Chapter 7

Well-Posedness of the Cauchy Problem

Ma per seguir virtute e conoscenza.[1]
— Dante Alighieri (1265–1321), La Divina Commedia

The goal of this chapter is to show that the limit found by front tracking, that is, the weak solution of the initial value problem

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \tag{7.1}$$

is stable in L^1 with respect to perturbations in the initial data. In other words, if $v = v(x,t)$ is another solution found by front tracking, then

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_1 \le C\,\|u_0 - v_0\|_1$$

for some constant C. Furthermore, we shall show that under some mild extra entropy conditions, every weak solution coincides with the solution constructed by front tracking.

◇ Example 7.1 (A special system)
As an example for this chapter we shall consider the special 2×2 system

$$\begin{aligned} u_t + \left(vu^2\right)_x &= 0, \\ v_t + \left(uv^2\right)_x &= 0. \end{aligned} \tag{7.2}$$

For simplicity assume that $u > 0$ and $v > 0$. The Jacobian matrix reads

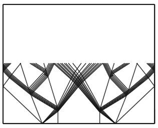
$$\begin{pmatrix} 2uv & u^2 \\ v^2 & 2uv \end{pmatrix}, \tag{7.3}$$

with eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 &= uv, & r_1 &= \begin{pmatrix} -u/v \\ 1 \end{pmatrix}, \\ \lambda_2 &= 3uv, & r_2 &= \begin{pmatrix} u/v \\ 1 \end{pmatrix}. \end{aligned} \tag{7.4}$$

[1] Hard to comprehend? It means "[but to] pursue virtue and knowledge."

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_7



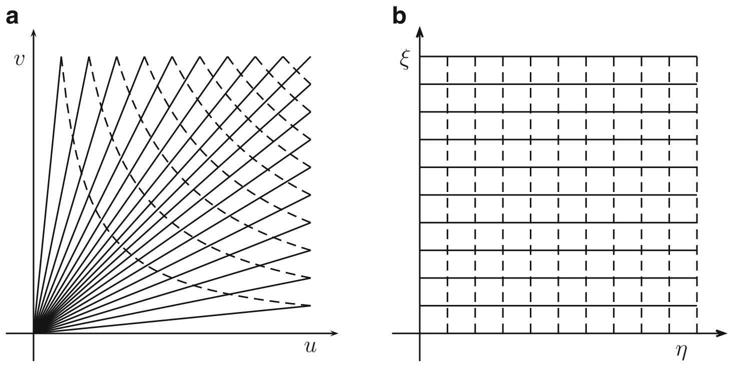


Fig. 7.1 The curves W in (u, v) coordinates (**a**) and (η, ξ) coordinates (**b**)

The system is clearly strictly hyperbolic. Observe that

$$\nabla\lambda_1 \cdot r_1 = 0,$$

and hence the first family is linearly degenerate. The corresponding wave curve $W_1(u_l, v_l) = C_1(u_l, v_l)$ is given by (cf. Theorem 5.7)

$$\frac{du}{dv} = -\frac{u}{v}, \quad u(v_l) = u_l,$$

or (see Fig. 7.1)

$$W_1(u_l, v_l) = C_1(u_l, v_l) = \{(u, v) \mid uv = u_l v_l\}.$$

The corresponding eigenvalue λ_1 is constant along each hyperbola.

With the chosen normalization of r_2 we find that

$$\nabla\lambda_2 \cdot r_2 = 6u,$$

and hence the second-wave family is genuinely nonlinear. The rarefaction curves of the second family are solutions of

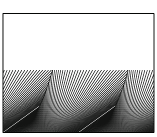
$$\frac{du}{dv} = \frac{u}{v}, \quad u(v_l) = u_l,$$

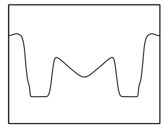
and thus

$$\frac{u}{v} = \frac{u_l}{v_l}.$$

We see that these are straight lines emanating from the origin, and λ_2 increases as u increases. Consequently, R_2 consists of the ray

$$v = u\frac{v_l}{u_l}, \quad u \geq u_l.$$





The rarefaction speed is given by

$$\lambda_2(u; u_l, v_l) = 3u^2 \frac{v_l}{u_l}.$$

To find the shocks in the second family, we use the Rankine–Hugoniot relation

$$\begin{aligned} s(u - u_l) &= vu^2 - v_l u_l^2, \\ s(v - v_l) &= v^2 u - v_l^2 u_l, \end{aligned}$$

which implies

$$\frac{u}{u_l} = \frac{1}{2}\left(\frac{v}{v_l} + \frac{v_l}{v} \pm \left(\frac{v}{v_l} - \frac{v_l}{v}\right)\right) = \begin{cases} v_l/v, \\ v/v_l. \end{cases}$$

(Observe that the solution with $u/u_l = v_l/v$ coincides with the wave curve of the linearly degenerate first family.) The shock part of this curve S_2 consists of the line

$$S_2(u_l, v_l) = \left\{(u, v) \mid v = u\frac{v_l}{u_l}, \quad 0 < u \le u_l\right\}.$$

The shock speed is given by

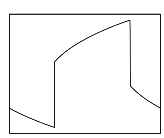
$$s := \mu_2(u; u_l, v_l) = \left(u^2 + uu_l + u_l^2\right)\frac{v_l}{u_l}.$$

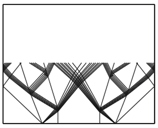
Hence the Hugoniot locus and rarefaction curves coincide for this system. Systems with this property are called *Temple class systems* after Temple [177]. Furthermore, the system is linearly degenerate in the first family and genuinely nonlinear in the second. Summing up, the solution of the Riemann problem for (7.2) is as follows: First the middle state is given by

$$u_m = \sqrt{u_l u_r \frac{v_l}{v_r}}, \quad v_m = \sqrt{v_l v_r \frac{u_l}{u_r}}.$$

If $u_l/v_l \le u_r/v_r$, the second wave is a rarefaction wave, and the solution can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} u_l \\ v_l \end{pmatrix} & \text{for } x/t \le u_l v_l, \\ \begin{pmatrix} u_m \\ v_m \end{pmatrix} & \text{for } u_l v_l < x/t \le 3u_m v_m, \\ \sqrt{\frac{x}{3t}\frac{v_m}{u_m}}\begin{pmatrix} u_m/v_m \\ 1 \end{pmatrix} & \text{for } 3u_m v_m < x/t \le 3u_r v_r, \\ \begin{pmatrix} u_r \\ v_r \end{pmatrix} & \text{for } 3u_r v_r < x/t. \end{cases} \tag{7.5}$$





In the shock case, that is, when $u_l/v_l > u_r/v_r$, the solution reads

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} u_l \\ v_l \end{pmatrix} & \text{for } x/t \le u_l v_l, \\ \begin{pmatrix} u_m \\ v_m \end{pmatrix} & \text{for } u_l v_l < x/t \le \mu_2(u_r; u_m, v_m), \\ \begin{pmatrix} u_r \\ v_r \end{pmatrix} & \text{for } \mu_2(u_r; u_m, v_m) < x/t. \end{cases} \tag{7.6}$$

If we set

$$\eta = uv, \quad \xi = \frac{u}{v},$$

and thus

$$u = \sqrt{\eta\xi}, \quad v = \sqrt{\eta/\xi},$$

the solution of the Riemann problem will be especially simple in (η, ξ) coordinates. See Fig. 7.1. Given left and right states (η_l, ξ_l), (η_r, ξ_r), the middle state is given by (η_l, ξ_r). Consequently, measured in (η, ξ) coordinates, the total variation of the solution of the Riemann problem equals the total variation of the initial data. This means that we do not need the Glimm functional to show that a front-tracking approximation to the solution of (7.2) has bounded total variation. With this in mind it is easy to show (using the methods of the previous chapters) that there exists a weak solution to the initial value problem for (7.2) whenever the total variation of the initial data is bounded.

We may use these variables to parameterize the wave curves as follows:

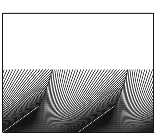
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_l v_l/\eta \\ \eta \end{pmatrix} \text{ (first family)},$$
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_l \eta/v_l \\ \eta \end{pmatrix} \text{ (second family)}.$$

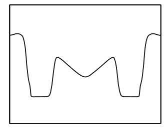
For future use we note that the rarefaction and shock speeds are as follows:

$$\lambda_1(\eta) = \mu_1(\eta) = \eta,$$
$$\lambda_2(\eta) = 3\eta, \quad \text{and} \quad \mu_2(\eta_l, \eta_r) = \left(\eta_l + \sqrt{\eta_l \eta_r} + \eta_r\right). \qquad \diamond$$

As a reminder we now summarize some properties of the front-tracking approximation for a fixed δ.

1. For all positive times t, $u^\delta(x,t)$ has finitely many discontinuities, each having position $x_i(t)$. These discontinuities can be of two types: shock fronts or approximate rarefaction fronts. Furthermore, only finitely many interactions between discontinuities occur for $t \ge 0$.





2. Along each shock front, the left and right states

$$u_{l,r} = u^\delta\left(x_i \mp, t\right) \tag{7.7}$$

are related by

$$u_r = S_{\hat{\imath}}\left(\epsilon_i\right) u_l + e_i,$$

where ϵ_i is the strength of the shock and $\hat{\imath}$ is the family of the shock. The "error" e_i is a vector of small magnitude. Furthermore, the speed of the shock, $\dot{x}$, satisfies

$$\left|\dot{x} - \mu_{\hat{\imath}}\left(u_l, u_r\right)\right| \le \mathcal{O}\left(1\right)\delta, \tag{7.8}$$

where $\mu_{\hat{\imath}}\left(u_l, u_r\right)$ is the $\hat{\imath}$th eigenvalue of the averaged matrix

$$M\left(u_l, u_r\right) = \int_0^1 df\left(\left(1-\alpha\right)u_l + \alpha u_r\right)\, d\alpha;$$

cf. (5.76)–(5.77).

3. Along each rarefaction front, the values u_l and u_r are related by

$$u_r = R_{\hat{\imath}}\left(\epsilon_i\right) u_l + e_i. \tag{7.9}$$

Also,

$$\left|\dot{x} - \lambda_{\hat{\imath}}\left(u_r\right)\right| \le \mathcal{O}\left(1\right)\delta \quad \text{and} \quad \left|\dot{x} - \lambda_{\hat{\imath}}\left(u_l\right)\right| \le \mathcal{O}\left(1\right)\delta, \tag{7.10}$$

where $\lambda_{\hat{\imath}}\left(u\right)$ is the $\hat{\imath}$th eigenvalue of $df\left(u\right)$.

4. The total magnitude of all errors is small:

$$\sum_i \left|e_i\right| \le \delta. \tag{7.11}$$

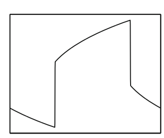
Also, recall that for a suitable constant C_0 the Glimm functional

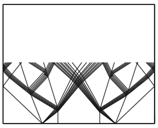
$$G\left(u^\delta(\cdot, t)\right) = T\left(u^\delta(\cdot, t)\right) + C_0 Q\left(u^\delta(\cdot, t)\right)$$

is nonincreasing for each collision of fronts, where T and Q are defined by (6.23) and (6.22), respectively, and that the interaction potential

$$Q\left(u^\delta(\cdot, t)\right)$$

is strictly decreasing for each collision of fronts.





7.1 Stability

Details are always vulgar.
— Oscar Wilde, The Picture of Dorian Gray (1891)

Now let v^δ be another front-tracking solution with initial condition v_0. To compare u^δ and v^δ in the L^1-norm, i.e., to estimate $\left\| u^\delta - v^\delta \right\|_1$, we introduce the vector $q = q(x,t) = (q_1, \ldots, q_n)$ by

$$v^\delta(x,t) = H_n(q_n)\, H_{n-1}(q_{n-1}) \cdots H_1(q_1)\, u^\delta(x,t) \tag{7.12}$$

and the intermediate states ω_i,

$$\omega_0 = u^\delta(x,t), \quad \omega_i = H_i(q_i)\, w_{i-1}, \quad \text{for } 1 \le i \le n, \tag{7.13}$$

with velocities

$$\mu_i = \mu_i(\omega_{i-1}, \omega_i). \tag{7.14}$$

As in Chapt. 5, $H_k(\epsilon)u$ denotes the kth Hugoniot curve through u, parameterized such that

$$\frac{d}{d\epsilon} H_k(\epsilon)\, u\Big|_{\epsilon=0} = r_k(u).$$

Note that in the definition of q we use *both* parts of this curve, not only the part where $\epsilon < 0$. The vector q represents a "solution" of the Riemann problem with left state u^δ and right state v^δ using only shocks. (For $\epsilon > 0$ these will be weak solutions; that is, they satisfy the Rankine–Hugoniot condition. However, they will not be Lax shocks.)

Later in this section we shall use the fact that genuine nonlinearity implies that $\mu_k(u, H_k(\epsilon)u)$ will be increasing in ϵ, i.e.,

$$\frac{d}{d\epsilon} \mu_k(u, H_k(\epsilon)u) \ge c > 0,$$

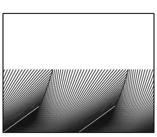
for some constant c depending only on f.

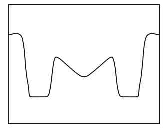
As our model problem showed, the L^1 distance is more difficult to control than the "q-distance." However, it turns out that even the q-distance is not quite enough, and we need to introduce a weighted form. We let $\mathcal{D}(u^\delta)$ and $\mathcal{D}(v^\delta)$ denote the sets of all discontinuities in u and v, respectively, and define the functional $\Phi(u^\delta, v^\delta)$ as

$$\Phi(u^\delta, v^\delta) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} |q_k(x)|\, W_k(x)\, dx. \tag{7.15}$$

Here the weights W_k are defined as

$$W_k = 1 + \kappa_1 A_k + \kappa_2 \left(Q\left(u^\delta\right) + Q\left(v^\delta\right) \right), \tag{7.16}$$





where $Q\left(u^\delta\right)$ and $Q\left(v^\delta\right)$ are the interaction potentials of u^δ and v^δ, respectively; cf. (6.22). The quantity A_k is the *total strength of all waves in* u^δ *or* v^δ *that approach the* k*-wave* $q_k(x)$. More precisely, if the kth field is linearly degenerate, then

$$A_k(x) = \sum_{\substack{i,\, x_i<x \\ \hat{\imath}>k}} |\epsilon_i| + \sum_{\substack{i,\, x>x_i \\ \hat{\imath}<k}} |\epsilon_i| \,. \tag{7.17}$$

The summation is over all discontinuities $x_i \in \mathcal{D}\left(u^\delta\right) \cup \mathcal{D}(v^\delta)$. If the kth field is genuinely nonlinear, we must also account for waves of the same family approaching each other, and define

$$\begin{aligned} A_k(x) &= \sum_{\substack{i,\, x_i<x \\ \hat{\imath}>k}} |\epsilon_i| + \sum_{\substack{i,\, x>x_i \\ \hat{\imath}<k}} |\epsilon_i| \\ &\quad + \begin{cases} \displaystyle\sum_{\substack{i\in\mathcal{D}(u^\delta) \\ \hat{\imath}=k,\, x_i<x}} |\epsilon_i| + \sum_{\substack{i\in\mathcal{D}(v^\delta) \\ \hat{\imath}=k,\, x<x_i}} |\epsilon_i| & \text{if } q_k(x)<0, \\ \displaystyle\sum_{\substack{i\in\mathcal{D}(v^\delta) \\ \hat{\imath}=k,\, x_i<x}} |\epsilon_i| + \sum_{\substack{i\in\mathcal{D}(u^\delta) \\ \hat{\imath}=k,\, x<x_i}} |\epsilon_i| & \text{if } q_k(x)>0. \end{cases} \end{aligned} \tag{7.18}$$

In plain words, a q_k shock is approached by k-waves in u^δ from the left, and k-waves in v^δ from the right. Similarly, a q_k rarefaction wave is approached by k-waves in v^δ from the left and k-waves in u^δ from the right.

Once the values of the constants κ_1 and κ_2 are determined, we will assume that the total variations of u^δ and v^δ are so small that

$$1 \le W_k(x) \le 2. \tag{7.19}$$

In this case we see that Φ is equivalent to the L^1 norm; i.e., there exists a finite constant C_1 such that

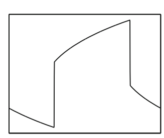
$$\frac{1}{C_1}\left\|u^\delta - v^\delta\right\|_1 \le \Phi\left(u^\delta, v^\delta\right) \le C_1\left\|u^\delta - v^\delta\right\|_1. \tag{7.20}$$

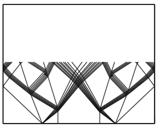
We can also define, with obvious modifications, $\Phi(u^{\delta_1}(t), v^{\delta_2}(t))$ with two different parameters δ_1 and δ_2. Our first goal will be to show that

$$\Phi\left(u^{\delta_1}(t), v^{\delta_2}(t)\right) - \Phi\left(u^{\delta_1}(s), v^{\delta_2}(s)\right) \le C_2(t-s)\big(\delta_1 \vee \delta_2\big), \tag{7.21}$$

for all $0 \le t \le s$. Once this inequality is in place, we can show that the sequence of front-tracking approximations is a Cauchy sequence in L^1 for

$$\begin{aligned} \left\|u^{\delta_1}(t) - u^{\delta_2}(t)\right\|_1 &\le C_1\Phi\left(u^{\delta_1}(t), u^{\delta_2}(t)\right) \\ &\le C_1\Phi\left(u^{\delta_1}(0), u^{\delta_2}(0)\right) + C_1C_2t\big(\delta_1 \vee \delta_2\big) \\ &\le C_1^2\left\|u^{\delta_1}(0) - u^{\delta_2}(0)\right\|_1 + C_1C_2t\big(\delta_1 \vee \delta_2\big). \end{aligned}$$





Letting δ_1 and δ_2 tend to zero, we have the convergence of the whole sequence, and not only a subsequence.

The first step in order to prove (7.21) is to choose κ_2 so large that the weights W_k do not increase when fronts in u^{δ_1} or v^{δ_2} collide. This is possible, since the total variations of both u^{δ_1} and v^{δ_2} are uniformly small; hence the terms $\kappa_1 A_k$ are uniformly bounded, and by the interaction estimate, Q decreases for all collisions. This ensures the inequalities (7.19).

Then we must examine how Φ changes between collisions. Observe that $\Phi(t)$ is piecewise linear and continuous in t. Let

$$\mathcal{D} = \mathcal{D}\left(u^{\delta_1}\right) \cup \mathcal{D}\left(v^{\delta_2}\right).$$

We differentiate Φ and find that

$$\begin{aligned}
\frac{d}{dt}\Phi\left(u^{\delta_1}, v^{\delta_2}\right) &= \sum_{i\in\mathcal{D}}\sum_{k=1}^{n}\left\{\left|q_k\left(x_i-\right)\right| W_k\left(x_i-\right) - \left|q_k\left(x_i+\right)\right| W_k\left(x_i+\right)\right\}\dot{x}_i \\
&= \sum_{i\in\mathcal{D}}\sum_{k=1}^{n}\left\{\left|q_k^{i,+}\right| W_k^{i,+}\left(\mu_k^{i,+} - \dot{x}_i\right) - \left|q_k^{i,-}\right| W_k^{i,-}\left(\mu_k^{i,-} - \dot{x}_i\right)\right\}, \\
&=: \sum_{i\in\mathcal{D}}\sum_{k=1}^{n} E_{i,k},
\end{aligned} \tag{7.22}$$

where

$$\begin{aligned}
\mu_k^{i,\pm} &= \mu_k\left(x_i\pm\right), \quad \mu_k(x) = \mu_k\left(\omega_{k-1}(x), \omega_k(x)\right), \\
q_k^{i,\pm} &= q_k\left(x_i\pm\right), \quad \text{and} \quad W_k^{i,\pm} = W_k\left(x_i\pm\right).
\end{aligned}$$

The second equality in (7.22) is obtained by adding terms

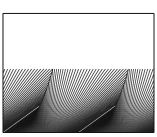
$$\left|q_k^{i,-}\right| W_k^{i,-}\mu_k^{i,-} - \left|q_k^{(i-1),+}\right| W_k^{(i-1),+}\mu_k^{(i-1),+} = 0,$$

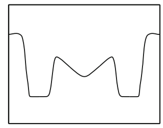
and observing that there is only a finite number of terms in the sum in (7.22).

◇ Example 7.2 (Example 7.1 (cont'd.))
Let us check how this works for our special system. The two front-tracking approximations are denoted by u and v, and for simplicity we omit the superscript δ. These are made by approximating a rarefaction wave between $\eta_l = n\delta$ and $\eta_r = m\delta$, $m > n$, by a series of discontinuities with speed $3j\delta$, $j = n, \ldots, m-1$. In other words, we use the characteristic speed to the left of the discontinuity. The functions u and v are well defined by standard techniques.

Since we managed this far without the interaction potential, we define the weights also without these (they are needed only to bound the weights, anyway). Hence for the example we use

$$W_k(x) = 1 + \kappa A_k(x). \tag{7.23}$$





Now we shall estimate

$$\frac{d}{dt}\Phi\,(u,v)=\sum_{i\in\mathcal{D}}(E_{i,1}+E_{i,2}). \tag{7.24}$$

To this end we consider a fixed discontinuity at x (to simplify the notation we do not use a subscript on this discontinuity) in one of the functions, say v. This discontinuity gives a contribution to the right-hand side of (7.24), denoted by E_1+E_2, where

$$E_j=W_j^+\left|q_j^+\right|\left(\mu_j^+-\dot{x}\right)-W_j^-\left|q_j^-\right|\left(\mu_j^--\dot{x}\right),\quad j=1,2.$$

For this 2×2 system we have

$$A_1(x)=\sum_{x_i<x,\,\hat{\imath}=2}|\epsilon_i|\,,$$

$$A_2(x)=\sum_{x_i>x,\,\hat{\imath}=1}|\epsilon_i| + \begin{cases}\displaystyle\sum_{\substack{\hat{\imath}=2,\,x_i<x\\ x_i\in\mathcal{D}(u)}}|\epsilon_i|+\sum_{\substack{\hat{\imath}=2,\,x_i>x\\ x_i\in\mathcal{D}(v)}}|\epsilon_i| & \text{if } q_2<0,\\ \displaystyle\sum_{\substack{\hat{\imath}=2,\,x_i<x\\ x_i\in\mathcal{D}(v)}}|\epsilon_i|+\sum_{\substack{\hat{\imath}=2,\,x_i>x\\ x_i\in\mathcal{D}(u)}}|\epsilon_i| & \text{if } q_2>0.\end{cases}$$

To estimate E_1+E_2 we study several cases.

Case 1 Assume first that the jump at x is a *contact discontinuity*, that is, of the first family, in which case

$$A_1^+=A_1^-,$$

and consequently,

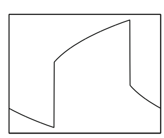
$$W_1^+=W_1^-. \tag{7.25}$$

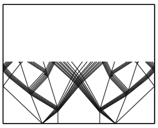
Furthermore,

$$q_1^+=q_1^-+\epsilon \quad\text{and}\quad \mu_1^+=\mu_1^-=\dot{x}-q_2^-.$$

Then

$$\begin{aligned}E_1&=W_1^+\left|q_1^+\right|\left(\mu_1^+-\dot{x}\right)-W_1^-\left|q_1^-\right|\left(\mu_1^--\dot{x}\right)\\&=W_1^-\left\{\left|q_1^-+\epsilon\right|-\left|q_1^-\right|\right\}\left(-q_2^-\right)\\&\le W_1^-\left|q_2^-\right|\left|\epsilon\right|.\end{aligned} \tag{7.26}$$





For the weights of the second family we find that

$$A_2^+ = A_2^- - |\epsilon|\,, \quad W_2^+ = W_2^- - \kappa\,|\epsilon|\,, \quad q_2^+ = q_2^-\,, \quad \mu_2^- = \mu_2^+.$$

To estimate $\mu_2^- - \dot{x}$ we exploit that μ_2^- is a discontinuity of the second family, while $\dot{x}$ is a contact discontinuity of the first family. Thus we can estimate from below their difference by the smallest difference in speeds between waves in the first- and second-wave families. We find that $\mu_2^- - \dot{x} \geq c = \min_{u,v}\{\eta\} > 0$. Hence

$$\begin{aligned} E_2 &= W_2^+ \left|q_2^+\right| \left(\mu_2^+ - \dot{x}\right) - W_2^- \left|q_2^-\right| \left(\mu_2^- - \dot{x}\right) \\ &= \left|q_2^-\right| \left(\mu_2^- - \dot{x}\right) \left(-\kappa\,|\epsilon|\right) \\ &\leq -\kappa c \left|q_2^-\right| |\epsilon|\,. \end{aligned} \tag{7.27}$$

Then

$$E_1 + E_2 = \left|q_2^-\right| |\epsilon| \left(W_1^- - \kappa c\right) \leq 0 \tag{7.28}$$

if $\kappa c \geq \sup_x W_1(x)$. (Throughout this argument we will choose larger and larger κ.) This inequality (7.28) is the desired estimate when x is a contact discontinuity.

Case 2 The case that x is a *genuinely nonlinear wave*, that is, belongs to the second family, is more complicated. There are two distinct cases, that of an (approximate) rarefaction wave and that of a shock wave. First we treat the term E_1, which is common to the two cases. Here

$$\begin{aligned} &A_1^+ = A_1^- + |\epsilon|\,, \quad W_1^+ = W_1^- + \kappa\,|\epsilon|\,, \quad q_1^+ = q_1^-, \\ &\mu_1^+ = \mu_1^-, \quad \text{and} \quad \mu_1^- - \dot{x} < -c. \end{aligned}$$

Consequently,

$$\begin{aligned} E_1 &= W_1^+ \left|q_1^+\right| \left(\mu_1^+ - \dot{x}\right) - W_1^- \left|q_1^-\right| \left(\mu_1^- - \dot{x}\right) \\ &= \kappa\,|\epsilon| \left|q_1^-\right| \left(\mu_1^- - \dot{x}\right) \\ &\leq -\kappa c \left|q_1^-\right| |\epsilon| \leq 0. \end{aligned} \tag{7.29}$$

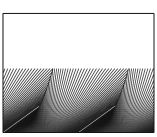
We split the estimate for E_2 into several cases.

Case 2a (rarefaction wave) First we consider the case that x is an approximate rarefaction wave. By the construction of v we have

$$\epsilon = \delta > 0 \quad \text{and} \quad q_2^+ = q_2^- + \epsilon.$$

The speeds appearing in E_2 are given by

$$\begin{aligned} \mu_2^+ &= 2\eta_u + q_2^- + \epsilon + \sqrt{\eta_u\left(\eta_u + q_2^- + \epsilon\right)}, \\ \mu_2^- &= 2\eta_u + q_2^- + \sqrt{\eta_u\left(\eta_u + q_2^-\right)}, \\ \dot{x} &= 3\left(\eta_u + q_2^-\right). \end{aligned}$$



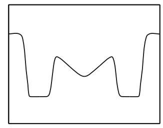
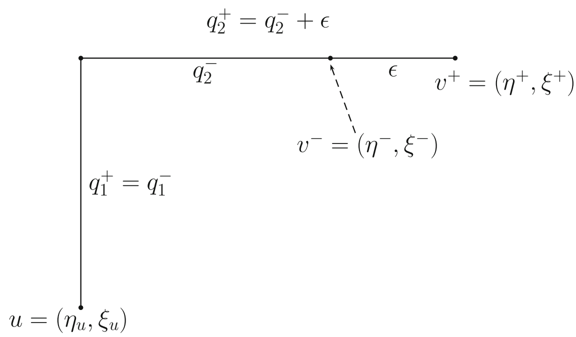


Fig. 7.2 $q_2^- > 0$



We define the auxiliary speed

$$\tilde{\mu} = \mu_2\left(v^-, v^+\right) = 2\eta_u + 2q_2^- + \epsilon + \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)}.$$

It is easily seen that

$$0 \leq \epsilon \leq \tilde{\mu} - \dot{x} \leq 2\epsilon.$$

We have several subcases. First we assume that $q_2^- > 0$, in which case $q_2^+ > 0$ as well; see Fig. 7.2.

In this case $A_2^+ = A_2^- + |\epsilon|$. Hence

$$\begin{aligned} E_2 &= W_2^+ \left|q_2^+\right| \left(\mu_2^+ - \dot{x}\right) - \left(W_2^+ - \kappa\,|\epsilon|\right) \left|q_2^-\right| \left(\mu_2^- - \dot{x}\right) \\ &= W_2^+ \left\{\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right\} \\ &\quad + W_2^+ \left(q_2^+ - q_2^-\right)(\tilde{\mu} - \dot{x}) + \kappa\,|\epsilon|\left|q_2^-\right|(\mu_2^- - \dot{x}). \end{aligned}$$

We need to estimate the term $\left\{\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right\}$. This estimate is contained in Lemma 7.4 in the general case, and it is verified directly for this model right after the proof of Lemma 7.4. We obtain

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right| \leq \mathcal{O}\left(1\right)|\epsilon|\left|q_2^-\right|\left(\left|q_2^-\right| + |\epsilon|\right),$$

and thus

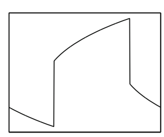
$$\begin{aligned} E_2 &\leq \mathcal{O}\left(1\right)|\epsilon|\left|q_2^-\right|\left(\left|q_2^-\right| + |\epsilon|\right) + W_2^+\,|\epsilon|\,|\tilde{\mu} - \dot{x}| + \kappa\,|\epsilon|\left|q_2^-\right|(\mu_2^- - \dot{x}) \\ &\leq \mathcal{O}\left(1\right)|\epsilon|\left|q_2^-\right|\left(\left|q_2^-\right| + |\epsilon|\right) + 2W_2^+\,|\epsilon|^2 + \kappa\,|\epsilon|\left|q_2^-\right|(\mu_2^- - \dot{x}). \end{aligned}$$

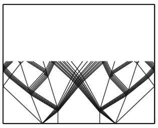
We estimate $\mu_2^- - \dot{x} \leq -q_2^- \leq 0$, and hence

$$E_2 \leq |\epsilon|\left|q_2^-\right|^2(\mathcal{O}\left(1\right) - \kappa) + \mathcal{O}\left(1\right)|\epsilon|^2\left|q_2^-\right| + \mathcal{O}\left(1\right)|\epsilon|^2 \leq M\,|\epsilon|\,\delta,$$

for some constant M if we choose κ big enough. We have used that W_2^+ is bounded. Therefore,

$$E_1 + E_2 \leq M\,|\epsilon|\,\delta.$$





Now for the case $q_2^- < 0$. Here we have two further subcases, $q_2^+ < 0$ and $q_2^+ > 0$. First we assume that $q_2^+ < 0$, and thus both q_2^- and q_2^+ are negative. Note that

$$\left|q_2^+\right| = |q_2^-| - |\epsilon|\,, 0 \le -q_2^- \le \mu_2^- - \dot{x} \le -2q_2^-, \quad \text{and} \quad A_2^+ = A_2^- - |\epsilon|\,.$$

Thus

$$\begin{aligned} E_2 &= \left(W_2^- - \kappa\,|\epsilon|\right)\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - W_2^-\,|q_2^-|\left(\mu_2^- - \dot{x}\right) \\ &= W_2^-\left\{\left(q_2^+ - \epsilon\right)\left(\mu_2^- - \tilde{\mu}\right) - q_2^+\left(\mu_2^+ - \tilde{\mu}\right)\right\} \\ &\quad - W_2^-\,|\epsilon|\,(\tilde{\mu} - \dot{x}) - \kappa\,|\epsilon|\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) \\ &\le \mathcal{O}(1)\,|\epsilon|\left|q_2^+\right|\left(\left|q_2^+\right| + |\epsilon|\right) + \mathcal{O}(1)\,|\epsilon|^2 - \kappa\,|\epsilon|\left|q_2^+\right|^2 \\ &\le |\epsilon|\,|q_2^-|^2\,(\mathcal{O}(1) - \kappa) + \mathcal{O}(1)\,|\epsilon|^2 \\ &\le M\,|\epsilon|\,\delta, \end{aligned}$$

where we have used Lemma 7.4 (with $\varepsilon = \epsilon$, $\varepsilon' = q_2^+$) and chosen κ sufficiently large. Thus we conclude that $E_1 + E_2 \le M\,|\epsilon|\,\delta$ in this case as well.

Now for the last case in which $\epsilon > 0$, namely $q_2^- < 0 < q_2^+$. Since $q_2^+ = q_2^- + \epsilon$, we have

$$\left|q_2^+\right| \le \delta, \qquad |q_2^-| \le \delta.$$

Furthermore, $A_2^+ = A_2^-$, and thus $W_2^+ = W_2^-$. We see that

$$0 \le -q_2^- \le \mu_2^- - \dot{x} \le -2q_2^-, \qquad \mu_2^+ - \dot{x} \le 2\epsilon - q_2^-,$$

and hence

$$\begin{aligned} E_2 &= W_2^+\left\{q_2^+\left(\mu_2^+ - \dot{x}\right) + |q_2^-|\left(\mu_2^- - \dot{x}\right)\right\} \\ &\le W_2^+\left\{q_2^+\left(2\,|\epsilon| + q_2^-\right) + |q_2^-|\;2\;|q_2^-|\right\} \\ &\le M\,|\epsilon|\,\delta, \end{aligned}$$

for some constant M.

Case 2b (shock wave) When x is a shock front, we have $\epsilon < 0$. In this case,

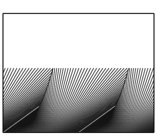
$$\dot{x} = \tilde{\mu} = \mu_2\left(v^-, v^+\right) = 2\eta_u + 2q_2^- + \epsilon + \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)}.$$

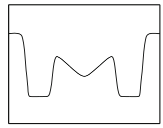
We first consider the case $q_2^- < 0$. Then

$$q_2^+ = q_2^- + \epsilon < 0, \quad \left|q_2^+\right| = |q_2^-| + |\epsilon|\,, \quad \text{and} \quad A_2^+ = A_2^- - |\epsilon|\,,$$

and we obtain

$$\begin{aligned} E_2 &= \left(W_2^- - \kappa\,|\epsilon|\right)\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - W_2^-\,|q_2^-|\left(\mu_2^- - \dot{x}\right) \\ &= -W_2^-\left((q_2^- + \epsilon)(\mu_2^+ - \dot{x}) - q_2^-(\mu_2^- - \dot{x})\right) \\ &\quad - \kappa\,|\epsilon|\,(|q_2^-| + |\epsilon|)(\mu_2^+ - \dot{x}) \\ &\le \mathcal{O}(1)\,|\epsilon|\,|q_2^-|\,(|q_2^-| + |\epsilon|) - \kappa\,|\epsilon|\,(|q_2^-| + |\epsilon|)\,|q_2^-| \\ &\le |\epsilon|\,|q_2^-|\,(|q_2^-| + |\epsilon|)\,(\mathcal{O}(1) - \kappa) \le 0. \end{aligned}$$





Lemma 7.4 (with $\varepsilon' = \epsilon$, $\varepsilon = q_2^-$) implies

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \dot{x}\right) - q_2^-\left(\mu_2^- - \dot{x}\right)\right| \leq \mathcal{O}(1)\,|\epsilon|\,|q_2^-|\left(|q_2^-| + |\epsilon|\right).$$

Furthermore,

$$\begin{aligned}
\mu_2^+ - \dot{x} &= -q_2^- + \sqrt{\eta_u\left(\eta_u + q_2^- + \epsilon\right)} \\
&\qquad - \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)} \\
&= -q_2^-\left(1 + \frac{\sqrt{\eta_u + q_2^+}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) \\
&\geq -q_2^- = |q_2^-|.
\end{aligned}$$

If $q_2^- > 0$, then there are two further cases to be considered, depending on the sign of q_2^+. We first consider the case $q_2^+ < 0$, and thus $q_2^+ < 0 < q_2^-$. Now $A_2^+ = A_2^-$. Furthermore,

$$\begin{aligned}
&\mu_2^- - \dot{x} \geq -2q_2^- \geq 0, \\
&\mu_2^+ - \dot{x} = -q_2^-\left(1 + \frac{\sqrt{\eta_u + q_2^+}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) < -|q_2^-|.
\end{aligned}$$

Thus

$$\mu_2^+ < \dot{x} < \mu_2^-,$$

and we easily obtain

$$E_2 = W_2^-\left\{\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - |q_2^-|\left(\mu_2^- - \dot{x}\right)\right\} < 0.$$

This leaves the final case $q_2^\pm > 0$. In this case we have that $A_2^+ = A_2^- + |\epsilon|$. We still have

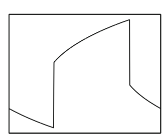
$$\mu_2^- - \dot{x} = -q_2^+\left(1 + \frac{\sqrt{\eta_u + q_2^-}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) \leq -q_2^+ < 0,$$

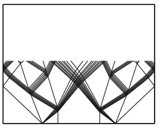
and thus

$$|\dot{x} - \mu_2^-| \geq q_2^+.$$

Furthermore, by Lemma 7.4, we have that

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \dot{x}\right) - q_2^-\left(\mu_2^- - \dot{x}\right)\right| \leq \mathcal{O}(1)\left|q_2^+\right||\epsilon|\left(\left|q_2^+\right| + |\epsilon|\right).$$





Then we calculate

$$\begin{aligned}
E_2 &= W_2^+ \left|q_2^+\right| \left(\mu_2^+ - \dot{x}\right) - \left(W_2^+ - \kappa \left|\epsilon\right|\right) \left|q_2^-\right| \left(\mu_2^- - \dot{x}\right) \\
&= W_2^+ \left((q_2^- + \epsilon)(\mu_2^+ - \dot{x}) - q_2^- (\mu_2^- - \dot{x})\right) + \kappa \left|\epsilon\right| \left|q_2^-\right| \left(\mu_2^- - \dot{x}\right) \\
&\le W_2^+ \left|q_2^+ \left(\mu_2^+ - \dot{x}\right) - q_2^- \left(\mu_2^- - \dot{x}\right)\right| - \kappa \left|\epsilon\right| \left|\mu_2^- - \dot{x}\right| \left|q_2^-\right| \\
&\le \mathcal{O}\left(1\right) \left|\epsilon\right| \left|q_2^-\right| \left(\left|q_2^-\right| + \left|\epsilon\right|\right) - \kappa \left|\epsilon\right| \left|q_2^-\right| \left|q_2^+\right| \\
&\le \mathcal{O}\left(1\right) \left|\epsilon\right|^2 + \left|\epsilon\right| \left|q_2^-\right|^2 \left(\mathcal{O}\left(1\right) - \kappa\right) \\
&\le M \left|\epsilon\right| \delta
\end{aligned}$$

if κ is sufficiently large. This is the last case.

Now we have shown that in all cases,

$$E_1 + E_2 \le M \left|\epsilon\right| \delta.$$

Summing over all discontinuities in u and v we conclude that

$$\frac{d}{dt}\Phi(u, v) \le C'\delta,$$

for some finite constant C' independent of δ.

We shall now show that

$$\sum_{k=1}^{n} E_{i,k} \le \mathcal{O}(1) \left|\epsilon_i\right| \left(\delta_1 \vee \delta_2\right) + \mathcal{O}\left(1\right) \left|e_i\right|, \tag{7.30}$$

and this estimate is easily seen to imply (7.21). To prove (7.30) we shall need some preliminary results:

Lemma 7.3 *Assume that the vectors $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, $\epsilon' = \left(\epsilon_1', \dots, \epsilon_n'\right)$, and $\epsilon'' = \left(\epsilon_1'', \dots, \epsilon_n''\right)$ satisfy*

$$H\left(\epsilon\right) u = H\left(\epsilon''\right) H\left(\epsilon'\right) u$$

for some vector u, where

$$H\left(\epsilon\right) = H_n\left(\epsilon_n\right) H_{n-1}\left(\epsilon_{n-1}\right) \cdots H_1\left(\epsilon_1\right).$$

Then

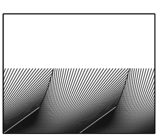
$$\sum_{k=1}^{n} \left|\epsilon_k - \epsilon_k' - \epsilon_k''\right| = \mathcal{O}\left(1\right) \left(\sum_j \left|\epsilon_j' \epsilon_j''\right| \left(\left|\epsilon_j'\right| + \left|\epsilon_j''\right|\right) + \sum_{\substack{k,l \\ k \ne l}} \left|\epsilon_j' \epsilon_l''\right|\right). \tag{7.31}$$

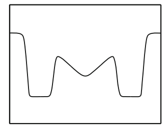
If the scalar ϵ and the vector $\epsilon' = \left(\epsilon_1', \dots, \epsilon_n'\right)$ satisfy

$$R_l\left(\epsilon\right) u = H\left(\epsilon'\right) u,$$

where R_l denotes the lth rarefaction curve, then

$$\left|\epsilon - \epsilon_l'\right| + \sum_{k \ne l} \left|\epsilon_k'\right| = \mathcal{O}\left(1\right) \left|\epsilon\right| \left(\left|\epsilon_l'\right| \left(\left|\epsilon\right| + \left|\epsilon_l'\right|\right) + \sum_{k \ne l} \left|\epsilon_k'\right|\right). \tag{7.32}$$





Proof The proof of this lemma is a straightforward modification of the proof of the interaction estimate (6.18). □

Lemma 7.4 *Let $\bar{\omega} \in \Omega$ be sufficiently small, and let ε and ε' be real numbers. Define*

$$\begin{aligned} \omega &= H_k(\varepsilon)\bar{\omega}, & \mu &= \mu_k\left(\bar{\omega},\omega\right), \\ \omega' &= H_k\left(\varepsilon'\right)\omega, & \mu' &= \mu_k\left(\omega,\omega'\right), \\ \omega'' &= H_k\left(\varepsilon+\varepsilon'\right)\bar{\omega}, & \mu'' &= \mu_k\left(\bar{\omega},\omega''\right). \end{aligned}$$

Then one has

$$|(\varepsilon+\varepsilon')(\mu''-\mu')-\varepsilon(\mu-\mu')| \leq \mathcal{O}(1)\,|\varepsilon\varepsilon'|\left(|\varepsilon|+|\varepsilon'|\right). \tag{7.33}$$

Proof The proof of this is again in the spirit of the proof of the interaction estimate, equation (6.13). Let the function Ψ be defined as

$$\Psi\left(\varepsilon,\varepsilon'\right) = \left(\varepsilon+\varepsilon'\right)\mu'' - \varepsilon\mu - \varepsilon'\mu'.$$

Then Ψ is at least twice differentiable, and satisfies

$$\Psi\left(\varepsilon,0\right) = \Psi\left(0,\varepsilon'\right) = 0, \qquad \frac{\partial^2\Psi}{\partial\varepsilon\partial\varepsilon'}(0,0) = 0.$$

Consequently,

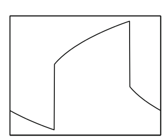
$$\Psi\left(\varepsilon,\varepsilon'\right) = \int_0^{\varepsilon}\int_0^{\varepsilon'} \frac{\partial^2\Psi}{\partial\varepsilon\partial\varepsilon'}(r,s)\,ds\,dr = \mathcal{O}(1)\int_0^{|\varepsilon|}\int_0^{|\varepsilon'|}\left(|r|+|s|\right)\,dr\,ds.$$

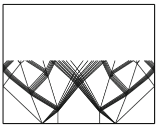
From this the lemma follows. □

◇ Example 7.5 (Lemma 7.4 for Example 7.1)
If $k=2$, let $\bar{\omega}$, ω', and ω'' denote the η-coordinate, since only this will influence the speeds. Then a straightforward calculation yields

$$\begin{aligned} &\left|(\varepsilon+\varepsilon')(\mu''-\mu')-\varepsilon(\mu-\mu')\right| \\ &\quad = |\varepsilon|\,|\varepsilon'|\left(|\varepsilon|+|\varepsilon'|\right) \\ &\qquad \times \frac{\sqrt{\bar{\omega}}+\sqrt{\omega'}+\sqrt{\omega''}}{\bar{\omega}\left(\sqrt{\omega'}+\sqrt{\omega''}\right)+\omega'\left(\sqrt{\bar{\omega}}+\sqrt{\omega''}\right)+\omega''\left(\sqrt{\bar{\omega}}+\sqrt{\omega'}\right)+2\sqrt{\bar{\omega}\omega'\omega''}} \\ &\quad \leq \frac{|\varepsilon|\,|\varepsilon'|\left(|\varepsilon|+|\varepsilon'|\right)}{\min\left\{\bar{\omega},\omega',\omega''\right\}}, \end{aligned}$$

verifying the lemma in this case. ◇





If the kth characteristic field is genuinely nonlinear, then the characteristic speed $\lambda_k\left(H_k(\epsilon)\omega\right)$ is increasing in ϵ, and we can even choose the parameterization such that

$$\lambda_k\left(H_k(\epsilon)\omega\right) - \lambda_k(\omega) = \epsilon,$$

for all sufficiently small ϵ and ω. This also implies that $\mu_k(\omega, H_k(\epsilon)\omega)$ is strictly increasing in ϵ. However, the Hugoniot locus through the point ω does not in general coincide with the Hugoniot locus through the point $H_k(q)\omega$. Therefore, it is not so straightforward comparing speeds defined on different Hugoniot loci. When proving (7.30) we shall need to do this, and we repeatedly use the following lemma:

Lemma 7.6 *For some state ω define*

$$\Psi(q) = \mu_k\left(H_k(q)\omega, H_k(\epsilon)H_k(q)\omega\right) - \mu_k\left(\omega, H_k(\epsilon + q)\omega\right).$$

Then Ψ is at least twice differentiable for all $k = 1, \dots, n$. Furthermore, if the kth characteristic field is genuinely nonlinear, then for sufficiently small $|q|$ and $|\epsilon|$,

$$\Psi'(q) \geq c > 0, \tag{7.34}$$

where c depends only on f for all sufficiently small $|\omega|$.

Proof Let the vector ϵ' be defined by $\mathcal{H}(\epsilon')\omega = H_k(\epsilon)H_k(q)\omega$. Then by Lemma 7.3,

$$\left|\epsilon'_k - (q+\epsilon)\right| + \sum_{i \neq k} |\epsilon'_i| \leq \mathcal{O}(1)\,|q\epsilon|\left(|\epsilon| + |q|\right).$$

Consequently,

$$H_k(\epsilon + q)\omega = H_k(\epsilon)H_k(q)\omega + \mathcal{O}(1)\,|q\epsilon|\left(|\epsilon| + |q|\right).$$

Using this we find that

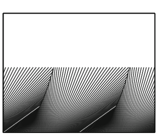
$$\left|\frac{H_k(\epsilon)H_k(q)\omega - H_k(\epsilon)\omega}{q}\right| = \left|\frac{H_k(\epsilon+q)\omega - H_k(\epsilon)\omega}{q}\right| + \mathcal{O}(1)\,|\epsilon|\left(|\epsilon| + |q|\right).$$

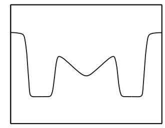
Therefore,

$$\frac{d}{dq}\left\{H_k(\epsilon)H_k(q)\omega\right\}\Big|_{q=0} = \frac{d}{d\epsilon}\left\{H_k(\epsilon)\omega\right\} + \mathcal{O}(1)\,|\epsilon|^2. \tag{7.35}$$

Hence, we compute

$$\begin{aligned}
\Psi'(0) &= \nabla_1\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot r_k(\omega) \\
&\quad - \nabla_2\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot\left(\frac{d}{d\epsilon}\left\{H_k(\epsilon)\omega\right\} - \frac{d}{dq}\left\{H_k(\epsilon)H_k(q)\omega\right\}\Big|_{q=0}\right) \\
&= \nabla_1\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot r_k(\omega) + \mathcal{O}(1)\,|\epsilon|^2 \\
&\geq c' > 0,
\end{aligned}$$





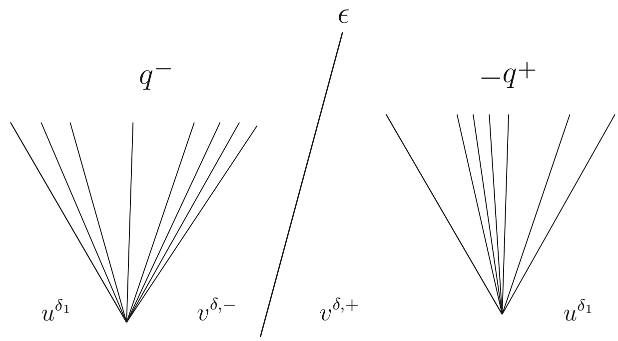


Fig. 7.3 The setting in the proof of (7.30)

for sufficiently small $|\epsilon|$. The value of the constant c' (and its existence) depends on the genuine nonlinearity of the system and hence on f. Since Ψ' is continuous for small $|q|$, the lemma follows. □

We shall prove (7.30) in the case that the front at x_i is a front in v^{δ_2}; the case in which it is a front in u^{δ_1} is completely analogous. We therefore fix i, and study the relation between q_k^- and q_k^+. Since the front is going to be fixed from now on, we drop the subscript i. For simplicity we write $\delta = \delta_2$. Assume the the family of the front x is l and the front has strength ϵ. The situation is as in Fig. 7.3.

A key observation is that we can regard the waves q_k^+ as the result of an interaction between the waves q_k^- and ϵ; similarly, the waves $-q_k^-$ are the result of an interaction between ϵ and $-q_k^+$.

Regarding the weights, from (7.16) and (7.18) we find that

$$W_k^+ - W_k^- = \begin{cases} \kappa_1 |\epsilon| & \text{if } k < l, \\ -\kappa_1 |\epsilon| & \text{if } k > l, \end{cases} \tag{7.36}$$

while for $k = l$ we obtain

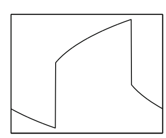
$$W_l^+ - W_l^- = \begin{cases} \kappa_1 |\epsilon| & \text{if } \min\left\{q_l^-, q_l^+\right\} > 0, \\ -\kappa_1 |\epsilon| & \text{if } \max\left\{q_l^-, q_l^+\right\} < 0, \\ \mathcal{O}(1) & \text{if } q_l^- q_l^+ < 0. \end{cases} \tag{7.37}$$

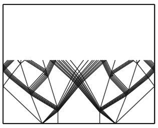
The proof of (7.30) is a study of cases. We split the estimate into two subgroups, depending on whether the front at x is an approximate rarefaction wave or a shock. Within each subgroup we discuss three subcases depending on the signs of $q_l^{\pm}$. In all cases we discuss the terms E_k $(k \neq l)$ and E_l separately. For $k \neq l$ we write E_k (recall that we dropped the subscript i) as

$$\begin{aligned} E_k &= \left(\left|q_k^+\right| - |q_k^-|\right) W_k^+ \left(\mu_k^+ - \dot{x}\right) \\ &\quad + |q_k^-| \left(W_k^+ - W_k^-\right)\left(\mu_k^+ - \dot{x}\right) + |q_k^-| \, W_k^- \left(\mu_k^+ - \mu_k^-\right). \end{aligned} \tag{7.38}$$

By the strict hyperbolicity of the system, we have that

$$\begin{aligned} \mu_k^+ - \dot{x} &\le -c < 0, \quad \text{for } k < l, \\ \mu_k^+ - \dot{x} &\ge c > 0, \quad \text{for } k > l, \end{aligned}$$





where c is some fixed constant depending on the system. Thus we always have that

$$\left(W_k^+ - W_k^-\right)\left(\mu_k^+ - \dot{x}\right) \leq -c\kappa_1 \left|\epsilon\right|, \quad k \neq l. \tag{7.39}$$

We begin with the case that the front at x is an approximate rarefaction wave ($\epsilon > 0$). In this case,

$$R_l(\epsilon)v^{\delta,-} + e = H\left(q^+\right)u^{\delta_1} = H\left(q^+\right)H\left(-q^-\right)v^{\delta,-} = H\left(\tilde{q}\right)v^{\delta,-}$$

for some vector $\tilde{q}$. Hence

$$H\left(-q^-\right)v^{\delta,-} = H\left(-q^+\right)H\left(\tilde{q}\right)v^{\delta,-}, \tag{7.40}$$

$$R_l(\epsilon)v^{\delta,-} + e = H\left(\tilde{q}\right)v^{\delta,-}. \tag{7.41}$$

From (7.31) and (7.40) we obtain

$$\sum_k \left|q_k^+ - q_k^- - \tilde{q}_k\right| \mathcal{O}\left(1\right)\Big(\sum_k \left|q_k^+\tilde{q}_k\right|\left(\left|q_k^+\right| + \left|\tilde{q}_k\right|\right) + \sum_{\substack{k,j\\k\neq j}} \left|q_k^+\tilde{q}_j\right|\Big), \tag{7.42}$$

and from (7.32) and (7.41) we obtain

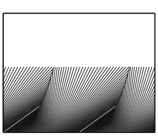
$$\left|\tilde{q}_l - \epsilon\right| + \sum_{k\neq l}\left|\tilde{q}_k\right| = \mathcal{O}\left(1\right)\left|\epsilon\right|\Big(\left|\tilde{q}_l\right|\left(\left|\tilde{q}_l\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|\tilde{q}_k\right|\Big) + \mathcal{O}\left(1\right)\left|e\right|.$$

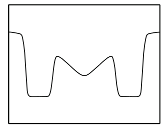
This implies that

$$\begin{aligned} \left|\tilde{q}_l - \epsilon\right| &\leq \mathcal{O}\left(1\right)\left|\epsilon\right| + \mathcal{O}\left(1\right)\left|e\right|, \\ \sum_{k\neq l}\left|\tilde{q}_k\right| &\leq \mathcal{O}\left(1\right)\left|\epsilon\right| + \mathcal{O}\left(1\right)\left|e\right|. \end{aligned} \tag{7.43}$$

Furthermore, since ϵ is an approximate rarefaction, $0 \leq \epsilon \leq \delta$. Therefore, we can replace $\tilde{q}_l$ with ϵ and $\tilde{q}_k$ ($k \neq l$) with zero on the right-hand side of (7.42), making an error of $\mathcal{O}\left(1\right)\delta$. Indeed,

$$\begin{aligned} &\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k\neq l}\left|q_k^+ - q_k^-\right| \\ &\quad\leq \sum_k \left|q_k^+ - q_k^- - \epsilon\right| + \left|\tilde{q}_l - \epsilon\right| + \sum_{k\neq l}\left|\tilde{q}_k\right| \\ &\quad\leq \mathcal{O}\left(1\right)\Bigg(\sum_k \left|q_k^+\tilde{q}_k\right|\left(\left|q_k^+\right| + \left|\tilde{q}_k\right|\right) + \sum_{\substack{k,j\\k\neq j}}\left|q_k^+\tilde{q}_j\right|\Bigg) \\ &\qquad + \mathcal{O}\left(1\right)\left|\epsilon\right|\Big(\left|\tilde{q}_l\right|\left(\left|\tilde{q}_l\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|\tilde{q}_k\right|\Big) + \mathcal{O}\left(1\right)\left|e\right|. \end{aligned}$$





Using (7.43) and the fact that $\epsilon \leq \delta$, we conclude that

$$\begin{aligned} &\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\ &\quad = \mathcal{O}(1)\,|\epsilon| \left(\delta + \left|q_l^+\right| \left(\left|q_l^+\right| + |\epsilon|\right) + \sum_{k \neq l} \left|q_k^+\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.44}$$

Similarly,

$$\begin{aligned} &\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\ &\quad = \mathcal{O}(1)\,|\epsilon| \left(\delta + \left|q_l^-\right| \left(\left|q_l^-\right| + |\epsilon|\right) + \sum_{k \neq l} \left|q_k^-\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.45}$$

Since in this case $0 \leq \epsilon \leq \delta$, and the total variation is small, we can assume that the right-hand sides of (7.44)–(7.45) are smaller than $\epsilon + \mathcal{O}(1)\,|e|$. Also, the error e is small; cf. (7.11). Then

$$0 < q_l^+ - q_l^- < 2\epsilon + \mathcal{O}(1)\,|e| \leq 2\delta + \mathcal{O}(1)\,|e|\,. \tag{7.46}$$

We can also use the estimates (7.44) and (7.45) to make a simplifying assumption throughout the rest of our calculations. Since the total variation of $u - v$ is uniformly bounded, we can assume that the right-hand sides of (7.44) and (7.45) are bounded by

$$\frac{1}{2}\,|\epsilon| + \mathcal{O}(1)\,|e|\,.$$

In particular, we then find that

$$\epsilon - \frac{1}{2}\,|\epsilon| - \mathcal{O}(1)\,|e| \leq q_\ell^+ - q_\ell^- \leq \epsilon + \frac{1}{2}\,|\epsilon| + \mathcal{O}(1)\,|e|\,.$$

Hence if $\epsilon > 0$, from the left inequality we find that

$$q_\ell^+ > q_\ell^-$$

or

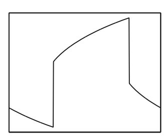
$$|\epsilon| \leq \mathcal{O}(1)\,|e|\,,$$

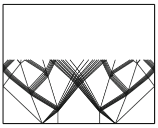
and if $\epsilon < 0$, from the right inequality above,

$$q_\ell^+ < q_\ell^-$$

or

$$|\epsilon| \leq \mathcal{O}(1)\,|e|\,.$$





If $\epsilon > 0$ and $q_\ell^- \geq q_\ell^+$ or $\epsilon < 0$ and $q_\ell^+ \geq q_\ell^-$, then $|\epsilon| \leq \mathcal{O}(1)\,|e|$. In this case we find for $k \neq l$, or $k = l$ and $q_\ell^- q_\ell^+ > 0$, that

$$\begin{aligned} E_k &= \left\{|q_k^-|\left(W_k^- - W_k^+\right) + W_k^+\left(|q_k^-| - |q_k^+|\right)\right\}\dot{x} \\ &\leq \left\{|q_k^-|\,\kappa_1\,|\epsilon| + \left|W_k^+\right|\left(|\epsilon|/2 + \mathcal{O}(1)\,|e|\right)\right\}|\dot{x}| \\ &\leq \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.47}$$

If $k = l$ and $q_\ell^- q_\ell^+ < 0$, then for $\epsilon > 0$ we have that $q_\ell^+ - q_\ell^- \geq \mathcal{O}(1)\,|e|$, so if $q_\ell^+ < q_\ell^-$, we must have that

$$\left|q_\ell^+\right| \leq \mathcal{O}(1)\,|e| \quad \text{and} \quad q_\ell^- \leq \mathcal{O}(1)\,|e|\,.$$

Similarly, if $\epsilon < 0$ and $q_\ell^+ > q_\ell^-$, we obtain

$$q_\ell^+ < \mathcal{O}(1)\,|e| \quad \text{and} \quad |q_\ell^-| \leq \mathcal{O}(1)\,|e|\,.$$

Then we find that

$$E_l = \left\{|q_\ell^-|\,W_l^- - \left|q_\ell^+\right|W_l^+\right\}\dot{x} \leq \mathcal{O}(1)\,|e|\,. \tag{7.48}$$

These observations imply that if $|\epsilon| = \mathcal{O}(1)\,|e|$, we have that

$$\sum_k E_k = \mathcal{O}(1)\,|e|\,,$$

which is what we want to show. Thus in the following we can assume that either

$$\epsilon > 0 \quad \text{and} \quad q_\ell^+ > q_\ell^-\,,$$

or

$$\epsilon < 0 \quad \text{and} \quad q_\ell^+ < q_\ell^-\,. \tag{7.49}$$

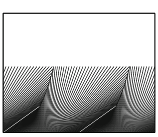
Now follows a discussion of several different cases, depending on whether the front is an approximate rarefaction wave or a shock wave, and on the signs of q_ℓ^- and q_ℓ^+.

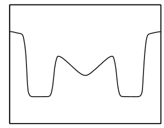
Case R1 $0 < q_l^- < q_l^+$, $\epsilon > 0$.
For $k \neq l$ we recall (7.38) that

$$\begin{aligned} E_k &= \left(\left|q_k^+\right| - |q_k^-|\right)W_k^+\left(\mu_k^+ - \dot{x}\right) \\ &\quad + |q_k^-|\left(W_k^+ - W_k^-\right)\left(\mu_k^+ - \dot{x}\right) + |q_k^-|\,W_k^-\left(\mu_k^+ - \mu_k^-\right). \end{aligned} \tag{7.50}$$

The second term in (7.50) is less than or equal to (cf. (7.39))

$$-c\kappa_1\,|q_k^-|\,|\epsilon|\,.$$





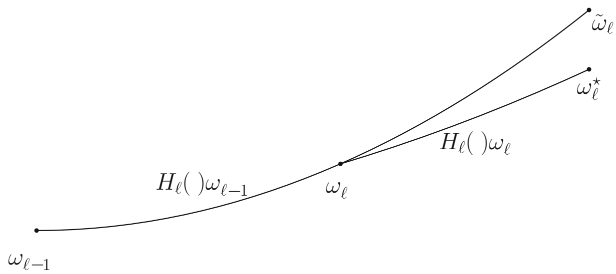


Fig. 7.4 The situation for $0 < q_l^- < q_l^+, \epsilon > 0$, and $k = l$

Furthermore, by (7.45),

$$\left|q_k^+\right| - \left|q_k^-\right| \le \mathcal{O}(1)\,|\epsilon| \left(\delta + \left|q_l^-\right| \left(\left|q_l^-\right| + |\epsilon|\right) + \sum_{k \ne l} \left|q_k^-\right|\right) + \mathcal{O}(1)\,|e|\,.$$

By the continuity of μ_k,

$$\left|\mu_k^+ - \mu_k^-\right| = \mathcal{O}(1)\left(|\epsilon| + |e|\right).$$

Hence from (7.38), we find that

$$\begin{aligned}
E_k &\le \mathcal{O}(1)\,|\epsilon| \left(\delta + \left|q_l^-\right| \left(\left|q_l^-\right| + |\epsilon|\right) + \sum_{\tilde{k} \ne l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e| - c\kappa_1 \left|q_k^-\right| |\epsilon| \\
&\le \mathcal{O}(1)\,|\epsilon| \left(\delta + \sum_{\tilde{k} \ne l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e| \\
&\quad - c\kappa_1\,|\epsilon| \left|q_k^-\right| + \mathcal{O}(1)\,|\epsilon| \left|q_l^-\right| \left(\left|q_l^-\right| + |\epsilon|\right).
\end{aligned} \tag{7.51}$$

For $k = l$ the situation is more complicated. We define states and speeds

$$\begin{aligned}
\tilde{\omega}_\ell &= H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-, & \tilde{\mu}_\ell &= \mu_l\left(\omega_{l-1}^-, \tilde{\omega}_\ell\right), \\
\omega_\ell^\star &= H_l(\epsilon)\,\omega_l^-, & \mu_\ell^\star &= \mu_l\left(\omega_l^-, \omega_\ell^\star\right);
\end{aligned} \tag{7.52}$$

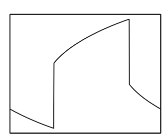
see Fig. 7.4.

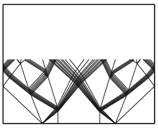
Recall that

$$\mu_l^\pm = \mu_l\left(\omega_{l-1}^\pm, \omega_l^\pm\right).$$

Now by Lemma 7.4, with $\omega = \omega_{l-1}^-$, $\varepsilon = q_\ell^-$, and $\varepsilon'' = q_\ell^- + \epsilon$,

$$\left|\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^-\left(\mu_l^- - \mu_\ell^\star\right)\right| = \mathcal{O}(1)\left|q_\ell^-\right| |\epsilon| \left(\left|q_\ell^-\right| + |\epsilon|\right). \tag{7.53}$$





We also find that (cf. (7.10) and the fact that $\mu_l(u,u) = \lambda_l(u)$)

$$
\begin{aligned}
\left|\mu_\ell^\star - \dot{x}\right| &\le \left|\mu_l\left(\omega_l^-, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,-}, v^{\delta,-}\right)\right| + \mathcal{O}(1)\,\delta \\
&= \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l\left(\omega_n^-, \omega_n^-\right)\right| + \mathcal{O}(1)\,\delta \\
&\le \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l(\omega_l^-, \omega_l^-)\right| \\
&\quad + \left|\mu_l(\omega_l^-, \omega_l^-) - \mu_l(\omega_l^-, \omega_{l+1}^-)\right| \\
&\quad + \left|\mu_l(\omega_l^-, \omega_{l+1}^-) - \mu_l(\omega_{l+1}^-, \omega_{l+2}^-)\right| + \cdots \\
&\quad + \left|\mu_l(\omega_{n-1}^-, \omega_n^-) - \mu_l(\omega_n^-, \omega_n^-)\right| + \mathcal{O}(1)\,\delta \\
&\le \mathcal{O}(1)\left(|\epsilon| + \left|\omega_l^- - \omega_{l+1}^-\right| + \cdots + \left|\omega_{n-1}^-, \omega_n^-\right|\right) + \mathcal{O}(1)\,\delta \\
&\le \mathcal{O}(1)\left(|\delta| + |q_l^-| + \sum_{k>l} |q_k^-|\right).
\end{aligned}
\tag{7.54}
$$

Furthermore,

$$
\begin{aligned}
\left|\mu_l^+ - \tilde{\mu}_\ell\right| &= \left|\mu_l\left(\omega_{l-1}^+, H_l\left(q_\ell^+\right)\omega_{l-1}^+\right) - \mu_l\left(\omega_{l-1}^-, H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right)\right| \\
&\le \left|\mu_l\left(\omega_{l-1}^+, H_l\left(q_\ell^+\right)\omega_{l-1}^+\right) - \mu_l\left(H_l\left(q_\ell^+\right)\omega_{l-1}^+, \omega_{l-1}^-\right)\right| \\
&\quad + \left|\mu_l\left(H_l\left(q_\ell^+\right)\omega_{l-1}^+, \omega_{l-1}^-\right) - \mu_l\left(\omega_{l-1}^+, H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right)\right| \\
&\le \mathcal{O}(1)\left(\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|H_l\left(q_\ell^+\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right|\right) \\
&\le \mathcal{O}(1)\left(\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|H_l\left(q_\ell^+\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^+\right|\right. \\
&\quad \left. + \left|H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right|\right) \\
&\le \mathcal{O}(1)\left(\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\
&\le \mathcal{O}(1)\left(\left|q_{l-2}^+ - q_{l-2}^-\right| + \cdots + \left|q_1^+ - q_1^-\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\
&= \mathcal{O}(1)\,\epsilon\left(\delta + |q_\ell^-|\left(|q_\ell^-| + |\epsilon|\right) + \sum_{k\ne l} |q_k^-|\right) + \mathcal{O}(1)\,|e|\,.
\end{aligned}
\tag{7.55}
$$

Since the lth field is genuinely nonlinear, then by Lemma 7.6,

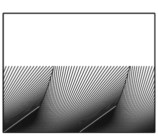
$$
\mu_\ell^\star - \tilde{\mu}_\ell \ge c\,|q_\ell^-| \tag{7.56}
$$

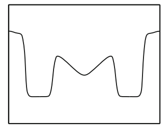
for some constant $c > 0$ depending only on the system. Recall that in this case,

$$
W_\ell^+ = W_\ell^- + \kappa_1\,|\epsilon|\,.
$$

Moreover, ϵ, q_ℓ^+, and q_ℓ^- are positive. Using the above inequalities, we compute

$$
\begin{aligned}
E_l &= W_\ell^+ q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- q_\ell^-\left(\mu_\ell^- - \dot{x}\right) \\
&= \left(W_\ell^- + \kappa_1\,|\epsilon|\right) q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- q_\ell^-\left(\mu_\ell^- - \dot{x}\right) \\
&= \kappa_1 \epsilon q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \epsilon\left\{\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - \left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right\} \\
&\quad + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \epsilon\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \epsilon\left\{\left(q_\ell^- + \epsilon\right)\left(\mu_\ell^+ - \dot{x} - \left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right) + \left(q_\ell^+ - q_\ell^- - \epsilon\right)\left(\mu_\ell^+ - \dot{x}\right)\right\} \\
&\quad + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\}
\end{aligned}
$$





$$
\begin{aligned}
&\leq \kappa_1\epsilon\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + \kappa_1\epsilon\left(q_\ell^- + \epsilon\right)\left(\left|\mu_\ell^+ - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right)\\
&\quad + \kappa_1\epsilon\left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot{x}\right| + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\}\\
&\leq -c\kappa_1 q_\ell^-\epsilon\left(q_\ell^- + \epsilon\right)\\
&\quad + \kappa_1\epsilon\left(q_\ell^- + \epsilon\right)\Bigg(\mathcal{O}(1)\,\epsilon\Big(\delta + q_\ell^-\left(q_\ell^- + \epsilon\right) + \sum_{k\neq l}\left|q_\ell^-\right|\Big)\\
&\qquad\qquad + \delta + \mathcal{O}(1)\sum_{k>l}\left|q_\ell^-\right| + \mathcal{O}(1)\left|e\right|\Bigg)\\
&\quad + \mathcal{O}(1)\,\kappa_1\epsilon^2\Big(\delta + q_\ell^-\left(q_\ell^- + \epsilon\right) + \sum_{k\neq l}\left|q_k^-\right|\Big) + \mathcal{O}(1)\left|e\right|\\
&\quad + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\}\\
&\leq -c\kappa_1 q_\ell^-\epsilon\left(q_\ell^- + \epsilon\right) + \mathcal{O}(1)\,\kappa_1\epsilon\Big(\delta + q_\ell^-\left(q_\ell^- + \epsilon\right) + \sum_{k\neq l}\left|q_k^-\right|\Big) + \mathcal{O}(1)\left|e\right|\\
&\quad + W_\ell^-\Big\{\left|\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^-\left(\mu_\ell^- - \mu_\ell^\star\right)\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot{x}\right|\\
&\qquad\quad \epsilon\left|\mu_\ell^\star - \dot{x}\right| + \left(q_\ell^- + \epsilon\right)\left|\mu_\ell^+ - \tilde{\mu}_\ell\right|\Big\}\\
&\leq -c\kappa_1\left|q_\ell^-\right|\left|\epsilon\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \mathcal{O}(1)\left|\epsilon\right|\Big(\delta + \left|q_\ell^-\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|q_k^-\right|\Big)\\
&\quad + \mathcal{O}(1)\left|e\right|\\
&\leq \mathcal{O}(1)\left|\epsilon\right|\Big(\delta + \sum_{k\neq l}\left|q_k^-\right|\Big) + \mathcal{O}(1)\left|e\right| + \left|\epsilon\right|\,\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\left(\mathcal{O}(1) - c\kappa_1\right).
\end{aligned}
\tag{7.57}
$$

Adding (7.57) and (7.51), we obtain

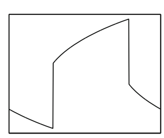
$$
\begin{aligned}
\sum_k E_k = E_l + \sum_{k\neq l} E_k
&\leq \mathcal{O}(1)\,\epsilon\delta + \mathcal{O}(1)\left|e\right| + \epsilon\sum_{k\neq l}\left|q_k^-\right|\left(\mathcal{O}(1) - c\kappa_1\right)\\
&\quad + \epsilon\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + \epsilon\right)\left(\mathcal{O}(1) - c\kappa_1\right)\\
&\leq \mathcal{O}(1)\,\epsilon\delta + \mathcal{O}(1)\left|e\right|,
\end{aligned}
\tag{7.58}
$$

which holds for sufficiently large κ_1. This implies (7.30) in Case R1.

Case R2 $q_l^- < q_l^+ < 0, \epsilon > 0.$

Writing E_k as in (7.38), and using (7.44) (instead of (7.45) as in the previous case), we find for $k \neq l$ that

$$
E_k \leq \mathcal{O}(1)\Big(\delta + \left|q_l^+\right|\left(\left|q_l^+\right| + \left|\epsilon\right|\right) + \sum_{\tilde{k}\neq l}\left|q_{\tilde{k}}^+\right|\Big) + \mathcal{O}(1)\left|e\right| - c\kappa_1\left|q_k^+\right|\left|\epsilon\right|.
\tag{7.59}
$$



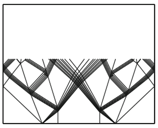
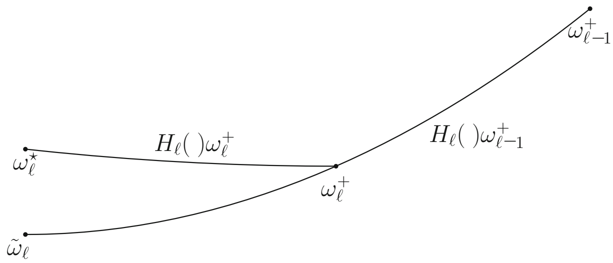


Fig. 7.5 The situation for $q_l^- < q_l^+ < 0$, $\epsilon > 0$, and $k = l$



For $k = l$ the situation is similar to the previous case. We define auxiliary states and speeds

$$\begin{aligned} \tilde{\omega}_\ell &= H_l\left(q_\ell^+ - \epsilon\right)\omega_{l-1}^+, & \tilde{\mu}_\ell &= \mu_l\left(\omega_{l-1}^+, \tilde{\omega}_\ell\right), \\ \omega_\ell^\star &= H_l\left(-\epsilon\right)\omega_l^+, & \mu_\ell^\star &= \mu_l\left(\omega_l^+, \omega_\ell^\star\right); \end{aligned} \tag{7.60}$$

see Fig. 7.5.

Recall that

$$\omega_\ell^+ = H_l\left(q_\ell^+\right)\omega_{l-1}^+ \quad \text{and} \quad \mu_\ell^+ = \mu_l\left(\omega_{l-1}^+, \omega_\ell^+\right).$$

In this case we use (7.33) with $\bar{\omega} = \omega_\ell^+$, $\varepsilon = q_\ell^+$, and $\varepsilon' = -\epsilon$. This gives

$$\left|\left(q_\ell^+ - \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^+\left(\mu_\ell^+ - \mu_\ell^\star\right)\right| = \mathcal{O}(1)\left|q_\ell^+\right|\,|\epsilon|\left(\left|q_\ell^+\right| + |\epsilon|\right). \tag{7.61}$$

As in (7.54), we find that

$$\begin{aligned} \left|\mu_\ell^\star - \dot{x}\right| &\le \left|\mu_l\left(\omega_\ell^+, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| + \mathcal{O}(1)\,\delta \\ &= \left|\mu_l\left(\omega_l^+, H_l(-\epsilon)\omega_l^+\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| + \mathcal{O}(1)\,\delta \\ &\le \mathcal{O}(1)\,\delta + \mathcal{O}(1)\left(\left|\omega_\ell^+ - \omega_n^+\right| + \epsilon\right) \\ &\le \mathcal{O}(1)\,\delta + \mathcal{O}(1)\sum_{k \ne l}\left|q_k^+\right|. \end{aligned} \tag{7.62}$$

We also obtain the analogue of (7.55), namely,

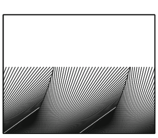
$$\begin{aligned} \left|\mu_l^- - \tilde{\mu}_\ell\right| &= \left|\mu_l\left(\omega_{l-1}^-, H_l\left(q_\ell^-\right)\omega_{l-1}^-\right) - \mu_l\left(\omega_{l-1}^+, H_l\left(q_l^+ - \epsilon\right)\omega_{l-1}^+\right)\right| \\ &= \mathcal{O}(1)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\ &= \mathcal{O}(1)\left(\delta + \left|q_\ell^+\right|\left(\left|q_\ell^+\right| + |\epsilon|\right) + \sum_{k \ne l}\left|q_k^+\right|\right) + \mathcal{O}(1)\,|e|. \end{aligned} \tag{7.63}$$

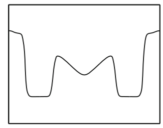
By genuine nonlinearity, using Lemma 7.6, we find that

$$\tilde{\mu}_\ell - \mu_\ell^\star > c\left|q_\ell^+\right|, \tag{7.64}$$

for some constant c. Now

$$W_\ell^- = W_\ell^+ + \kappa_1\,|\epsilon|.$$





Using the above estimates (7.61)–(7.64), we compute

$$
\begin{aligned}
E_l &= W_\ell^+ \left|q_\ell^+\right| \left(\mu_\ell^+ - \dot{x}\right) - \left(W_\ell^+ + \kappa_1 \epsilon\right) \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) \\
&= -\kappa_1 \epsilon \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -\kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\left|\mu_\ell^- - \tilde{\mu}_\ell\right| + \left|\dot{x} - \mu_\ell^\star\right|\right) \\
&\quad + \kappa_1 \epsilon \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \\
&\quad + \mathcal{O}(1)\, \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \ne l} \left|q_k^+\right| + |e|\right) \\
&\quad - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \\
&\quad + \mathcal{O}(1)\, \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \ne l} \left|q_k^+\right| + |e|\right) \\
&\quad + W_\ell^+ \Big\{\left|\, \left|q_\ell^+\right| \left(\mu_\ell^+ - \mu_\ell^\star\right) - \left(\left|q_\ell^+\right| + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right| \\
&\qquad\qquad + \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| + \epsilon \left|\mu_\ell^\star - \dot{x}\right| + \left(\left|q_\ell^+\right| + \epsilon\right) \left|\mu_\ell^- - \tilde{\mu}_\ell\right|\Big\} \\
&\le -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \mathcal{O}(1)\, \epsilon \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \ne l} \left|q_k^+\right|\right) \\
&\qquad + \mathcal{O}(1)\, |e| \\
&\le \mathcal{O}(1)\, \epsilon \left(\delta + \sum_{k \ne l} \left|q_k^+\right|\right) + \mathcal{O}(1)\, |e| + \epsilon \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \left(\mathcal{O}(1) - c\kappa_1\right).
\end{aligned}
\tag{7.65}
$$

Finally,

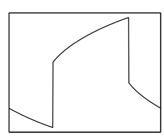
$$
\begin{aligned}
\sum_k E_k &= E_l + \sum_{k \ne l} E_k \\
&\le \mathcal{O}(1)\, \epsilon\, \delta + \mathcal{O}(1)\, |e| + \epsilon \sum_{k \ne l} \left|q_k^+\right| \left(\mathcal{O}(1) - c\kappa_1\right) \\
&\quad + \epsilon \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \left(\mathcal{O}(1) - c\kappa_1\right) \\
&\le \mathcal{O}(1)\, \epsilon\, \delta + \mathcal{O}(1)\, |e|
\end{aligned}
\tag{7.66}
$$

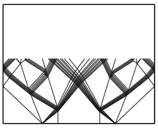
by choosing κ_1 larger if necessary. Hence (7.30) holds in this case as well.

Case R3 $q_l^- < 0 < q_l^+, \epsilon > 0$.

Since the front at x is a rarefaction front, both estimates (7.51) and (7.59) hold. Moreover, we have that

$$
q_\ell^+ - q_\ell^- = \left|q_\ell^+\right| + \left|q_\ell^-\right| < 2\epsilon \le 2\delta.
$$





Then from $AD + BC \leq (A + B)(D + C)$ for positive A, B, C, and D, we obtain

$$
\begin{aligned}
E_l &= W_\ell^+ \left|q_\ell^+\right| \left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) \\
&\leq \mathcal{O}(1) \left(\left|q_\ell^+\right| + \left|q_\ell^-\right|\right) \left(\left|\mu_\ell^+ - \dot{x}\right| + \left|\mu_\ell^- - \dot{x}\right|\right) \\
&\leq \mathcal{O}(1)\,\epsilon \left(\left|\mu_\ell^+ - \dot{x}\right| + \left|\mu_\ell^- - \dot{x}\right|\right) \\
&= \mathcal{O}(1)\,\epsilon \Big(\left|\mu_l\left(\omega_{l-1}^+, \omega_\ell^+\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| \\
&\qquad\qquad + \left|\mu_l\left(\omega_{l-1}^-, \omega_\ell^-\right) - \mu_l\left(v^{\delta,-}, v^{\delta,-}\right)\right|\Big) \\
&= \mathcal{O}(1)\,\epsilon \Big(\delta + \left|q_\ell^+\right| + \left|q_\ell^-\right| + \sum_{k>l} \left|q_k^+\right| + \sum_{k<l} \left|q_k^-\right|\Big) \\
&= \mathcal{O}(1)\,\epsilon \Big(\delta + \sum_{k>l} \left|q_k^+\right| + \sum_{k<l} \left|q_k^-\right|\Big).
\end{aligned}
\tag{7.67}
$$

Using (7.51) for $k < l$ and (7.59) for $k > l$, and choosing κ_1 sufficiently large, we obtain (7.30).

Now we shall study the cases in which the front at x is a shock front. Also, here we prove (7.30) in three cases depending on q_ℓ^- and q_ℓ^+. If the front at x is a shock front, then by the construction of the front-tracking approximation, we have

$$H_l(\epsilon) v^{\delta,-} = v^{\delta,+} + e,$$

or

$$H_l(\epsilon) H\left(q^-\right) u^{\delta_1} = H\left(q^+\right) u^{\delta_1} + e,$$

where $q^\pm = \left(q_1^\pm, \ldots, q_n^\pm\right)$, and e is the error of the front at x. Then we can use (7.31) and continuity of the mapping H to find that

$$
\begin{aligned}
&\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\
&\quad = \mathcal{O}(1) \left|\epsilon\right| \Big(\left|q_l^-\right| \left(\left|q_l^-\right| + \left|\epsilon\right|\right) + \sum_{k \neq l} \left|q_k^-\right|\Big) + \mathcal{O}(1) \left|e\right|.
\end{aligned}
\tag{7.68}
$$

We also have that

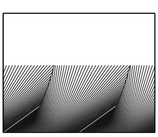
$$u^{\delta_1} = H\left(-q^+\right) v^{\delta,+} = H\left(-q^+\right) \left(H_l(\epsilon) v^{\delta,-} + e\right),$$

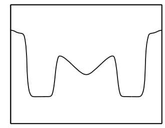
or

$$H\left(-q^-\right) v^{\delta,-} = H\left(-q^+\right) H_l(\epsilon) v^{\delta,-} + \mathcal{O}(1) \left|e\right|,$$

by the continuity of H. From this we obtain

$$
\begin{aligned}
&\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\
&\quad = \mathcal{O}(1) \left|\epsilon\right| \Big(\left|q_l^+\right| \left(\left|q_l^+\right| + \left|\epsilon\right|\right) + \sum_{k \neq l} \left|q_k^+\right|\Big) + \mathcal{O}(1) \left|e\right|.
\end{aligned}
\tag{7.69}
$$





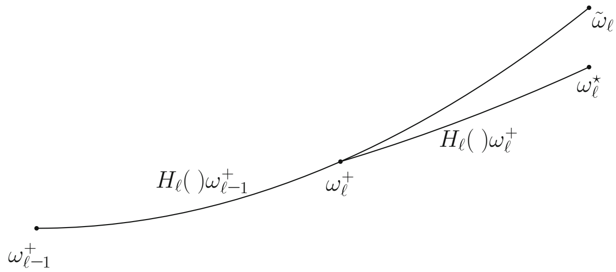


Fig. 7.6 The situation for $0 < q_\ell^+ < q_\ell^-, \epsilon < 0$, and $k = l$

Case S1 $0 < q_\ell^+ < q_\ell^-, \epsilon < 0$.

If $k \neq l$, then we can write E_k as (7.38) and use the arguments leading to (7.51) and the estimate (7.69) to obtain

$$E_k \leq \mathcal{O}(1)\,|\epsilon|\left(\left|q_l^+\right|\left(\left|q_l^+\right| + |\epsilon|\right) + \sum_{\tilde{k}\neq l}\left|q_{\tilde{k}}^+\right|\right) + \mathcal{O}(1)\,|e| - c\kappa_1\left|q_k^+\right||\epsilon|\,. \tag{7.70}$$

For $k = l$ we define the auxiliary states and speeds as in (7.60); see Fig. 7.6.

Then the estimate (7.61) holds. Also, using (7.69) we find that

$$\begin{aligned}|\mu_l^- - \tilde{\mu}_\ell| &= \mathcal{O}(1)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right)\\ &= \mathcal{O}(1)\,|\epsilon|\left(\left|q_\ell^+\right|\left(\left|q_\ell^+\right| + |\epsilon|\right) + \sum_{k\neq l}\left|q_k^+\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.71}$$

Moreover,

$$\begin{aligned}|\mu_\ell^\star - \dot{x}| &= \left|\mu_l\left(\omega_\ell^+, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,-}, v^{\delta,+}\right)\right|\\ &\leq \left|\mu_l\left(\omega_\ell^+, H_l(-\epsilon)\omega_\ell^+\right) - \mu_l\left(v^{\delta,+}, H_l\left(-\epsilon\right)v^{\delta,+}\right)\right| + \mathcal{O}(1)\,|e|\\ &= \mathcal{O}(1)\left(\left|\omega_\ell^+ - \omega_n^+\right|\right) + \mathcal{O}(1)\,|e|\\ &= \mathcal{O}(1)\left(\sum_{k>l}\left|q_k^+\right| + |e|\right). \end{aligned} \tag{7.72}$$

By Lemma 7.6, we have

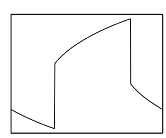
$$\mu_\ell^\star - \tilde{\mu}_\ell > cq_\ell^+. \tag{7.73}$$

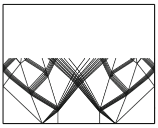
In this case

$$W_\ell^+ = W_\ell^- + \kappa_1\,|\epsilon|\,, \tag{7.74}$$

and

$$\epsilon < 0 < q_\ell^+ < q_\ell^-.$$





We estimate

$$
\begin{aligned}
E_l &= W_\ell^+ q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - \left(W_\ell^+ - \kappa_1 \left|\epsilon\right|\right) q_\ell^- \left(\mu_\ell^- - \dot{x}\right) \\
&= \kappa_1 \left|\epsilon\right| q_\ell^- \left(\mu_\ell^- - \dot{x}\right) + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \left|\epsilon\right| \left\{\left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mu_\ell^- - \mu_\ell^\star\right) + q_\ell^- \left(\mu_\ell^- - \dot{x}\right) - \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mu_\ell^- - \mu_\ell^\star\right)\right\} \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left\{\left(q_\ell^+ + \left|\epsilon\right|\right)\left(\left(\mu_\ell^- - \dot{x}\right) - \left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right)\right. \\
&\qquad\qquad \left. + \left(q_\ell^- - q_\ell^+ - \left|\epsilon\right|\right)\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\left|\mu_\ell^- - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\kappa_1 \left(q_\ell^+ + \left|\epsilon\right|\right)\left|\epsilon\right| q_\ell^+ \\
&\quad + \mathcal{O}\left(1\right)\kappa_1 \left|\epsilon\right|^2 \left(q_\ell^+ \left(q_\ell^+ + \left|\epsilon\right|\right) + \sum_{k\neq l} \left|q_k^+\right|\right) + \mathcal{O}\left(1\right)\left|e\right| \\
&\quad + \mathcal{O}\left(1\right)\kappa_1 \left|\epsilon\right| \left(\sum_{k>l} \left|q_k^+\right|\right) + \mathcal{O}\left(1\right)\left|e\right| \\
&\quad + W_\ell^+ \left\{\left|q_\ell^+ \left(\mu_\ell^+ - \mu_\ell^\star\right) - \left(q_\ell^+ - \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right|\right. \\
&\quad \left. + \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| + \left|\epsilon\right| \left|\mu_\ell^\star - \dot{x}\right| + \left(q_\ell^+ + \left|\epsilon\right|\right)\left|\mu_\ell^- - \tilde{\mu}_\ell\right|\right\} \\
&\le -c\kappa_1 \left(q_\ell^+ + \left|\epsilon\right|\right)\left|\epsilon\right| q_\ell^+ + \mathcal{O}\left(1\right)\left|\epsilon\right| \left(q_\ell^+ \left(q_\ell^+ + \left|\epsilon\right|\right) + \sum_{k\neq l} \left|q_k^+\right|\right) \\
&\quad + \mathcal{O}\left(1\right)\left|e\right| \\
&\le \mathcal{O}\left(1\right)\sum_{k\neq l} \left|q_k^+\right| + \left|\epsilon\right| \left|q_\ell^+\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mathcal{O}\left(1\right) - c\kappa_1\right) + \mathcal{O}\left(1\right)\left|e\right|.
\end{aligned} \tag{7.75}
$$

As before, setting κ_1 sufficiently large, (7.75) and (7.70) imply

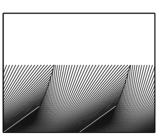
$$
\sum_k E_k = E_l + \sum_{k\neq l} E_k \le \mathcal{O}\left(1\right)\left|e\right|, \tag{7.76}
$$

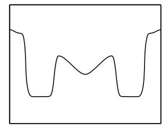
which is (7.30).

Case S2 $q_\ell^+ < q_\ell^- < 0, \epsilon < 0.$

In this case we proceed as in Case S1, but using (7.68) instead of (7.69). For $k \neq l$ this gives the estimate

$$
E_k \le \mathcal{O}\left(1\right)\left|\epsilon\right| \left(\left|q_\ell^-\right| \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{\tilde{k}\neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}\left(1\right)\left|e\right| - c\kappa_1 \left|q_k^-\right| \left|\epsilon\right|. \tag{7.77}
$$





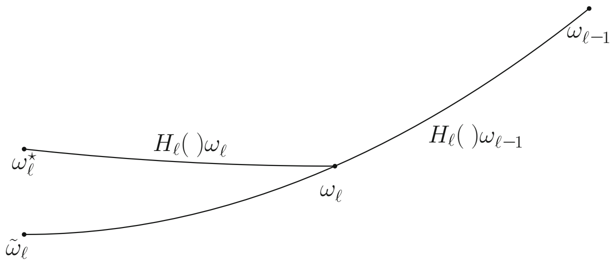


Fig. 7.7 The situation for $q_\ell^+ < q_\ell^- < 0$, $\epsilon < 0$, and $k = l$

We now define the intermediate states $\tilde\omega_\ell$, $\omega_\ell^\star$ and the speeds $\tilde\mu_\ell$ and $\mu_\ell^\star$ as in (7.52); see Fig. 7.7.

Then the estimate (7.53) holds. As in Case R1, we compute

$$\begin{aligned}\left|\mu_\ell^+ - \tilde\mu_\ell\right| &= \mathcal{O}(1)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right)\\ &= \mathcal{O}(1)\,|\epsilon|\left(\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + |\epsilon|\right) + \sum_{k\neq l}\left|q_k^-\right|\right) + \mathcal{O}(1)\,|e| \qquad (7.78)\end{aligned}$$

and

$$\begin{aligned}\left|\mu_\ell^\star - \dot x\right| &\le \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l\left(v^{\delta,-}, H_l(\epsilon)v^{\delta,-}\right)\right| + \mathcal{O}(1)\,|e|\\ &\le \mathcal{O}(1)\,\delta + \mathcal{O}(1)\left|\omega_l^- - \omega_0^-\right| + \mathcal{O}(1)\,|e|\\ &\le \mathcal{O}(1)\,|e| + \mathcal{O}(1)\sum_{k<l}\left|q_k^-\right|. \qquad (7.79)\end{aligned}$$

In this case, genuine nonlinearity and Lemma 7.6 imply that

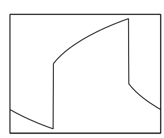
$$\tilde\mu_\ell - \mu_\ell^\star > c\left|q_\ell^-\right|, \qquad (7.80)$$

with $c > 0$. Moreover, now

$$W_\ell^+ = W_\ell^- - \kappa_1\,|\epsilon|\,.$$

Now we can use the (by now) familiar technique of estimating E_l:

$$\begin{aligned}E_l &= \left(W_\ell^- - \kappa_1\,|\epsilon|\right)\left|q_\ell^+\right|\left(\mu_\ell^+ - \dot x\right) - W_\ell^-\left|q_\ell^-\right|\left(\mu_\ell^- - \dot x\right)\\ &\le -\kappa_1\,|\epsilon|\left(\left|q_\ell^-\right| + |\epsilon|\right)\left(\tilde\mu_\ell - \mu_\ell^\star\right)\\ &\quad + \kappa_1\,|\epsilon|\left(\left|q_\ell^-\right| + |\epsilon|\right)\left(\left|\mu_\ell^+ - \tilde\mu_\ell\right| + \left|\mu_\ell^\star - \dot x\right|\right)\\ &\quad + \kappa_1\,|\epsilon|\left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot x\right|\\ &\quad + W_\ell^-\left\{\left|q_\ell^+\right|\left(\mu_\ell^+ - \dot x\right) - \left|q_\ell^-\right|\left(\mu_\ell^- - \dot x\right)\right\}\\ &\le -c\kappa_1\left|q_\ell^-\right||\epsilon|\left(\left|q_\ell^-\right| + |\epsilon|\right)\\ &\quad + \mathcal{O}(1)\,\kappa_1\,|\epsilon|\left(\left|q_\ell^-\right| + |\epsilon|\right)\left(\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + |\epsilon|\right) + \sum_{k\neq l}\left|q_k^-\right|\right)\\ &\quad + W_\ell^-\Big\{\left|q_\ell^-\left(\mu_\ell^- - \mu_\ell^\star\right) - \left(q_\ell^- + \epsilon\right)\left(\tilde\mu_\ell - \mu_\ell^\star\right)\right|\\ &\qquad + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot x\right|\\ &\qquad + |\epsilon|\left|\mu_\ell^\star - \dot x\right| + \left(\left|q_\ell^-\right| + |\epsilon|\right)\left|\mu_\ell^+ - \tilde\mu_\ell\right|\Big\} + \mathcal{O}(1)\,|e|\end{aligned}$$



$$\begin{aligned}
&\leq -c\kappa_1 \left|q_\ell^-\right| |\epsilon| \left(\left|q_\ell^-\right| + |\epsilon|\right) + \mathcal{O}(1) \left(\left|q_\ell^-\right| \left(\left|q_\ell^-\right| + |\epsilon|\right) + \sum_{k \neq l} \left|q_k^-\right|\right) \\
&\quad + \mathcal{O}(1) |e| \\
&\leq \mathcal{O}(1) \sum_{k \neq l} \left|q_k^-\right| + |\epsilon| \left|q_\ell^-\right| \left(\left|q_\ell^-\right| + |\epsilon|\right) \left(\mathcal{O}(1) - c\kappa_1\right) + \mathcal{O}(1) |e| . \qquad (7.81)
\end{aligned}$$

Combining (7.81) and (7.77), we obtain

$$\sum_k E_k = E_l + \sum_{k \neq l} E_k \leq \mathcal{O}(1) |e| , \qquad (7.82)$$

which is (7.30).

Case S3 $q_\ell^+ < 0 < q_\ell^-, \epsilon < 0$.

For $k \neq l$, the estimate (7.77) remains valid.

Next we consider the case $k = l$. The $\mathcal{O}(1)$ that multiplies $|\epsilon|$ in (7.69) (or (7.69)) is proportional to the total variation of the initial data. Hence we can assume that this is arbitrarily small by choosing T.V. (u_0) sufficiently small. Since all terms $q_j^\pm$ are bounded, we can and will assume that

$$\left|q_l^+ - q_l^- - \epsilon\right| \leq \frac{1}{2} |\epsilon| + \mathcal{O}(1) |e| . \qquad (7.83)$$

Without loss of generality we may assume that $\left|q_l^+\right| \geq \left|q_l^-\right|$. This implies that

$$\left|q_l^+ - q_l^- - \epsilon\right| \geq \left|q_l^- - q_l^+\right| - |\epsilon| = q_l^- - q_l^+ + \epsilon \geq 2q_l^- + \epsilon. \qquad (7.84)$$

Thus

$$2q_l^- + \epsilon \leq \frac{1}{2} |\epsilon| + \mathcal{O}(1) |e| , \qquad (7.85)$$

or

$$q_l^- + \epsilon \leq -\frac{1}{4} |\epsilon| + \mathcal{O}(1) |e| , \qquad (7.86)$$

which can be rewritten as

$$\left|q_l^- + \epsilon - \mathcal{O}(1) |e|\right| \geq \frac{1}{4} |\epsilon| . \qquad (7.87)$$

From this we conclude that

$$\left|q_l^- + \epsilon\right| \geq \frac{1}{4} |\epsilon| - \mathcal{O}(1) |e| . \qquad (7.88)$$

We define the auxiliary states $\tilde{\omega}_\ell$, $\omega_\ell^\star$ and the speeds $\tilde{\mu}_\ell$ and $\mu_\ell^\star$ as in (7.52); see Fig. 7.8. Then estimates (7.78) and (7.79) hold.

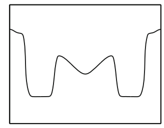
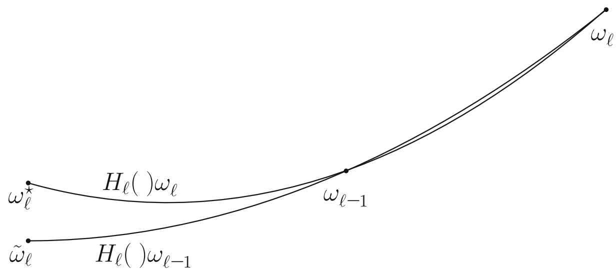


Fig. 7.8 The situation for $q_\ell^+ < 0 < q_\ell^-, \epsilon < 0$, and $k = l$



By Lemma 7.6 we have that

$$\tilde\mu_\ell - \mu_\ell^\star \le 0, \tag{7.89}$$
$$\mu_\ell^- - \mu_\ell^\star \ge c\left|q_\ell^- + \epsilon\right|, \tag{7.90}$$

for a positive constant c. Recalling that $W_\ell^- \ge 1$, and using (7.89), (7.90), and the estimates (7.78) and (7.79) (which remain valid in this case), we compute

$$\begin{aligned}
E_l &= W_\ell^+\left|q_\ell^+\right|\left(\mu_\ell^+ - \dot x\right) - W_\ell^-\left|q_\ell^-\right|\left(\mu_\ell^- - \dot x\right)\\
&\le W_\ell^+\left|q_\ell^+\right|\left(\tilde\mu_\ell - \mu_\ell^\star\right) - W_\ell^-\left|q_\ell^-\right|\left(\mu_\ell^- - \mu_\ell^\star\right)\\
&\qquad + W_\ell^+\left|q_\ell^+\right|\left(\left|\mu_\ell^+ - \tilde\mu_\ell\right| + \left|\mu_\ell^\star - \dot x\right|\right) + W_\ell^-\left|q_\ell^-\right|\left|\mu_\ell^\star - \dot x\right|\\
&\le -\left|q_\ell^-\right| c\left|q_\ell^- + \epsilon\right| + \mathcal{O}(1)\left|\epsilon\right|\left(q_\ell^-\left(q_\ell^- + |\epsilon|\right) + \sum_{\tilde k\ne l}\left|q_{\tilde k}^-\right|\right) + \mathcal{O}(1)\left|e\right|\\
&\le \frac{-c}{4}\left|q_\ell^-\right|\left|\epsilon\right| + \mathcal{O}(1)\left|\epsilon\right|\left(q_\ell^-\left(q_\ell^- + |\epsilon|\right) + \sum_{\tilde k\ne l}\left|q_{\tilde k}^-\right|\right) + \mathcal{O}(1)\left|e\right|.
\end{aligned} \tag{7.91}$$

Now (7.77) and (7.91) are used to balance the terms containing the factor $\sum_{k\ne l}\left|q_k^-\right|$. The remaining term,

$$\left|q_\ell^-\right|\left|\epsilon\right|\left(-\frac{1}{4}c + \mathcal{O}(1)\left(q_\ell^- + |\epsilon|\right)\right),$$

can be made negative by choosing T.V. (u_0) (and hence $\mathcal{O}(1)$) sufficiently small.

Hence also in this case (7.30) holds.

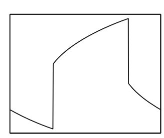
Finally, if q_ℓ^- or q_ℓ^+ is zero, (7.30) can easily be shown to be a limit of one of the previous cases.

Summing up, we have proved the following theorem:

Theorem 7.7 *Let u^{δ_1} and v^{δ_2} be front-tracking approximations, with accuracies defined by δ_1, δ_2,*

$$G\left(u^{\delta_1}(t)\right) < M, \quad \text{and} \quad G\left(v^{\delta_2}(t)\right) < M, \qquad \text{for } t \ge 0. \tag{7.92}$$

For sufficiently small M there exist constants κ_1, κ_2, and C_2 such that the functional Φ defined by (7.15) and (7.16) satisfies (7.21). Furthermore, there exists



a constant C *(independent of* δ_1 *and* δ_2*) such that*

$$\left\| u^{\delta_1}(t) - v^{\delta_2}(t) \right\|_1 \leq C \left\| u^{\delta_1}(0) - v^{\delta_2}(0) \right\|_1 + C t \left(\delta_1 \vee \delta_2 \right). \tag{7.93}$$

To state the next theorem we need the following definition. Let the domain $\mathcal{D}$ be defined as the L^1 closure of the set

$$\mathcal{D}_0 = \left\{ u \in L^1(\mathbb{R}; \mathbb{R}^n) \mid u \text{ is piecewise constant and } G(u) < M \right\}; \tag{7.94}$$

that is, $\mathcal{D} = \overline{\mathcal{D}_0}$. Since the total variation is small, we will assume that all possible values of u are in a (small) neighborhood $\Omega \subset \mathbb{R}^n$.

Theorem 7.8 *Let* $f_j \in C^2(\mathbb{R}^n)$*,* $j = 1, \dots, n$*. Consider the strictly hyperbolic equation* $u_t + f(u)_x = 0$*. Assume that each wave family is either genuinely nonlinear or linearly degenerate. For all initial data* u_0 *in* $\mathcal{D}$*, defined by* (7.94)*, every sequence of front-tracking approximations* u^δ *converges to a unique limit* u *as* $\delta \to 0$*. Furthermore, let* u *and* v *denote solutions*

$$u_t + f(u)_x = 0,$$

with initial data u_0 *and* v_0*, respectively, obtained as a limit of a front-tracking approximation. Then*

$$\| u(t) - v(t) \|_1 \leq C \| u_0 - v_0 \|_1. \tag{7.95}$$

Proof First we use (7.93) to conclude that every front-tracking approximation u^δ has a unique limit u as $\delta \to 0$. Then we take the limit $\delta \to 0$ in (7.93) to conclude that (7.95) holds. □

Note that this also gives an error estimate for front tracking for systems. If we denote the limit of the sequence $\left\{ u^\delta \right\}$ by u and $v^{\delta_2} = u^\delta$, then by letting $\delta_2 \to 0$ in (7.93)

$$\left\| u^\delta(\cdot, t) - u(\cdot, t) \right\|_1 \leq C \left(\left\| u_0^\delta - u_0 \right\|_1 + \delta t \right) = \mathcal{O}(1)\, \delta$$

for some finite constant C. Hence front tracking for systems is a first-order method.

7.2 Uniqueness

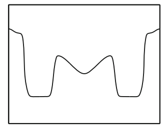
Let S_t denote the map that maps initial data u_0 into the solution u of

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0$$

at time t, that is, $u = S_t u_0$. In Chapt. 6 we showed the existence of the semigroup S_t, and in the previous section its stability for initial data in the class $\mathcal{D}$ as limits of approximate solutions obtained by front tracking. Thus we know that it satisfies

$$S_0 u = u, \qquad S_t S_s u = S_{t+s} u,$$
$$\| S_t u - S_s v \|_1 \leq L \left(|t - s| + \| u - v \|_1 \right)$$

for all $t, s \geq 0$ and u, v in $\mathcal{D}$.



In this section we prove uniqueness of solutions that have initial data in $\mathcal{D}$.

We want to demonstrate that every other solution u coincides with this semigroup. To do this we will basically need three assumptions. The first is that u is a weak solution, the second is that it satisfies Lax's entropy conditions across discontinuities, and the third is that it has locally bounded variation on a certain family of curves. Concretely, we define an *entropy solution* of

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0,$$

to be a bounded measurable function $u = u(x,t)$ of bounded total variation satisfying the following conditions:

A The function $u = u(x,t)$ is a weak solution of the Cauchy problem (7.1) taking values in $\mathcal{D}$, i.e.,

$$\int_0^T \int_{\mathbb{R}} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{\mathbb{R}} \varphi(x,0)u_0(x)\,dx = 0 \tag{7.96}$$

for all test functions φ whose support is contained in the strip $[0,T\rangle$.

B Assume that u has a jump discontinuity at some point (x,t), i.e., there exist states $u_{l,r} \in \Omega$ and speed σ such that if we let

$$U(y,s) = \begin{cases} u_l & \text{for } y < x + \sigma(s-t), \\ u_r & \text{for } y \geq x + \sigma(s-t), \end{cases} \tag{7.97}$$

then

$$\lim_{\rho\to 0} \frac{1}{\rho^2} \int_{t-\rho}^{t+\rho} \int_{x-\rho}^{x+\rho} |u(y,s) - U(y,s)|\,dy\,ds = 0. \tag{7.98}$$

Furthermore, there exists k such that

$$\lambda_k(u_l) \geq \sigma \geq \lambda_k(u_r). \tag{7.99}$$

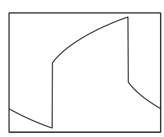
C There exists a $\theta > 0$ such that for all Lipschitz functions γ with Lipschitz constant not exceeding θ, the total variation of $u(x,\gamma(x))$ is locally bounded.

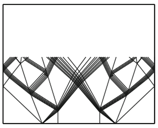
Remark 7.9 One can prove, see Exercise 7.1, that the front-tracking solution constructed in the previous chapter is an entropy solution of the conservation law.

There is a direct argument showing that any weak solution, whether it is a limit of a front-tracking approximation or not, satisfies a Lipshitz continuity in time of the spatial L^1-norm, as long as the solution has a uniform bound on the total variation. We present that argument here.

Theorem 7.10 *Let $u_0 \in \mathcal{D}$, and let u denote any weak solution of* (7.1) *such that* $\text{T.V.}(u(t)) \leq C$. *Then*

$$\|u(\cdot,t) - u(\cdot,s)\|_1 \leq C\,\|f\|_{\text{Lip}}\,|t-s|\,, \quad s,t \geq 0. \tag{7.100}$$





Proof Let $0 < s < t < T$, and let α_h be a smooth approximation to the characteristic function of the interval $[s, t]$, so that

$$\lim_{h \to 0} \alpha_h = \chi_{[s,t]}.$$

Furthermore, define

$$\varphi_h(y, \tau) = \alpha_h(\tau)\phi(y),$$

where ϕ is any smooth function with compact support. If we insert this into the weak formulation

$$\int_0^T \int_{\mathbb{R}} (u\varphi_{h,t} + f(u)\varphi_{h,x})\, dx\, dt + \int_{\mathbb{R}} \varphi_h(x, 0)u(x, 0)\, dx = 0, \tag{7.101}$$

and let $h \to 0$, we obtain

$$\int \phi(y)\,(u(y, t) - u(y, s))\; dy + \int_s^t \int \phi_y\, f(u)\, dy\, ds = 0.$$

From this we obtain

$$\begin{aligned}
\|u(\,\cdot\,, t) - u(\,\cdot\,, s)\|_1 &= \sup_{|\phi| \le 1} \int \phi(y)\,(u(y, t) - u(y, s))\; dy \\
&= -\sup_{|\phi| \le 1} \int_s^t \int \phi(y)_y\, f(u)\, dy\, ds \\
&\le \int_s^t \text{T.V.}\,(f(u))\; ds \\
&\le C\, \|f\|_{\text{Lip}}(t - s),
\end{aligned}$$

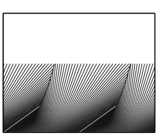
which proves the claim. Here we first used Exercise A.1, Theorem A.4, subsequently the definition (A.1) for T.V. (f), and finally the Lipschitz continuity of f and the bound on the total variation on u. □

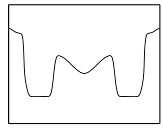
Remark 7.11 This argument provides an alternative to the proof of the Lipschitz continuity in Theorem 2.15 in the scalar case.

Before we can compare an arbitrary entropy solution to the semigroup solution, we need some preliminary results. Firstly, Theorem 7.10 says that every function $u(\,\cdot\,, t)$ taking values in $\mathcal{D}$ and satisfying **A** is L^1 Lipschitz continuous:

$$\|u(\,\cdot\,, t) - u(\,\cdot\,, s)\|_1 \le L(t - s),$$

for $t \ge s$.





Furthermore, by the structure theorem for functions of bounded variation [193, Theorem 5.9.6], u is continuous almost everywhere. For the sake of definiteness, we shall assume that all functions in $\mathcal{D}$ are right continuous. Also, there exists a set $\mathcal{N}$ of zero Lebesgue measure in the interval $[0, T]$ such that for $t \in [0, T] \backslash \mathcal{N}$, the function $u(\,\cdot\,, t)$ either is continuous at x or has a jump discontinuity there. Intuitively, the set $\mathcal{N}$ can be thought of as the set of times when collisions of discontinuities occur.

Lemma 7.12 *If* (7.96)–(7.98) *hold, then*

$$u_l = \lim_{y \to x-} u(y, t), \qquad u_r = \lim_{y \to x+} u(y, t),$$
$$\textit{and} \quad \sigma\,(u_l - u_r) = f\,(u_l) - f\,(u_r)\,.$$

Proof Let P_λ denote the parallelogram

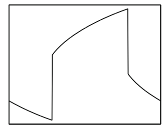
$$P_\lambda = \big\{(y, s) \mid |t - s| \le \lambda,\ |y - x - \sigma(s - t)| \le \lambda\big\}.$$

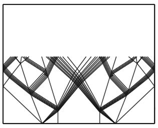
Integrating the conservation law over P_λ, we obtain

$$\begin{aligned}
&\Big(\int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y, t+\lambda)\,dy - \int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y, t-\lambda)\,dy\Big) \\
&\quad + \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x + \lambda + \sigma(s - t), s)\,ds \\
&\quad - \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x - \lambda + \sigma(s - t), s)\,ds = 0.
\end{aligned}$$

If we furthermore integrate this with respect to λ from $\lambda = 0$ to $\lambda = \rho$, and divide by ρ^2, we obtain

$$\begin{aligned}
&\frac{1}{\rho^2}\Bigg(\int_0^\rho \int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y, t+\lambda)\,dy\,d\lambda - \int_0^\rho \int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y, t-\lambda)\,dy\,d\lambda\Bigg) \\
&\quad + \frac{1}{\rho^2}\Big(\int_0^\rho \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x + \lambda + \sigma(s - t), s)\,ds\,d\lambda \\
&\quad - \int_0^\rho \int_{t-\sigma}^{t+\sigma} (f(u) - \sigma u)(x - \lambda + \sigma(s - t), s)\,ds\,d\lambda\Big) = 0.
\end{aligned}$$





Now let $\rho \to 0$. Then

$$\frac{1}{\rho^2}\int_0^\rho \int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y,t+\lambda)\,dy\,d\lambda \to \frac{1}{2}(u_l+u_r),$$

$$\frac{1}{\rho^2}\int_0^\rho \int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y,t-\lambda)\,dy\,d\lambda \to \frac{1}{2}(u_l+u_r),$$

$$\frac{1}{\rho^2}\int_0^\rho \int_{t-\lambda}^{t+\lambda} (f(u)-\sigma u)(x+\lambda+\sigma(s-t),s)\,ds\,d\lambda \to f(u_r)-\sigma u_r,$$

$$\frac{1}{\rho^2}\int_0^\rho \int_{t-\lambda}^{t+\lambda} (f(u)-\sigma u)(x-\lambda+\sigma(s-t),s)\,ds\,d\lambda \to f(u_l)-\sigma u_l.$$

Hence

$$\frac{1}{2}(u_l+u_r)-\frac{1}{2}(u_l+u_r)+(f(u_r)-\sigma u_r)-(f(u_l)-\sigma u_l)=0.$$

This concludes the proof of the lemma. □

The next lemma states that if u satisfies **C**, then the discontinuities cannot cluster too tightly together.

Lemma 7.13 *Assume that* $u\colon[0,T]\to\mathcal{D}$ *satisfies* **C**. *Let* $t\in[0,T]$ *and* $\varepsilon>0$. *Then the set*

$$B_{t,\varepsilon}=\Big\{x\in\mathbb{R}\mid \limsup_{s\to t+,\,y\to x}|u(x,t)-u(y,s)|>\varepsilon\Big\} \tag{7.102}$$

has no limit points.

Proof Assume that $B_{t,\varepsilon}$ has a limit point, denoted by x_0. Then there is a monotone sequence $\{x_i\}_{i=1}^\infty$ in $B_{t,\varepsilon}$ converging to x_0. Without loss of generality we assume that the sequence is decreasing. Since $u(x,t)$ is right continuous, we can find a point z_i in (x_i,x_{i-1}) such that

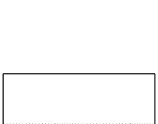
$$|u(z_i,t)-u(x_i,t)|\le\frac{\varepsilon}{2}.$$

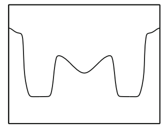
Now choose $s_i>t$ and $y_i\in(z_{i+1},z_i)$ such that

$$|u(y_i,s_i)-u(x_i,t)|\ge\varepsilon,\quad |s_i-t|\le\theta\max\{|y_i-z_i|\,,\,|y_i-z_{i+1}|\}.$$

We define a curve $\gamma(x)$ for $x\in[x_0,x_1]$ passing through all the points (z_i,t) and (y_i,s_i) by

$$\gamma(x)=\begin{cases} t & \text{for } x=x_0 \text{ or } x\ge z_1,\\ s_i-(x-y_i)\,\frac{s_i-t}{z_i-y_i} & \text{for } x\in[y_i,z_i],\\ t+(x-z_{i+1})\,\frac{s_i-t}{y_i-z_{i+1}} & \text{for } x\in[z_{i+1},y_i]. \end{cases} \tag{7.103}$$





Then γ is Lipschitz continuous with Lipschitz constant θ, and we have that

$$|u(y_i, s_i) - u(z_i, t)| \geq \frac{\varepsilon}{2}$$

for all $i \in \mathbb{N}$. This means that the total variation of $u(x, \gamma(x))$ is infinite, violating **C**, concluding the proof of the lemma. □

In the following, we let $\sigma^\star$ be a number strictly larger than the absolute value of every characteristic speed, and we also demand that $\sigma^\star \geq 1/\theta$, where θ is the constant in **C**. The next lemma says that if u satisfies **C**, then discontinuities cannot propagate faster than $\sigma^\star$. Precisely, we have the following result.

Lemma 7.14 *Assume that* $u\colon [0, T] \to \mathcal{D}$ *satisfies* **C**. *Then for* $(x, t) \in (0, T) \times \mathbb{R}$,

$$\lim_{\substack{s \to t^+,\, y \to x\pm \\ |x-y| > \sigma^\star(s-t)}} u(y, s) = u(x\pm, t). \tag{7.104}$$

Proof We assume that the lemma does not hold. Then, for some (x_0, t) there exist decreasing sequences $s_j \to t$ and $y_j \to x_0$ such that

$$\left|y_j - x_0\right| \geq \sigma^\star\left(s_j - t\right), \quad \left|u\left(y_j, s_j\right) - u\left(x_0, t\right)\right| \geq \varepsilon$$

for some $\varepsilon > 0$ and $j \in \mathbb{N}$. Now let

$$z_0 = y_1 + \frac{s_1 - t}{\theta},$$

where as before θ is defined by **C**. Now we define a subsequence of $\left\{(y_j, s_j)\right\}$ as follows. Set $j_1 = 1$ and for $i \geq 1$ define

$$\begin{cases} z_i = y_{j_i} - \frac{s_{j_i} - t}{\theta}, \\ j_{i+1} = \min\left\{k \;\middle|\; s_k \leq t - \theta\left(y_k - z_i\right)\right\}. \end{cases}$$

Then

$$y_{j_i} \in (z_{i+1}, z_i) \quad \text{and} \quad \left|s_{j_i} - t\right| \leq \theta \max\left\{\left|y_{j_i} - z_i\right|, \left|y_{j_i} - z_{i+1}\right|\right\}$$

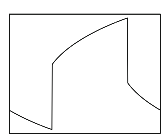
for all i. Let γ be the curve defined in (7.103). Since we have that $z_i \to x_0$, we have that

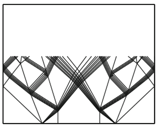
$$|u(z_i, t) - u(x_0, t)| \leq \frac{\varepsilon}{2}$$

for sufficiently large i. Consequently,

$$\left|u(z_i, t) - u\left(y_{j_i}, s_{j_i}\right)\right| \geq \frac{\varepsilon}{2},$$

and the total variation of $u(x, \gamma(x))$ is infinite, contradicting **C**. □





The next lemma concerns properties of the semigroup S_t. We assume that u is a continuous function $u\colon [0,T] \to \mathcal{D}$, and wish to estimate $S_T u(0) - u(T)$. Let h be a small number such that $Nh = T$. Then we can calculate

$$\begin{aligned} \|S_T u(0) - u(T)\|_1 &\leq \sum_{i=1}^{N} \left\| S_{T-(i-1)h} u((i-1)h) - S_{T-ih} u(ih) \right\|_1 \\ &\leq L \sum_{i=1}^{N} \left\| \frac{1}{h}\big(u(ih) - S_h u((i-1)h)\big) \right\|_1 h. \end{aligned}$$

Letting h tend to zero, we obtain the following lemma:

Lemma 7.15 *Assume that $u\colon [0,T] \to \mathcal{D}$ is Lipschitz continuous in the L^1-norm. Then for every interval $[a,b]$, we have*

$$\begin{aligned} &\|S_T u(0) - u(T)\|_{L^1([a+\sigma^\star T, b-\sigma^\star T]\,;\,\mathbb{R}^n)} \\ &\leq \mathcal{O}(1) \int_0^T \left\{ \liminf_{h\to 0+} \frac{1}{h} \|S_h u(t) - u(t+h)\|_{L^1([a+\sigma^\star(t+h), b-\sigma^\star(t+h)]\,;\,\mathbb{R}^n)} \right\} dt. \end{aligned}$$

Proof For ease of notation we set

$$\|\cdot\| = \|\cdot\|_{L^1([a+\sigma^\star(t+h), b-\sigma^\star(t+h)]\,;\,\mathbb{R}^n)}.$$

Observe that by finite speed of propagation, we can define $u(x,0)$ to be zero outside of $[a,b]$, and the Lipschitz continuity of the semigroup will look identical written in the norm $\|\cdot\|$ to how it looked before. Let

$$\phi(t) = \liminf_{h\to 0+} \frac{1}{h} \|u(t+h) - S_h u(t)\|.$$

Note that ϕ is measurable, and for all $h > 0$, the function

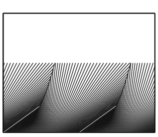
$$\phi_h(t) = \frac{1}{h} \|u(t+h) - S_h u(t)\|$$

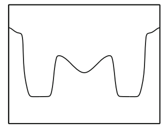
is continuous. Hence we have that

$$\phi(t) = \lim_{\varepsilon\to 0+} \inf_{h\in\mathbb{Q}\cap[0,\varepsilon]} \phi_h(t),$$

and therefore ϕ is Borel measurable. Define functions

$$\begin{aligned} \Psi(t) &= \|S_{T-t} u(t) - S_T u(0)\|, \\ \psi(t) &= \Psi(t) - L \int_0^t \phi(s)\, ds. \end{aligned} \tag{7.105}$$





The function ψ is a Lipschitz function, and hence

$$\psi(T) = \int_0^T \psi'(s)\,ds. \tag{7.106}$$

Furthermore, Rademacher's theorem[2] implies that there exists a null set $\mathcal{N}_1 \subseteq [0,T]$ such that Ψ and ψ are differentiable outside $\mathcal{N}_1$. Furthermore, using that Lebesgue measurable functions are approximately continuous almost everywhere (see [64, p. 47]), we conclude that there exists another null set $\mathcal{N}_2$ such that ϕ is continuous outside $\mathcal{N}_2$. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$. Outside $\mathcal{N}$ we have

$$\psi'(t) = \lim_{h\to 0} \frac{1}{h}\big(\Psi(t+h) - \Psi(t)\big) - L\psi(t). \tag{7.107}$$

Using properties of the semigroup we infer

$$\begin{aligned}
\Psi(t+h) - \Psi(t) &= \|S_{T-t-h}u(t+h) - S_T u(0)\| - \|S_{T-t}u(t) - S_T u(0)\| \\
&\le \|S_{T-t-h}u(t+h) - S_{T-t}u(t)\| \\
&= \|S_{T-t-h}u(t+h) - S_{T-t-h}S_h u(t)\| \\
&\le L\,\|u(t+h) - S_h u(t)\|\,,
\end{aligned}$$

which implies

$$\lim_{h\to 0} \frac{1}{h}\big(\Psi(t+h) - \Psi(t)\big) \le L \liminf_{h\to 0} \frac{1}{h}\,\|u(t+h) - S_h u(t)\| = L\phi(t).$$

Thus $\psi' \le 0$ almost everywhere, and we conclude that

$$\psi(T) \le 0, \tag{7.108}$$

which proves the lemma. □

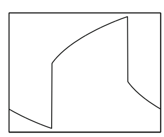
The next two lemmas are technical results valid for functions satisfying (7.97) and (7.98).

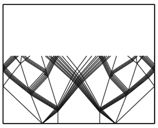
Lemma 7.16 *Assume that $u\colon [0,T] \to \mathcal{D}$ is Lipschitz continuous, and that for some (x,t) equations (7.97) and (7.98) hold. Then for all positive α we have*

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_0^\alpha |u(x+\lambda h+\rho y, t+h) - u_r|\,dy = 0, \tag{7.109}$$

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_{-\alpha}^0 |u(x+\lambda h+\rho y, t+h) - u_l|\,dy = 0. \tag{7.110}$$

[2] Rademacher's theorem states that a Lipschitz function is differentiable almost everywhere; see [64, p. 81].





Proof We assume that the limit in (7.109) is not zero. Then there exist sequences $\rho_i \to 0$ and $|h_i| < \rho_i$ and a $\delta > 0$ such that

$$\int_0^\alpha |u\left(x + \lambda h_i + \rho_i y, t + h_i\right) - u_r|\, dy > \delta \tag{7.111}$$

for all i. Without loss of generality we assume that $h_i > 0$, and let

$$v(z,h) = u(x + \lambda h + z, t + h).$$

Then the map $h \mapsto v(\cdot, h)$ is Lipschitz continuous with respect to the L^1 norm, since

$$\begin{aligned}
\|v(\cdot,h) - v(\cdot,\eta)\|_1 &= \int |u(z,t+h) - u(\lambda(\eta - h) + z, t + \eta)|\, dz \\
&\le \int |u(z,t+h) - u(z,t+\eta)|\, dz \\
&\quad + \int |u(z,t+\eta) - u(\lambda(\eta - h) + z, t+\eta)|\, dz \\
&\le M\,|h - \eta| + \lambda\,|\eta - h|\,\mathrm{T.V.}\,(u(t+\eta)) \\
&\le \widetilde{M}\,|\eta - h|\,.
\end{aligned}$$

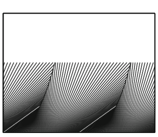
From (7.111) we obtain

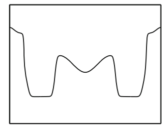
$$\begin{aligned}
&\int_0^{\alpha\rho_i} |u(x + \lambda h + z, t + h) - u_r|\, dz \\
&\quad \ge \int_0^{\alpha\rho_i} |u\left(x + \lambda h_i + z, t + h_i\right) - u_r|\, dz \\
&\qquad - \int_0^{\alpha\rho_i} |u\left(x + \lambda h_i + z, t + h_i\right) - u(x + \lambda h + z, t + h)|\, dz \\
&\quad \ge \delta\rho_i - \widetilde{M}\,|h_i - h|\,.
\end{aligned}$$

We can (safely) assume that $\delta/\widetilde{M} < 1$ (if this is not so, then (7.111) will hold for smaller δ as well). We integrate the last inequality with respect to h, for h in $[-\rho_i, \rho_i]$. Since $\left[h_i - \rho_i\delta/\widetilde{M}, h_i\right] \subset [-\rho_i, \rho_i]$, we obtain

$$\begin{aligned}
\int_{-\rho_i}^{\rho_i}\int_0^{\alpha\rho_i} |u(x + \lambda h + z, t + h) - u_r|\, dz\, dh &\ge \int_{h_i - \rho_i\delta/\widetilde{M}}^{h_i} \left(\delta\rho_i - \widetilde{M}(h_i - h)\right) dh \\
&= (\delta^2\rho_i^2)/(2\widetilde{M}).
\end{aligned}$$

Comparing this with (7.97) and (7.98) yields a contradiction. The limit (7.111) is proved similarly. □





For the next lemma, recall that a (signed) Radon measure is a (signed) regular Borel measure[3] that is finite on compact sets.

Lemma 7.17 *Assume that w is in $L^1((a,b);\mathbb{R}^n)$ such that for some Radon measure μ, we have that*

$$\left|\int_{x_1}^{x_2} w(x)\,dx\right| \le \mu([x_1,x_2]) \quad \textit{for all } a < x_1 < x_2 < b. \tag{7.112}$$

Then

$$\int_a^b |w(x)|\,dx \le \mu((a,b)). \tag{7.113}$$

Proof First observe that the assumptions of the lemma also hold if the closed interval on the right-hand side of (7.112) is replaced by an open interval. We have that

$$\begin{aligned}\left|\int_{x_1}^{x_2} w(x)\,dx\right| &= \lim_{\varepsilon\to 0}\left|\int_{x_1+\varepsilon}^{x_2-\varepsilon} w(x)\,dx\right| \\ &\le \lim_{\varepsilon\to 0}\mu([x_1+\varepsilon, x_2-\varepsilon]) = \mu((x_1,x_2)).\end{aligned}$$

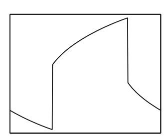
Secondly, since w is in L^1, it can be approximated by piecewise constant functions. Let v be a piecewise constant function with discontinuities located at $a = x_0 < x_1 < \cdots < x_N = b$, and

$$\int_a^b |w(x) - v(x)|\,dx \le \varepsilon.$$

Then we have

$$\begin{aligned}\int_a^b |w(x)|\,dx &\le \int_a^b |w(x)-v(x)| + \int_a^b |v(x)|\,dx \\ &\le \varepsilon + \sum_i \int_{x_{i-1}}^{x_i} |v(x)|\,dx \\ &= \varepsilon + \sum_i \left|\int_{x_{i-1}}^{x_i} v(x)\,dx\right|\end{aligned}$$

[3] A Borel measure μ is regular if it is outer regular on all Borel sets (i.e., $\mu(B) = \inf\{\mu(A) \mid A \supseteq B, A \text{ open}\}$ for all Borel sets B) and inner regular on all open sets (i.e., $\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}$ for all open sets U).



$$\begin{aligned}
&\leq \varepsilon + \sum_i \left| \int_{x_{i-1}}^{x_i} (v(x) - w(x)) \, dx \right| + \sum_i \left| \int_{x_{i-1}}^{x_i} w(x) \, dx \right| \\
&\leq \varepsilon + \int_a^b |v(x) - w(x)| \, dx + \sum_i \mu\left((x_{i-1}, x_i)\right) \\
&\leq 2\varepsilon + \mu\left((a, b)\right).
\end{aligned}$$

Since ε can be made arbitrarily small, this proves the lemma. □

Next we need two results that state how well the semigroup is approximated firstly by the solution of a Riemann problem with states that are close to the initial state for the semigroup, and secondly by the solution of the linearized equation. To define this precisely, let ω_0 be a function in $\mathcal{D}$, fix a point x on the real line (which will remain fixed throughout the next lemma and its proof), and let $\omega(y, t)$ be the solution of the Riemann problem

$$\omega_t + f(\omega)_y = 0, \qquad \omega(y, 0) = \begin{cases} \omega_0(x-) & \text{for } y < 0, \\ \omega_0(x+) & \text{for } y \geq 0. \end{cases}$$

(If ω_0 is continuous at x, then $\omega(y, t) = \omega_0(x)$ is constant.) Define $\tilde{A} = df(\omega_0(x+))$, and let $\tilde{u}$ be the solution of the linearized equation

$$\tilde{u}_t + \tilde{A}\tilde{u}_y = 0, \qquad \tilde{u}(y, 0) = \omega_0(y). \tag{7.114}$$

Furthermore, define $\hat{u}(y, t)$ by

$$\hat{u}(y, t) = \begin{cases} \omega(y - x, t) & \text{for } |y - x| \leq \sigma^\star t, \\ \omega_0(y) & \text{otherwise.} \end{cases} \tag{7.115}$$

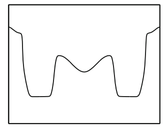
Then we can state the following lemma.

Lemma 7.18 *Let $\omega_0 \in \mathcal{D}$. Then we have*

$$\frac{1}{h} \int_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} \left|(S_h\omega_0)(y) - \hat{u}(y, h)\right| \, dy = \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}(\omega_0), \tag{7.116}$$

$$\frac{1}{h} \int_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} \left|(S_h\omega_0)(y) - \tilde{u}(y, h)\right| \, dy = \mathcal{O}(1)\left(\mathrm{T.V.}_{(x-\rho,x+\rho)}(\omega_0)\right)^2, \tag{7.117}$$

for all x and all positive h and ρ such that $x - \rho + h\sigma^\star < x + \rho - h\sigma^\star$.



Proof We first prove (7.117). In the proof of this we shall need the following general result:

Let $\bar{v}$ be the solution of $\bar{v}_t + f(\bar{v})_y = 0$ with Riemann initial data

$$\bar{v}(y,0) = \begin{cases} u_l & \text{for } y < 0, \\ u_r & \text{for } y \geq 0, \end{cases}$$

for some states $u_{l,r} \in \Omega$. We have that this Riemann problem is solved by waves separating constant states $u_l = v_0, v_1, \ldots, v_n = u_r$. Let u^c be a constant in Ω and set $A^c = df(u^c)$. Assume that u_l and u_r satisfy

$$A^c\,(u_l - u_r) = \lambda_k^c\,(u_l - u_r)\,;$$

i.e., λ_k^c is the kth eigenvalue and $u_l - u_r$ is the kth eigenvector of A^c. Let $\tilde{v}$ be defined by

$$\tilde{v}(y,t) = \begin{cases} u_l & \text{for } y < \lambda_k^c t, \\ u_r & \text{for } y \geq \lambda_k^c t \end{cases}$$

($\tilde{v}$ solves $u_t + A^c u_y = 0$ with a single jump at $y = 0$ from u_l to u_r as initial data). We wish to estimate

$$I = \frac{1}{t} \int_{-\sigma^\star t}^{\sigma^\star t} \left|\bar{v}(y,t) - \tilde{v}(y,t)\right|\,dy.$$

Note that since $\bar{v}$ and $\tilde{v}$ are equal outside the range of integration, the limits in the integral can be replaced by $\mp\infty$.

Due to the hyperbolicity of the system, the vectors $\{r_j(u)\}_{j=1}^n$ form a basis in $\mathbb{R}^n$, and hence we can find unique numbers $\bar{\varepsilon}_j^{l,r}$ such that

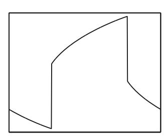
$$u_r - u_l = \sum_{j=1}^{n} \bar{\varepsilon}_j^l r_j(u_l) = \sum_{j=n}^{1} \bar{\varepsilon}_j^r r_j(u_r). \tag{7.118}$$

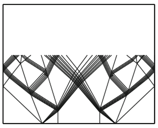
From $u_r - u_l = \varepsilon^c r_k(u^c)$ for some ε^c it follows that

$$\begin{aligned} \bar{\varepsilon}_i^l &= l_i(u_l) \cdot \sum_{j=1}^{n} \bar{\varepsilon}_j^l r_j(u_l) \\ &= l_i(u_l) \cdot (u_l - u_r) \\ &= (l_i(u_l) - l_i(u^c)) \cdot (u_l - u_r) + l_i(u^c) \cdot (u_l - u_r) \\ &= (l_i(u_l) - l_i(u^c)) \cdot (u_l - u_r) + \varepsilon^c l_i(u^c) \cdot r_k(u^c) \\ &= (l_i(u_l) - l_i(u^c)) \cdot (u_l - u_r), \quad i \neq k. \end{aligned}$$

Thus we conclude (using an identical argument for the right state) that

$$\begin{aligned} \left|\bar{\varepsilon}_i^l\right| &\leq C\ |u_l - u_r|\ |u_l - u^c|, \quad i \neq k, \\ \left|\bar{\varepsilon}_i^r\right| &\leq C\ |u_l - u_r|\ |u_r - u^c|, \quad i \neq k. \end{aligned} \tag{7.119}$$





Let ε_i denote the strength of the ith wave in $\bar{v}$. Then, by construction of the solution of the Riemann problem, for $i < k$ we have that

$$\left|\varepsilon_i - \bar{\varepsilon}_i^l\right| \leq C\left(\left|v_{i-1} - u_l\right|^2 + \left|v_i - u_l\right|^2\right) \leq C\left|u_l - u_r\right|^2,$$

while for $i > k$ we find that

$$\left|\varepsilon_i - \bar{\varepsilon}_i^r\right| \leq C\left|u_l - u_r\right|^2,$$

for some constant C. Assume that the k-wave in $\bar{v}$ moves with speed in the interval $[\underline{\lambda}_k, \bar{\lambda}^k]$; i.e., if the k-wave is a shock, then $\underline{\lambda}_k = \bar{\lambda}^k = \mu_k(v_{k-1}, v_k)$, and if the wave is a rarefaction wave, then $\underline{\lambda}_k = \lambda_k(v_{k-1})$ and $\bar{\lambda}_k = \lambda_k(v_k)$. Set $\underline{s} = \min(\underline{\lambda}_k, \tilde{\lambda}_k)$ and $\bar{s} = \max(\bar{\lambda}_k, \tilde{\lambda}_k)$. Then we can write I as

$$\begin{aligned} I &= \frac{1}{t}\Bigg(\int_{-\infty}^{\underline{s}} \left|u_l - \bar{v}(y,t)\right| \, dy \\ &\qquad + \int_{\underline{s}}^{\bar{s}} \left|\hat{v}(y,t) - \bar{v}(y,t)\right| \, dy + \int_{\bar{s}}^{\infty} \left|u_r - \bar{v}(y,t)\right| \, dy\Bigg) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Next we note that the first integral above can be estimated as

$$I_1 \leq C\sum_{i=1}^{k-1}\left|v_i - u_l\right| \leq C\sum_{i=1}^{k-1}\left|\varepsilon_i\right| \leq C\left(\sum_{i=1}^{k-1}\left|\bar{\varepsilon}_i^l\right| + \left|u_r - u_l\right|^2\right),$$

and similarly,

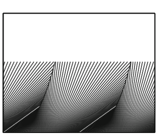
$$I_3 \leq C\left(\sum_{i=k+1}^{n}\left|\bar{\varepsilon}_i^r\right| + \left|u_l - u_r\right|^2\right).$$

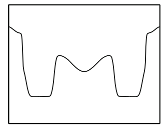
Using (7.119), we obtain

$$\begin{aligned} I_1 + I_3 &\leq C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right| + \left|u_l - u_r\right|\right) \\ &\leq C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right|\right), \end{aligned} \tag{7.120}$$

for some constant C. It remains to estimate I_2. We first assume that the k-wave in $\bar{v}$ is a shock wave and that $\lambda_k^c > \mu_k(v_{k-1}, v_k)$. Then

$$\begin{aligned} I_3 &= \left(\lambda_k^c - \mu_k(v_{k-1}, v_k)\right)\left|u_l - v_k\right| \\ &\leq C\left|u_l - v_k\right|\left(\left|u^c - v_{k-1}\right| + \left|u^c - v_k\right|\right) \\ &\leq C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right| + \left|v_k - u_r\right| + \left|v_{k-1} - u_l\right|\right), \\ &\leq C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right| + C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right|\right)\right) \\ &\leq C\left|u_l - u_r\right|\left(\left|u_l - u^c\right| + \left|u_r - u^c\right|\right) \end{aligned} \tag{7.121}$$





by the above estimates for $|v_k - u_r|$ and $|v_{k-1} - u_l|$. If $\lambda_k^c \le \mu_k(v_{k-1}, v_k)$ or the k-wave is a rarefaction wave, we similarly establish (7.121). Combining this with (7.120), we find that

$$I \le C\,|u_l - u_r|\,(|u_l - u^c| + |u_r - u^c|)\,. \tag{7.122}$$

Having established this preliminary estimate, we turn to the proof of (7.117). Let $\bar{\omega}_0$ be a piecewise constant approximation to ω_0 such that

$$\begin{gathered}\bar{\omega}_0(x\pm) = \omega_0(x\pm), \quad \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\,dy \le \epsilon, \\ \text{T.V.}_{(x-\rho,x+\rho)}\left(\bar{\omega}_0\right) \le \text{T.V.}_{(x-\rho,x+\rho)}\left(\omega_0\right). \end{gathered} \tag{7.123}$$

Furthermore, let v be the solution of the linear hyperbolic problem

$$v_t + \tilde{A} v_y = 0, \quad v(y,0) = \bar{\omega}_0(y),$$

where again $\tilde{A} = df(\omega_0(x+))$. Let the eigenvalues and the right and left eigenvectors of $\tilde{A}$ be denoted by $\tilde{\lambda}_k$, $\tilde{r}_k$, and $\tilde{l}_k$, respectively, for $k = 1, \dots, n$, normalized so that

$$\left|\tilde{l}_k\right| = 1, \quad \tilde{l}_k \cdot \tilde{r}_j = \begin{cases} 0 & \text{for } j \ne k, \\ 1 & \text{for } j = k. \end{cases} \tag{7.124}$$

Then it is not too difficult to verify (see Sect. 1.1) that $v(y,t)$ is given by

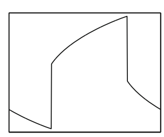
$$v(y,t) = \sum_k \left(\tilde{l}_k \cdot \bar{\omega}_0(y - \tilde{\lambda}_k t)\right)\tilde{r}_k. \tag{7.125}$$

We can also construct v by front tracking. Since the eigenvalues are constant and the initial data piecewise constant, front tracking will give the exact solution. Hence v will be piecewise constant with a finite number of jumps occurring at $x_i(t)$, where we have that

$$\begin{gathered}\frac{d}{dt}x_i(t) = \tilde{\lambda}_k, \\ \left(\tilde{A} - \tilde{\lambda}_k I\right)\left(v(x_i(t)+,t) - v(x_i(t)-,t)\right) = 0,\end{gathered}$$

for all t where we do not have a collision of fronts, that is, for all but a finite number of t's. Now we apply the estimate (7.122) to each individual front x_i. Then we obtain, introducing $v_i^\pm = v(x_i(t)\pm, t)$,

$$\begin{aligned}\int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} & \left|\left(S_\varepsilon v(\,\cdot\,,\tau)\right)(y) - v(y,\tau+\varepsilon)\right|\,dy \\ &\le \varepsilon\,\mathcal{O}(1)\sum_i \left|v_i^+ - v_i^-\right|\left(\left|v_i^+ - \omega_0(x+)\right| + \left|v_i^- - \omega_0(x+)\right|\right) \\ &\le \varepsilon\,\mathcal{O}(1)\,\text{T.V.}_{(x-\rho,x+\rho)}\left(\bar{\omega}_0\right)\sum_i \left|v_i^+ - v_i^-\right| \\ &\le \varepsilon\,\mathcal{O}(1)\left(\text{T.V.}_{(x-\rho,x+\rho)}\left(\omega_0\right)\right)^2. \end{aligned} \tag{7.126}$$



Recall that $\tilde{A} = df(\omega_0(x+))$ and that $\tilde{u}$ was defined by (7.114), that is,

$$\tilde{u}_t + \tilde{A}\tilde{u}_y = 0, \qquad \tilde{u}(y,0) = \omega_0(y). \tag{7.127}$$

In analogy to formula (7.125) we have that $\tilde{u}$ satisfies

$$\tilde{u}(y,t) = \sum_k \big(\tilde{l}_k \cdot \omega_0(y - \tilde{\lambda}_k t)\big)\tilde{r}_k. \tag{7.128}$$

Regarding the difference between $\tilde{u}$ and v, we find that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |v(y,h) - \tilde{u}(y,h)|\, dy \tag{7.129}$$

$$\begin{aligned} &= \int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} \left| \sum_k \Big(\tilde{l}_k \cdot (\bar{\omega}_0 - \omega_0)\,(y - \tilde{\lambda}_k h)\Big)\, \tilde{r}_k \right| dy \\ &\le \mathcal{O}(1) \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \\ &\le \mathcal{O}(1)\,\epsilon. \end{aligned} \tag{7.130}$$

By the Lipschitz continuity of the semigroup we have that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |S_h\bar{\omega}_0(y) - S_h\omega_0(y)|\, dy \le L \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \le L\epsilon. \tag{7.131}$$

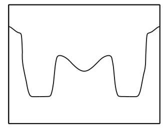
Furthermore, by Lemma 7.15 with $a = x - \rho$, $b = x + \rho$, $T = h$, and $t = 0$, and using (7.126), we obtain

$$\begin{aligned} &\frac{1}{h} \int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |(S_h\bar{\omega}_0)\,(y) - v(y,h)|\, dy \\ &\le \frac{\mathcal{O}(1)}{h} \int_0^h \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} |(S_\varepsilon v\,(\,\cdot\,,\tau))\,(y) - v(y,\tau+\varepsilon)|\, dy\, d\tau \\ &\le \mathcal{O}(1)\left(\mathrm{T.V.}_{(x-\rho,x+\rho)}\,(\omega_0)\right)^2. \end{aligned} \tag{7.132}$$

Consequently, using (7.132), (7.131), and (7.130), we find that

$$\begin{aligned} &\frac{1}{h} \int_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} |(S_h\omega_0)\,(y) - \tilde{u}(y,h)|\, dy \\ &\qquad \le \mathcal{O}(1)\left(\mathrm{T.V.}_{(x-\rho,x+\rho)}\,(\omega_0)\right)^2 + \frac{L\epsilon}{h} + \mathcal{O}(1)\,\frac{\epsilon}{h}. \end{aligned}$$

Since ϵ is arbitrary, this proves (7.117).



Now we turn to the proof of (7.116). First we define z to be the function

$$z(y,t) = \begin{cases} u_l & \text{for } y < \lambda t, \\ u_r & \text{for } y \geq \lambda t, \end{cases}$$

where $|\lambda| \leq \sigma^\star$. Recall that $\bar{v}(y,t)$ denotes the solution of $\bar{v}_t + f(\bar{v})_y = 0$ with Riemann initial data

$$\bar{v}(y,0) = \begin{cases} u_l & \text{for } y < 0, \\ u_r & \text{for } y \geq 0. \end{cases}$$

Then trivially we have that

$$\int_{-\sigma^\star t}^{\sigma^\star t} |z(y,t) - \bar{v}(y,t)| \, dy \leq t\, \mathcal{O}(1) \, |u_l - u_r| \,. \tag{7.133}$$

Let $\bar{\omega}_0$ be as (7.123) but replacing the TV bound by

$$\text{T.V.}_{(x-\rho,x)\cup(x,x+\rho)} (\bar{\omega}_0) \leq \text{T.V.}_{(x-\rho,x)\cup(x,x+\rho)} (\omega_0) \,.$$

Recall that $\hat{u}(y,t)$ was defined in (7.115) by

$$\hat{u}(y,t) = \begin{cases} \omega(y-x,t) & \text{for } |y-x| \leq \sigma^\star t, \\ \omega_0(y) & \text{otherwise.} \end{cases}$$

Let J_h be the set

$$J_h = \left\{ y \mid h\sigma^\star < |y-x| < \rho - h\sigma^\star \right\},$$

and let $\hat{v}$ be the function defined by

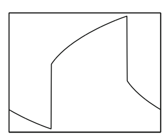
$$\hat{v}(y,t) = \begin{cases} \hat{u}(y,t) & \text{for } |x-y| \leq \sigma^\star t, \\ \bar{\omega}_0(y) & \text{otherwise.} \end{cases}$$

Then we have that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |\hat{v}(y,h) - \hat{u}(y,h)| \, dy \leq \int_{J_h} |\bar{\omega}_0(y) - \omega_0(y)| \, dy \leq \epsilon. \tag{7.134}$$

Note that the bound (7.131) remains valid. We need a replacement for (7.126). In this case we wish to estimate

$$I = \int_{x-\rho+\sigma^\star \varepsilon}^{x+\rho-\sigma^\star \varepsilon} |(S_\varepsilon v(\,\cdot\,,\tau))\,(y) - \bar{v}(y,\tau+\varepsilon)| \, dy.$$



For $|x-y| > \sigma^\star t$, the function $\bar{v}(y,t)$ is discontinuous across lines located at x_i. In addition, it may be discontinuous across the lines $|x-y| = \sigma^\star t$. Inside the region $|x-y| \le \sigma^\star t$, v is an exact entropy solution, coinciding with the semigroup solution. Using this and (7.133), we find that

$$
\begin{aligned}
I &= \left(\int\limits_{x-\rho+\sigma^\star\varepsilon}^{x-\sigma^\star\tau} + \int\limits_{x+\sigma^\star\tau}^{x+\rho-\sigma^\star\varepsilon} \right) \left| \left(S_{\tau+\varepsilon}\bar{\omega}_0\right)(y) - \bar{\omega}_0(y) \right| \, dy \\
&\quad + \int\limits_{x-\sigma^\star\tau}^{x+\sigma^\star\tau} \left| \left(S_\varepsilon \hat{u}(\,\cdot\,,\tau)\right)(y) - \hat{u}(y,\tau+\varepsilon) \right| \, dy \\
&\le \varepsilon\, \mathcal{O}(1) \left(\sum_{|x_i - x| < \sigma^\star\tau} \left| \bar{\omega}_0(x_i+) - \bar{\omega}_0(x_i-) \right| \right) \\
&\quad + L \left(\int\limits_{x-2\sigma^\star\tau}^{x} \left| \bar{\omega}_0(y) - u_l \right| \, dy + \int\limits_{x}^{x+2\sigma^\star\tau} \left| \bar{\omega}_0(y) - u_r \right| \, dy \right) \\
&\le \varepsilon\, \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right).
\end{aligned}
\tag{7.135}
$$

Now using Lemma 7.15, we find that

$$
\begin{aligned}
&\frac{1}{h} \int\limits_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} \left| \left(S_h \bar{\omega}_0\right)(y) - \bar{v}(y,t) \right| \, dy \\
&\quad \le \frac{\mathcal{O}(1)}{h} \int\limits_0^h \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} \left| \left(S_\varepsilon \bar{v}(\,\cdot\,,\tau)\right)(y) - \bar{v}(y,\tau+\varepsilon) \right| \, dy\, d\tau \\
&\quad \le \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right).
\end{aligned}
\tag{7.136}
$$

As before, since ϵ is arbitrary, (7.131), (7.134), and (7.136) imply (7.116). □

Remark 7.19 Note that if ω_0 is continuous at x, then Lemma 7.18 and (7.117) say that the linearized equation gives a good local approximation of the action of the semigroup. If ω_0 has a discontinuity at x, then

$$
\mathrm{T.V.}_{\langle x-\rho,x+\rho\rangle}\left(\omega_0\right)
$$

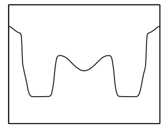
does not become small as ρ tends to zero; hence we must resort to (7.116) in this case. Since the total variation of every function in $\mathcal{D}$ is small, (7.117) is a much stronger estimate than (7.116).

Now that the preliminary technicalities are out of the way, we can set about proving that an entropy solution coincides with the semigroup.

Let u be an entropy solution. To prove that $u(\,\cdot\,,t) = S_t u_0$, it suffices to show, applying Lemma 7.15, that

$$
\liminf_{h\to 0} \frac{1}{h} \left\| S_h u(\,\cdot\,,t) - u(\,\cdot\,,t+h) \right\|_{L^1([a,b])} = 0, \tag{7.137}
$$

for all $a < b$, and for all $t \in [0,T] \setminus \mathcal{N}$.



Assume therefore that $t \notin \mathcal{N}$. Then by the structure theorem, see [193, Theorem 5.9.6], there exists a null set $\mathcal{N} \subset [0,T]$ such that outside that set, u either is continuous or has a jump discontinuity (as a function of x). Therefore, we split the argument into two cases, one in which u has a jump discontinuity, and one in which u is continuous or has a small jump in the sense that it is not in the set $B_{t,\varepsilon}$.

Consider first a point (x,t) where u has jump discontinuity.[4] By condition **B** there exist $u_{l,r} \in \Omega$ and σ such that the limit (7.98) holds when U is defined by (7.97). Using a change of variables, we find that

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |u(y,t+h) - U(y,t+h)|\,dy \\
&\quad= \lim_{h\to 0^+} \sigma^\star \Bigg[\int_{-1-\lambda/\sigma^\star}^{0} |u\,(x+\lambda h + \sigma^\star h y, t+h) - u_l|\,dy \\
&\qquad\qquad + \int_{0}^{1-\lambda/\sigma^\star} |u\,(x+\lambda h + \sigma^\star h y, t+h) - u_r|\,dy \Bigg] = 0,
\end{aligned}$$

by Lemma 7.16. Hence for small positive h, we have that

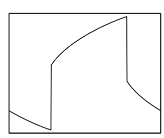
$$\frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |u(y,t+h) - U(y,t+h)|\,dy \le \tilde{\varepsilon}, \tag{7.138}$$

for some small $\tilde{\varepsilon}$ to be determined later. By Lemma 7.14 we have $U(y,s) = \hat{u}(y,s-t)$, where $\hat{u}$ is defined by (7.115) with $\omega_0(y) = u(y,t)$, and U is defined by (7.97), in some forward neighborhood of (x,t). Then using (7.138) and (7.116), we obtain

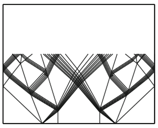
$$\begin{aligned}
&\frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y,t+h)|\,dy \\
&\qquad \le \tilde{\varepsilon} + \frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - U(y,t+h)|\,dy \\
&\qquad \le \tilde{\varepsilon} + \mathcal{O}\,(1)\,\mathrm{T.V.}_{(x-2\sigma^\star h,x)\cup(x,x+2\sigma^\star h)}\,(u(\,\cdot\,,t)) \\
&\qquad \le 2\tilde{\varepsilon},
\end{aligned} \tag{7.139}$$

for all h sufficiently small, since we compute the total variation on a shrinking interval excluding the jump in u at x.

Now we consider points (x,t) where u either is continuous or has a small jump discontinuity. Hence we can choose an interval $\langle c,d\rangle$ centered at x such that $B_{t,\varepsilon} \cap$



[4] The following argument is valid for every jump discontinuity, but will be applied only to jumps in $B_{t,\varepsilon}$.



$(c,d) = \emptyset$. Recall that $B_{t,\varepsilon}$, defined in (7.102), is the set of points where $u(\,\cdot\,,t)$ has a jump larger than ε. Let the family of trapezoids Γ_h be defined by

$$\Gamma_h = \left\{(y,s) \mid s \in [t,t+h],\ \ y \in (c+\sigma^\star(s-t), d-\sigma^\star(s-t))\right\}.$$

Now we claim that for h sufficiently small, we have that for all $(y,s) \in \Gamma_h$,

$$|u(y,s) - u(x,t)| \leq 2\varepsilon + \mathrm{T.V.}_{(c,d)}\left(u(\,\cdot\,,t)\right). \tag{7.140}$$

To prove this, we argue as follows: By Lemma 7.14, discontinuities in u cannot propagate faster than $\sigma^\star$; hence $u(\,\cdot\,,t)$ is continuous in the lower corners of Γ_h, and the estimate surely holds for (y,s) located there. We must prove (7.140) for (y,s) in a region $[c+h', d-h'] \times [t,t+h]$, where h' is given and we can be free to choose h small. Now also $[c+h', d-h'] \cap B_{\varepsilon,t} = \emptyset$; hence for each $y \in [c+h', d-h']$ we can find ξ_y, h_y such that the estimate (7.140) is valid for

$$(y,s) \in (y-\xi_y, y+\xi_y) \times \left[t, t+h_y\right].$$

Now we can cover the compact interval $[c+h', d-h']$ with a finite number of intervals of the form $(y_i - \xi_{y_i}, y_i + \xi_{y_i})$, and choose

$$h = \min_i h_{y_i}.$$

Then we obtain (7.140) for (y,s) in $[c+h', d-h'] \times [t,t+h]$.

Now we must compare u and $\tilde{u}$ near (x,t). The eigenvectors of $\tilde{A} = df(u(x,t))$ are normalized according to (7.124). Observe that trivially

$$u = \sum_k \left(\tilde{l}_k \cdot u\right) \tilde{r}_k.$$

Then

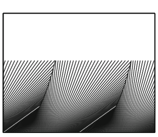
$$\begin{aligned}
&\int_{c+\sigma^\star h}^{d-\sigma^\star h} |u(y,t+h) - \tilde{u}(y,t+h)|\, dy \\
&\qquad \leq \sum_k \int_{c+\sigma^\star h}^{d-\sigma^\star h} \left|\tilde{l}_k \cdot \left(u(y-\tilde{\lambda}_k h, t) - u(y,t+h)\right)\right| dy.
\end{aligned} \tag{7.141}$$

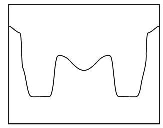
To aid us here we use Lemma 7.17. Let $x_1 < x_2$ be in the interval $(c+\sigma^\star h, d-\sigma^\star h)$. Then we shall estimate

$$E_k = \int_{x_1}^{x_2} \tilde{l}_k \cdot \left(u(y,t+h) - u(y-\tilde{\lambda}_k h, t)\right) dy.$$

If we integrate the conservation law over the region

$$\left\{(y,s) \mid y \in [x_1 - (s-(t+h))\tilde{\lambda}_k, x_2 + (s-(t+h))\tilde{\lambda}_k],\ s \in [t,t+h]\right\},$$





we find that

$$\int_{x_1}^{x_2} u(y,t+h)\,dy - \int_{x_1-\tilde{\lambda}_k h}^{x_2+\tilde{\lambda}_k h} u(y,t)\,dy + \int_t^{t+h} (f(u)-\tilde{\lambda}_k u)(x_2+(s-(t+h))\tilde{\lambda}_k, s)\,ds$$
$$- \int_t^{t+h} (f(u)-\tilde{\lambda}_k u)(x_1-(s-(t+h))\tilde{\lambda}_k, s)\,ds = 0.$$

Taking the inner product with $\tilde{l}_k$, we obtain

$$\begin{aligned} E_k &= \int_t^{t+h} \tilde{l}_k \cdot (f(u)-\tilde{\lambda}_k u)(x_2+(s-(t+h))\tilde{\lambda}_k, s)\,ds \\ &\quad - \int_t^{t+h} \tilde{l}_k \cdot (f(u)-\tilde{\lambda}_k u)(x_1-(s-(t+h))\tilde{\lambda}_k, s)\,ds \\ &= \int_t^{t+h} \tilde{l}_k \cdot \big(f(u_2)-f(u_1)-\tilde{\lambda}_k(u_2-u_1)\big)\,ds, \end{aligned} \tag{7.142}$$

where we have defined

$$u_1 = u\big(x_1-(s-(t+h))\tilde{\lambda}_k, s\big), \quad u_2 = u\big(x_2+(s-(t+h))\tilde{\lambda}_k, s\big).$$

Let $A^\star$ denote the matrix

$$A^\star = \int_0^1 df\big(su_2+(1-s)u_1\big)\,ds - \tilde{A}.$$

Then

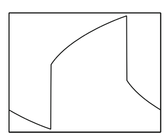
$$\begin{aligned} \tilde{l}_k \cdot \big(f(u_2)-f(u_1)-\tilde{\lambda}_k(u_2-u_1)\big) &= \tilde{l}_k \cdot \big(A^\star(u_2-u_1) \\ &\qquad + \tilde{A}(u_2-u_1) - \tilde{\lambda}_k\,(u_2-u_1)\big) \\ &= \tilde{l}_k \cdot A^\star(u_2-u_1). \end{aligned} \tag{7.143}$$

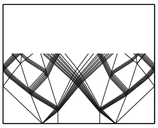
Since

$$\|A^\star\| \le \mathcal{O}(1)\left(|u_1-u(x,t)|+|u_2-u(x,t)|\right),$$

(7.142) and (7.143) yield

$$\begin{aligned} |E_k| &\le \mathcal{O}(1) \int_t^{t+h} \left(|u_1-u(x,t)|+|u_2-u(x,t)|\right)\,|u_2-u_1|\,ds \\ &\le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s)-u(x,t)| \\ &\qquad \times \int_t^{t+h} \mathrm{T.V.}_{(x_1-(s-(t+h))\tilde{\lambda}_k,\, x_2+(s-(t+h))\tilde{\lambda}_k)}\left(u(\,\cdot\,,s)\right)\,ds. \end{aligned}$$





Therefore,

$$
\begin{aligned}
\left|\int_{x_1}^{x_2} (u(y,t+h) - \tilde{u}(y,t+h))\, dy\right|
&\le \sum_k |E_k| \\
&\le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s) - u(x,t)| \\
&\quad \times \int_t^{t+h} \sum_k \mathrm{T.V.}_{(x_1-(s-(t+h))\tilde{\lambda}_k,\, x_2+(s-(t+h))\tilde{\lambda}_k)} (u(\,\cdot\,,s))\, ds. \qquad (7.144)
\end{aligned}
$$

Returning to (7.141) and using Lemma 7.17, we find that

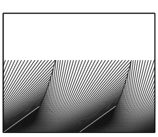
$$
\begin{aligned}
\int_{c+\sigma^\star h}^{d-\sigma^\star h} \left|u(y,t+h) - \tilde{u}(y,t+h)\right| dy
&\le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s) - u(x,t)| \qquad (7.145)\\
&\quad \times \int_t^{t+h} \mathrm{T.V.}_{[c+\sigma^\star(s-t),\, d-\sigma^\star(s-t)]} (u(\,\cdot\,,s))\, ds.
\end{aligned}
$$

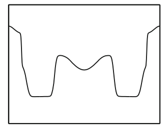
Now we use (7.117), (7.145), and (7.140) to obtain

$$
\begin{aligned}
&\int_{c+\sigma^\star h}^{d-\sigma^\star h} |(S_h u(\,\cdot\,,t))(y) - u(y,t+h)|\, dy \\
&\quad \le \int_{c+\sigma^\star h}^{d-\sigma^\star h} \left(|(S_h u(\,\cdot\,,t))(y) - \tilde{u}(y,t+h)| + |\tilde{u}(y,t+h) - u(y,t+h)| \right) dy \\
&\quad \le \mathcal{O}(1)\, h \left(\mathrm{T.V.}_{(c,d)} (u(\,\cdot\,,t))\right)^2 \\
&\qquad + \mathcal{O}(1) \left(2\varepsilon + \mathrm{T.V.}_{[c,d]} (u(\,\cdot\,,t))\right) \\
&\qquad \times \int_t^{t+h} \mathrm{T.V.}_{[c+\sigma^\star(s-t),\, d-\sigma^\star(s-t)]} (u(\,\cdot\,,s))\, ds. \qquad (7.146)
\end{aligned}
$$

By Lemma 7.13, the set $B_{t,\varepsilon}$ contains only a finite number of points; $x_1 < x_2 < \dots < x_N$, where $u(\,\cdot\,,t)$ has a discontinuity larger than ε. We can cover the set $[a,b] \setminus \cup_i \{x_i\}$ by a finite number of open intervals (c_j,d_j), $j = 1,\dots,M$, such that:

(a) $x_i \notin \cup_j (c_j,d_j) = \emptyset$ for $i = 1,\dots,N$.
(b) $\mathrm{T.V.}_{(c_j,d_j)} (u\,(\,\cdot\,,t)) \le 2\varepsilon$ for $j = 1,\dots,M$.
(c) Every $x \in [a,b]$ is contained in at most two distinct intervals (c_i,d_i).





We have established that for sufficiently small h,

$$\frac{1}{h}\int_{x_i-\sigma^\star h}^{x_i+\sigma^\star h} |(S_h u(\,\cdot\,,t))(y) - u(y,t+h)| \le \frac{\varepsilon}{N},$$

by (7.139) choosing $\tilde{\varepsilon} = \varepsilon/(2N)$. Also,

$$\begin{aligned}\int_{c_j+\sigma^\star h}^{d_j-\sigma^\star h} &|(S_h u(\,\cdot\,,t))(y) - u(y,t+h)|\, dy \\ &\le \mathcal{O}(1)\,\varepsilon \int_t^{t+h} \mathrm{T.V.}_{(c_j+\sigma^\star(s-t),d_j-\sigma^\star(s-t))}\,(u(\,\cdot\,,s))\, ds \\ &\quad + \mathcal{O}(1)\, h\varepsilon \mathrm{T.V.}_{(c_j,d_j)}\,(u(\,\cdot\,,t))\end{aligned}$$

for all i, j, and $\varepsilon > 0$. Combining this, we find that

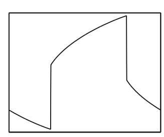
$$\begin{aligned}\frac{1}{h}\int_a^b &|(S_h u(\,\cdot\,,t))(y) - u(y,t+h)|\, dy \\ &\le \sum_i \frac{1}{h}\int_{x_i-\sigma^\star h}^{x_i+\sigma^\star h} |(S_h u(\,\cdot\,,t))(y) - u(y,t+h)|\, dy \\ &\quad + \sum_j \int_{c_j+\sigma^\star h}^{d_j-\sigma^\star h} |(S_h u(\,\cdot\,,t))(y) - u(y,t+h)|\, dy \\ &\le \varepsilon + \mathcal{O}(1)\,\frac{\varepsilon}{h}\int_t^{t+h} \mathrm{T.V.}\,(u(\,\cdot\,,s))\, ds + \varepsilon \mathrm{T.V.}\,(u(\,\cdot\,,t)) \\ &\le \mathcal{O}(1)\,\varepsilon.\end{aligned}$$

Since ε can be arbitrarily small, (7.137) holds, and we have proved the following theorem:

Theorem 7.20 *Let $f_j \in C^2(\mathbb{R}^n)$, $j = 1, \dots, n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$. Assume that each wave family is either genuinely nonlinear or linearly degenerate. For every $u_0 \in \mathcal{D}$, defined by (7.94), the initial value problem*

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x),$$

*has a unique weak entropy solution satisfying conditions **A**–**C**, see Sect. 7.2. Furthermore, this solution can be found by the front-tracking construction.*



7.3 Notes

The material in Sect. 7.1 is taken almost entirely from the fundamental result of Bressan, Liu, and Yang [33]; there is really only an $\mathcal{O}(|e|)$ difference.

Stability of front-tracking approximations to systems of conservation laws was first proved by Bressan and Colombo in [28], in which they used a pseudopolygon technique to "differentiate" the front-tracking approximation with respect to the initial location of the fronts. This approach was later used to prove stability for many special systems; see [47], [8], [3], [4].

The same results as those in Sect. 7.1 of this chapter have also been obtained by Bressan, Crasta, and Piccoli, using a variant of the pseudopolygon approach [29]. This leads to *many* technicalities, and [29] is heavy reading indeed!

The material in Sect. 7.2 is taken from the works of Bressan [23–26] and coworkers, notably Lewicka [32], Goatin [30], and LeFloch [31].

There are few earlier results on uniqueness of solutions to systems of conservation laws; most notable are those by Bressan [20], where uniqueness and stability are obtained for Temple class systems where every characteristic field is linearly degenerate, and in [22] for more general Temple class systems.

Continuity in L^1 with respect to the initial data was also proved by Hu and LeFloch [100] using a variant of Holmgren's technique. See also [77].

Stability for some non-strictly hyperbolic systems of conservation laws (these are really only "quasisystems") has been proved by Winther and Tveito [185] and Klingenberg and Risebro [114].

We end this chapter with a suitable quotation:

This is really easy:

$$|\textit{what you have}| \le |\textit{what you want}| + |\textit{what you have} - \textit{what you want}|$$

— Rinaldo Colombo, private communication

7.4 Exercises

7.1 Show that the solution of the Cauchy problem obtained by the front-tracking construction of Chapt. 6 is an entropy solution in the sense of conditions **A**–**C** in Sect. 7.2.

7.2 The proof of Theorem 7.8 was carried out in detail only in the genuinely nonlinear case. Do the necessary estimates in the case of a linearly degenerate wave family.