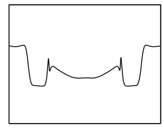


Chapter 5

The Riemann Problem for Systems

> *Diese Untersuchung macht nicht darauf Anspruch, der experimentellen Forschung nützliche Ergebnisse zu liefern; der Verfasser wünscht sie nur als einen Beitrag zur Theorie der nicht linearen partiellen Differentialgleichungen betrachtet zu sehen.*[1]
> *— G. F. B. Riemann [156]*

We return to the conservation law (1.2), but now study the case of systems, i.e.,

$$u_t + f(u)_x = 0, \tag{5.1}$$

where $u = u(x,t) = (u_1, \dots, u_n)$ and $f = f(u) = (f_1, \dots, f_n) \in C^2$ are vectors in $\mathbb{R}^n$. (We will not distinguish between row and column vectors, and use whatever is more convenient.) Furthermore, in this chapter we will consider only systems on the line; i.e., the dimension of the underlying physical space is still one. In Chapt. 2 we proved existence, uniqueness, and stability of the Cauchy problem for the scalar conservation law in one space dimension, i.e., well-posedness in the sense of Hadamard. However, this is a more subtle question in the case of systems of hyperbolic conservation laws. We will here first discuss the basic concepts for systems: fundamental properties of shock waves and rarefaction waves. In particular, we will discuss various entropy conditions to select the right solutions of the Rankine–Hugoniot relations.

Using these results, we will eventually be able to prove well-posedness of the Cauchy problem for systems of hyperbolic conservation laws with small variation in the initial data.

5.1 Hyperbolicity and Some Examples

Before we start to define the basic properties of systems of hyperbolic conservation laws we discuss some important and interesting examples. The first example is a model for shallow-water waves and will be used throughout this chapter as both a motivation and an example in which all the basic quantities will be explicitly computed.

[1] The present work does not claim to lead to results in experimental research; the author asks only that it be considered as a contribution to the theory of nonlinear partial differential equations.

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_5

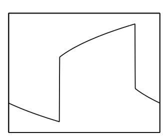
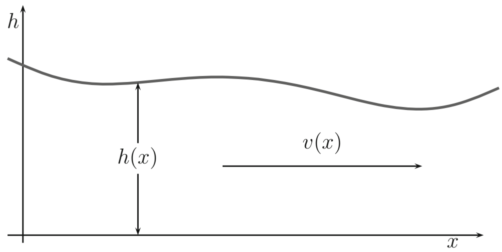


Fig. 5.1 A shallow channel



◇ Example 5.1 (Shallow water)

Water shapes its course according to the nature of the ground over which it flows.
— Sun Tzu, The Art of War (6th–5th century BC)

We will now give a brief derivation of the equations governing shallow-water waves in one space dimension, or, if we want, the long-wave approximation.[2] Consider a one-dimensional channel along the x-axis with a perfect, inviscid fluid with constant density ρ, and assume that the bottom of the channel is horizontal.

In the long-wave or shallow-water approximation we assume that the fluid velocity v is a function only of time and the position along the channel measured along the x-axis. Thus we assume that there is no vertical motion in the fluid. The distance of the surface of the fluid from the bottom is denoted by $h = h(x,t)$. The fluid flow is governed by *conservation of mass* and *conservation of momentum*.

Consider first the conservation of mass of the system. Let $x_1 < x_2$ be two points along the channel. The change of mass of fluid between these points is given by

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho\, dy\, dx = -\int_0^{h(x_2,t)} \rho v(x_2,t)\, dy + \int_0^{h(x_1,t)} \rho v(x_1,t)\, dy.$$

Assuming smoothness of the functions and domains involved, we may rewrite the right-hand side as an integral of the derivative of $\rho v h$. We obtain

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho\, dy\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho v(x,t)h(x,t)\right)\, dx,$$

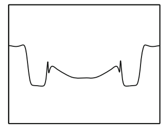
or

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\left(\rho h(x,t)\right) + \frac{\partial}{\partial x}\left(\rho v(x,t)h(x,t)\right)\right] dx = 0.$$

Dividing by $(x_2 - x_1)\rho$ and letting $x_2 - x_1 \to 0$, we obtain the familiar

$$h_t + (vh)_x = 0. \tag{5.2}$$

[2] A word of warning. There are several different equations that are called the shallow-water equations. Also the name Saint-Venant equation is used.



Observe the similarity in the derivations of (5.2) and (1.26). In fact, in the derivation of (1.26) we started by considering individual cars before we made the continuum assumption corresponding to high traffic densities, thereby obtaining (1.26), while in the derivation of (5.2) we simply assumed a priori that the fluid constituted a continuum, and formulated mass conservation directly in the continuum variables.

For the derivation of the equation describing the conservation of momentum we have to assume that the fluid is in hydrostatic balance. For that we introduce the pressure $P = P(x, y, t)$ and consider a small element of the fluid $[x_1, x_2] \times [y, y + \Delta y]$. Hydrostatic balance means that the pressure exactly balances the effect of gravity, or

$$(P(\tilde{x}, y + \Delta y, t) - P(\tilde{x}, y, t))\,(x_2 - x_1) = -(x_2 - x_1)\rho g \Delta y$$

for some $\tilde{x} \in [x_1, x_2]$, where g is the acceleration due to gravity. Dividing by $(x_2 - x_1)\Delta y$ and taking $x_1, x_2 \to x$, $\Delta y \to 0$, we find that

$$\frac{\partial P}{\partial y}(x, y, t) = -\rho g.$$

Integrating and normalizing the pressure to be zero at the fluid surface, we conclude that

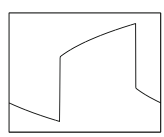
$$P(x, y, t) = \rho g(h(x, t) - y). \tag{5.3}$$

Consider again the fluid between two points $x_1 < x_2$ along the channel. According to Newton's second law, the rate of change of momentum of this part of the fluid is balanced by the net momentum inflow $(\rho v)v = \rho v^2$ across the boundaries $x = x_1$ and $x = x_2$ plus the forces exerted by the pressure at the boundaries. Thus we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho v(x,t)\,dy\,dx = &-\int_0^{h(x_2,t)} P(x_2, y, t)\,dy + \int_0^{h(x_1,t)} P(x_1, y, t)\,dy \\ &-\int_0^{h(x_2,t)} \rho v(x_2, t)^2\,dy + \int_0^{h(x_1,t)} \rho v(x_1, t)^2\,dy.\end{aligned}$$

In analogy with the derivation of the equation for conservation of mass, we may rewrite this, using (5.3), as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho v h\,dx = &-\rho g\left(h(x_2, t)^2 - \frac{1}{2}h(x_2, t)^2\right) \\ &+ \rho g\left(h(x_1, t)^2 - \frac{1}{2}h(x_1, t)^2\right) - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho h v^2\right)\,dx \\ = &-\rho g \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{1}{2}h^2\right)\,dx - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho v^2 h\right)\,dx.\end{aligned}$$



Dividing again by $(x_2 - x_1)\rho$ and letting $x_2 - x_1 \to 0$, scaling g to unity, we obtain

$$(vh)_t + \left(v^2 h + \frac{1}{2}h^2\right)_x = 0. \tag{5.4}$$

To summarize, we have the following system of conservation laws:

$$h_t + (vh)_x = 0, \quad (vh)_t + \left(v^2 h + \frac{1}{2}h^2\right)_x = 0, \tag{5.5}$$

where h and v denote the height (depth) and velocity of the fluid, respectively. Introducing the variable q defined by

$$q = vh, \tag{5.6}$$

we may rewrite the shallow-water equations as

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}_x = 0, \tag{5.7}$$

which is the form we will study in detail later on in this chapter. We note in passing that we can write the equation for v as

$$v_t + v v_x + h_x = 0 \tag{5.8}$$

by expanding the second equation in (5.5), and then using the first equation in (5.5).

A different derivation is based on the incompressible Navier–Stokes equations.[3] Consider gravity waves of an incompressible two-dimensional fluid governed by the Navier–Stokes equations

$$\begin{aligned} \bar{v}_{\bar{t}} + (\bar{v} \cdot \nabla)\bar{v} &= \bar{g} - \frac{\bar{p}}{\rho} + \nu \Delta \bar{v}, \\ \nabla \cdot \bar{v} &= 0. \end{aligned} \tag{5.9}$$

Here ρ, $\bar{p}$, $\bar{v} = (\bar{v}_1, \bar{v}_2)$, ν denote the density, pressure, velocity, and viscosity of the fluid, respectively. The first equation describes the momentum conservation, and the second is the incompressibility assumption. We let the y-direction point upward, and thus the gravity $\bar{g}$ is a vector with length equal to g, the acceleration due to gravity, and direction in the negative y-direction. Let L and H denote typical wavelengths of the surface wave and water depth, respectively. The shallow-water assumption (or long-wave assumption) is the following

$$\varepsilon = \frac{H}{L} \ll 1. \tag{5.10}$$

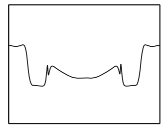
We introduce scaled variables

$$\begin{aligned} x &= L\bar{x}, \quad & y &= H\bar{y}, \quad & t &= T\bar{t}, \\ v &= U\bar{v}_1, \quad & u &= V\bar{v}_2, \quad & p &= \rho g H \bar{p}. \end{aligned} \tag{5.11}$$

The following relations are natural:

$$UT = L, \quad VT = H, \quad U^2 = gH. \tag{5.12}$$

[3] Thanks to Harald Hanche-Olsen.



In addition, we introduce the dimensionless Reynolds number $\mathrm{Re} = UH/\nu$. In the new variables we obtain

$$\begin{aligned} v_t + vv_x + uv_y &= -p_x + \frac{1}{\varepsilon \mathrm{Re}}(\varepsilon^2 v_{xx} + v_{yy}), \\ \varepsilon^2(vu_t + vuv_x + uu_y) &= -1 - p_y + \frac{\varepsilon}{\mathrm{Re}}(\varepsilon^2 u_{xx} + u_{yy}), \\ u_x + v_y &= 0. \end{aligned} \tag{5.13}$$

For typical waves we have $\mathrm{Re} \gg 1$, yet $\varepsilon \mathrm{Re} \gg 1$.[4] Hence a reasonable approximation reads

$$\begin{aligned} v_t + vv_x + uv_y &= -p_x, \\ p_y &= -1, \\ v_x + u_y &= 0. \end{aligned} \tag{5.14}$$

We assume that the bottom is flat and normalize the pressure to vanish at the surface of the fluid, given by $y = h(x,t)$. Hence the pressure equation integrates in the y-direction to yield $p = h(x,t) - y$.

Next we claim that if the horizontal velocity v is independent of y initially, it will remain so, and thus $v_y = 0$. Namely, for a given fluid particle we have that

$$\begin{aligned} \frac{d^2x}{dt^2} = \frac{dv}{dt} &= v_t + v_x \frac{dx}{dt} + v_y \frac{dy}{dt} \\ &= v_t + vv_x + uv_y = -p_x. \end{aligned} \tag{5.15}$$

Since the right-hand side is independent of y, the claim is proved. We can then write

$$v_t + vv_x + h_x = 0. \tag{5.16}$$

A fluid particle at the surface satisfies $y = h(x,t)$, or

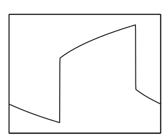
$$u = h_x v + h_t, \text{ whenever } y = h(x,t). \tag{5.17}$$

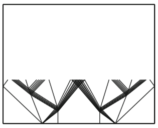
Consider the fluid contained in a domain R between two fixed points x_1 and x_2. By applying Green's theorem on the domain R and on $v_x + u_y = 0$, we obtain

$$\begin{aligned} 0 = \iint\limits_R (v_x + u_y)\, dx\, dy &= \int\limits_{\partial R} (-u\, dx + v\, dy) \\ &= \int_{x_1}^{x_2} \big((h_x v + h_t)\, dx - v h_x\, dx\big) \\ &\quad + v(x_2,t)h(x_2,t) - v(x_1,t)h(x_1,t) \\ &= \int_{x_1}^{x_2} \big(h_t + (vh)_x\big)\, dx, \end{aligned} \tag{5.18}$$

or $h_t + (vh)_x = 0$, where we used that $v\, dy = v h_x\, dx$ along the curve $y = h(x,t)$.

[4] In tidal waves, say in the North Sea, we have $H \approx 100\,\mathrm{m}$, $T = 6\,\mathrm{h}$, $\nu = 10^{-6}\,\mathrm{m^2 s^{-1}}$, which yields $\varepsilon \approx 2 \cdot 10^{-4}$ and $\mathrm{Re} \approx 3 \cdot 10^9$.





From this we conclude that the shallow-water equations read

$$\begin{aligned} h_t + (vh)_x &= 0, \\ v_t + vv_x + h_x &= 0, \end{aligned} \tag{5.19}$$

in nonconservative form. ◇

◇ Example 5.2 (The wave equation)
Let $\phi = \phi(x,t)$ denote the transverse position away from equilibrium of a one-dimensional string. If we assume that the amplitude of the transversal waves is small, we obtain the wave equation

$$\phi_{tt} = (c^2\phi_x)_x, \tag{5.20}$$

where c denotes the wave speed. Introducing new variables $u = \phi_x$ and $v = \phi_t$, we find that (5.20) may be written as the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t - \begin{pmatrix} v \\ c^2 u \end{pmatrix}_x = 0. \tag{5.21}$$

If c is constant, we recover the classical linear wave equation $\phi_{tt} = c^2\phi_{xx}$. See also Example 1.14. ◇

◇ Example 5.3 (The p-system)
The p-system is a classical model of an isentropic gas, where one has conservation of mass and momentum, but not of energy. In Lagrangian coordinates it is described by

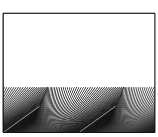
$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0. \tag{5.22}$$

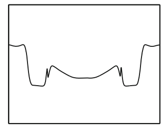
Here v denotes specific volume, that is, the inverse of the density; u is the velocity; and p denotes the pressure. ◇

◇ Example 5.4 (The Euler equations)
The Euler equations are commonly used to model gas dynamics. They can be written in several forms, depending on the physical assumptions used and variables selected to describe them. Let it suffice here to describe the case in which ρ denotes the density, v velocity, p pressure, and E the energy. Conservation of mass and momentum give $\rho_t + (\rho v)_x = 0$ and $(\rho v)_t + (\rho v^2 + p)_x = 0$, respectively. The total energy can be written as $E = \frac{1}{2}\rho v^2 + \rho e$, where e denotes the specific internal energy. Furthermore, we assume that there is a relation between this quantity and the density and pressure, namely $e = e(\rho, p)$. Conservation of energy now reads $E_t + (v(E + p))_x = 0$, yielding finally the system

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0. \tag{5.23}$$

We will return to this system at length in Sect. 5.6. ◇





We will have to make assumptions on the (vector-valued) function f so that many of the properties of the scalar case carry over to the case of systems. In order to have finite speed of propagation, which characterizes hyperbolic equations, we have to assume that the Jacobian of f, denoted by df, has n real eigenvalues

$$df(u)r_j(u) = \lambda_j(u)r_j(u), \quad \lambda_j(u) \in \mathbb{R}, \quad j = 1, \dots, n. \tag{5.24}$$

(We will later normalize the eigenvectors $r_j(u)$.) Furthermore, we order the eigenvalues

$$\lambda_1(u) \le \lambda_2(u) \le \cdots \le \lambda_n(u). \tag{5.25}$$

A system with a full set of eigenvectors with real eigenvalues is called *hyperbolic*, and if all the eigenvalues are distinct, we say that the system is *strictly hyperbolic*.

Let us look at the shallow-water model to see whether that system is hyperbolic.

◇ Example 5.5 (Shallow water (cont'd.))
In the case of the shallow-water equations (5.7) we easily find that

$$\lambda_1(u) = \frac{q}{h} - \sqrt{h} < \frac{q}{h} + \sqrt{h} = \lambda_2(u), \tag{5.26}$$

with corresponding eigenvectors

$$r_j(u) = \begin{pmatrix} 1 \\ \lambda_j(u) \end{pmatrix}, \tag{5.27}$$

and thus the shallow-water equations are strictly hyperbolic away from $h = 0$. ◇

5.2 Rarefaction Waves

Natura non facit saltus.[5]
— Carl Linnaeus, Philosophia Botanica (1751)

Let us consider smooth solutions for the initial value problem

$$u_t + f(u)_x = 0, \tag{5.28}$$

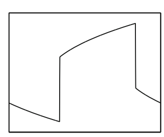
with Riemann initial data

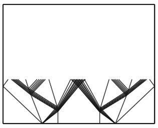
$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \ge 0. \end{cases} \tag{5.29}$$

First we observe that since both the initial data and the equation are scale-invariant or self-similar, i.e., invariant under the map $x \mapsto kx$ and $t \mapsto kt$, the solution should also have that property. Let us therefore search for solutions of the form

$$u(x,t) = w(x/t) = w(\xi), \quad \xi = x/t. \tag{5.30}$$

[5] Nature does not make jumps.





Inserting this into the differential equation (5.28), we find that

$$-\frac{x}{t^2}\dot{w} + \frac{1}{t}df(w)\dot{w} = 0, \tag{5.31}$$

or

$$df(w)\dot{w} = \xi\dot{w}, \tag{5.32}$$

where $\dot{w}$ denotes the derivative of w with respect to the one variable ξ. Hence we observe that $\dot{w}$ is an eigenvector for the Jacobian $df(w)$ with eigenvalue ξ. From our assumptions on the flux function we know that $df(w)$ has n eigenvectors given by $r_1, \dots, r_n$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. This implies

$$\dot{w}(\xi) = r_j(w(\xi)), \quad \lambda_j(w(\xi)) = \xi, \tag{5.33}$$

for a value of j. Assume in addition that

$$w(\lambda_j(u_l)) = u_l, \quad w(\lambda_j(u_r)) = u_r. \tag{5.34}$$

Thus for a fixed time t, the function $w(x/t)$ will continuously connect the given left state u_l to the given right state u_r. This means that ξ is increasing, and hence $\lambda_j(w(x/t))$ has to be increasing. If this is the case, we have a solution of the form

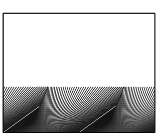
$$u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ w(x/t) & \text{for } t\lambda_j(u_l) \le x \le t\lambda_j(u_r), \\ u_r & \text{for } x \ge t\lambda_j(u_r), \end{cases} \tag{5.35}$$

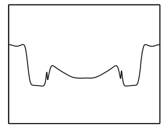
where $w(\xi)$ satisfies (5.33) and (5.34). We call these solutions *rarefaction waves*, a name that comes from applications to gas dynamics. Furthermore, we observe that the normalization of the eigenvector $r_j(u)$ also is determined from (5.33), namely,

$$\nabla\lambda_j(u) \cdot r_j(u) = 1, \tag{5.36}$$

which follows by taking the derivative with respect to ξ. But this also imposes an extra condition on the eigenvector fields, since we clearly have to have a nonvanishing scalar product between $r_j(u)$ and $\nabla\lambda_j(u)$ to be able to normalize the eigenvector properly. It so happens that in most applications this can be done. However, the Euler equations of gas dynamics have the property that in one of the eigenvector families, the eigenvector and the gradient of the corresponding eigenvalue are orthogonal. We say that the jth family is *genuinely nonlinear* if $\nabla\lambda_j(u) \cdot r_j(u) \neq 0$ and *linearly degenerate* if $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$ for all u under consideration. We will not discuss mixed cases whereby a wave family is linearly degenerate only in certain regions in phase space, e.g., along curves or at isolated points.

Before we discuss these two cases separately, we will make a slight but important change in point of view. Instead of considering given left and right states as in (5.29), we will assume only that u_l is given, and consider those states u_r for which we have a rarefaction wave solution. From (5.33) and (5.35) we see that for each





point u_l in phase space there are n curves emanating from u_l on which u_r can lie allowing a solution of the form (5.35). Each of these curves is given as integral curves of the vector fields of eigenvectors of the Jacobian $df(u)$. Thus our phase space is now the u_r space.

We may sum up the above discussion in the genuinely nonlinear case by the following theorem.

Theorem 5.6 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$ and assume that the equation is genuinely nonlinear in the jth wave family in D. Let the jth eigenvector $r_j(u)$ of $df(u)$ with corresponding eigenvalue $\lambda_j(u)$ be normalized so that $\nabla\lambda_j(u) \cdot r_j(u) = 1$ in D.*

Let $u_l \in D$. Then there exists a curve $R_j(u_l)$ in D, emanating from u_l, such that for each $u_r \in R_j(u_l)$ the initial value problem (5.28), (5.29) has weak solution

$$u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ w(x/t) & \text{for } \lambda_j(u_l)t \le x \le \lambda_j(u_r)t, \\ u_r & \text{for } x \ge \lambda_j(u_r)t, \end{cases} \tag{5.37}$$

where w satisfies $\dot{w}(\xi) = r_j(w(\xi))$, $\lambda_j(w(\xi)) = \xi$, $w(\lambda_j(u_l)) = u_l$, and $w(\lambda_j(u_r)) = u_r$.

Proof The discussion preceding the theorem gives the key computation and the necessary motivation behind the following argument. Assume that we have a strictly hyperbolic, genuinely nonlinear conservation law with appropriately normalized jth eigenvector. Due to the assumptions on f, the system of ordinary differential equations

$$\dot{w}(\xi) = r_j(w(\xi)), \quad w(\lambda_j(u_l)) = u_l \tag{5.38}$$

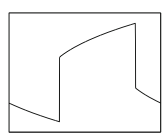
has a solution for all $\xi \in [\lambda_j(u_l), \lambda_j(u_l) + \eta)$ for some $\eta > 0$. For this solution we have

$$\frac{d}{d\xi}\lambda_j(w(\xi)) = \nabla\lambda_j(w(\xi)) \cdot \dot{w}(\xi) = 1, \tag{5.39}$$

proving the second half of (5.33). We denote the orbit of (5.38) by $R_j(u_l)$. If we define $u(x,t)$ by (5.37), a straightforward calculation shows that u indeed satisfies both the equation and the initial data. □

Observe that we can also solve (5.38) for ξ less than $\lambda_j(u_l)$. However, in that case $\lambda_j(u)$ will be decreasing. We remark that the solution u in (5.37) is continuous, but not necessarily differentiable, and hence is not necessarily a regular, but rather a weak, solution.

We will now introduce a different parameterization of the rarefaction curve $R_j(u_l)$, which will be convenient in Section 5.5 when we construct the wave curves for the solution of the Riemann problem. From (5.39) we see that $\lambda_j(u)$ is increasing along $R_j(u_l)$, and hence we may define the positive parameter ϵ by



$\epsilon := \xi - \xi_l = \lambda_j(u) - \lambda_j(u_l) > 0$. We denote the corresponding u by $u_{j,\epsilon}$, that is, $u_{j,\epsilon} = w(\xi) = w(\lambda_j(u)) = w(\epsilon + \lambda_j(u_l))$. Clearly,

$$\left.\frac{du_{j,\epsilon}}{d\epsilon}\right|_{\epsilon=0} = r_j(u_l). \tag{5.40}$$

Assume now that the system is linearly degenerate in the family j, i.e., $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$. Consider the system of ordinary differential equations

$$\frac{du}{d\epsilon} = r_j(u), \quad u|_{\epsilon=0} = u_l, \tag{5.41}$$

with solution $u = u_{j,\epsilon}$ for $\epsilon \in (-\eta, \eta)$ for some $\eta > 0$. We denote this orbit by $C_j(u_l)$, along which $\lambda_j(u_{j,\epsilon})$ is constant, since

$$\frac{d}{d\epsilon}\lambda_j(u_{j,\epsilon}) = \nabla\lambda_j(u_{j,\epsilon}) \cdot r_j(u_{j,\epsilon}) = 0.$$

Furthermore, the Rankine–Hugoniot condition is satisfied on $C_j(u_l)$ with speed $\lambda_j(u_l)$, because

$$\begin{aligned}\frac{d}{d\epsilon}(f(u_{j,\epsilon}) - \lambda_j(u_l)u_{j,\epsilon}) &= df(u_{j,\epsilon})\frac{du_{j,\epsilon}}{d\epsilon} - \lambda_j(u_l)\frac{du_{j,\epsilon}}{d\epsilon} \\ &= (df(u_{j,\epsilon}) - \lambda_j(u_l))r_j(u_{j,\epsilon}) \\ &= (df(u_{j,\epsilon}) - \lambda_j(u_{j,\epsilon}))r_j(u_{j,\epsilon}) = 0,\end{aligned}$$

which implies that $f(u_{j,\epsilon}) - \lambda_j(u_l)u_{j,\epsilon} = f(u_l) - \lambda_j(u_l)u_l$.

Let $u_r \in C_j(u_l)$, i.e., $u_r = u_{j,\epsilon_0}$ for some ϵ_0. It follows that

$$u(x,t) = \begin{cases} u_l & \text{for } x < \lambda_j(u_l)t, \\ u_r & \text{for } x \geq \lambda_j(u_l)t, \end{cases}$$

is a weak solution of the Riemann problem (5.28), (5.29). We call this solution a *contact discontinuity*.

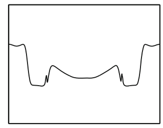
We sum up the above discussion concerning linearly degenerate waves in the following theorem.

Theorem 5.7 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Assume that the equation is linearly degenerate in the jth wave family in D, i.e., $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$ in D, where $r_j(u)$ is the jth eigenvector of $df(u)$ with corresponding eigenvalue $\lambda_j(u)$.*

Let $u_l \in D$. Then there exists a curve $C_j(u_l)$ in D, passing through u_l, such that for each $u_r \in C_j(u_l)$ the initial value problem (5.28), (5.29) has solution

$$u(x,t) = \begin{cases} u_l & \text{for } x < \lambda_j(u_l)t, \\ u_r & \text{for } x \geq \lambda_j(u_l)t, \end{cases} \tag{5.42}$$

where u_r is determined as follows: Consider the function $\epsilon \mapsto u_\epsilon$ determined by $\frac{du}{d\epsilon} = r_j(u)$, $u|_{\epsilon=0} = u_l$. Then $u_r = u_{\epsilon_0}$ for some ϵ_0.



◇ Example 5.8 (Shallow water (cont'd.))
Let us now consider the actual computation of rarefaction waves in the case of shallow-water waves. Recall that

$$u = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(u) = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix},$$

with eigenvalues $\lambda_j = \frac{q}{h} + (-1)^j \sqrt{h}$, and corresponding eigenvectors $r_j(u) = \binom{1}{\lambda_j(u)}$. With this normalization of r_j, we obtain

$$\nabla\lambda_j(u) \cdot r_j(u) = \frac{3(-1)^j}{2\sqrt{h}}, \tag{5.43}$$

and hence we see that the shallow-water equations are genuinely nonlinear in both wave families. From now on we will renormalize the eigenvectors to satisfy (5.36):

$$r_j(u) = \frac{2}{3}(-1)^j \sqrt{h} \begin{pmatrix} 1 \\ \lambda_j(u) \end{pmatrix}. \tag{5.44}$$

For the 1-family we have that

$$\begin{pmatrix} \dot{h} \\ \dot{q} \end{pmatrix} = -\frac{2}{3}\sqrt{h} \begin{pmatrix} 1 \\ \frac{q}{h} - \sqrt{h} \end{pmatrix}, \tag{5.45}$$

implying that

$$\frac{dq}{dh} = \lambda_1 = \frac{q}{h} - \sqrt{h},$$

which can be integrated to yield

$$q = q(h) = q_l \frac{h}{h_l} - 2h\big(\sqrt{h} - \sqrt{h_l}\,\big). \tag{5.46}$$

Since $\lambda_1(u)$ has to increase along the rarefaction wave, we see from (5.26) (inserting the expression (5.46) for q) that we have to use $h \le h_l$ in (5.46).

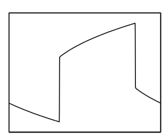
For the second family we again obtain

$$\frac{dq}{dh} = \lambda_2 = \frac{q}{h} + \sqrt{h},$$

yielding

$$q = q(h) = q_l \frac{h}{h_l} + 2h\big(\sqrt{h} - \sqrt{h_l}\,\big). \tag{5.47}$$

In this case we see that we have to use $h \ge h_l$. Observe that (5.46) and (5.47) would follow for *any* normalization of the eigenvector $r_j(u)$. See Fig. 5.2.



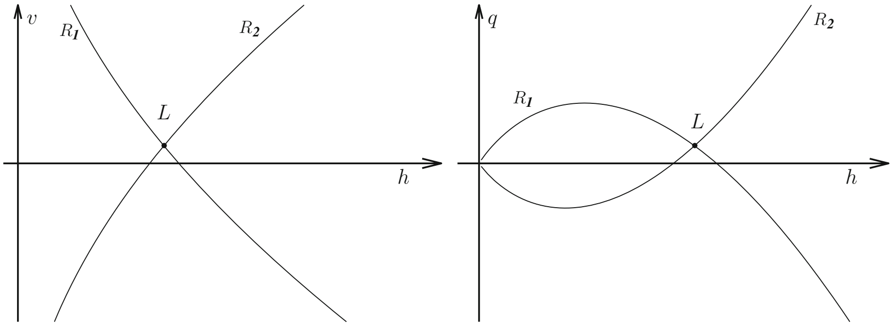


Fig. 5.2 Rarefaction curves in the (h, v)- and (h, q)-planes. We have illustrated the full solution of (5.38) for the shallow-water equations. Only the part given by (5.48) and (5.49) will be actual rarefaction curves

Summing up, we obtain the following rarefaction waves expressed in terms of h:

$$R_1: \quad q = R_1(h; u_l) := q_l \frac{h}{h_l} - 2h\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \in (0, h_l], \tag{5.48}$$

$$R_2: \quad q = R_2(h; u_l) := q_l \frac{h}{h_l} + 2h\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \geq h_l. \tag{5.49}$$

Alternatively, in the (h, v) variables (with $v = q/h$) we have the following:

$$R_1: \quad v = R_1(h; u_l) := v_l - 2\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \in (0, h_l], \tag{5.50}$$

$$R_2: \quad v = R_2(h; u_l) := v_l + 2\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \geq h_l. \tag{5.51}$$

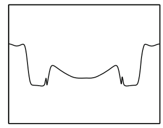
However, if we want to compute the rarefaction curves in terms of the parameter ξ or ϵ, we have to use the proper normalization of the eigenvectors given by (5.44). Consider first the 1-family. We obtain

$$\dot{h} = -\frac{2}{3}\sqrt{h}, \quad \dot{q} = \frac{2}{3}\left(-\frac{q}{\sqrt{h}} + h\right). \tag{5.52}$$

Integrating the first equation directly and inserting the result into the second equation, we obtain

$$\begin{aligned} w_1(\xi) = \begin{pmatrix} h_1 \\ q_1 \end{pmatrix}(\xi) &= R_1(\xi; u_l) \\ &:= \begin{pmatrix} \frac{1}{9}\big(v_l + 2\sqrt{h_l} - \xi\big)^2 \\ \frac{1}{27}\big(v_l + 2\sqrt{h_l} + 2\xi\big)\big(v_l + 2\sqrt{h_l} - \xi\big)^2 \end{pmatrix} \end{aligned} \tag{5.53}$$

for $\xi \in \big[v_l - \sqrt{h_l}, v_l + 2\sqrt{h_l}\big)$.



Similarly, for the second family we obtain

$$w_2(\xi) = \begin{pmatrix} h_2 \\ q_2 \end{pmatrix}(\xi) = R_2(\xi; u_l)$$
$$:= \begin{pmatrix} \frac{1}{9}(-v_l + 2\sqrt{h_l} + \xi)^2 \\ \frac{1}{27}(v_l - 2\sqrt{h_l} + 2\xi)(-v_l + 2\sqrt{h_l} + \xi)^2 \end{pmatrix} \tag{5.54}$$

for $\xi \in \big[\lambda_2(u_l), \infty\big)$. Hence the actual solution reads

$$u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ R_j(x/t; u_l) & \text{for } \lambda_j(u_l)t \le x \le \lambda_j(u_r)t, \\ u_r & \text{for } x \ge \lambda_j(u_r)t. \end{cases} \tag{5.55}$$

In the (h, v) variables (with $v = q/h$) we obtain

$$v_1(\xi) = \frac{1}{3}\big(v_l + 2\sqrt{h_l} + 2\xi\big) \tag{5.56}$$

and

$$v_2(\xi) = \frac{1}{3}\big(v_l - 2\sqrt{h_l} + 2\xi\big), \tag{5.57}$$

for the first and the second families, respectively.

In terms of the parameter ϵ we may write (5.53) as

$$u_{1,\epsilon} = \begin{pmatrix} h_{1,\epsilon} \\ q_{1,\epsilon} \end{pmatrix} = R_{1,\epsilon}(u_l) := \begin{pmatrix} (\sqrt{h_l} - \frac{\epsilon}{3})^2 \\ (v_l + \frac{2\epsilon}{3})(\sqrt{h_l} - \frac{\epsilon}{3})^2 \end{pmatrix} \tag{5.58}$$

for $\epsilon \in \big[0, 3\sqrt{h_l}\,\big)$, and (5.54) as

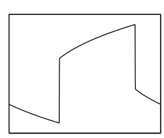
$$u_{2,\epsilon} = \begin{pmatrix} h_{2,\epsilon} \\ q_{2,\epsilon} \end{pmatrix} = R_{2,\epsilon}(u_l) := \begin{pmatrix} \big(\sqrt{h_l} + \frac{\epsilon}{3}\big)^2 \\ \big(v_l + \frac{2\epsilon}{3}\big)\big(\sqrt{h_l} + \frac{\epsilon}{3}\big)^2 \end{pmatrix} \tag{5.59}$$

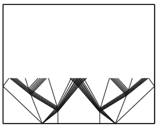
for $\epsilon \in [0, \infty)$. ◇

5.3 The Hugoniot Locus: The Shock Curves

God lives in the details.
— Johannes Kepler (1571–1630)

The discussion in Chapt. 1 concerning weak solutions, and in particular the Rankine–Hugoniot condition (1.27), carries over to the case of systems without restrictions. However, the concept of entropy is considerably more difficult for systems and is still an area of research. Our main concern in this section is the





characterization of solutions of the Rankine–Hugoniot relation. Again, we will take the point of view introduced in the previous section, assuming the left state u_l to be fixed, and consider possible right states u that satisfy the Rankine–Hugoniot condition

$$s(u - u_l) = f(u) - f(u_l), \tag{5.60}$$

for some speed s. We introduce the jump in a quantity ϕ as

$$[\![\phi]\!] = \phi_r - \phi_l,$$

and hence (5.60) takes the familiar form

$$s\,[\![u]\!] = [\![f(u)]\!]\,.$$

The solutions of (5.60), for a given left state u_l, form a set, which we call the *Hugoniot locus* and write $H(u_l)$, i.e.,

$$H(u_l) := \left\{u \mid \exists s \in \mathbb{R} \text{ such that } s\,[\![u]\!] = [\![f(u)]\!]\right\}. \tag{5.61}$$

We start by computing the Hugoniot locus for the shallow-water equations.

◇ Example 5.9 (Shallow water (cont'd.))
The Rankine–Hugoniot condition reads

$$\begin{aligned} s(h - h_l) &= q - q_l, \\ s(q - q_l) &= \left(\frac{q^2}{h} + \frac{h^2}{2}\right) - \left(\frac{q_l^2}{h_l} + \frac{h_l^2}{2}\right), \end{aligned} \tag{5.62}$$

where s as usual denotes the shock speed between the left state $u_l = \binom{h_l}{q_l}$ and right state $u = \binom{h}{q}$:

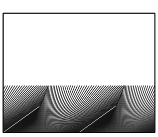
$$\begin{pmatrix} h \\ q \end{pmatrix}(x,t) = \begin{cases} \binom{h_l}{q_l} & \text{for } x < st, \\ \binom{h}{q} & \text{for } x \geq st. \end{cases} \tag{5.63}$$

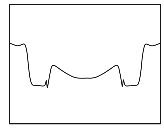
In the context of the shallow-water equations such solutions are called *bores*. Eliminating s in (5.62), we obtain the equation

$$[\![h]\!]\left(\left[\!\!\left[\frac{q^2}{h}\right]\!\!\right] + \frac{1}{2}\,[\![h^2]\!]\right) = [\![q]\!]^2\,. \tag{5.64}$$

Introducing the variable v, given by $q = vh$, equation (5.64) becomes

$$[\![h]\!]\left([\![hv^2]\!] + \frac{1}{2}\,[\![h^2]\!]\right) = [\![vh]\!]^2,$$





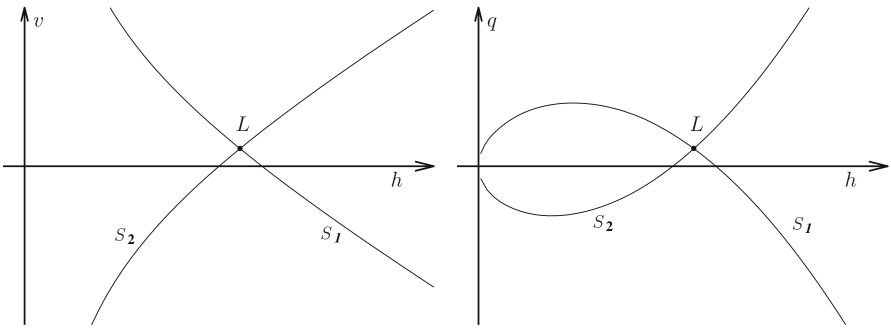


Fig. 5.3 Shock curves in the (h, v)- and (h, q)-planes. Slow (S_1) and fast (S_2) shocks indicated; see Sect. 5.4

with solution

$$v = v_l \pm \frac{1}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}}, \tag{5.65}$$

or alternatively,

$$q = vh = q_l \frac{h}{h_l} \pm \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}}. \tag{5.66}$$

See Fig. 5.3. For later use, we will also obtain formulas for the corresponding shock speeds. We find that

$$\begin{aligned} s = \frac{[\![vh]\!]}{[\![h]\!]} &= \frac{v(h - h_l) + (v - v_l)h_l}{h - h_l} \\ &= v + \frac{[\![v]\!]}{[\![h]\!]}h_l = v \pm \frac{h_l}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}, \end{aligned} \tag{5.67}$$

or

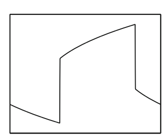
$$s = v + \frac{[\![v]\!]}{[\![h]\!]}h_l = v_l + [\![v]\!] + \frac{[\![v]\!]}{[\![h]\!]}h_l = v_l \pm \frac{h}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}. \tag{5.68}$$

When we want to indicate the wave family, we write

$$\begin{aligned} s_j = s_j(h; v_l) &= v_l + (-1)^j \frac{h}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}} \\ &= v + (-1)^j \frac{h_l}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}. \end{aligned} \tag{5.69}$$

Thus we see that through a given left state u_l there are two curves on which the Rankine–Hugoniot relation holds, namely,

$$H_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \,\middle|\, h > 0 \right\} \tag{5.70}$$



and

$$H_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}} (h - h_l) \sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \,\middle|\, h > 0 \right\}. \tag{5.71}$$

We call the corresponding shocks slow shocks (or 1-shocks) and fast shocks (or 2-shocks), respectively. The Hugoniot locus now reads

$$H(u_l) = \left\{ u \mid \exists s \in \mathbb{R} \text{ such that } s [\![u]\!] = [\![f(u)]\!] \right\} = H_1(u_l) \cup H_2(u_l). \quad \diamond$$

We will soon see that the basic features of the Hugoniot locus of the shallow-water equations carry over to the general case of strictly hyperbolic systems at least for small shocks where u is near u_l. The problem to be considered is to solve implicitly the system of n equations

$$\mathcal{H}(s, u; u_l) := s(u - u_l) - (f(u) - f(u_l)) = 0 \tag{5.72}$$

for the $n + 1$ unknowns $u_1, \ldots, u_n$ and s for u close to the given u_l. The major problem comes from the fact that we have one equation fewer than the number of unknowns, and that $\mathcal{H}(s, u_l; u_l) = 0$ for all values of s. Hence the implicit function theorem cannot be used without first removing the singularity at $u = u_l$.

Let us first state the relevant version of the implicit function theorem that we will use.

Theorem 5.10 (Implicit function theorem) *Let the function*

$$\Phi = (\Phi_1, \ldots, \Phi_p) : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p \tag{5.73}$$

be C^1 in a neighborhood of a point (x_0, y_0), $x_0 \in \mathbb{R}^q$, $y_0 \in \mathbb{R}^p$ with $\Phi(x_0, y_0) = 0$. Assume that the $p \times p$ matrix

$$\frac{\partial \Phi}{\partial y} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial y_1} & \cdots & \frac{\partial \Phi_1}{\partial y_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_p}{\partial y_1} & \cdots & \frac{\partial \Phi_p}{\partial y_p} \end{pmatrix} \tag{5.74}$$

is nonsingular at the point (x_0, y_0).

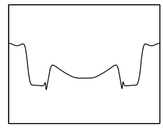
Then there exist a neighborhood N of x_0 and a unique differentiable function $\phi: N \to \mathbb{R}^p$ such that

$$\Phi(x, \phi(x)) = 0, \quad \phi(x_0) = y_0. \tag{5.75}$$

We will rewrite equation (5.72) into an eigenvalue problem that we can study locally around each eigenvalue $\lambda_j(u_l)$. This removes the singularity, and hence we can apply the implicit function theorem.

Theorem 5.11 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Let $u_l \in D$.*

Then there exist n smooth curves $H_1(u_l), \ldots, H_n(u_l)$ locally through u_l on which the Rankine–Hugoniot relation is satisfied.



Proof Writing

$$
\begin{aligned}
f(u) - f(u_l) &= \int_0^1 \frac{\partial}{\partial \alpha} f((1-\alpha)u_l + \alpha u)\, d\alpha \\
&= \int_0^1 df((1-\alpha)u_l + \alpha u)(u - u_l)\, d\alpha \\
&= M(u, u_l)(u - u_l),
\end{aligned}
\tag{5.76}
$$

where $M(u, u_l)$ is the averaged Jacobian

$$
M(u, u_l) = \int_0^1 df((1-\alpha)u_l + \alpha u)\, d\alpha,
$$

we see that (5.72) takes the form

$$
\mathcal{H}(s, u, u_l) = (s - M(u, u_l))(u - u_l) = 0. \tag{5.77}
$$

Here $u - u_l$ is an eigenvector of the matrix M with eigenvalue s. The matrix $M(u_l, u_l) = df(u_l)$ has n distinct eigenvalues $\lambda_1(u_l), \dots, \lambda_n(u_l)$, and hence we know that there exists an open set N such that the matrix $M(u, u_l)$ has twice-differentiable eigenvectors and distinct eigenvalues, namely,

$$
\left(\mu_j(u, u_l) - M(u, u_l)\right) v_j(u, u_l) = 0, \tag{5.78}
$$

for all $u, u_l \in N$.[6] Let $w_j(u, u_l)$ denote the corresponding left eigenvectors normalized so that

$$
w_k(u, u_l) \cdot v_j(u, u_l) = \delta_{jk}. \tag{5.79}
$$

In this terminology u and u_l satisfy the Rankine–Hugoniot relation with speed s if and only if there exists a j such that

$$
w_k(u, u_l) \cdot (u - u_l) = 0 \text{ for all } k \neq j, \quad s = \mu_j(u, u_l), \tag{5.80}
$$

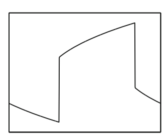
and $w_j(u, u_l) \cdot (u - u_l)$ is nonzero. We define functions $F_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$
F_j(u, \epsilon) = \left(w_1(u, u_l) \cdot (u - u_l) - \epsilon \delta_{1j}, \dots, w_n(u, u_l) \cdot (u - u_l) - \epsilon \delta_{nj}\right). \tag{5.81}
$$

The Rankine–Hugoniot relation is satisfied if and only if $F_j(u, \epsilon) = 0$ for some ϵ and j. Furthermore, $F_j(u_l, 0) = 0$. A straightforward computation shows that

$$
\frac{\partial F_j}{\partial u}(u_l, 0) = \begin{pmatrix} l_1(u_l) \\ \vdots \\ l_n(u_l) \end{pmatrix},
$$

[6] The properties of the eigenvalues follow from the implicit function theorem used on the determinant of $\mu I - M(u, u_l)$, and for the eigenvectors by considering the one-dimensional eigenprojections $\int (M(u, u_l) - \mu)^{-1}\, d\mu$ integrated around a small curve enclosing each eigenvalue $\lambda_j(u_l)$.



which is nonsingular. Hence the implicit function theorem implies the existence of a unique solution $u_j(\epsilon)$ of

$$F_j(u_j(\epsilon), \epsilon) = 0 \tag{5.82}$$

for ϵ small. □

Occasionally, in particular in Chapt. 7, we will use the notation

$$H_j(\epsilon)u_l = u_j(\epsilon).$$

We will have the opportunity later to study in detail properties of the parameterization of the Hugoniot locus. Let it suffice here to observe that by differentiating each component of $F_j(u_j(\epsilon), \epsilon) = 0$ at $\epsilon = 0$, we find that

$$l_k(u_l) \cdot u_j'(0) = \delta_{jk} \tag{5.83}$$

for *all* $k = 1, \dots, n$, showing that indeed

$$u_j'(0) = r_j(u_l). \tag{5.84}$$

From the definition of M we see that $M(u, u_l) = M(u_l, u)$, and this symmetry implies that

$$\begin{aligned} \mu_j(u, u_l) &= \mu_j(u_l, u), \quad & \mu_j(u_l, u_l) &= \lambda_j(u_l), \\ v_j(u, u_l) &= v_j(u_l, u), \quad & v_j(u_l, u_l) &= r_j(u_l), \\ w_j(u, u_l) &= w_j(u_l, u), \quad & w_j(u_l, u_l) &= l_j(u_l). \end{aligned} \tag{5.85}$$

Let $\nabla_k h(u_1, u_2)$ denote the gradient of a function $h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with respect to the kth variable $u_k \in \mathbb{R}^n$, $k = 1, 2$. Then the symmetries (5.85) imply that

$$\nabla_1 \mu_j(u, u_l) = \nabla_2 \mu_j(u, u_l). \tag{5.86}$$

Hence

$$\nabla \lambda_j(u_l) = \nabla_1 \mu_j(u_l, u_l) + \nabla_2 \mu_j(u_l, u_l) = 2\nabla_1 \mu_j(u_l, u_l). \tag{5.87}$$

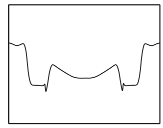
For a vector-valued function $\phi(u) = (\phi_1(u), \dots, \phi_n(u))$ we let $\nabla \phi(u)$ denote the Jacobian matrix,

$$\nabla \phi(u) = \begin{pmatrix} \nabla \phi_1 \\ \vdots \\ \nabla \phi_n \end{pmatrix}. \tag{5.88}$$

Now the symmetries (5.85) imply that

$$\nabla l_k(u_l) = 2\nabla_1 w_k(u_l, u_l) \tag{5.89}$$

in obvious notation.



5.4 The Entropy Condition

... and now remains
That we find out the cause of this effect,
Or rather say, the cause of this defect ...
— W. Shakespeare, Hamlet (1603)

Having derived the Hugoniot loci for a general class of conservation laws in the previous section, we will have to select the parts of these curves that give admissible shocks, i.e., satisfy an entropy condition. This will be considerably more complicated in the case of systems than in the scalar case. To guide our intuition we will return to the example of shallow-water waves.

◇ Example 5.12 (Shallow water (cont'd.))
Let us first study the points on $H_1(u_l)$; a similar analysis will apply to $H_2(u_l)$. We will work with the variables h, v rather than h, q. Consider the Riemann problem where we have a high-water bank at rest to the left of the origin and a lower-water bank to the right of the origin, with a positive velocity; or in other words, the fluid from the lower-water bank moves away from the high-water bank. More precisely, for $h_l > h_r$ we let

$$\begin{pmatrix} h \\ v \end{pmatrix}(x,0) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq 0, \end{cases}$$

where we have chosen initial data so that the right state is on $H_1(u_l)$, i.e., the Rankine–Hugoniot is already satisfied for a certain speed s. This implies that

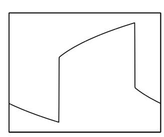
$$\begin{pmatrix} h \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < st, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq st, \end{cases}$$

for $h_l > h_r$, where the negative shock speed s given by

$$s = -\frac{h_r \sqrt{h_r^{-1} + h_l^{-1}}}{\sqrt{2}}$$

is a perfectly legitimate weak solution of the initial value problem. However, we see that this is not at all a reasonable solution, since the solution predicts a high-water bank being pushed by a lower one! See Fig. 5.4.

If we change the initial conditions so that the right state is on the other branch of $H_1(u_l)$, i.e., we consider a high-water bank moving into a lower-water bank at rest,



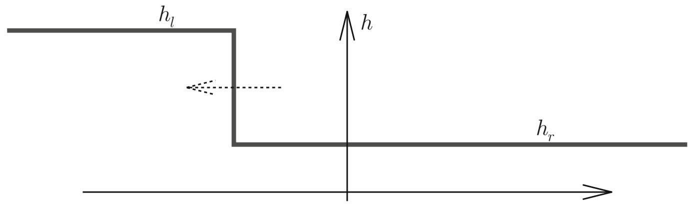


Fig. 5.4 Unphysical solution

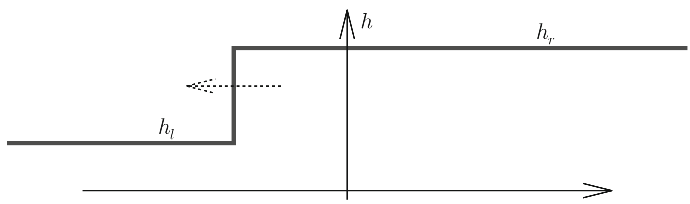


Fig. 5.5 Reasonable solution

or

$$\begin{pmatrix} h \\ v \end{pmatrix}(x,0) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq 0, \end{cases}$$

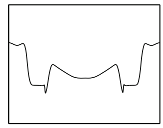
for $h_l < h_r$, we see that the weak solution

$$\begin{pmatrix} h \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < st, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq st, \end{cases}$$

for $h_l < h_r$ and with speed $s = -h_r \sqrt{h_r^{-1} + h_l^{-1}}/\sqrt{2}$ is reasonable physically, since the high-water bank now is pushing the lower one. See Fig. 5.5

If you are worried about the fact that the shock is preserved, i.e., that there is no deformation of the shock profile, this is due to the fact that the right state is carefully selected. In general we will have both a shock wave and a rarefaction wave in the solution. This will be clear when we solve the full Riemann problem.

Let us also consider the above examples with energy conservation in mind. In our derivation of the shallow-water equations we used conservation of mass and momentum only. For smooth solutions of these equations, conservation of mechanical energy will follow. Indeed, the kinetic energy of a vertical section of the shallow-water system at a point x is given by $h(x,t)v(x,t)^2/2$ in dimensionless variables, and the potential energy of the same section is given by $h(x,t)^2/2$, and hence the total mechanical energy reads $(h(x,t)v(x,t)^2 + h(x,t)^2)/2$. Consider now a section



of the channel between points $x_1 < x_2$ and assume that we have a smooth (classical) solution of the shallow-water equations. The rate of change of mechanical energy is given by the net energy flow across x_1 and x_2, i.e., $\frac{1}{2}(hv^2 + h^2)v = \frac{1}{2}(hv^3 + h^2 v)$, plus the work done by the pressure. Energy conservation yields

$$
\begin{aligned}
0 &= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + \frac{1}{2}h^2 v\right)dx \\
&\quad + \int_0^{h(x_2,t)} P(x_2,y,t)v(x_2,t)\,dy - \int_0^{h(x_1,t)} P(x_1,y,t)v(x_1,t)\,dy \\
&= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + \frac{1}{2}h^2 v\right)dx \\
&\quad + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}h^2 v\right)dx \\
&= \int_{x_1}^{x_2}\frac{\partial}{\partial t}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + h^2 v\right)dx,
\end{aligned}
$$

where we have used that $P(x,y,t) = h(x,t) - y$ in dimensionless variables. Hence we conclude that

$$
\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)_t + \left(\frac{1}{2}hv^3 + h^2 v\right)_x = 0.
$$

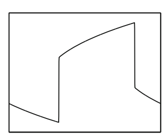
This equation follows easily directly from (5.5) for smooth solutions.

However, for weak solutions, mechanical energy will in general not be conserved. Due to dissipation we expect an energy loss across a bore. Let us compute this change in energy ΔE across the bore in the two examples above, for a time t such that $x_1 < st < x_2$. We obtain

$$
\begin{aligned}
\Delta E &= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \left(\frac{1}{2}hv^3 + h^2 v\right)\Big|_{x_1}^{x_2} \\
&= -s\left[\!\left[\frac{1}{2}hv^2 + \frac{1}{2}h^2\right]\!\right] + \left[\!\left[\frac{1}{2}hv^3 + h^2 v\right]\!\right] \\
&= \frac{1}{2}h_r\delta(\left[\!\left[h\right]\!\right]^2\delta^2 h_r + h_r^2 - h_l^2) + (-\left[\!\left[h\right]\!\right]^3\delta^3 h_r - 2\left[\!\left[h\right]\!\right]\delta h_r^2) \\
&= -\frac{1}{4}\left[\!\left[h\right]\!\right]^3\delta,
\end{aligned} \tag{5.90}
$$

where we have introduced

$$
\delta := \frac{\sqrt{h_r^{-1} + h_l^{-1}}}{\sqrt{2}} = \left(\frac{h_r + h_l}{2h_r h_l}\right)^{1/2}. \tag{5.91}
$$



(Recall that $v_l = 0$ and $v_r = [\![v]\!] = -[\![h]\!]\,\delta$ from the Rankine–Hugoniot condition.) Here we have used that we have a smooth solution with energy conservation on each interval $[x_1, st]$ and $[st, x_2]$. In the first case, where we had a low-water bank pushing a high-water bank with $h_r < h_l$, we find indeed that $\Delta E > 0$, while in the other case we obtain the more reasonable $\Delta E < 0$.

From these two simple examples we get a hint that only one branch of $H_1(u_l)$ is physically acceptable. We will now translate this into conditions on existence of viscous profiles and conditions on the eigenvalues of $df(u)$ at $u = u_l$ and $u = u_r$, conditions we will use in cases where our intuition will be more blurred.

In Chapt. 2 we discussed the notion of traveling waves. Recall from (2.7) that a shock between two fixed states u_l and u_r with speed s,

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{5.92}$$

admits a viscous profile if $u(x,t)$ is the limit as $\epsilon \to 0$ of $u^\epsilon(x,t) = U((x - st)/\epsilon) = U(\xi)$ with $\xi = (x - st)/\epsilon$, which satisfies

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon.$$

Integrating this equation, using $\lim_{\epsilon \to 0} U(\xi) = u_l$ if $\xi < 0$, we obtain

$$\dot{U} = A(h,q) := f(U) - f(u_l) - s(U - u_l), \tag{5.93}$$

where the differentiation is with respect to ξ. We will see that it is possible to connect the left state with a viscous profile to a right state only for the branch with $h_r > h_l$ of $H_1(u_l)$, i.e., the physically correct solution.

Computationally it will be simpler to work with viscous profiles in the (h, v) variables rather than (h, q). Using $\dot{q} = \dot{v}h + v\dot{h}$ and (5.93), we find that there is a viscous profile in (h, q) if and only if (h, v) satisfies

$$\begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix} = B(h,v) := \begin{pmatrix} vh - v_l h_l - s(h - h_l) \\ (v - v_l)(v_l - s)\frac{h_l}{h} + \frac{h^2 - h_l^2}{2h} \end{pmatrix}. \tag{5.94}$$

Consider now a slow shock with $s = v_l - h_r\delta$, cf. (5.69). We can write

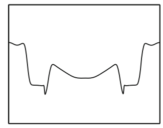
$$B(h,v) = \begin{pmatrix} vh - v_l h_l - s(h - h_l) \\ (v - v_l)\frac{h_l h_r}{h}\delta + \frac{h^2 - h_l^2}{2h} \end{pmatrix}. \tag{5.95}$$

We will analyze the vector field $B(h, v)$ carefully. The Jacobian of B reads

$$dB(h,v) = \begin{pmatrix} v - s & h \\ \frac{h^2 + h_l^2}{2h^2} - (v - v_l)\frac{h_l h_r}{h^2}\delta & \frac{h_l h_r}{h}\delta \end{pmatrix}. \tag{5.96}$$

At the left state u_l we obtain

$$dB(h_l, v_l) = \begin{pmatrix} v_l - s & h_l \\ 1 & h_r\delta \end{pmatrix} = \begin{pmatrix} h_r\delta & h_l \\ 1 & h_r\delta \end{pmatrix}, \tag{5.97}$$



using the value of the shock speed s, equation (5.68). The eigenvalues of $dB(h_l, v_l)$ are $h_r\delta \pm \sqrt{h_l}$, both of which are easily seen to be positive when $h_r > h_l$; thus (h_l, v_l) is a source. Similarly, we obtain

$$dB(h_r, v_r) = \begin{pmatrix} h_l\delta & h_r \\ 1 & h_l\delta \end{pmatrix}, \tag{5.98}$$

with eigenvalues $h_l\delta \pm \sqrt{h_r}$. In this case, one eigenvalue is positive and one negative, and thus (h_r, v_r) is a saddle point. However, we still have to establish an orbit connecting the two states. To this end we construct a region K with (h_l, v_l) and (h_r, v_r) at the boundary of K such that a connecting orbit has to connect the two points within K. The region K will have two curves as boundaries where the first and second components of B vanish, respectively. The first curve, denoted by C_h, is defined by the first component being zero,

$$vh - v_l h_l - s(h - h_l) = 0, \quad h \in [h_l, h_r],$$

which can be simplified to yield

$$v = v_l - (h - h_l)\frac{h_r}{h}\delta, \quad h \in [h_l, h_r]. \tag{5.99}$$

For the second curve, C_v, we have

$$(v - v_l)(v_l - s)\frac{h_l}{h} + \frac{h^2 - h_l^2}{2h} = 0, \quad h \in [h_l, h_r],$$

which can be rewritten as

$$v = v_l - \frac{h^2 - h_l^2}{2h_l h_r\delta}, \quad h \in [h_l, h_r]. \tag{5.100}$$

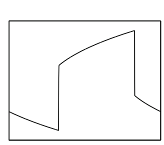
Let us now study the behavior of the second component of B along the curve C_h where the first component vanishes, i.e.,

$$\begin{aligned} &\Big[(v - v_l)\frac{h_l h_r}{h}\delta + \frac{h^2 - h_l^2}{2h}\Big]\Big|_{C_h} \\ &\qquad = -\frac{h_l}{2h^2}(h_r - h)(h - h_l)(1 + \frac{h + h_r}{h_l}) < 0. \end{aligned} \tag{5.101}$$

Similarly, for the first component of B along C_v, we obtain

$$\begin{aligned} &\big[vh - v_l h_l - s(h - h_l)\big]\big|_{C_v} \\ &\qquad = \frac{h - h_l}{2h_r h_l\delta}\left(h_r(h_l + h_r) - h(h + h_l)\right) > 0, \end{aligned} \tag{5.102}$$

which is illustrated in Fig. 5.6. The flow of the vector field is leaving the region K along the curves C_h and C_v. Locally, around (h_r, v_r) there has to be an orbit entering



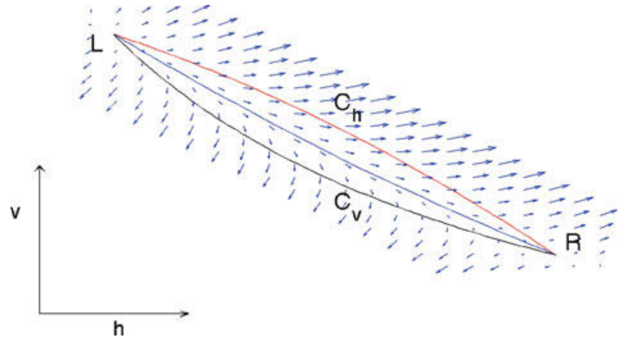


Fig. 5.6 The vector field B, the curves C_v and C_h, as well as the orbit connecting the left and the right states

K as ξ decreases from ∞. This curve cannot escape K and has to connect to a curve coming from (h_l, v_l). Consequently, we have proved existence of a viscous profile.

We saw that the relative values of the shock speed and the eigenvalues of the Jacobian of B, and hence of A, at the left and right states were crucial for this analysis to hold. Let us now translate these assumptions into assumptions on the eigenvalues of dA. The Jacobian of A reads

$$dA(h,q) = \begin{pmatrix} -s & 1 \\ h - \frac{q^2}{h^2} & \frac{2q}{h} - s \end{pmatrix}.$$

Hence the eigenvalues are $-s + \frac{q}{h} \pm \sqrt{h} = -s + \lambda(u)$. At the left state both eigenvalues are positive, and thus u_l is a source, while at u_r one is positive and one negative, and hence u_r is a saddle. We may write this as

$$\lambda_1(u_r) < s < \lambda_1(u_l), \quad s < \lambda_2(u_r). \tag{5.103}$$

We call these the *Lax inequalities*, and say that a shock satisfying these inequalities is a *Lax 1-shock* or a *slow Lax shock*. We have proved that for the shallow-water equations with $h_r > h_l$ there exists a viscous profile, and that the Lax shock conditions are satisfied.

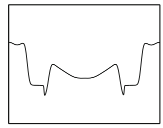
Let us now return to the unphysical shock "solution." In this case we had $u_r \in H_1(u_l)$ with $h_r < h_l$ with the eigenvalues at the left state (h_l, v_l) of different signs. Thus (h_l, v_l) is a saddle. However, for the right state (h_r, v_r) both eigenvalues are positive, and hence that point is a source. Accordingly, there cannot be any orbit connecting the left state with the right state.

A similar analysis can be performed for $H_2(u_l)$, giving that there exists a viscous profile for a shock satisfying the Rankine–Hugoniot relation if and only if the following *Lax entropy conditions* are satisfied:

$$\lambda_2(u_r) < s < \lambda_2(u_l), \quad s > \lambda_1(u_l). \tag{5.104}$$

In that case we have a *fast Lax shock*, or *Lax 2-shock*.

We may sum up the above argument as follows. A shock has a viscous profile if and only if the Lax shock conditions are satisfied. We call such shocks *admissible* and denote the part of the Hugoniot locus where the Lax j conditions are satisfied



by S_j. In the case of shallow-water equations we obtain

$$S_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \middle| \, h \geq h_l \right\}, \tag{5.105}$$

$$S_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \middle| \, 0 < h \leq h_l \right\}. \tag{5.106}$$

(These curves are depicted in Sect. 5.3.) We may also want to parameterize the admissible shocks differently. For the slow Lax shocks let

$$h_{1,\epsilon} := h_l - \frac{2}{3}\sqrt{h_l}\,\epsilon, \quad \epsilon < 0. \tag{5.107}$$

This gives

$$q_{1,\epsilon} := q_l \left(1 - \frac{2\epsilon}{3\sqrt{h_l}}\right) + \frac{\epsilon}{9}\sqrt{2h_l \left(6\sqrt{h_l} - 2\epsilon\right)\left(3\sqrt{h_l} - 2\epsilon\right)} \tag{5.108}$$

such that

$$\frac{d}{d\epsilon}\begin{pmatrix} h_{1,\epsilon} \\ q_{1,\epsilon} \end{pmatrix}\bigg|_{\epsilon=0} = r_1(u_l), \tag{5.109}$$

where $r_1(u_l)$ is given by (5.44). Similarly, for the fast Lax shocks let

$$h_{2,\epsilon} := h_l + \frac{2}{3}\sqrt{h_l}\,\epsilon, \quad \epsilon < 0. \tag{5.110}$$

Then

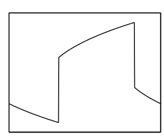
$$q_{2,\epsilon} := q_l \left(1 + \frac{2\epsilon}{3\sqrt{h_l}}\right) + \frac{\epsilon}{9}\sqrt{2h_l \left(6\sqrt{h_l} + 2\epsilon\right)\left(3\sqrt{h_l} + 2\epsilon\right)}, \tag{5.111}$$

such that

$$\frac{d}{d\epsilon}\begin{pmatrix} h_{2,\epsilon} \\ q_{2,\epsilon} \end{pmatrix}\bigg|_{\epsilon=0} = r_2(u_l), \tag{5.112}$$

where $r_2(u_l)$ is given by (5.44). ◇

In the above example we have seen the equivalence between the existence of a viscous profile and the Lax entropy conditions for the shallow-water equations. This analysis has yet to be carried out for general systems. We will use the above example as a motivation for the following definition, stated for general systems.



Definition 5.13 We say that a shock

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{5.113}$$

is a Lax j-shock if the shock speed s satisfies the Rankine–Hugoniot condition $s\,[\![u]\!] = [\![f]\!]$ and

$$\lambda_{j-1}(u_l) < s < \lambda_j(u_l), \quad \lambda_j(u_r) < s < \lambda_{j+1}(u_r). \tag{5.114}$$

(Here $\lambda_0 = -\infty$ and $\lambda_{n+1} = \infty$.)

Observe that for strictly hyperbolic systems, for which the eigenvalues are distinct, it suffices to check the inequalities $\lambda_j(u_r) < s < \lambda_j(u_l)$ for small Lax j-shocks if the eigenvalues are continuous in u.

The following result follows from Theorem 5.11.

Theorem 5.14 *Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ in a domain D in $\mathbb{R}^n$. Assume that $\nabla\lambda_j \cdot r_j = 1$. Let $u_l \in D$. A state $u_{j,\epsilon} \in H_j(u_l)$ is a Lax j-shock near u_l if $|\epsilon|$ is sufficiently small and ϵ negative. If ϵ is positive, the shock is not a Lax j-shock.*

Proof Using the ϵ parameterization of the Hugoniot locus, we see that the shock is a Lax j-shock if and only if

$$\lambda_{j-1}(0) < s(\epsilon) < \lambda_j(0), \quad \lambda_j(\epsilon) < s(\epsilon) < \lambda_{j+1}(\epsilon), \tag{5.115}$$

where for simplicity we write $u(\epsilon) = u_{j,\epsilon}$, $s(\epsilon) = s_{j,\epsilon}$, and $\lambda_k(\epsilon) = \lambda_k(u_{j,\epsilon})$. The observation following the definition of Lax shocks shows that it suffices to check the inequalities

$$\lambda_j(\epsilon) < s(\epsilon) < \lambda_j(0). \tag{5.116}$$

Assume first that $u(\epsilon) \in H_j(u_l)$ and that ϵ is negative. We know from the implicit function theorem that $s(\epsilon)$ tends to $\lambda_j(0)$ as ϵ tends to zero. From the fact that also $\lambda_j(\epsilon) \to \lambda_j(0)$ as $\epsilon \to 0$, and

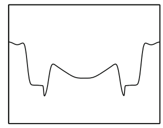
$$\left.\frac{d\lambda_j(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \nabla\lambda_j(0) \cdot r_j(u_l) = 1,$$

it suffices to prove that $0 < s'(0) < 1$. We will in fact prove that $s'(0) = \frac{1}{2}$. Recall from (5.80) that s is an eigenvalue of the matrix $M(u, u_l)$, i.e., $s(\epsilon) = \mu_j(u(\epsilon), u_l)$. Then

$$s'(0) = \nabla_1\mu_j(u_l, u_l) \cdot u'(0) = \frac{1}{2}\nabla\lambda_j(u_l) \cdot r_j(u_l) = \frac{1}{2}, \tag{5.117}$$

using the symmetry (5.87) and the normalization of the right eigenvalue.

If $\epsilon > 0$, we immediately see that $s(\epsilon) > s(0) = \lambda_j(0)$, and in this case we cannot have a Lax j-shock. □



5.5 The Solution of the Riemann Problem

Wie für die Integration der linearen partiellen Differentialgleichungen die fruchtbarsten Methoden nicht durch Entwicklung des allgemeinen Begriffs dieser Aufgabe gefunden worden, sondern vielmehr aus der Behandlung specieller physikalischer Probleme hervorgegangen sind, so scheint auch die Theorie der nichtlinearen partiellen Differentialgleichungen durch eine eingehende, alle Nebenbedingungen berücksichtigende, Behandlung specieller physikalischer Probleme am meisten gefördert zu werden, und in der That hat die Lösung der ganz speciellen Aufgabe, welche den Gegenstand dieser Abhandlung bildet, neue Methoden und Auffassungen erfordert, und zu Ergebnissen geführt, welche wahrscheinlich auch bei allgemeineren Aufgaben eine Rolle spielen werden.[7]
— G. F. B. Riemann [156]

In this section we will combine the properties of the rarefaction waves and shock waves from the previous sections to derive the unique solution of the Riemann problem for small initial data. Our approach will be the following. Assume that the left state u_l is given, and consider the space of all right states u_r. For each right state we want to describe the solution of the corresponding Riemann problem. (We could, of course, reverse the picture and consider the right state as fixed and construct the solution for all possible left states.)

To this end we start by defining *wave curves*. If the jth wave family is genuinely nonlinear, we define

$$W_j(u_l) := R_j(u_l) \cup S_j(u_l), \tag{5.118}$$

and if the jth family is linearly degenerate, we let

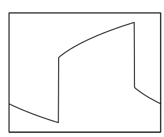
$$W_j(u_l) := C_j(u_l). \tag{5.119}$$

Recall that we have parameterized the shock and rarefaction curves separately with a parameter ϵ such that ϵ positive (negative) corresponds to a rarefaction (shock) wave solution in the case of a genuinely nonlinear wave family. The important fact about the wave curves is that they almost form a local coordinate system around u_l, and this will make it possible to prove existence of solutions of the Riemann problem for u_r close to u_l.

We will commence from the left state u_l and connect it to a nearby intermediate state $u_{m_1} = u_{1,\epsilon_1} \in W_1(u_l)$ using either a rarefaction wave solution ($\epsilon_1 > 0$) or a shock wave solution ($\epsilon_1 < 0$) if the first family is genuinely nonlinear. If the first family is linearly degenerate, we use a contact discontinuity for all ϵ_1. From this state we find another intermediate state $u_{m_2} = u_{2,\epsilon_2} \in W_2(u_{m_1})$. We continue in this way until we have reached an intermediate state $u_{m_{n-1}}$ such that $u_r = u_{n,\epsilon_n} \in W_n(u_{m_{n-1}})$. The problem is to show existence of a unique n-tuple of $(\epsilon_1, \dots, \epsilon_n)$ such that we "hit" u_r starting from u_l using this construction.

As usual, we will start by illustrating the above discussion for the shallow-water equations. This example will contain the fundamental description of the solution, which in principle will carry over to the general case.

[7] The theory of nonlinear equations can, it seems, achieve the most success if our attention is directed to special problems of physical content with thoroughness and with a consideration of all auxiliary conditions. In fact, the solution of the very special problem that is the topic of the current paper requires new methods and concepts and leads to results which probably will also play a role in more general problems.



◇ Example 5.15 (Shallow water (cont'd.))
Fix u_l. For each right state u_r we have to determine one middle state u_m on the first-wave curve through u_l such that u_r is on the second-wave curve with left state u_m, i.e., $u_m \in W_1(u_l)$ and $u_r \in W_2(u_m)$. (In the special case that $u_r \in W_1(u_l) \cup W_2(u_l)$ no middle state u_m is required.) For 2×2 systems of conservation laws it is easier to consider the "backward" second-wave curve $W_2^-(u_r)$ consisting of states u_m that can be connected to u_r on the right with a *fast* wave. The Riemann problem with left state u_l and right state u_r has a unique solution if and only if $W_1(u_l)$ and $W_2^-(u_r)$ have a unique intersection. In that case, clearly the intersection will be the middle state u_m. The curve $W_1(u_l)$ is given by

$$v = v(h) = \begin{cases} v_l - 2\left(\sqrt{h} - \sqrt{h_l}\right) & \text{for } h \in [0, h_l], \\ v_l - \frac{h-h_l}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}} & \text{for } h \geq h_l, \end{cases} \tag{5.120}$$

and we easily see that $W_1(u_l)$ is strictly decreasing, unbounded, and starting at $v_l + 2\sqrt{h_l}$. Using (5.49) and (5.106), we find that $W_2^-(u_r)$ reads

$$v = v(h) = \begin{cases} v_r + 2(\sqrt{h} - \sqrt{h_r}) & \text{for } h \in [0, h_r], \\ v_r + \frac{h-h_r}{\sqrt{2}}\sqrt{h^{-1} + h_r^{-1}} & \text{for } h \geq h_r, \end{cases} \tag{5.121}$$

which is strictly increasing, unbounded, with minimum $v_r - 2\sqrt{h_r}$. Thus we conclude that the Riemann problem for shallow water has a unique solution in the region where

$$v_l + 2\sqrt{h_l} \geq v_r - 2\sqrt{h_r}. \tag{5.122}$$

To obtain explicit equations for the middle state u_m we have to make case distinctions, depending on the type of wave curves that intersect, i.e., rarefaction waves or shock curves. This gives rise to four regions, denoted by I, . . . , IV. See Fig. 5.7. For completeness we give the equations for the middle state u_m in all cases.

Assume first that $u_r \in$ I. We will determine a unique intermediate state $u_m \in S_1(u_l)$ such that $u_r \in R_2(u_m)$. These requirements give the following equations to be solved for h_m, v_m such that $u_m = (h_m, q_m) = (h_m, h_m v_m)$:

$$v_m = v_l - \frac{1}{\sqrt{2}}(h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}}, \quad v_r = v_m + 2\left(\sqrt{h_r} - \sqrt{h_m}\right).$$

Summing these equations, we obtain the equation

$$\sqrt{2}\,[\![v]\!] = 2\sqrt{2}\left(\sqrt{h_r} - \sqrt{h_m}\right) - (h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}} \quad \text{(I)} \tag{5.123}$$

to determine h_m. Consider next the case with $u_r \in$ III. Here $u_m \in R_1(u_l)$ and $u_r \in S_2(u_m)$, and in this case we obtain

$$\sqrt{2}\,[\![v]\!] = (h_r - h_m)\sqrt{\frac{1}{h_r} + \frac{1}{h_m}} - 2\sqrt{2}\left(\sqrt{h_m} - \sqrt{h_l}\right), \quad \text{(III)} \tag{5.124}$$

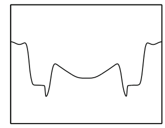
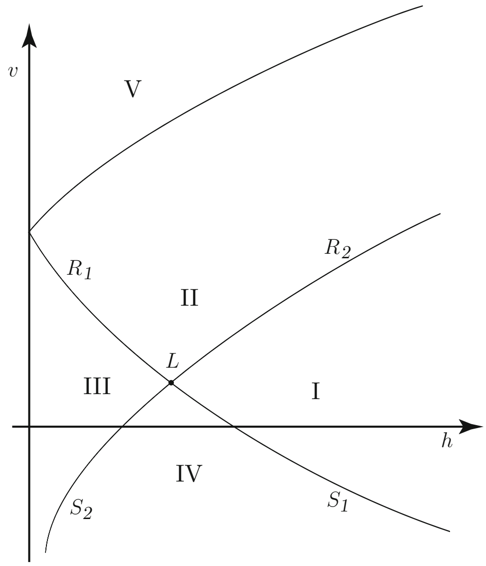


Fig. 5.7 The partition of the (h, v)-plane; see (5.127) and (5.146)



while in the case $u_r \in \mathrm{IV}$, we obtain (here $u_m \in S_1(u_l)$ and $u_r \in S_2(u_m)$)

$$\sqrt{2}\,[\![v]\!] = (h_r - h_m)\sqrt{\frac{1}{h_r} + \frac{1}{h_m}} - (h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}}. \quad \text{(IV)} \tag{5.125}$$

The case $u_r \in \mathrm{II}$ is special. Here $u_m \in R_1(u_l)$ and $u_r \in R_2(u_m)$. The intermediate state u_m is given by

$$v_m = v_l - 2\left(\sqrt{h_m} - \sqrt{h_l}\right), \quad v_r = v_m + 2\left(\sqrt{h_r} - \sqrt{h_m}\right),$$

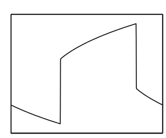
which can easily be solved for h_m to yield

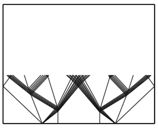
$$\sqrt{h_m} = \frac{2\left(\sqrt{h_r} + \sqrt{h_l}\right) - [\![v]\!]}{4}. \quad \text{(II)} \tag{5.126}$$

This equation is solvable only for right states such that the right-hand side of (5.126) is nonnegative. Observe that this is consistent with what we found above in (5.122). Thus we find that for

$$u_r \in \left\{u \in (0, \infty) \times \mathbb{R} \mid 2(\sqrt{h} + \sqrt{h_l}) \geq [\![v]\!]\right\} \tag{5.127}$$

the Riemann problem has a unique solution consisting of a slow wave followed by a fast wave. Let us summarize the solution of the Riemann problem for the shallow-water equations. First of all, we were not able to solve the problem globally, but only locally around the left state. Secondly, the general solution consists of a composition of elementary waves. More precisely, let $u_r \in \left\{u \in (0, \infty) \times \mathbb{R} \mid 2\left(\sqrt{h} + \sqrt{h_l}\right) \geq [\![v]\!]\right\}$. Let $w_j(x/t; h_m, h_l)$ denote the





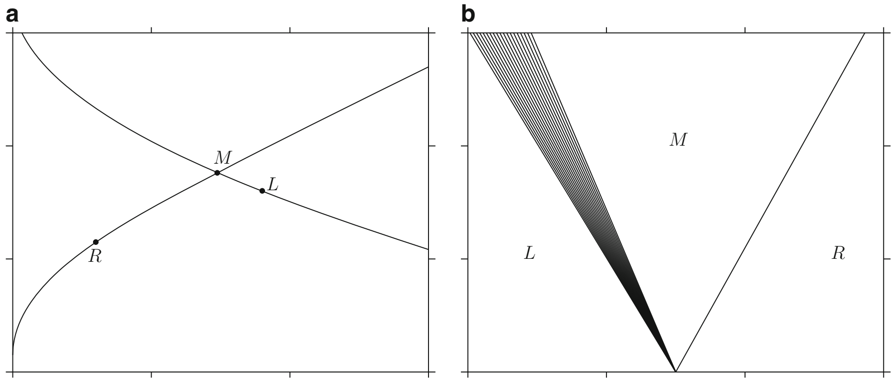


Fig. 5.8 The solution of the Riemann problem in phase space (**a**) and in (x,t)-space (**b**)

solution of the Riemann problem for $u_m \in W_j(u_l)$; here, as in most of our calculations on the shallow-water equations, we use h rather than ϵ as the parameter. We will introduce the notation $\sigma_j^{\pm}$ for the slowest and fastest wave speeds in each family to simplify the description of the full solution. Thus we have that for $j = 1$ $(j = 2)$ and $h_r < h_l$ $(h_r > h_l)$, w_j is a rarefaction-wave solution with slowest speed $\sigma_j^- = \lambda_j(u_l)$ and fastest speed $\sigma_j^+ = \lambda_j(u_r)$. If $j = 1$ $(j = 2)$ and $h_r > h_l$ $(h_r < h_l)$, then w_j is a shock-wave solution with speed $\sigma_j^- = \sigma_j^+ = s_j(h_r, h_l)$. The solution of the Riemann problem reads (see Fig. 5.8)

$$u(x,t) = \begin{cases} u_l & \text{for } x < \sigma_1^- t, \\ w_1(x/t; u_m, u_l) & \text{for } \sigma_1^- t \le x \le \sigma_1^+ t, \\ u_m & \text{for } \sigma_1^+ t < x \le \sigma_2^- t, \\ w_2(x/t; u_r, u_m) & \text{for } \sigma_2^- t \le x \le \sigma_2^+ t, \\ u_r & \text{for } x \ge \sigma_2^+ t. \end{cases} \tag{5.128}$$

We will show later in this chapter how to solve the Riemann problem globally for the shallow-water equations. ◇

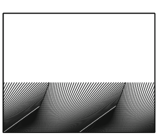
Before we turn to the existence and uniqueness theorem for solutions of the Riemann problem, we will need a certain property of the wave curves that we can explicitly verify for the shallow-water equations.

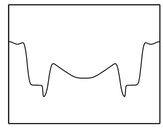
Recall from (5.84) and (5.40) that $\left.\frac{du_\epsilon}{d\epsilon}\right|_{\epsilon=0} = r_j(u_l)$; thus $W_j(u_l)$ is at least differentiable at u_l. In fact, one can prove that $W_j(u_l)$ has a continuous second derivative across u_l.

We introduce the following notation for the *directional derivative* of a quantity $h(u)$ in the direction r (not necessarily normalized) at the point u, which is defined as

$$D_r h(u) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(h(u + \epsilon r) - h(u) \right) = (\nabla h \cdot r)(u). \tag{5.129}$$

(When h is a vector, ∇h denotes the Jacobian.)





Theorem 5.16 *The wave curve $W_j(u_l)$ has a continuous second derivative across u_l. In particular,*

$$u_{j,\epsilon} = u_l + \epsilon r_j(u_l) + \frac{1}{2}\epsilon^2 D_{r_j} r_j(u_l) + \mathcal{O}\left(\epsilon^3\right).$$

Proof In our proof of the admissibility of parts of the Hugoniot loci, Theorem 5.14, we derived most of the ingredients required for the proof of this theorem. The rarefaction curve $R_j(u_l)$ is the integral curve of the right eigenvector $r_j(u)$ passing through u_l, and thus we have (when for simplicity we have suppressed the j-dependence in the notation for u, and write $u(\epsilon) = u_{j,\epsilon}$, etc.)

$$u(0+) = u_l, \quad u'(0+) = r_j(u_l), \quad u''(0+) = \nabla r_j(u_l) r_j(u_l). \tag{5.130}$$

(Here $\nabla r_j(u_l) r_j(u_l)$ denotes the product of the $n \times n$ matrix $\nabla r_j(u_l)$, cf. (5.88), and the (column) vector $r_j(u_l)$.) Recall that the Hugoniot locus is determined by the relation (5.82), i.e.,

$$w_k(u(\epsilon), u_l) \cdot (u(\epsilon) - u_l) = \epsilon \delta_{jk}, \quad k = 1, \dots, n. \tag{5.131}$$

We know already from (5.84) that $u'(0-) = r_j(u_l)$. To find the second derivative of $u(\epsilon)$ at $\epsilon = 0$, we have to compute the second derivative of (5.131). Here we find that[8]

$$2 r_j(u_l) \nabla_1 w_k(u_l, u_l) r_j(u_l) + w_k(u_l, u_l) \cdot u''(0-) = 0, \quad k = 1, \dots, n. \tag{5.132}$$

(A careful differentiation of each component may be helpful here; at least we thought so.) In the first term, the matrix $\nabla_1 w_k(u_l, u_l)$ is multiplied from the right by the (column) vector $r_j(u_l)$ and by the (row) vector $r_j(u_l)$ from the left. Using (5.89), i.e., $\nabla_1 w_k(u_l, u_l) = \frac{1}{2} \nabla l_k(u_l)$, we find that

$$r_j(u_l) \cdot \nabla l_k(u_l) r_j(u_l) + l_k(u_l) \cdot u''(0-) = 0. \tag{5.133}$$

The orthogonality of the left and the right eigenvectors, $l_k(u_l) \cdot r_j(u_l) = \delta_{jk}$, shows that

$$r_j(u_l) \nabla l_k(u_l) = -l_k(u_l) \nabla r_j(u_l). \tag{5.134}$$

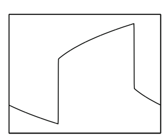
Inserting this into (5.133), we obtain

$$l_k(u_l) \cdot u''(0-) = l_k(u_l) \nabla r_j(u_l) r_j(u_l) \text{ for all } k = 1, \dots, n.$$

From this we conclude that also $u''(0-) = \nabla r_j(u_l) r_j(u_l)$, thereby proving the theorem. □

We will now turn to the proof of the classical Lax theorem about existence of a unique entropy solution of the Riemann problem for small initial data. The assumption of strict hyperbolicity of the system implies the existence of a full set of linearly independent eigenvectors. Furthermore, we have proved that the wave curves are C^2, and hence intersect transversally at the left state. This shows, in

[8] Lo and behold; the second derivative of $w_k(u(\epsilon), u_l)$ is immaterial, since it is multiplied by $u(\epsilon) - u_l$ at $\epsilon = 0$.



a heuristic way, that it is possible to solve the Riemann problem locally. Indeed, we saw that we could write the solution of the corresponding problem for the shallow-water equations as a composition of individual elementary waves that do not interact, in the sense that the fastest wave of one family is slower than the slowest wave of the next family. This will enable us to write the solution in the same form in the general case. In order to do this, we introduce some notation. Let $u_{j,\epsilon_j} = u_{j,\epsilon_j}(x/t; u_r, u_l)$ denote the unique solution of the Riemann problem with left state u_l and right state u_r that consists of a single elementary wave (i.e., shock wave, rarefaction wave, or contact discontinuity) of family j with strength ϵ_j. Furthermore, we need to define notation for speeds corresponding to the fastest and slowest waves of a fixed family. Let

$$\begin{aligned} &\sigma_j^+ = \sigma_j^- = s_{j,\epsilon_j} && \text{if } \epsilon_j < 0, \\ &\left.\begin{aligned} \sigma_j^- &= \lambda_j(u_{j-1,\epsilon_{j-1}}) = \lambda_j(u_{m_{j-1}}), \\ \sigma_j^+ &= \lambda_j(u_{j,\epsilon_j}) = \lambda_j(u_{m_j}) \end{aligned}\right\} && \text{if } \epsilon_j > 0, \end{aligned} \tag{5.135}$$

if the jth wave family is genuinely nonlinear, and

$$\sigma_j^+ = \sigma_j^- = \lambda_j(u_{j,\epsilon_j}) = \lambda_j(u_{m_j}) \tag{5.136}$$

if the jth wave family is linearly degenerate. With these definitions we are ready to write the solution of the Riemann problem as

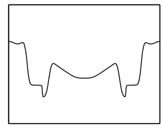
$$u(x,t) = \begin{cases} u_l & \text{for } x < \sigma_1^- t, \\ u_{1,\epsilon_1}(x/t; u_{m_1}, u_l) & \text{for } \sigma_1^- t \le x \le \sigma_1^+ t, \\ u_{m_1} & \text{for } \sigma_1^+ t \le x < \sigma_2^- t, \\ u_{2,\epsilon_2}(x/t; u_{m_2}, u_{m_1}) & \text{for } \sigma_2^- t \le x \le \sigma_2^+ t, \\ u_{m_2} & \text{for } \sigma_2^+ t \le x < \sigma_3^- t, \\ \vdots & \\ u_{n,\epsilon_n}(x/t; u_r, u_{m_{n-1}}) & \text{for } \sigma_n^- t \le x \le \sigma_n^+ t, \\ u_r & \text{for } x \ge \sigma_n^+ t. \end{cases} \tag{5.137}$$

Theorem 5.17 (Lax's theorem) *Assume that $f_j \in C^2(\mathbb{R}^n)$, $j = 1, \dots, n$. Let D be a domain in $\mathbb{R}^n$ and consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Assume that each wave family is either genuinely nonlinear or linearly degenerate.*

Then for $u_l \in D$ there exists a neighborhood $\tilde{D} \subset D$ of u_l such that for all $u_r \in \tilde{D}$ the Riemann problem

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \ge 0, \end{cases} \tag{5.138}$$

has a unique solution in $\tilde{D}$ consisting of up to n elementary waves, i.e., rarefaction waves, shock solutions satisfying the Lax entropy condition, or contact discontinuities. The solution is given by (5.137).



Proof Consider the map $W_{j,\epsilon}: u \mapsto u_{j,\epsilon} \in W_j(u)$. We may then write the solution of the Riemann problem using the composition

$$W_{(\epsilon_1,\dots,\epsilon_n)} = W_{n,\epsilon_n} \circ \cdots \circ W_{1,\epsilon_1} \tag{5.139}$$

as

$$W_{(\epsilon_1,\dots,\epsilon_n)} u_l = u_r, \tag{5.140}$$

and we want to prove the existence of a unique vector $(\epsilon_1, \dots, \epsilon_n)$ (near the origin) such that (5.140) is satisfied for $|u_l - u_r|$ small. In our proof we will need the two leading terms, i.e., up to the linear term, in the Taylor expansion for W. For later use we expand to the quadratic term in the next lemma.

Lemma 5.18 *We have*

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_n)}(u_l) &= u_l + \sum_{i=1}^{n} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{n} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right). \end{aligned} \tag{5.141}$$

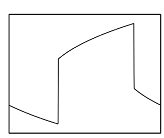
Proof (of Lemma 5.18) We shall show that for $k = 1, \dots, n$,

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l) &= u_l + \sum_{i=1}^{k} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{k} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right) \end{aligned} \tag{5.142}$$

by induction on k. It is clearly true for $k = 1$; cf. Theorem 5.16. Assume (5.142). Now,

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_{k+1},0,\dots,0)}\,(u_l) &= W_{k+1,\epsilon_{k+1}}\left(W_{(\epsilon_1,\dots,\epsilon_k)}(u_l)\right) \\ &= u_l + \sum_{i=1}^{k} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{k} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \epsilon_{k+1} r_{k+1}\left(W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l)\right) \\ &\quad + \frac{1}{2}\epsilon_{k+1}^2 D_{r_{k+1}} r_{k+1}\left(W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l)\right) + \mathcal{O}\left(|\epsilon|^3\right) \\ &= u_l + \sum_{i=1}^{k+1} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k+1} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{k+1} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right) \end{aligned}$$

by Theorem 5.16. □



Let $u_l \in D$ and define the map

$$\mathcal{L}(\epsilon_1, \dots, \epsilon_n, u) = W_{(\epsilon_1,\dots,\epsilon_n)} u_l - u. \tag{5.143}$$

This map $\mathcal{L}$ satisfies

$$\mathcal{L}(0, \dots, 0, u_l) = 0, \quad \nabla_\epsilon \mathcal{L}(0, \dots, 0, u_l) = (r_1(u_l), \dots, r_n(u_l)),$$

where the matrix $\nabla \mathcal{L}$ has the right eigenvectors r_j evaluated at u_l as columns. This matrix is nonsingular by the strict hyperbolicity assumption.

The implicit function theorem then implies the existence of a neighborhood N around u_l and a unique differentiable function $(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1(u), \dots, \epsilon_n(u))$ such that $\mathcal{L}(\epsilon_1, \dots, \epsilon_n, u) = 0$. If $u_r \in N$, then there exists unique $(\epsilon_1, \dots, \epsilon_n)$ with $W_{(\epsilon_1,\dots,\epsilon_n)} u_l = u_r$, which proves the theorem. □

Observe that we could rephrase the Lax theorem as saying that we may use $(\epsilon_1, \dots, \epsilon_n)$ to measure distances in phase space, and that we indeed have

$$A\,|u_r - u_l| \le \sum_{j=1}^{n} \left|\epsilon_j\right| \le B\,|u_r - u_l| \tag{5.144}$$

for constants A and B.

Let us now return to the shallow-water equations and prove the existence of a global solution of the Riemann problem.

◇ Example 5.19 (Shallow water (cont'd.))
We will construct a global solution of the Riemann problem for the shallow-water equations for all left and right states in $D = \left\{(h, v) \mid h \in [0, \infty), v \in \mathbb{R}\right\}$. Of course, we will maintain the same solution in the region where we already have constructed a solution, so it remains to construct a solution in the region

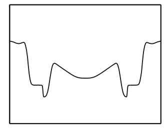
$$u_r \in V := \left\{u_r \in D \mid 2\left(\sqrt{h_r} + \sqrt{h_l}\right) < [\![v]\!]\right\} \cup \{h = 0\}. \tag{5.145}$$

We will work in the (h, v) variables rather than (h, q). Assume first that $u_r = (h_r, v_r)$ in V with h_r positive. We first connect u_l, using a slow rarefaction wave, with a state u_m on the "vacuum line" $h = 0$. This state is given by

$$v_m = v_l + 2\sqrt{h_l}, \tag{5.146}$$

using (5.50). From this state we jump to the unique point v^* on $h = 0$ such that the fast rarefaction starting at $h^* = 0$ and v^* hits u_r. Thus we see from (5.51) that $v^* = v_r - 2\sqrt{h_r}$, which gives the following solution (see Fig. 5.9):

$$u(x,t) = \begin{cases} \binom{h_l}{v_l} & \text{for } x < \lambda_1(u_l)t, \\ R_1(x/t; u_l) & \text{for } \lambda_1(u_l)t < x < (2\sqrt{h_l} + v_l)t, \\ \binom{0}{\tilde{v}(x,t)} & \text{for } (2\sqrt{h_l} + v_l)t < x < v^* t, \\ R_2(x/t; (0, v^*)) & \text{for } v^* t < x < \lambda_2(u_r)t, \\ \binom{h_r}{v_r} & \text{for } x > \lambda_2(u_r)t. \end{cases} \tag{5.147}$$



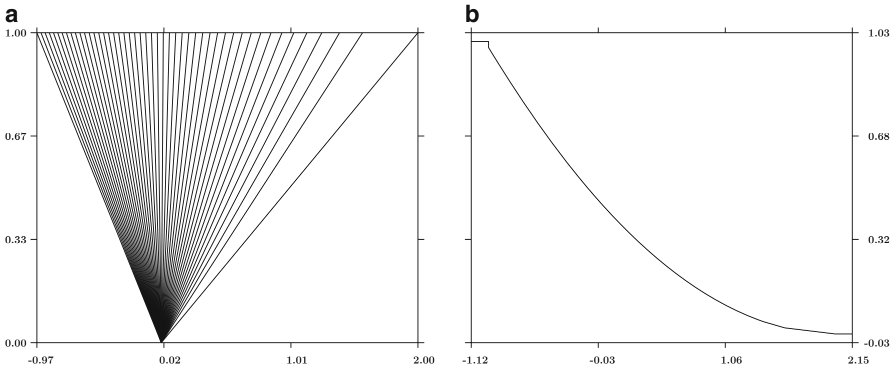


Fig. 5.9 The solution of the dam-breaking problem in (x, t)-space (**a**), and the h-component (**b**)

Physically, it does not make sense to give a value of the speed v of the water when there is no water, i.e., $h = 0$, and mathematically we see that any v will satisfy the equations when $h = 0$. Thus we do not have to associate any value with $\tilde{v}(x, t)$.

If u_r is on the vacuum line $h = 0$, we still connect to a state u_m on $h = 0$ using a slow rarefaction, and subsequently we connect to u_r along the vacuum line. By considering a nearby state $\tilde{u}_r$ with $\tilde{h} > 0$, we see that with this construction we have continuity in the data.

Finally, we have to solve the Riemann problem with the left state on the vacuum line $h = 0$. Now let $u_l = (0, v_l)$, and let $u_r = (h_r, v_r)$ with $h_r > 0$. We now connect u_l to an intermediate state u_m on the vacuum line given by $v_m = v_r - 2\sqrt{h_r}$ and continue with a fast rarefaction to the right state u_r. ◇

We will apply the above theory to one old and two ancient problems:

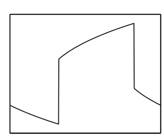
◇ Example 5.20 (Dam breaking)
For this problem we consider Riemann initial data of the form (in (h, v) variables)

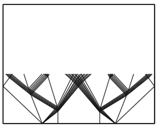
$$u(x, 0) = \begin{pmatrix} h(x,0) \\ v(x,0) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < 0, \\ \binom{0}{0} & \text{for } x \geq 0. \end{cases}$$

From the above discussion we know that the solution consists of a slow rarefaction (see Fig. 5.10); thus

$$u(x, t) = \begin{pmatrix} h(x,t) \\ v(x,t) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < -\sqrt{h_l}\,t, \\ \begin{pmatrix} \frac{1}{9}\left(2\sqrt{h_l} - \frac{x}{t}\right)^2 \\ \frac{2}{3}\left(\sqrt{h_l} + \frac{x}{t}\right) \end{pmatrix} & \text{for } -\sqrt{h_l}\,t < x < 2\sqrt{h_l}\,t, \\ \binom{0}{0} & \text{for } x > 2\sqrt{h_l}\,t. \end{cases} \quad \diamond$$

We shall call the two ancient problems Moses's first and second problems.





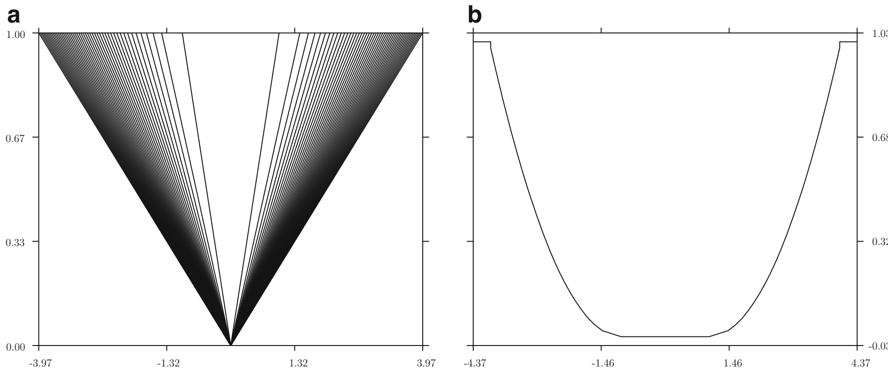


Fig. 5.10 The solution of Moses's first problem in (x, t)-space (**a**), and the h-component (**b**)

◇ Example 5.21 (Moses's first problem)

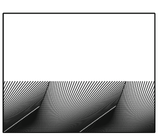
And Moses stretched out his hand over the sea; and the Lord caused the sea to go back by a strong east wind all that night, and made the sea dry land, and the waters were divided. And the children of Israel went into the midst of the sea upon the dry ground: and the waters were a wall unto them on their right hand, and on their left.
— Exodus (*14:21–22*)

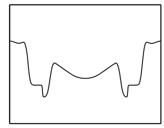
For the first problem we consider initial data of the form (in (h, v) variables)

$$u(x,0) = \begin{cases} \binom{h_0}{-v_0} & \text{for } x < 0, \\ \binom{h_0}{v_0} & \text{for } x \geq 0, \end{cases}$$

for a positive speed v_0. By applying the above analysis, we find that in this case we connect to an intermediate state u_1 on the vacuum line using a slow rarefaction. This state is connected to another state u_2 also on the vacuum line, which subsequently is connected to the right state using a fast rarefaction wave. More precisely, the state u_1 is determined by $v_1 = v(x_1, t_1)$, where $h(x,t) = \frac{1}{9}\big(-v_0 + 2\sqrt{h_0} - \frac{x}{t}\big)^2$ along the slow rarefaction wave (cf. (5.53)) and $h(x_1, t_1) = 0$. We find that $x_1 = \big(2\sqrt{h_0} - v_0\big)t_1$ and thus $v_1 = 2\sqrt{h_0} - v_0$. The second intermediate state u_2 is such that a fast rarefaction wave with left state u_2 hits u_r. This implies that $v_0 = v_2 + 2\sqrt{h_0}$ from (5.51), or $v_2 = v_0 - 2\sqrt{h_0}$. In order for this construction to be feasible, we will have to assume that $v_2 > v_1$ or $v_0 \geq 2\sqrt{h_0}$. If this condition does not hold, we will not get a region without water, and thus the original problem of Moses will not be solved. Combining the above waves in one solution, we obtain

$$h(x,t) = \begin{cases} h_0 & \text{for } x < -\big(v_0 + \sqrt{h_0}\big)t, \\ \frac{1}{9}\big(-v_0 + 2\sqrt{h_0} - \frac{x}{t}\big)^2 & \text{for } -\big(v_0 + \sqrt{h_0}\big)t < x < \big(2\sqrt{h_0} - v_0\big)t, \\ 0 & \text{for } \big(2\sqrt{h_0} - v_0\big)t < x < \big(v_0 - 2\sqrt{h_0}\big)t, \\ \frac{1}{9}\big(v_0 - 2\sqrt{h_0} - \frac{x}{t}\big)^2 & \text{for } \big(v_0 - 2\sqrt{h_0}\big)t < x < \big(v_0 + \sqrt{h_0}\big)t, \\ h_0 & \text{for } x > \big(v_0 + \sqrt{h_0}\big)t, \end{cases}$$





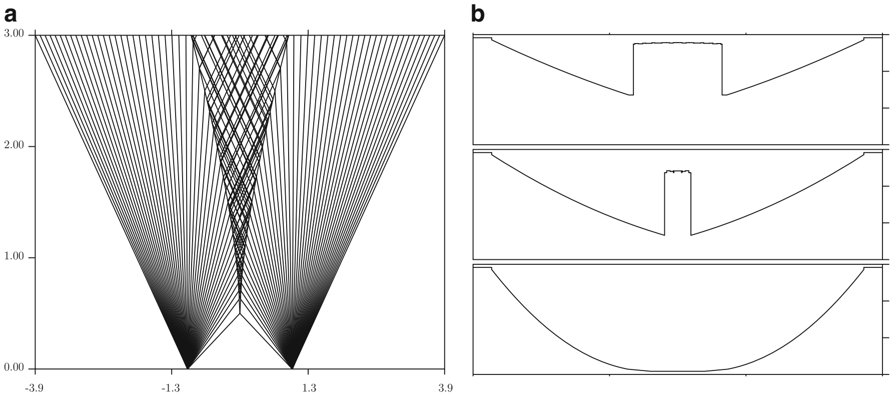


Fig. 5.11 The solution of Moses's second problem in (x,t)-space (**a**), and the h-component (**b**)

$$v(x,t) = \begin{cases} -v_0 & \text{for } x < -\left(v_0 + \sqrt{h_0}\right)t, \\ \frac{1}{3}\left(-v_0 + 2\sqrt{h_0} + 2\frac{x}{t}\right) & \text{for } -\left(v_0 + \sqrt{h_0}\right)t < x < \left(2\sqrt{h_0} - v_0\right)t, \\ 0 & \text{for } \left(2\sqrt{h_0} - v_0\right)t < x < \left(v_0 - 2\sqrt{h_0}\right)t, \\ \frac{1}{3}\left(v_0 - 2\sqrt{h_0} + 2\frac{x}{t}\right) & \text{for } \left(v_0 - 2\sqrt{h_0}\right)t < x < \left(v_0 + \sqrt{h_0}\right)t, \\ v_0 & \text{for } x > \left(v_0 + \sqrt{h_0}\right)t. \end{cases}$$

◇

◇ Example 5.22 (Moses's second problem)

And Moses stretched forth his hand over the sea, and the sea returned to his strength when the morning appeared; and the Egyptians fled against it; and the Lord overthrew the Egyptians in the midst of the sea.
— Exodus (14:27)

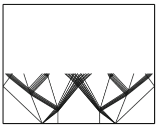
Here we study the multiple Riemann problem given by (in (h, v) variables)

$$u(x,0) = \begin{cases} \binom{h_0}{0} & \text{for } x < 0, \\ \binom{0}{0} & \text{for } 0 < x < L, \\ \binom{h_0}{0} & \text{for } x > L. \end{cases}$$

For small times t, the solution of this problem is found by patching together the solution of two dam-breaking problems. The left problem is solved by a fast rarefaction wave, and the right by a slow rarefaction. At some positive time, these rarefactions will interact, and thereafter explicit computations become harder.

In place of explicit computation we therefore present the numerical solution constructed by front tracking. This method is a generalization of the front-tracking method presented in Chapt. 2, and will be the subject of the next chapter.

In the left part of Fig. 5.11 we see the fronts in (x,t)-space. These fronts are similar to the fronts for the scalar front tracking, and the approximate solution is discontinuous across the lines shown in the figure. Looking at the figure, it is not hard to see why explicit computations become difficult as the two rarefaction waves



interact. The right part of the figure shows the water level as it engulfs the Egyptians. The lower figure shows the water level before the two rarefaction waves interact, and the two upper ones show that two shock waves result from the interaction of the two rarefaction waves. ◇

5.6 The Riemann Problem for the Euler Equations

The Euler equations are often used as a simplification of the Navier–Stokes equations as a model of the flow of a gas. In one space dimension these represent the conservation of mass, momentum, and energy, and read

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0. \tag{5.148}$$

Here ρ denotes the density of the gas, v the velocity, p the pressure, and E the energy. To close this system, i.e., to reduce the number of unknowns to the number of equations, one can add a constitutive "law" relating these. Such laws are often called *equations of state* and are deduced from thermodynamics. For a so-called ideal polytropic gas the equation of state takes the form

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2,$$

where $\gamma > 1$ is a constant spesific to the gas. For air, $\gamma \approx 1.4$. Solving for p, we get

$$p = (\gamma - 1)\, E - \frac{\gamma - 1}{2}\rho v^2 = (\gamma - 1)\, E - \frac{\gamma - 1}{2}\frac{q^2}{\rho}, \tag{5.149}$$

where the momentum q equals ρv. Inserting this in the Euler equations yields

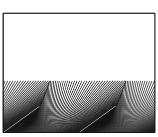
$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \frac{\gamma-3}{2}\rho v^2 + (\gamma - 1)E \\ v\left(\gamma E - \frac{\gamma-1}{2}\rho v^2\right) \end{pmatrix}_x = 0.$$

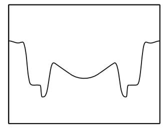
In the conserved variables ρ, q, and E, this system of conservation laws reads

$$\begin{pmatrix} \rho \\ q \\ E \end{pmatrix}_t + \begin{pmatrix} q \\ \left(\frac{3-\gamma}{2}\right)\frac{q^2}{\rho} + (\gamma - 1)\, E \\ \gamma \frac{Eq}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^3}{\rho^2} \end{pmatrix}_x = 0. \tag{5.150}$$

Set

$$u = \begin{pmatrix} \rho \\ q \\ E \end{pmatrix} \qquad \text{and} \qquad f(u) = \begin{pmatrix} q \\ \left(\frac{3-\gamma}{2}\right)\frac{q^2}{\rho} + (\gamma - 1)\, E \\ \gamma \frac{Eq}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^3}{\rho^2} \end{pmatrix}.$$





Then the Jacobian $df(u)$ reads

$$df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \left(\frac{\gamma-3}{2}\right)\frac{q^2}{\rho^2} & (3-\gamma)\frac{q}{\rho} & \gamma-1 \\ -\gamma\frac{Eq}{\rho^2} + (\gamma-1)\frac{q^3}{\rho^3} & \gamma\frac{E}{\rho} - \frac{3(\gamma-1)}{2}\frac{q^2}{\rho^2} & \gamma\frac{q}{\rho} \end{pmatrix}.$$

Introducing the *enthalpy* as

$$H = \frac{E+p}{\rho} = \gamma\frac{E}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^2}{\rho^2} = \frac{\gamma}{\rho}\left(\frac{p}{\gamma-1}\right) + \frac{1}{2}v^2,$$

the Jacobian can be rewritten as

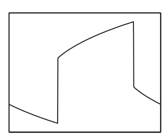
$$df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \left(\frac{\gamma-3}{2}\right)v^2 & (3-\gamma)\,v & \gamma-1 \\ \left(\frac{\gamma-1}{2}\right)v^3 - vH & H-(\gamma-1)\,v^2 & \gamma v \end{pmatrix}.$$

To find its eigenvalues, we compute the determinant

$$\begin{aligned}
\det(\lambda I - df(u)) &= \lambda\left[(\lambda-(3-\gamma)v)\,(\lambda-\gamma v) + (\gamma-1)\left((\gamma-1)\,v^2 - H\right)\right] \\
&\quad + \frac{3-\gamma}{2}v^2\,(\lambda-\gamma v) + (\gamma-1)\left(vH - \frac{\gamma-1}{2}v^3\right) \\
&= \lambda\left[\lambda^2 - 3v\lambda + \gamma(3-\gamma)v^2 + (\gamma-1)^2v^2 + (\gamma-1)H\right] \\
&\quad + \frac{3-\gamma}{2}v^2\lambda - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)Hv \\
&= \lambda\left[\lambda^2 - 3v\lambda + 2v^2 + \frac{1}{2}(\gamma+1)\,v^2 - (\gamma-1)H\right] \\
&\quad - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)vH \\
&= \lambda\Big[(\lambda-v)(\lambda-2v) + \frac{1}{2}(\gamma+1)v^2 - (\gamma-1)H\Big] \\
&\quad - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)vH \\
&= (\lambda-v)\left[\lambda(\lambda-2v) + \frac{1}{2}(\gamma+1)v^2 - (\gamma-1)H\right] \\
&= (\lambda-v)\left[(\lambda-v)^2 - \left(v^2 - \frac{1}{2}(\gamma+1)v^2 + (\gamma-1)H\right)\right] \\
&= (\lambda-v)\left[(\lambda-v)^2 - \left(\frac{\gamma-1}{2}(2H-v^2)\right)\right].
\end{aligned}$$

This can be simplified further by introducing the *sound speed* c, by

$$c^2 = \frac{\gamma p}{\rho}.$$



We then calculate

$$2H - v^2 = 2\gamma\frac{E}{\rho} - (\gamma-1)v^2 - v^2 = 2\gamma\frac{E}{\rho} - \gamma v^2 = \gamma\left(\frac{2E}{\rho} - v^2\right)$$
$$= \frac{\gamma}{\rho}\left(2E - \rho v^2\right) = \frac{\gamma}{\rho}\frac{2p}{\gamma-1}.$$

Therefore

$$\det(\lambda I - df(u)) = (\lambda - v)\left[(\lambda - v)^2 - c^2\right].$$

Thus the eigenvalues of the Jacobian are

$$\lambda_1(u) = v - c,\ \ \lambda_2(u) = v,\ \ \lambda_3(u) = v + c. \tag{5.151}$$

As for the corresponding eigenvectors, we write these as $r_i = (1, y_i, z_i)$,[9] and we see that $y_i = \lambda_i$, and

$$z_i = \frac{1}{\gamma-1}\left(\lambda_i^2 - \frac{1}{2}(\gamma-3)v^2 + \lambda_i(\gamma-3)v\right).$$

For $i = 1$ we find that

$$\begin{aligned}
z_1 &= \frac{1}{\gamma-1}\left(v^2 - \frac{1}{2}(\gamma-3)v^2 + v(\gamma-3)v\right)\\
&\quad + \frac{1}{\gamma-1}\left(c^2 - 2cv - (\gamma-3)cv\right)\\
&= \frac{1}{2}v^2 + \frac{c^2}{\gamma-1} - cv\\
&= \left(\frac{1}{2}v^2 + \frac{\gamma p}{\rho(\gamma-1)}\right) - cv\\
&= H - cv.
\end{aligned}$$

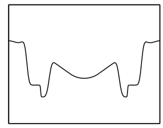
For $i = 3$ we similarly calculate

$$z_3 = H + cv,$$

and for $i = 2$ it is straightforward to see that $z_2 = v^2/2$. Summing up, we have the following eigenvalues and eigenvectors:

$$\begin{aligned}
\lambda_1(u) &= v - c, \quad & r_1(u) &= \begin{pmatrix} 1 \\ v - c \\ H - cv \end{pmatrix},\\
\lambda_2(u) &= v, & r_2(u) &= \begin{pmatrix} 1 \\ v \\ \frac{1}{2}v^2 \end{pmatrix},\\
\lambda_3(u) &= v + c, & r_3(u) &= \begin{pmatrix} 1 \\ v + c \\ H + cv \end{pmatrix}.
\end{aligned} \tag{5.152}$$

[9] Recall that this is in (ρ, q, E) coordinates.



It is important to observe that the second family is linearly degenerate, since

$$\nabla \lambda_2(u) \cdot r_2(u) \equiv 0, \tag{5.153}$$

and hence the solution of the Riemann problem in this family will consist of a contact discontinuity. The first and the third families are both genuinely nonlinear, and we encounter the familiar shock and rarefaction waves.

At this point it is convenient to introduce the concept of an *i-Riemann invariant*. (See Exercise 5.8.) An i-Riemann invariant is a function $R = R(\rho, q, E)$ such that R is constant along the integral curves of r_i. In other words, an i-Riemann invariant satisfies

$$\nabla R(u) \cdot r_i = 0.$$

The usefulness of this is that if we can find for each of the three eigenvectors, two Riemann invariants $R(u)$ and $\tilde{R}(u)$, then we can possibly solve the equations

$$R(\rho, q, E) = R(\rho_l, q_l, E_l), \qquad \tilde{R}(\rho, q, E) = \tilde{R}(\rho_l, q_l, E_l)$$

to obtain a formula for the rarefaction waves. This is equivalent to finding an implicit solution of the ordinary differential equation $\dot{u} = r(u)$ defining the rarefaction curves.

It turns out that we have the following Riemann invariants (see Exercise 5.12):

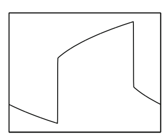
$$\begin{aligned} i &= 1, \quad \text{Riemann invariants:} \begin{cases} S, \\ v + \frac{2c}{\gamma-1}, \end{cases} \\ i &= 2, \quad \text{Riemann invariants:} \begin{cases} v, \\ p, \end{cases} \\ i &= 3, \quad \text{Riemann invariants:} \begin{cases} S, \\ v - \frac{2c}{\gamma-1}, \end{cases} \end{aligned} \tag{5.154}$$

where we have introduced the *entropy* S by

$$S = -\log\left(\frac{p}{\rho^\gamma}\right). \tag{5.155}$$

Now we can try to obtain solution formulas for the rarefaction curves. For $i = 1$, this curve is given by

$$p = p_l \left(\frac{\rho}{\rho_l}\right)^\gamma, \qquad v = v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right).$$



This curve is parameterized by ρ. We must check which half of the curve to use. This will be the part where $\lambda_1 = v - c$ is increasing. On the curve we have

$$\begin{aligned} v(\rho) - c(\rho) &= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - \left(\frac{\gamma p(\rho)}{\rho}\right)^{1/2} \\ &= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - \left(\frac{\gamma p_l}{\rho_l}\right)^{1/2}\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2} \\ &= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - c_l\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2} \\ &= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \frac{\gamma+1}{2}\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right). \end{aligned}$$

Since $\gamma > 1$, we see that $v(\rho) - c(\rho)$ is decreasing in ρ, and for the 1-rarefaction wave we must use $\rho < \rho_l$. Since $p(\rho)$ is increasing in ρ, this also means that we use the part where $p < p_l$. Therefore we can use p as a parameter in the curve for v and write the 1-rarefaction curve as

$$v_1(p) = v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{p}{p_l}\right)^{(\gamma-1)/(2\gamma)}\right), \quad p \le p_l.$$

The general theory tells us that (at least for p close to p_l) this curve can be continued smoothly as a 1-shock curve.

To find the rarefaction curve of the third family, we adopt the viewpoint that u_r is fixed, and we wish to find u as a function of u_r (cf. the solution of the Riemann problem for the shallow-water equations). In the same way as for v_1 this leads to the formula

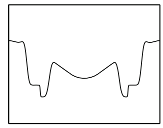
$$v_3(p) = v_r + \frac{2c_r}{\gamma - 1}\left(1 - \left(\frac{p}{p_r}\right)^{(\gamma-1)/(2\gamma)}\right), \quad p \le p_r.$$

To find how the density varies along the rarefaction curves, we can use that the entropy S is constant, leading to

$$\frac{\rho}{\rho_l} = \left(\frac{p}{p_l}\right)^{1/\gamma}.$$

Now we turn to the computation of the Hugoniot loci. We view the left state u_l as fixed, and try to find the right state u; recall the notation $[\![u]\!] = u - u_l$. The Rankine–Hugoniot relations for (5.148) are

$$\begin{aligned} s\,[\![\rho]\!] &= [\![\rho v]\!], \\ s\,[\![\rho v]\!] &= [\![\rho v^2 + p]\!], \\ s\,[\![E]\!] &= [\![v(E+p)]\!], \end{aligned} \tag{5.156}$$



where s denotes the speed of the discontinuity. Now we introduce new variables by

$$w = v - s \quad \text{and} \quad m = \rho w.$$

Then the first equation in (5.156) reads

$$s\rho - s\rho_l = \rho w + s\rho - \rho_l w_l - s\rho_l,$$

which implies that $[\![m]\!] = 0$. Similarly, the second equation reads

$$s\rho w + s^2\rho - s\rho w_l - s^2\rho_l = \rho(w+s)^2 - \rho_l(w_l+s)^2 + [\![p]\!],$$

or

$$s\,[\![m]\!] + s^2\,[\![\rho]\!] = \rho w^2 + 2\rho w + s^2\rho - \rho_l w_l^2 - 2\rho_l w_l - s^2\rho_l + [\![p]\!],$$

and subsequently

$$s^2\,[\![\rho]\!] = [\![\rho w^2 + p]\!] + s^2\,[\![\rho]\!].$$

Hence $[\![mw + p]\!] = 0$. Finally, the third equation in (5.156) reads

$$sE - sE_l = Ew + Es + pw + ps - E_l w_l - E_l s - p_l w_l - p_l s,$$

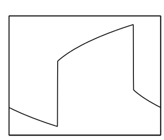
which implies

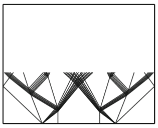
$$\begin{aligned}
0 &= \left(\frac{E}{\rho} - \frac{E_l}{\rho_l}\right) m + pw - p_l w_l + s\,[\![p]\!] \\
&= \left(\frac{E}{\rho} - \frac{E_l}{\rho_l} + \frac{p}{\rho} - \frac{p_l}{\rho_l}\right) m - sm\,[\![w]\!] \\
&= m\left[\!\!\left[\frac{E+p}{\rho} - sw\right]\!\!\right] \\
&= m\left[\!\!\left[\frac{c^2}{\gamma-1} + \frac{1}{2}(w+s)^2 - sw\right]\!\!\right] \\
&= m\left[\!\!\left[\frac{c^2}{\gamma-1} + \frac{1}{2}w^2\right]\!\!\right].
\end{aligned}$$

Hence the Rankine–Hugoniot conditions are equivalent to

$$\begin{aligned}
[\![m]\!] &= 0, \\
[\![mw + p]\!] &= 0, \\
m\left[\!\!\left[\frac{c^2}{\gamma-1} + \frac{1}{2}w^2\right]\!\!\right] &= 0.
\end{aligned} \tag{5.157}$$

We immediately find one solution by setting $m = 0$, which implies $[\![p]\!] = 0$ and $[\![v]\!] = 0$. This is the contact discontinuity. Hence we assume that $m \neq 0$ to find the other Hugoniot loci.





Now we introduce auxiliary parameters

$$\pi = \frac{p}{p_l} \quad \text{and} \quad z = \frac{\rho}{\rho_l}.$$

Using these, we have that

$$\frac{c^2}{c_l^2} = \frac{\pi}{z} \quad \text{and} \quad \frac{w}{w_l} = \frac{1}{z}. \tag{5.158}$$

Then the third equation in (5.157) reads

$$\frac{c_l^2}{\gamma - 1} + \frac{1}{2} w_l^2 = \frac{c_l^2}{\gamma - 1} \frac{\pi}{z} + \frac{1}{2} w_l^2 \frac{1}{z^2},$$

which can be rearranged as

$$c_l^2 \frac{2}{\gamma - 1} \left(1 - \frac{\pi}{z}\right) = w_l^2 \left(\frac{1}{z^2} - 1\right),$$

so that

$$\left(\frac{w_l}{c_l}\right)^2 = \frac{2}{\gamma - 1} \frac{z(z - \pi)}{1 - z^2}. \tag{5.159}$$

Next recall that $p = \rho c^2/\gamma$. Using this, the second equation in (5.157) reads

$$\frac{\rho c^2}{\gamma} + \rho w^2 = \frac{\rho_l c_l^2}{\gamma} + \rho_l w_l^2,$$

or

$$z \left(\frac{c^2}{\gamma} + w^2\right) = \frac{c_l^2}{\gamma} + w_l^2,$$

which again can be rearranged as

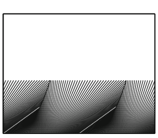
$$z \left(\frac{c_l^2 \pi}{\gamma z} + w_l^2 \frac{1}{z^2}\right) = \frac{c_l^2}{\gamma} + w_l^2.$$

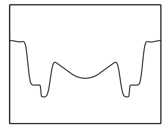
Dividing by c_l^2, we can solve for $(w_l/c_l)^2$:

$$\left(\frac{w_l}{c_l}\right)^2 = \frac{1}{\gamma} \frac{z(\pi - 1)}{z - 1}. \tag{5.160}$$

Equating (5.160) and (5.159) and solving for z yields

$$z = \frac{\beta \pi + 1}{\pi + \beta}, \tag{5.161}$$





where

$$\beta = \frac{\gamma + 1}{\gamma - 1}. \tag{5.162}$$

Using this expression for z in (5.159), we get

$$\begin{aligned}\left(\frac{w_l}{c_l}\right)^2 &= \frac{2}{\gamma - 1} \frac{\frac{\pi\beta+1}{\pi+\beta}\left(\frac{\pi\beta+1}{\pi+\beta} - \pi\right)}{1 - \frac{(\pi\beta+1)^2}{(\pi+\beta)^2}} \\ &= \frac{2}{\gamma - 1} \frac{(\pi\beta + 1)(1 - \pi^2)}{(\pi^2 - 1)(1 - \beta^2)} \\ &= \frac{2}{\gamma - 1} \frac{\pi\beta + 1}{\beta^2 - 1}.\end{aligned}$$

Note that $\gamma > 1$ implies $\beta > 1$, so that this is always well defined. Since $w_l = v_l - s$, we can use this to get an expression for the shock speed,

$$s = v_l \mp c_l \sqrt{\frac{2}{\gamma - 1} \frac{\beta\pi + 1}{\beta^2 - 1}}, \tag{5.163}$$

where we use the minus sign for the first family and the plus sign for the third.

Next, using (5.158), we get

$$\frac{v - s}{v_l - s} = \frac{1}{z},$$

which can be used to express v as a function of π:

$$\begin{aligned} v &= v_l \mp c_l \sqrt{\frac{2}{\gamma - 1} \frac{\beta\pi + 1}{(\beta^2 - 1)}} \pm \frac{\pi + \beta}{\pi\beta + 1} c_l \sqrt{\frac{2}{\gamma - 1} \frac{\pi\beta + 1}{(\beta^2 - 1)}} \\ &= v_l \mp c_l \sqrt{\frac{2}{\gamma - 1} \frac{1}{(\beta^2 - 1)} (\pi\beta + 1)} \left(\frac{(\beta - 1)(\pi - 1)}{\pi\beta + 1}\right) \\ &= v_l \mp 2c_l \frac{1}{\sqrt{2\gamma(\gamma - 1)}} \frac{\pi - 1}{(\pi\beta + 1)^{1/2}}, \end{aligned}$$

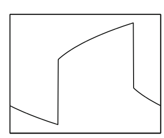
where we take the minus sign for the first family and the plus sign for the third. To see how the density varies along the Hugoniot loci, we use that $\rho = \rho_l z$, or

$$\rho = \rho_l \frac{\pi\beta + 1}{\pi + \beta}, \tag{5.164}$$

which holds for both the first and third families.

Next we have to verify the Lax entropy condition, Definition 5.13. Consider the Lax 1-shock condition

$$s < \lambda_1(u_l), \quad \lambda_1(u) < s < \lambda_2(u).$$



Since the shock speed $s = s(u)$ given by (5.163) satisfies

$$s(u_l) = v_l - c_l = \lambda_1(u_l)$$

and is a decreasing function in π, we infer that $s < \lambda_1(u_l)$ holds when $p \geq p_l$, that is, $\pi > 1$. As for the inequality involving the right state, it is advantageous to rewrite the shock speed (5.163) in terms of the right state (see Exercise 5.13); thus

$$s = v \mp c\sqrt{\frac{2}{\gamma-1}\frac{\beta/\pi+1}{\beta^2-1}}. \tag{5.165}$$

Since $\pi > 1$, we see that

$$\sqrt{\frac{2}{\gamma-1}\frac{\beta/\pi+1}{\beta^2-1}} < 1,$$

thereby proving $\lambda_1(u) < s < \lambda_2(u)$. This shows that the part of the Hugoniot locus with $p \geq p_l$ satisfies the Lax 1-shock condition. A similar argument applies to the third family.

This means that the whole solution curve for waves of the first family is given by

$$v_1(p) = v_l + 2c_l \begin{cases} \frac{1}{\gamma-1}\left(1-\left(\frac{p}{p_l}\right)^{(\gamma-1)/(2\gamma)}\right), & p \leq p_l, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1-\frac{p}{p_l}\right)\left(1+\beta\frac{p}{p_l}\right)^{-1/2}, & p \geq p_l. \end{cases} \tag{5.166}$$

To find the density along this solution curve, we have the formula

$$\rho_1(p) = \rho_l \begin{cases} \left(\frac{p}{p_l}\right)^{1/\gamma}, & p \leq p_l, \\ \frac{1+\beta\frac{p}{p_l}}{\beta+\frac{p}{p_l}}, & p \geq p_l. \end{cases} \tag{5.167}$$

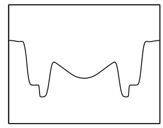
In terms of the parameter $\pi = p/p_l$, the wave curve of the first family reads

$$\begin{aligned} \rho_1(\pi) &= \rho_l \begin{cases} \pi^{1/\gamma}, & \pi \leq 1, \\ \frac{1+\beta\pi}{\beta+\pi}, & \pi \geq 1, \end{cases} \\ v_1(\pi) &= v_l + 2c_l \begin{cases} \frac{1}{\gamma-1}\left(1-\pi^{(\gamma-1)/(2\gamma)}\right), & \pi \leq 1, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}(1-\pi)\left(1+\beta\pi\right)^{-1/2}, & \pi \geq 1. \end{cases} \end{aligned} \tag{5.168}$$

Similar formulas can also be computed for the variables q and E.

Since the second family is linearly degenerate, we can use the whole integral curve of r_2. Using the Riemann invariants, this is given simply as

$$v = v_l, \quad p = p_l, \tag{5.169}$$



and thus only the density ρ varies. The contact discontinuity is often called a slip line.

For the third family, we take the same point of view as for the shallow-water equations; we keep u_r fixed and look for states u such that the Riemann problem

$$u(x,0) = \begin{cases} u & x < 0, \\ u_r & x > 0, \end{cases}$$

is solved by a wave (shock or rarefaction) of the third family. By much the same calculations as for the first family we end up with

$$v_3(p) = v_r - 2c_r \begin{cases} \frac{1}{\gamma-1}\left(1 - \left(\frac{p}{p_r}\right)^{(\gamma-1)/(2\gamma)}\right), & p \le p_r \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1 - \frac{p}{p_r}\right)\left(1 + \beta\frac{p}{p_r}\right)^{-1/2}, & p \ge p_r, \end{cases} \tag{5.170}$$

where the rarefaction part is for $p \le p_r$ and the shock part for $p \ge p_r$. Regarding the density along this curve, it will change according to

$$\rho_3(p) = \rho_r \begin{cases} \left(\frac{p}{p_r}\right)^{1/\gamma}, & p \le p_r, \\ \left(\frac{1+\beta\frac{p}{p_r}}{\beta+\frac{p}{p_r}}\right), & p \ge p_r. \end{cases} \tag{5.171}$$

In terms of the parameter $\pi_r = p/p_r$, the wave curve of the third family reads

$$\begin{aligned} \rho_3(\pi_r) &= \rho_r \begin{cases} \pi^{1/\gamma}, & \pi_r \le 1, \\ \frac{1+\beta\pi}{\beta+\pi}, & \pi_r \ge 1, \end{cases} \\ v_3(\pi_r) &= v_r - 2c_r \begin{cases} \frac{1}{\gamma-1}\left(1-\pi_r\right)^{(\gamma-1)/(2\gamma)}, & \pi_r \le 1, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1-\pi_r\right)\left(1+\beta\pi_r\right)^{-1/2}, & \pi_r \ge 1. \end{cases} \end{aligned} \tag{5.172}$$

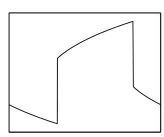
Now for every ρ_l, the curve $v_1(p)$ is a strictly decreasing function of p (or π) for nonnegative density p taking values in the set $(-\infty, v_l + 2c_l/(\gamma-1)]$. Similarly, for every ρ_r, we have that $v_3(p)$ is a strictly increasing function of p (or π_r) taking values in the set $[v_r - 2c_r/(\gamma-1), \infty)$. It follows that these curves will intersect in one point (p_m, v_m) if

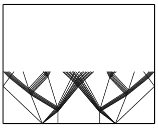
$$v_r - \frac{2c_r}{\gamma-1} \le v_l + \frac{2c_l}{\gamma-1},$$

or

$$\frac{1}{2}(\gamma-1)\,[\![v]\!] \le c_l + c_r.$$

In this case we obtain a unique solution of the Riemann problem as the pressure jumps from the value to the left of the slip line to the value on the right-hand side, while the pressure p and velocity v remain unchanged and equal to p_m and v_m, respectively, across the slip line. If this does not hold, then v_1 does not intersect v_3, and we have a solution with vacuum.





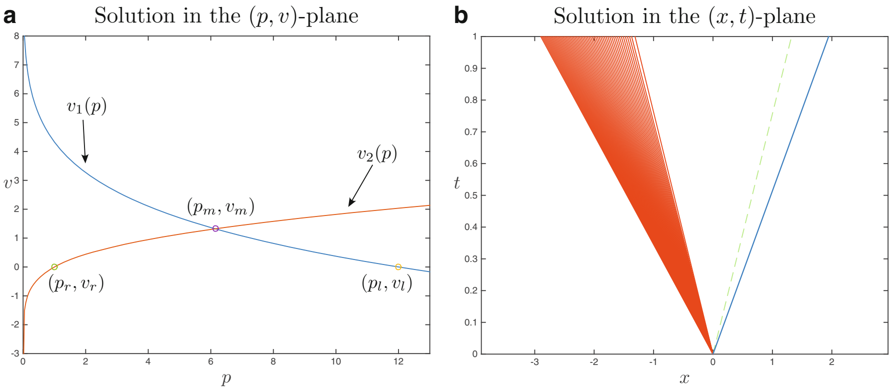


Fig. 5.12 The solution of the Riemann problem (5.173)

◇ Example 5.23 (Sod's shock tube problem)
We consider an initial value problem similar to the dam-breaking problem for shallow water. The initial velocity is everywhere zero, but the pressure to the left is higher than the pressure on the right. Specifically, we set

$$p(x,0) = \begin{cases} 12 & x < 0, \\ 1 & x \geq 0, \end{cases} \qquad v(x,0) = 0, \qquad \rho(x,0) = 2. \tag{5.173}$$

We have used $\gamma = 1.4$.

In Fig. 5.12 we show the solution to this Riemann problem in the (p, v)-plane and in the (x, t)-plane. We see that the solution consists of a leftward-moving rarefaction wave of the first family, followed by a contact discontinuity and a shock wave of the third family. In Fig. 5.13 we show the pressure, velocity, density, and the Mach number as functions of x/t. The Mach number is defined to be $|v|/c$, so that if this is larger than 1, the flow is called supersonic. The solution found here is actually supersonic between the contact discontinuity and the shock wave. ◇

The Euler Equations and Entropy

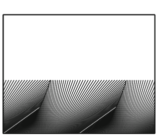
We shall show that the physical entropy is in fact also a mathematical entropy for the Euler equations, in the sense that

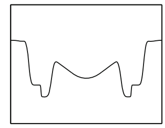
$$(\rho S)_t + (v\rho S)_x \leq 0, \tag{5.174}$$

weakly for every weak solution $u = (\rho, q, E)$ that is the limit of solutions to the viscous approximation.

To this end, it is convenient to introduce the *internal specific energy*, defined by

$$e = \frac{1}{\rho}\left(E - \frac{1}{2}\rho v^2\right).$$





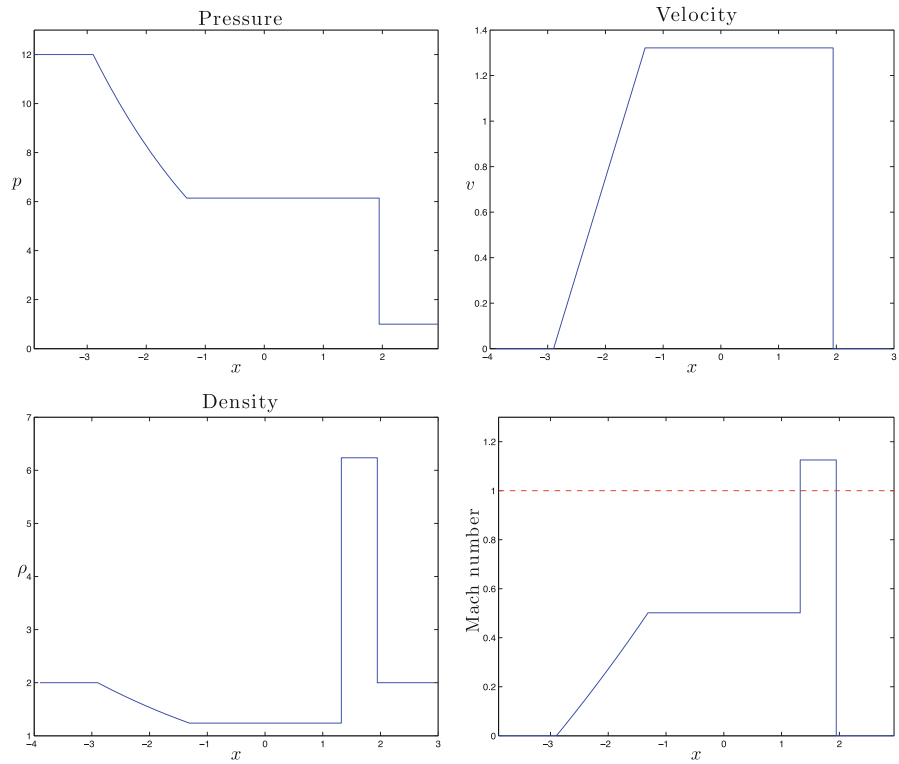


Fig. 5.13 Pressure, velocity, density, and the Mach number for the solution of (5.173)

Then the Euler equations read

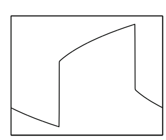
$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + \left(\rho v^2 + p\right)_x &= 0, \\ \left(\rho\left(e + \frac{1}{2}v^2\right)\right)_t + \left(\frac{1}{2}\rho v^2 + \rho e v + p v\right)_x &= 0. \end{aligned} \tag{5.175}$$

For classical solutions, this is equivalent to the nonconservative form (see Exercise 5.12)

$$\begin{aligned} \rho_t + v\rho_x + \rho v_x &= 0, \\ v_t + v v_x + \frac{1}{\rho} p_x &= 0, \\ e_t + v e_x + \frac{p}{\rho} v_x &= 0. \end{aligned} \tag{5.176}$$

We have that

$$\begin{aligned} S &= -\log\left(\frac{p}{\rho^\gamma}\right) \\ &= -\log\left(\frac{(\gamma-1)e}{\rho^{\gamma-1}}\right) \\ &= (\gamma-1)\log(\rho) - \log(e) - \log(\gamma-1). \end{aligned} \tag{5.177}$$



Thus we see that

$$S_\rho = \frac{\gamma - 1}{\rho} > 0 \ \text{ and } \ S_e = -\frac{1}{e} < 0.$$

These inequalities are general, and thermodynamic mumbo jumbo implies that they hold for every equation of state, not only for polytropic gases.

For classical solutions we can compute

$$\begin{aligned} S_t &= S_\rho \rho_t + S_e e_t \\ &= -\frac{\gamma - 1}{\rho}(v\rho_x + \rho v_x) + \frac{1}{e}\left(v e_x + \frac{p}{\rho} v_x\right) \\ &= -\left((\gamma - 1) - \frac{p}{e\rho}\right) v_x - \left((\gamma - 1)\frac{\rho_x}{\rho} - \frac{e_x}{e}\right) v \\ &= -v S_x. \end{aligned}$$

Therefore

$$S_t + v S_x = 0$$

for smooth solutions to the Euler equations. This states that the entropy of a "particle" of the gas remains constant as the particle is transported with velocity v. Furthermore,

$$\begin{aligned} (\rho S)_t &= \rho_t S + \rho S_t \\ &= -(\rho v)_x S - \rho v S_x \\ &= -(v\rho S)_x. \end{aligned}$$

Thus for smooth solutions the *specific entropy* $\eta(u) = \rho S(u)$ is conserved:

$$(\rho S)_t + (\rho v S)_x = 0. \tag{5.178}$$

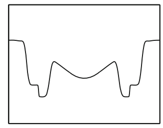
The existence of such an entropy/entropy flux pair is rather exceptional for a system of three hyperbolic conservation laws; see Exercise 5.10. Of course, combining this with (5.175) and viewing the entropy as an independent unknown, we have four equations for three unknowns, so we cannot automatically expect to have a solution. Sometimes one considers models in which the energy is not conserved but the entropy is, so-called *isentropic* flow. In models of isentropic flow the third equation in (5.175) is replaced by the conservation of entropy (5.178).

To show that (5.174) holds for viscous limits, we first show that the map

$$u \mapsto \eta(u) = \rho S(\rho, e(u))$$

is convex. We have that η is convex if its Hessian $d^2\eta$ is a positive definite matrix. For the moment we use the convention that all vectors are column vectors, and for a vector a, a^T denotes its transpose. We first obtain

$$\begin{aligned} \nabla\eta &= S\nabla\rho + \rho\nabla S \\ &= S\nabla\rho + \rho(S_\rho\nabla\rho + S_e\nabla e) \\ &= (S + \rho S_\rho)\nabla\rho + \rho S_e \nabla e. \end{aligned}$$



Trivially we have that $\nabla\rho = (1,0,0)^T$. Furthermore,

$$e(u) = \frac{E}{\rho} - \frac{1}{2}\frac{q^2}{\rho^2},$$

so we have

$$\nabla e = \left(-\frac{E}{\rho^2} + \frac{q^2}{\rho^3}, -\frac{q}{\rho^2}, \frac{1}{\rho}\right)^T = \frac{1}{\rho}\left(-e + \frac{1}{2}v^2, -v, 1\right)^T.$$

Next we compute

$$\begin{aligned} d^2\eta &= d^2\left(\rho S(\rho,e)\right) \\ &= \nabla\rho\,(\nabla S)^T + \nabla S\,(\nabla\rho)^T + \rho d^2 S \\ &= \nabla\rho\left(S_\rho\nabla\rho + S_e\nabla e\right)^T + \left(S_\rho\nabla\rho + S_e\nabla e\right)(\nabla\rho)^T + \rho d^2 S \\ &= 2S_\rho\nabla\rho(\nabla\rho)^T + S_e\left(\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T\right) + \rho d^2 S. \end{aligned}$$

To compute the Hessian of S we first compute its gradient:

$$\nabla S(\rho,e) = S_\rho\nabla\rho + S_e\nabla e.$$

Thus[10]

$$\begin{aligned} d^2 S(\rho,e) &= \nabla(S_\rho\nabla\rho) + \nabla(S_e\nabla e) \\ &= \nabla\rho(\nabla S_\rho)^T + \nabla e(\nabla S_e)^T + S_e d^2 e \\ &= \nabla\rho(S_{\rho\rho}\nabla\rho + S_{\rho e}\nabla e)^T + \nabla e(S_{e\rho}\nabla\rho + S_{ee}\nabla e)^T + S_e d^2 e \\ &= S_{\rho\rho}\nabla\rho(\nabla\rho)^T + S_{\rho e}\left(\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T\right) + S_{ee}\nabla e(\nabla e)^T + S_e d^2 e. \end{aligned}$$

If we use this in the previous equation, we end up with

$$\begin{aligned} d^2\eta(u) = &\left(\rho S_{\rho\rho} + 2S_\rho\right)\nabla\rho\,(\nabla\rho)^T \\ &+ \rho S_{\rho e}\left(\nabla\rho\,(\nabla e)^T + \nabla e\,(\nabla\rho)^T\right) + \rho S_{ee}\nabla e\,(\nabla e)^T - S_e C, \end{aligned}$$

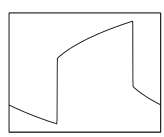
where C is given by

$$C = -\left(\rho d^2 e + \nabla\rho\,(\nabla e)^T + \nabla e\,(\nabla\rho)^T\right).$$

The Hessian of e is given by

$$d^2 e = \begin{pmatrix} 2\frac{E}{\rho^3} - 3\frac{q^2}{\rho^4} & 2\frac{q}{\rho^3} & -\frac{1}{\rho^2} \\ 2\frac{q}{\rho^3} & -\frac{1}{\rho^2} & 0 \\ -\frac{1}{\rho^2} & 0 & 0 \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} 2e - 2v^2 & 2v & -1 \\ 2v & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

[10] In our notation we have $\nabla(f(u)V(u)) = V(u)(\nabla f(u))^T + f(u)\nabla V(u)$, where f is a scalar-valued function and V is (column) vector-valued. The result $\nabla(fV)$ is a matrix.



Next,

$$\begin{aligned}\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T &= \frac{1}{\rho}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}-e+\frac{1}{2}v^2, & -v, & 1\end{pmatrix}\\ &\quad + \frac{1}{\rho}\begin{pmatrix}-e+\frac{1}{2}v^2\\ -v\\ 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\\ &= \frac{1}{\rho}\begin{pmatrix}-2e+v^2 & -v & 1\\ -v & 0 & 0\\ 1 & 0 & 0\end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}C &= -\frac{1}{\rho}\begin{pmatrix}2e-2v^2 & 2v & -1\\ 2v & -1 & 0\\ -1 & 0 & 0\end{pmatrix} - \frac{1}{\rho}\begin{pmatrix}-2e+v^2 & -v & 1\\ -v & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\\ &= \frac{1}{\rho}\begin{pmatrix}v^2 & -v & 0\\ -v & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.\end{aligned}$$

Now introduce the matrix D by

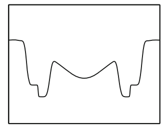
$$D = \begin{pmatrix}1 & v & \frac{1}{2}v^2+e\\ 0 & \rho & \rho v\\ 0 & 0 & \rho\end{pmatrix}.$$

We have that D is invertible, and thus $d^2\eta$ is positive definite if and only if $Dd^2\eta D^T$ is positive definite. Then

$$\begin{aligned}Dd^2\eta(u)D^T &= \left(\rho S_{\rho\rho} + 2S_\rho\right) D\nabla\rho\,(D\nabla\rho)^T\\ &\quad + \rho S_{\rho e}\left(D\nabla\rho\,(D\nabla e)^T + D\nabla e\,(D\nabla\rho)^T\right)\\ &\quad + \rho S_{ee} D\nabla e\,(D\nabla e)^T - S_e DCD^T.\end{aligned}$$

We compute

$$D\nabla\rho = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad D\nabla e = \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad DCD^T = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$



and using this,

$$Dd^2\eta(u)D^T = \left(\rho S_{\rho\rho} + 2S_\rho\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \rho S_{\rho e}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} + \rho S_{ee}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} - S_e\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}\rho S_{\rho\rho} + 2S_\rho & 0 & S_{\rho e}\\ 0 & -S_e & 0\\ S_{\rho e} & 0 & S_{ee}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\gamma-1}{\rho} & 0 & 0\\ 0 & \frac{1}{e} & 0\\ 0 & 0 & \frac{1}{e^2}\end{pmatrix}.$$

Hence $Dd^2\eta(u)D^T$ has three positive eigenvalues and is positive definite. Therefore, also $d^2\eta$ is positive definite, and η is convex. From the general identity

$$\eta(u)_{xx} = (u_x)^T d^2\eta(u)u_x + (\nabla\eta(u))^T u_{xx}, \quad u = u(x) = (u_1, \ldots, u_n), \tag{5.179}$$

we get from the convexity of $d^2\eta$ that

$$\eta(u)_{xx} \geq (\nabla\eta(u))^T u_{xx}. \tag{5.180}$$

Consider now a smooth solution of the regularized Euler equations

$$u^\varepsilon_t + f(u^\varepsilon)_x = \epsilon u^\varepsilon_{xx}. \tag{5.181}$$

We multiply from the left by $(\nabla\eta)^T$, which yields[11]

$$\begin{aligned}0 &= (\nabla\eta(u^\varepsilon))^T u^\varepsilon_t + (\nabla\eta(u^\varepsilon))^T df(u^\varepsilon)u^\varepsilon_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\ &= \eta(u^\varepsilon)_t + \left(\nabla(v^\epsilon\eta(u^\varepsilon))\right)^T u^\varepsilon_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\ &= \eta(u^\varepsilon)_t + \left(v^\epsilon\eta(u^\varepsilon)\right)_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\ &\geq \eta(u^\varepsilon)_t + \left(v^\epsilon\eta(u^\varepsilon)\right)_x - \epsilon\eta(u^\varepsilon)_{xx}.\end{aligned}$$

By assuming that $u^\varepsilon \to u$ as $\epsilon \to 0$, we see that

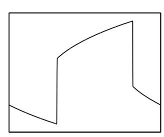
$$\eta_t + (v\eta)_x \leq 0$$

holds in the weak sense (cf. (2.15)). Hence we conclude that (5.174), that is,

$$(\rho S)_t + (v\rho S)_x \leq 0, \tag{5.182}$$

holds weakly.

[11] A word of caution: To show that $(\nabla\eta(u))^T df(u) = (\nabla(v\eta(u)))^T$ is strenuous. It is better done in nonconservative coordinates; see Exercise 5.12.



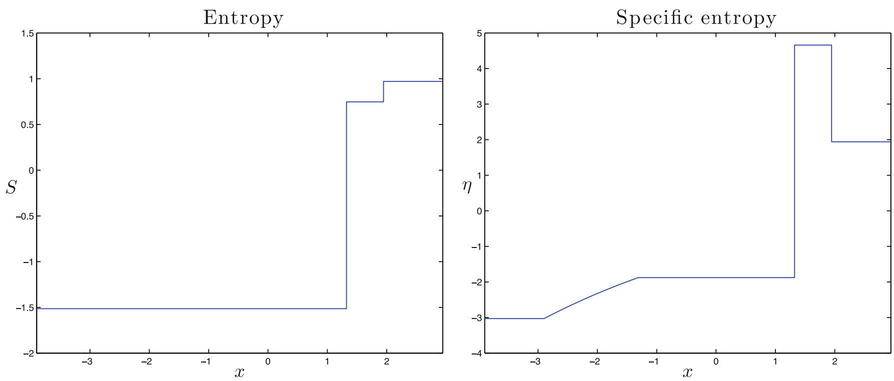


Fig. 5.14 The entropy and specific entropy for the solution of the Riemann problem (5.173)

In Fig. 5.14 we show the entropy and the specific entropy for the solution of Riemann problem (5.173). The entropy decreases as the shock and the contact discontinuity pass, while it is constant across the rarefaction wave.

Analogously to the shallow-water equations, we can also check whether (5.174) holds for the solution of the Riemann problem. We know that this will hold if and only if

$$-s\,[\![\rho S]\!] + [\![\rho v S]\!] \le 0.$$

Using the expression giving the shock speed, (5.163), we calculate

$$\begin{aligned}
-s\,[\![\rho S]\!] + [\![\rho v S]\!] &= S\left(-s\,[\![\rho]\!] + [\![\rho v]\!]\right) + \rho_l\left(-s\,[\![S]\!] + v_l\,[\![S]\!]\right) \\
&= \pm \rho_l c_l \sqrt{\frac{2}{\gamma-1}\frac{\beta\pi+1}{\beta^2-1}}\,[\![S]\!]\,,
\end{aligned}$$

where we use the plus sign for the first family and the minus sign for the second. Hence the entropy will decrease if and only if $[\![S]\!] < 0$ for the first family, and $[\![S]\!] > 0$ for the third family.

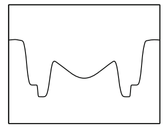
Note in passing that for the contact discontinuity, $s = v$, and thus

$$-s\,[\![\rho S]\!] + [\![\rho v S]\!] = -v\,[\![\rho S]\!] + v\,[\![\rho S]\!] = 0.$$

Therefore, as expected, entropy is conserved across a contact discontinuity.

We consider shocks of the first family, and view $[\![S]\!]$ as a function of $\pi = p/p_l$. Recall that for these shocks, we have $\pi > 1$. Thus

$$\begin{aligned}
[\![S]\!] &= S - S_l \\
&= \log\left(\frac{\rho^\gamma}{\rho_l^\gamma}\right) - \log\left(\frac{p}{p_l}\right) \\
&= \gamma\log(z) - \log(\pi) \\
&= \gamma\log\left(\frac{\beta\pi+1}{\pi+\beta}\right) - \log(\pi) \\
&=: h(\pi).
\end{aligned}$$



To check whether $h(\pi) < 0 = h(1)$, we differentiate, using (5.162):

$$\begin{aligned}
h'(\pi) &= \gamma \frac{\beta^2 - 1}{(\pi + \beta)(\beta\pi + 1)} - \frac{1}{\pi} \\
&= \frac{1}{\pi(\pi + \beta)(\beta\pi + 1)} \left(\gamma(\beta^2 - 1)\pi - (\pi + \beta)(\beta\pi + 1)\right) \\
&= \frac{1}{\pi(\pi + \beta)(\beta\pi + 1)} \left(\frac{\beta + 1}{\beta - 1}(\beta^2 - 1)\pi - (\pi + \beta)(\beta\pi + 1)\right) \\
&= \frac{\beta}{\pi(\pi + \beta)(\beta\pi + 1)}(2\pi - \pi^2 - 1) \\
&= -\frac{\beta}{\pi(\pi + \beta)(\beta\pi + 1)}(\pi - 1)^2 < 0.
\end{aligned}$$

Thus S is monotonically decreasing along the Hugoniot locus of the first family. We see also that (5.174) holds only if $p \geq p_l$ for waves of the first family.

For shocks of the third family, an identical computation shows that (5.174) holds only if $p \leq p_l$.

5.7 Notes

The fundamentals of the Riemann problem for systems of conservation laws were presented in the seminal paper by Lax [125], where also the Lax entropy condition was introduced. We refer to Smoller [169] as a general reference for this chapter. Our proof of Theorem 5.11 follows Schatzman [165]. This also simplifies the proof of the classical result that $s'(0) = \frac{1}{2}$ in Theorem 5.14. The parameterization of the Hugoniot locus introduced in Theorem 5.11 makes the proof of the smoothness of the wave curves, Theorem 5.16, quite simple.

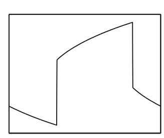
We have used shallow-water equations as our prime example in this chapter. This model can be found in many sources; a good presentation is in Kevorkian [112]. Our treatment of the vacuum for these equations can be found in Liu and Smoller [138].

There is extensive literature on the Euler equations; see, e.g., [51], [169], [167], and [42]. The computations on the Euler equations and entropy are taken from [85].

Our version of the implicit function theorem, Theorem 5.10, was taken from Cheney [40]. See Exercise 5.11 for a proof.

5.8 Exercises

5.1 In this exercise we consider the shallow-water equations in the case of a variable bottom. Make the same assumptions regarding the fluid as in Example 5.1 except that the bottom is given by the function $\bar{y} = \bar{b}(\bar{x}, \bar{t})$. Assume that the characteristic depth of the water is given by H and the characteristic depth



of the bottom is A. Let $\delta = A/H$. Show that the shallow-water equations read

$$\begin{aligned} h_t + (vh)_x &= 0, \\ (vh)_t + (v^2 h + \frac{1}{2} h^2 + \delta h b)_x &= 0. \end{aligned} \tag{5.183}$$

5.2 What assumption on p is necessary for the p-system to be hyperbolic?

5.3 Solve the Riemann problem for the p-system in the case $p(v) = 1/v$. For what left and right states does this Riemann problem have a solution?

5.4 Repeat Exercise 5.3 in the general case where $p = p(v)$ is such that p' is negative and p'' is positive.

5.5 Solve the following Riemann problem for the shallow-water equations:

$$u(x,0) = \begin{pmatrix} h(x,0) \\ v(x,0) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < 0, \\ \binom{h_r}{0} & \text{for } x \geq 0, \end{cases}$$

with $h_l > h_r > 0$.

5.6 Let $w = (u, v)$ and let $\varphi(w)$ be a smooth scalar function. Consider the system of conservation laws

$$w_t + (\varphi(w)w)_x = 0. \tag{5.184}$$

(a) Find the characteristic speeds λ_1 and λ_2 and the associated eigenvectors r_1 and r_2 for the system (5.184).

(b) Let $\varphi(w) = |w|^2/2$. Then find the solution of the Riemann problem for (5.184).

(c) Now let

$$\varphi(w) = \frac{1}{1 + u + v},$$

and assume that u and v are positive. Find the solution of the Riemann problem of (5.184) in this case.

5.7 Let us consider the Lax–Friedrichs scheme for systems of conservation laws. As in Chapt. 3 we write this as

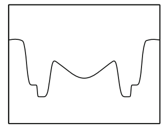
$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\lambda}{2}\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right),$$

where $\lambda = \Delta t/\Delta x$, and we assume that the CFL condition

$$\lambda \leq \max_k |\lambda_k|$$

holds. Let $v_j^n(x,t)$ denote the solution of the Riemann problem with initial data

$$\begin{cases} u_{j-1}^n & \text{for } x < j\Delta x, \\ u_{j+1}^n & \text{for } x \geq j\Delta x. \end{cases}$$



Show that

$$u_j^{n+1} = \frac{1}{2\Delta x} \int_{(j-1)\Delta x}^{(j+1)\Delta x} v_j^n(x, \Delta t)\, dx.$$

5.8 A smooth function $w : \mathbb{R}^n \to \mathbb{R}$ is called a k-*Riemann invariant* if

$$\nabla w(u) \cdot r_k(u) = 0,$$

where r_k is the kth right eigenvector of the Jacobian matrix df, which is assumed to be strictly hyperbolic.

(a) Show that locally there exist precisely $(n-1)$ k-Riemann invariants whose gradients are linearly independent.
(b) Let $R_k(u_l)$ denote the kth rarefaction curve through a point u_l. Then show that all $(n-1)$ k-Riemann invariants are constant on $R_k(u_l)$. This gives an alternative definition of the rarefaction curves.
(c) We say that we have a coordinate system of Riemann invariants if there exist n scalar-valued functions $w_1, \dots, w_n$ such that w_j is a k-Riemann invariant for $j, k = 1, \dots, n$, $j \neq k$, and

$$\nabla w_j(u) \cdot r_k(u) = \gamma_j(u)\delta_{j,k}, \tag{5.185}$$

for some nonzero function g_j. Why cannot we expect to find such a coordinate system if $n > 2$?
(d) Find the Riemann invariants for the shallow-water system, and verify parts **b** and **c** in this case.

5.9 We study the p-system with $p(v) = 1/v$ as in Exercise 5.3.

(a) Find the two Riemann invariants w_1 and w_2 in this case.
(b) Introduce coordinates

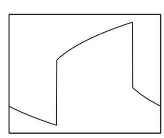
$$\mu = w_1(v, u) \quad \text{and} \quad \tau = w_2(v, u),$$

and find the wave curves in (μ, τ) coordinates.
(c) Find the solution of the Riemann problem in (μ, τ) coordinates.
(d) Show that the wave curves W_1 and W_2 are *stiff* in the sense that if a point (μ, τ) is on a wave curve through (μ_l, τ_l), then the point $(\mu + \Delta\mu, \tau + \Delta\tau)$ is on a wave curve through $(\mu_l + \Delta\mu, \tau_l + \Delta\tau)$. Hence the solution of the Riemann problem can be said to be translation-invariant in (μ, τ) coordinates.
(e) Show that the 2-shock curve through a point (μ_l, τ_l) is the reflection about the line $\mu - \mu_l = \tau - \tau_l$ of the 1-shock curve through (μ_l, τ_l).

5.10 As for scalar equations, we define an entropy/entropy flux pair (η, q) as scalar functions of u such that for smooth solutions,

$$u_t + f(u)_x = 0 \quad \Rightarrow \quad \eta_t + q_x = 0,$$

and η is supposed to be a convex function.



(a) Show that η and q are related by

$$\nabla_u q = \nabla_u \eta \, df. \tag{5.186}$$

(b) Why cannot we expect to find entropy/entropy flux pairs if $n > 2$?
(c) Find an entropy/entropy flux pair for the p-system if $p(v) = 1/v$.
(d) Find an entropy/entropy flux pair for the shallow-water equations.

5.11 This exercise outlines a proof of the implicit function theorem, Theorem 5.10.

(a) Define T to be a mapping $\mathbb{R}^p \to \mathbb{R}^p$ such that for y_1 and y_2,

$$|T(y_1) - T(y_2)| \le c\,|y_1 - y_2|\,, \quad \text{for some constant } c < 1.$$

Such mappings are called contractions. Show that there exists a unique y such that $T(y) = y$.

(b) Let $u : \mathbb{R}^p \to \mathbb{R}^p$, and assume that u is C^1 in some neighborhood of a point y_0, and that $du(y_0)$ is nonsingular. We are interested in solving the equation

$$u(y) = u(y_0) + v \tag{5.187}$$

for some v where $|v|$ is sufficiently small. Define

$$T(y) = y - du(y_0)^{-1}\left(u(y) - u(y_0) - v\right).$$

Show that T is a contraction in a neighborhood of y_0, and consequently that (5.187) has a unique solution $x = \varphi(v)$ for small v, and that $\varphi(0) = y_0$.

(c) Now let $\Phi(x, y)$ be as in Theorem 5.10. Show that for x close to x_0 we can find $\varphi(x, v)$ such that

$$\Phi(x, \varphi(x, v)) = \Phi(x, y_0) + v$$

for small v.

(d) Choose a suitable $v = v(x)$ to conclude the proof of the theorem.

5.12 Many calculations for the Euler equations become simpler in nonconservative variables. Introduce $w = (\rho, v, e)$, where

$$e = \frac{1}{\rho}\left(E - \frac{1}{2}\rho v^2\right)$$

is the internal specific energy.

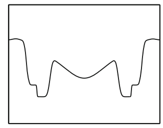
(a) Show that in these variables we have

$$p = (\gamma - 1)e\rho, \quad c^2 = \gamma(\gamma - 1)e. \tag{5.188}$$

(b) Show that w satisfies an equation of the form

$$w_t + A(w)w_x = 0, \tag{5.189}$$

and determine A.



(c) Compute the eigenvalues and eigenvectors for A and determine whether the wave families are linearly degenerate or genuinely nonlinear.

(d) Compute the Riemann invariants in these variables.

(e) Show that

$$\left(\frac{\partial \rho S}{\partial w}\right)^T A(w) = \left(\frac{\partial \rho v S}{\partial w}\right)^T, \tag{5.190}$$

where S denotes the entropy and is given by (5.155) or (5.177). Here

$$\left(\frac{\partial f}{\partial w}\right)^T = (f_\rho, f_v, f_e)$$

for any scalar function f.

5.13 Prove (5.165).