Big[\Big(s_1\cos\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}+s_2\sin\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2}\Big)\sin\alpha\\
&\qquad\qquad +2\ell\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\cos\alpha\Big]\,. \qquad (3.111)
\end{aligned}$$

From (3.108) $\varphi_{\rm res}$ is determined (sign arbitrary), from (3.109) $\mathbf{n}_{\rm res}$, from (3.110) $s_{\rm res}$ and from (3.111) $\mathbf{w}_{\rm res}$. A sign change of $\varphi_{\rm res}$ results in a change of signs of all other quantities. For the screw displacement this has no effect. The quantities s_1, s_2 and $s_{\rm res}$ appear in linear form only.

For specifying the location of the resultant screw axis the perpendicular $\mathbf{u}=\mathbf{n}_{\rm res}\times\mathbf{w}_{\rm res}$ from point 0 onto the screw axis is needed. The cross-product of the vectors in (3.109) and (3.111) is $2\mathbf{u}\sin^2\varphi_{\rm res}/2$ where $\sin^2\varphi_{\rm res}/2$ is determined by (3.108). The result of this multiplication is[4]

$$\begin{aligned}
\mathbf{u}\sin^2\frac{\varphi_{\rm res}}{2}&=-\mathbf{e}_1\sin^2\frac{\alpha}{2}\Big[\Big(s_1\sin^2\frac{\varphi_2}{2}-s_2\sin^2\frac{\varphi_1}{2}\Big)\cos\frac{\alpha}{2}\\
&\qquad +\ell\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\sin\frac{\alpha}{2}\Big]\\
&\quad +\mathbf{e}_2\cos^2\frac{\alpha}{2}\Big[\Big(s_1\sin^2\frac{\varphi_2}{2}+s_2\sin^2\frac{\varphi_1}{2}\Big)\sin\frac{\alpha}{2}\\
&\qquad +\ell\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\alpha}{2}\Big]\\
&\quad +\frac{1}{4}\mathbf{e}_3\Big[(s_2\sin\varphi_1-s_1\sin\varphi_2)\sin\alpha+\ell(\cos\varphi_1-\cos\varphi_2)\Big]\,. \qquad (3.112)
\end{aligned}$$

In accordance with Fig. 3.7 this equation shows that, in general, the resultant screw axis does not intersect the common perpendicular $\mathbf{e}_3$ of the screw axes 1 and 2. The resultant screw displacement has scalar measures $p_{\rm D}$ and $p_{\rm P}$ defined by (3.18) and (3.19). They are written in the forms

$$p_{\rm D}=\frac{s_{\rm res}}{\sin\varphi_{\rm res}}=\frac{s_{\rm res}\sin\frac{\varphi_{\rm res}}{2}}{2\sin^2\frac{\varphi_{\rm res}}{2}\cos\frac{\varphi_{\rm res}}{2}}\,,\qquad p_{\rm P}=p_{\rm D}\cos^2\frac{\varphi_{\rm res}}{2}\,. \qquad (3.113)$$

With (3.108) and (3.110) both measures are expressed in terms of s_1, φ_1, s_2, φ_2, α and ℓ. The quantities s_1 and s_2 appear only in the numerator expressions. If $p_{{\rm D}_i}$ and $p_{{\rm P}_i}$ $(i=1,2)$ denote the corresponding measures of

[4] Alternative forms for the factors of ℓ in (3.112):
$\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}\sin\frac{\varphi_1\pm\varphi_2}{2}=\frac{1}{4}\left[\sin\varphi_2\pm\sin\varphi_1\mp\sin(\varphi_1\pm\varphi_2)\right]$

the screw displacements 1 and 2, p_{D} is a linear function of p_{D_1} and p_{D_2}, and p_{P} is a linear function of p_{P_1} and p_{P_2}. The coefficients are functions of φ_1 and φ_2.

In what follows, special cases are investigated.
Special case $s_1 = s_2 = s$, $\varphi_1 = \varphi_2 = \varphi$: Equations (3.108) – (3.112) become

$$\left.\begin{aligned} \cos\frac{\varphi_{\mathrm{res}}}{2} &= \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\,\cos\alpha\,,\\ \mathbf{n}_{\mathrm{res}}\sin\frac{\varphi_{\mathrm{res}}}{2} &= \mathbf{e}_1\sin\varphi\,\cos\frac{\alpha}{2} - \mathbf{e}_3\sin^2\frac{\varphi}{2}\,\sin\alpha\,,\\ s_{\mathrm{res}}\sin\frac{\varphi_{\mathrm{res}}}{2} &= s\,\sin\varphi(1+\cos\alpha) - \ell(1-\cos\varphi)\sin\alpha\,,\\ \mathbf{u}\,\sin^2\frac{\varphi_{\mathrm{res}}}{2} &= \mathbf{e}_2\cos^2\frac{\alpha}{2}\sin^2\frac{\varphi}{2}\Big(2s\,\sin\frac{\alpha}{2} + \ell\sin\varphi\cos\frac{\alpha}{2}\Big)\,. \end{aligned}\right\} \tag{3.114}$$

Special case $\varphi_2 = 0$: The second screw displacement is the pure translation $s_2\mathbf{n}_2$. The screw axis $\hat{\mathbf{n}}_2$, the common perpendicular of length ℓ and the origin 0 of the $\mathbf{e}_{1,2,3}$-system are not uniquely defined. Equations (3.108) – (3.112) reduce to

$$\left.\begin{aligned} &\varphi_{\mathrm{res}} = \varphi_1\,,\qquad \mathbf{n}_{\mathrm{res}} = \mathbf{n}_1\,,\qquad s_{\mathrm{res}} = s_1 + s_2\cos\alpha\,,\\ &\mathbf{u} = -\frac{1}{2}\ell\mathbf{e}_3 + \frac{1}{2}s_2\sin\alpha\left(\mathbf{e}_1\,\sin\frac{\alpha}{2} + \mathbf{e}_2\,\cos\frac{\alpha}{2} + \mathbf{e}_3\,\cot\frac{\varphi_1}{2}\right)\\ &\quad = -\frac{1}{2}\ell\mathbf{e}_3 + \frac{1}{2}s_2\sin\alpha\left(\mathbf{e}_3\times\mathbf{n}_1 + \mathbf{e}_3\,\cot\frac{\varphi_1}{2}\right)\,. \end{aligned}\right\} \tag{3.115}$$

The leading term $(-\ell/2)\mathbf{e}_3$ is the perpendicular vector from the arbitrarily chosen origin 0 onto the screw axis $\hat{\mathbf{n}}_1$. If an arbitrary point on $\hat{\mathbf{n}}_1$ is chosen as origin 0,

$$\mathbf{u} = \frac{1}{2}s_2\sin\alpha\left(\mathbf{e}_3\times\mathbf{n}_1 + \mathbf{e}_3\,\cot\frac{\varphi_1}{2}\right)\,. \tag{3.116}$$

The absolute value is

$$|\mathbf{u}| = \left|\frac{\frac{s_2}{2}}{\sin\frac{\varphi_1}{2}}\,\sin\alpha\right|\,. \tag{3.117}$$

Special case $s_1 = s_2 = 0$ (resultant of pure rotations about nonintersecting axes; see Fig. 3.8): Equations (3.108) and (3.109) remain valid without change. Equations (3.110) and (3.112) reduce to

$$s_{\mathrm{res}}\sin\frac{\varphi_{\mathrm{res}}}{2} = -2\ell\sin\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\sin\alpha\,, \tag{3.118}$$

$$\mathbf{u}\sin^2\frac{\varphi_{\rm res}}{2}=\frac{\ell}{4}\Big\{-\mathbf{e}_1\sin^3\frac{\alpha}{2}\Big[\sin\varphi_2-\sin\varphi_1+\sin(\varphi_1-\varphi_2)\Big] + \mathbf{e}_2\cos^3\frac{\alpha}{2}\Big[\sin\varphi_2+\sin\varphi_1-\sin(\varphi_1+\varphi_2)\Big] + \mathbf{e}_3(\cos\varphi_1-\cos\varphi_2)\Big\} . \tag{3.119}$$

These equations govern the even more special case of the resultant of two 180°-rotations about skew axes. With $\varphi_1 = -\pi$ (equivalent to $\varphi_1 = \pi$) and with $\varphi_2 = \pi$ they yield $\varphi_{\rm res} = 2\alpha$, $s_{\rm res} = 2\ell$, $\mathbf{n}_{\rm res} = \mathbf{e}_3$, $\mathbf{u} = \mathbf{0}$. Hence the resultant is the screw displacement about the common perpendicular of the two rotation axes with rotation angle 2α and translation 2ℓ . This is the statement made by Halphen's theorem.

Example: In Theorem 3.4 the problem is posed: Decompose a screw displacement given by $s_{\rm res}$, $\varphi_{\rm res}$ and by its axis $\mathrm{S}_{\rm res}$ into two subsequent pure rotations φ_1 about S_1 (first rotation) and φ_2 about S_2 of which either only S_2 or only S_1 is given. To be determined are φ_1 , φ_2 and the axis not given.
Solution: Let S_2 be given. The unknown first rotation is the resultant of the given screw displacement followed by the inverse of the second rotation. From this it follows that (3.108) – (3.112) are valid if the following changes are made.
1. The basis $\mathbf{e}_{1,2,3}$ is placed at the midpoint of the common perpendicular g_2 of the given axes $\mathrm{S}_{\rm res}$ and S_2 . The quantities ℓ and α specify the relative location of these axes.
2. $(s_{\rm res}, \varphi_{\rm res})$, (s_1, φ_1) and (s_2, φ_2) are replaced by $(0, \varphi_1)$, $(s_{\rm res}, \varphi_{\rm res})$ and $(0, -\varphi_2)$, respectively. Following these changes (3.108), (3.109), (3.110) and (3.112) read:

$$\cos\frac{\varphi_1}{2}=\cos\frac{\varphi_{\rm res}}{2}\cos\frac{\varphi_2}{2}+\sin\frac{\varphi_{\rm res}}{2}\sin\frac{\varphi_2}{2}\cos\alpha , \tag{3.120}$$

$$\mathbf{n}_1\sin\frac{\varphi_1}{2}=\mathbf{e}_1\sin\frac{\varphi_{\rm res}-\varphi_2}{2}\cos\frac{\alpha}{2}-\mathbf{e}_2\sin\frac{\varphi_{\rm res}+\varphi_2}{2}\sin\frac{\alpha}{2} +\mathbf{e}_3\sin\frac{\varphi_{\rm res}}{2}\sin\frac{\varphi_2}{2}\sin\alpha , \tag{3.121}$$

$$0=s_{\rm res}\Big(\sin\frac{\varphi_{\rm res}}{2}\cos\frac{\varphi_2}{2}-\cos\frac{\varphi_{\rm res}}{2}\sin\frac{\varphi_2}{2}\cos\alpha\Big) +2\ell\sin\frac{\varphi_{\rm res}}{2}\sin\frac{\varphi_2}{2}\sin\alpha , \tag{3.122}$$

$$\begin{aligned}\mathbf{u}\sin^2\frac{\varphi_1}{2} = &-\mathbf{e}_1\sin\frac{\varphi_2}{2}\,\sin^2\frac{\alpha}{2}\Big[s_{\mathrm{res}}\sin\frac{\varphi_2}{2}\cos\frac{\alpha}{2}\\ &\qquad -\ell\sin\frac{\varphi_{\mathrm{res}}}{2}\,\sin\frac{\varphi_{\mathrm{res}}+\varphi_2}{2}\,\sin\frac{\alpha}{2}\Big]\\ &+\mathbf{e}_2\sin\frac{\varphi_2}{2}\,\cos^2\frac{\alpha}{2}\Big[s_{\mathrm{res}}\sin\frac{\varphi_2}{2}\sin\frac{\alpha}{2}\\ &\qquad -\ell\sin\frac{\varphi_{\mathrm{res}}}{2}\,\sin\frac{\varphi_{\mathrm{res}}-\varphi_2}{2}\,\cos\frac{\alpha}{2}\Big]\\ &+\frac{1}{4}\,\mathbf{e}_3\Big[s_{\mathrm{res}}\sin\varphi_2\sin\alpha+\ell(\cos\varphi_{\mathrm{res}}-\cos\varphi_2)\Big]\,. \end{aligned}\qquad (3.123)$$

Equation (3.122) determines φ_2 :

$$\tan\frac{\varphi_2}{2} = \frac{s_{\mathrm{res}}}{s_{\mathrm{res}}\cot\dfrac{\varphi_{\mathrm{res}}}{2}\cos\alpha - 2\ell\sin\alpha}\,. \qquad (3.124)$$

With this angle φ_2 (3.120) and (3.121) determine $\cos\varphi_1/2$ and $\mathbf{n}_1\sin\varphi_1/2$. The vector $\mathbf{u}$ determined by (3.123) is the perpendicular from the midpoint of g_2 in Fig. 3.8 onto the first rotation axis. The equations fail in the case $\sin\alpha = 0$ (axes S_2 and $\mathrm{S}_{\mathrm{res}}$ parallel).

When instead of S_2 the axis S_1 is given, φ_1 , φ_2 and S_2 are determined as follows. The unknown second rotation is the resultant of the inverse of the first rotation followed by the given screw displacement. In (3.108) – (3.112) the following changes are made.

1. The basis $\mathbf{e}_{1,2,3}$ is placed at the midpoint of the common perpendicular g_3 of the given axes $\mathrm{S}_{\mathrm{res}}$ and S_1 . The quantities ℓ and α specify the relative location of these axes.
2. $(s_{\mathrm{res}}, \varphi_{\mathrm{res}})$, (s_1, φ_1) and (s_2, φ_2) are replaced by $(0, \varphi_2)$, $(0, -\varphi_1)$ and $(s_{\mathrm{res}}, \varphi_{\mathrm{res}})$, respectively. The modified Eq.(3.110) leads to

$$\tan\frac{\varphi_1}{2} = \frac{s_{\mathrm{res}}}{s_{\mathrm{res}}\cot\dfrac{\varphi_{\mathrm{res}}}{2}\cos\alpha - 2\ell\sin\alpha}\,. \qquad (3.125)$$

This is formally identical with (3.124). End of example.

Special case $\alpha = 0$ (parallel screw axes; $\mathbf{n}_1 = \mathbf{n}_2$): The case $\varphi_2 = -\varphi_1$ has to be distinguished from the general case $\varphi_2 \neq -\varphi_1$. This general case is considered first. Equations (3.108) – (3.112) reduce to

$$\varphi_{\mathrm{res}} = \varphi_1+\varphi_2 \neq 0\,,\qquad s_{\mathrm{res}} = s_1+s_2\,,\qquad \mathbf{n}_{\mathrm{res}} = \mathbf{n}_1 = \mathbf{n}_2\,, \qquad (3.126)$$

$$\mathbf{u}\sin\frac{\varphi_1+\varphi_2}{2} = \ell\Big(\mathbf{e}_2\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2} - \mathbf{e}_3\,\frac{1}{2}\sin\frac{\varphi_1-\varphi_2}{2}\Big)\,. \qquad (3.127)$$

The last equation is rewritten in the form

$$\frac{\ell}{2}\mathbf{e}_3+\mathbf{u}=\ell\Big(\mathbf{e}_2\sin\frac{\varphi_1}{2}+\mathbf{e}_3\cos\frac{\varphi_1}{2}\Big)\,\frac{\sin\frac{\varphi_2}{2}}{\sin\frac{\varphi_1+\varphi_2}{2}}\;. \tag{3.128}$$

This equation proves that the parallel screw axes $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_{\text{res}}$, seen in projection along the axes, form the triangle (P_1,P_2,P_3) shown in Fig. 3.14a. It has internal angles $\varphi_1/2$ and $\varphi_2/2$ at P_1 and P_2, respectively, and the external angle $\varphi_{\text{res}}/2$ at P_3. The vector $(\ell/2)\mathbf{e}_3+\mathbf{u}$ is $\overrightarrow{\text{P}_1\text{P}_3}$, and $(\mathbf{e}_2\sin\frac{\varphi_1}{2}+\mathbf{e}_3\cos\frac{\varphi_1}{2})$ is the unit vector in the direction of $\overrightarrow{\text{P}_1\text{P}_3}$. The equation expresses the sine law in the triangle.

In the special case $\varphi_2=-\varphi_1$, (3.108) yields $\varphi_{\text{res}}=0$. This indicates that the resultant of the two screw displacements is a translation. No further information is obtained from (3.109) – (3.112). Both magnitude and direction of the translation are obtained from Fig. 3.7. In the case of parallel screw axes $\mathbf{n}_1=\mathbf{n}_2$ and with $\varphi_2=-\varphi_1$, the lines g_2 and g_3 are parallel. In Fig. 3.14b the screw axes and the lines are shown in projection along the axes as in Fig. 3.14a. The component $(s_1+s_2)\mathbf{e}_1$ of the displacement is normal to the plane. The in-plane component is illustrated by the displacement of the point which prior to the first screw displacement is located at A. It is displaced via B to C. The total translatory displacement vector is

$$\mathbf{s}_{\text{res}}=(s_1+s_2)\mathbf{e}_1+\overrightarrow{\text{AC}}=(s_1+s_2)\mathbf{e}_1+\ell\,[-\sin\varphi_1\,\mathbf{e}_2+(1-\cos\varphi_1)\mathbf{e}_3]\;. \tag{3.129}$$

Equations (3.126) – (3.129) remain valid in the case $s_1=s_2=0$. In this case, the equations determine the resultant of two rotations about parallel axes. In Sects. 14.3 and 14.4 this case is investigated in more detail.

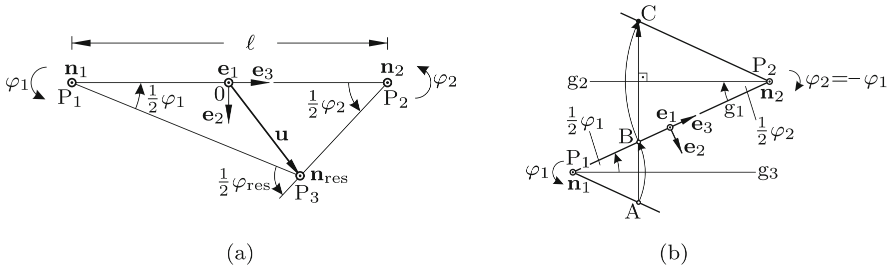

Fig. 3.14 Parallel screw axes $\mathbf{n}_1=\mathbf{n}_2$; $\varphi_2\neq-\varphi_1$ **(a)** and $\varphi_2=-\varphi_1$ **(b)**

3.12 Equations for the Screw Triangle

The quaternion formulation of the resultant of two rotations led to the rotation triangle shown in Fig. 1.6. Three rotations $(\mathbf{n}_{12},\varphi_{12})$, $(\mathbf{n}_{23},\varphi_{23})$,

$(\mathbf{n}_{31}, \varphi_{31})$ executed in this order or in any order produced by cyclic permutation carry a body via two intermediate positions back into its initial position. Each rotation is the inverse of the resultant of the previous two. Application of the sine and cosine laws led to (1.134):

$$\tan\frac{\varphi_{31}}{2} = \frac{\mathbf{n}_{12}\times\mathbf{n}_{23}\cdot\mathbf{n}_{31}}{(\mathbf{n}_{12}\times\mathbf{n}_{31})\cdot(\mathbf{n}_{23}\times\mathbf{n}_{31})}\;. \tag{3.130}$$

Analogously, two successive screw displacements followed by the inverse of the resultant of these two carry a body via two intermediate positions back into its initial position. The figure analogous to the rotation triangle is the spatial hexagon shown in Fig. 3.7 with φ_{res} and s_{res} replaced by $-\varphi_{\text{res}}$, $-s_{\text{res}}$. This analogy explains the name screw triangle of the hexagon.

Let the three screw displacements be newly labeled 12, 23 and 31. Then, according to the principle of transference, (3.130) is valid in the form

$$\tan\frac{\hat{\varphi}_{31}}{2} = \frac{\hat{\mathbf{n}}_{12}\times\hat{\mathbf{n}}_{23}\cdot\hat{\mathbf{n}}_{31}}{(\hat{\mathbf{n}}_{12}\times\hat{\mathbf{n}}_{31})\cdot(\hat{\mathbf{n}}_{23}\times\hat{\mathbf{n}}_{31})} \tag{3.131}$$

with

$$\hat{\varphi}_{ij} = \varphi_{ij} + \varepsilon s_{ij}\;,\quad \hat{\mathbf{n}}_{ij} = \mathbf{n}_{ij} + \varepsilon\mathbf{w}_{ij}\;,\quad \mathbf{n}_{ij}^2 = 1\;,\quad \mathbf{n}_{ij}\cdot\mathbf{w}_{ij} = 0 \tag{3.132}$$

$(ij) = (12),\ (23),\ (31)$. The vectors $\mathbf{n}_{ij}$ and $\mathbf{w}_{ij}$ are the Plücker vectors of the screw axis ij in a reference frame with arbitrary origin 0. The perpendicular from 0 onto the screw axis is $\mathbf{n}_{ij}\times\mathbf{w}_{ij}$. From the dual part of the equation an expression for the translatory displacement s_{31} is developed. The dual part of the left-hand side of the equation is

$$\begin{aligned}\frac{s_{31}}{2}\,\frac{1}{\cos^2\frac{\varphi_{31}}{2}} &= \frac{s_{31}}{2}\Big(1+\tan^2\frac{\varphi_{31}}{2}\Big)\\ &= \frac{s_{31}}{2}\,\frac{[(\mathbf{n}_{12}\times\mathbf{n}_{31})\cdot(\mathbf{n}_{23}\times\mathbf{n}_{31})]^2 + (\mathbf{n}_{12}\times\mathbf{n}_{23}\cdot\mathbf{n}_{31})^2}{[(\mathbf{n}_{12}\times\mathbf{n}_{31})\cdot(\mathbf{n}_{23}\times\mathbf{n}_{31})]^2}\;.\end{aligned} \tag{3.133}$$

The numerator is

$$\begin{aligned}&[(\mathbf{n}_{12}\times\mathbf{n}_{31})\cdot(\mathbf{n}_{23}\times\mathbf{n}_{31})]^2 + (\mathbf{n}_{12}\times\mathbf{n}_{23}\cdot\mathbf{n}_{31})^2\\ &= (\mathbf{n}_{12}\times\mathbf{n}_{31})^2(\mathbf{n}_{23}\times\mathbf{n}_{31})^2\;.\end{aligned} \tag{3.134}$$

This is proved as follows. With the abbreviations $\mathbf{a} = \mathbf{n}_{12}\times\mathbf{n}_{31}$ and $\mathbf{b} = \mathbf{n}_{23}\times\mathbf{n}_{31}$ and with $\beta = \sphericalangle(\mathbf{a},\mathbf{b})$ the equation reads
$\mathbf{a}^2\mathbf{b}^2\cos^2\beta + (\mathbf{n}_{12}\cdot\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2$ or

$$\begin{aligned}&(\mathbf{n}_{12}\cdot\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2\sin^2\beta = (\mathbf{a}\times\mathbf{b})^2 = [(\mathbf{n}_{12}\times\mathbf{n}_{31})\times\mathbf{b}]^2\\ &= [(\mathbf{n}_{12}\cdot\mathbf{b})\mathbf{n}_{31} - (\mathbf{n}_{31}\cdot\mathbf{b})\mathbf{n}_{12}]^2\;.\end{aligned} \tag{3.135}$$

This is, indeed, true since $\mathbf{n}_{31} \cdot \mathbf{b} = 0$. Thus, the dual part of the left-hand side of (3.131) is

$$\frac{s_{31}}{2} \frac{(\mathbf{n}_{12} \times \mathbf{n}_{31})^2 (\mathbf{n}_{23} \times \mathbf{n}_{31})^2}{[(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31})]^2} . \tag{3.136}$$

The dual part of the right-hand side is calculated as follows. With Eqs.(3.132) for $\hat{\mathbf{n}}_{ij}$ the numerator has the form $N + \varepsilon N_\mathrm{d}$ with

$$\left.\begin{aligned} N &= \mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31} , \\ N_\mathrm{d} &= \mathbf{n}_{23} \times \mathbf{n}_{31} \cdot \mathbf{w}_{12} + \mathbf{n}_{31} \times \mathbf{n}_{12} \cdot \mathbf{w}_{23} + \mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{w}_{31} . \end{aligned}\right\} \tag{3.137}$$

The denominator has the form $D + \varepsilon D_\mathrm{d}$ with

$$\left.\begin{aligned} D &= (\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31}) , \\ D_\mathrm{d} &= (\mathbf{w}_{12} \times \mathbf{n}_{31} + \mathbf{n}_{12} \times \mathbf{w}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31}) \\ &\quad +(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{w}_{23} \times \mathbf{n}_{31} + \mathbf{n}_{23} \times \mathbf{w}_{31}) \\ &= -[(\mathbf{n}_{23} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}] \cdot \mathbf{w}_{12} - [(\mathbf{n}_{12} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}] \cdot \mathbf{w}_{23} \\ &\quad +[(\mathbf{n}_{23} \times \mathbf{n}_{31}) \times \mathbf{n}_{12} + (\mathbf{n}_{12} \times \mathbf{n}_{31}) \times \mathbf{n}_{23}] \cdot \mathbf{w}_{31} . \end{aligned}\right\} \tag{3.138}$$

In these terms the right-hand side of (3.131) is

$$\frac{N + \varepsilon N_\mathrm{d}}{D + \varepsilon D_\mathrm{d}} = \frac{N}{D} + \varepsilon \frac{D N_\mathrm{d} - N D_\mathrm{d}}{D^2} . \tag{3.139}$$

The dual part equals the expression in (3.136). This yields for s_{31} the expression

$$\frac{s_{31}}{2} = \frac{D N_\mathrm{d} - N D_\mathrm{d}}{(\mathbf{n}_{12} \times \mathbf{n}_{31})^2 (\mathbf{n}_{23} \times \mathbf{n}_{31})^2} . \tag{3.140}$$

The numerator is

$$D N_\mathrm{d} - N D_\mathrm{d} = \mathbf{v}_{12} \cdot \mathbf{w}_{12} + \mathbf{v}_{23} \cdot \mathbf{w}_{23} + \mathbf{v}_{31} \cdot \mathbf{w}_{31} \tag{3.141}$$

with vectors

$$\left.\begin{aligned} \mathbf{v}_{12} &= \Big[(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31})\Big] \mathbf{n}_{23} \times \mathbf{n}_{31} \\ &\quad +(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})\Big[(\mathbf{n}_{23} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}\Big] , \\ \mathbf{v}_{23} &= -\Big[(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31})\Big] \mathbf{n}_{12} \times \mathbf{n}_{31} \\ &\quad +(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})\Big[(\mathbf{n}_{12} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}\Big] , \\ \mathbf{v}_{31} &= \Big[(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31})\Big] \mathbf{n}_{12} \times \mathbf{n}_{23} \\ &\quad -(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})\Big[(\mathbf{n}_{23} \times \mathbf{n}_{31}) \times \mathbf{n}_{12} + (\mathbf{n}_{12} \times \mathbf{n}_{31}) \times \mathbf{n}_{23}\Big] . \end{aligned}\right\} \tag{3.142}$$

These vectors are simplified as follows. First, the vector $\mathbf{v}_{12}$. Obviously, $\mathbf{v}_{12} \cdot \mathbf{n}_{31} = 0$. When the multiple products are simplified, it turns out that also $\mathbf{v}_{12} \cdot \mathbf{n}_{12} = 0$. Hence $\mathbf{v}_{12}$ has the form $\mathbf{v}_{12} = A\mathbf{n}_{12} \times \mathbf{n}_{31}$ with an unknown scalar A. It is determined by dot-multiplying this equation by $(\mathbf{n}_{12} \times \mathbf{n}_{31})$. Taking into account (3.134) this results in the equation

$$\begin{aligned} A(\mathbf{n}_{12} \times \mathbf{n}_{31})^2 &= [(\mathbf{n}_{12} \times \mathbf{n}_{31}) \cdot (\mathbf{n}_{23} \times \mathbf{n}_{31})]^2 + (\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})^2 \\ &= (\mathbf{n}_{12} \times \mathbf{n}_{31})^2 (\mathbf{n}_{23} \times \mathbf{n}_{31})^2 \,. \end{aligned} \tag{3.143}$$

Hence

$$\mathbf{v}_{12} = (\mathbf{n}_{23} \times \mathbf{n}_{31})^2 \, \mathbf{n}_{12} \times \mathbf{n}_{31} \,. \tag{3.144}$$

The same arguments lead to

$$\mathbf{v}_{23} = (\mathbf{n}_{12} \times \mathbf{n}_{31})^2 \, \mathbf{n}_{23} \times \mathbf{n}_{31} \,. \tag{3.145}$$

The product $\mathbf{v}_{31} \cdot \mathbf{w}_{31}$ in (3.141) eliminates the (nonzero) component of $\mathbf{v}_{31}$ in the direction of $\mathbf{n}_{31}$. For determining the relevant components the ansatz is made: $\mathbf{v}_{31} = A\mathbf{n}_{12} \times \mathbf{n}_{31} + B\mathbf{n}_{23} \times \mathbf{n}_{31} + C\mathbf{n}_{31}$. Scalar multiplication with $(\mathbf{n}_{23} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}$ eliminates B and C, and scalar multiplication with $(\mathbf{n}_{12} \times \mathbf{n}_{31}) \times \mathbf{n}_{31}$ eliminates A and C. The first multiplication yields

$$\begin{aligned} A(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31}) = \Big\{ & [(\mathbf{n}_{12} \cdot \mathbf{n}_{23}) - (\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{n}_{23} \cdot \mathbf{n}_{31})]\mathbf{n}_{12} \times \mathbf{n}_{23} \\ & -(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})[2(\mathbf{n}_{12} \cdot \mathbf{n}_{23})\mathbf{n}_{31} - (\mathbf{n}_{12} \cdot \mathbf{n}_{31})\mathbf{n}_{23} \\ & -(\mathbf{n}_{23} \cdot \mathbf{n}_{31})\mathbf{n}_{12}] \Big\} \cdot [(\mathbf{n}_{23} \cdot \mathbf{n}_{31})\mathbf{n}_{31} - \mathbf{n}_{23}] \\ = & -(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})(\mathbf{n}_{12} \cdot \mathbf{n}_{31})[1 - (\mathbf{n}_{23} \cdot \mathbf{n}_{31})^2] \\ = & -(\mathbf{n}_{12} \times \mathbf{n}_{23} \cdot \mathbf{n}_{31})(\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{n}_{23} \times \mathbf{n}_{31})^2 \,. \end{aligned} \tag{3.146}$$

Hence $A = -(\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{n}_{23} \times \mathbf{n}_{31})^2$. In the same way
$B = (\mathbf{n}_{23} \cdot \mathbf{n}_{31})(\mathbf{n}_{12} \times \mathbf{n}_{31})^2$. Hence

$$\begin{aligned} \mathbf{v}_{31} = & -(\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{n}_{23} \times \mathbf{n}_{31})^2 \, \mathbf{n}_{12} \times \mathbf{n}_{31} \\ & +(\mathbf{n}_{23} \cdot \mathbf{n}_{31})(\mathbf{n}_{12} \times \mathbf{n}_{31})^2 \, \mathbf{n}_{23} \times \mathbf{n}_{31} + C\mathbf{n}_{31} \,. \end{aligned} \tag{3.147}$$

The expressions obtained for $\mathbf{v}_{12}$, $\mathbf{v}_{23}$ and $\mathbf{v}_{31}$ are substituted into (3.141). Further substitution into (3.140) yields for s_{31} the final result

$$\begin{aligned} \frac{s_{31}}{2} = & \frac{1}{(\mathbf{n}_{23} \times \mathbf{n}_{31})^2} [\mathbf{n}_{31} \cdot \mathbf{n}_{23} \times \mathbf{w}_{23} + (\mathbf{n}_{23} \cdot \mathbf{n}_{31})\mathbf{n}_{23} \cdot \mathbf{n}_{31} \times \mathbf{w}_{31}] \\ & - \frac{1}{(\mathbf{n}_{12} \times \mathbf{n}_{31})^2} [\mathbf{n}_{31} \cdot \mathbf{n}_{12} \times \mathbf{w}_{12} + (\mathbf{n}_{12} \cdot \mathbf{n}_{31})\mathbf{n}_{12} \cdot \mathbf{n}_{31} \times \mathbf{w}_{31}] \,. \end{aligned} \tag{3.148}$$

The vectors $\mathbf{n}_{12} \times \mathbf{w}_{12}$, $\mathbf{n}_{23} \times \mathbf{w}_{23}$ and $\mathbf{n}_{31} \times \mathbf{w}_{31}$ are the perpendiculars from the reference point onto the three screw axes. The scalar products of unit vectors can be expressed through the angles α_1 and α_3 in Fig. 1.6:

$$\left.\begin{array}{ll}\mathbf{n}_{12}\cdot\mathbf{n}_{31}=\cos\alpha_1\ , & (\mathbf{n}_{12}\times\mathbf{n}_{31})^2=\sin^2\alpha_1\ ,\\ \mathbf{n}_{23}\cdot\mathbf{n}_{31}=\cos\alpha_3\ , & (\mathbf{n}_{23}\times\mathbf{n}_{31})^2=\sin^2\alpha_3\ .\end{array}\right\} \tag{3.149}$$

Tsai and Roth [48] deduced (3.148) geometrically from Fig. 3.7. See also Bottema/Roth [5].

3.13 Resultant of two Infinitesimal Screw Displacements. Cylindroid

In this section Eqs.(3.109) – (3.112) for the resultant of two screw displacements are evaluated in the special case of infinitesimal screw displacements. Let p_1, p_2 and p_{res} be the pitches of the three screw displacements so that

$$s_i=p_i\varphi_i\quad(i=1,2)\ ,\qquad s_{\mathrm{res}}=p_{\mathrm{res}}\varphi_{\mathrm{res}}\ . \tag{3.150}$$

In what follows, the index res is omitted.

For Eq.(3.109) a Taylor series expansion up to 1st-order terms is made. When (3.107) is taken into account, this results in the parallelogram rule for small rotations (see Fig. 3.15):

$$\mathbf{n}\varphi=\mathbf{e}_1(\varphi_1+\varphi_2)\cos\frac{\alpha}{2}+\mathbf{e}_2(\varphi_2-\varphi_1)\sin\frac{\alpha}{2} \tag{3.151}$$

$$=\mathbf{n}_1\varphi_1+\mathbf{n}_2\varphi_2\ . \tag{3.152}$$

In (3.110) and (3.112) $s_i=p_i\varphi_i$ $(i=1,2)$ and $s=p\varphi$ are substituted. Following this, Taylor series expansions are made up to 2nd-order terms. This results in the equations

$$p\,\varphi^2=p_1\varphi_1^2+p_2\varphi_2^2+[(p_1+p_2)\cos\alpha-\ell\sin\alpha]\varphi_1\varphi_2\ , \tag{3.153}$$

$$\mathbf{u}\,\varphi^2=\mathbf{e}_3\Big[(p_2-p_1)\varphi_1\varphi_2\sin\alpha+\frac{1}{2}\ell(\varphi_2^2-\varphi_1^2)\Big]\ . \tag{3.154}$$

From the latter equation it follows that $\mathbf{u}$ has the form $\mathbf{u}=u\mathbf{e}_3$. This means that the resultant screw axis intersects the common perpendicular $\mathbf{e}_3$ of the screw axes 1 and 2 orthogonally at the point u given by the equation

$$u\,\varphi^2=(p_2-p_1)\varphi_1\varphi_2\sin\alpha+\frac{1}{2}\ell(\varphi_2^2-\varphi_1^2)\ . \tag{3.155}$$

Let ψ be the angle of the resultant screw axis in the $\mathbf{e}_1,\mathbf{e}_2$-plane against the $\mathbf{e}_1$-axis as shown in Fig. 3.15. The sine law applied to the triangles in this figure yields the expressions

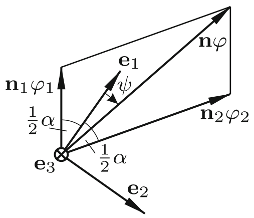

Fig. 3.15 Triangle of infinitesimal rotations

$$\varphi_1 = \varphi\,\frac{\sin\left(\dfrac{\alpha}{2} - \psi\right)}{\sin\alpha}\;, \qquad \varphi_2 = \varphi\,\frac{\sin\left(\dfrac{\alpha}{2} + \psi\right)}{\sin\alpha}\;, \tag{3.156}$$

from which it follows that

$$\left.\begin{aligned} \varphi_{1,2}^2 &= \varphi^2\,\frac{1 - \cos 2\psi\cos\alpha \mp \sin 2\psi\sin\alpha}{2\sin^2\alpha}\;, \\ \varphi_1\varphi_2 &= \varphi^2\,\frac{\cos 2\psi - \cos\alpha}{2\sin^2\alpha}\;. \end{aligned}\right\} \tag{3.157}$$

Substitution into (3.153) and (3.155) results in explicit expressions for p and u:

$$p = \frac{1}{2}(p_1 + p_2 + \ell\cot\alpha) - \frac{\ell\cos 2\psi + (p_1 - p_2)\sin 2\psi}{2\sin\alpha}\;, \tag{3.158}$$

$$u = \frac{1}{2}(p_1 - p_2)\cot\alpha \qquad + \frac{\ell\sin 2\psi - (p_1 - p_2)\cos 2\psi}{2\sin\alpha}\;. \tag{3.159}$$

These expressions are simpler if the transition is made to an x, y-system of principal axes which is rotated against the $\mathbf{e}_1, \mathbf{e}_2$-system through the angle ψ_0 given by

$$\cos 2\psi_0 = \frac{\ell}{\sqrt{(p_1 - p_2)^2 + \ell^2}}\;, \qquad \sin 2\psi_0 = \frac{p_1 - p_2}{\sqrt{(p_1 - p_2)^2 + \ell^2}}\;. \tag{3.160}$$

More precisely, ψ_0 is the angle of the x-axis against the $\mathbf{e}_1$-axis. Then

$$\left.\begin{aligned} p &= p_0 - h\cos 2\chi\;, \\ z &= \quad\;\; h\sin 2\chi \end{aligned}\right\} \tag{3.161}$$

with new variables

$$\chi = \psi - \psi_0\;, \qquad z = u - u_0 \tag{3.162}$$

and with constants

$$p_0 = \frac{1}{2}(p_1 + p_2 + \ell \cot\alpha)\ , \quad u_0 = \frac{1}{2}(p_1 - p_2)\cot\alpha\ , \quad h = \frac{\sqrt{(p_1 - p_2)^2 + \ell^2}}{2\sin\alpha}\ . \tag{3.163}$$

Equations (3.161) – (3.163) determine a one-parametric manifold of resultant screw displacements with the angle χ between screw axis and x-axis as parameter. In what follows, statements are made about this manifold. Elimination of χ from Eqs.(3.161) results in the equation of a circle relating p and z:

$$(p - p_0)^2 + z^2 \equiv h^2\ . \tag{3.164}$$

Every value of z in the interval $|z| \le |h|$ occurs at two angles χ_1 and $\chi_2 = \pi/2 - \chi_1$ which are located symmetrically with respect to $\chi = \pi/4$ as well as to $\chi = -\pi/4$. These two angles specify the directions of two screw axes which intersect at z on the $\mathbf{e}_3$-axis. The two screw axes intersecting at $z = +h$ coincide $(\chi_1 = \chi_2 = \pi/4)$. Likewise, the two screw axes intersecting at $z = -h$ coincide $(\chi_1 = \chi_2 = -\pi/4)$. The pitch associated with these screw axes is p_0.

The two screw axes intersecting at $z = 0$ are the principal x, y-axes $(\chi_1 = 0,\ \chi_2 = \pi/2)$. The associated principal pitches are the extremal pitches

$$p_\mathrm{x} = p_0 - h\ , \qquad p_\mathrm{y} = p_0 + h\ . \tag{3.165}$$

In terms of principal pitches the constants p_0 and h are

$$p_0 = \frac{1}{2}(p_\mathrm{x} + p_\mathrm{y})\ , \qquad h = \frac{1}{2}(p_\mathrm{y} - p_\mathrm{x})\ , \tag{3.166}$$

and Eqs.(3.161) have the forms

$$\left.\begin{aligned} p &= p_\mathrm{x}\cos^2\chi + p_\mathrm{y}\sin^2\chi\ , \\ z &= -(p_\mathrm{x} - p_\mathrm{y})\sin\chi\cos\chi\ . \end{aligned}\right\} \tag{3.167}$$

The first equation has the two forms $p - p_\mathrm{x} = (p_\mathrm{y} - p_\mathrm{x})\sin^2\chi$ and $p - p_\mathrm{y} = -(p_\mathrm{y} - p_\mathrm{x})\cos^2\chi$. Together with the second Eq.(3.167) this yields

$$\left.\begin{aligned} (p_\mathrm{x} - p)\cos\chi \qquad &+ z\sin\chi = 0\ , \\ z\cos\chi\ &-(p_\mathrm{y} - p)\sin\chi = 0\ . \end{aligned}\right\} \tag{3.168}$$

This is an eigenvalue problem with eigenvalue $p(z)$ and with the associated eigenvector $[\cos\chi(z) \quad \sin\chi(z)]$. The characteristic equation for p is Eq.(3.164).

Every pitch p in the interval between the extremal pitches p_x and p_y occurs at two angles $\pm\chi$. Definition: Two screws of equal pitch are called *conjugate screws*. The pair of conjugate screws at angles ± 0 coincides in the principal screw with pitch p_x. Likewise, the pair of conjugate screws at angles $\pm\pi$ coincides in the principal screw with pitch p_y. For a given pitch

p the first Eq.(3.167) and (3.164) determine the associated angles $\pm\chi$ and coordinates z . Example: For the pair of conjugate screws with pitch $p = 0$, i.e., pure rotations, the equations yield

$$\cos 2\chi_{\text{rot}} = \frac{p_0}{h} = \frac{p_{\text{y}} + p_{\text{x}}}{p_{\text{y}} - p_{\text{x}}} , \qquad z_{\text{rot}} = \pm\sqrt{h^2 - p_0^2} = \pm\sqrt{-p_{\text{x}}p_{\text{y}}} . \quad (3.169)$$

Real solutions exist only if $p_{\text{x}}p_{\text{y}} \leq 0$.

The one-parametric manifold of screw axes defines a ruled surface. With an additional parameter λ this ruled surface has the parameter representation $\mathbf{u}(\chi) + \lambda\mathbf{n}(\chi)$. Its coordinate form is

$$x = \lambda\cos\chi , \qquad y = \lambda\sin\chi , \qquad z = (p_{\text{y}} - p_{\text{x}})\sin\chi\cos\chi . \quad (3.170)$$

The parameters λ and χ are eliminated by forming $x^2 + y^2 = \lambda^2$ and $xy = \lambda^2\sin\chi\cos\chi$. Combining these equations with the third Eq.(3.170) results in the third-order equation

$$z(x^2 + y^2) = (p_{\text{y}} - p_{\text{x}})xy . \quad (3.171)$$

This ruled surface is called *cylindroid*. In the theory of ruled surfaces a line which is intersected by every generator is called *directrix*. The z-axis is a double directrix because it is intersected by two generators at every point $|z| \leq |\frac{1}{2}(p_{\text{y}} - p_{\text{x}})|$. From (2.59) the distribution parameter δ is calculated as function of χ . For this purpose the following change of notation has to be made (see (2.51)). The role of the parameter u is played by χ . The curve $\mathbf{r}(u)$ is the directrix $\mathbf{u}(\chi) = (p_{\text{y}} - p_{\text{x}})\sin\chi\cos\chi\mathbf{e}_3$, and the unit vector $\mathbf{e}(u)$ is $\mathbf{n}(\chi)$. According to (3.170) $\dot{\mathbf{n}} = \mathbf{e}_3 \times \mathbf{n}$ and $\dot{\mathbf{n}}^2 \equiv 1$. Hence with $\dot{\mathbf{u}}(\chi) = (p_{\text{y}} - p_{\text{x}})\cos 2\chi\mathbf{e}_3$ the distribution parameter is $\delta(\chi) = \dot{\mathbf{n}} \cdot \dot{\mathbf{u}} \times \mathbf{n} = (p_{\text{y}} - p_{\text{x}})\cos 2\chi$. The two pairs of coinciding generators at $z = -\frac{1}{2}(p_{\text{y}} - p_{\text{x}})$ and at $z = \frac{1}{2}(p_{\text{y}} - p_{\text{x}})$ are torsal lines ($\delta = 0$) . In Fig. 3.16 the cylindroid is represented in the x, y, z-system by an orthogonal net of lines $\chi =$ const and $\lambda =$ const. The curved lines $\lambda =$ const are kinematically insignificant. Their only purpose is to show the shape of the surface more clearly. What matters are the straight lines $\chi =$ const, i.e., the screw axes. These axes extend to $\pm\infty$. The symmetry with respect to the two planes each spanned by the directrix and by one torsal line is clearly shown.

Let $\mathbf{n}$ and $\mathbf{w}$ be the first and the second Plücker vector of the screw axis associated with the angle χ . The x, y, z-coordinates of these vectors are

$$\left.\begin{aligned} &\mathbf{n}: \ [\cos\chi \quad \sin\chi \quad 0] , \\ &\mathbf{w}: \ (p_{\text{y}} - p_{\text{x}})\sin\chi\cos\chi\,[-\sin\chi \quad \cos\chi \quad 0] . \end{aligned}\right\} \quad (3.172)$$

The results obtained so far are summarized as follows. The resultant of two arbitrarily oriented infinitesimal screw displacements with constant pa-

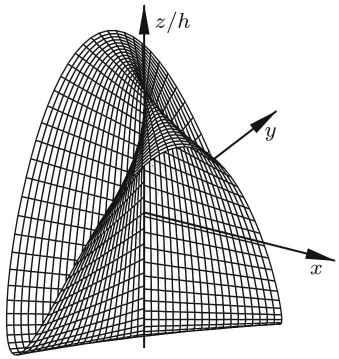

Fig. 3.16 Cylindroid

rameters α, ℓ, p_1 and p_2 and with variable parameters φ_1 and φ_2 is the manifold of screw displacements on the cylindroid defined by its axis, by u_0 and by its principal pitches p_x and p_y associated with mutually orthogonal principal axes x and y. The same manifold of screw displacements is obtained if the two principal screws are used as starting point. This is the special case $\alpha = \pi/2$ and $\ell = 0$. Equations (3.163) yield $u_0 = 0$ (this is in accordance with the statement preceding (3.165)) and $p_0 = p_\mathrm{x} + p_\mathrm{y}$, $h = p_\mathrm{y} - p_\mathrm{x}$ (as before; see Eqs.(3.165)). Every screw on the cylindroid is resultant of two principal screws about the principal axes. Also the following is true. Every screw on the cylindroid is resultant of two (arbitrary) conjugate screws. The quantities u_0 and h determining the cylindroid depend on the difference $p_1 - p_2$ only, whereas p_0 and, hence, $p(\chi)$ depend on the sum $p_1 + p_2$. If p_1 and p_2 (arbitrary) are increased by one and the same amount (arbitrary), the cylindroid remains the same, whereas $p(\chi)$ is increased by the same amount independent of χ.

The cylindroid was discovered independently by Hamilton [18] (1830), Plücker [38] (1865) (see also [39]), Battaglini [3] (1869) and Ball [2] (1900). Geometrical properties of the cylindroid see in Zindler [57]. In the present book the cylindroid is met again in the next Sect. 3.14 and in Sect. 12.6.2.

The end of the previous Sect. 3.11 on the resultant of two finite screw displacements was devoted to cases which require special considerations (Eqs.(3.118) – (3.129)). The same cases are considered here, and results are developed from the said equations. As before, the index $_\mathrm{res}$ is omitted.

Special case $\varphi_2 = 0$: Since the pitch p_2 is not defined the infinitesimal displacement s_2 is expressed in the form $s_2 = \mu\varphi_1$. As origin 0 of the $\mathbf{e}_{1,2,3}$-system an arbitrary point on the screw axis $\hat{\mathbf{n}}_1$ is chosen (see the text following (3.115)). Substituting $s_2 = \mu\varphi_1$ and $s_1 = p_1\varphi_1$ into (3.115) and (3.116) and making a Taylor series expansion for $\mathbf{u}$ results in the equations

$$\varphi = \varphi_1 \;, \quad \mathbf{n} = \mathbf{n}_1 \;, \quad p = p_1 + \mu \cos \alpha \;, \quad \mathbf{u} = \mathbf{e}_3 \mu \sin \alpha \;. \tag{3.173}$$

These equations show that the resultant screw axis has the direction of $\mathbf{n}_1$, and that it intersects the $\mathbf{e}_3$-axis orthogonally at the point $u = \mu \sin \alpha$.

Special case $\alpha = 0$ (parallel screw axes) and $\varphi_2 \neq -\varphi_1$: Equations (3.126) become (with a Taylor series expansion for $\mathbf{u}$)

$$\left.\begin{array}{ll} \varphi = \varphi_1 + \varphi_2 \neq 0 \;, & p = p_1 + p_2 \;, \\ \mathbf{n} = \mathbf{n}_1 = \mathbf{n}_2 \;, & \mathbf{u} = -\mathbf{e}_3 \dfrac{\ell}{2} \dfrac{\varphi_1 - \varphi_2}{\varphi_1 + \varphi_2} \;. \end{array}\right\} \tag{3.174}$$

This means that the resultant screw axis intersects the $\mathbf{e}_3$-axis orthogonally at a point u which depends on the ratio φ_2/φ_1. The points P_1, P_2 and P_3 in Fig. 3.14a are collinear.

Special case $\alpha = 0$ (parallel screw axes) and $\varphi_2 = -\varphi_1$: According to (3.129) the resultant displacement is a pure translation $\mathbf{s}$. With $s_1 = p_1\varphi_1$ and $s_2 = p_2\varphi_2 = -p_2\varphi_1$ it is

$$\mathbf{s} = [(p_1 - p_2)\mathbf{e}_1 - \ell\mathbf{e}_2]\varphi_1 \;. \tag{3.175}$$

This displacement is normal to $\mathbf{e}_3$.

3.14 Screw Displacements Effecting a Prescribed Line Displacement

In Fig. 3.17 two skew lines are defined by their unit line vectors $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. The line vector $\hat{\mathbf{r}}_2$ is produced from $\hat{\mathbf{r}}_1$ by the screw displacement $(\hat{\mathbf{n}}_3, \hat{\alpha})$ with the dual unit vector $\hat{\mathbf{n}}_3$ along the common perpendicular and with the dual screw angle $\hat{\alpha} = \alpha + \varepsilon\ell$ between the two lines. Without loss of generality, it is assumed that $0 < \alpha < \pi$ whereas ℓ may be positive, zero or negative. The dual unit vector $\hat{\mathbf{n}}_3$ is one of the basis vectors $\hat{\mathbf{n}}_i$ $(i = 1, 2, 3)$ of a dual basis which has its origin at the midpoint 0 of the common perpendicular. The basis vector $\hat{\mathbf{n}}_1$ is bisecting the angle α when seen in the projection along $\hat{\mathbf{n}}_3$. The line vector $\hat{\mathbf{r}}_2$ is produced from $\hat{\mathbf{r}}_1$ not only by the screw displacement $(\hat{\mathbf{n}}_3, \hat{\alpha})$, but also by the screw displacement $(\hat{\mathbf{n}}_1, \pm\pi)$. Both screw displacements carry the point A fixed on line 1 to the point B fixed on line 2. These observations stimulate an investigation of the following problems.

Problem 1: Determine the manifold of all screw displacements $(\hat{\mathbf{n}}, \hat{\varphi})$ of a rigid body which carry a body-fixed directed line from position $\hat{\mathbf{r}}_1$ into position $\hat{\mathbf{r}}_2$. In this manifold determine the submanifold of all screw displace-

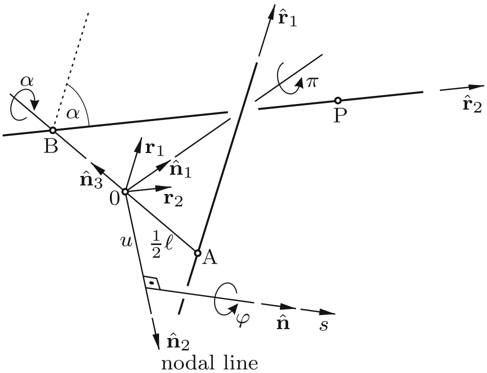

Fig. 3.17 Quantities associated with screw displacements $(\hat{\mathbf{n}}, \hat{\varphi})$ carrying the line $\hat{\mathbf{r}}_1$ into the line $\hat{\mathbf{r}}_2$

ments which are pure rotations. The problem statement does not require that point A of $\hat{\mathbf{r}}_1$ is carried to a specified point on $\hat{\mathbf{r}}_2$.

Problem 2 : From the manifold of all screw displacements solving Problem 1 determine the submanifold of all screw displacements carrying point A of $\hat{\mathbf{r}}_1$ to an arbitrarily prescribed point P on $\hat{\mathbf{r}}_2$. Consider the special case P=B .

Both problems have a practical background. Let the body-fixed line be the axis of a cylindrical workpiece which is displaced by a robot from a box into the chuck of a machine. In the box the axis has position $\hat{\mathbf{r}}_1$, and in the chuck it has position $\hat{\mathbf{r}}_2$. The robot is equipped with a single cylindrical joint so that it can execute screw displacements only. The angular position of the cylindrical workpiece in the chuck is of no interest. If the depth of insertion into the chuck is not prescribed either, every screw displacement solving Problem 1 is acceptable, otherwise every screw displacement solving Problem 2 .

Solution to Problem 1 : The solution is deduced by means of the principle of transference from the solution to the following rotation problem solved in Sect. 1.15.9 . Determine all rotations $(\mathbf{n}, \varphi)$ about a fixed point 0 which carry a body-fixed line passing through 0 from a given position $\mathbf{r}_1$ into another given position $\mathbf{r}_2$. The lines $\mathbf{r}_1$ and $\mathbf{r}_2$ and the angle α between them are shown in Fig. 1.11a . Figure 1.11b explains cartesian basis vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. They are related to $\mathbf{r}_1$ and $\mathbf{r}_2$ through (1.220) and (1.221):

$$\mathbf{n}_1 = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2\cos\frac{\alpha}{2}}\,, \qquad \mathbf{n}_3 = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sin\alpha}\,, \qquad \mathbf{n}_2 = \mathbf{n}_3 \times \mathbf{n}_1\,, \tag{3.176}$$

$$\mathbf{r}_{1,2} = \mathbf{n}_1 \cos\frac{\alpha}{2} \mp \mathbf{n}_2 \sin\frac{\alpha}{2}\,. \tag{3.177}$$

The notation in Fig. 3.17 is chosen such that in the case $\ell = 0$ the dual angle $\hat{\alpha}$ is the angle α of Fig. 1.11, and that the dual vectors $\hat{\mathbf{r}}_i$ $(i = 1, 2)$ and $\hat{\mathbf{n}}_i$ $(i = 1, 2, 3)$ are the vectors $\mathbf{r}_i$ $(i = 1, 2)$ and $\mathbf{n}_i$ $(i = 1, 2, 3)$, respectively. The solution to the rotation problem of Fig. 1.11 is a one-parametric manifold of rotations $(\mathbf{n}, \varphi)$. With a free parameter ψ it is given in (1.222) and (1.228) in the form

$$\mathbf{n} = \mathbf{n}_1 \cos\psi - \mathbf{n}_3 \sin\psi \; , \qquad \cot\frac{\varphi}{2} = -\cot\frac{\alpha}{2}\sin\psi \; . \tag{3.178}$$

All vectors and angles are transferred into dual form by defining dual parts as follows:

$$\hat{\alpha} = \alpha + \varepsilon\ell \; , \qquad \hat{\varphi} = \varphi + \varepsilon s \; , \qquad \hat{\psi} = \psi + \varepsilon u \; , \tag{3.179}$$

$$\hat{\mathbf{n}}_i = \mathbf{n}_i \quad (i = 1, 2, 3) \; , \quad \hat{\mathbf{r}}_i = \mathbf{r}_i + \varepsilon\mathbf{w}_i \quad (i = 1, 2) \; , \quad \hat{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{w} \; . \tag{3.180}$$

The identity $\hat{\mathbf{n}}_i = \mathbf{n}_i$ holds true because the second Plücker vectors are zero with respect to the origin 0 of the dual basis. The second Plücker vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ of $\hat{\mathbf{r}}_i$ are

$$\mathbf{w}_{1,2} = \mp\frac{1}{2}\ell\mathbf{n}_3 \times \mathbf{r}_{1,2} = -\frac{1}{2}\ell\Big(\mathbf{n}_1 \sin\frac{\alpha}{2} \pm \mathbf{n}_2 \cos\frac{\alpha}{2}\Big) \; . \tag{3.181}$$

The same formulas are obtained by dual differentiation of (3.177). The quantities ψ and u are two independent parameters of the manifold of solutions. To be determined as functions of ψ and u are the dual parts s and $\mathbf{w}$ of the screw displacement. Unknown, too, is the geometrical meaning of u. The unknowns are determined as follows. Equations (3.178) are transferred into the dual forms

$$\hat{\mathbf{n}} = \mathbf{n}_1 \cos\hat{\psi} - \mathbf{n}_3 \sin\hat{\psi} \; , \qquad \cot\frac{\hat{\varphi}}{2} = -\cot\frac{\hat{\alpha}}{2}\sin\hat{\psi} \; . \tag{3.182}$$

The dual part of the first equation yields

$$\mathbf{w} = -u(\mathbf{n}_1 \sin\psi + \mathbf{n}_3 \cos\psi) \; . \tag{3.183}$$

This together with the first Eq.(3.178) yields for the perpendicular $\mathbf{n} \times \mathbf{w}$ from 0 onto the screw axis the expression

$$\mathbf{n} \times \mathbf{w} = u\mathbf{n}_2 \; . \tag{3.184}$$

From this it follows, first, that all screw axes intersect the line $\hat{\mathbf{n}}_2$ at right angles and, second, that the free parameter u (positive, zero or negative) represents the length of the perpendicular from 0 onto the screw axis. This is shown in Fig. 3.17. The line $\hat{\mathbf{n}}_2$ is called *nodal line* of the two lines $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$.

The dual part of the second Eq.(3.182) is an equation for s:

$$\frac{-s/2}{\sin^2\frac{\varphi}{2}} = \frac{\ell/2}{\sin^2\frac{\alpha}{2}}\,\sin\psi - u\cot\frac{\alpha}{2}\,\cos\psi\,. \tag{3.185}$$

For $\sin^2\varphi/2$ and for other functions of φ the following expressions in terms of ψ are known from (1.227) and (1.226):

$$\sin^2\frac{\varphi}{2} = \frac{\sin^2\frac{\alpha}{2}}{1-\cos^2\frac{\alpha}{2}\cos^2\psi}\,,\qquad \sin\frac{\varphi}{2}\cos\frac{\varphi}{2} = \frac{-\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\psi}{1-\cos^2\frac{\alpha}{2}\cos^2\psi}\,, \tag{3.186}$$

$$1-\cos\varphi = \frac{1-\cos\alpha}{1-\cos^2\frac{\alpha}{2}\cos^2\psi}\,,\qquad \sin\varphi = \frac{-\sin\alpha\sin\psi}{1-\cos^2\frac{\alpha}{2}\cos^2\psi}\,. \tag{3.187}$$

With the first Eq.(3.186) s becomes a function of u and ψ:

$$s = \frac{u\sin\alpha\cos\psi - \ell\sin\psi}{1-\cos^2\frac{\alpha}{2}\cos^2\psi}\,. \tag{3.188}$$

With Eqs.(3.178) for $\mathbf{n}$ and φ and with the equations for $\mathbf{w}$, $\mathbf{n}\times\mathbf{w}$ and s the two-parametric manifold of screw displacements $(\hat{\mathbf{n}},\hat{\varphi})$ solving Problem 1 is uniquely determined. The special screw displacements $(\hat{\mathbf{n}}_3,\hat{\alpha})$ and $(\hat{\mathbf{n}}_1,\pm\pi)$ shown in Fig. 3.17 belong to this manifold. The associated parameter values are $\psi=-\pi/2$, $u=0$ for the first and $\psi=0$, $u=0$ for the second.

Pure rotations satisfy the condition $s=0$, i.e.,

$$u = \frac{\ell\tan\psi}{\sin\alpha}\,. \tag{3.189}$$

This condition defines the one-parametric submanifold of rotations in the two-parametric manifold of screw displacements. The special rotation with parameter values $\psi=0$, $u=0$ belongs to this submanifold. The manifold of all rotation axes defines a ruled surface. Let x, u, z be the coordinates of points of this ruled surface along $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, respectively. From the primary part of (3.182) and from (3.189) it follows that $z/x=-\tan\psi=-(u/\ell)\sin\alpha$ or $ux=-z\ell/\sin\alpha$. This is the equation of an equilateral hyperbolic paraboloid. In a ξ,η-system rotated through $45°$ about the $\mathbf{n}_2$-axis the transformation is $x=(\xi+\eta)\sqrt{2}/2$ and $u=(-\xi+\eta)\sqrt{2}/2$ and consequently $ux=-(\xi^2-\eta^2)/2$. Hence the principal-axes equation of the hyperbolic paraboloid is $z=(\xi^2-\eta^2)/a$ with $a=2\ell/\sin\alpha$. This ends the solution of Problem 1.

Solution to Problem 2: According to the problem statement the point A on $\hat{\mathbf{r}}_1$ at the foot of the common perpendicular is displaced to some prescribed point P on $\hat{\mathbf{r}}_2$. This displacement is achieved by a one-parametric submanifold of the screw displacements solving Problem 1. An appropriate measure of displacement of point A is the quantity $(\mathbf{r}_\mathrm{P} - \mathbf{r}_\mathrm{B}) \cdot \mathbf{r}_2 = (\mathbf{r}_\mathrm{P} - \mathbf{r}_\mathrm{A}) \cdot \mathbf{r}_2$. The analysis to follow shows that the essential measure of displacement is the quantity $\sigma = (\mathbf{r}_\mathrm{P} - \mathbf{r}_\mathrm{B}) \cdot \mathbf{r}_2 \cos\alpha/2$.

Further below σ is prescribed. However, before doing so, σ is determined for the screw displacements solving Problem 1 as function of ψ and u. Let $\boldsymbol{\varrho}$ be the vector leading from the point of intersection of a screw axis $\hat{\mathbf{n}}$ with the nodal line $\mathbf{n}_2$ to point A. Figure 3.17 yields the expression

$$\boldsymbol{\varrho} = -\Big(u\mathbf{n}_2 + \frac{1}{2}\ell\mathbf{n}_3\Big) . \tag{3.190}$$

The associated measure of displacement is

$$\sigma = \Big\{(1-\cos\varphi)[(\mathbf{n}\cdot\boldsymbol{\varrho})\mathbf{n} - \boldsymbol{\varrho}] + \sin\varphi\,\mathbf{n}\times\boldsymbol{\varrho} + s\mathbf{n}\Big\} \cdot \mathbf{r}_2 \cos\frac{\alpha}{2} . \tag{3.191}$$

The term $s\mathbf{n}$ is due to translation, and the remaining terms are copied from (1.37). The scalar products are expressed in terms of ψ and u with the help of (3.190), (3.177) and (3.178):

$$\left.\begin{aligned} &\mathbf{n}\cdot\boldsymbol{\varrho} = \frac{1}{2}\ell\sin\psi , \quad \mathbf{n}\cdot\mathbf{r}_2 = \cos\frac{\alpha}{2}\cos\psi , \quad \boldsymbol{\varrho}\cdot\mathbf{r}_2 = -u\sin\frac{\alpha}{2} , \\ &\mathbf{n}\times\boldsymbol{\varrho}\cdot\mathbf{r}_2 = \frac{1}{2}\ell\sin\frac{\alpha}{2}\cos\psi - u\cos\frac{\alpha}{2}\sin\psi . \end{aligned}\right\} \tag{3.192}$$

For s, for $(1-\cos\varphi)$ and for $\sin\varphi$ the expressions (3.188) and (3.187), respectively, are substituted. This yields σ as function of ψ and u. The result is written in the two forms

$$\sigma = \frac{2u\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \ell\cos^2\frac{\alpha}{2}\sin\psi\cos\psi}{1-\cos^2\frac{\alpha}{2}\cos^2\psi} = \frac{4u\sin\alpha - \ell(1+\cos\alpha)\sin 2\psi}{4-(1+\cos\alpha)(1+\cos 2\psi)} . \tag{3.193}$$

In what follows, the manifold of all screw displacements is determined for which σ is an arbitrarily prescribed constant $-\infty < \sigma < \infty$. Then (3.193) represents a constraint equation for ψ and u. As independent parameter ψ is chosen. The solution for u as function of ψ is written in the two forms

$$u\sin\alpha = \sigma + \cos^2\frac{\alpha}{2}\cos\psi(\ell\sin\psi - \sigma\cos\psi) , \tag{3.194}$$

$$u = \frac{\sigma(3-\cos\alpha)}{4\sin\alpha} + \frac{1}{4}\cot\frac{\alpha}{2}(\ell\sin 2\psi - \sigma\cos 2\psi) . \tag{3.195}$$

For further simplification the x, z-system of principal axes is introduced which is rotated against the $\mathbf{n}_1, \mathbf{n}_3$-axes through the angle ψ_0 defined by

$$\cos 2\psi_0 = \frac{\ell}{\sqrt{\ell^2+\sigma^2}} , \qquad \sin 2\psi_0 = \frac{\sigma}{\sqrt{\ell^2+\sigma^2}} . \tag{3.196}$$

With new variables

$$y = u - u_0 \ , \qquad \chi = \psi - \psi_0 \tag{3.197}$$

and with the constants

$$u_0 = \frac{\sigma(3 - \cos\alpha)}{4\sin\alpha} \ , \qquad h = \frac{1}{4}\cot\frac{\alpha}{2}\sqrt{\ell^2 + \sigma^2} \tag{3.198}$$

(3.195) becomes

$$y = h\sin 2\chi \ . \tag{3.199}$$

The vector $u(\chi)\mathbf{n}_2$ is the perpendicular from 0 onto the screw axis. The rotation angle φ and the translation s are determined through (3.178) and (3.188) as functions of $\psi = \chi + \psi_0$.

The one-parametric manifold of screw axes with parameter χ defines a ruled surface. With an additional free parameter λ this ruled surface has the parameter equations

$$x = \lambda\cos\chi \ , \qquad z = -\lambda\sin\chi \ , \qquad y = h\sin 2\chi \ . \tag{3.200}$$

Compare this with Eqs.(3.170). Except for an interchange of y and z both sets of equations are identical. Hence the ruled surface is a cylindroid the directrix of which is the nodal line. The parameter-free equation of the cylindroid is (cf. (3.171))

$$y(x^2 + z^2) = -2hxz \ . \tag{3.201}$$

Every prescribed value of σ determines a one-parametric manifold of screw displacements with the associated cylindroid of screw axes.

Following (3.171) some important geometrical properties of the cylindroid were listed (see Fig. 3.16). At every point y in the interval $|y| \leq |h|$ the directrix is orthogonally intersected by two screw axes. At $y = 0$ these screw axes lie in the x-axis and in the z-axis, respectively. These are the principal screw axes. At $y = -h$ and at $y = +h$ the two intersecting screw axes coincide. The angles associated with these axes are $\chi = -\pi/4$ and $\chi = +\pi/4$, respectively. These screw axes are torsal lines of the cylindroid. Each of the two planes spanned by the directrix and by a torsal line is a plane of symmetry of the cylindroid. Every pair of screw axes intersecting the directrix in one point is located symmetrically with respect to each of the planes of symmetry, i.e., under angles χ_1 and χ_2 related by the equation $\chi_2 = \pi/2 - \chi_1$. The relationship between the corresponding angles $\psi_1 = \chi_1 + \psi_0$ and $\psi_2 = \chi_2 + \psi_0$ is $\psi_2 = \pi/2 - \psi_1 + 2\psi_0$.

It was shown that the screw axes under the angles $\psi = 0$ and $\psi = -\pi/2$ belong to special screw displacements. The first one is characterized by $\varphi = \pm\pi$ and the second by $\varphi = \alpha$ and $s = \ell$. It will be seen that also the screw displacements with screw axes located symmetrically to these screw axes have special properties. The associated angles are $\psi = \pi/2 + 2\psi_0$ and $\psi = 2\psi_0$, respectively.

First, it is investigated which screw axis $\hat{\mathbf{n}}$ intersects one or the other of the two lines $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. The screw axis $\hat{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{w}$ has the Plücker vectors $\mathbf{n}$ and $\mathbf{w}$ given in (3.182) and (3.183), respectively. The lines $\hat{\mathbf{r}}_i = \mathbf{r}_i + \varepsilon\mathbf{w}_i$ $(i = 1, 2)$ have the Plücker vectors $\mathbf{r}_i$ and $\mathbf{w}_i$ given in (3.177) and (3.181). The condition of intersection is $\mathbf{n} \cdot \mathbf{w}_i + \mathbf{r}_i \cdot \mathbf{w} = 0$ (see (2.18)). It turns out that this condition has for both lines the same form

$$\ell \sin\frac{\alpha}{2}\cos\psi + 2u\cos\frac{\alpha}{2}\sin\psi = 0\ . \tag{3.202}$$

This equation is multiplied by $\sin\alpha/2$, and then $u\sin\alpha$ is replaced by the expression in (3.194). Simple reformulation produces the final form

$$(\ell\cos\psi + \sigma\sin\psi)\Big(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\sin^2\psi\Big) = 0\ . \tag{3.203}$$

The results are summarized as follows. A screw axis which intersects one of the two lines intersects also the other, and there is only one such screw axis. Its angle ψ is given by $\tan\psi = -\ell/\sigma$ or, because of (3.196), by $\tan\psi = -\cot 2\psi_0$. Hence $\psi = 2\psi_0 + \pi/2$. This shows that the screw axis is the one which is located symmetrically with respect to $\chi = \pi/4$ to the screw axis associated with $\psi = 0$, $\varphi = \pi$.

Every screw axis on the cylindroid, i.e., every angle ψ or χ, respectively, is associated with a rotation angle φ and a translation s. The angle is determined by (3.178). The two screw axes along the principal axes and the two screw axes along the torsal lines are examples of pairs of mutually orthogonal screw axes (with or without intersection point). All pairs of mutually orthogonal screw axes satisfy the identity (1.229): $\cot^2(\varphi/2) + \cot^2(\varphi^*/2) \equiv \cot^2\alpha/2$.

The quantity s in the screw displacements solving Problem 2 remains to be formulated. It is obtained by substituting in (3.188) for $u\sin\alpha$ the expression from (3.194). The result is

$$s = \sigma\cos\psi - \ell\sin\psi\ . \tag{3.204}$$

Comparison with (3.195) yields the relationship

$$s(\psi) = -4\tan\frac{\alpha}{2}\,[u(\psi/2) - u_0]\ . \tag{3.205}$$

Let s_1 and s_2 be the values of s associated with the principal axes x $(\psi = \psi_0)$ and z $(\psi = \psi_0 - \pi/2)$. With (3.196)

$$s_{1,2} = \sqrt{\frac{1}{2}\Big(\ell^2 + \sigma^2 \mp \ell\sqrt{\ell^2 + \sigma^2}\Big)}\ . \tag{3.206}$$

From (3.204) it follows that there is only a single screw displacement which is a pure rotation $(s = 0)$. The direction of its axis is determined by $\tan\psi = \sigma/\ell = \tan 2\psi_0$. Hence $\psi = 2\psi_0$. This shows that the rotation axis is the screw axis which is located symmetrically with respect to $\chi = \pi/4$ to the screw axis associated with $\psi = -\pi/2$, $\varphi = \alpha$ and $s = \ell$.

The results developed up to this point were obtained first by Moshammer [33], Pelisěk [35], Rath [40] and Lilienthal [28]. See also Bottema [4]. None of the cited papers made use of the principle of transference.

Equation (3.19) defined for screw displacements the scalar measure $p_{\mathrm{P}} = (s/2)/\tan\varphi/2$. With the second Eq.(3.178) and with (3.204)

$$p_{\mathrm{P}} = \frac{1}{2}\cot\frac{\alpha}{2}\sin\psi\,(\ell\sin\psi - \sigma\cos\psi) = \frac{1}{4}\cot\frac{\alpha}{2}\,[\ell(1-\cos 2\psi) - \sigma\sin 2\psi]\,. \tag{3.207}$$

The angle $\psi = 0$ is associated with $s = \sigma$ and $p_{\mathrm{P}} = 0$. Transformation of (3.207) into the principal-axes system by means of (3.198) results in the equation

$$p_{\mathrm{P}} = p_{\mathrm{P0}} - h\cos 2\chi\,, \qquad p_{\mathrm{P0}} = \frac{1}{4}\ell\cot\frac{\alpha}{2}\,. \tag{3.208}$$

The mean value p_{P0} of p_{P} occurs in the torsal lines. The extremal values $p_{\mathrm{P0}} \pm h$ have opposite signs. They occur in the principal axes. For every pair of screw axes which is located symmetrically with respect either to $\chi = \pi/4$ or to $\chi = -\pi/4$ both screw axes are associated with one and the same measure p_{P}. Equations (3.208) and (3.199) reveal the remarkable relationship

$$(p_{\mathrm{P}} - p_{\mathrm{P0}})^2 + y^2 \equiv h^2\,. \tag{3.209}$$

It is formally identical with (3.164).

Special case $\sigma = 0$ (A *is displaced to* B)
The formulas for $\hat{\mathbf{n}}$ and φ are independent of σ. Equations (3.195) – (3.198), (3.204) and (3.206) become

$$\left.\begin{array}{lll} u = \dfrac{\ell}{4}\cot\dfrac{\alpha}{2}\sin 2\psi\,, & \chi \equiv \psi\,, & u_0 = 0\,, \\ s = -\ell\sin\psi\,, & s_1 = 0\,, & s_2 = \ell\,. \end{array}\right\} \tag{3.210}$$

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