Formal Analysis of Epistemic Modalities and Conditionals Based on Logic of Belief Structures

Yasuo Nakayama(✉)

Graduate School of Human Sciences, Osaka University, Suita, Japan
nakayama@hus.osaka-u.ac.jp

Abstract. There is a strong context dependency in meaning of modalities in natural languages. Kratzer [9] demonstrates how to deal with this problem within possible world semantics. In this paper, we propose to interpret epistemic modalities in background of an epistemic state. Our analysis is a meta-linguistic one and we extensively use the proof-theoretic consequence relation. We define, then, a *belief structure* and introduce a *belief structure revision operator*. We call this framework *Logic of Belief Structures* (LBS). Then, we apply LBS to formalization of belief revision and interpretation of conditionals and investigate the relationship between belief revision and conditionals. Furthermore, we propose two types of conditionals, *epistemic* and *causal conditionals*.

Keywords: Conditionals · Epistemic modality · Belief structure · Belief revision · AGM theory · Sphere system

1 Logic for Epistemic Modalities

According to von Fintel [2], we can distinguish six kinds of modal meaning. They are *alethic*, *epistemic*, *deontic*, *bouletic*, *circumstantial*, and *teleological modality*. He characterized *epistemic modality* as the modality that is based on epistemic state:

(1a) [Epistemic modality] Epistemic modality concerns what is possible or necessary, given what is known and what the available evidence is.
(1b) [Example for epistemic modality] It has to be raining. [After observing people coming inside with wet umbrellas.]

Kratzer [9, pp. 4–6] proposes to explain the varieties of modalities in terms of the distinction of views. According to Kratzer, the core meaning of *must* can be interpreted as *must in view of*. This *must in view of* takes two arguments, namely *modal restriction* and *modal scope*. Then, we have the following schema for modal sentences:

must in view of (modal restriction, modal scope).

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To demonstrate how to use this schema, let us take an example for epistemic modality:

(2a) [Example for epistemic modality] The ancestors of the Maoris must have arrived from Tahiti.
(2b) [*must-in-view-of* Interpretation] In view of what is known, the ancestors of the Maoris must have arrived from Tahiti.
(2c) [Application of Kratzer's schema] *must in view of* (what is known, the ancestors of the Maoris arrived from Tahiti).

Kratzer [9, pp. 10–11] defines a possible world semantics for *must in view of*; her definition is restricted to propositional logic.

Definition 1. (3a) *A proposition* p *is true in a world* w *in* W *iff* $w \in p$.
(3b) *The meaning of* ***must in view of*** *is a function* ν *that satisfies the following conditions:*
1. *The domain of* ν *is the set of all pairs* $\langle p, f\rangle$ *such that* $p \in P(W)$ *and* f *is a function from* W *to* $P(P(W))$.
2. *For any* p *and* f *such that* $\langle p, f\rangle$ *is in the domain of* ν: $\nu(p, f) = \{w \in W : \bigcap f(w) \subseteq p\}$.

The modal scope denotes a proposition p and the modal restriction denotes an individual concept f. The meaning of *must in view of* is a function that maps pairs consisting of a proposition and a function of the same type as f to another proposition. When we apply (3b) to (2a), (2a) is true in those worlds w such that it follows from what is known in w that the ancestors of the Maoris arrived from Tahiti.

Recently, I proposed a formal framework in which the epistemic and the deontic modality are relativized by an accepted epistemic and a deontic theory [13–15]. The framework is called *Logic for Normative Systems* (LNS). In this paper, we concentrate on the epistemic part of LNS and show that Krazer's view can be rewritten within our framework.

Logic for Epistemic Modalities (LEM) is a framework expressed in a meta-language of *First-order Logic* (FOL). We define LEM-sentences as follows:

Definition 2. (4a) *All FO-sentences (i.e., sentences in FOL) are LEM-sentences.*
(4b) *If* p *is a FO-sentence and* T *is a set of FO-sentences, then* $MUST_T\ p$, $MIGHT_T\ p$, $KNOWN_T\ p$, *and* $BEL_T^{inf}\,p$ *are LEM-sentences. In this paper, we use small letters* p, q, ... *to denote FO-sentences.*
(4c) *If* ϕ *and* ψ *are LEM-sentences, then not* ϕ, $\phi \& \psi$, ϕ *or* ψ, $\phi \Rightarrow \psi$, *and* $\phi \Leftrightarrow \psi$ *are LEM-sentences, where logical connectives, not, &, or,* $\Rightarrow$, *and* $\Leftrightarrow$ *belong to the meta-language.*
(4d) *If* ϕ *is a LEM-sentence, then* ϕ *satisfies (4a) or (4b) or (4c).*

Definition 2 indicates that no iteration of modal operators is allowed in LEM. The meaning of epistemic modalities is defined as follows.

Definition 3. *Let T be a set of FO-sentences and p be a FO-sentence. We use $cons(T)$ as an abbreviation of $\langle T$ is consistent$\rangle$. We call T in the following definitions $\langle$belief base$\rangle$. A belief base represents what is explicitly believed.*

(5a) $MUST_T\ p$ *iff* $(T \vdash p\ \&\ cons(T))$.
(5b) $MIGHT_T\ p$ *iff* $cons(T \cup \{p\})$.
(5c) *[Knowledge as Explicit Belief]* $KNOWN_T\ p$ *iff* $(p \in T\ \&\ cons(T))$.
(5d) *[Inferential Belief]* $BEL_T^{inf} p$ *iff* $(MUST_T\ p\ \&$ *not* $KNOWN_T\ p)$.
(5e) $mod(T) = \{M : M \models T\}$.

Explicit belief and inferential belief play an important role for analysis of epistemic modalities (see Sect. 2). The semantics of LEM can be given in the same way as for FO-sentences.

To demonstrate the relationship to Krazer's approach, we introduce the following notations.

Definition 4. *Let T be a set of PL-formulas and p be a PL-formula. Let W be a set of possible worlds.*

(6a) v_W *is a function from PL-formulas to* $P(W)$.
(6b) $v_W^s(T) = \bigcap\{v_W(p) : p \in T\}$.
(6c) W *is a maximal set of worlds iff for any consistent set T of PL-formulas there is w such that* $w \in W\ \&\ w \in v_W^s(T)$.

Now, from Definitions 3 and 4, Propositions 5 and 6 immediately follow.

Proposition 5. *Let T be a set of FO-sentences and p be a FO-sentence.*

(7a) $MUST_T\ p \Rightarrow MIGHT_T\ p$.
(7b) $MUST_T\ (p \rightarrow q) \Rightarrow (MUST_T\ p \Rightarrow MUST_T\ q)$.
(7c) $(T_1 \subseteq T_2\ \&\ MIGHT_{T_2}(p \rightarrow p)) \Rightarrow (MUST_{T_1}\ p \Rightarrow MUST_{T_2}\ p)$.[1]
(7d) $(T_1 \subseteq T_2 \Rightarrow (MIGHT_{T_2}\ p \Rightarrow MIGHT_{T_1}\ p)$.
(7e) $KNOWN_T\ p \Rightarrow MUST_T\ p$.
(7f) $cons(T) \Rightarrow (KNOWN_T\ p \Leftrightarrow p \in T)$.
(7g) $BEL_T^{inf} p \Rightarrow p \notin T$.
(7h) $MUST_T\ p$ *iff* $(mod(T) \subseteq mod(\{p\})\ \&\ mod(T) \neq \emptyset)$.
(7i) $MIGHT_T\ p$ *iff* $mod(T \cup \{p\}) \neq \emptyset$.

Proposition 6. *Let T be a set of PL-formulas and p be a PL-formula. Let W be a maximal set of worlds.*

(8a) $MUST_T\ p$ *iff* $(v_W^s(T) \subseteq v_W(p)\ \&\ v_W^s(T) \neq \emptyset)$.
(8b) $MIGHT_T\ p$ *iff* $v_W^s(T \cup \{p\}) \neq \emptyset$.

In LEM, Krazer's *modal restriction* can be imitated by the *restriction given by a belief base*. We interpret, then, the modality not as a relation but as an operator restricted by a belief base: [*must in view of* (T)] (*proposition*).

Now, let us reconsider Krazer's example (2a). We interpret it as (2d).

[1] Because $(p \rightarrow p)$ is a FOL-theorem, it holds: $MIGHT_{T_2}(p \rightarrow p)$ iff T_2 is consistent.

(2c) [Application of Kratzer's schema] *must in view of* (what is known, the ancestors of the Maoris arrived from Tahiti).
(2d) $MUST_T$ tr(the ancestors of the Maoris arrived from Tahiti).[2]

Thus, we are justified to say that theory T in $MUST_T$ expresses *the view of what is known*. In this context, $MUST_T$ can be understood as *must in view of what is known*.

2 Evidential Aspects of Epistemic Modalities

The interpretation of *must* as *must in view of what is known*, proposed in the previous section, is still inappropriate as an interpretation of *epistemic must*, because it ignores evidential aspects of *epistemic must*. According to von Fintel and Gilles [3, p. 357], *epistemic must* presupposes the presence of indirect inference rather than a direct observation. Karttunen [7] observed problems connected with the traditional interpretation of *epistemic must*. When one considers which of the answers to the question (9a) conveys more confidence, it is natural to feel that epistemic modal sentence (9c) is less forceful than simple sentence (9b).

(9a) Where are the keys?
(9b) They are in the kitchen drawer.
(9c) They must be in the kitchen drawer.

According to Karttunen [7], modal semantics predicts that (9c) is a stronger answer to the question than (9b), but our intuition goes the other way. To respect this intuition, we propose to analyze (9b) as (9d) and (9c) as (9e). Here, we presuppose that belief base T_{9b} represents *what is known* by the speaker of (9b) and that belief base T_{9c} represents *what is known* by the speaker of (9c). Let $p_{keys} = tr$(The keys are in the kitchen drawer).

(9d) Felicitous condition: $KNOWN_{T_{9b}}\ p_{keys}$; Claim: p_{keys}.
(9e) Felicitous condition: $BEL^{inf}_{T_{9c}}\ p_{keys}$; Claim: $MUST_{T_{9c}}\ p_{keys}$.

It will be appropriate to interpret the situation described by (9a) $\sim$ (9c) as follows: Sentence (9b) is uttered by a person who is convinced that p_{keys}, while sentence (9c) is uttered by a person who has evidences for p_{keys} and accepts this proposition based on these evidences. According to our interpretation, (9b) is stronger than (9c) in the sense that the felicitous condition for (9b) implies $\langle$*must* (9b)$\rangle$.[3]

The bearer of T_{9b} knows that p_{keys}, while the bearer of T_{9c} does not know that p_{keys} and his belief of p_{keys} is supported by his inference based on his evidences.[4] Our interpretation fully supports the following observation of von Fintel and Gillies [3, p. 354]:

[2] Here, function tr is the translation function from English sentences to FO-sentences.
[3] Note that it holds: $KNOWN_{T_{9b}}\ p_{keys} \Rightarrow MUST_{T_{9b}}\ p_{keys}$. See (7e).
[4] Note that our interpretation agrees with Willet's taxonomy of evidential categories [16]. Willet interpret epistemic modalities as makers of indirect inference [3, p. 354].

epistemic modals are also evidential markers: they signal that the prejacent was reached through an inference rather than on the basis of direct observation or trustworthy reports.

What does this signaling means? We propose to interpret it as a felicitous condition (abbreviated as FC). This is described in Table 1.

S's utterance of p is felicitous iff S believes that S knows p.
S's utterance of $Must\ p$ is felicitous iff S accepts p based on an indirect inference.

von Fintel and Gillies argue for the thesis that epistemic modalities signal not weakness but indirect inference. This observation agrees with our interpretation of epistemic modalities (See Table 1).

Table 1. Interpretation of simple sentences and epistemic modalities

	Claim	FC	Formal representation of FC
Simple sentence	p	$KNOWN_{T_1}\ p$	$p \in T_1$ & $cons(T_1)$
Epistemic *must*	$MUST_{T_2}\ p$	$BEL^{inf}_{T_2} p$	$p \notin T_2$ & $T_2 \vdash p$ & $cons(T_2)$

Let us consider some additional examples from von Fintel and Gillies [3, p. 372]:

(9f) Seeing the pouring rain, Billy says: *It's raining.*
(9g) Seeing people coming inside with wet umbrellas, Billy says: *It must be raining.*

We assume that $p_{rain} = tr$(it is raining) and $p_{umbrellas} = tr$(people coming inside have wet umbrellas). Because of (9f) and (9g), it holds: $p_{rain} \in T_{9f}$ & $p_{umbrellas} \in T_{9g}$ & $p_{rain} \notin T_{9g}$. In this case, the situation can be described as Table 2.

Table 2. Examples for simple sentences and epistemic modalities

	Claim	FC
Simple sentence	p_{rain}	$KNOWN_{T_{9f}}\ p_{rain}$
Epistemic *must*	$MUST_{T_{9g}}\ p_{rain}$	$BEL^{inf}_{T_{9g}} p_{rain}$

We see that both (9f) and (9g) are appropriate, because felicitous conditions for both cases are satisfied in these situations.

As von Fintel and Gillies [3] discuss, there are several semantic approaches for epistemic modalities. Our approach is proof-theoretic and very straightforward. It is directly based on the following observation: Epistemic modalities are used in a situation in which the speaker has no direct but only indirect evidences for the prejacent.

3 Logic of Belief Structures

To describe semantics for conditionals, we propose to represent an epistemic state by a belief structure, which is a linearly ordered set of consistent sets of FO-sentences. In this section, we define a logical framework for such belief structures and call it *Logic of Belief Structures* (LBS).

Definition 7. (10a) *[Belief structure BS]* $BS = \langle ST, > \rangle$ *is a belief structure, when the following three conditions are satisfied:*
1. $ST = \{T_i : 1 \leq i \leq n$ *&* T_i *is a consistent set of FO-sentences*$\}$,
2. $>$ *is a total order on* ST *and* $T_1 > ... > T_n$, *and*
3. *for all* $T_i \in ST$ *and* $T_j \in ST$, $T_i \cap T_j = \emptyset$.
(10b) *[k first fragment of BS]* $top(BS, k) = \bigcup\{T_i : 1 \leq i \leq k$ *and* $T_i \in ST\}$. *In other words, k first fragment of BS is the union of the first k elements of BS. We can also define* $top(BS, k)$ *recursively as follows:*
1. $top(BS, 1) = T_1$.
2. $top(BS, k) = top(BS, k-1) \cup T_k$.
(10c) *[Consistent maximum of BS]* $top(BS, k)$ *is the consistent maximum of BS (abbreviated as cons-max*(BS)*) iff* $(cons(top(BS, k))$ *& not* $cons(top(BS, k+1)))$. *We call k the consistent maximum number of BS (abbreviated as* $cmn(BS)$*), when* $top(BS, k) =$ *cons-max(BS).*
(10d) *[Deductive closure]* $Cn(T) = \{p : T \vdash p\}$.
(10e) *[Belief set for BS] We call* Cn*(cons-max*(BS)*) the belief set for BS.*

Based on Definition 7, we can define some modal operators and some notions related to sphere systems.

Definition 8. *Let* $BS = \langle ST, > \rangle$ *be a belief structure with* $T_1 > ... > T_n$. *Let p and q be FO-sentences.*

(11a) $MUST^*_{BS}\ p$ *iff* $MUST_{cons\text{-}max(BS)}\ p$.
(11b) $MIGHT^*_{BS}\ p$ *iff* $MIGHT_{cons\text{-}max(BS)}\ p$.
(11c) *[Probability Order]* $MORE\text{-}PROBABLE_{BS}(p, q)$ *iff (there are* $T_i \in ST$ *and* $T_j \in ST$ *such that* $(p \in T_i$ *&* $q \in T_j$ *&* $T_i > T_j)$*).*
(11d) $PROBABLY_{BS}\ p$ *iff* $(MIGHT^*_{BS}\ p$ *& not* $MUST^*_{BS}\ p$ *&* $p \in top(BS, n)$ *&* $(\neg p \in top(BS, n) \Rightarrow MORE\text{-}PROBABLE_{BS}(p, \neg p)))$.
(11e) $MUST\text{-}min(BS, k, p)$ *iff* $(MUST_{top(BS,k)}\ p$ *& not* $MUST_{top(BS,k-1)}\ p)$.
(11f) $p \preccurlyeq_{BS} q$ *iff there are k and m such that* $(k \leq m \leq cmn(BS)$ *&* $MUST\text{-}min(BS, k, p)$ *&* $MUST\text{-}min(BS, m, q))$.
(11g) $p \approx_{BS} q$ *iff* $(p \preccurlyeq_{BS} q$ *&* $q \preccurlyeq_{BS} p)$.
(11h) $p \prec_{BS} q$ *iff* $(p \preccurlyeq_{BS} q$ *& not* $(p \approx_{BS} q))$.
(11i) *[Sphere Model System]*
SMS_{BS} *is a sphere model system for BS iff*
1. $SMS_{BS} = \{S_{cmn(BS)}, ..., S_1\}$, *and*
2. $S_k = mod(top(BS, k))$ *for k with* $1 \leq k \leq cmn(BS)$.
(11j) *[Sphere System] Let W be a maximal set of worlds. Let ST be a set of PL-formulas.*
SS_{BS} *is a sphere system for BS iff*

1. $SS_{BS} = \{S_{cmn(BS)}, ..., S_1\}$, *and*
2. $S_k = v^s_W(top(BS, k))$ *for* k *with* $1 \leq k \leq cmn(BS)$.[5]

$p \preccurlyeq_{BS} q$ is read as ⟨Based on BS, it is at least as possible that p as it is that q⟩. $p \approx_{BS} q$ is read as ⟨Based on BS, it is equally possible that p and that q⟩. $p \prec_{BS} q$ is read as ⟨Based on BS, it is more possible that p than that q⟩.[6] From the view of belief change, we may read $p \prec_{BS} q$ as ⟨In BS, p is more entrenched than q⟩. Based on Definition 8, Propositions 9 and 10 can be easily shown.

Proposition 9. *Let* BS *be a belief structure with* $T_1 > ... > T_n$. *Let* T_k $(1 \leq k \leq n)$ *be a set of FO-sentences.*

(12a) $k \leq m \leq n \Rightarrow top(BS, k) \subseteq top(BS, m)$.
(12b) $k \leq m \leq cmn(BS) \Rightarrow Cn(top(BS, k)) \subseteq Cn(top(BS, m))$.
(12c) $k \leq m \leq cmn(BS) \Rightarrow mod(top(BS, m)) \subseteq mod(top(BS, k))$.
(12d) *If* SMS_{BS} *is a sphere model system for* BS, *then* SMS_{BS} *satisfies the following four requirements:*
  1. SMS_{BS} *is centered on* $S_{cmn(BS)}$, *i.e., for all* $S_k \in SMS_{BS}$, $S_{cmn(BS)} \subseteq S_k$.
  2. SMS_{BS} *is nested, i.e., for all* $Si, Sj \in SMS_{BS}$, $(S_i \subseteq Sj$ *or* $S_j \subseteq S_i)$.
  3. SMS_{BS} *is closed under unions, i.e.,* $X \subseteq SMS_{BS} \Rightarrow \bigcup X \in SMS_{BS}$.
  4. SMS_{BS} *is closed under (nonempty) intersections, i.e.,* $(X \subseteq SMS_{BS}$ *&* $X \neq \emptyset) \Rightarrow \bigcap X \in SMS_{BS}$.

Proof. (12a) follows from (10b). (12b) follows from (10d) and (12a). (12c) follows from (5e) and (12a). (12d) 1, 2, 3, and 4 follow from (11i) and (12c). Q.E.D.

Proposition 10. *Let* T_k $(1 \leq k)$ *be a set of PL-formulas and* W *be a maximal set of worlds. Let* BS *be a belief structure.*

(13a) $k \leq m \leq cmn(BS) \Rightarrow v^s_W(top(BS, m)) \subseteq v^s_W(top(BS, k))$.
(13b) *If* SS_{BS} *is a sphere system for* BS, *then* SS_{BS} *is centered on* $S_{cmn(BS)}$, *nested, closed under unions, and closed under (nonempty) intersections.*

Proof. (13a) follows from (6b) and (12a). (13b) follows from (11j) and (13a). Q.E.D.

Lewis defined a sphere system in [11, p. 14]. (12d) shows that only the first characterization is different from his definition. Lewis required that a sphere system is centered on a singleton $\{w_0\}$, where the intended reference of w_0 is the actual world. Our interpretation of the center of a sphere system is epistemic. The center, $S_{cmn(BS)}$, denotes the set of worlds (or the set of models) in which all of what are consistently believed are true.

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[5] According to definition of v^s_W, $v^s_W(top(BS, k)) = \{w \in W$: all formulas in $top(BS, k)$ are true in $w\}$.
[6] These orders are a modification of *comparative possibility* in Lewis [11, p. 52].

Proposition 11. *Let BS be a belief structure.*

(14a) $\preccurlyeq_{BS}$ *is transitive.*[7]
(14b) $\approx_{BS}$ *is symmetric and transitive.*[8]
(14c) $PROBABLY_{BS}\ p \Rightarrow (MIGHT^*_{BS}\ p$ *& not* $MUST^*_{BS}\ p)$.

Proof. To show (14a), suppose that $p \preccurlyeq_{BS} q$ & $q \preccurlyeq_{BS} r$. Then from (11f), there are k, l, m such that ($k \leq l \leq m \leq cmn(BS)$ & $MUST$-$min(BS, k, p)$ & $MUST$-$min(BS, l, q)$ & $MUST$-$min(BS, m, r)$). Thus, from (11f), $p \preccurlyeq_{BS} r$. Therefore, transitivity holds for $\preccurlyeq_{BS}$. (14b) follows from (11g) and (14a). (14c) follows from (11d). Q.E.D.

4 Belief Revision Based on Logic of Belief Structures

We can divide a belief structure BS into two parts, namely the *consistent part*, $top(BS, k)$ with $k \leq cmn(BS)$, and the *inconsistent part*, $top(BS, k)$ with $cmn(BS) < k \leq n$. Now, let us define the *belief structure revision* and *expansion*.

Definition 12. *Let H be a consistent set of FO-sentences. Let BS be a belief structure with $T_1 > ... > T_n$.*

(15a) *We define $ext(H, BS)$ as the belief structure with $H > T_1 > ... > T_n$. In other words, the extended belief structure of BS by H is the belief structure that can be obtained from BS by adding H as the most reliable element.*
(15b) *[Belief structure revision]* $bsR(BS, H) = Cn(cons\text{-}max(ext(H, BS)))$.
(15c) *[Belief structure expansion]* $bsEX(BS, H) = Cn(cons\text{-}max(BS) \cup H)$.

We can show that our revision operator bsR satisfies all of postulates for the belief revision operator * in AGM-theory, if $H = \{p\}$ and p is a consistent FO-sentence.[9] Because the AGM-theory is a theory for propositional representation and our revision operator is defined for FO-sentences, our approach is broader than the AGM approach. The AGM postulates for belief revision can be defined as described in [6].

Definition 13. *Let p and q be PL-formulas and K be a set of PL-formulas. Let $K + p = Cn(K \cup p)$.*

(16a) *[Closure]* $K^*p = Cn(K^*p)$.
(16b) *[Success]* $p \in K^*p$.
(16c) *[Inclusion]* $K^*p \subseteq K + p$.
(16d) *[Vacuity] If* $\neg p \notin K$, *then* $K^*p = K + p$.
(16e) *[Consistency]* K^*p *is consistent if* p *is consistent.*
(16f) *[Extensionality] If* p *and* q *are logically equivalent, then* $K^*p = K^*q$.

[7] In domain $cons\text{-}max(BS)$, $\preccurlyeq_{BS}$ is also reflexive and connected.
[8] In domain $cons\text{-}max(BS)$, $\approx_{BS}$ is also reflexive. Thus, in $cons\text{-}max(BS)$, $\approx_{BS}$ is an equivalence relation.
[9] For AGM-theory, consult Gärdenfors [4, Sect. 3.3] and Hansson [6].

(16g) *[Superexpansion]* $K^*(p \wedge q) \subseteq (K^*p) + q$.
(16h) *[Subexpansion] If* $\neg q \notin K^*p$, *then* $(K^*p) + q \subseteq K^*(p \wedge q)$.

The following theorem shows that belief structure revision operator bsR satisfies all of the AGM postulates with the restriction that the revising FO-sentence is consistent.

Theorem 14. *Let* p, q, *and* $p \wedge q$ *be consistent FO-sentences.*

(17a) *[Closure]* $bsR(BS, \{p\})$ *is a belief set.*
(17b) *[Success]* $p \in bsR(BS, \{p\})$.
(17c) *[Inclusion]* $bsR(BS, \{p\}) \subseteq bsEX(BS, \{p\})$.
(17d) *[Vacuity]* $\neg p \notin Cn(\textit{cons-max}(BS)) \Rightarrow bsR(BS, \{p\}) = bsEX(BS, \{p\})$.
(17e) *[Consistency]* $bsR(BS, \{p\})$ *is consistent.*
(17f) *[Extensionality] If* p *and* q *are logically equivalent, then* $bsR(BS, \{p\}) = bsR(BS, \{q\})$.
(17g) *[Superexpansion]* $bsR(BS, \{p \wedge q\}) \subseteq bsEX(bsR(BS, \{p\}), \{q\})$.
(17h) *[Subexpansion]* $\neg q \notin bsR(BS, \{p\}) \Rightarrow bsEX(bsR(BS, \{p\}), \{q\}) \subseteq bsR(BS, \{p \wedge q\})$.

Proof. We assume that p, q and $p \wedge q$ are consistent FO-sentences. Then, (17a) holds because of (15a), (15b), and Definition 7. Because $\{p\}$ is consistent, (17b) follows from Definitions 7 and 12. From Definitions 7 and 12 follows: $\textit{cons-max}(\textit{ext}(H, BS)) \subseteq \textit{cons-max}(BS) \cup H$. Then, (17c) holds because of Definition 12. To show (17d), suppose $\neg p \notin Cn(\textit{cons-max}(BS))$. Then, $\textit{cons-max}(BS) \cup \{p\}$ is consistent. Thus, $\textit{cons-max}(\textit{ext}(\{p\}, BS)) = \textit{cons-max}(BS) \cup \{p\}$. Hence, (17d) holds based on Definition 12. (17e) holds because of (15b). (17f) holds based on (15b) and inference rules of FOL. To show (17g), we assume: $k = \textit{cmn}(\textit{ext}(\{p \wedge q\}, BS)) - 1$ and $m = \textit{cmn}(\textit{ext}(\{p\}, BS)) - 1$. Then, from Definitions 7 and 12: $\textit{top}(BS, k) \subseteq \textit{top}(BS, m)$. In FOL, it holds: $T_1 \subseteq T_2 \Rightarrow Cn(Cn(T_1 \cup \{p \wedge q\}) \cup \{q\}) \subseteq Cn(Cn(T_2 \cup \{p\}) \cup \{q\})$. Because $Cn(Cn(T_1 \cup \{p \wedge q\}) \cup \{q\}) = Cn(T_1 \cup \{p \wedge q\})$, (17g) holds based on Definition 12. To show (17h), we assume $\neg q \notin bsR(BS, \{p\})$. In FOL, we can prove: If $T \cup \{p\} \nvdash \neg q$, then [$\textit{cons}(T \cup \{p\})$ iff $\textit{cons}(T \cup \{p \wedge q\})$]. Thus, $bsR(BS, \{p\}) = bsR(BS, \{p \wedge q\})$. Therefore, $(bsR(BS, \{p\}) \cup \{q\}) = (bsR(BS, \{p \wedge q\}) \cup \{q\})$. However, because q follows from $p \wedge q$, $Cn(bsR(BS, \{p \wedge q\}) \cup \{q\}) = bsR(BS, \{p \wedge q\})$. From these: $bsEX(bsR(BS, \{p\}), \{q\}) = bsR(BS, \{p \wedge q\})$. Thus, (17h) holds. Q.E.D.

AGM-theory is a standard framework for belief revision. Thus, Theorem 14 suggests the adequacy of our definition of belief structure revision. In fact, our approach provides a useful tool for belief revision, because it only requires a linearly order sets of FO-sentences. The original AGM requirements for the entrenchment relation are rather unnatural and difficult to use.[10]

[10] However, AGM-theory has a nice correspondence with the probability theory [4, Chap. 5]. Our approach is difficult to relate with a probability theory.

5 Conditionals and Belief Revision

Our analysis of conditionals in this paper is based on Ramsey Test [4, p. 147]:

> [RT] Accept the sentence of the form ⟨If A, then C⟩ in a state of belief K if and only if the minimal change of K needed to accept A also requires accepting C.

This idea can be roughly expressed as follows: ⟨If A, then C⟩ is acceptable with respect to K iff *minimal-change*(K, A) implies C.

This idea can be combined with Kratzer's approach to counterfactual conditionals. Kratzer [9, p. 64] suggests that there are (at least) three forms of conditionals: (*If* ...), (*necessarily*/*possibly*/*probably*). According to this observation, we have two types of operators in counterfactual conditionals (*If* p, $Modal$ q). The operator *If* characterizes the considered situation, and the operator $Modal$ makes a modal statement. The antecedent [*If* p] brings us to imagine a situation in which p is true, where the situation is described by T. Then, we consider whether the modal claim in the consequence $[MODAL_T\ q]$ holds in the imagined situation. Based on this idea, we propose to interpret *If*-operator as a *belief structure revision operator* and p as the *revising consistent FO-sentence*.

Definition 15. *Let BS be a belief structure and H be a consistent set of FO-sentences. Let Modal ∈ {Must, Might, Known} and MODAL ∈ {MUST, MIGHT, KNOWN}.*

(18a) $IF_{BS}(H) = \textit{cons-max}(ext(H, BS))$.
(18b) $[If^{BS}p](Modal\ q)$ *iff*
$(\text{not } cons(top(BS, 1) \cup \{p\}) \text{ or } (T = IF_{BS}(\{p\})\ \&\ MODAL_T\ q))$.
(18c) $[If^{BS}p](Probably\ q)$ *iff*
$(\text{not } cons(top(BS, 1) \cup \{p\}) \text{ or } PROBABLY_{ext(\{p\},BS)}\ q)$.

From Definition (18b) follows: If $cons(top(BS, 1) \cup \{p\})$, then $[If^{BS}\ p](Must\ q)$ holds iff the minimal change of *cons-max*(BS) needed to accept p also requires accepting q. This formulation roughly corresponds to [RT]. Based on Definition 15, we can prove Proposition 16.

Proposition 16. *Let BS be a belief structure and H be a consistent set of FO-sentences.*

(19a) $bsR(BS, H) = Cn(IF_{BS}(H))$.
(19b) $[If^{BS}\ p](Must\ q)$ *iff* $(IF_{BS}(\{p\}) = \{p\} \text{ or } q \in bsR(BS, \{p\}))$.
(19c) $MIGHT_{cons-max(BS)}\, p \Rightarrow$
$([If^{BS}\, p](Must\, q) \Rightarrow MUST_{cons-max(BS)}\, (p \rightarrow q))$.
(19d) $[If^{BS}\, p](Must\, q) \Rightarrow$
$(mod(top(BS, 1) \cup \{p\}) = \emptyset \text{ or } mod(IF_{BS}(\{p\})) \subseteq mod(\{q\}))$.

Proof. (19a) follows from (15b) and (18a). (19b) follows from (18a), (18b), and (19a). To show (19c), suppose that $MIGHT_{cons\text{-}max(BS)}\ p$ holds. Then, because of (5b), $cons\text{-}max(BS) \cup \{p\}$ is consistent. Thus, according to (15a) and (18a), $IF_{BS}(\{p\}) = cons\text{-}max(ext(\{p\}, BS) = cons\text{-}max(BS) \cup \{p\}$. Now, suppose that $[If^{BS}\ p](Must\ q)$ holds. Then, from (18b), $MUST_{cons\text{-}max(ext(\{p\},BS))}\ q$. Thus, $MUST_{cons\text{-}max(BS)\cup\{p\}}\ q$. Then, because of (5a) and the deduction theorem of FOL, $MUST_{cons\text{-}max(BS)}\ (p \rightarrow q)$ holds. Hence, (19c) holds. (19d) follows from (7h) and (18b). Q.E.D.

(19a) and (19b) show that our definition of counterfactual conditional is based on the belief structure revision. According to (19c), a material conditional follows from a counterfactual conditional, when no change is required to accept its antecedent. (19d) expresses the idea that the antecedent of a counterfactual conditional determines the range of models in which the consequent is evaluated.

Let us apply LBS to an example from Kratzer [9, p. 94].

(20a) If a wolf entered the house, he must have eaten grandma, since she was bedridden. He might have eaten the girl with the red cap, too. In fact, that's rather likely. The poor little thing wouldn't have been able to defend herself.

We assume that there are appropriate translations of sentences in (16a) into FO-sentences:

p: tr(a wolf entered the house).
q: tr(the wolf ate grandma).
r: tr(the grandma was bedridden).
s: tr(the wolf ate the girl with the red cap).
t: tr(the girl was not able to defend herself).

Now, we can express story (20a) within LBS as follows:

(20b) $[If^{BS}\, p](Must\, q)$ & $([If^{BS}\, p](Known\, r)$ & $not\ MUST_{\{p\}}\, r)$ & $[If^{BS}\, p]$ $((Might\, s)$ & $(Probably\, s)$ & $(Must\, t))$.

Here, $([If^{BS}\ p](Known\ r)$ & $not\ MUST_{\{p\}}\ r)$ expresses that r belongs to the part of BS that is kept in its consistent maximum after acceptance of p and that r is independent from p. When we assume $cons(top(BS, 1) \cup \{p\})$, from (20b) follows (20c).

(20c) $T = IF_{BS}(\{p\})$ & $MUST_T\ q$ & $(KNOWN_T\ r$ & $not\ MUST_{\{p\}}\ r)$ & $MIGHT_T\ s$ & $PROBABLY_{ext(\{p\},BS)}\ s$ & $MUST_T\ t$.

(20c) roughly means the following: (Suppose p. Then, (it must be q, because it is known that r & it might be s & it is probable that s & it must be t)). As Kratzer [9, p. 94] points out, the *if-clause* determines the evaluation range of modal operators in long stretches of subsequent discourse.[11].

[11] For interpretation of (20a), it would be more appropriate to deal with anaphoric relation. This can be done by using Skolem-symbols [12,15].

6 Interpretation of Conditionals

In this section, we examine the relationship between our interpretation of conditionals and the standard interpretation. Lewis [11, p. 1] explains the standard interpretation as follows [8, p. 428]:

> A possible world in which the antecedent of a counterfactual is true is called an "antecedent-world." One can state the theory (in a somewhat simplified form) by saying that a counterfactual is true just in case its consequent is true in those antecedent-worlds that are most similar to the actual world.

Based on this idea, Lewis [11, p. 16] defines the truth condition for a counterfactual conditional as follows:[12]

> [LEWIS] $p \mapsto q$ is true at a world i (according to a system of spheres SS) iff either
>
> **1.** no p-world belongs to any sphere S in SS_i, or
> **2.** some sphere S in SS_i does contain at least one p-world, and $p \rightarrow q$ holds at every world in S.

Now, we examine the relationship between our interpretation and Lewis's standard interpretation. Actually, it turns out that our interpretation is very similar to [LEWIS]. The main difference lies in the notion of *center of a sphere system*, namely Lewis accepts only a singleton as the center. We can prove a proposition that is very close to [LEWIS].

Proposition 17. *Let BS be a belief structure with $T_1 > \ldots > T_n$.*

(21a) *Let $T_1, \ldots, T_n$ be sets of FO-sentences.*
$[If^{BS}\, p](Must\, q)$ iff either
1. *no p-model belongs to any sphere S in SMS_{BS}, or*
2. *some sphere S in SMS_{BS} does contain at least one p-model, and $p \rightarrow q$ holds in every model in S.*

(21b) *Let $T_1, \ldots, T_n$ be sets of PL-formulas and W be a maximal set of worlds.*
$[If^{BS}\, p](Must\, q)$ iff either
1. *no p-model belongs to any sphere S in SS_{BS}, or*
2. *some sphere S in SS_{BS} does contain at least one p-model, and $p \rightarrow q$ holds in every model in S.*

Proof. It can be easily shown: $mod(T_1 \cup \{p\}) = \emptyset$ iff (21a.1). Now, we consider cases in which $mod(T_1 \cup \{p\}) \neq \emptyset$. Let $k = cmn(ext(\{p\}, BS)) - 1$ and $S_k = mod(top(BS, k))$. Because $top(BS, k) \cup \{p\}$ is consistent, there is a model in S_k that makes p true. Furthermore, $IF_{BS}(\{p\}) = cons\text{-}max(ext(p, BS)) = top(BS, k) \cup \{p\}$. According to (5a) and (11i), ($T = IF_{BS}(\{p\})$ & $MUST_T\ q$)

[12] Here, we represent counterfactual conditional with $\mapsto$.

iff $mod(IF_{BS}(\{p\})) \subseteq mod(\{q\})$. Then, because of the deduction theorem in FOL: $mod(top(BS, k) \cup \{p\}) \subseteq mod(q)$ iff $mod(top(BS, k)) \subseteq mod(\{p \to q\})$ iff $S_k \subseteq mod(\{p \to q\})$. Hence, $(T = IF_{BS}(\{p\})$ & $MUST_T\ q)$ iff (21a.2). Then, because of (18b), (21a) holds. (21b) can be proved in the same way as the proof of (21a). Q.E.D.

This result shows that our interpretation of conditionals is very similar to the standard one. In fact, with respect to the determination of *spheres*, our approach is more explicit than Lewis's approach (see (11i) and (11j)).

Grove [5] proposes sphere-semantics for theory change and shows that this semantics satisfies AGM postulates for the belief revision and that it is very similar to the sphere-semantics for counterfactual logic proposed by Lewis [11]. Thus, our results are similar to results in [5]. It is Grove's motivation for his investigation to connect the sphere semantics with the treatment of theory change.[13] His interest shares with ours. In fact, LBS is applicable to description of theory change in scientific activities.[14]

The main difference between two approaches lies in generality. Grove requires that the language is compact [5, p.157], while we deal with full FO-languages. Thus, our approach is broader than Grove's.

7 Two Types of Conditionals

Williams [17] points out a semantic difference between indicative and counterfactual conditionals.

(22a) [Indicative conditional] If Oswald didn't shoot Kennedy, someone else did.
(22b) [Counterfactual conditional] If Oswald hadn't shot Kennedy, someone else would have.

According to Williams, (22a) is true, while (22b) is false. This means that the meaning of indicative conditionals and that of counterfactual conditionals are different. We usually accept (22c) instead of (22b).

(22c) If Oswald hadn't shot Kennedy, Kennedy might not have been killed.

Williams explained this difference through a slight modification of the standard interpretation of conditionals proposed by Lewis [11]. Instead, we propose to distinguish both cases through a different relationship to causal dependencies.

[13] Gärdenfors [4, Sect. 4.5] gives an insightful description of Grove's system.

[14] Some parts of Lakatos' discussion on scientific research programs in [10] can be described within LBS. In belief structures of scientists, basic theories are more trusted than their auxiliary hypotheses ($BT > AH$). Suppose that the set nO of observation data is consistent with BT but inconsistent with $BT \cup AH$. In such a case, scientists would try to find the set nAH of new auxiliary hypotheses such that $nO \cup BT \cup nAH$ is consistent. In this way, a basic theory can be protected against new anomalies.

It is usual to distinguish two types of conditionals [1, Sect. 1]. We propose that one type, like case (22a), is *epistemic* and the other type, like case (22c), is concerned with causal effects. The second type has the form "if-had A, then-would B", where the occurrence of B is causally dependent on the occurrence of A. In such a case, the shift of temporal perspective is often required. In (22a), our temporal view is fixed in the present and we assume that we know that Kennedy was killed. In this situation, we think about the possibility of Oswald's innocence. However, when we utter (22b) or (22c), our temporal viewpoint is shifted to the situation just before Kennedy was shot and we imagine what could happen after that situation. In this paper, we call the first type of conditionals *epistemic conditionals* and the second type *causal conditionals*.

Now, we define *casual dependency* as follows.

Definition 18. *Let BS be a belief structure with $T_1 > \ldots > T_n$. Let CT be a theory of causality that implies causal laws.*

A fact expressed by q is causally dependent on a fact expressed by p with respect to (wrt) BS iff there are i, j, and k such that ($i < j < k \leq cmn(BS)$ & $CT \subseteq T_i$ & $T_j = \{p\}$ & $MUST\text{-}min(BS, j, p)$ & $MUST\text{-}min(BS, k, q)$ & not $MUST_{top(BS,k)-CT}\ q$ & not $MUST_{top(BS,k)-T_j}\ q$).

To explain this distinction of conditionals, let us consider examples (22a), (22b), and (22c). We use some abbreviations to improve readability:

killed: Kennedy was killed [$\exists t(killed(Kennedy, t) \wedge t <_t now)$];
someone: Someone shot Kennedy [$\exists t \exists x(shoot(x, Kennedy, t) \wedge t <_t now)$];
oswald: Oswald shot Kennedy [$\exists t(shoot(Oswald, Kennedy, t) \wedge t <_t now)$];
someone-else: Someone else shot Kennedy [$\exists t \exists x(shoot(x, Kennedy, t) \wedge x \neq Oswald \wedge t <_t now)$].

For the sake of simplicity, we assume that BS_1 is a belief structure with $CT > \{someone\} > \{killed\} > \{oswald\}$ and that BS_2 is a belief structure with $CT > \{oswald\} > \{killed\}$. We assume also that $CT \cup \{oswald\} \cup \{killed\}$ is consistent. Because ($oswald \rightarrow someone$) is a FOL-theorem, it holds $Cn(BS_1) = Cn(BS_2)$. Furthermore, according to Definition 18, the fact expressed by *killed* is causally dependent on the fact expressed by *oswald* only wrt BS_2. Because ($IF_{BS_1}(\{\neg oswald\}) = \{\neg oswald\} \cup CT \cup \{someone\} \cup \{killed\}$ & $IF_{BS2}(\{\neg oswald\}) = \{\neg oswald\} \cup CT$) and ($\neg oswald \wedge someone \rightarrow someone - else$) is a FOL-theorem, we obtain: $[If^{BS_1} \neg oswald](Must\ someone\text{-}else)$ & *not* $[If^{BS_2} \neg oswald](Must\ someone\text{-}else)$ & $[If^{BS_2} \neg oswald](Might\ \neg killed)$. This result can be summarized as follows:

BS_1: $CT > \{someone\} > \{killed\} > \{oswald\}$.
BS_2: $CT > \{oswald\} > \{killed\}$.
The fact expressed by *killed* is causally dependent on the fact expressed by *oswald* only wrt BS_2.
$[If^{BS_1} \neg oswald](Must\ someone\text{-}else)$.
not $[If^{BS_2} \neg oswald](Must\ someone\text{-}else)$.
$[If^{BS_2} \neg oswald](Might\ \neg killed)$.

To evaluate a causal conditional ⟨If p, then q⟩, we, at first, reformulate our belief structure, so that it reflects the causal dependency expressed by the conditional. Then, we imagine a situation in which p holds. Let us call this reformulated belief structure BS_c. The determination of the imagined situation can be achieved by calculating $[If^{BS_c}\, p]$. After that, we examine whether $MUST_T\, q$ holds, where $T = IF_{BS_c}(\{p\})$. In contrast, we do not need any reformulation of belief structures, when we evaluate epistemic conditionals.

8 Concluding Remarks

In the first part of this paper, we proposed *Logic for Epistemic Modalities* (LRM). LEM is based on the consequence relation of FOL. We have shown how to express in LEM some evidential features of epistemic modalities.

In the second part, we extended LEM to *Logic of Belief Structures* (LBS). Then, we defined a belief structure revision operator *bsR* based on LBS. A belief structure can be roughly understood as a linearly ordered set of sets of FO-sentences. We proved that *bsR* satisfies all postulates for belief revision in AGM-theory. Then, we defined the truth condition of counterfactual conditionals; we interpreted that the consequent of a conditional describes a modal state after a belief revision invoked by acceptance of the antecedent. We have also shown that *sphere semantics* can be defined for our treatment of conditionals. The characteristic feature of our approach lies in its explicitness. Instead of similarity relation among worlds, we use a reliability order among sets of FO-sentences. An example of causal interpretation of counterfactual conditionals demonstrated how LBS-approach can be used for describing truth conditions of modal statements in natural languages.[15]

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