Towards the Graph Minor Theorems for Directed Graphs

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Abstract. Two key results of Robertson and Seymour's graph minor theory are:

1. a structure theorem stating that all graphs excluding some fixed graph as a minor have a tree decomposition into pieces that are almost embeddable in a fixed surface.
2. the *k-disjoint paths problem* is tractable when k is a fixed constant: given a graph G and k pairs (s_1, t_1), ..., (s_k, t_k) of vertices of G, decide whether there are k mutually vertex disjoint paths of G, the ith path linking s_i and t_i for $i = 1, \dots, k$.

In this talk, we shall try to look at the corresponding problems for digraphs.

Concerning the first point, the grid theorem, originally proved in 1986 by Robertson and Seymour in Graph Minors V, is the basis (even for the whole graph minor project). In the mid-90s, Reed and Johnson, Robertson, Seymour and Thomas (see [13,26]), independently, conjectured an analogous theorem for directed graphs, i.e. the existence of a function $f : \mathbb{N} \to \mathbb{N}$ such that every digraph of directed treewidth at least $f(k)$ contains a directed grid of order k. In an unpublished manuscript from 2001, Johnson, Robertson, Seymour and Thomas give a proof of this conjecture for planar digraphs. But for over a decade, this was the most general case proved for the conjecture.

We are finally able to confirm the Reed, Johnson, Robertson, Seymour and Thomas conjecture in full generality. As a consequence of our results we are able to improve results in Reed et al. in 1996 [27] to disjoint cycles of length at least l. This would be the first but a significant step toward the structural goals for digraphs (hence towards the first point).

Concerning the second point, in [19] we contribute to the disjoint paths problem using the directed grid theorem. We show that the following can be done in polynomial time:

Suppose that we are given a digraph G and k terminal pairs $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$, where k is a fixed constant. In polynomial time, either

– we can find k paths $P_1, \dots, P_k$ such that P_i is from s_i to t_i for $i = 1, \dots, k$ and every vertex in G is in at most four of the paths, or

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– we can conclude that G does not contain disjoint paths $P_1, \ldots, P_k$ such that P_i is from s_i to t_i for $i = 1, \ldots, k$.

To the best of our knowledge, this is the first positive result for the general directed disjoint paths problem (and hence for the second point). Note that the directed disjoint paths problem is NP-hard even for $k = 2$. Therefore, this kind of results is the best one can hope for.

We also report some progress on the above two points.

Keywords: Directed graphs · Grid minor · The directed disjoint paths problem

1 Introduction

One of the deepest and the most far-reaching theories of the recent 20 years in discrete mathematics (and theoretical computer science as well) is Graph Minor Theory developed by Robertson and Seymour in a series of over 20 papers spanning the last 20 years [28]. Their theory leads to "structural graph theory", which has proved to be a powerful tool for coping with computational intractability. It provides a host of results that can be used to design efficient (approximation or exact) algorithms for many NP-hard problems on specific classes of graphs that occurs naturally in applications.

Two key results of Robertson and Seymour's graph minor theory are:

1. a structure theorem stating that all graphs excluding some fixed graph as a minor have a tree decomposition into pieces that are almost embeddable in a fixed surface.
2. the *k-disjoint paths problem* is tractable when k is a fixed constant: given a graph G and k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G, decide whether there are k mutually vertex disjoint paths of G, the ith path linking s_i and t_i for $i = 1, \ldots, k$.

In order to solve these two problems, of particular importance is the concept of *treewidth*, introduced by Robertson and Seymour. Treewidth has gained immense attention ever since, especially because many NP-hard problems can be handled efficiently on graphs of bounded treewidth [1]. In fact, all problems that can be defined in monadic second-order logic are solvable on graphs of bounded treewidth [4].

A keystone in the proof of the above two results (and many other theorems) is a grid theorem [29]: any graph of treewidth at least some $f(r)$ is guaranteed to have the $r \times r$ grid graph as a minor. This gird theorem played a key role in the k-disjoint paths problem [17,30]. It also played a key role for some other deep applications (e.g., [12,21,22]).

This grid theorem has also played a key role for many algorithmic applications, in particular via bidimensionality theory (e.g., [6–8]), including many approximation algorithms, PTASs, and fixed-parameter algorithms. These include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, dominating set, edge dominating set, R-dominating set, connected dominating set, connected edge dominating set, connected R-dominating set, and unweighted TSP tour.

The grid theorem of [29] has been extended, improved, and re-proved by Robertson, Seymour, and Thomas [31], Reed [25], Diestel, Jensen, Gorbunov, and Thomassen [10], Kawarabayashi and Kobayashi [16] and Leaf and Seymour [23]. Very recently, this has been improved to be polynomial [3]. On the other side, the best known lower bound is $\Omega(r^2 \log r)$.

A linear upper bound has been shown for planar graphs [31] and for bounded genus graphs [7]. Recently this min-max relation is also established for graphs excluding any fixed minor H: every H-minor-free graph of treewidth at least $c_H r$ has an $r \times r$ grid minor for some constant c_H [9]. The bound is now explicitly described as $|H|^{|H|}$ [16] This bound leads to many powerful algorithmic results on H-minor-free graphs [7,9] that are previously not known.

2 What about Digraphs?

The structural techniques discussed in graph minor theory all relate to undirected graphs. What about directed graphs? Given the enormous success for problems of width parameters (c.f., treewidth) defined on undirected graphs, it is quite natural to ask whether they can also be extended to analyze the structure of digraphs. In principle by ignoring the direction of edges, it is possible to apply many techniques for undirected graphs to directed graphs. However, we would have an information loss and might fail to properly distinguish between simple and hard input instances. For example, the k-disjoint paths problem for digraphs is NP-complete even when we consider the fixed value $k = 2$ (Fortune, Hopcroft and Wylie [11]), but it is polynomially solvable for all fixed k for undirected graphs [15,30]. Hence, for computational problems whose instances are directed graphs, many methods for undirected graphs may be less useful.

As a first step (but also a significant step) towards overcoming such a difficulty, Reed in 1999 and Johnson, Robertson, Seymour and Thomas [13] proposed a concept of *directed treewidth* and showed that the k-disjoint paths problem is solvable in polynomial time for any fixed k on any class of graphs of bounded directed treewidth [13]. Reed and Johnson et al. also conjectured a directed analogue of the grid theorem.

Conjecture 1 (Reed; Johnson, Robertson, Seymour, Thomas [13]). There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every digraph of directed treewidth at least $f(k)$ contains a cylindrical grid of order k as a butterfly minor.

Actually, according to [13], this conjecture was formulated by Robertson, Seymour and Thomas, together with Alon and Reed at a conference in Annecy, France in 1995. Here, a *cylindrical grid* consists of k concentric directed cycles and $2k$ paths connecting the cycles in alternating directions. A *butterfly minor* of a digraph G is a digraph obtained from a subgraph of G by contracting edges which are either the only outgoing edge of their tail or the only incoming edge of their head. All details for these notations can be found in appendix.

Let us now report progress on the conjecture. In an unpublished manuscript, Johnson et al. [14] proved the conjecture for planar digraphs. In [18], this result was generalised to all classes of directed graphs excluding a fixed undirected graph as an undirected minor. For instance, this includes classes of digraphs of bounded genus. Another related result was established in [19], where a half-integral grid theorem was proved (for the definition of a "half-integral directed grid", we refer the reader to [19]).

Very recently, we finally confirm this conjecture [20]. We believe that this is a first but an important step towards a more general structure theory for directed graphs based on directed treewidth, similar to the grid theorem for undirected graphs being the basis of more general structure theorems (including the main graph minor structure theorem).

3 Algorithmic Contributions

Our main algorithmic interest is the directed k-disjoint paths problem. Recall that for undirected graphs the problem is solvable in polynomial time for any fixed number k. For directed graphs, the situation is much worse since the problem is NP-complete even for only two such pairs.

Theorem 1 (Fortune, Hopcroft, and Wyllie [11]). *The following problem is NP-complete even for $k = 2$:*

DIRECTED DISJOINT PATHS *Input:* A digraph G and terminals $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$. *Problem:* Find k (vertex) disjoint paths $P_1, \ldots, P_k$ such that P_i is from s_i to t_i for $i = 1, \ldots, k$.

Therefore much work has gone into finding polynomial time algorithms for solving this problem on restricted classes of digraphs. See e.g. [5,32] for work in this direction.

In this talk, we are not so much interested in solving disjoint paths problems on special classes of digraphs, but rather in obtaining algorithms working on all directed graphs. We therefore have to relax some of the conditions. Indeed, we allow each vertex of the graph to be contained in small number of paths linking the source/terminal pairs.

Using the directed grid minor, the following is shown in [19].

Theorem 2. *For every fixed $k \geq 1$ there is a polynomial time algorithm for deciding the following problem.*

QUARTER- INTEGRAL DISJOINT PATHS *Input:* A digraph G and terminals $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$. *Problem:* – Find k paths $P_1, \ldots, P_k$ such that P_i is from s_i to t_i for $i = 1, \ldots, k$ and every vertex in G is in at most four of the paths, or – conclude that G does not contain disjoint paths $P_1, \ldots, P_k$ such that P_i is from s_i to t_i for $i = 1, \ldots, k$.

As far as we are aware, this is the first result that establishes a positive result, and gives a polynomial time algorithm for the variant of the disjoint paths problems on the class of all digraphs. Note that this result is best possible in a sense. Indeed, Slivkins [33] proved that the directed disjoint paths problem is W[1]-hard already on acyclic digraphs and it is not hard to extend this result to the half- or quarter-integral case. Hence in terms of running time our algorithm is optimal in the sense that it cannot be improved to $\mathcal{O}(f(k)n^c)$ for any fixed constant c.

As we said, the key is to use a cylindrical grid. The following theorem tells us why a "directed" grid minor is important.

Theorem 3. *Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be (not necessarily distinct) $2k$ vertices in a digraph G. Suppose that G has a cylindrical grid W of order $8k^3$. Let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$. Suppose furthermore that*

1. *there is no separation* (A_1, B_1) *of order at most* k *such that* A_1 *contains* S *and* B_1 *contains all but at most* k *vertices* Q_1 *of in-degree or out-degree at least two in* W*, and there is no path from* S *to* Q_1 *in* $G - (A_1 \cap B_1)$*, and*
2. *there is no separation* (A'_1, B'_1) *of order at most* k *such that* A'_1 *contains* T *and* B'_1 *contains all but at most* k *vertices* Q_2 *of in-degree or out-degree at least two in* W*, and there is no path from* Q_2 *to* T *in* $G - (A'_1 \cap B'_1)$.

Then in polynomial time, we can find k *paths* $P_1, \dots, P_k$ *in* G *such that endpoints of* P_i *are* s_i, t_i *for* $i = 1, \dots, k$*, and moreover each vertex in* G *is used in at most two of these paths.*

Using Theorem 3, we are currently working on the following conjecture.

Conjecture 2. For a fixed constant k, there is a polynomial time algorithm for the following problem:

DIRECTED HALF-DISJOINT PATHS
Input: A digraph G and terminals $s_1, t_1, s_2, t_2, \dots, s_k, t_k$.
Problem: Find k paths $P_1, \dots, P_k$ such that P_i connects from s_i to t_i for $i = 1, \dots, k$ and every vertex in G is in at most two of the paths.

As pointed out above, this is the best we can hope.

4 Additional Notations

An $r \times r$ *grid* is a graph which is isomorphic to the graph W_r obtained from Cartesian product of paths of length $r-1$, with vertex set $V(W_r) = \{(i, j) \mid 1 \le i \le r,\ 1 \le j \le r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$.

The (4×5)-grid, as well as the (8×5)-wall (which can be defined in a similar way) are shown in Figure 1.

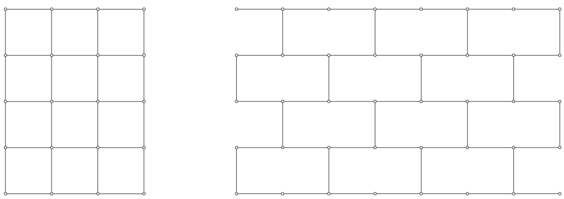

Fig. 1. The (4×5)-grid and the (8×5)-wall

A *tree decomposition* of a graph G is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W}$ is a family $\{W_t \mid t \in V(T)\}$ of vertex sets $W_t \subseteq V(G)$, such that the following two properties hold:

(1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t.

(2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of a tree decomposition $(T, \mathcal{W})$ is $\max_{t \in V(T)} |W_t| - 1$. The *treewidth* of a graph G is the minimum width over all possible tree decompositions of G.

Robertson and Seymour developed the first polynomial time algorithm for constructing a tree decomposition of a graph of bounded width [30], and eventually came up with an algorithm which runs in $O(n^2)$ time, for this problem. Reed [24] developed an algorithm for the problem which runs in $O(n \log n)$ time, and then Bodlaender [2] developed a linear time algorithm.

Directed Treewidth We briefly recall the definition of directed treewidth from [13].

By an *arborescence* we mean a directed graph R such that R has a vertex r_0, called the *root* of R, with the property that for every vertex $r \in V(R)$ there is a unique directed path from r_0 to r. Thus every arborescence arises from a tree by selecting a root and directing all edges away from the root. If $r, r' \in V(R)$ we write $r' > r$ if $r' \neq r$ and there exists a directed path in R with initial vertex r and terminal vertex r'. If $e \in E(R)$ we write $r' > e$ if either $r' = r$ or $r' > r$, where r is the head of e. Let G be a digraph, and let $Z \subseteq V(G)$. We say that a set $S \subseteq (V(G) - Z)$ is Z-*normal* if there is no directed walk in $G - Z$ with the first and the last vertex in S that uses a vertex of $G - (Z \cup S)$. It follows that every Z-normal set is the union of the vertex-sets of strong components of $G - Z$. As one readily checks, a set S is Z-normal if and only if the vertex-sets of the strong components of $G - Z$ can be numbered $S_1, S_2, \ldots, S_d$ in such a way that

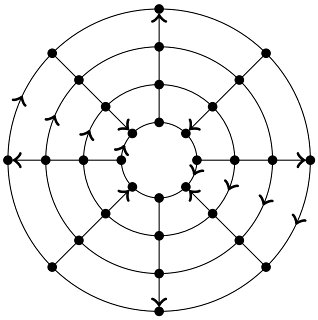

Fig. 2. Cylindrical grid G_4.

1. if $1 \leq i < j \leq d$, then no edge of G has head in S_i and tail in S_j, and
2. either $S = \emptyset$, or $S = S_i \cup S_{i+1} \cup \cdots \cup S_j$ for some integers i, j with $1 \leq i \leq j \leq d$.

Definition 1. *A* directed tree-decomposition *of a digraph G is a triple (R, X, W), where R is an arborescence, and $X = (X_e : e \in E(R))$ and $W = (W_r : r \in V(R))$ are sets of vertices of G that satisfy*

1. *$(W_r : r \in V(R))$ is a partition of $V(G)$ into nonempty sets, and*
2. *if $e \in E(R)$, then $\bigcup(W_r : r \in V(R), r > e)$ is X_e-normal.*

The width *of (R, X, W) is the least integer w such that for all $r \in V(R)$, $|W_r \cup \bigcup_e X_e| \leq w + 1$, where e is taken over all edges incident to r. The* directed treewidth *of G is the least integer w such that G has a directed tree-decomposition of width w.*

Sometimes, we call W_r or X_e a *bag* for $r \in V(R)$ and $e \in E(R)$. It is easy to see that the directed tree-width of a subdigraph of G is at most the tree-width of G.

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