Determining Knots by Minimizing the Second Derivative

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Abstract. In constructing a parametric curve interpolating a set of data points, one of the key problems is to specify a parameter value (node) for each data point. A new method of choosing knots is presented. For each data points, the new method constructs a quadratic polynomial curve by three adjacent data points. The node parameters of the quadratic curve are determined by minimizing the second derivative of the quadratic curve. And the knot interval between two adjacent data points is determined by two quadratic curves associated with the two adjacent data points. Experiments showed that the curves constructed using the knots by the new method generally have better interpolation precision.

Keywords: Polynomial curve · Determining knots · Second derivative · Minimizing

1 Introduction

The problem of describing objects in three dimensional is a fundamental issue and a core problem in computer aided geometric design/modeling, scientific computing and computer graphics. The key to 3D objects' expression is the construction of curves and curved surfaces. Having the advantage of easily geometric expression, efficient computation, convenience display and control, and geometrical invariability, parametric curves and curved surfaces become the most common way in 3D expressions. The constructed curves and curved surfaces are often required to have a certain smoothness or higher fitting accuracy [1–6]. In the constructing of curves and curved surfaces, the various objective functions (such as minimizing the energy of the curve, length of the curves, or integral of second derivative's square, etc.) are often used as constraints, so that the constructed curves/curved surfaces have the higher accuracy or the shape suggested by the data points.

Each data point's parameter value (node value) should be known before constructing a parameter curve. In practice application, however, these values are usually not given. Therefore, to construct a nice parameter curve not only need a good interpolation method, but also a good method to determine the appropriate parameter node. The simplest way to choose node is the uniform parameterization. But if the distance between the data points is uneven, the uniform parametrization generally leads to unsatisfactory results. For non-uniform distribution data points, there are usually three

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ways of parameterization, which are the chord length method, the Foley's method [7] and the centripetal method [8]. But our experiments show that, in terms of interpolation error, none of these methods has a distinct advantage over the others. Although these methods are widely used to construct the parametric curve, in some cases, none of them can produce a satisfactory result. In papers [9], a new method for choosing knots is presented. The knots are chosen using a global method. The chosen knots can be used to construct interpolants which reproduce parametric quadratic curves, if the interpolation scheme has quadratic polynomials. In papers [10–13], choosing knots by optimized methods has been discussed. Theses methods choosing knots by objective function that are defined. There are also articles discussing parameterization problems of spatial data points [14–16], the parameterized results are used to construct a parametric surface.

For given planar data points, a new method of choosing knots is presented in this paper. The new method is a local method, thus it is easily to modify the curves alternately, and hence provides conveniences for the designers. New method assumes that the curve between the data points can be approximated by a quadratic polynomial curve. And the node parameters of the quadratic curve are determined by minimizing the square of the curves' second derivative.

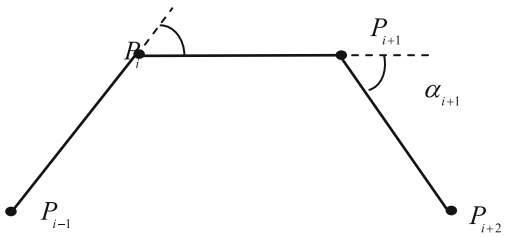

Fig. 1. Four points and α_i.

2 Basic Idea

Let $P_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$ be a set of distinct data points. The goal is to determine a knot t_i for each point, P_i, $i = 1, 2, \ldots, n$. After the knots being determined for the data points, existing methods for constructing parameter fitting curve could be used to construct a parameter curve $P(t)$ with $P_i = (x_i, y_i)$, $1 \le i \le n$.

Suppose the data points are taken from a parameter curve, the part of it between P_i and P_{i+1} could be approximate by a curve segment $P_i(t)$ passing four points, P_{i-1}, P_i, P_{i+1} and P_{i+2}, as shown in Fig. 1. So, for P_i and P_{i+1}, the corresponding knots t_i and t_{i+1} could be computed by $P_i(t)$ approximately. The knots t_i and t_{i+1} will be determined by the following process. To make the process of computing t_i and t_{i+1} easier, $P_i(t)$ is approximated by two quadratic curves $Q_i(t)$ and $Q_{i+1}(t)$. $Q_j(t)$, $j = i, i + 1$ is constructed by passing P_{j-1}, P_j and P_{j+1}. With $Q_i(t)$ and $Q_{i+1}(t)$, two groups of t_i and t_{i+1} can be determined, the combination of the two groups of t_i and t_{i+1} is used to determine their

end values. First of all, how to construct quadratic curve $Q_i(t)$ that interpolates P_{i-1}, P_i and P_{i+1} will be discussed.

For P_{i-1}, P_i and P_{i+1}'s three knots, t_{i-1}, t_i and t_{i+1}, there is only one free degree between them. To reduce the constructing process of $Q_i(t)$, following parameter transformation will be done to t.

$$t = t_{i-1} + (t_{i+1} - t_{i-1})s \tag{1}$$

So the knots t_{i-1}, t_i and t_{i+1} will be transformed into 0, s_i and 1, where

$$s_i = (t_i - t_{i-1})/(t_{i+1} - t_{i-1}) \tag{2}$$

Therefore, quadratic Lagrangian curve $Q_i(s) = (x_i(s), y_i(s))$ that interpolates P_{i-1}, P_i and P_{i+1} can be defined by

$$Q_i(s) = \frac{(s - s_i)(s - 1)}{s_i}(P_{i-1} - P_i) + \frac{s(s - s_i)}{1 - s_i}(P_{i+1} - P_i) + P_i, \tag{3}$$

where s_i is a unknown to be determined, which satisfies $0 < s_i < 1$.

Physically, the curve $Q_i(s)$ can be viewed as a path that a proton moves from point P_{i-1} through P_i to P_{i+1}. $Q_i'(s)$ and $Q_i''(s)$ are the proton's velocity and acceleration, respectively. As s_i is a variable, $Q_i(s)$ is actually a family of curves. Our aim is to select a perfect curve from the family. If the proton's speed in point P_{i-1} is fixed, one of the conditions that it moves from point P_{i-1} through P_i to P_{i+1} smoothly is that its acceleration should be as small as possible. The smaller the acceleration of protons, the velocity of the proton will be more close to the constant. And its orbit will be closer to the polygon composed of $P_{i-1}\,P_i\,P_{i+1}$. The movement path of $Q_i(s)$ will also be shorter, which means $Q_i(s)$ has the shape suggested by $\overline{P_{i-1}P_iP_{i+1}}$. So, $Q_i(s)$ is the perfect shape. Based on the above discussion, we'll use a method that minimizes the acceleration to select the perfect curve from $Q_i(s)$'s family of curves. That is, s_i in $Q_i(s)$ will be determined by minimizing the following objective function

$$|Q_i''(s)| = 2\sqrt{\frac{d_{i-1}^2}{s_i^2} - \frac{2\cos\alpha_i}{s_i(1 - s_i)}d_{i-1}d_i + \frac{d_i^2}{(1 - s_i)^2}} \tag{4}$$

where $d_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$.

3 Determining s_i

Since $|Q_i''(s)|$ and $|Q_i''(s)|^2$ have the same extreme point, $|Q_i''(s)|$'s extreme point will be solved by minimizing

$$|Q_i''(s_i)|^2 = 4\left(\frac{d_{i-1}^2}{s_i^2} - \frac{2\cos\alpha_i}{s_i(1-s_i)}d_{i-1}d_i + \frac{d_i^2}{(1-s_i)^2}\right) \tag{5}$$

So, s_i is the function of d_{i-1}, d_i and a_i, which could be expressed as

$$F(d_{i-1}, d_i, \alpha_i, s_i) = 0 \tag{6}$$

For any d_{i-1}, d_i and a_i, computing s_i in (6) involves solving nonlinear equations, and hence there is no explicit function for defining s_i. We'll seek its approximation function by the following way. Firstly, for $a_i = 0, \pi/2, \pi$, three particular values of s_i are computed, which are used to construct an interpolant which is used as an approximation to s_i. How to compute the three particular values of s_i in (6) will be discussed below.

Case 1. When $a_i = 0$, (5) can be written as

$$||Q_i''(s)||^2 = 4\left(\frac{d_{i-1}}{s_i} - \frac{d_i}{1-s_i}\right)^2 \tag{7}$$

It follows from $\frac{d_{i-1}}{s_i} - \frac{d_i}{1-s_i} = 0$ that

$$s_i = \frac{d_{i-1}}{d_{i-1} + d_i} \tag{8}$$

This means that if s_i is defined by (8), the acceleration of the proton is zero.

Case 2. When $a_i = \pi/2$, (5) can be written as

$$||Q_i''(s)||^2 = 4\left(\frac{d_{i-1}^2}{s_i^2} + \frac{d_i^2}{(1-s_i)^2}\right) \tag{9}$$

the solution of (6) is

$$s_i = \frac{{d_{i-1}}^{\frac{2}{3}}}{{d_{i-1}}^{\frac{2}{3}} + d_i^{\frac{2}{3}}} \tag{10}$$

Case 3. When $a_i = \pi$, (5) can be written as

$$||Q_i''(s)||^2 = 4\left(\frac{d_{i-1}}{s_i} + \frac{d_i}{1-s_i}\right)^2 \tag{11}$$

the solution of (6) is

$$s_i = \frac{d_{i-1}^{\frac{1}{2}}}{d_{i-1}^{\frac{1}{2}} + d_i^{\frac{1}{2}}} \tag{12}$$

It follows from (8), (10) and (12) that s_i in (6) could be expressed in the general form as

$$s_i = \frac{d_{i-1}^{\rho}}{d_{i-1}^{\rho} + d_i^{\rho}} \tag{13}$$

where ρ is a function of a_i.

Now, we are in the position on constructing the function ρ. For ρ in (13), we have computed three points $(\rho, \alpha_i) = \{(1,0), (2/3, \pi/2), (1/2, \pi)\}$. These three points will be used to construct quadratic polynomial to define ρ. By Lagrange formula, ρ which passes the three points can be defined by

$$\rho = \frac{1}{3} * \frac{(\alpha_i - \pi/2)(\alpha_i - \pi)}{\pi * \pi/2} - \frac{1}{6} * \frac{\alpha_i(\alpha_i - \pi/2)}{\pi * \pi/2} + \frac{2}{3} \tag{14}$$

4 Merging Local Knots Sequences

So far we have computed the local knots 0, s_i, 1 for the three points P_{i-1}, P_i, P_{i+1}, and 0, s_{i+1}, 1 for the three points P_i, P_{i+1}, P_{i+2} with respect to the two locally interpolating quadratic curves $Q_i(s)$ and $Q_{i+1}(s)$. These knots define the knot interval [0, s_i] between P_{i-1} and P_i, and the knot interval [s_i, 1] between P_i and P_{i+1}; we will associate the lengths of these two intervals, i.e., s_i and $1 - s_i$, with the point P_i, and still call them knot intervals. Even when all the data points P_i are taken from the same quadratic curve, the knot intervals associated with different points may not be equal due to the different linear scales of different parameterizations. In this section, we introduce a normal form of a quadratic curve and use it to merge all the knot intervals associated with different points P_i, $i = 2, 3, \ldots n - 1$, into a consistent global knot sequence with respect to the same parameterization of a quadratic curve.

Let $P_i(s) = (x_i(s), y_i(s))$ be a quadratic curve, which is defined by

$$\begin{aligned} x_i(s) &= X_{i,2}s^2 + X_{i,1}s + X_{i,0} \\ y_i(s) &= Y_{i,2}s^2 + Y_{i,1}s + Y_{i,0} \end{aligned} \tag{15}$$

where $X_{i,2} \neq 0$ or $Y_{i,2} \neq 0$.

By a rigid transformation and a linear reparameterization, $P_i(s)$ can be transformed into the following normal form

$$\begin{aligned} \bar{x}_i(t) &= t^2 + \bar{X}_1 t + \bar{X}_0 \\ \bar{y}_i(t) &= t^2 + \bar{Y}_1 t + \bar{Y}_0 \end{aligned} \tag{16}$$

Where

$$\begin{aligned}
\bar{X}_0 &= \cos\beta_i X_{i,0} + \sin\beta_i Y_{i,0} \\
\bar{Y}_0 &= -\sin\beta_i X_{i,0} + \cos\beta_i Y_{i,0} \\
\bar{X}_1 &= \frac{\cos\beta_i X_{i,1} + \sin\beta_i Y_{i,1}}{\sqrt{\cos\beta_i X_{i,2} + \sin\beta_i Y_{i,2}}} \\
\bar{Y}_1 &= \frac{-\sin\beta_i X_{i,1} + \cos\beta_i Y_{i,1}}{\sqrt{\cos\beta_i X_{i,2} + \sin\beta_i Y_{i,2}}}
\end{aligned} \tag{17}$$

The transformation and reparameterization required are

$$\begin{aligned}
\bar{x} &= x\cos\beta_i + y\sin\beta_i \\
\bar{y} &= -x\sin\beta_i + y\cos\beta_i
\end{aligned} \tag{18}$$

where

$$\cos\beta_i = \frac{X_{i,2} + Y_{i,2}}{\sqrt{X_{i,2}^2 + Y_{i,2}^2}}, \quad \sin\beta_i = \frac{Y_{i,2} - X_{i,2}}{\sqrt{X_{i,2}^2 + Y_{i,2}^2}}$$

and

$$t = \left(X_{i,2}^2 + Y_{i,2}^2\right)^{\frac{1}{4}} s \tag{19}$$

For P_{i-1}, P_i and P_{i+1}, $i = 2, 3, \ldots, n-1$, the parametric quadratic polynomial $Q_i(s) = (x_i(s), y_i(s))$ which interpolates P_{i-1}, P_i and P_{i+1} at 0, s_i and 1, respectively, is

$$\begin{aligned}
x_i(s) &= X_{i,2}s^2 + X_{i,1}s + x_{i-1} \\
y_i(s) &= Y_{i,2}s^2 + Y_{i,1}s + y_{i-1}
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
X_{i,2} &= \frac{x_{i-1} - x_i}{s_i} + \frac{x_{i+1} - x_i}{1 - s_i} \\
X_{i,1} &= -\frac{(x_{i-1} - x_i)(s_i + 1)}{s_i} - \frac{(x_{i+1} - x_i)s_i}{1 - s_i} \\
Y_{i,2} &= \frac{y_{i-1} - y_i}{s_i} + \frac{y_{i+1} - y_i}{1 - s_i} \\
Y_{i,1} &= -\frac{(y_{i-1} - y_i)(s_i + 1)}{s_i} - \frac{(y_{i+1} - y_i)s_i}{1 - s_i}
\end{aligned} \tag{21}$$

When we convert the quadratic curve $Q_i(s)$ in Eq. (20) to the normal form in Eq. (16), by the reparameterization (19), the knot intervals s_i and $1 - s_i$ associated with P_i become

$$\begin{aligned} \Delta_{i-1}^{i} &= \left(X_{i,2}^2 + Y_{i,2}^2\right)^{\frac{1}{4}} s_i \\ \Delta_{i}^{i} &= \left(X_{i,2}^2 + Y_{i,2}^2\right)^{\frac{1}{4}} (1 - s_i) \end{aligned} \tag{22}$$

Where $X_{i,2}$ and $Y_{i,2}$ are defined in (21).

If P_{i-1}, P_i and P_{i+1} are on a straight line, then with (7) and (8), it is easy to prove that

$$\begin{aligned} \Delta_{i-1}^{i} &= |P_{i-1}P_i| \\ \Delta_{i}^{i} &= |P_iP_{i+1}| \end{aligned} \tag{23}$$

Hence, by mapping each $Q_i(s)$ into the normal form, for each pair of consecutive points P_i and P_{i+1} there are two knot intervals Δ_i^i and Δ_i^{i+1}, $2 \leqslant i \leqslant n-1$. We have $\Delta_i^i = \Delta_i^{i+1}$ if all the data points are taken from the same quadratic curve. But in general, $\Delta_i^i \neq \Delta_i^{i+1}$. Furthermore, for end data points, there is only one knot interval, Δ_1^1, for the pair P_1 and P_2; and there is one knot interval, Δ_{n-1}^{n-2}, for the pair P_{n-1} and P_n.

We average the two sequences of knot intervals, $\left\{\Delta_i^i\right\}$ and $\left\{\Delta_i^{i+1}\right\}$, into a single sequence of knot intervals, $\{\Delta_i\}$, $i = 1, 2, \ldots, n-1$, as follows.

$$\begin{aligned} \Delta_1 &= \Delta_1^1, \\ \Delta_i &= \frac{2\Delta_i^i \Delta_i^{i+1}}{\Delta_i^i + \Delta_i^{i+1}}, i = 2, 3, \ldots, n-2, \\ \Delta_{n-1} &= \Delta_{n-1}^{n-2} \end{aligned} \tag{24}$$

From the knot intervals $\{\Delta_i\}$, we compute the global knot sequence $\{t_i\}$, $i = 1, 2, \ldots, n$, as follows.

$$\begin{aligned} t_1 &= 0; \\ t_{i+1} &= t_i + \Delta_i, i = 1, 2, \ldots, n-1 \end{aligned} \tag{25}$$

5 Experiments

In this section, the new method is compared with the chord length method, the Foley's method and the centripetal method by experiments. The data points used in comparison are taken from the given primitive curves. The four methods are used to compute knots to construct cubic spline curves which interpolate the data points, the comparison is

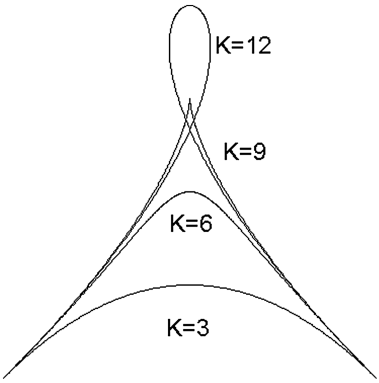

Fig. 2. F(K, t).

carried out by comparing the interpolation precision of the cubic spline curves. The primitive curves are a set of cubic curves, $F(K, t) = (x(F, t), y(K, t))$, which is defined by

$$\begin{aligned} x(K,t) &= t(t-1)(2t-1)K + 3t^2(3-2t) \\ y(K,t) &= t(1-t)K \end{aligned} \tag{26}$$

where $K = 1, 2, \ldots 12$.

The cubic curve $F(K, t)$ has the following properties: it is convex when $K = 1, 2, 3, 4$, it has two inflection points when $K = 5, 6, 7, 8$, it has one cusp when $K = 9$, and it has one loop when $K = 10, 11, 12$. When $K = 3, 6, 9, 12$, the figures of $F(K, t)$ on the region $[0, 1]$ are shown in Fig. 2.

The interval $[0, 1]$ used in comparison is divided into 20 subinterval to define the data points $P_i = F(K, t_i)$, $i = 0, 1, 2, \ldots 20$, where t_i is defined by

$$t_i = [i + \lambda \sin((20 - i)i)]/20 \quad i = 0, 1, 2, \ldots, 20 \tag{27}$$

where $0 < \lambda \leq 0.2$ to make the data points satisfying non-uniform distribution.

The tangent vectors of $F(K, t)$ at the end points $t = 1$ and $t = 1$ are used to construct the cubic splines. The four methods are evaluated in terms of absolute error curve $E(K, t)$ which is defined by

$$\begin{aligned} E(K,t) &= |P(s) - F(K,t)| = \min\{|P_i(s) - F(K,t)|,\ s_i \leq s \leq s_{i+1}\} \\ & i = 0, 1, 2, \ldots, 19 \end{aligned} \tag{28}$$

where $P(s)$ denotes one of the cubic spline curves constructed by the four methods. $F(K, t)$ is defined by (25), and $Pi(s)$ denotes the part of $P(s)$ on $[s_i, s_{i+1}]$, $|P(s) - F(K, t)|$ the distance from $P(s)$ to $F(K, t)$.

When $\lambda = 0.1, 0.2$ in (26), the maximum values of the error curve $E(t)$ generated by the four methods are shown in Table 1, the enlarged error curve $E(t)$'s by the four methods are shown in Figs. 3, 4, 5 and 6.

Table 1. The maximum errors by the four methods for $\lambda = 0.1$

$E(K, t)$	New	Centripetal	Foley	Chord length
K = 1	1.90e-004	3.16e-004	1.65e-004	5.16e-004
K = 2	3.47e-005	3.37e-004	6.58e-005	4.99e-005
K = 3	2.13e-005	4.53e-004	1.08e-004	4.19e-005
K = 4	6.23e-005	5.96e-004	2.12e-004	1.08e-004
K = 5	1.36e-004	7.10e-004	3.31e-004	2.19e-004
K = 6	2.50e-004	7.11e-004	4.83e-004	4.82e-004
K = 7	5.54e-004	8.50e-004	6.07e-004	1.15e-003
K = 8	6.17e-004	7.86e-004	4.37e-004	2.54e-003
K = 9	3.93e-004	6.43e-004	4.67e-004	4.80e-004
K = 10	5.80e-004	1.35e-003	1.04e-003	3.01e-003
K = 11	4.18e-004	1.62e-003	1.41e-003	2.00e-003
K = 12	4.29e-004	1.64e-003	1.55e-003	1.58e-003

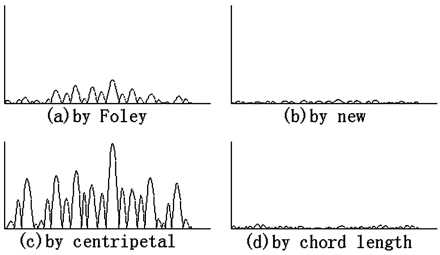

Fig. 3. 1300E(t) by four methods for K = 3.

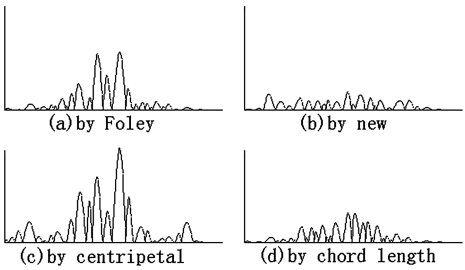

Fig. 4. 900E(t) by four methods for K = 6.

Results in Table 1 and Figs. 3, 4, 5 and 6 showed that compared with the three methods, the precision of the curves constructed by new method is higher. The four methods have also been compared on data points which divide [0, 1] into 10, 40, … etc. subintervals. The results are basically the same as those shown in Table 1 and Figs. 3, 4, 5 and 6.

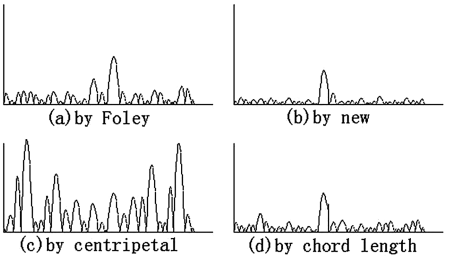

Fig. 5. 1000E(t) by four methods for K = 9.

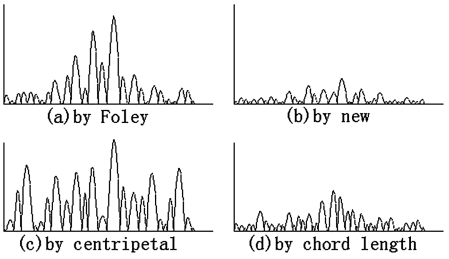

Fig. 6. 400E(t) by four methods for K = 12.

6 Conclusion

The new method for choosing knots in this paper presents is a local one. It allows the designers to change the shape of the curve through the adjusting the nodes, which is very important in producing and engineering design. Therefore, the new method is particularly suitable for interactive design of the parametric curve, thus has a high value in the construction of the curve. The innovations of the new method are as follows: for each two adjacent data points, the new method assumes that the curve between data points can be approximated by quadratic polynomial curve, and the node of the quadratic curve is determined by minimizing the curve's second derivative. So the new method is based on better mathematical foundation.

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