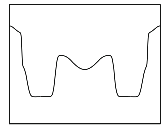

Chapter 8

Conservation Laws with Discontinuous Flux Functions

Of course it is happening inside your head, Harry,
but why on earth should that mean it is not real?
— Albus Dumbledore, in Harry Potter and the Deathly Hallows

The aim of this chapter is to give a brief introduction to scalar conservation laws with a space-dependent flux function, where the spatial dependence of the flux can have discontinuities. We shall restrict ourselves to one spatial dimension, both for reasons of simplicity and because the theory is more complete in one dimension.

In one spatial dimension, a conservation law with a space-dependent flux can be written

$$u_t + f(x,u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{8.1}$$

Since the interpretation of f is the flux of u at the point x, there are many applications where the flux depends on the location. We give some simple examples that are modeled by such conservation laws.

◇ Example 8.1
Traffic flow is a simple model in whcih conservation laws with space-dependent coefficients arise naturally. We refer to Example 1.6, and repeat the necessary notation here.

Let ρ denote the density of cars on a long "one way" road. We normalize units, so that $\rho = 1$ if the cars are packed bumper to bumper. Assume that the speed of the cars is a decreasing function of the density $v = v(\rho)$. The speed of the cars on an empty road ($\rho = 0$) is governed by the road conditions and the speed limits, so that $v(0) = \gamma$, where γ is a function of the position on the road. Furthermore, it is reasonable to assume that $v(1) = 0$. For simplicity we can then define v as $v(\rho) = \gamma(1-\rho)$. If the road conditions, and thereby γ, vary with the position x, then we end up with the conservation law

$$\rho_t + (\gamma(x)\rho(1-\rho))_x = 0, \tag{8.2}$$

which is an example of a conservation law with an x-dependent flux function. On the scale of continuum traffic, where the natural lengths are *many* times that of a single car, the road conditions often vary discontinuously. ◇

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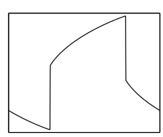
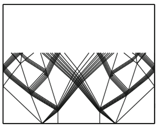




◇ Example 8.2
When extracting oil from an oil reservoir, one often injects water in order to maintain the pressure, and thereby to force out more oil. Assuming that we have two phases, oil and water, present, the mass conservation of oil and water reads,

$$\phi s_t + u_x = 0 \quad \text{and} \quad \phi(1-s)_t - v_x = 0,$$

where (the unknown) s denotes the saturation, i.e., the fraction of the available pore space occupied by oil, and u and v are the Darcy velocities of oil and water respectively. The factor ϕ denotes the fraction of the void space in the material, commonly called the *porosity*. One often assumes that Darcy's law holds,

$$u = -k\lambda_{\text{oil}} P'_{\text{oil}} - g\rho_{\text{oil}} \quad \text{and} \quad v = -k\lambda_{\text{water}} P'_{\text{water}} - g\rho_{\text{water}},$$

where k denotes the absolute permeability of the medium, g the gravitational constant, λ_{phase} the mobility, P_{phase} the pressure, and ρ_{phase} the density. Here the subscript "phase" denotes either water or oil. If we assume that the two pressures are the same, and that the total velocity $q = u + v$ is constant (incompressibilty), we can add the two conservation equations to obtain

$$\phi s_t + \left(\frac{\lambda_{\text{oil}}(s)}{\lambda_{\text{oil}}(s) + \lambda_{\text{water}}(s)} \left(q - k(x) g \lambda_{\text{water}}(s) \Delta\rho\right) \right)_x = 0, \tag{8.3}$$

where $\Delta\rho = \rho_{\text{water}} - \rho_{\text{oil}}$. The absolute permeability of the rock is often modeled as a piecewise constant function of x, and therefore this is another example of a conservation law in which the flux function varies discontinuously. ◇

◇ Example 8.3
Since oil is much more viscous than water, water injection can lead to the formation of thin "fingers" of water. In order to prevent this, one sometimes injects a mixture of polymer and water instead of water only. This polymer is passively transported with the water. In a "one-dimensional" homogeneous oil reservoir, conservation of water and polymer is expressed through the system of conservation laws

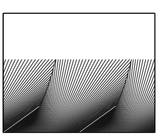
$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc)_t + (cf(s,c))_x &= 0, \end{aligned} \tag{8.4}$$

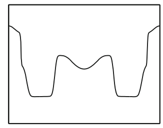
where c denotes the concentration of the polymer in the water, and the flux function $f(s,c)$ is a known function of the type in (8.3), where λ_{water} is now a function of both s and c. Introducing new coordinates (y,τ) by

$$\frac{\partial y}{\partial x} = s, \quad \frac{\partial y}{\partial t} = -f(s,c), \quad \frac{\partial \tau}{\partial x} = 0, \quad \text{and} \quad \frac{\partial \tau}{\partial t} = 1,$$

the system (8.4) reads

$$\begin{aligned} \left(\frac{1}{s}\right)_\tau - \left(\frac{f(s,c)}{s}\right)_y &= 0, \\ c_\tau &= 0. \end{aligned} \tag{8.5}$$





This change of independent variables is valid only for differentiable (classical) solutions, whereas we know that we cannot expect solutions of conservation laws to be even continuous. Therefore, we must interpret solutions in the *weak* sense. Nevertheless, by [187, Thm. 2] there is a one-to-one correspondence between weak solutions of (8.4) and weak solutions of (8.5). Hence if the initial polymer concentration is discontinuous, (8.5) is another example of a conservation law with a flux function depending discontinuously on the spatial location. ◇

We can always view an x-dependent flux as a flux function depending on a parameter γ that in turn depends on x. In this way we write (8.1) as a system

$$u_t + f(\gamma, u)_x = 0, \quad \gamma_t = 0. \tag{8.6}$$

This is a hyperbolic system with a Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial \gamma} \\ 0 & 0 \end{pmatrix},$$

which has the eigenvalues

$$\lambda_1 = \frac{\partial f}{\partial u}, \quad \lambda_2 = 0.$$

So if $\frac{\partial f}{\partial u} = 0$ for some values of γ and u, the system is not strictly hyperbolic. This is the cause of many difficulties when one is working with conservation laws with x-dependent fluxes. In [176], Temple exhibited a simple example of a sequence of approximate solutions without any uniform bound on the variation. This means that when studying conservation laws of the type (8.6), one must use more powerful (and complicated) tools. The "z-transform" used in this chapter is perhaps the simplest (and least powerful) example of such a tool. Recently, compensated compactness and variants of the "div-curl" lemma have taken the place of the "z-transform" in proving convergence of approximations; see [107] for a recent example.

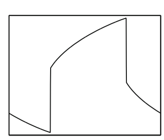
We emphasize that this chapter is meant to be an introduction to this topic and does not contain the most general results.

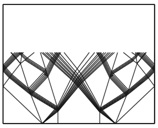
8.1 The Riemann Problem

In this section we shall study the Riemann problem, that is, the initial value problem in which the initial values consist of two constants separated by a jump discontinuity. More precisely, this is the problem

$$\begin{cases} u_t + f(\gamma_l, u)_x = 0, & u(x,0) = u_l, \quad \text{for } x < 0, \\ u_t + f(\gamma_r, u)_x = 0, & u(x,0) = u_r, \quad \text{for } x > 0, \end{cases} \tag{8.7}$$

where γ_l, γ_r, u_l, and u_r are constants. Riemann problems for conservation laws have the simplest solutions that are not constant. Furthermore, by studying the





solution of Riemann problems, we gain insight into the local behavior of typical solutions. It turns out that solutions of Riemann problems can be used as a building block in many numerical methods, in particular front tracking.

By a solution of (8.7) we mean a weak solution in the usual sense, i.e., $u \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0,\infty))$ is called a weak solution if for every test function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$,

$$\int_0^\infty \Big(\int_{-\infty}^0 \big(u\varphi_t + f(\gamma_l, u)\,\varphi_x\big)\,dx + \int_0^\infty \big(u\varphi_t + f(\gamma_r, u)\,\varphi_x\big)\,dx \Big)\,dt + \int_{\mathbb{R}} u(x,0)\varphi(x,0)\,dx = 0. \tag{8.8}$$

Now we shall first show that under reasonable assumptions on f, weak solutions exist, and that if we require that weak solutions satisfy an additional entropy condition, then there exists only one weak solution.

Existence of a Solution

To show the existence of a solution, we start by observing that for x negative, $u(x,t)$ must be the solution of a scalar conservation law

$$v_t + f(\gamma_l, v)_x = 0, \tag{8.9}$$

with $v(x,0)$ given by

$$v(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_l' & \text{for } x = 0, \end{cases}$$

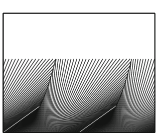
where u_l' is a value to be determined. Similarly, for x positive, u must be the solution of the scalar initial value problem

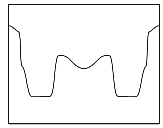
$$w_t + f(\gamma_r, w)_x = 0, \qquad w(x,0) = \begin{cases} u_r' & \text{for } x = 0, \\ u_r & \text{for } x > 0, \end{cases} \tag{8.10}$$

where u_r' is to be determined. Setting

$$u(x,t) = \begin{cases} v(x,t) & \text{for } x < 0, \\ w(x,t) & \text{for } x > 0, \end{cases} \tag{8.11}$$

provided that $v(0-,t)$ and $w(0+,t)$ satisfy some extra condition, we find that this will give a weak solution, since both v and w are weak solutions. Therefore, to find a weak solution, we must find solutions of scalar Riemann problems v and w such that this construction is possible.





Now recall, or reread Sect. 2.2, that the solution to the scalar Riemann problem

$$v_t + g(v)_x = 0, \quad v(x,0) = \begin{cases} v_l & x < 0, \\ v_r & x \geq 0, \end{cases}$$

is found by constructing the lower convex (if $v_l < v_r$) or upper concave (if $v_l > v_r$) envelope of g between v_l and v_r; cf. Sect. 2.2. To make the notation less cumbersome we introduce

$$\bar{g}\,(v; v_l, v_r) = \begin{cases} g_{\frown}(v; v_l, v_r) & \text{if } v_r < v_l, \\ g_{\smile}(v; v_l, v_r) & \text{if } v_l < v_r. \end{cases} \tag{8.12}$$

In this notation the entropy solution v is given by

$$v(x,t) = \bar{g}'^{-1}\left(\frac{x}{t}; v_l, v_r\right), \qquad t > 0. \tag{8.13}$$

Now we turn to the Riemann problem (8.7). The left and right parts of u are v, given by (8.9), and w, given by (8.10). If we are to form u by gluing together v and w, then v must equal u'_l for $x > 0$, and w must equal u'_r for $x < 0$. In other words, v must contain only waves of nonpositive speed, and w only waves of nonnegative speed. To utilize these observations, we introduce the notation

$$f_l(u) = f\,(\gamma_l, u) \quad \text{and} \quad f_r(u) = f\,(\gamma_r, u)$$

and define $\bar{f}_l(u; u_l, \tilde{u})$ and $\bar{f}_r(u; \tilde{u}, u_r)$ analogously to (8.12).

Since v contains only waves of nonpositive speed, we must choose u'_l from the set

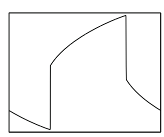
$$H_l\,(u_l) = \left\{\tilde{u} \mid \bar{f}_l'(u; u_l, \tilde{u}) \leq 0 \quad \text{for all } u \text{ between } u_l \text{ and } \tilde{u}\right\}. \tag{8.14}$$

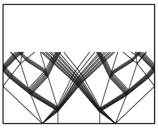
Similarly, since w must contain waves of nonnegative speed, we must choose u'_r from the set

$$H_r\,(u_r) = \left\{\tilde{u} \mid \bar{f}_r'(u; \tilde{u}, u_r) \geq 0 \quad \text{for all } u \text{ between } u_r \text{ and } \tilde{u}\right\}. \tag{8.15}$$

There is another characterization of the admissible sets H_l and H_r that will be useful. Let h_l be defined by

$$h_l\,(u; u_l) = \begin{cases} \inf\left\{h(u) \;\middle|\; \begin{matrix} h(u) \geq f_l(u), & h'(u) \leq 0, \\ \text{and} & h(u_l) = f_l(u_l) \end{matrix}\right\} & \text{if } u \leq u_l, \\[2ex] \sup\left\{h(u) \;\middle|\; \begin{matrix} h(u) \leq f_l(u), & h'(u) \leq 0, \\ \text{and} & h(u_l) = f_l(u_l) \end{matrix}\right\} & \text{if } u \geq u_l, \end{cases} \tag{8.16}$$





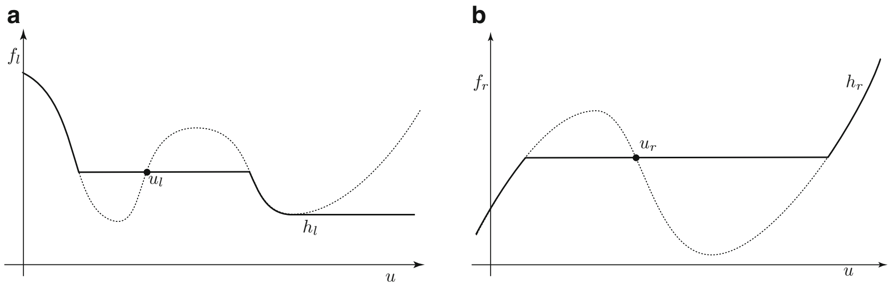


Fig. 8.1 **a** h_l (*solid line*) and f_l (*dotted line*). **b** h_r (*solid line*) and f_r (*dotted line*)

and define h_r by

$$h_r\left(u;u_r\right) = \begin{cases} \sup\left\{h(u) \;\middle|\; \begin{matrix} h(u) \le f_r(u), & h'(u) \ge 0, \\ \text{and} & h(u_r) = f_r(u_r) \end{matrix}\right\} & \text{if } u \le u_r, \\ \inf\left\{h(u) \;\middle|\; \begin{matrix} h(u) \ge f_r(u), & h'(u) \le 0, \\ \text{and} & h(u_l) = f_l(u_l) \end{matrix}\right\} & \text{if } u \ge u_l. \end{cases} \tag{8.17}$$

In these definitions, the function h appearing in the infima and suprema is assumed to be continuous. In Fig. 8.1 we show an example of h_l and h_r. Using h_l and h_r we can use the following alternative definition of the admissible sets H_l and H_r, namely

$$H_l\left(u_l\right) = \left\{u \mid h_l\left(u;u_l\right) = f_l(u)\right\}, \tag{8.18}$$

$$H_r\left(u_r\right) = \left\{u \mid h_r\left(u;u_r\right) = f_r(u)\right\}. \tag{8.19}$$

Since the jump in u at $x = 0$ is a discontinuity with zero speed, the Rankine–Hugoniot condition says that for every weak solution we must have

$$f\left(\gamma_l, u_l'\right) = f\left(\gamma_r, u_r'\right) =: f^{\times}. \tag{8.20}$$

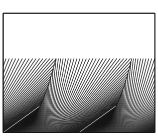
We now have $u_l' \in H_l(u_l)$ and $u_r' \in H_r(u_r)$, using (8.18) and (8.19). This can be restated as

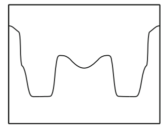
$$h_l\left(u_l', u_l\right) = h_r\left(u_r', u_r\right). \tag{8.21}$$

Since the mapping $u \mapsto h_l(u;u_l)$ is nonincreasing and $u \mapsto h_r(u;u_r)$ is nondecreasing, the above equality, (8.21), will hold for at most one h value. Therefore, if the graphs of h_l and h_r intersect, the flux value at $x = 0$ is determined by the flux value at this intersection point. We label this flux value $f^{\times}$.

From these observations it also follows that if the graph of h_l does *not* intersect the graph of h_r, we cannot hope to find a weak solution to the Riemann problem (8.7). For instance, if

$$f_l(u) = e^{-u^2} \quad \text{and} \quad f_r(u) = 2 + e^{-u^2},$$





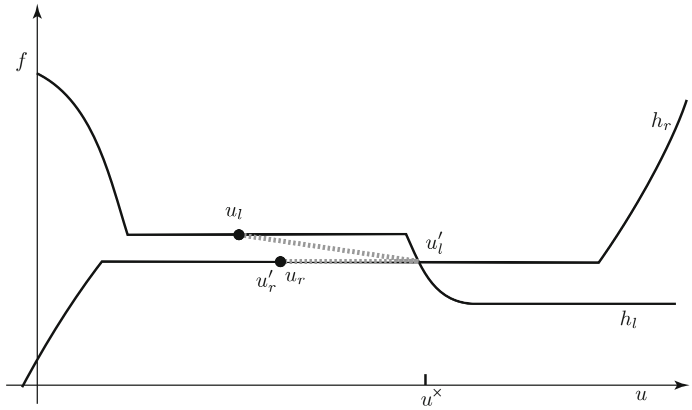


Fig. 8.2 An example showing how to solve a Riemann problem of the type (8.7)

we cannot find any weak solution. Another important example for which we cannot find any solution to the Riemann problem is

$$f_l'(u) \geq 0 \quad \text{and} \quad f_r'(u) \leq 0.$$

In this case $h_l(u; u_l) = f_l(u_l)$ and $h_r(u; u_r) = f_r(u_r)$, so unless these happen to be equal, we cannot find any solution.

Furthermore, even if the flux value at the intersection is uniquely determined, the actual values u_l' and u_r' need not be. This is so since for $u \notin H_l(u_l)$ we have $h_l'(u; u_l) = 0$, and similarly, if $u \notin H_r(u_r)$, then $h_r'(u; u_r) = 0$. In other words, h_l and h_r may both be constant on the interval where their graphs cross. In order to resolve this nonuniqueness problem, we propose that u_l' and u_r' be chosen such that the variation of the solution u is minimal, subject to the above restrictions.

To be more concrete, we choose u_l' to be the unique value such that

$$|u_l - u_l'| \text{ is minimized, provided } u_l' \in H_l(u_l) \text{ and } h_l(u_l'; u_l) = f^\times. \qquad (8.22)$$

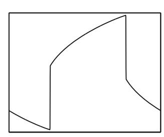
Similarly, we choose u_r' to be the unique value such that

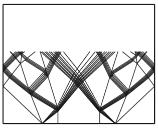
$$|u_r - u_r'| \text{ is minimized, provided } u_r' \in H_r(u_r) \text{ and } h_l(u_r'; u_r) = f^\times. \qquad (8.23)$$

These criteria for choosing u_l' and u_r' are called the *minimal jump entropy condition*.

It is perhaps instructive to examine this condition in a little more detail. If the graphs of h_l and h_r intersect in a single point $u^\times$, then $u^\times \in H_l(u_l)$ or $u^\times \in H_r(u_r)$. If $u^\times \in H_l(u_l)$, then $u_l' = u^\times$, and if $u^\times \in H_r(u_r)$, then $u_r' = u^\times$. Assuming for definiteness that $u_l < u^\times$ and $u^\times \notin H_l(u_l)$, then there will be a smallest point $\tilde{u}$ in the interval $[u_l, u^\times]$ such that the interval $(\tilde{u}, u^\times]$ is not contained in $H_l(u_l)$, and $\tilde{u} \in H_l(u_l)$. It is clear that according to (8.22) we must choose $u_l' = \tilde{u}$.

In Fig. 8.2 we show how the Riemann problem from Fig. 8.1 is solved in this way. Here $u^\times \in H_l(u_l)$ so $u_l' = u^\times$. Also the point minimizing $|u_r' - u_r|$ is clearly u_r, so that $u_r' = u_r$. Finally the Riemann problem is solved by a shock of negative





speed from u_l to u_l', and then by a discontinuity at $x = 0$ from u_l' to u_r. There is some more important information to be extracted from the minimal jump entropy condition. Since the Riemann problem with $u_l = u_l'$ and $u_r = u_r'$ is solved by a single stationary discontinuity, in the interval spanned by u_l' and u_r', we must have

$$h_l\left(u; u_l'\right) = f^\times, \text{ or } h_r\left(u; u_r'\right) = f^\times. \tag{8.24}$$

If $u_l' < u_r'$, since $h_l(\cdot\,; u_l')$ is the largest nonincreasing continuous function less than or equal to f_l such that $h_l(u_l'; u_l') = f_l(u_l')$, then

$$h_l\left(u; u_l'\right) = f^\times \quad\Rightarrow\quad f_l(u) > f^\times \text{ for } u \in (u_l', u_r')$$

and

$$h_r\left(u; u_r'\right) = f^\times \quad\Rightarrow\quad f_r(u) > f^\times \text{ for } u \in (u_l', u_r'),$$

since $h_r(\cdot\,; u_r')$ is the largest continuous nondecreasing function smaller than or equal to f_r. Similarly, if $u_r' < u_l'$, then

$$h_l\left(u; u_l'\right) = f^\times \quad\Rightarrow\quad f_l(u) < f^\times \text{ for } u \in (u_r', u_l')$$

and

$$h_r\left(u; u_r'\right) = f^\times \quad\Rightarrow\quad f_r(u) < f^\times \text{ for } u \in (u_r', u_l').$$

Summing up, we have

$$u_l' \le u_r' \Rightarrow \begin{cases} f_l(u) \ge f_l(u_l') & \text{for all } u \in [u_l', u_r'] \text{ or} \\ f_r(u) \ge f_r(u_r') & \text{for all } u \in [u_l', u_r'], \end{cases} \tag{8.25}$$

$$u_r' \le u_l' \Rightarrow \begin{cases} f_l(u) \le f_l(u_l') & \text{for all } u \in [u_r', u_l'] \text{ or} \\ f_r(u) \le f_r(u_r') & \text{for all } u \in [u_r', u_l']. \end{cases} \tag{8.26}$$

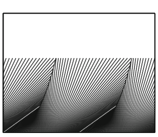
Furthermore, the implications (8.25) and (8.26) actually imply that u_l' and u_r' are chosen according to the minimal jump entropy condition.

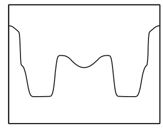
Lemma 8.4 *If the values u_l' and u_r' are chosen according to the minimal jump entropy condition* (8.22), (8.23)*, then for every constant c,*

$$F_r\left(u_r', c\right) - F_l\left(u_l', c\right) \le \left|f_r(c) - f_l(c)\right|, \tag{8.27}$$

where F_l and F_r are the Kružkov entropy fluxes. Thus

$$F_l(u, c) = \operatorname{sign}(u - c)\left(f_l(u) - f_l(c)\right),$$
$$F_r(u, c) = \operatorname{sign}(u - c)\left(f_r(u) - f_r(c)\right).$$





Proof If $\operatorname{sign}\left(u_l' - c\right) = \operatorname{sign}\left(u_r' - c\right)$, then the left-hand side of (8.27) equals

$$\operatorname{sign}\left(u_l' - c\right)\left(f_r\left(u_r'\right) - f_r(c) - f_l\left(u_l'\right) + f_l(c)\right) = \operatorname{sign}\left(u_l' - c\right)\left(f_l(c) - f_r(c)\right),$$

and the inequality clearly holds.

If $u_l' \le c \le u_r'$, then (8.27) reads

$$2 f^\times - f_l(c) - f_r(c) \le |f_r(c) - f_l(c)|,$$

or

$$\begin{aligned} 2 f^\times - \max\{f_l(c), f_r(c)\} - \min\{f_l(c), f_r(c)\} \\ \le \max\{f_l(c), f_r(c)\} - \min\{f_l(c), f_r(c)\}. \end{aligned}$$

In other words, (8.27) is the same as

$$f^\times \le \max\{f_l(c), f_r(c)\},$$

and it is immediate that (8.25) implies this.

If $u_r' \le c \le u_l'$, then (8.27) reads

$$f^\times \ge \min\{f_l(c), f_r(c)\},$$

which is implied by (8.26). □

From the proof of Lemma 8.4 it is also transparent that the condition (8.27) does *not* imply the minimal jump entropy condition (8.25) and (8.26). However, define the pair of "constants" c_l and c_r (these numbers depend on u_l' and u_r') by requiring

$$c_l\left(u_l', u_r'\right) = \begin{cases} \min \arg_{[u_l', u_r']} f_l(u) & \text{if } u_l' \le u_r', \\ \max \arg_{[u_r', u_l']} f_l(u) & \text{if } u_l' \ge u_r', \end{cases} \tag{8.28}$$

$$c_r\left(u_l', u_r'\right) = \begin{cases} \min \arg_{[u_l', u_r']} f_r(u) & \text{if } u_l' \le u_r', \\ \max \arg_{[u_r', u_l']} f_r(u) & \text{if } u_l' \ge u_r'. \end{cases} \tag{8.29}$$

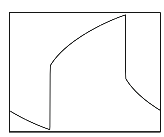
Using the arguments of the proof of Lemma 8.4, it readily follows that the minimal jump entropy condition is equivalent to

$$F_r\left(u_r', c_r\right) - F_l\left(u_l', c_l\right) \le |f_r(c_r) - f_l(c_l)|. \tag{8.30}$$

Furthermore, for every c between u_l' and u_r', the inequality

$$F_r\left(u_r', c\right) - F_l\left(u_l', c\right) \le F_r\left(u_r', c_r\right) - F_l\left(u_l', c_l\right),$$

holds.



Remark 8.5 In a special case (8.27) actually implies that the values u'_l and u'_r are chosen according to the minimal jump entropy condition. Assume that there is a value $\hat{u}$ such that both $f_l(u)$ and $f_r(u)$ have a global maximum (minimum) at $\hat{u}$, and that $f_{l,r}$ is increasing (decreasing) for $u < \hat{u}$ and decreasing (increasing) for $u > \hat{u}$. To see this, we recall that (8.27) holds trivially if c is not between u'_l and u'_r,while if c is between these values, (8.27) reads

$$\begin{cases} f^{\times} \leq \max\left\{f_l(c), f_r(c)\right\}, & \text{if } u'_l < u'_r, \\ f^{\times} \geq \max\left\{f_l(x), f_r(c)\right\}, & \text{if } u'_l > u'_r. \end{cases} \tag{8.31}$$

By assuming that $f_l(u'_l) = f_r(u'_r)$, that the above holds, and that the flux functions $f_{l,r}$ have a single common maximum, the reader can check that (8.31) implies (8.25) and (8.26). Actually, this implication holds for more general flux functions as well; cf. the notorious "crossing condition" in [110].

Although it seemingly has nothing to do with the solution of the Riemann problem, at this point it is convenient to state and prove the following lemma, which will play an important role in proving well-posedness in Sect. 8.3.

Lemma 8.6 *Assume that the pairs (u'_l, u'_r) and (v'_l, v'_r) are both chosen according to the minimal jump entropy condition. Then*

$$Q = F_r\left(u'_r, v'_r\right) - F_l\left(u'_l, v'_l\right) \leq 0. \tag{8.32}$$

Proof Since $f_l(v'_l) = f_r(v'_r)$ and $f_l(u'_l) = f_r(u'_r)$, if

$$\operatorname{sign}\left(u'_l - v'_l\right) = \operatorname{sign}\left(u'_r - v'_r\right),$$

then $Q = 0$. Assume therefore that

$$\operatorname{sign}\left(u'_l - v'_l\right) = -1 \quad \text{and} \quad \operatorname{sign}\left(u'_r - v'_r\right) = 1.$$

In this case,

$$\begin{aligned} Q &= \left[f_r\left(u'_r\right) - f_r\left(v'_r\right)\right] + \left[f_l\left(u'_l\right) - f_l\left(v'_l\right)\right] \\ &= 2\left(f_r\left(u'_r\right) - f_r\left(v'_r\right)\right) \qquad (8.33) \\ &= 2\left(f_l\left(u'_l\right) - f_l\left(v'_l\right)\right), \qquad (8.34) \end{aligned}$$

since $f_l(v'_l) = f_r(v'_r)$ and $f_l(u'_l) = f_r(u'_r)$. Moreover

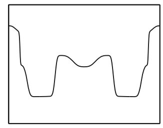
$$u'_l \leq v'_l \quad \text{and} \quad v'_r \leq u'_r.$$

Then either u'_l and u'_r are both in the interval $[v'_r, v'_l]$ (case **a**), or v'_l and v'_r are in the interval $[u'_l, u'_r]$ (case **b**), or $v'_r \leq u'_l \leq v'_l \leq u'_r$ (case **c**), or $u'_l \leq v'_r \leq u'_r \leq v'_l$ (case **d**).

If case **a** holds, then (8.26) for v'_l and v'_r gives that either

$$f_l\left(u'_l\right) \leq f_l\left(v'_l\right) \quad \text{or} \quad f_r\left(u'_r\right) \leq f_r\left(v'_r\right).$$

It is easy to see that this coupled with either (8.33) or (8.34) will give $Q \leq 0$.



If case **b** holds, then (8.26) for u gives that either

$$f_l\left(v_l'\right) \geq f_l\left(u_l'\right) \quad \text{or} \quad f_r\left(v_r'\right) \geq f_r\left(u_r'\right).$$

So again $Q \leq 0$.

Recall that case **c** is defined to hold if

$$v_r' \leq u_l' \leq v_l' \quad \text{and} \quad u_l' \leq v_l' \leq u_r'.$$

Using the first inequality and (8.26) for v, we find that

$$f_l\left(u_l'\right) \leq f_l\left(v_l'\right) \quad \text{or} \quad f_r\left(u_r'\right) \leq f_r\left(v_r'\right),$$

both of which give the desired conclusion.

Finally, in case **d**, we have

$$u_l' \leq v_r' \leq u_r' \quad \text{and} \quad v_r' \leq u_r' \leq v_l'.$$

Using the first inequality with (8.25) gives

$$f_l\left(v_l'\right) \geq f_l\left(u_l'\right) \quad \text{or} \quad f_r\left(v_r'\right) \geq f_r\left(u_r'\right),$$

thereby completing the proof. □

◇ Example 8.7

Now we pause to consider two examples. First consider the Riemann problem for the equation

$$u_t + \left(\frac{1}{2}u^2 + \gamma\right)_x = 0, \tag{8.35}$$

where

$$u_0(0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0. \end{cases}$$

If $u_l \leq 0$, then

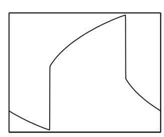
$$H_l\left(u_l\right) = (-\infty, 0],$$

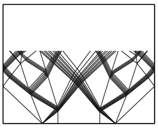
and if $u_l \geq 0$, then

$$H_l\left(u_l\right) = \left(-\infty, -u_l\right] \cup \left\{u_l\right\}.$$

Similarly, if $u_r \leq 0$, then

$$H_r\left(u_r\right) = \left\{-u_r\right\} \cup \left[-u_r, \infty\right),$$





and if $u_r \geq 0$, then

$$H_r(u_r) = [0, \infty).$$

Now it is easy to construct the solution for any initial data and any γ. Assume that $\gamma_l = -1$, $\gamma_r = 1$, $u_l = 1$, and $u_r = 1$. Then

$$h_l(u; -1) = \begin{cases} \frac{1}{2}u^2 - 1 & \text{if } u \leq -1, \\ -\frac{1}{2} & \text{if } u \geq -1, \end{cases} \quad \text{and} \quad h_r(u; 1) = \begin{cases} 1 & \text{if } u \leq 0, \\ \frac{1}{2}u^2 + 1 & \text{if } u \geq 0. \end{cases}$$

The graphs of h_l and h_r intersect in a single point where the flux equals 1 and $u < 0$. Thus we obtain u_l' as the solution of

$$h_l\left(u_l'; -1\right) = 1, \quad u_l' < 0,$$

and thus $u_l' = -2$. Following the general construction, we see that $u_r' = 0$. The complete solution thus consists of the solution of a scalar Riemann problem for the equation

$$v_t + \left(\frac{1}{2}v^2\right)_x = 0, \quad v(x,0) = \begin{cases} 1 & \text{for } x \leq 0, \\ -2 & \text{for } x \geq 0, \end{cases}$$

glued together with the solution of the scalar Riemann problem

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0, \quad w(x,0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

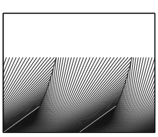
From the general solution procedure for scalar Riemann problems, i.e., taking envelopes, we see that

$$v(x,t) = \begin{cases} 1 & \text{for } x \leq -t/2, \\ -2 & \text{for } x > -t/2, \end{cases} \quad \text{and} \quad w(x,t) = \begin{cases} 0 & \text{for } x \leq 0, \\ x/t & \text{for } 0 < x \leq t, \\ 1 & \text{for } t < x. \end{cases}$$

Finally, we set

$$u(x,t) = \begin{cases} v(x,t) & \text{for } x < 0, \\ w(x,t) & \text{for } x > 0. \end{cases}$$

This solution is depicted in Fig. 8.3. To the left we see the solution path in the (u, f)-plane, and to the right $u(x,t)$. Perhaps the most important lesson to be learned from this example is that the variation of the solution u is *not* bounded by the variation of the initial data $u(x,0)$. Even though this is so, it is natural to ask whether the variation of u is bounded by the variation of u_0 plus the variation of γ. From the construction of the solution of the Riemann problem, the total variation of $f(\gamma, u)$ is bounded by the total variation of $f(\gamma, u_0)$. Nevertheless, an explicit example shows that it may happen that the total variation of u_0 is finite, yet for a finite $T > 0$, we have T.V. $(u(\cdot, T)) = \infty$; see [1]. We shall return to these observations in a later section. ◇



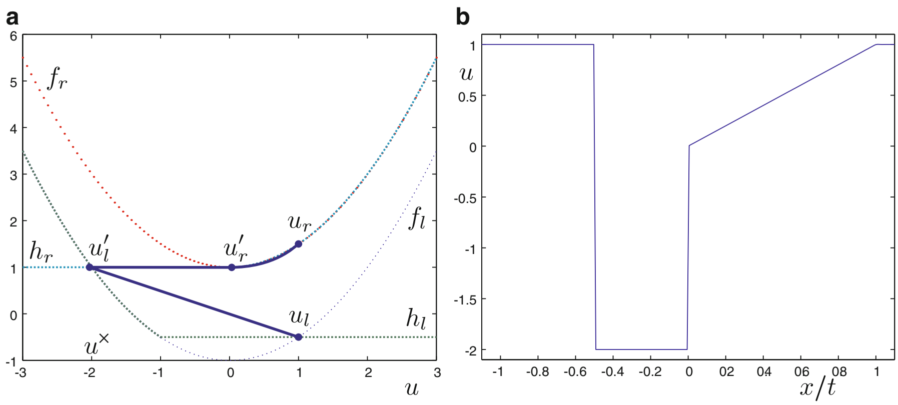
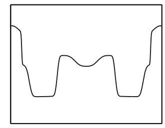


Fig. 8.3 An example of the solution of a Riemann problem. **a** The solution path in (u, f) space. **b** $u(x, t)$



◇ Example 8.8

As a second example we study the traffic flow model

$$u_t + (\gamma(x)4u(1-u))_x = 0, \tag{8.36}$$

where

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x \geq 0. \end{cases}$$

For simplicity, we assume that γ_l and γ_r are positive. Now

$$H_l(u_l) = \begin{cases} \{u_l\} \cup [1-u_l, \infty) & \text{if } u_l \leq 1/2, \\ [1/2, \infty) & \text{if } u_l \geq 1/2, \end{cases}$$

and

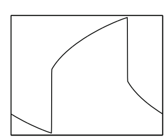
$$H_r(u_r) = \begin{cases} (-\infty, 1/2] & \text{if } u_r \leq 1/2, \\ (-\infty, 1-u_r] \cup \{u_r\} & \text{if } u_r \geq 1/2. \end{cases}$$

We shall now detail the complete solution of the Riemann problem in this case. This is instructive, since (8.36) exhibits many of the features of Riemann solutions for general flux functions.

We describe the solution by listing what happens in various cases, depending on γ_l, γ_r, u_l, and u_r. Note first that $f(\gamma, u)$ has a maximum at $u = 1/2$ for all γ and that $f(\gamma, 1/2) = \gamma$. We start by assuming that

$$u_l \leq \frac{1}{2}. \tag{8.37}$$

In this case the structure of the solution will depend on whether $\gamma_l < \gamma_r$. We start by examining the case $\gamma_l < \gamma_r$ and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$. The situation is depicted in Fig. 8.4. Here we show the h_l and h_r functions as dotted lines, and



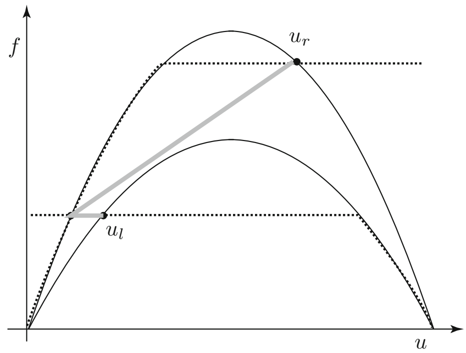


Fig. 8.4 The solution of the Riemann problem if $u_l < 1/2$, $\gamma_l < \gamma_r$, and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \le 1/2$

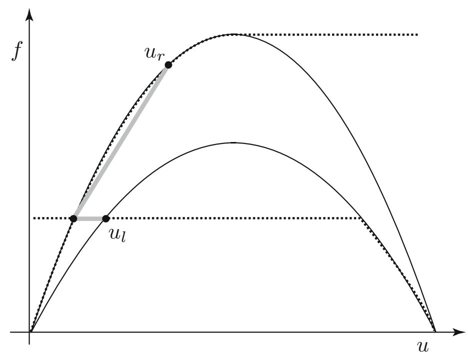


Fig. 8.5 The solution of the Riemann problem if $u_l < 1/2$, $\gamma_l < \gamma_r$, and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \le 1/2$

the solution path as a gray line. In this case $u_l' = u_l$, and u_r' is the solution of

$$f\left(\gamma_r, u_r'\right) = f\left(\gamma_l, u_l\right), \quad u_r' < \frac{1}{2}.$$

In our case, this means that

$$u_r' = \frac{1}{2}\left(1 - \sqrt{1 - \frac{\gamma_l}{\gamma_r} 4u_l\left(1 - u_l\right)}\right).$$

The solution consists of a stationary discontinuity separating (u_l', γ_l) and (u_r', γ_r), which we shall call a γ-wave, followed by a shock in u moving to the right. This we call a u-wave. For clarity we also show the solution if $u_r \le 1/2$ in Fig. 8.5.

Next, we turn to the case that $\gamma_l < \gamma_r$ and $f(\gamma_l, u_l) \ge f(\gamma_r, u_r)$, depicted in Fig. 8.6. The solution consists of a u-wave with negative speed followed by a γ-wave separating u_l' and u_r. In other words, we have $u_r' = u_r$, and u_l' is the solution of

$$f\left(\gamma_l, u_l'\right) = f\left(\gamma_r, u_r\right), \quad u_l' \ge \frac{1}{2}.$$

In the next case we assume that $u_l \ge 1/2$. In this case, if $u_r \le 1/2$, or $f(\gamma_r, u_r) > f(\gamma_l, 1/2)$, then $u_l' = 1/2$, and u_r' solves

$$f\left(\gamma_r, u_r'\right) = f\left(\gamma_l, u_l'\right) = \gamma_l, \quad u_r' < \frac{1}{2}.$$

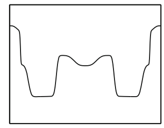


Fig. 8.6 The solution of the Riemann problem if $u_l < 1/2, \gamma_l < \gamma_r$, $f(\gamma_l, u_l) \geq f(\gamma_r, u_r)$, and $u_r \geq 1/2$

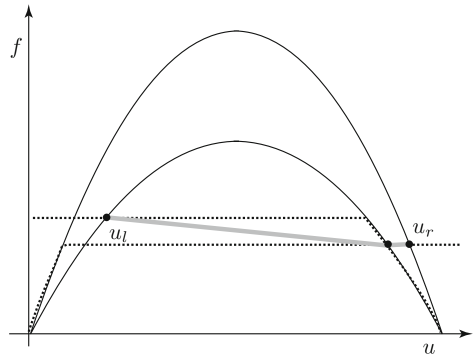


Fig. 8.7 The solution of the Riemann problem if $u_l \geq 1/2, \gamma_l < \gamma_r$, and $f(\gamma_l, 1/2) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$

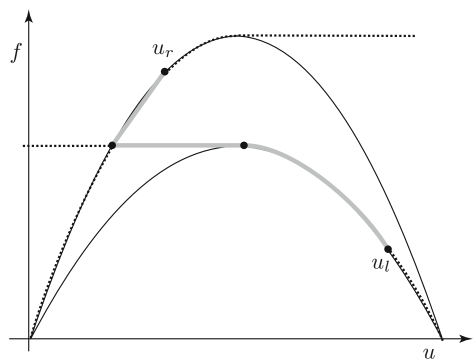
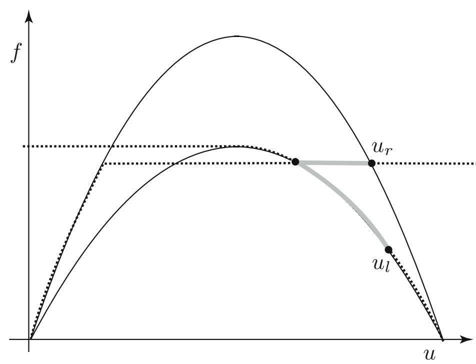


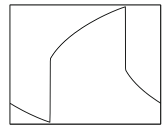
Fig. 8.8 The solution of the Riemann problem if $u_l \geq 1/2, \gamma_l < \gamma_r$, $f(1/2, u_l) \geq f(\gamma_r, u_r)$, and $u_r > 1/2$

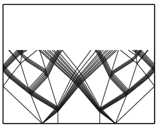


This is illustrated in Fig. 8.7. Now the solution consists of a u-wave moving to the left, this u-wave is a rarefaction wave, followed by a γ-wave. The last wave is a u-wave moving to the right; this is a shock wave.

Next, if $u_l \geq 1/2$, $u_r \geq 1/2$, and $f(\gamma_r, u_r) \leq f(\gamma_l, 1/2)$, the solution is shown in Fig. 8.8. In this case u consists of a leftward moving u-wave followed by a γ-wave. This exhausts the case $\gamma_l < \gamma_r$.

The case $\gamma_l > \gamma_r$ is analogous, and we show the four different possibilities in Fig. 8.9.





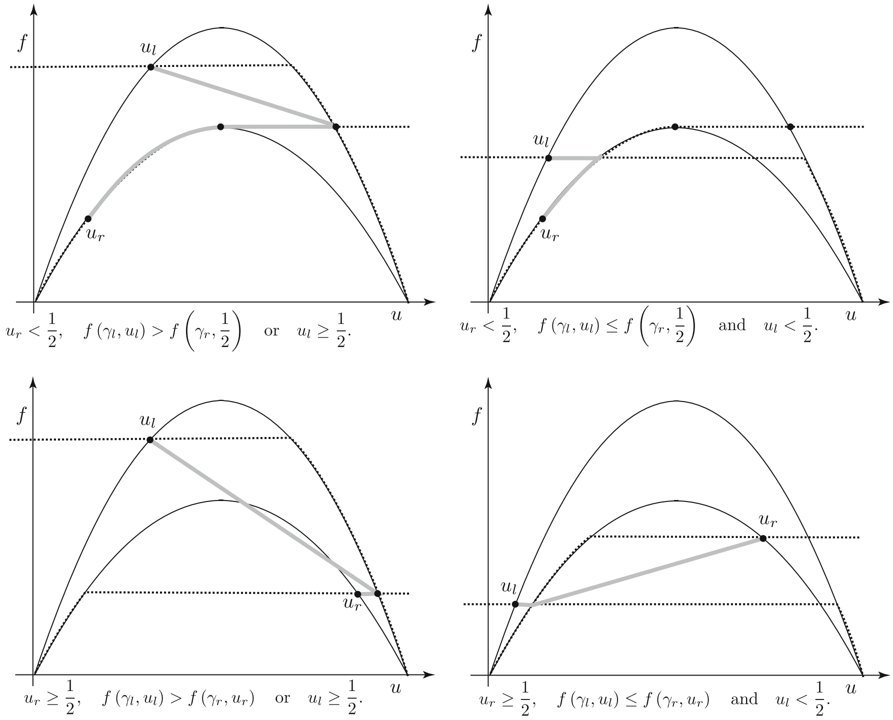


Fig. 8.9 The different possibilities for a solution of the Riemann problem if $u_r \geq 1/2$. The solution path is the *gray line*

In order to determine a particular solution, follow the gray path from u_l to u_r. If the path follows the graph of f_l or f_r, the wave is a rarefaction wave, and, if not, it is a shock wave. The horizontal segments joining f_l and f_r are γ-waves. In these figures, the dotted lines indicate the functions h_l and h_r.

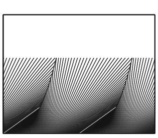
From the above diagrams, we observe that if u_l and u_r are in the interval [0, 1], then also the solution $u(x,t)$ will take values in [0, 1]. In many applications involving similar conservation laws, u is interpreted as a density; hence it is natural to require that u be between 0 and 1.

There is another and much more compact way to depict the solution of the general Riemann problem for this conservation law. Let $z = z(\gamma, u)$ be defined as

$$\begin{aligned} z(\gamma, u) &= \operatorname{sign}\left(\frac{1}{2} - u\right)\left[f(\gamma, u) - f\left(\gamma, \frac{1}{2}\right)\right] \\ &= \gamma \operatorname{sign}\left(u - \frac{1}{2}\right)(2u-1)^2 \\ &= \int_{1/2}^{u} \left|\frac{\partial f}{\partial u}(\gamma, \xi)\right| d\xi. \end{aligned} \tag{8.38}$$

This mapping takes the rectangle $[\gamma_1, \gamma_2] \times [0, 1]$ into the region

$$\{(z, \gamma) \mid \gamma_1 \leq \gamma \leq \gamma_2 \text{ and } -\gamma \leq z \leq \gamma\}.$$



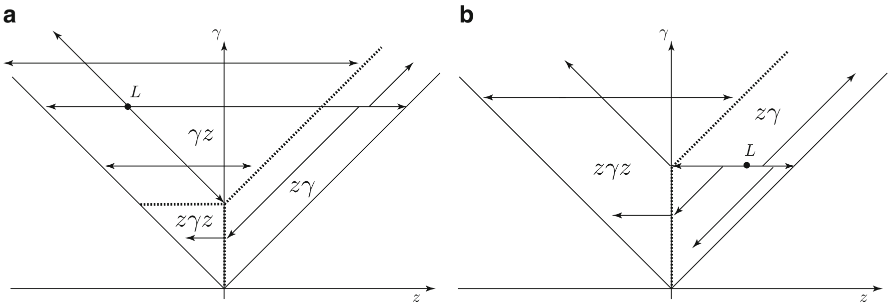
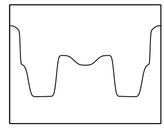


Fig. 8.10 The solution of the Riemann problem. **a** $z_l \leq 0$. **b** $z_l \geq 0$



Furthermore, $u \mapsto z(\gamma, u)$ is injective, and strictly increasing. In (z, γ) coordinates, γ-waves are straight lines of slope -1 if $u \leq 1/2$ and slope 1 if $u \geq 1/2$, and u-waves are horizontal lines. In Fig. 8.10 we show how the solution looks in the various cases in the (z, γ)-plane. To read the diagram, start at the point $L = (z(u_l, \gamma_l), \gamma_l)$ and follow the arrows to the right location. The dotted lines mark the boundaries where the solution type is constant. Since we are working with (z, γ) coordinates, we call u-waves z-waves, and the solution types are $z\gamma$, $z\gamma z$, and γz. If a solution type is, e.g., γz, this means that the solution consists of a z-wave (u-wave) followed by a γ-wave. This finishes the second example. ◇

Actually, our two examples are more similar than it might seem at a first glance. The inverse of the mapping (8.38) is

$$u = \frac{1}{2}\left(1 + \operatorname{sign}(z)\sqrt{\frac{|z|}{\gamma}}\right),$$

and

$$f(\gamma, u) = |z| + \gamma.$$

Inserting this into equation (8.36), we find that

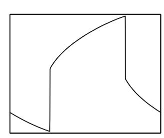
$$\left(\frac{1}{2}\left(1 + \operatorname{sign}(z)\sqrt{\frac{|z|}{\gamma}}\right)\right)_t + (|z| + \gamma)_x = 0.$$

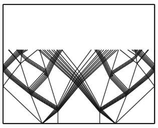
Since γ is independent of t, we can rearrange this as

$$\left(\operatorname{sign}(z)\sqrt{|z|}\right)_t + 2\sqrt{\gamma}\,(|z| + \gamma)_x = 0.$$

If we now introduce $w = \operatorname{sign}(z)\sqrt{|z|}$ and a new time coordinate τ such that $\partial/\partial\tau = \sqrt{2\gamma}\,\partial/\partial t$, then

$$w_\tau + \left(\frac{1}{2}w^2 + \gamma\right)_x = 0.$$





Now we return to the discussion of the Riemann problem for the general conservation law; cf. (8.7). We have seen that we cannot always find a weak solution to this problem, but if the graphs of the functions $H_l(\,\cdot\,; u_l)$ and $H_r(\,\cdot\,; u_r)$ intersect, we can choose a unique pair (u_l', u_r'), which in turn gives us a unique solution of the Riemann problem. We call this solution, satisfying the minimal jump entropy condition, an *entropy solution* of the Riemann problem.

It seems complicated to give a general criterion for f_l and f_r to guarantee the intersection of h_l and h_r, but for two important classes of flux functions we always have an intersection.

Lemma 8.9 *Consider the Riemann problem*

$$u_t + f(\gamma, u)_x = 0, \quad t > 0,$$
$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0. \end{cases} \tag{8.39}$$

(i) *Let $f = f(\gamma, u)$ be a continuously differentiable function on the set*

$$(\gamma, u) \in [\gamma_1, \gamma_2] \times [u_1, u_2] = \Omega.$$

Assume that

$$\frac{\partial f}{\partial \gamma}(\gamma, u_1) = \frac{\partial f}{\partial \gamma}(\gamma, u_2) = 0,$$

so that $f(\gamma, u_1) = C_1$ and $f(\gamma, u_2) = C_2$ for some constants C_1 and C_2. Then the Riemann problem (8.39) has a unique entropy solution for all (γ_l, u_l) and (γ_r, u_r) in Ω. Furthermore, $u(x,t) \in \Omega$ for all x and t.

(ii) *Let $f = f(\gamma, u)$ be a locally Lipschitz continuous function for $\gamma \in [\gamma_1, \gamma_2]$ and $u \in \mathbb{R}$. Assume that*

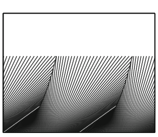
$$\lim_{u \to \pm\infty} f(\gamma, u) = \infty \quad \text{or} \quad \lim_{u \to \pm\infty} f(\gamma, u) = -\infty,$$

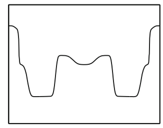
for all $\gamma \in [\gamma_1, \gamma_2]$. Then the Riemann problem (8.39) has a unique entropy solution for all (γ_l, u_l) and (γ_r, u_r) in $[\gamma_1, \gamma_2] \times \mathbb{R}$.

Our first example is of the second type mentioned in the lemma, and the second example is of the first type. This lemma is proved simply by constructing the functions h_l and h_r in the two cases.

Vanishing Viscosity and Smoothing

We would like to motivate the minimal jump entropy condition. In our construction of the solution of the Riemann problem, it emerged naturally as a candidate for finding a unique solution. In this section we shall give two partial motivations for





this entropy condition. Both of these motivations are based on the study of equations that formally have (8.7) as a limit, but whose solutions, or the equations themselves, possess more regularity than the conservation law with a discontinuous coefficient. When doing this, we hope that the minimal jump condition will be a consequence of requiring that the solutions to the perturbed equations tend to the solution of the Riemann problem as the size of the perturbations tends to zero.

It is common to motivate entropy conditions for conservation laws by requiring that the solution of Riemann problems be limits of traveling wave solutions to the singularly perturbed equation

$$v_t + f(v)_x = \varepsilon v_{xx},$$

as $\varepsilon \downarrow 0$. For a scalar equation in which the flux function does not depend on x, the "lower convex envelope" criterion is indeed a consequence of such an approach. We also say that the weak solution found by taking envelopes satisfies the *vanishing viscosity* entropy condition; see Sects. 2.1 and 2.2.

Let now u^ε be a traveling wave solution of the initial value problem

$$u^\varepsilon_t + f(\gamma, u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \quad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0 \end{cases} \tag{8.40}$$

(with $\gamma_l \neq \gamma_r$). We hope that

$$\lim_{x \to -\infty} u^\varepsilon(x,t) = u'_l, \text{ and } \lim_{x \to \infty} u^\varepsilon(x,t) = u'_r \tag{8.41}$$

for some values u'_l, u'_r. Since γ depends on x, we cannot expect to find a traveling wave solution, i.e., a solution that depends on $(x - st)/\varepsilon$, unless it is stationary, that is, $s = 0$. Thus we consider a function that depends on space only, $u^\varepsilon(x,t) = u(x/\varepsilon)$. Introduce $\xi = x/\varepsilon$, to obtain

$$\dot{f}(\gamma, u) = \ddot{u},$$

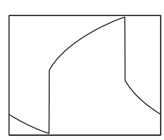
where $\dot{f} = df/d\xi$. The equation can be integrated once, and if we assume that the limits (8.41) are reached in a suitable manner, we get

$$\dot{u} = f(\gamma, u) - f\left(\gamma_l, u'_l\right) = f(\gamma, u) - f\left(\gamma_r, u'_r\right),$$

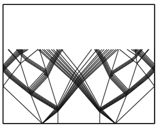
which also gives us the Rankine–Hugoniot condition

$$f\left(\gamma_l, u'_l\right) = f\left(\gamma_r, u'_r\right) =: f^\times. \tag{8.42}$$

Summing up, we say that the discontinuity separating (γ_l, u'_l) and (γ_r, u'_r) admits a viscous profile, or that this discontinuity satisfies the viscous profile entropy conditions, if the ordinary differential equation



$$\frac{du}{d\xi} = \begin{cases} f\left(\gamma_l, u\right) - f^\times & \text{if } \xi < 0, \\ f\left(\gamma_r, u\right) - f^\times & \text{if } \xi > 0, \end{cases} \tag{8.43}$$



has a (at least one) solution $u(\xi)$ such that either

$$\lim_{\xi\to-\infty} u(\xi) = u_l' \quad \text{and} \quad u(\bar{\xi}) = u_r'$$

or

$$u(\bar{\xi}) = u_r' \quad \text{and} \quad \lim_{\xi\to\infty} u(\xi) = u_r',$$

where $\bar{\xi}$ can be finite or infinite. This means that one of two alternatives must hold: *Either* the ordinary differential equation

$$\dot{v} = f(\gamma_l, v) - f^{\times}, \quad \xi < 0, \quad v(0) = u_r',$$

has a solution such that

$$\lim_{\xi\to-\infty} v(\xi) = u_l',$$

in which case we say that v is a left viscous profile, *or* the equation

$$\dot{w} = f(\gamma_r, u) - f^{\times}, \quad \xi > 0, \quad w(0) = u_l',$$

has a solution such that

$$\lim_{\xi\to\infty} w(\xi) = u_r',$$

in which case we call w a right viscous profile.
Hence the discontinuity satisfies the viscous profile entropy condition if there exists a left or right viscous profile.

If $u_l' < u_r'$, we will have a left viscous profile if and only if

$$f(\gamma_l, u) > f\left(\gamma_l, u_l'\right), \quad \text{for all } u \in (u_l', u_r').$$

Similarly, we will have a right viscous profile if and only if

$$f(\gamma_r, u) > f\left(\gamma_r, u_r'\right), \quad \text{for all } u \in (u_l', u_r').$$

Also, if $u_l' > u_r'$, we will have a left viscous profile if and only if

$$f(\gamma_l, u) < f\left(\gamma_l, u_l'\right), \quad \text{for all } u \in (u_l', u_r').$$

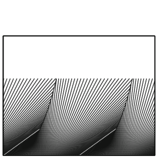
Similarly, we will have a right viscous profile if and only if

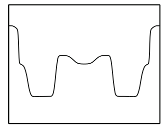
$$f(\gamma_r, u) < f\left(\gamma_r, u_r'\right), \quad \text{for all } u \in (u_l', u_r').$$

Summing up, the viscous profile entropy condition is equivalent to

$$u_l' \le u_r' \Rightarrow \begin{cases} f(\gamma_l, u) > f^{\times} & \text{for all } u \in (u_l', u_r') \text{ or} \\ f(\gamma_r, u) > f^{\times} & \text{for all } u \in (u_l', u_r'), \end{cases} \tag{8.44}$$

$$u_r' \le u_l' \Rightarrow \begin{cases} f(\gamma_l, u) < f^{\times} & \text{for all } u \in (u_r', u_l') \text{ or} \\ f(\gamma_r, u) < f^{\times} & \text{for all } u \in (u_r', u_l'). \end{cases} \tag{8.45}$$





This condition implies the minimal jump entropy condition, and thus provides a motivation.

If the coefficient γ is a continuous function of x, then the classical theory of scalar conservation laws applies, and the initial value problem has a unique weak solution. If we let γ^ε denote a smooth approximation to

$$\gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0, \end{cases}$$

such that $\gamma^\varepsilon \to \gamma$ as $\varepsilon \to 0$, and let u^ε denote the weak solution to

$$u_t^\varepsilon + f\,(\gamma^\varepsilon, u^\varepsilon)_x = 0, \quad u^\varepsilon(x,0) = \begin{cases} u_l' & \text{for } x < 0, \\ u_r' & \text{for } x > 0, \end{cases} \tag{8.46}$$

it is natural to ask whether u^ε tends to the minimal jump entropy solution as $\varepsilon \to 0$.

◇ Example 8.10

We shall consider this in an example. Define

$$\begin{aligned} f_l(u) &= 4 - (u+1)^2, \\ f_r(u) &= 4 - (u-1)^2, \\ f(\gamma, u) &= (1-\gamma) f_l(u) + \gamma f_r(u), \end{aligned}$$

and consider the Riemann problem

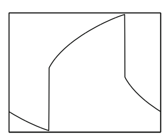
$$u_t + f(\gamma, u)_x = 0, \quad u(x,0) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases} \quad \gamma(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

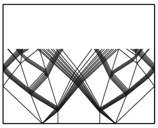
In this case we find that

$$\begin{aligned} h_l(u; -1) &= \begin{cases} 4 & \text{if } u < -1, \\ 4 - (u+1)^2 & \text{if } u \ge -1, \end{cases} \\ h_r(u; 1) &= \begin{cases} 4 - (u+1)^2 & \text{if } u \le 1, \\ 0 & \text{if } u > 1. \end{cases} \end{aligned}$$

Furthermore, the discontinuity separating the u and γ values $(-1, 0)$ and $(1, 1)$ satisfies the minimal jump entropy condition, and hence $u(x, 0)$ is a weak solution satisfying the minimal jump entropy condition. Now set

$$\gamma^\varepsilon(x) = \begin{cases} 0 & \text{for } x \le -\varepsilon, \\ \frac{x+\varepsilon}{2\varepsilon} & \text{for } -\varepsilon < x < \varepsilon, \\ 1 & \text{for } \varepsilon \le x, \end{cases}$$





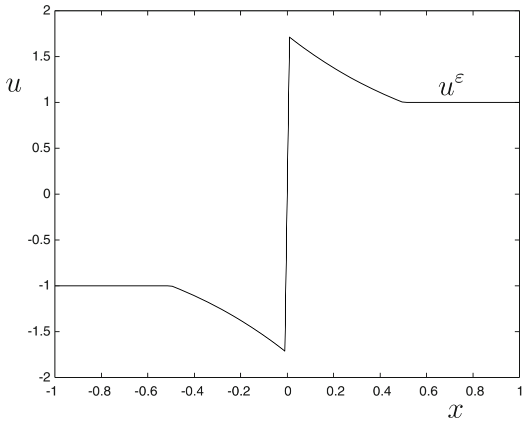


Fig. 8.11 The stationary solution of (8.46), $\varepsilon = 1/2$, and the discontinuity at $x = 0$

and let u^ε denote the stationary solution to (8.46) with $u_l' = -1$ and $u_r' = 1$. We have that u^ε satisfies

$$f(\gamma^\varepsilon, u^\varepsilon)_x = 0,$$

and thus

$$f(\gamma^\varepsilon, u^\varepsilon) = f(0, -1) = 0.$$

Solving this for u^ε, we find that

$$u^\varepsilon(x) = 1 - 2\gamma^\varepsilon(x) \pm \sqrt{(1 - 2\gamma^\varepsilon(x))^2 + 3}.$$

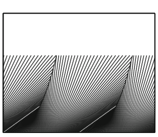
Since $u^\varepsilon = -1$ for $x \le -\varepsilon$ and $u^\varepsilon = 1$ for $x \ge \varepsilon$, we can choose the negative sign for x close to $-\varepsilon$ and the positive sign for x close to ε. Furthermore, since for every (fixed) γ, $f(\gamma, u)$ is concave in u, we can jump from the negative to the positive solution if this will give a shock with zero speed (recall that u^ε is stationary). But since $f(\gamma^\varepsilon, u^\varepsilon)$ is constant, we can jump at any value of x! For instance, we can choose to jump at $x = 0$, giving

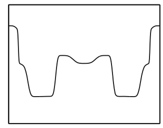
$$u^\varepsilon = \begin{cases} -1 & \text{for } x \le -\varepsilon, \\ 1 - \frac{2x}{\varepsilon} - \sqrt{\left(1 - \frac{2x}{\varepsilon}\right)^2 + 3} & \text{for } -\varepsilon < x < 0, \\ 1 - \frac{2x}{\varepsilon} + \sqrt{\left(1 - \frac{2x}{\varepsilon}\right)^2 + 3} & \text{for } 0 < x < \varepsilon, \\ 1 & \text{for } \varepsilon \le x. \end{cases}$$

We show a plot of this solution in Fig. 8.11, and we note that although $u^\varepsilon \to u$, the variation of the approximate solution is larger than that of u. ◇

This example readily generalizes to the following case. Assume that the map

$$u \mapsto f(\gamma, u)$$





has a single global maximum for all γ, and

$$\lim_{u\to-\infty} f(\gamma,u) = -\infty \quad \text{and} \quad \lim_{u\to\infty} f(\gamma,u) = -\infty.$$

Let $u^{\pm}(\gamma, y)$ denote the two solutions of

$$y = f\left(\gamma, u^{\pm}\right),$$

such that $u^- \le u^+$. As before, let u^ε denote the stationary solution of (8.46), where

$$\gamma^\varepsilon(x) = \gamma_l + \frac{x+\varepsilon}{2\varepsilon}\left(\gamma_r - \gamma_l\right), \quad -\varepsilon < x < \varepsilon.$$

Then it is possible to find a weak solution u^ε if and only if

$$u^-\left(\gamma_l, f\left(\gamma_l, u_l'\right)\right) = u_l' \quad \text{or} \quad u^+\left(\gamma_r, f\left(\gamma_r, u_r'\right)\right) = u_r'. \tag{8.47}$$

Recall that we are always assuming that u_l' and u_r' satisfy the Rankine–Hugoniot condition, i.e., $f(\gamma_l, u_l') = f(\gamma_r, u_r') = f^\times$. If both of the conditions in (8.47) hold, then this solution is given by

$$u^\varepsilon(x) = \begin{cases} u_l' & \text{for } x < -\varepsilon, \\ u^-\left(\gamma^\varepsilon(x), f^\times\right) & \text{for } -\varepsilon \le x \le x_J, \\ u^+\left(\gamma^\varepsilon(x), f^\times\right) & \text{for } x_J < x \le \varepsilon, \\ u_r' & \text{for } \varepsilon < x, \end{cases} \tag{8.48}$$

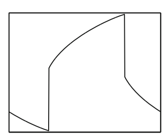
for every $x_J \in [-\varepsilon, \varepsilon]$. Since we are jumping from u^- to u^+, this jump is allowed since $u^- \le u^+$ and $f(\gamma, u) > f^\times$ in the interval (u^-, u^+). If only one of the conditions in (8.47) holds, then we stay on u^+ or u^- throughout the interval $[-\varepsilon, \varepsilon]$. If

$$u_l' = u^+\left(\gamma_l, f\left(\gamma_l, u_l'\right)\right) \quad \text{and} \quad u_r' = u^-\left(\gamma_r, f\left(\gamma_r, u_r'\right)\right),$$

we must at some point jump *from* u^+ *to* u^-, and this will give an entropy-violating weak solution. Looking at the shapes of the graphs of $f(\gamma_l, u)$ and $f(\gamma_r, u)$, we see that (8.47) is equivalent to the minimal jump entropy condition in this case. Hence, if (u_l', u_r') satisfies the minimal jump entropy condition, there exist entropy solutions u^ε of (8.46) such that u^ε tends to the minimal jump entropy condition when $\varepsilon \to 0$ (if the flux f has the properties assumed above).

Remark 8.11 The minimal jump entropy condition is not always reasonable. Entropy conditions are based on extra information, such as physics or common sense. To illustrate this, consider the equation

$$\begin{gathered} u_t + \left(\frac{1}{2}(u+\gamma)^2\right)_x = 0, \\ \gamma(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases} \quad u(x,0) = 0. \end{gathered} \tag{8.49}$$



In this case,

$$h_l(u;0) = \begin{cases} \frac{1}{2}(u-1)^2 & \text{if } u \leq 1, \\ 0 & \text{if } u > 1, \end{cases} \quad h_r(u;0) = \begin{cases} 0 & \text{if } u \leq -1, \\ \frac{1}{2}(u+1)^2 & \text{if } u > -1. \end{cases}$$

We see that there is a unique crossing value $f^\times = 1/2$, and the minimal jump entropy condition gives the solution $u(x,t) = 0$.

One can also try to find a solution of (8.49) by making the substitution $w = u + \gamma$, which turns (8.49) into

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0, \quad w(x,0) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

The entropy solution to this, found by taking the lower convex envelope, reads

$$w(x,t) = \begin{cases} -1 & \text{for } x < -t, \\ x/t & \text{for } -t \leq x \leq t, \\ 1 & \text{for } x > t. \end{cases}$$

Since $u = w - \gamma$, we obtain the alternative solution

$$\tilde{u}(x,t) = \begin{cases} 0 & \text{for } |x| > t, \\ \frac{x}{t} - \operatorname{sign}(x) & \text{otherwise.} \end{cases} \tag{8.50}$$

So which of these solutions shall we choose? We have already seen that the minimal jump solution, $u = 0$, is the limit of the viscous approximations u^ε satisfying

$$u_t^\varepsilon + \left(\frac{1}{2}(u^\varepsilon + \gamma)^2\right)_x = \varepsilon u_{xx}^\varepsilon. \tag{8.51}$$

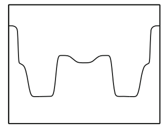
We know that w is the limit of the viscous approximation w^ε satisfying

$$w_t^\varepsilon + \left(\frac{1}{2}w^{\varepsilon 2}\right)_x = \varepsilon w_{xx}^\varepsilon.$$

This means that $\tilde{u}$ is the limit of $\tilde{u}^\varepsilon$, where $\tilde{u}^\varepsilon$ and γ^ε satisfy the viscous approximation for the system (8.6), i.e.,

$$\begin{aligned} \tilde{u}_t^\varepsilon + \left(\frac{1}{2}(\tilde{u}^\varepsilon + \gamma^\varepsilon)^2\right)_x &= \varepsilon \tilde{u}_{xx}^\varepsilon, \\ \gamma_t^\varepsilon &= \varepsilon \gamma_{xx}^\varepsilon. \end{aligned} \tag{8.52}$$

Therefore, it is reasonable to choose $u = 0$ if (8.49) is an approximation of (8.51) and $\tilde{u}$ if (8.49) is an approximation of (8.52).



8.2 The Cauchy Problem

In this section we shall demonstrate the existence of an entropy solution to the conservation law where the flux function depends on a discontinuous coefficient. To be concrete, this is the initial value problem

$$\begin{cases} u_t + f(\gamma, u)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{8.53}$$

where $\gamma = \gamma(x)$ is a function of bounded variation. Fix an arbitrary $T > 0$, and set $\Pi_T = \mathbb{R} \times [0,T)$. By a solution of (8.53) we mean a weak solution, that is, a function u in $L^1_{\mathrm{loc}}(\Pi_T) \cap C([0,T); L^1_{\mathrm{loc}}(\mathbb{R}))$ such that

$$\iint\limits_{\mathbb{R}\times(0,\infty)} \big(u\varphi_t + f(\gamma,u)\varphi_x\big)\,dt\,dx + \int\limits_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx = 0, \tag{8.54}$$

for all test functions $\varphi \in C^1_0(\Pi_T)$. In order to demonstrate existence we shall assume that f and γ have additional properties; for instance, we must be assured that the Riemann problem has a solution for all relevant initial data.

To show that a solution exists, we shall construct it as a limit of a sequence of approximations. This can be done using difference approximations, front-tracking approximations, or the limits of parabolic regularizations, but we shall use front tracking.

Front Tracking for the Model Equation

In this section we will restrict ourselves to the model equation with $f(\gamma,u) = 4\gamma u(1-u)$, i.e.,

$$u_t + (4\gamma u(1-u))_x = 0, \quad u(x,0) = u_0(x). \tag{8.55}$$

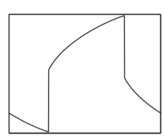
We assume that $\gamma\colon \mathbb{R} \to \mathbb{R}$ is a function of bounded variation that is continuously differentiable on a finite set of intervals. In particular, we assume that there exists a finite number of intervals

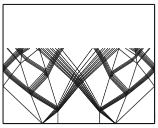
$$I_m = (\xi_m, \xi_{m+1}) \quad \text{for } m = 0,\dots,M,$$

where $\xi_0 = -\infty$, $\xi_{M+1} = \infty$, such that

$$\gamma'\,\Big|_{I_m} \quad \text{is continuous and bounded for } m = 0,\dots,M. \tag{8.56}$$

For the moment, we also assume that the initial function u_0 is of bounded variation and such that $u_0(x) \in [0,1]$ for all x. Now we shall design a front-tracking scheme to approximate solutions of (8.55).





In order to prove convergence of the front-tracking approximations in the scalar case, we used that the variation of $\{u^\delta\}_{\delta>0}$ was uniformly bounded. As Example 8.13 will show, such a bound does not exist if γ is not constant.

In order to circumvent this obstacle, we shall work with the variable z defined by (8.38). The reason that this is a good idea is outlined in the remark below.

Remark 8.12 Assume that u^ε and v^ε are smooth solutions of the regularized equations

$$u_t^\varepsilon + f(\gamma, u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad v_t^\varepsilon + f(\gamma, v^\varepsilon)_x = \varepsilon v_{xx}^\varepsilon,$$

with smooth initial data u_0^ε and v_0^ε, respectively. Let η be a smooth convex function. We subtract these equations and multiply the result by $\eta'(u^\varepsilon - v^\varepsilon)$ to obtain

$$\begin{aligned}
\eta(u^\varepsilon - v^\varepsilon)_t &= -\eta'(u^\varepsilon - v^\varepsilon)\left[f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)\right]_x \\
&\quad + \varepsilon\eta(u^\varepsilon - v^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon - v^\varepsilon)(u^\varepsilon - v^\varepsilon)_x^2 \\
&\leq -\left[\eta'(u^\varepsilon - v^\varepsilon)\left(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)\right)\right]_x \\
&\quad + \varepsilon\eta(u^\varepsilon - v^\varepsilon)_{xx} + \eta'(u^\varepsilon - v^\varepsilon)_x\left(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)\right).
\end{aligned}$$

Now we let $\eta = \eta_\kappa$ be a continuously differentiable approximation to $|\cdot|$, explicitly

$$\eta_\kappa(u) = \int_0^u \max\left(-1, \min\left(\frac{v}{\kappa}, 1\right)\right)\, dv.$$

Assuming that $u^\varepsilon - v^\varepsilon$ has compact support in x, we can integrate the above inequality over $x \in \mathbb{R}$, and get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \eta_\kappa(u^\varepsilon - v^\varepsilon)\, dx &\leq \int_{\mathbb{R}} \eta_\kappa''(u^\varepsilon - v^\varepsilon)\left(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)\right)(u^\varepsilon - v^\varepsilon)_x\, dx \\
&\leq L \int_{|u^\varepsilon - v^\varepsilon| < \kappa} \left|(u^\varepsilon - v^\varepsilon)_x\right|\, dx,
\end{aligned}$$

where $L = \sup|f_u|$, since

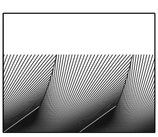
$$\eta_\kappa''(u) = \begin{cases} \frac{1}{\kappa} & \text{for } |u| \leq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

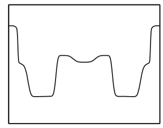
By Lemma B.5,

$$\lim_{\kappa \to 0} \int_{|u^\varepsilon - v^\varepsilon| < \kappa} \left|(u^\varepsilon - v^\varepsilon)_x\right|\, dx = 0.$$

Thus we can send κ to zero, and obtain for any two solutions of the regularized equation

$$\|u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0^\varepsilon - v_0^\varepsilon\|_{L^1(\mathbb{R})}. \tag{8.57}$$





Now we can set $v^\varepsilon(\cdot,t) = u^\varepsilon(\cdot,t+\tau)$ in (8.57), then divide by τ and let $\tau \to 0$, to deduce that

$$\|u_t^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_t^\varepsilon(\cdot,0+)\|_{L^1(\mathbb{R})} = \left|f\left(\gamma,u_0^\varepsilon\right)\right|_{BV}. \tag{8.58}$$

Without loss of generality we can construct u_0^ε so that $\left|f(\gamma,u_0^\varepsilon)\right|_{BV} \le |f(\gamma,u_0)|_{BV}$. This means that the total variation of $f(\gamma,u^\varepsilon)$ is bounded independently of ε, i.e.,

$$|f(\gamma,u^\varepsilon(\cdot,t))|_{BV} \le |f(\gamma,u_0)|_{BV}. \tag{8.59}$$

If $f_u(\gamma,u) \ge c > 0$ for all γ and u, then this would imply that also u^ε had uniformly bounded variation.[1] For the flux function in our example, $f_u(\gamma,1/2) = 0$, so we cannot deduce that u^ε is of bounded variation. This is precisely where the z-mapping comes to the rescue. We write (8.38) as

$$z(\gamma,u) = \operatorname{sign}\left(u-\frac{1}{2}\right)\left(f(\gamma,u) - f\left(\gamma,\frac{1}{2}\right)\right).$$

Now

$$\begin{aligned}|z(\gamma,u^\varepsilon)|_{BV} &\le |f(\gamma,u^\varepsilon)|_{BV} + \left\|f_\gamma\right\|_{L^\infty}|\gamma|_{BV}\\ &\le |f(\gamma,u_0)|_{BV} + \left\|f_\gamma\right\|_{L^\infty}|\gamma|_{BV}.\end{aligned}$$

Thus $z^\varepsilon = z(\gamma,u^\varepsilon)$ has uniformly bounded variation, and the mapping $u \mapsto z(\gamma,u)$ is continuous and invertible. The next step in this strategy is to attempt to show that $\{z^\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(\mathbb{R}\times[0,\infty))$, and thus (for a subsequence) $z^\varepsilon \to \bar{z}$ as $\varepsilon \to 0$. Then we define

$$u = z^{-1}(\gamma,\bar{z}) = \lim_{\varepsilon\to 0} z^{-1}(\gamma,z^\varepsilon) = \lim_{\varepsilon\to 0} u^\varepsilon.$$

The final step will then be to show that the limit u is a weak solution. See, e.g., [114] for an example where this strategy has been carried out.

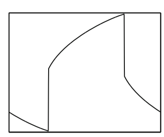
This remark is meant to indicate how the z-mapping could be used to show existence via viscous regularizations, and to motivate the use of the z-mapping also for front-tracking approximations.

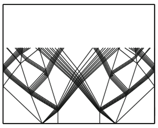
As in the case without a coefficient, we start with a discussion of an approximate solution to the Riemann problem, or rather with the exact solution of the Riemann problem for an approximate equation. In the simple scalar case, we saw that the exact solution of the Riemann problem was piecewise constant in x/t if the flux function was piecewise linear. We shall now define an approximate flux function g^δ such that $g^\delta(\gamma,u) \approx 4\gamma u(1-u)$ and the solution of the Riemann problem with flux g^δ is piecewise constant.

From Sect. 8.1 we saw that the solution of the Riemann problem consisted of a sequence of straight lines in the (z,γ)-plane, where

$$z(\gamma,u) = \operatorname{sign}\left(u-\frac{1}{2}\right)\gamma\,(1-2u)^2. \tag{8.60}$$

[1] This assumption excludes resonances, i.e., coinciding eigenvalues.





There were z-waves, over which γ is constant, and γ-waves, over which γ was not constant. Now fix a (small) positive number δ, and set

$$\gamma_i = i\delta, \quad i > 0, \quad i \in \mathbb{N}, \tag{8.61}$$

and for integers j such that $-i \le j \le i$, $z_{i,j} = j\delta$, and

$$u_{i,j} = z^{-1}\left(\gamma_i, z_{i,j}\right) = \frac{1}{2}\left(1 + \operatorname{sign}\left(z_{i,j}\right)\sqrt{\frac{\left|z_{i,j}\right|}{\gamma_i}}\right). \tag{8.62}$$

Note that the set $\left\{(z_{i,j}, \gamma_i)\right\}$ defines a grid in the (z,γ)-plane. We define g^δ to be the linear interpolation to f on this grid, i.e.,

$$g^\delta\left(\gamma_i, u\right) = f_{i,j} + \left(u - u_{i,j}\right)\frac{f_{i,j+1} - f_{i,j}}{u_{i,j+1} - u_{i,j}}, \quad \text{for } u \in [u_{i,j}, u_{i,j+1}], \tag{8.63}$$

where $f_{i,j} = f(\gamma_i, u_{i,j}) = 4\gamma_i u_{i,j}(1 - u_{i,j})$. For each fixed i, $g^\delta(\gamma_i, u)$ will be a concave function with a maximum for $u = 1/2$. Therefore the solution of the Riemann problem

$$\begin{gathered} u_t + g^\delta\left(\gamma(x), u\right)_x = 0, \\ u(x,0) = \begin{cases} u_{i,j} & \text{for } x < 0, \\ u_{m,n} & \text{for } x > 0, \end{cases} \quad \gamma(x) = \begin{cases} \gamma_i & \text{for } x < 0, \\ \gamma_m & \text{for } x > 0, \end{cases} \end{gathered} \tag{8.64}$$

can be found from the diagrams in Fig. 8.10. Furthermore, since g^δ is piecewise linear in u, this solution will be piecewise constant in x/t. Also, by our choice of interpolation points in constructing g^δ, *all* the intermediate values of $u(x,t)$ will be grid points, i.e.,

$$z(\gamma(x), u(x,t)) = \left(z_{i',j'}, \gamma_{i'}\right), \quad \text{where } i' = i \text{ or } i' = m.$$

We label the grid points in the (u,γ)-plane, or when there is no danger of misunderstanding, in the (z,γ)-plane $\mathcal{U}^\delta$. Hence, the solution of the Riemann problem takes pointwise values in $\mathcal{U}^\delta$ if the "initial" states $(u(x,0), \gamma(x))$ take values in $\mathcal{U}^\delta$.

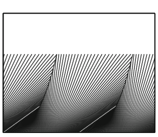
Once we have the solution of the approximate Riemann problem (8.64), we can use this to design a front-tracking scheme. To this end, let $\left\{u_0^\delta\right\}_{\delta>0}$ and $\left\{\gamma^\delta\right\}_{\delta>0}$ be two sequences of piecewise constant functions such that

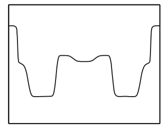
$$\left(u_0^\delta(x), \gamma^\delta(x)\right) \in \mathcal{U}^\delta \quad \text{for all but a finite number of } x\text{-values.}$$

Furthermore, we demand that

$$\lim_{\delta\to 0}\left\|u_0^\delta - u_0\right\|_1 = 0, \tag{8.65}$$

$$\lim_{\delta\to 0}\left\|\gamma^\delta - \gamma\right\|_1 = 0. \tag{8.66}$$





We label the discontinuity points of γ^δ by $y_1 < \cdots < y_N$. Of course, these depend on δ, but we suppress this dependency in our notation. At each point of discontinuity of either u_0^δ or γ^δ, we have a Riemann problem whose solution will give a sequence of z-waves and γ-waves. We define the front-tracking approximation as in the scalar case, by following discontinuities, called fronts, and solve the Riemann problems (using the approximate flux g^δ) defined by their collisions. We call the resulting piecewise constant function u^δ. As in the scalar case, in order to show that we can define $u^\delta(\cdot\,, t)$ for every $t > 0$, we must study the interaction of fronts.

The front-tracking solution u^δ has two types of fronts, z-fronts and γ-fronts, where z-fronts are those fronts whose left and right γ-values are equal. Regarding the collision of two or more z-fronts, we have seen that such a collision always results in *one* z-front. Hence, the number of fronts in u^δ decreases when z-fronts collide.

Moreover, γ-fronts have zero speed (recall that these are the discontinuities of γ^δ), and therefore two γ-fronts will never collide. It remains to study collisions between z-fronts and γ-fronts. This turns out to be complicated, and simple examples show that we can have such collisions that result in three outgoing fronts. Furthermore, even if such collisions always result in two outgoing fronts, it is in general not possible to bound the total variation of u^δ independently of δ, as the next example shows.

◇ Example 8.13
Assume for the moment that

$$u_0(x) = \frac{1}{2}, \quad \gamma(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 + x & \text{for } 0 < x \le 2, \\ 2 & \text{for } 2 < x. \end{cases} \tag{8.67}$$

In this case $z(\gamma(x), u_0(x)) = 0$, and we can set

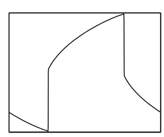
$$\gamma^\delta(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 + i\delta & \text{for } i\delta < x \le (i+1)\delta,\ i = 0, \dots, 2/(\delta - 1), \\ 2 & \text{for } 2 < x. \end{cases}$$

The z-component of the solution of each of the Riemann problems defined by $(u_0^\delta, \gamma^\delta)$ at $x = i\delta$ reads

$$(z, \gamma) = \begin{cases} (0, 1 + (i-1)\delta) & \text{for } x < i\delta, \\ (-\delta, 1 + i\delta) & \text{for } i\delta \le x < ts_i + i\delta, \\ (0, 1 + i\delta) & \text{for } i\delta + ts_i \le x, \end{cases}$$

where

$$s_i = \sqrt{\delta(1 + i\delta)}.$$



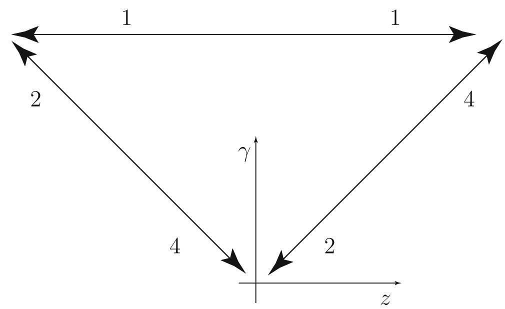


Fig. 8.12 The weights in the Temple functional, (8.68)

This follows from the diagram in Fig. 8.10. Hence, before any interaction of fronts, the total variation of u^δ reads

$$\left|u^\delta\right|_{BV} = \sum_{i=1}^{1/\delta} \sqrt{\frac{\delta}{1+i\delta}} \geq \sum_{i=1}^{1/\delta} \sqrt{\frac{\delta}{2}} = \frac{1}{\delta}\sqrt{\frac{\delta}{2}} = \frac{1}{\sqrt{2\delta}} \to \infty \quad \text{as } \delta \to 0.$$

Despite this, since $\gamma(x)$ is Lipschitz continuous, the total variation of the exact solution to this problem is uniformly bounded for $t < T$ for every finite time T; see, e.g., Kružkov [118] or Karlsen and Risebro [109]. As an indication of things to come, we observe in passing that

$$\left|z^\delta\right|_{BV} = \sum_{i=1}^{1/\delta} |\delta| = 1,$$

where $z^\delta = z(\gamma^\delta, u^\delta)$. So the total variation of the transformed variable z is uniformly bounded for this example, at least until the first interaction. ◇

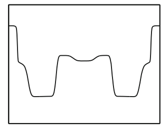
For reasons outlined in the above example and in Remark 8.12, we shall work with the z variable instead of u. In the above example, it was trivial to show that the variation of z was bounded independently of δ, but this becomes more cumbersome in general, so to help us we use the Temple functional.[2] For a single front, which we label $\mathfrak{f}$, this is defined as

$$T(\mathfrak{f}) = \begin{cases} |\Delta z| & \text{if } \mathfrak{f} \text{ is a } z\text{-front}, \\ 4\,|\Delta z| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l < z_r, \\ 2\,|\Delta z| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l > z_r, \end{cases} \tag{8.68}$$

where z_l is the z value to the left of the front, z_r the value to the right, and $\Delta z = z_r - z_l$. Fig. 8.12 will perhaps be useful later. The figure shows the weights given to $|\Delta z|$ in the various cases. Recall also that if $\mathfrak{f}$ is a γ-front, then

$$|\Delta z| = |\Delta\gamma|,$$

[2] This, or rather a similar functional, was first used in the paper of Temple [176].



and thus an alternative definition of T is

$$T(\mathfrak{f}) = \begin{cases} |\Delta z| & \text{if } \mathfrak{f} \text{ is a } z\text{-front,} \\ 4\,|\Delta\gamma| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l < z_r, \\ 2\,|\Delta\gamma| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l > z_r. \end{cases}$$

For a sequence of fronts, we define T additively, and with a slight abuse of notation we write

$$T\left(u^\delta\right) = \sum_{\mathfrak{f}\in u^\delta} T\left(\mathfrak{f}\right).$$

With this definition of T we have the obvious inequalities

$$\left|z^\delta\right|_{BV} \le T\left(u^\delta\right) \le 4\left(\left|z^\delta\right|_{BV} + \left|\gamma^\delta\right|_{BV}\right). \tag{8.69}$$

We also have for every front $\mathfrak{f} \in u^\delta$ that

$$T\left(\mathfrak{f}\right) \ge \delta.$$

With a further abuse of notation we shall write $T(t) = T(u^\delta(\,\cdot\,,t))$.

Lemma 8.14 *If* $0 < s < t$, *then*

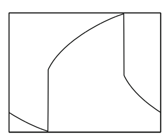
$$T(t) \le T(s). \tag{8.70}$$

Hence $\left|z^\delta(\,\cdot\,,t)\right|_{BV} \le T(0+)$.

Proof The value of T will change only when fronts collide. From the analysis of collisions of z-fronts, we have established that T does not increase at such collisions. To prove the lemma, it therefore remains to study collisions between z-fronts and γ-fronts. We say that a γ-front is nonpositive if it connects points in the half-plane $z \le 0$, and similarly, we say that it is nonnegative if it connects points in the half-plane $z \ge 0$.

We shall study the collision between z-fronts and a γ-fronts, and we thus have three points in the (z,γ)-plane, (z_l,γ_l), (z_m,γ_m), and (z_r,γ_r), which lie to the left of, in between, and to the right of the colliding fronts respectively. If we have more than one z-front colliding with the γ-front, we can reduce to the two-front collision type as follows. If we have several z-fronts colliding with the γ-front from the same side, then we can resolve the collision between the z-fronts first, and then the collision between the (single) resulting z-front and the γ-front.

Therefore, we consider the case that we have two z-fronts colliding with one γ-front. One z-front collides from the left, the other from the right. We label the states to the left of the left z-front $L = (z_l,\gamma_l)$, the one to the left of the γ-front $M_- = (z_-,\gamma_l)$, the state to the left of the right z-front $M_+ = (z_+,\gamma_r)$, and finally, the state to the right of this z-front $R = (z_r,\gamma_r)$. Of course we may have $z_l = z_-$ or $z_+ = z_r$, in which case we have only two colliding fronts. In order to study how T



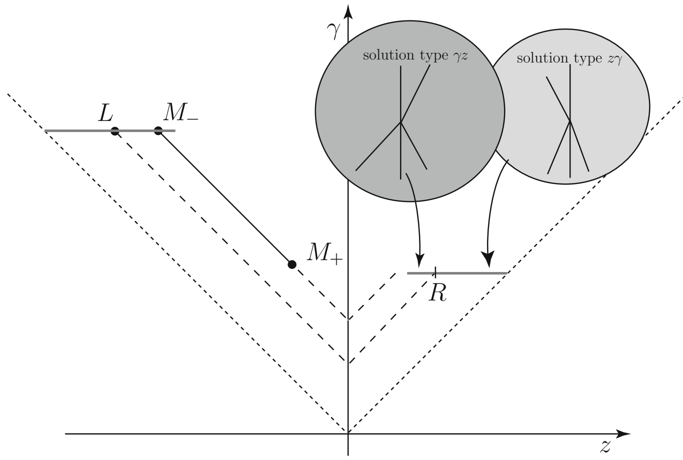


Fig. 8.13 The possible locations of L and R if the γ-front is nonpositive and $\gamma_l > \gamma_r$

changes by this collision, we study a number of cases. These are distinguished by whether the γ-front lies in the left (it is *nonpositive*) or the right (it is *nonnegative*) half-spaces and by whether $\gamma_l < \gamma_r$.

Case 1: The γ-front is nonpositive and $\gamma_l > \gamma_r$. Consult Fig. 8.13 in what follows. Now we regard the z-front, and hence M_- and M_+, as fixed. Since the γ-front is negative, $z_+ \leq 0$, and since $\gamma_l > \gamma_r$, $z_- \leq -\delta$. The z-front between z_l and z_- moves with positive speed, and it is the solution of the Riemann problem defined by these two states with a flux function $f^\delta(\gamma_l, \cdot)$. Hence z_l cannot be larger than "one breakpoint to the right" of z_-. If it were, then the solution would contain more than one front. Furthermore, $u_l = z^{-1}(\gamma_l, z_l) \geq 0$, which is the same as $z_l \geq -\gamma_l$. Thus

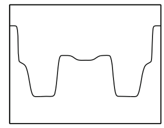
$$z_l \in [-\gamma_l, z_- + \delta].$$

This interval is indicated by the upper left horizontal gray line in Fig. 8.13. Reasoning in the same way, we see that the right z-front must have negative speed and thus that

$$z_r \in \{z_+\} \cup [-z_+ + \delta, \gamma_r].$$

This interval is indicated by the lower right horizontal gray line in Fig. 8.13. We have two alternatives. First if $-z_l + \gamma_l \geq z_r + \gamma_r$, then the solution of the Riemann problem defined by (z_l, γ_l) and (z_r, γ_r) is of type γz, and if $-z_l + \gamma_l < z_r + \gamma_r$, then this Riemann problem has a solution of type $z\gamma$. This is indicated in Fig. 8.13, where the dashed line passing through L is the line where $|z| + \gamma = -z_l + \gamma_l$.

If $z_l = z_-$, i.e., we have a collision between a γ-front and a z-front from the right, then the solution type is always $z\gamma$. In other words, the wave is transmitted. Consulting Fig. 8.12, we see that if $z_l \leq z_-$, then T is unchanged by the collision. If $z_l = z_- + \delta$ (which is the maximum value for z_l), and the solution type is $z\gamma$, then T decreases by 2δ. Otherwise, T is unchanged. In the special case that $z_r = z_- = 0$ and $z_l = z_- + \delta$, the z-front is reflected. Thus we see that a reflection results in a decrease of T by 2δ. The reader is urged to check these statements.



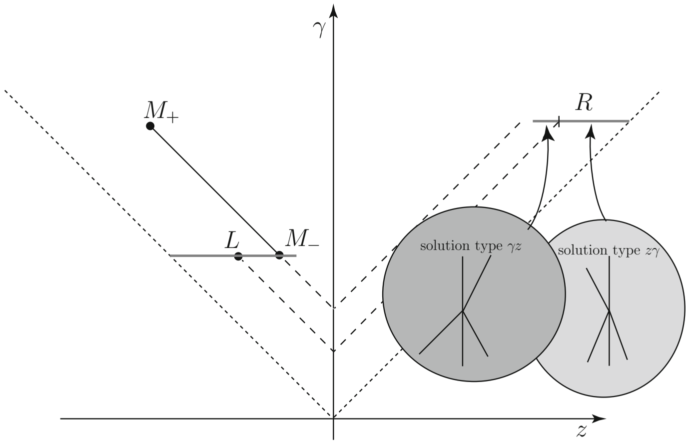


Fig. 8.14 The possible locations of L and R if the γ-front is nonpositive and $\gamma_l < \gamma_r$

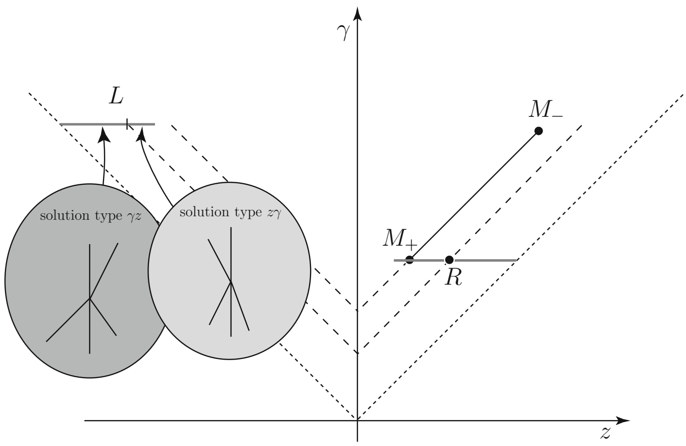
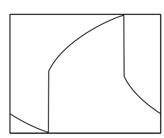


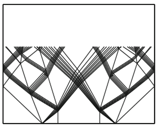
Fig. 8.15 The possible locations of L and R if the γ-front is nonnegative and $\gamma_l > \gamma_r$

Case 2: The γ-front is nonpositive and $\gamma_l < \gamma_r$. Consult Fig. 8.14 in what follows. Since the fronts are colliding, the speed of the left z-front is positive and that of the right z-front is negative. Hence $z_l \in [-\gamma_l, z_- + \delta]$ and $z_r \in \{z_+\} \cup [-z_+ + \delta, \gamma_r]$. These intervals are indicated in Fig. 8.14 by the lower left and upper right horizontal lines. If $z_r + \gamma_r < -z_l + \gamma_l$, then the solution type is $z\gamma$, and if $z_r + \gamma_r \geq -z_l + \gamma_l$, the solution type is γz. In both of these cases T is unchanged. If $z_r = z_+$, then the solution type is γz, and if $z_l = z_-$, then the solution type is $z\gamma$. Thus there are no reflected fronts in this case.

Case 3: The γ-front is nonnegative and $\gamma_l > \gamma_r$. Consult Fig. 8.15 in what follows. This case is similar to Case 2 above. By considering the speeds of the colliding fronts, we find that

$$z_l \in [-z_- - \delta, -\gamma_l] \cup \{z_-\} \quad \text{and} \quad z_r \in [z_+ - \delta, \gamma_r].$$





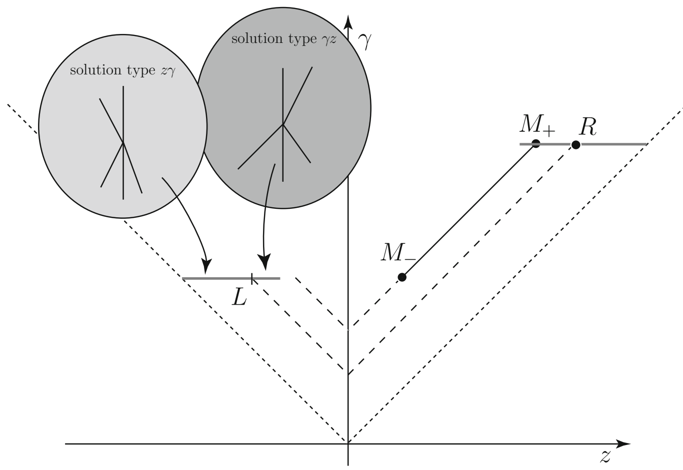


Fig. 8.16 The possible locations of L and R if the γ-front is nonnegative and $\gamma_l < \gamma_r$

If $|z_l| + \gamma_l < z_r + \gamma_r$, then the solution is of type γz, and if $|z_l| + \gamma_l \geq z_r + \gamma_r$, the solution is of type $z\gamma$. Note that if $z_r = z_+$, then the solution type is γz, while if $z_l = z_-$, the solution type is $z\gamma$. So also in this case a front cannot be reflected. Furthermore, T is unchanged.

Case 4: The γ-front is nonnegative and $\gamma_l < \gamma_r$. Consult Fig. 8.16 in what follows. This case is similar to Case 1 above. We find that

$$z_l \in [-z_- - \delta, -\gamma_l] \cup \{z_-\} \quad \text{and} \quad z_r \in [z_+ - \delta, \gamma_r].$$

If $|z_l| + \gamma_l > z_r + \gamma_r$, then the solution type is $z\gamma$, while if $|z_l| + \gamma_l \leq z_r + \gamma_r$, the type is γz. If $z_r = z_+ - \delta$ and the solution type is γz, then T decreases by 2δ; otherwise, it is unchanged. If $z_+ = z_r$, then the solution type is $z\gamma$, while if $z_l = z_-$ and $z_r = z_+ - \delta$, we have a reflection, and in this case T decreases by 2δ.

This finishes the proof of Lemma 8.14. □

Remark 8.15 Recall that we have used the term "reflection" for a collision between a z-front and a γ-front if the z-front collides from the left and the solution of the Riemann problem is of type $z\gamma$, or if the z-front collides from the right and the solution type is γz. From the proof of the above lemma, it is clear that whenever we have a reflection, T decreases by 2δ. Hence, if $T(0+)$ is finite, we can have only a finite number of reflections in u^δ.

One immediate consequence of Lemma 8.14 and (8.69) is the following result.

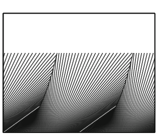
Corollary 8.16 *If*

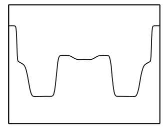
$$\left|\gamma^\delta\right|_{BV} \leq |\gamma|_{BV} \quad \textit{and} \quad \left|z(u_0^\delta, \gamma^\delta)\right|_{BV} \leq |z(u_0, \gamma)|_{BV}\,, \tag{8.71}$$

then for $t \geq 0$,

$$\left|z^\delta(\,\cdot\,, t)\right|_{BV} \leq |z(u_0, \gamma)|_{BV} + 4\,|\gamma|_{BV}\,,$$

and thus $\left|z^\delta(\,\cdot\,, t)\right|_{BV}$ *is bounded independently of* δ *and* t.





Note that this corollary in itself does not imply that the front-tracking construction u^δ can be defined up to an arbitrary time t. In order to show this, we have to do some more work. For a z-front $\mathfrak{f}_z$ let $\mathcal{A}(\mathfrak{f}_z)$ be the set of γ-fronts $\mathfrak{f}_\gamma$ that approach $\mathfrak{f}_z$, i.e.,

$$\mathfrak{f}_\gamma \in \mathcal{A}(\mathfrak{f}_z) \quad \text{if} \quad \begin{cases} x(\mathfrak{f}_z) < x(\mathfrak{f}_\gamma) & \text{and } s(\mathfrak{f}_z) \geq 0 \text{ or} \\ x(\mathfrak{f}_z) > x(\mathfrak{f}_\gamma) & \text{and } s(\mathfrak{f}_z) \leq 0, \end{cases}$$

where $x(\mathfrak{f})$ denotes the position of $\mathfrak{f}$, and $s(\mathfrak{f})$ its speed. For every z-front $\mathfrak{f}_z$ define

$$J(\mathfrak{f}_z) = \sum_{\mathfrak{f}_\gamma \in \mathcal{A}(\mathfrak{f}_z)} |\Delta\gamma|, \tag{8.72}$$

where $\Delta\gamma$ denotes the difference in γ over the front.

Lemma 8.17 *Assume that (8.71) holds. Then for each fixed δ, the functional*

$$F(t) = \delta \sum_{\mathfrak{f}_z} J(\mathfrak{f}_z) + T(t)\, |\gamma|_{BV} \tag{8.73}$$

is nonincreasing, and it decreases by at least δ^2 when a z-front collides with a γ-front.

Proof Let $N_\mathfrak{f}(t)$ denote the number of fronts in u^δ at time t. For each front we have $|\Delta z| \geq \delta$, and thus

$$N_\mathfrak{f} \leq \frac{\left|z^\delta\right|_{BV}}{\delta}.$$

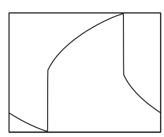
Recall that T is bounded and $J(\mathfrak{f}_z) \leq |\gamma|_{BV}$. Hence F is bounded by

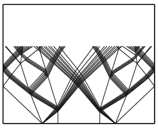
$$\begin{aligned} F(t) &\leq \delta\, |\gamma|_{BV}\, N_\mathfrak{f} + 2T(0+)\, |\gamma|_{BV} \\ &\leq 4\, |\gamma|_{BV} \left(|z(u_0, \gamma)|_{BV} + 4\, |\gamma|_{BV}\right). \end{aligned} \tag{8.74}$$

Thus F is bounded independently of δ and t. We must show that F is decreasing by at least δ^2 for collisions between z-fronts and γ-fronts, and nonincreasing when z-fronts collide.

First consider a collision between one (or two) z-front(s) and a γ-front. From the proof of Lemma 8.14 we saw that either (**a**) a z-front "passes through" the γ-front in the collision, or (**b**) we have a reflection, and T decreases by 2δ. If (**a**) holds, then the sum in (8.73) will "lose" at least one term (two terms if one z-front is lost in the collision) of size $|\Delta\gamma|$, and the second term in (8.73), does not increase. Thus F decreases by at least $\delta\, |\Delta\gamma| \geq \delta^2$. If (**b**) holds, then T decreases by 2δ, and the sum increases by at most $|\gamma|_{BV}$. Hence F decreases by a least $\delta\, |\gamma|_{BV} \geq \delta^2$.

Next we consider a collision between two (or more) z-fronts. Recall that this collision will result in one z-front. If more than two fronts collide, we can consider this as several collisions between two fronts occurring at the same point. Therefore,





we consider a collision between two z-fronts, $\mathfrak{f}_l$ and $\mathfrak{f}_r$, separating values z_l, z_m and z_r. We label the resulting front $\mathfrak{f}$. If z_m is between z_l, and z_r, then T does not change by the collision. However, the speed of $\mathfrak{f}$ is between the speeds of $\mathfrak{f}_l$ and $\mathfrak{f}_r$. If the speed of $\mathfrak{f}$ is different from 0, then $\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_l)$ or $\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_r)$. Hence the sum in (8.73) loses one term, and F decreases by at least δ^2. If the speed of $\mathfrak{f}$ is 0, then the speed of $\mathfrak{f}_l$ is positive, and the speed of $\mathfrak{f}_r$ negative, whence

$$\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_l) \cup \mathcal{A}(\mathfrak{f}_r),$$

and thus F is constant.

If z_m is not between z_l and z_r, then either $z_r = z_m - \delta$ or $z_l = z_m + \delta$. This is so because g^δ is convex. In this case T decreases by δ, and the first term in equation (8.73) increases by at most $\delta \left|\gamma^\delta\right|_{BV}$. This concludes the proof of the lemma. □

Note that an immediate consequence of equation (8.74) and Lemma 8.17 is that for a fixed δ, the number of collisions of z-fronts and γ-fronts is bounded by

$$4\left|\gamma\right|_{BV} \frac{\left|z(u_0,\gamma)\right|_{BV} + 4\left|\gamma\right|_{BV}}{\delta^2}.$$

Also, the *smallest* absolute value of the speed of any z-front having speed different from zero is bounded below by

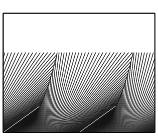
$$\sqrt{\min\left(\gamma^\delta\right)\delta}.$$

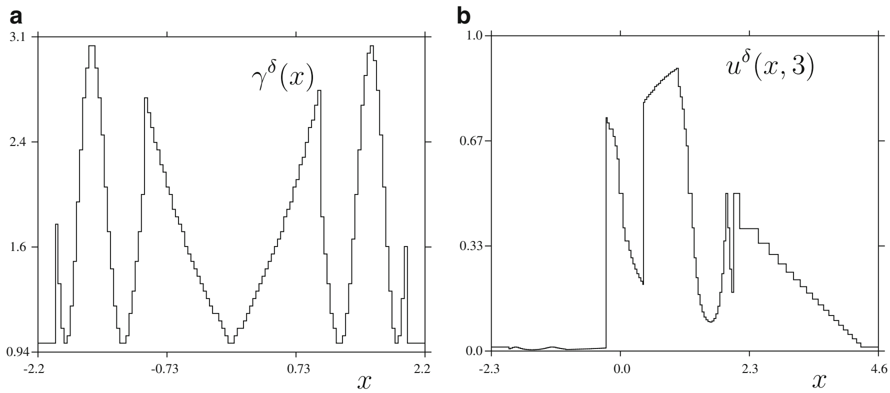
Hence, after some *finite* time T_1, collisions between z-fronts and γ-fronts cannot occur. This means that there must be a time $T_2 \geq T_1$ such that all z-fronts in the interval (y_1, y_N) (recall that γ^δ has discontinuities at $y_1, \dots, y_N$) have zero speed, that all z-fronts to the left of y_1 have nonpositive speed and that the z-fronts to the right of y_N have nonnegative speed for all $t > T_2$. Outside the interval $[y_1, y_N]$, u^δ is the front-tracking approximation to a scalar conservation law with a constant coefficient, and there can be only a finite number of collisions between fronts in u^δ there. Therefore, there exists a finite time $T_3 \geq T_2$ such that there will be no further collisions between fronts in u^δ for $t > T_3$. Thus, the front-tracking method is hyperfast.

◇ Example 8.18

Now we pause for a moment in order to exhibit an example of how front tracking looks in practice. We wish to find the front-tracking approximation to the initial value problem

$$\begin{aligned} &u_t + [4\gamma(x)u(1-u)]_x = 0, \quad t > 0, \\ &\gamma(x) = \begin{cases} e^{|x|} & \text{for } -1 \leq x \leq 1, \\ \sin(\pi x^2) + 2 & \text{for } 1 < |x| < 2, \\ 1 & \text{otherwise,} \end{cases} \\ &u(x,0) = \begin{cases} \frac{1}{2}(1 + e^{-|x|}) & \text{for } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.75}$$





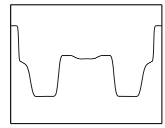


Fig. 8.17 **a** $\gamma^\delta(x)$. **b** $u^\delta(x,3)$ for $\delta = 0.05$

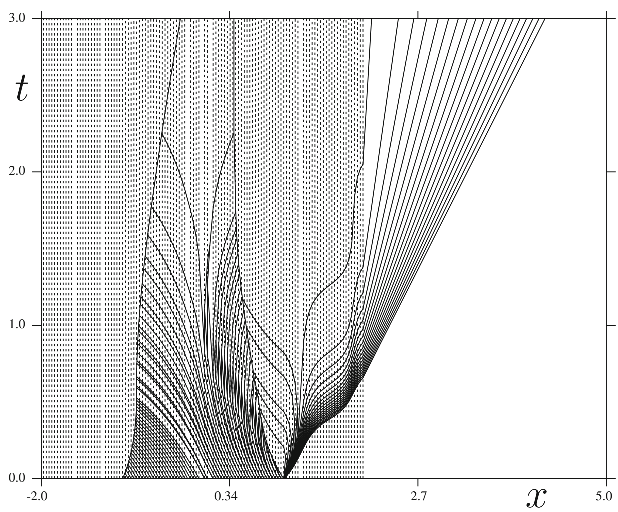


Fig. 8.18 The fronts in the (x,t)-plane for Example 8.18

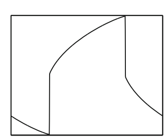
In Fig. 8.17, we show the approximation γ^δ for $\delta = 0.05$, and $u^\delta(\cdot,3)$. In Fig. 8.18, we show the fronts in u^δ in the (x,t)-plane. Here z-fronts are marked with solid lines, and γ-fronts with dashed lines. We see that the number of fronts decreases rapidly, and there do not seem to be many collisions after $t = 3$. ◇

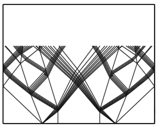
Returning now to the more general case, we claim that the sequence $\{z^\delta\}_{\delta>0}$ satisfies the following bounds:

$$\left\|z^\delta\right\|_{L^\infty(\mathbb{R})} \le \left\|\gamma^\delta\right\|_{L^\infty(\mathbb{R})} \le C, \tag{8.76}$$

$$\left\|z^\delta(\cdot,t)\right\|_{L^1_{\mathrm{loc}}} \le C, \quad t < T, \tag{8.77}$$

$$\|z(\cdot,t) - z(\cdot,s)\|_{L^1(\mathbb{R})} \le C(t-s), \tag{8.78}$$





where the constant C does not depend on t or on δ. The first bound (8.76) follows by the definition of z, (8.60), and the fact that u^δ takes values in the interval $[0,1]$. Regarding (8.77), we have that u^δ is a weak solution of

$$u_t^\delta + g^\delta\left(\gamma^\delta, u^\delta\right)_x = 0, \quad u^\delta(x,0) = u_0^\delta(x). \tag{8.79}$$

Thus we can repeat the argument used in the proof of Theorem 7.10, to obtain

$$\begin{aligned}\left\|u^\delta(\cdot,t) - u^\delta(\cdot,s)\right\|_{L^1(\mathbb{R})} &\le \max_{\tau\in[s,t]} \left|g^\delta\left(\gamma^\delta, u^\delta(\cdot,\tau)\right)\right|_{BV}(t-s) \\ &\le \max_{\tau\in[s,t]} \left|z^\delta(\cdot,\tau)\right|_{BV}(t-s) \\ &\le C(t-s),\end{aligned} \tag{8.80}$$

for some constant not depending on t, s, or δ. Setting $s=0$, we obtain

$$\left\|u^\delta(\cdot,t)\right\|_{L^1(\mathbb{R})} \le \left\|u_0^\delta\right\|_{L^1(\mathbb{R})} + Ct, \tag{8.81}$$

and thus $u^\delta(\cdot,t)$ is in $L^1(\mathbb{R})$ for all finite t. Now

$$\begin{aligned}\left|z\left(u^\delta,\gamma^\delta\right)\right| &= \left|z\left(0,\gamma^\delta\right) + z_u\left(\xi,\gamma^\delta\right)u^\delta\right| \\ &\le \left|\gamma^\delta\right| + C\left|u^\delta\right|,\end{aligned}$$

for some positive constant C, where ξ is in the interval $[0,u^\delta]$. Since γ^δ is in L^1_{loc}, equation (8.77) follows. Actually, in our case, since $u^\delta(x,t)\in[0,1]$, we have that

$$\left\|u^\delta(\cdot,t)\right\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} u^\delta(x,t)\,dx = \int_{\mathbb{R}} u_0^\delta(x)\,dx = \left\|u_0^\delta\right\|_1,$$

which is stronger than (8.81).

To prove (8.78) we use the equality

$$\begin{aligned}z^\delta(x,t) - z^\delta(x,s) &= z\left(u^\delta(x,t),\gamma^\delta\right) - z\left(u^\delta(x,s),\gamma^\delta\right) \\ &= z_u\left(\xi,\gamma^\delta\right)\left(u^\delta(x,t) - u^\delta(x,s)\right).\end{aligned}$$

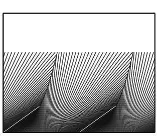
Since z_u is bounded, by (8.80) the bound (8.78) holds.

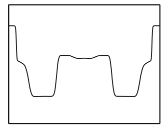
Hence, by standard techniques as in the case with constant coefficients, it follows that there exists a subsequence of $\{\delta\}$ (which we also label $\{\delta\}$) and a function $z\in L^1_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))\cap L^\infty((0,\infty);BV(\mathbb{R}))$ such that

$$\lim_{\delta\to 0} z^\delta = z \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T]). \tag{8.82}$$

Since $z^\delta = z(u^\delta,\gamma^\delta)$, it also follows that there is a function $u\in L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T])$ such that $u^\delta\to u$, and $u = z^{-1}(z,\gamma)$. Furthermore, for this subsequence also $g^\delta(\gamma^\delta,u^\delta)\to f(\gamma,u)$. Thus

$$\begin{aligned}\lim_{\delta\to 0}&\iint\left(u^\delta\varphi_t + g^\delta(\gamma^\delta,u^\delta)\varphi_x\right)dx\,dt \\ &= \iint\left(u\varphi_t + f(\gamma,u)\varphi_x\right)dx\,dt,\end{aligned}$$





and by construction,

$$\lim_{\delta\to 0}\int_{\mathbb{R}} u^\delta(x,0)\varphi(x,0)\,dx = \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx.$$

Since u^δ is a weak solution to (8.79), it follows from this that u is a weak solution to (8.53).

Furthermore, it is transparent that although we performed the analysis for $f(\gamma,u) = 4\gamma u(1-u)$, our results could be (slightly) extended to include flux functions that are similar to f. To be precise, assume that:

A.1 There is an interval $[a,b]$ such that $f(\gamma,a) = f(\gamma,b) = C$ for all γ.
A.2 There is a point $u^\star(\gamma) \in (a,b)$ such that $f_u(\gamma,u) > 0$ for $a < u < u^\star(\gamma)$ and $f_u(\gamma,u) < 0$ for $u^\star(\gamma) < u < b$.
A.3 The map $\gamma \mapsto f(\gamma,u)$ is strictly monotone for all $u \in (a,b)$.
A.4 The flux function f belongs to $C^2(\mathbb{R}\times[a,b])$.

If f satisfies these assumptions, we can define the mapping z as

$$z(\gamma,u) = \operatorname{sign}\left(u-u^\star(\gamma)\right)\left(f(\gamma,u^\star(\gamma)) - f(\gamma,u)\right), \tag{8.83}$$

and use this to show that the front-tracking approximation is well defined. This analysis is only a slight modification of the analysis in the case $f(\gamma,u) = 4\gamma u(1-u)$. Hence, mutatis mutandis, we have proved the following theorem.

Theorem 8.19 *Let f be a function satisfying A.1–A.4, and assume that $u_0(x)$ is a function in L^1_{loc} taking values in the interval $[a,b]$, and that γ is a function in $BV(\mathbb{R}) \cup L^1_{\mathrm{loc}}(\mathbb{R})$. Then there exists a weak solution to the initial value problem*

$$u_t + f(\gamma,u)_x = 0, \quad x\in\mathbb{R} \quad t>0, \qquad u(x,0) = u_0(x).$$

Furthermore, this solution is the limit of a sequence of front-tracking approximations.

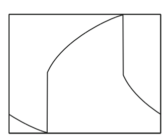
An Entropy Inequality

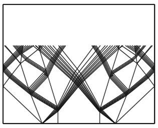
Now we shall show that the limit of every front-tracking approximation to the general conservation law (8.53) satisfies a Kružkov-type entropy condition. Thus we let u^δ be a weak solution to the approximate problem

$$\begin{cases} u^\delta_t + g^\delta\left(\gamma^\delta,u^\delta\right)_x = 0, & x\in\mathbb{R} \quad t>0,\\ \quad u^\delta(x,0) = u^\delta_0(x), & x\in\mathbb{R}, \end{cases} \tag{8.84}$$

where $g^\delta(\gamma,\cdot)$ is a piecewise linear continuous approximation of $f(\gamma,u)$ such that $g^\delta \to f$ as $\delta\to 0$. Here γ^δ is a piecewise constant approximation to γ, such that $\gamma^\delta\to\gamma$ in L^1 as $\delta\to 0$. We *assume* that u^δ can be constructed by front tracking, and that for each fixed $T>0$,

$$u^\delta \to u \text{ in } L^1(\mathbb{R}\times[0,T]) \text{ as } \delta\to 0. \tag{8.85}$$





Furthermore, we let

$$z(\gamma, u) = \int_0^u |f_u(\gamma, v)| \, dv, \tag{8.86}$$

and set $z^\delta = z(\gamma^\delta, u^\delta)$. We shall also assume that for each t the family $\{z^\delta(\,\cdot\,, t)\}$ is a sequence of uniformly bounded variation in x and satisfies the three basic estimates (8.76), (8.77), and (8.78), so that we have convergence of z^δ along a subsequence.

Using that u^δ is a weak solution to (8.84), it is not hard to show that u is a weak solution to (8.53) if $u_0^\delta \to u$ as $\delta \to 0$. We would like to show that the limit u satisfies a generalization of the Kružkov entropy condition. Recall that if γ is continuous, then an entropy solution to (8.53) in the strip $\Pi_T = \mathbb{R} \times [0, T]$ satisfies

$$\begin{aligned} &\iint_{\Pi_T} \left(|u - c| \, \varphi_t + F(\gamma, u, c)\varphi_x \right) dx \, dt \\ &\quad - \iint_{\Pi_T} \operatorname{sign}(u - c) \, \partial_x f(\gamma, c)\varphi \, dx \, dt + \int_{\mathbb{R}} |u_0(x) - c| \, \varphi(x, 0) \, dx \geq 0, \end{aligned} \tag{8.87}$$

for all constants c and all nonnegative test functions φ such that $\varphi(\,\cdot\,, T) = 0$. Here F is the Kružkov entropy flux defined by

$$F(\gamma, u, c) = \operatorname{sign}(u - c) \left(f(\gamma, u) - f(\gamma, c) \right). \tag{8.88}$$

We would like to show that the front-tracking limit u satisfies (8.87) if γ is continuous, and if γ has discontinuities, find a suitable generalization that is satisfied by the front-tracking limit. The condition (8.87) does not make sense for discontinuous γ's, since the second integral is undefined.

We shall assume that γ is piecewise continuous on a *finite* number of intervals, i.e., that γ has a finite number of discontinuities. We call this set of discontinuities $\mathcal{D}_\gamma = \{\xi_0, \dots, \xi_N\}$, and we assume that $\gamma(x)$ is continuously differentiable for $x \notin \mathcal{D}_\gamma$. Thus γ and γ' have left and right limits at each discontinuity point $\xi_i \in \mathcal{D}_\gamma$.

Next, we shall require that the approximation $\gamma^\delta(x)$ also have discontinuity points for all $x \in \mathcal{D}_\gamma$ for all relevant δ. In addition to these discontinuities, for a fixed δ, γ^δ has discontinuities at $\{y_{i,j}\}$. These are ordered so that

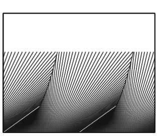
$$\xi_i = y_{i,0} < y_{i,1} < \cdots < y_{i,N_i} < y_{i,N_i+1} = \xi_{i+1},$$

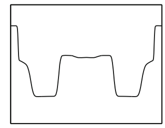
for $i = 0, \dots, N$. Let $\gamma_{i,j+1/2}$ denote the value of γ^δ in the interval $(y_{i,j}, y_{i,j+1})$, and set

$$\Delta x_{i,j} = \frac{1}{2} \left(y_{i,j+1} - y_{i,j-1} \right), \quad j = 1, \dots, N_i.$$

Of course, these quantities all depend on δ, but for simplicity we omit this in our notation. We also assume that

$$\lim_{\delta \to 0} \frac{g^\delta\left(\gamma_{i,j+1/2}, c\right) - g^\delta\left(\gamma_{i,j-1/2}, c\right)}{\Delta x_{i,j}} \chi_{I_{i,j}}(x) = \frac{\partial f\left(\gamma(x), c\right)}{\partial x}, \tag{8.89}$$





where $\chi_{I_{i,j}}$ denotes the characteristic function of the interval

$$I_{i,j} = \left(\frac{y_{i,j-1} + y_{i,j}}{2}, \frac{y_{i,j} + y_{i,j+1}}{2} \right).$$

This is not unreasonable, since γ is continuously differentiable in (ξ_i, ξ_{i+1}). In what follows, we let $u_i^{\mp}$ and $u_{i,j}^{\mp}$ denote the left and right limits of u^δ at the points ξ_i and $y_{i,j}$, respectively. Since $u^\delta(\,\cdot\,, t)$ is piecewise constant, these limits exist.

In each interval $(y_{i,j}, y_{i,j+1})$ the function u^δ is an entropy solution of the conservation law

$$u_t^\delta + g^\delta(\gamma_{i,j+1/2}, u^\delta)_x = 0,$$

and hence

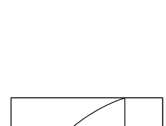
$$\begin{aligned} -\int_0^T \int_{y_{i,j}}^{y_{i,j+1}} &\left(\left| u^\delta - c \right| \varphi_t + F^\delta \left(\gamma_{i,j+1/2}, u^\delta, c \right) \varphi_x \right) dx\, dt \\ &+ \int_0^T \left(F^\delta \left(\gamma_{i,j+1/2}, u_{i,j+1}^-, c \right) \varphi \left(y_{i,j+1}, t \right) - F^\delta \left(\gamma_{i,j+1/2}, u_{i,j}^+, c \right) \varphi \left(y_{i,j}, t \right) \right) dt \\ &\qquad - \int_{y_{i,j}}^{y_{i,j+1}} \left| u^\delta(x,0) - c \right| \varphi(x,0)\, dx \le 0, \end{aligned} \tag{8.90}$$

where

$$F^\delta(\gamma, u, c) = \operatorname{sign}(u - c) \left(g^\delta(\gamma, u) - g^\delta(\gamma, c) \right).$$

Summing this for $j = 0, \dots, N_i$, we find that

$$\begin{aligned} -\int_0^T \int_{\xi_i}^{\xi_{i+1}} &\left(\left| u^\delta - c \right| \varphi_t + F^\delta \left(\gamma^\delta, u^\delta, c \right) \varphi_x \right) dx\, dt - \int_{\xi_i}^{\xi_{i+1}} \left| u^\delta(x,0) - c \right| \varphi(x,0)\, dx \\ &+ \int_0^T \left(F^\delta \left(\gamma_{i,N_i+1/2}, u_{i+1}^-, c \right) \varphi(\xi_i, t) - F^\delta \left(\gamma_{i,1/2}, u_i^+, c \right) \varphi(\xi_{i+1}, t) \right) dt \\ &- \int_0^T \sum_{j=1}^{N_i} \left[F^\delta \left(\gamma_{i,j+1/2}, u_{i,j}^+, c \right) - F^\delta \left(\gamma_{i,j-1/2}, u_{i,j}^-, c \right) \right] \varphi(y_{i,j}, t)\, dt \\ &\le 0. \end{aligned} \tag{8.91}$$



Regarding the terms in the integrand in the last term in (8.91), we can write

$$F^\delta\left(\gamma_{i,j+1/2},u^+_{i,j},c\right)-F^\delta\left(\gamma_{i,j-1/2},u^-_{i,j},c\right)$$
$$=\begin{cases} -\operatorname{sign}\left(u^+_{i,j}-c\right)\left[f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right] \\ \qquad +\left\{\operatorname{sign}\left(u^+_{i,j}-c\right)-\operatorname{sign}\left(u^-_{i,j}-c\right)\right\}\left(f^\times_{i,j}-f\left(\gamma_{i,j-1/2}\right)\right) \\ \text{or} \\ -\operatorname{sign}\left(u^-_{i,j}-c\right)\left[f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right] \\ \qquad +\left\{\operatorname{sign}\left(u^+_{i,j}-c\right)-\operatorname{sign}\left(u^-_{i,j}-c\right)\right\}\left(f^\times_{i,j}-f\left(\gamma_{i,j+1/2}\right)\right), \end{cases}$$

where $f^\times_{i,j}=f(\gamma_{i,j+1/2},u^+_{i,j})=f(\gamma_{i,j-1/2},u^-_{i,j})$. If $\operatorname{sign}(u^+_{i,j}-c)=\operatorname{sign}(u^-_{i,j}-c)$, the last terms in the above expressions are zero, while if $u^-_{i,j}\le c\le u^+_{i,j}$, then since these values are chosen according to the minimal jump entropy condition (8.25), we have that

$$\operatorname{sign}\left(u^+_{i,j}-c\right)-\operatorname{sign}\left(u^-_{i,j}-c\right)=2 \quad\text{and}\quad \begin{cases} f(\gamma_{i,j-1/2},c)\ge f^\times_{i,j} & \text{or} \\ f(\gamma_{i,j+1/2},c)\ge f^\times_{i,j}, \end{cases}$$

and thus in this case one of the last terms must be nonpositive. If $u^+_{i,j}<c<u^-_{i,j}$, we use (8.26) to conclude that

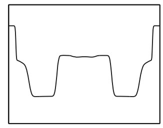
$$\operatorname{sign}\left(u^+_{i,j}-c\right)-\operatorname{sign}\left(u^-_{i,j}-c\right)=-2 \quad\text{and}\quad \begin{cases} f(\gamma_{i,j-1/2},c)\le f^\times_{i,j} & \text{or} \\ f(\gamma_{i,j+1/2},c)\le f^\times_{i,j}, \end{cases}$$

and again we find that one of the last terms is nonpositive. If the first of these last terms is nonpositive for c between $u^-_{i,j}$ and $u^+_{i,j}$, we define $u_{i,j}=u^\delta(y_{i,j},t)=u^+_{i,j}$. Otherwise, we define $u_{i,j}=u^\delta(y_{i,j},t)=u^-_{i,j}$. Using these observations, we find that

$$\begin{aligned} -\int_0^T\int_{\xi_i}^{\xi_{i+1}}&\left(\left|u^\delta-c\right|\varphi_t+F^\delta\left(\gamma^\delta,u^\delta,c\right)\varphi_x\right)dx\,dt-\int_{\xi_i}^{\xi_{i+1}}\left|u^\delta(x,0)-c\right|\varphi(x,0)\,dx \\ &+\int_0^T\left(F^\delta\left(\gamma_{i,N_i+1/2},u^-_{i+1},c\right)\varphi\left(\xi_i,t\right)-F^\delta\left(\gamma_{i,1/2},u^+_i,c\right)\varphi\left(\xi_{i+1},t\right)\right)dt \\ &+\int_0^T\sum_{j=1}^{N_i}\operatorname{sign}\left(u_{i,j}-c\right)\left[f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right]\varphi\left(y_{i,j},t\right)\,dt \\ \le 0. \end{aligned} \tag{8.92}$$

Now $u_{i,j}=u^\delta(y^-_{i,j},\cdot)$ or $u_{i,j}=u^\delta(y^+_{i,j},\cdot)$; hence if we define $\bar{u}^\delta(x,t)=u_{i,j}(t)\chi_{I_{i,j}}(x)$, and set $\bar{z}^\delta=z^\delta_{i,j}(t)\chi_{I_{i,j}}$, we have that

$$\bar{z}^\delta\left(y_{i,j},t\right)=z^\delta\left(y_{i,j},t\right).$$



Now we claim that the sequence $\{\bar{z}^\delta\}$ is compact in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T])$. Trivially we have that

$$\left\|\bar{z}^\delta\right\|_{L^\infty(\mathbb{R})} \le \left\|z^\delta\right\|_{L^\infty(\mathbb{R})} < C \tag{8.93}$$

and

$$\left|\bar{z}^\delta(\,\cdot\,,t)\right|_{BV} \le \left|z^\delta(\,\cdot\,,t)\right|_{BV} \le C. \tag{8.94}$$

Furthermore,

$$\begin{aligned}
\left\|\bar{z}^\delta(\,\cdot\,,t)-z^\delta(\,\cdot\,,t)\right\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \left|\bar{z}^\delta(x,t)-z^\delta(x,t)\right|\,dx\\
&= \sum_{i,j}\int_{y_{i,j-1/2}}^{y_{i,j+1/2}} \left|z^\delta\left(y_{i,j},t\right)-z^\delta(y,t)\right|\,dy\\
&\le \sum_{i,j}\int_{y_{i,j-1/2}}^{y_{i,j+1/2}}\int_{y}^{y_{i,j}} \left|z^\delta_x(x,t)\right|\,dx\,dy\\
&\le \max_{i,j}\left|\Delta x_{i,j}\right|\left|z^\delta(\,\cdot\,,t)\right|_{BV}.
\end{aligned}$$

Setting $\Delta x=\max_{i,j}\Delta x_{i,j}$, we therefore find that

$$\begin{aligned}
\left\|\bar{z}^\delta(\,\cdot\,,t)-\bar{z}^\delta(\,\cdot\,,s)\right\|_{L^1(\mathbb{R})} &\le \left\|z^\delta(\,\cdot\,,t)-z^\delta(\,\cdot\,,s)\right\|_{L^1(\mathbb{R})} + 2\Delta x\left|z^\delta(\,\cdot\,,t)\right|_{BV}\\
&\le C((t-s)+\Delta x).
\end{aligned} \tag{8.95}$$

By the bounds (8.93), (8.94), and (8.95), the sequence $\{\bar{z}^\delta\}$ converges along a subsequence (also labeled δ), and

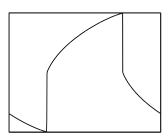
$$\lim_{\delta\to 0}\bar{z}^\delta = \lim_{\delta\to 0} z^\delta = z.$$

Therefore, also $\lim_{\delta\to 0}\bar{u}^\delta = u$. Now define

$$\Delta_x g^\delta(x,c) = \frac{1}{\Delta x_{i,j}}\left(f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right),\quad \text{for } x\in I_{i,j}.$$

Using this notation, the inequality (8.92) reads

$$\begin{aligned}
&-\int_0^T\int_{\xi_i}^{\xi_{i+1}}\left(\left|u^\delta-c\right|\varphi_t + F^\delta\left(\gamma^\delta,u^\delta,c\right)\varphi_x\right)dx\,dt - \int_{\xi_i}^{\xi_{i+1}}\left|u^\delta(x,0)-c\right|\varphi(x,0)\,dx\\
&\qquad -\int_0^T\left(F^\delta\left(\gamma_i^+,u_i^+,c\right)\varphi\left(\xi_i,t\right)-F^\delta\left(\gamma_{i+1}^-,u_{i+1}^-,c\right)\varphi\left(\xi_{i+1},t\right)\right)dt\\
&\qquad +\int_0^T\int_{\xi_i}^{\xi_{i+1}}\operatorname{sign}\left(\bar{u}^\delta-c\right)\Delta_x g^\delta(y,c)\sum_{j=1}^{N_i}\varphi\left(y_{i,j},t\right)\chi_{I,j}(y)\,dy\,dt\\
&\quad\le 0.
\end{aligned} \tag{8.96}$$



Now we can add this for $i = 0, \dots, M$ to obtain

$$
\begin{aligned}
&-\iint\limits_{\Pi_T} \left(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta, u^\delta, c\right)\varphi_x \right) dx\,dt - \int\limits_{\mathbb{R}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\,dx \\
&\qquad - \int_0^T \sum_{i=1}^{M} \left[F^\delta\left(\gamma_i^+, u_i^+, c\right) - F^\delta\left(\gamma_i^-, u_i^-, c\right) \right] \varphi\left(\xi_i, t\right)\,dt \\
&\qquad + \int_0^T \sum_{i=0}^{M} \int_{\xi_i}^{\xi_{i+1}} \operatorname{sign}\left(\bar{u}^\delta - c\right) \Delta_x g^\delta(y,c) \sum_{j=1}^{N_i} \varphi\left(y_{i,j}, t\right) \chi_{I_{i,j}}(y)\,dy\,dt \\
&\quad \le 0. \qquad (8.97)
\end{aligned}
$$

At this point it is convenient to state the following general lemma.

Lemma 8.20 *Let $\Omega \in \mathbb{R}$ be a bounded open set, $g \in L^1(\Omega)$, and suppose that $g_n(x) \to g(x)$ almost everywhere. Then there exists a set $\Theta \subseteq \mathbb{R}$, which is a most countable, such that for every $c \in \mathbb{R} \setminus \Theta$,*

$$\operatorname{sign}\left(g_n(x) - c\right) \to \operatorname{sign}\left(g(x) - c\right) \quad \textit{a.e. in } \Omega.$$

Furthermore, let $c \in \Theta$ and define

$$\mathcal{I}_c = \left\{ x \in \Omega \mid g(x) = c \right\}.$$

Then it is possible to define sequences $\{\underline{c}_m\}_{m=1}^\infty \subset \mathbb{R} \setminus \Theta$ and $\{\bar{c}_m\}_{m=1}^\infty \subset \mathbb{R} \setminus \Theta$ such that

$$\underline{c}_m \uparrow c \quad \textit{and} \quad \operatorname{sign}\left(g(x) - \underline{c}_m\right) \to \operatorname{sign}\left(g(x) - c\right) \quad \textit{a.e. in } \Omega \setminus \mathcal{I}_c, \qquad (8.98)$$

$$\bar{c}_m \downarrow c \quad \textit{and} \quad \operatorname{sign}\left(g(x) - \bar{c}_m\right) \to \operatorname{sign}\left(g(x) - c\right) \quad \textit{a.e. in } \Omega \setminus \mathcal{I}_c, \qquad (8.99)$$

as $m \to \infty$.

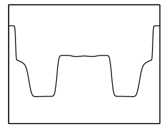
Proof Fix $c \in \mathbb{R}$ and a point $x \in \Omega$ such that $g_n(x) \to g(x)$ and $g(x) \neq c$. For sufficiently large n, $\operatorname{sign}(g_n(x) - c) = \operatorname{sign}(g(x) - c)$, i.e., $\operatorname{sign}(g_n(x) - c)$ is constant in n, and therefore converges to the correct limit. Thus for each $c \in \mathbb{R}$, $\operatorname{sign}(g_n(x) - c) \to \operatorname{sign}(g(x) - c)$ almost everywhere in $\Omega \setminus \mathcal{I}_c$. It remains to show that all but countably many of the sets $\mathcal{I}_c$ have zero measure. To this end, define

$$C_k = \left\{ c \in \mathbb{R} \mid \operatorname{meas}(\mathcal{I}_c) \ge \frac{1}{k} \right\}.$$

Since Ω is bounded, C_k contains only a finite number of points. Therefore, the set

$$\left\{ c \in \mathbb{R} \mid \operatorname{meas}(\mathcal{I}_c) > 0 \right\} = \bigcup_{k>0} C_k$$

is at most countable.



To prove (8.98), fix $c \in \Theta$. Since Θ is at most countable, we can find a sequence $\underline{c}_n \uparrow c$ such that $c_n \notin \Theta$. For $x \in \Omega \setminus \mathcal{E}_c$, we have that $g(x) \neq c$, and thus $\operatorname{sign}\left(g(x) - \underline{c}_n\right) = \operatorname{sign}\left(g(x) - c\right)$ for n sufficiently large. Thus (8.98) holds. The existence of $\{\bar{c}_n\}$ and (8.99) is proved in the same way. □

Now clearly

$$\Delta_x g^\delta(y,c) \sum_{j=1}^{N_i} \varphi\left(y_{i,j}, t\right) \chi_{I_{i,j}}(y) \to \partial_x f(\gamma(y), c)\varphi(y,t) \text{ as } \delta \to 0$$

in each interval (ξ_i, ξ_{i+1}). Furthermore, by Lemma 8.20,

$$\operatorname{sign}\left(\bar{u}^\delta - c\right) \to \operatorname{sign}\left(u - c\right),$$

for almost all (x,t) and all but at most a countable set of c's.

Regarding the middle term of (8.97), by Lemma 8.4 each summand is bounded by

$$\left| g^\delta\left(\gamma_i^+, c\right) - g^\delta\left(\gamma_i^-, c\right) \right|,$$

since (u_i^-, u_i^+) satisfies the minimal jump entropy condition. Therefore, by sending δ to 0 in (8.97), we find that

$$\begin{aligned}
&- \iint_{\Pi_T} \left(|u - c|\, \varphi_t + F(\gamma, u, c)\varphi_x \right) dx\, dt + \underbrace{\iint_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}\left(u - c\right) \partial_x f(\gamma, c)\varphi\, dx\, dt}_{I(c)} \\
&\quad - \int_0^T \sum_{x \in \mathcal{D}_\gamma} \left| f(\gamma(x^+), c) - f(\gamma(x^-), c) \right| \varphi(x,t)\, dt - \int_{\mathbb{R}} |u_0 - c|\, \varphi(x,0)\, dx \\
&\le 0
\end{aligned} \tag{8.100}$$

for all but a countable set of c's and all nonnegative test functions φ. This can be rewritten as

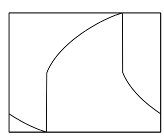
$$I(c) \le G(c),$$

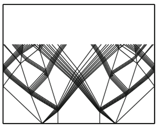
where G is a *continuous* function of c. Let Θ denote the set where the convergence of $\operatorname{sign}\left(\bar{u}^\delta - c\right) \to \operatorname{sign}\left(u - c\right)$ does not hold. Fix some $c \in \Theta$ and define the two sequences $\{\underline{c}_n\}$ and $\{\bar{c}_n\}$ as in Lemma 8.20. Set

$$\mathcal{E}_c = \left\{ (x,t) \;\middle|\; u(x,t) = c \right\}.$$

Since (8.100) holds for $\underline{c}_n$ and $\bar{c}_n$, we can write $I(c)$ as

$$\begin{aligned}
&\iint_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}\left(u - \underline{c}_n\right) \partial_x f(\gamma, u)\varphi\, dx\, dt \\
&\qquad + \iint_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \operatorname{sign}\left(u - \underline{c}_n\right) \partial_x f(\gamma, u)\varphi\, dx\, dt \le G(c),
\end{aligned} \tag{8.101}$$





where $\hat{\Pi}_T = \Pi_T \setminus \mathcal{D}_\gamma$. Since $\underline{c}_n < c$, the last integral can be rewritten as

$$\iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dx dt.$$

Since f is continuous, by sending n to ∞, we find that

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u - c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt + \iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c). \quad (8.102)$$

Similarly, using the sequence $\{\bar{c}_n\}$, we arrive at

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u - c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt - \iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c). \quad (8.103)$$

Adding (8.102) and (8.103) and dividing by 2, we find that

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u - c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c).$$

Since $\operatorname{sign}(0) = 0$, $\operatorname{sign}(u - c) = 0$ on $\mathcal{E}_c$, and therefore, we can conclude that

$$\iint\limits_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}(u - c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c) \quad (8.104)$$

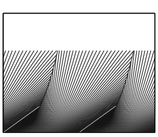
for *all* constants c. We have proved the following theorem.

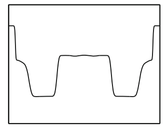
Theorem 8.21 *Assume that the flux function satisfies A.1–A.4, and let u^δ be a weak solution of (8.84), constructed by front tracking, such that u^δ converges to u in $L^1(\Pi_T)$. Then the entropy condition (8.100) holds for all constants c.*

8.3 Uniqueness of Entropy Solutions

Now we shall use the Kružkov entropy formulation, (8.100), to show that there exists at most one entropy solution. For convenience, we restate this condition,

$$\iint\limits_{\Pi_T} \left(|u - c|\, \varphi_t + F(\gamma, u, c)\varphi_x \right) dt\, dx - \iint\limits_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}(u - c)\, \partial_x f(\gamma, c)\varphi \, dt\, dx$$
$$+ \int_0^T \sum_i \left| f\left(\gamma_i^+, c\right) - f\left(\gamma_i^-, c\right) \right| \varphi\left(\xi_i, t\right)\, dt + \int_{\mathbb{R}} |u_0 - c|\, \varphi(x, 0)\, dx \ge 0, \quad (8.105)$$





for all nonnegative test functions $\varphi \in C_0^1(\mathbb{R} \times [0, T))$ and all real constants c, and where we write $\gamma_i^{\pm} = \gamma(\xi_i \pm)$.

In addition to satisfying this entropy inequality, we demand[3] that an entropy solution be a weak solution, i.e., that it satisfy (8.54) and be slightly more regular in the sense described below.

If $w \in L^\infty(\Pi_T)$, by the left and right traces of $w(\,\cdot\,, t)$ at a point x_0 we understand functions $t \mapsto w(x_0\pm, t) \in L^\infty([0, T])$ that satisfy a.e. $t \in [0, T)$,

$$\begin{aligned} \operatorname{ess\,lim}_{x \downarrow x_0} |w(x, t) - w(x_0+, t)| = 0, \\ \operatorname{ess\,lim}_{x \uparrow x_0} |w(x, t) - w(x_0-, t)| = 0. \end{aligned} \tag{8.106}$$

When comparing two entropy solutions, we shall need that they have traces at the points ξ_i, i.e., if u is an entropy solution, then we assume that the following traces exist:

$$u_i^{\pm}(t) = u\,(x_i \pm, t)\,, \tag{8.107}$$

in the sense of (8.106) for almost all t and for $i = 1, \ldots, N$.

An entropy solution of (8.53) is a function in $L^1_{\mathrm{loc}}(\Pi_T) \cap C([0, T); L^1_{\mathrm{loc}}(\mathbb{R}))$ such that (8.54), (8.105), and the regularity assumption (8.107) all hold.

We have already shown that an entropy solution exists for our model problem, since the existence of traces follows by noting that $z(\,\cdot\,, t) \in BV(\mathbb{R})$, which implies that z has traces. Since $u = z^{-1}(\gamma, z)$, the same applies to u.

Let now $w = w(x)$ be any function on $\mathbb{R}$, and fix a point y. We use the following notation:

$$\text{L-lim}_{x \downarrow y}\, w(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_y^{y+\varepsilon} w(x)\, dx,$$

$$\text{L-lim}_{x \uparrow y}\, w(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{y-\varepsilon}^{y} w(x)\, dx.$$

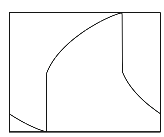
Lemma 8.22 *Let $w \in L^\infty(\Pi_T)$, and fix a point $x_0 \in \mathbb{R}$. If the left and right traces $t \mapsto w(x_0\pm, t)$ exist in the sense of* (8.106)*, then for a.e. $t \in [0, t)$ we have that*

$$\text{L-lim}_{x \downarrow x_0}\, w(x, t) = w(x_0+, t), \quad \text{L-lim}_{x \uparrow x_0}\, w(x, t) = w(x_0-, t).$$

Proof We prove the first limit as follows:

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} |w(x, t) - w(x_0+, t)|\; dx \\ &\qquad \le \frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} \operatorname{ess\,sup}_{y \in (x_0, x_0+\varepsilon)} |w(y, t) - w(x_0+, t)|\; dx \\ &\qquad = \operatorname{ess\,sup}_{y \in (x_0, x_0+\varepsilon)} |w(y, t) - w(x_0+, t)| \quad \to 0 \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$ □

[3] This does not follow easily from the entropy condition, which is in contrast to the case in which the flux function is space-independent.



As a consequence of this lemma, the following limits exist for every entropy solution u:

$$\begin{aligned} \text{L-lim}_{x\downarrow\xi_i} f\,(\gamma(x), u(x,t)) &= f\,(\gamma\,(\xi_i+)\,, u(\xi_i+,t))\,, \\ \text{L-lim}_{x\uparrow\xi_i} f\,(\gamma(x), u(x,t)) &= f\,(\gamma\,(\xi_i-)\,, u(\xi_i-,t))\,, \end{aligned} \tag{8.108}$$

and therefore, if v is another entropy solution,

$$\begin{aligned} \text{L-lim}_{x\downarrow\xi_i} F\,(\gamma(x), u(x,t), v(x,t)) &= F\,(\gamma\,(\xi_i+)\,, u(\xi_i+,t), v(\xi_i+,t))\,, \\ \text{L-lim}_{x\uparrow\xi_i} F\,(\gamma(x), u(x,t), v(x,t)) &= F\,(\gamma\,(\xi_i-)\,, u(\xi_i-,t), v(\xi_i-,t))\,, \end{aligned} \tag{8.109}$$

where F is the Kružkov entropy flux (8.88). Before we continue, let us define the following compactly supported Lipschitz continuous function:

$$\theta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}(\varepsilon + x) & \text{if } x \in (-\varepsilon, 0], \\ \frac{1}{\varepsilon}(\varepsilon - x) & \text{if } x \in [0, \varepsilon), \\ 0 & \text{otherwise.} \end{cases} \tag{8.110}$$

Lemma 8.23 *Let u be an entropy solution. Then for a.e. $t \in [0,t)$ and for all constants c,*

$$\begin{gathered} f\left(\gamma_i^+, u_i^+(t)\right) = f\left(\gamma_i^-, u_i^-(t)\right), \\ F\left(\gamma_i^+, u_i^+, c\right) - F\left(\gamma_i^-, u_i^-\right) \le \left| f\left(\gamma_i^+, c\right) - f\left(\gamma_i^-, c\right)\right|, \end{gathered}$$

where F is the Kružkov entropy flux (8.88).

Proof Since $u \in L^\infty(\Pi_T)$, a density argument shows that

$$\varphi(x,t) = \theta_\varepsilon\,(x - \xi_i)\,\psi(t),$$

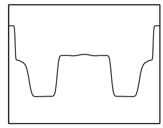
where $\psi \in C_0^1((0,T))$ is an admissible test function that can be used in the weak formulation (8.54). If $\varepsilon < \min_i \{\xi_{i+1} - \xi_i\}$, we get

$$\begin{aligned} &\iint\limits_{\Pi_T} u\theta_\varepsilon\,(x - \xi_i)\,\psi'(t)\,dx\,dt \\ &\qquad = \int_0^T \Big(\frac{1}{\varepsilon}\int_{\xi_i}^{\xi_i+\varepsilon} f(\gamma(x), u)\,dx - \frac{1}{\varepsilon}\int_{\xi_i-\varepsilon}^{\xi_i} f(\gamma(x), u)\,dx\Big)\psi(t)\,dt. \end{aligned}$$

By sending $\varepsilon \downarrow 0$ and using Lemma 8.23, we obtain

$$\int_0^T \left(f\left(\gamma_i^+, u_i^+\right) - f\left(\gamma_i^-, u_i^-\right)\right)\psi(t)\,dt = 0.$$

Since this holds for every test function ψ, the integrand must be zero.



To prove the inequality in the lemma, we choose the same test function, but restrict ψ to be nonnegative. By the entropy condition, (8.105), we get

$$\iint\limits_{\Pi_T} |u-c|\theta_\varepsilon(x-\xi_i)\,\psi'(t)\,dx\,dt$$
$$-\int_0^T\Big(\frac{1}{\varepsilon}\int_{\xi_i}^{\xi_i+\varepsilon} F(\gamma(x),u,c)\,dx-\frac{1}{\varepsilon}\int_{\xi_i-\varepsilon}^{\xi_i} F(\gamma(x),u,c)\,dx\Big)\psi(t)\,dt$$
$$-\iint\limits_{\Pi_T} \operatorname{sign}(u-c)\,\partial_x f(\gamma(x),c)\theta_\varepsilon(x-\xi_i)\,\psi(t)\,dx\,dt$$
$$+\int_0^T \left|f\left(\gamma_i^+,c\right)-f\left(\gamma_i^-,c\right)\right|\psi(t)\,dt\geq 0.$$

Again, by sending $\varepsilon\downarrow 0$,

$$\int_0^T \left|f\left(\gamma_i^+,c\right)-f\left(\gamma_i^-,c\right)\right|\psi(t)\,dt\geq\int_0^T \left(F\left(\gamma_i^+,u_i^+,c\right)-F\left(\gamma_i^-,u_i^-,c\right)\right)\psi(t)\,dt,$$

which implies the inequality. □

This has the following immediate corollary.

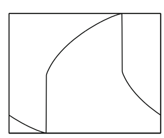
Corollary 8.24 *Assume that the flux function f satisfies A.1–A.4. If u is an entropy solution, then the pairs (u_i^-,u_i^+) satisfy the minimal jump entropy condition (8.25)–(8.26) for $i=1,\dots,N$.*

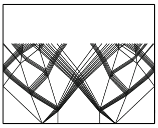
For any test function φ that has support away from $\mathcal{D}_\gamma$, we can double variables in the sense of Kružkov.

Lemma 8.25 *For every two entropy solutions u and v and nonnegative test function $\varphi\in C_0^1(\Pi_T\setminus\mathcal{D}_\gamma)$, we have that*

$$-\iint\limits_{\Pi_T}\left(|u-v|\,\varphi_t+F(\gamma,u,v)\varphi_x\right)dt\,dx$$
$$\leq C\iint\limits_{\Pi_T}|u-v|\,\varphi\,dt\,dx+\int\limits_{\mathbb{R}}|u_0-v_0|\,\varphi(x,0)\,dx, \tag{8.111}$$

where the constant C is zero if γ is piecewise constant.





Proof The proof is a classical doubling of variables argument. It uses exactly the same arguments as in Sect. 2.4, but adapted to our situation.

Let ϕ be a nonnegative test function in $C_0^1(\Pi_T \times \Pi_T)$. We use the notation $u = u(y,s)$, $v = v(x,t)$. Then using $c = u(y,s)$ as the constant in the entropy inequality for v and then integrating over (y,s), we get

$$
\begin{aligned}
-\iiiint\limits_{\Pi_T\times\Pi_T} &\left(|u-v|\,\phi_t + F(\gamma(x),u,v)\phi_x\right) dt\,dx\,ds\,dy \\
&+ \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\, f\,(\gamma(x),u)_x\,\phi\,dt\,dx\,dy\,ds \\
&\qquad \le \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |v_0-u|\,\phi(x,0,y,s)\,dx\,ds\,dy.
\end{aligned}
\tag{8.112}
$$

Similarly, starting with the entropy inequality for u and integrating over (x,t), we arrive at

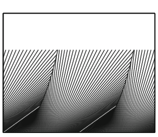
$$
\begin{aligned}
-\iiiint\limits_{\Pi_T\times\Pi_T} &\left(|u-v|\,\phi_s + F(\gamma(y),u,v)\phi_y\right) ds\,dy\,dt\,dx \\
&+ \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\, f\,(\gamma(y),v)_y\,\phi\,ds\,dy\,dt\,dx \\
&\qquad \le \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |u_0-v|\,\phi(x,t,y,0)\,dy\,dt\,dx.
\end{aligned}
\tag{8.113}
$$

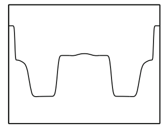
Since γ is differentiable outside $\mathcal{D}_\gamma$, for $(x,t) \in \Pi_T \setminus \mathcal{D}_\gamma$ we have

$$
\begin{aligned}
F\,(\gamma(x),v,u)\,\phi_x - &\operatorname{sign}(v-u)\,f\,(\gamma(x),u)_x\,\phi \\
&= \operatorname{sign}(v-u)\,(f\,(\gamma(x),v) - f\,(\gamma(y),u))\,\phi_x \\
&\quad - \operatorname{sign}(v-u)\,((f\,(\gamma(x),u) - f\,(\gamma(y),u))\,\phi)_x\,.
\end{aligned}
$$

Using this, we find that

$$
\begin{aligned}
&- \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \left(F\,(\gamma(x),v,u)\,\phi_x - \operatorname{sign}(v-u)\,f\,(\gamma(x),u)_x\,\phi\right) dt\,dx\,ds\,dy \\
&\quad = - \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\,(f\,(\gamma(x),v) - f\,(\gamma(y),u))\,\phi_x\,dt\,dx\,ds\,dy \\
&\qquad + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\,((f\,(\gamma(x),u) - f\,(\gamma(y),u))\,\phi)_x\;dt\,dx\,ds\,dy.
\end{aligned}
$$





We also have a similar equality for u,

$$
\begin{aligned}
&-\iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \left(F\left(\gamma(y),v,u\right)\phi_y - \operatorname{sign}(u-v)\, f\left(\gamma(y),v\right)_y \phi\right) ds\, dy\, dt\, dx \\
&\quad = -\iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\left(f\left(\gamma(y),u\right) - f\left(\gamma(x),v\right)\right)\phi_y \, ds\, dy\, dt\, dx \\
&\qquad + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\left(\left(f\left(\gamma(y),v\right) - f\left(\gamma(x),v\right)\right)\phi\right)_y \, ds\, dy\, dt\, dx.
\end{aligned}
$$

Now we introduce the notation

$$
\partial_{t+s} = \partial_t + \partial_s, \quad \partial_{x+y} = \partial_x + \partial_y.
$$

We use the above result and add (8.113) and (8.112) to obtain

$$
\begin{aligned}
&-\iiiint\limits_{\Pi_T\times\Pi_T} \Big(|v-u|\,\partial_{t+s}\phi \\
&\qquad\qquad + \operatorname{sign}(v-u)\left(f\left(\gamma(x),v\right) - f\left(\gamma(y),u\right)\right)\partial_{x+y}\phi\Big)\, dt\, dx\, ds\, dy \\
&+ \iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\Big[\left(\left(f\left(\gamma(x),u\right) - f\left(\gamma(y),u\right)\right)\phi\right)_x \\
&\qquad\qquad + \left(\left(f\left(\gamma(y),v\right) - f\left(\gamma(x),v\right)\right)\phi\right)_y\Big]\, dt\, dx\, ds\, dy \\
&\le \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |v_0 - u|\,\phi(x,0,y,s)\, dx\, ds\, dy \\
&\quad + \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |u_0 - v|\,\phi(x,t,y,0)\, dy\, dt\, dx.
\end{aligned} \tag{8.114}
$$

Now we shall choose a suitable test function. First let $\omega \in C_0^\infty(\mathbb{R})$ be a function such that $\omega(-a) = \omega(a)$, $\omega'(a) \le 0$ for $a > 0$, $|\omega'(a)| \le 2$, $\omega(a) = 0$ for $|a| \ge 1$, and $\int \omega(a)\, da = 1$. For positive ε, set

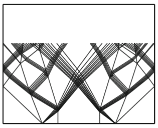
$$
\omega_\varepsilon(a) = \frac{1}{\varepsilon}\omega\left(\frac{a}{\varepsilon}\right).
$$

Let $\varphi(x,t)$ be a test function such that

$$
\varphi(x,t) = 0 \quad \text{for } |x - \xi_i| \le \varepsilon_0,\ i = 1,\dots,N,
$$

for some positive ε_0. Then we define

$$
\phi(x,t,y,s) = \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right), \tag{8.115}
$$



for $\varepsilon < \varepsilon_0$. We can easily check that $\phi \in C_0^1((\Pi_T \setminus \mathcal{D}_\gamma) \times (\Pi_T \setminus \mathcal{D}_\gamma))$. Furthermore, we have the useful identities

$$\partial_{t+s}\phi(x,t,y,s) = \partial_{t+s}\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right),$$
$$\partial_{x+y}\phi(x,t,y,s) = \partial_{x+y}\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right).$$

If we use these identities in (8.114), we find that

$$\begin{aligned} -\iiiint\limits_{\Pi_T \times \Pi_T} &\left(I_{\text{time}}(x,t,y,s) + I_{\text{conv}}(x,t,y,s)\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right) dt\,dx\,ds\,dy \\ &\le \iiiint\limits_{\Pi_T \times \Pi_T} \left(I^1_{\text{flux}}(x,t,y,s) + I^2_{\text{flux}}(x,t,y,s) + I^3_{\text{flux}}(x,t,y,s)\right) dt\,dx\,ds\,dy \\ &\quad + \underbrace{\iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |v_0 - u|\phi(x,0,y,s)\,dx\,ds\,dy + \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |u_0 - v|\phi(x,t,y,0)\,dy\,dt\,dx}_{J_{\text{init}}}, \end{aligned} \tag{8.116}$$

where

$$I_{\text{time}}(x,t,y,s) = |v-u|\,\partial_{t+s}\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$\begin{aligned} I_{\text{conv}}(x,t,y,s) &= \operatorname{sign}(v-u)\left[f(\gamma(x),v) - f(\gamma(y),u)\right] \\ &\qquad \times \partial_{x+y}\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right), \end{aligned}$$

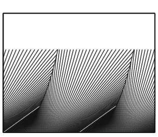
$$\begin{aligned} I^1_{\text{flux}}(x,t,y,s) &= -\operatorname{sign}(v-u)\,\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \\ &\qquad \times \left[\gamma'(x) f_\gamma(\gamma(x),u) - \gamma'(y) f_\gamma(\gamma(y),v)\right], \end{aligned}$$

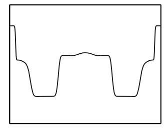
$$\begin{aligned} I^2_{\text{flux}}(x,t,y,s) &= -\operatorname{sign}(v-u)\,\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right) \\ &\quad \times \Big[\partial_x\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\left(f(\gamma(x),u) - f(\gamma(y),u)\right) \\ &\qquad + \partial_y\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\left(f(\gamma(x),v) - f(\gamma(y),v)\right)\Big], \end{aligned}$$

$$\begin{aligned} I^3_{\text{flux}}(x,t,y,s) &= \left[F(\gamma(x),v,u) - F(\gamma(y),v,u)\right] \\ &\quad \times \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)\partial_x\omega_\varepsilon\left(\frac{x-y}{2}\right). \end{aligned}$$

Introduce new variables

$$\tilde{x} = \frac{x+y}{2}, \quad z = \frac{x-y}{2}, \quad \tilde{t} = \frac{t+s}{2}, \quad \tau = \frac{t-s}{2},$$





which map $\Pi_T \times \Pi_T$ into

$$\Omega_T = \left\{ \left(\tilde{x}, \tilde{t}, z, \tau \right) \in \mathbb{R}^4 \mid 0 \le \tilde{t} \pm \tau \le T \right\},$$

and $(\Pi_T \setminus \mathcal{D}_\gamma) \times (\Pi_T \setminus \mathcal{D}_\gamma)$ into

$$\Omega_{T,\gamma} = \left\{ \left(\tilde{x}, \tilde{t}, z, \tau \right) \in \Omega_T \mid \tilde{x} \pm z \neq \xi_i,\ i = 1, \dots, N \right\}.$$

We start by estimating the terms in J_{init}:

$$\begin{aligned}
&\iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |v_0(x) - u(y,s)| \, \varphi\left(\frac{x+y}{2}, \frac{s}{2} \right) \omega_\varepsilon\left(\frac{x-y}{2} \right) \omega_\varepsilon\left(\frac{-s}{2} \right) dx\, ds\, dy \\
&\qquad = \int_0^\varepsilon \int\limits_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} \left| v_0(\tilde{x} + z) - u(\tilde{x} - z, \tilde{t} - \tau) \right| \varphi(\tilde{x}, \tau) \omega_\varepsilon(z) \omega_\varepsilon(\tau)\, dz\, d\tilde{x}\, d\tau \\
&\qquad \to \frac{1}{2} \int\limits_{\mathbb{R}} |v_0(x) - u(x,0)| \, \varphi(x,0)\, dx
\end{aligned}$$

as $\varepsilon \to 0$. Since $t \mapsto u(x,t)$ is L^1 continuous, we can replace $u(x,0)$ by $u_0(x)$. Similarly, we find that

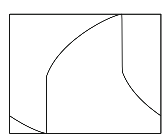
$$\iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |u_0 - v| \, \phi(x,t,y,0)\, dy\, dt\, dx \to \frac{1}{2} \int\limits_{\mathbb{R}} |u_0 - v_0| \, \varphi(x,0)\, dx$$

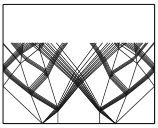
as $\varepsilon \to 0$, and thus

$$\lim_{\varepsilon \to 0} J_{\text{init}} = \int\limits_{\mathbb{R}} |u_0 - v_0| \, \varphi(x,0)\, dx. \tag{8.117}$$

In the transformed variables, we have

$$\begin{aligned}
I_{\text{time}}(\tilde{x}, \tilde{t}, z, \tau) &= \left| v(\tilde{x} + z, \tilde{t} + \tau) - u(\tilde{x} - z, \tilde{t} - \tau) \right| \partial_{\tilde{t}} \varphi\left(\tilde{x}, \tilde{t} \right), \\
I_{\text{conv}}(\tilde{x}, \tilde{t}, z, \tau) &= \operatorname{sign}\left(v(\tilde{x} + z, \tilde{t} + \tau) - u(\tilde{x} - z, \tilde{t} - \tau) \right) \partial_{\tilde{x}} \varphi\left(\tilde{x}, \tilde{t} \right) \\
&\qquad \times \Big[f\left(\gamma(\tilde{x} + z), v(\tilde{x} + z, \tilde{t} + \tau) \right) \\
&\qquad\qquad - f\left(\gamma(\tilde{x} - z), u(\tilde{x} - z, \tilde{t} - \tau) \right) \Big], \\
I^1_{\text{flux}}(\tilde{x}, \tilde{t}, z, \tau) &= \operatorname{sign}\left(v(\tilde{x} + z, \tilde{t} + \tau) - u(\tilde{x} - z, \tilde{t} - \tau) \right) \omega_\varepsilon(z)\, \omega_\varepsilon(\tau) \\
&\qquad \times \Big[\gamma'(\tilde{x} + z) f_\gamma(\gamma(\tilde{x} + z), u(\tilde{x} - z, \tilde{t} - \tau)) \\
&\qquad\qquad - \gamma'(\tilde{x} - z) f_\gamma(\gamma(\tilde{x} - z), v(\tilde{x} + z, \tilde{t} + \tau)) \Big] \varphi\left(\tilde{x}, \tilde{t} \right),
\end{aligned}$$





$$
\begin{aligned}
&I^2_{\text{flux}}(\tilde{x},\tilde{t},z,\tau) \\
&\quad = \operatorname{sign}\big(v(\tilde{x}+z,\tilde{t}+\tau)-u(\tilde{x}-z,\tilde{t}-\tau)\big)\,\omega_\varepsilon(z)\,\omega_\varepsilon(\tau)\,\partial_{\tilde{x}}\varphi\left(\tilde{x},\tilde{t}\right) \\
&\qquad \times\Big[\big(f(\gamma(\tilde{x}+z),u(\tilde{x}-z,\tilde{t}-\tau))-f(\gamma(\tilde{x}-z),u(\tilde{x}-z,\tilde{t}-\tau))\big) \\
&\qquad\quad +\big(f(\gamma(\tilde{x}+z),v(\tilde{x}+z,\tilde{t}+\tau))-f(\gamma(\tilde{x}-z),v(\tilde{x}+z,\tilde{t}+\tau))\big)\Big], \\
&I^3_{\text{flux}}(\tilde{x},\tilde{t},z,\tau) = \Big[F\left(\gamma(\tilde{x}+z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right) \\
&\qquad\qquad -F\left(\gamma(\tilde{x}-z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right)\Big] \\
&\qquad\qquad \times\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(t)\,\partial_z\omega_\varepsilon(z)\,.
\end{aligned}
$$

It is straightforward to deduce the limits

$$
\lim_{\varepsilon\to 0}\iiiint_{\Omega} I_{\text{time}}(\tilde{x},\tilde{t},z,\tau)\omega_\varepsilon(z)\,\omega_\varepsilon(t)\;d\tau\,dz\,d\tilde{t}\,d\tilde{x} = \iint_{\Pi_T}|u-v|\,\varphi_t\,dt\,dx, \tag{8.118}
$$

$$
\lim_{\varepsilon\to 0}\iiiint_{\Omega} I_{\text{conv}}(\tilde{x},\tilde{t},z,\tau)\omega_\varepsilon(z)\,\omega_\varepsilon(t)\;d\tau\,dz\,d\tilde{t}\,d\tilde{x} = \iint_{\Pi_T} F\left(\gamma(x),u,v\right)\varphi_x\,dt\,dx. \tag{8.119}
$$

Since γ is C^1 outside $\mathcal{D}_\gamma$, we deduce that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\iiiint_{\Omega_\gamma} I^1_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau dz &= \iint_{\Pi_T\setminus\mathcal{D}_\gamma}\gamma'(x)F_\gamma\left(\gamma(x),u,v\right)\,dt\,dx \\
&\le C\iint_{\Pi_T}|u-v|\;dt\,dx,
\end{aligned}
\tag{8.120}
$$

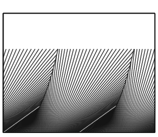
where

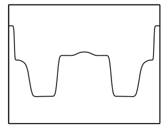
$$
C = \|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\left\|f_{u\gamma}\right\|_{L^\infty}.
$$

In particular, we observe that C can be chosen as zero if γ is a piecewise constant function.

Next we consider I^2_{flux}. Since φ vanishes near $\mathcal{D}_\gamma$, I^2_{flux} also vanishes near $\mathcal{D}_\gamma$. Hence γ is uniformly C^1 where $I^2_{\text{flux}}\neq 0$. Therefore,

$$
\begin{aligned}
&\left|I^2_{\text{flux}}(\tilde{x},\tilde{t},z,\tau)\right| \\
&\quad\le \omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right| \\
&\qquad\times\Big(\left|f\left(\gamma(\tilde{x}+z),u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right)-f\left(\gamma(\tilde{x}-z),u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right)\right| \\
&\qquad\quad+\left|f\left(\gamma(\tilde{x}+z),v\left(\tilde{x}+z,\tilde{t}+\tau\right)\right)-f\left(\gamma(\tilde{x}-z),v\left(\tilde{x}+z,\tilde{t}+\tau\right)\right)\right|\Big) \\
&\quad\le \omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right|2\left\|f_\gamma\right\|_{L^\infty}\left|\gamma(\tilde{x}+z)-\gamma(\tilde{x}-z)\right| \\
&\quad\le 4\left\|f_\gamma\right\|_{L^\infty(\mathbb{R})}\|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\,\omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right||z|\,.
\end{aligned}
$$





From this we conclude that

$$\lim_{\varepsilon\to 0}\left|\iiiint_{\Omega} I^2_{\mathrm{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz\right| \le \lim_{\varepsilon\to 0} C\int_{-\varepsilon}^{\varepsilon}|z|\,\omega_\varepsilon(z)\,dz = 0. \tag{8.121}$$

Finally, we turn to I^3_{flux}:

$$\begin{aligned}
\left|I^3_{\mathrm{flux}}(\tilde{x},\tilde{t},z,\tau)\right| &\le \varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\left|\partial_z\omega_\varepsilon(z)\right| \\
&\quad\times\Big|F\left(\gamma(\tilde{x}+z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right) \\
&\qquad - F\left(\gamma(\tilde{x}-z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right)\Big| \\
&\le \varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\left|\partial_z\omega_\varepsilon(z)\right| 2\left\|\gamma'\right\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}|z| \\
&\quad\times\left\|f_{\gamma u}\right\|_{L^\infty(\mathbb{R})}\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right| \\
&\le \left\|f_{\gamma u}\right\|_{L^\infty(\mathbb{R})}\left\|\gamma'\right\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\frac{8}{2\varepsilon}\chi_{\{z\,|\,|z|\le\varepsilon\}} \\
&\quad\times\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|.
\end{aligned}$$

Now set

$$h_\varepsilon(\tilde{x},\tilde{t}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\,d\tau\,dz.$$

By Lebesgue's differentiation theorem,

$$\lim_{\varepsilon\to 0} h_\varepsilon(x,t) = |v(x,t)-u(x,t)| \quad \text{a.e. } (x,t).$$

Therefore,

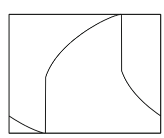
$$\lim_{\varepsilon\to 0}\left|\iiiint_{\Omega} I^3_{\mathrm{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz\right| \le \lim_{\varepsilon\to 0} C\iint_{\Pi_T}|u-v|\,\varphi\,dt\,dx, \tag{8.122}$$

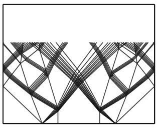
where the constant C is zero if γ is piecewise constant.

Combining (8.118), (8.119), (8.120), (8.121), and (8.122) we get (8.111). □

Equipped with Lemma 8.25, we can continue to prove uniqueness of entropy solutions. Define

$$\psi_\varepsilon(x) = \begin{cases} \frac{2}{\varepsilon}\left(\varepsilon+x\right) & \text{if } x\in[-\varepsilon,-\varepsilon/2], \\ 1 & \text{if } -\varepsilon/2 < x < \varepsilon/2, \\ \frac{2}{\varepsilon}(\varepsilon-x) & \text{if } x\in[\varepsilon/2,\varepsilon], \\ 0 & \text{otherwise,} \end{cases}$$





and set $\Psi_\varepsilon(x) = 1 - \sum_i^N \psi_\varepsilon(x - \xi_i)$. Observe that $\Psi_\varepsilon \to 1$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon \to 0$, and we consider only ε such that $\varepsilon < \min_i \{\xi_{i+1} - \xi_i\}$. Let φ be a nonnegative test function in $C_0^1(\Pi_T)$. Then $\phi = \varphi \Psi_\varepsilon$ is an admissible test function, as a density argument will show. Furthermore, ϕ has support away from $\mathcal{D}_\gamma$. With this test function, (8.111) takes the form

$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \Psi_\varepsilon \varphi_t + F(\gamma, u, v) \Psi_\varepsilon \varphi_x\big)\, dt\, dx - \iint\limits_{\Pi_T} F(\gamma, u, v) \Psi_\varepsilon' \varphi \, dt\, dx \\
\leq C \iint\limits_{\Pi_T} |u - v|\, \Psi_\varepsilon \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \Psi_\varepsilon \varphi(x, 0)\, dx.
\end{aligned}
$$

Set

$$
I_\varepsilon = \iint\limits_{\Pi_T} F(\gamma, u, v) \Psi_\varepsilon' \varphi \, dt\, dx,
$$

and let $\varepsilon \downarrow 0$. This yields

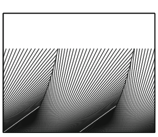
$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \varphi_t + F(\gamma, u, v) \varphi_x\big)\, dt\, dx \\
\leq C \iint\limits_{\Pi_T} |u - v|\, \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x, 0)\, dx + \lim_{\varepsilon \downarrow 0} I_\varepsilon.
\end{aligned}
$$

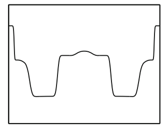
Now we use that (u_i^-, u_i^+) and (v_i^-, v^+) both satisfy the minimal jump entropy condition, and thus Lemma 8.6 applies at each discontinuity in γ. With this in mind, we calculate

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} I_\varepsilon &= \sum_i^N \lim_{\varepsilon \downarrow 0} \int_0^T \Bigg(\frac{2}{\varepsilon} \int\limits_{\xi_i + \varepsilon/2}^{\xi_i + \varepsilon} F(\gamma(x), u, v) \varphi \, dx \\
&\qquad\qquad - \frac{2}{\varepsilon} \int\limits_{\xi_i - \varepsilon}^{\xi_i - \varepsilon/2} F\left(\gamma(x), u, v\right) \varphi \, dx \Bigg) \, dt \\
&= \lim_{\varepsilon \downarrow 0} \sum_i^N \int_0^T \left(F\left(\gamma_i^+, u_i^+, v_i^+\right) - F\left(\gamma_i^-, u_i^-, v_i^-\right) \right) \varphi\left(\xi_i, t\right)\, dt \\
&\leq 0.
\end{aligned}
$$

Hence for every nonnegative test function, we have

$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \varphi_t + F(\gamma, u, v) \varphi_x\big)\, dt\, dx \\
\leq C \iint\limits_{\Pi_T} |u - v|\, \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x, 0)\, dx.
\end{aligned}
\tag{8.123}
$$





This equation is very similar to (2.60), the difference being that F replaces q and that F depends explicitly on x. What follows is therefore analogous to the arguments used after (2.60).

Now let $\alpha_r(x)$ be a smooth function taking values in $[0, 1]$ such that

$$\alpha_r(x) = \begin{cases} 1 & \text{if } |x| \le r, \\ 0 & \text{if } |x| \ge r + 1, \end{cases}$$

and $\max |\alpha_r'(x)| \le 2$. Then fix s_0 and s so that $0 < s_0 < s < T$. For all positive κ and τ such that $s_0 + \tau < s + \kappa < T$, let $\beta_{\kappa,\tau}(t)$ be a Lipschitz function that is linear on $[s_0, s_0 + \kappa]$ and on $[s, s + \tau]$ and satisfies

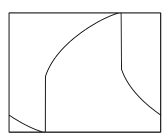
$$\beta_{\kappa,\tau}(t) = \begin{cases} 0 & \text{if } t < s_0 \text{ or } t > s + \kappa, \\ 1 & \text{if } s \in [s_0 + \tau, s]. \end{cases}$$

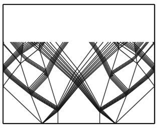
By density arguments, $\varphi = \alpha_r \beta_{\kappa,\tau}$ is an admissible test function, and using this in (8.123) gives

$$\begin{aligned} &\frac{1}{\kappa} \int_s^{s+\kappa} \int_{\mathbb{R}} |u - v| \, \alpha_r \, dx \, dt - \frac{1}{\tau} \int_{s_0}^{s_0+\tau} |u - v| \, \alpha_r \, dx \, dt \\ &\quad \le C \int_{s_0}^{s+\kappa} \int_{\mathbb{R}} |u - v| \, \alpha_r \, dx \, dt + 2 \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F(\gamma, u, v)| \, \beta_{\kappa,\tau} \, dx \, dt. \end{aligned}$$

Next, we let $s_0 \downarrow 0$ and use the triangle inequality to get

$$\begin{aligned} \frac{1}{\kappa} \int_s^{s+\kappa} \int_{\mathbb{R}} |u - v| \, \alpha_r \, dx \, dt \le &\int_{\mathbb{R}} |u_0 - v_0| \, \alpha_r \, dx \\ &+ \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |v(x,t) - v_0(x)| \, \alpha_r(x) \, dx \, dt \\ &+ \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |u(x,t) - u_0(x)| \, \alpha_r(x) \, dx \, dt \\ &+ C \int_{s_0}^{s+\kappa} \int_{\mathbb{R}} |u - v| \, \alpha_r \, dx \, dt \\ &+ 2 \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F(\gamma, u, v)| \, \beta_{\kappa,\tau} \, dx \, dt. \end{aligned}$$





We shall now send $\tau \downarrow 0$ and prove later that for every entropy solution u,

$$\lim_{\tau\downarrow 0} \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |u(x,t) - u_0(x)|\, \alpha_r(x)\, dx dt = 0. \tag{8.124}$$

Furthermore, by finite speed of propagation, if $u_0(x) = v_0(x)$ for $|x|$ large, then also $u(x,t) = v(x,t)$ for $|x|$ large. Hence $F(\gamma(x), u(x,t), v(x,t)) = 0$ for $|x|$ large. Thus

$$\lim_{r\to\infty} \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F\,(\gamma, u, v)|\, \beta_{\kappa,\tau}\, dx\, dt = 0.$$

Set

$$\mathcal{E}(t) = \int_{\mathbb{R}} |u(x,t) - v(x,t)|\; dx.$$

By sending $\tau \downarrow 0$ and then $r \uparrow \infty$, we obtain

$$\frac{1}{\kappa} \int_s^{s+\kappa} \mathcal{E}(t)\, dt \le \mathcal{E}(0) + C \int_0^{s+\kappa} \mathcal{E}(t)\, dt. \tag{8.125}$$

Let s be a Lebesgue point for the L^1 function $\mathcal{E}$. Sending $\kappa \downarrow 0$ yields

$$\mathcal{E}(s) \le \mathcal{E}(0) + C \int_0^{s} \mathcal{E}(t)\, dt.$$

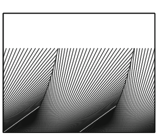
Since the set of Lebesgue points has full measure, we can use Gronwall's inequality to conclude that

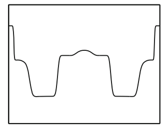
$$\mathcal{E}(t) \le e^{Ct}\mathcal{E}(0),$$

for almost every $t > 0$.

It remains to prove (8.124). To this end, define

$$\beta_\tau(t) = d \begin{cases} \frac{1}{\tau}(\tau - t) & \text{if } 0 \le t \le \tau, \\ 0 & \text{otherwise.} \end{cases}$$





We then use the test function $\omega_\varepsilon(x-y)\beta_\tau(t)\alpha_r(x)$ and the constant $c = u_0(y)$ in the entropy formulation (8.105). The result of this is

$$\begin{aligned}
&\iint\limits_{\Pi_\tau} |u(x,t) - u_0(y)|\, \omega_\varepsilon(x-y)\alpha_r(x)\beta_\tau'(t)\, dt\, dx \\
&\quad + \iint\limits_{\Pi_\tau} F\left(\gamma(x), u, u_0(y)\right) \left(\omega_\varepsilon(x-y)\alpha_r(x)\right)_x \beta_\tau(t)\, dt\, dx \\
&\quad - \iint\limits_{\Pi_\tau\setminus\mathcal{D}_\gamma} \operatorname{sign}\left(u - u_0(y)\right) \partial_x f\left(\gamma(x), u_0(y)\right) \omega_\varepsilon(x-y)\alpha_r(x)\beta_\tau(t)\, dt\, dx \\
&\quad + \int_0^\tau \sum_i \left| f\left(\gamma_i^+, u_0(y)\right) - f\left(\gamma_i^+, u_0(y)\right)\right| \omega_\varepsilon\left(\xi_i - y\right) \alpha_r\left(\xi_i\right) \beta_\tau(t)\, dt \\
&\quad + \int\limits_{\mathbb{R}} |u_0(x) - u_0(y)|\, \omega_\varepsilon(x-y)\alpha_r(x)\, dx \ge 0.
\end{aligned}$$

Since $u \in L^1_{\mathrm{loc}}$, on sending $\tau \downarrow 0$, all terms in the above expression containing β_τ will vanish. Recalling that $\beta_\tau'(t) = -1/\tau$ for $t \in (0,\tau)$, after an application of the triangle inequality and an integration over $y \in \mathbb{R}$, we find that

$$\begin{aligned}
&\lim_{\tau\downarrow 0} \frac{1}{\tau} \int_0^\tau \int\limits_{\mathbb{R}} |u(x,t) - u_0(x)|\, \alpha_r(x)\, dx\, dt \\
&\qquad\qquad \le 2 \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |u_0(x) - u_0(y)|\, \omega_\varepsilon(x-y)\alpha_r(x)\, dx\, dy.
\end{aligned}$$

Since $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$, we can send $\varepsilon \downarrow 0$ to prove (8.124).

We have now proved that the initial value problem (8.53) is well posed in L^1.

Theorem 8.26 *Assume that the flux function f satisfies the assumptions A.1–A.4, and that the initial value u_0 is in $L^1(\mathbb{R})$ and $f(\gamma, u_0) \in BV(\mathbb{R})$. Then there exists a weak entropy solution, in the sense of* (8.54) *and* (8.105)*, to the initial value problem* (8.53).

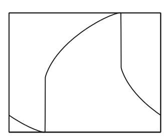
If v is another entropy solution with initial data v_0, then

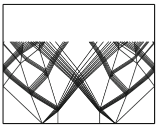
$$\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{Ct} \|v_0 - u_0\|_{L^1(\mathbb{R})},$$

where the constant C depends on $\gamma'(x)$ for $x \notin \mathcal{D}_\gamma$ and is zero if γ is piecewise constant.

8.4 Notes

The presentation here is based on [161]. Over that last twenty years, conservation laws with spatially discontinuous flux functions have been studied in several papers; a very incomplete list includes [2, 36, 59, 71, 110, 111, 166, 181, 182] and other references therein.





The solution of the Riemann problem presented in this chapter is based on [70]. Regarding the admissibility criteria for solutions of the Riemann problem, as already hinted at in the text, there exist many criteria for selecting unique solutions; see, e.g., [2, 59]. It turns out that all these recipes can be used to prove an estimate similar to (8.123), and thus give a unique solution to the Cauchy problem. How this is done is explained in [5]. Example 8.8 is taken from [143].

The convergence of the front-tracking algorithm is taken from [113]. In [114] the convergence of front tracking was shown for the polymer model (8.5). Existence proofs based on finite volume methods were first presented in [181], see also [182], and later extended to several dimensions in [107]. For a general overview we refer to [35].

8.5 Exercises

8.1 Solve the Riemann problem for the linear conservation law with discontinuous coefficients,

$$u_t + (a(x)u)_x = 0, \qquad a(x) = \begin{cases} a_l, & x < 0, \\ a_r, & x \geq 0. \end{cases}$$

8.2 Carry out the coordinate change transforming (8.4) into (8.5).