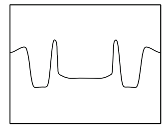


Chapter 4

Multidimensional Scalar Conservation Laws

Just send me the theorems, then I shall find the proofs.[1]
— Chrysippus told Cleanthes, 3rd century BC

Our analysis has so far been confined to scalar conservation laws in one dimension. Clearly, the multidimensional case is considerably more important. Luckily enough, the analysis in one dimension can be carried over to higher dimensions by essentially treating each dimension separately. This technique is called *dimensional splitting*. The final results are very much the natural generalizations one would expect.

The same splitting techniques of dividing complicated differential equations into several simpler parts can in fact be used to handle other problems. These methods are generally called *operator splitting methods* or *fractional steps methods*.

4.1 Dimensional Splitting Methods

We will show in this section how one can analyze scalar multidimensional conservation laws by dimensional splitting, which amounts to solving one space direction at a time. To be more concrete, let us consider the two-dimensional conservation law

$$u_t + f(u)_x + g(u)_y = 0, \qquad u(x, y, 0) = u_0(x, y). \tag{4.1}$$

If we let $S_t^{f,x} u_0$ denote the solution of

$$v_t + f(v)_x = 0, \qquad v(x, y, 0) = u_0(x, y)$$

(where y is a passive parameter), and similarly let $S_t^{g,y} u_0$ denote the solution of

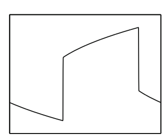
$$w_t + g(w)_y = 0, \qquad w(x, y, 0) = u_0(x, y)$$

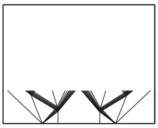
(x is a parameter), then the idea of dimensional splitting is to approximate the solution of (4.1) as follows:

$$u(x, y, n\Delta t) \approx \left[S_{\Delta t}^{g,y} \circ S_{\Delta t}^{f,x} \right]^n u_0. \tag{4.2}$$

[1] Lucky guy! Paraphrased from Diogenes Laertius, *Lives of Eminent Philosophers*, c. A.D. 200.

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_4





◇ Example 4.1 (A single discontinuity)
We first show how this works on a concrete example. Let

$$f(u) = g(u) = \frac{1}{2}u^2$$

and

$$u_0(x, y) = \begin{cases} u_l & \text{for } x < y, \\ u_r & \text{for } x \geq y, \end{cases}$$

with $u_r > u_l$. The solution in the x-direction for fixed y gives a rarefaction wave, the left and right parts moving with speeds u_l and u_r, respectively. With a quadratic flux, the rarefaction wave is a linear interpolation between the left and right states. Thus

$$u^{1/2} := S^{f,x}_{\Delta t} u_0 = \begin{cases} u_l & \text{for } x < y + u_l \Delta t, \\ (x - y)/\Delta t & \text{for } y + u_l \Delta t < x < y + u_r \Delta t, \\ u_r & \text{for } x > y + u_r \Delta t. \end{cases}$$

The solution in the y-direction for fixed x with initial state $u^{1/2}$ will exhibit a focusing of characteristics. The left state, which now equals u_r, will move with speed given by the derivative of the flux function, in this case u_r, and hence overtake the right state, given by u_l, which moves with smaller speed, namely u_l. The characteristics interact at a time t given by

$$u_r t + x - u_r \Delta t = u_l t + x - u_l \Delta t,$$

or $t = \Delta t$. At that time we are back to the original Riemann problem between states u_l and u_r at the point $x = y$. Thus

$$u^1 := S^{g,y}_{\Delta t} u^{1/2} = u_0.$$

Another application of $S^{f,x}_{\Delta t}$ will of course give

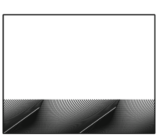
$$u^{3/2} := S^{f,x}_{\Delta t} u^1 = u^{1/2}.$$

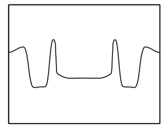
So we have that $u^n = u_0$ for all $n \in \mathbb{N}$. When we introduce coordinates

$$\xi = \frac{1}{\sqrt{2}}(x + y), \qquad \eta = \frac{1}{\sqrt{2}}(x - y),$$

the equation transforms into

$$u_t + \left(\frac{1}{\sqrt{2}} u^2\right)_\xi = 0, \quad u(\xi, \eta, 0) = \begin{cases} u_l & \text{for } \eta \leq 0, \\ u_r & \text{for } \eta > 0. \end{cases}$$





We see that $u(x, y, t) = u_0(x, y)$, and consequently $\lim_{\Delta t \to 0} u^n = u_0$ (where we keep $n\Delta t = t$ fixed). Thus the dimensional splitting procedure produces approximate solutions converging to the right solution in this case. ◇

We will state all results for the general case of arbitrary dimension, while proofs will be carried out in two dimensions only, to keep the notation simple. We first need to define precisely what is meant by a weak entropy solution of the initial value problem

$$u_t + \operatorname{div} f(u) = 0, \quad u|_{t=0} = u_0, \tag{4.3}$$

where $f = (f_1, \dots, f_m)$, and the spatial variables are denoted by $(x_1, \dots, x_m) \in \mathbb{R}^m$. Here we adopt the *Kružkov entropy condition* from Chapt. 2, and say that u is a (weak) Kružkov entropy solution of (4.3) for time $[0, T]$ if u is a bounded function that satisfies[2]

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^m} \Big(|u-k|\, \varphi_t + \operatorname{sign}(u-k) \sum_{j=1}^m \big(f_j(u) - f_j(k) \big) \varphi_{x_j} \Big)\, dx_1 \cdots dx_m\, dt \\ + \int_{\mathbb{R}^m} \Big(\varphi|_{t=0} \, |u_0 - k| - (|u-k|\,\varphi)|_{t=T} \Big)\, dx_1 \cdots dx_m \geq 0,\end{aligned} \tag{4.4}$$

for all constants $k \in \mathbb{R}$ and all nonnegative test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0, T])$. It certainly follows as in the one-dimensional case that u is a weak solution, i.e.,

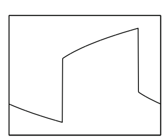
$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^m} \Big(u\varphi_t + f(u)\cdot\nabla\varphi \Big)\, dx_1 \cdots dx_m\, dt \\ + \int_{\mathbb{R}^m} \varphi|_{t=0} u_0 \, dx_1 \cdots dx_m = 0,\end{aligned} \tag{4.5}$$

for all test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0, \infty))$.

Our analysis aims at two different goals. We first show that the dimensional splitting indeed produces a sequence of functions that converges to a solution of the multidimensional equation (4.3). Our discussion will here be based on the more or less standard argument using Kolmogorov's compactness theorem. The argument is fairly short. In order to obtain stability in the multidimensional case in the sense of Theorem 2.14, we show that dimensional splitting preserves this stability. Furthermore, we show how one can use front tracking as our solution operator in one dimension in combination with dimensional splitting. Finally, we determine the appropriate convergence rate of this procedure. This analysis strongly uses Kuznetsov's theory from Sect. 3.3, but matters are more complicated and technical than in one dimension.

We shall now show that dimensional splitting produces a sequence that converges to the entropy solution u of (4.3); that is, the limit u should satisfy (4.4).

[2] If we want a solution for all time, we disregard the last term in (4.4) and integrate t over $[0, \infty)$.



As promised, our analysis will be carried out in the two-dimensional case only, i.e., for equation (4.1). Assume that u_0 is a function in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ (consult Definition A.2 for a definition of $BV(\mathbb{R}^2)$; see also (A.11)). Let $t_n = n\Delta t$ and $t_{n+1/2} = \left(n + \frac{1}{2}\right)\Delta t$. Define

$$u^0 = u_0, \qquad u^{n+1/2} = S^{f,x}_{\Delta t} u^n, \qquad u^{n+1} = S^{g,y}_{\Delta t} u^{n+1/2}, \tag{4.6}$$

for $n \in \mathbb{N}_0$. We shall also be needing an approximate solution for $t \neq t_n$. We want the approximation to be an exact solution to a one-dimensional conservation law in each interval $\left[t_j, t_{j+1/2}\right]$, $j = k/2$, and $k \in \mathbb{N}_0$. The way to do this is to make "time go twice as fast" in each such interval; i.e., let $u_{\Delta t}$ be defined by[3]

$$u_{\Delta t}(x,t) = \begin{cases} S^{f,x}_{2(t-t_n)} u^n & \text{for } t_n \le t \le t_{n+1/2}, \\ S^{g,y}_{2(t-t_{n+1/2})} u^{n+1/2} & \text{for } t_{n+1/2} \le t \le t_{n+1}. \end{cases} \tag{4.7}$$

We will use Theorem A.11, that is, we show that the sequence $\{u_{\Delta t}\}$ is compact. Since neither the operator $S^{f,x}$ nor $S^{g,y}$ increases the L^∞ norm, $u_{\Delta t}$ will be uniformly bounded, i.e.,

$$\|u_{\Delta t}\|_{L^\infty(\mathbb{R}^2)} \le \|u_0\|_{L^\infty(\mathbb{R}^2)} \tag{4.8}$$

independent of Δt.

Next we study the total variation. We start by considering

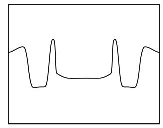
$$\begin{aligned} \int \mathrm{T.V.}_{\cdot y}\left(S^{f,x}_{\Delta t} u^n\right) dx &= \int \lim_{h\to 0} \frac{1}{h} \int \left|u^{n+1/2}(x, y+h) - u^{n+1/2}(x,y)\right| dy\, dx \\ &= \lim_{h\to 0} \frac{1}{h} \iint \left|u^{n+1/2}(x, y+h) - u^{n+1/2}(x,y)\right| dx\, dy \\ &\le \lim_{h\to 0} \frac{1}{h} \iint \left|u^{n}(x, y+h) - u^{n}(x,y)\right| dx\, dy \\ &= \int \lim_{h\to 0} \frac{1}{h} \int \left|u^{n}(x, y+h) - u^{n}(x,y)\right| dy\, dx \\ &= \int \mathrm{T.V.}_{\cdot y}\left(u^n\right) dx, \end{aligned} \tag{4.9}$$

where we used Lemma A.1 and the L^1-contractivity; cf. Theorem 2.15 (vi). The interchange of integrals and limits is justified using Lebesgue's dominated convergence theorem.

For the solution constructed from dimensional splitting we have

$$\begin{aligned} \mathrm{T.V.}_{x,y}\left(u^{n+1/2}\right) &= \int \mathrm{T.V.}_{x\cdot}\left(S^{f,x}_{\Delta t} u^n\right) dy + \int \mathrm{T.V.}_{\cdot y}\left(S^{f,x}_{\Delta t} u^n\right) dx \\ &\le \int \mathrm{T.V.}_{x\cdot}\left(u^n\right) dy + \int \mathrm{T.V.}_{\cdot y}\left(u^n\right) dx \\ &= \mathrm{T.V.}_{x,y}\left(u^n\right), \end{aligned} \tag{4.10}$$

[3] We will keep the ratio $\lambda = \Delta t/\Delta x$ fixed, and thus we index only with Δt.



using the TVD property of $S^{f,x}$ and (4.9). Similarly,

$$\mathrm{T.V.}_{x,y}\left(u^{n+1}\right) \le \mathrm{T.V.}_{x,y}\left(u^{n+1/2}\right),$$

and thus

$$\mathrm{T.V.}_{x,y}\left(u^{n}\right) \le \mathrm{T.V.}_{x,y}\left(u_0\right)$$

follows by induction. This extends to

$$\mathrm{T.V.}_{x,y}\left(u_{\Delta t}\right) \le \mathrm{T.V.}_{x,y}\left(u_0\right). \tag{4.11}$$

We now want to establish Lipschitz continuity in time of the L^1-norm, i.e.,

$$\|u_{\Delta t}(t) - u_{\Delta t}(s)\|_{L^1(\mathbb{R}^2)} \le C\,|t-s| \tag{4.12}$$

for some constant C. By repeated use of the triangle inequality it suffices to estimate

$$\begin{aligned}\|u_{\Delta t}(t_{n+1}) - u_{\Delta t}(t_n)\|_{L^1(\mathbb{R}^2)} &\le \left\|u^{n+1} - u^{n+1/2}\right\|_1 + \left\|u^{n+1/2} - u^n\right\|_{L^1(\mathbb{R}^2)} \\ &= \left\|S^{f,x}_{\Delta t}u^n - u^n\right\|_{L^1(\mathbb{R}^2)} \\ &\quad + \left\|S^{g,y}_{\Delta t}u^{n+1/2} - u^{n+1/2}\right\|_{L^1(\mathbb{R}^2)}.\end{aligned} \tag{4.13}$$

Using Theorem 2.15 (vi), we conclude that the first term in (4.13) is bounded by $\|f\|_{\mathrm{Lip}}\Delta t\ \mathrm{T.V.}_{x,y}\left(u^n\right)$. For the second term. we obtain, using in addition (4.9), the bound $\|g\|_{\mathrm{Lip}}\Delta t\ \mathrm{T.V.}_{x,y}\left(u^n\right)$. This proves

$$\|u_{\Delta t}(t_{n+1}) - u_{\Delta t}(t_n)\|_1 \le \Delta t \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right). \tag{4.14}$$

Using interpolation, we obtain the estimate

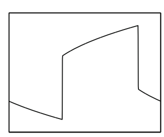
$$\begin{aligned}\|u_{\Delta t}(t) - u_{\Delta t}(s)\|_1 &\le \|u_{\Delta t}(t) - u_{\Delta t}(t_n)\|_1 \\ &\qquad + \|u_{\Delta t}(t_n) - u_{\Delta t}(t_m)\|_1 + \|u_{\Delta t}(s) - u_{\Delta t}(t_m)\|_1 \\ &\le \left(|t_n - t_m| + 2\Delta t\right)\max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right) \\ &\le \left(|t - s| + 4\Delta t\right)\max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right),\end{aligned} \tag{4.15}$$

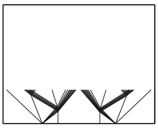
where $t \in [t_n, t_{n+1})$ and $s \in [t_m, t_{m+1})$.

Using Theorem A.11, we conclude the existence of a convergent subsequence, also labeled $\{u_{\Delta t}\}$, and set $u = \lim_{\Delta t \to 0} u_{\Delta t}$. Next we have to prove that the limit u is a weak entropy solution.

Let $\phi = \phi(x, y, t)$ be a nonnegative test function, and define φ by $\varphi(x, y, t) = \phi(x, y, \frac{1}{2}t + t_n)$. By defining $\tau = 2(t - n\Delta t)$, we have that for each y, the function $u_{\Delta t}$ is a weak solution in x on the strip $t \in [t_n, t_{n+1/2}]$ satisfying the inequality

$$\begin{aligned}&\int_0^{\Delta t}\int \left(|u_{\Delta t} - k|\,\varphi_\tau + q^f(u_{\Delta t}, k)\varphi_x\right)\,dx\,d\tau \\ &\qquad \ge \int \left|u^{n+1/2} - k\right|\varphi|_{t=\Delta t}\,dx - \int |u^n - k|\,\varphi|_{t=0}\,dx,\end{aligned} \tag{4.16}$$





for all constants k. Here $q^f(u,k) = \operatorname{sign}(u-k)\,(f(u)-f(k))$. Changing back to the t variable, we find that

$$2\int_{t_n}^{t_{n+1/2}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t + q^f(u_{\Delta t},k)\phi_x\right)dx\,dt \geq \int\left|u^{n+1/2}-k\right|\phi|_{t=t_{n+1/2}}\,dx - \int\left|u^n-k\right|\phi|_{t=t_n}\,dx. \tag{4.17}$$

Similarly,

$$2\int_{t_{n+1/2}}^{t_{n+1}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t + q^g(u_{\Delta t},k)\phi_y\right)dy\,dt \geq \int\left|u^{n+1}-k\right|\phi|_{t=t_{n+1}}\,dy - \int\left|u^{n+1/2}-k\right|\phi|_{t=t_{n+1/2}}\,dy. \tag{4.18}$$

Here q^g is defined similarly to q^f, using g instead of f. Integrating (4.17) over y and (4.18) over x and adding the two results and summing over n, we obtain

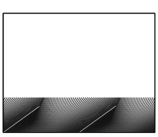
$$\begin{aligned}2\int_0^T\iint\bigg(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t &+ \sum_n \chi_n q^f(u_{\Delta t},k)\phi_x \\ &+ \sum_n \tilde{\chi}_n q^g(u_{\Delta t},k)\phi_y\bigg)\,dx\,dy\,dt \\ &\geq \iint\left(\left|u_{\Delta t}-k\right|\phi\right)|_{t=T}\,dx\,dy - \iint\left|u_0-k\right|\phi(0)\,dx\,dy,\end{aligned}$$

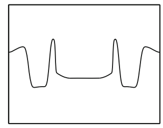
where χ_n and $\tilde{\chi}_n$ denote the characteristic functions of the strips $t_n \leq t \leq t_{n+1/2}$ and $t_{n+1/2} \leq t \leq t_{n+1}$, respectively. As Δt tends to zero, it follows that

$$\sum_n \chi_n \overset{*}{\rightharpoonup} \frac{1}{2}, \quad \sum_n \tilde{\chi}_n \overset{*}{\rightharpoonup} \frac{1}{2}.$$

Specifically, for continuous functions ψ we see that

$$\begin{aligned}\sum_n\int_0^T \chi_n\psi\,dt &= \sum_n\int_{t_n}^{t_{n+1/2}}\psi\,dt \\ &= \sum_n \psi(t_n^*)\frac{\Delta t}{2} \\ &= \frac{1}{2}\sum_n \psi(t_n^*)\Delta t \\ &\to \frac{1}{2}\int_0^T\psi\,dt \text{ as } \Delta t \to 0\end{aligned}$$





(where t_n^* is in $[t_n, t_{n+1/2}]$), by definition of the Riemann integral. The general case follows by approximation.

Letting $\Delta t \to 0$, we thus obtain

$$\int_0^T \iint \left(|u-k|\phi_t + q^f(u,k)\phi_x + q^g(u,k)\phi_y \right) dx\,dy\,dt \\ + \iint |u_0 - k|\,\phi|_{t=0}\,dx\,dy \ge \iint (|u-k|\,\phi)|_{t=T}\,dx\,dy,$$

which proves that $u(x,y,t)$ is a solution to (4.1) satisfying the Kružkov entropy condition.

Next, we want to prove uniqueness of solutions of multidimensional conservation laws. Let u and v be two Kružkov entropy solutions of the conservation law

$$u_t + f(u)_x + g(u)_y = 0 \tag{4.19}$$

with initial data u_0 and v_0, respectively. The argument in Sect. 2.4 leads, with no fundamental changes in the multidimensional case, to the same result (2.65), namely,

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^2)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^2)}, \tag{4.20}$$

thereby proving uniqueness. Using the fact that if every subsequence of a sequence has a further subsequence converging to the same limit, the whole sequence converges to that (unique) limit, we find that the whole sequence $\{u_{\Delta t}\}$ converges, not just a subsequence. We have proved the following result.

Theorem 4.2 *Let f_j be piecewise twice continuously differentiable functions, and furthermore, let u_0 be an integrable and bounded function in $BV(\mathbb{R}^m)$. Define the sequence of functions $\{u^n\}$ by $u^0 = u_0$ and*

$$u^{n+j/m} = S_{\Delta t}^{f_j,x_j} u^{n+(j-1)/m}, \quad j = 1,\dots,m, \quad n \in \mathbb{N}_0.$$

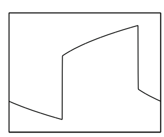
Introduce the function (where $t_r = r\Delta t$ for a rational number r)

$$u_{\Delta t}(x_1,\dots,x_m,t) = S_{m(t-t_{n+(j-1)/m})}^{f_j,x_j} u^{n+(j-1)/m},$$

for $t \in [t_{n+(j-1)/m}, t_{n+j/m}]$. Fix $T > 0$. Then for every sequence $\{\Delta t\}$ such that $\Delta t \to 0$, for all $t \in [0,T]$ the function $u_{\Delta t}(t)$ converges to the unique weak solution $u(t)$ of (4.3) satisfying the Kružkov entropy condition (4.4). The limit is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$.

To prove stability of the solution with respect to flux functions, we will show that the one-dimensional stability result (2.80) in Sect. 2.4 remains valid with obvious modifications in several dimensions. Let u and v denote the unique solutions of

$$u_t + f(u)_x + g(u)_y = 0, \quad u|_{t=0} = u_0,$$



and

$$v_t + \tilde{f}(v)_x + \tilde{g}(v)_y = 0, \quad v|_{t=0} = v_0,$$

respectively, that satisfy the Kružkov entropy condition. We want to estimate the L^1-norm of the difference between the two solutions. To this end, we first consider

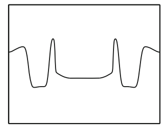
$$\begin{aligned}
\left\| u^{n+1/2} - v^{n+1/2} \right\|_{L^1(\mathbb{R}^2)} &= \iint \left| u^{n+1/2} - v^{n+1/2} \right| \, dx \, dy \\
&\le \int \Big(\int |u^n - v^n| \, dx \\
&\qquad + \Delta t \min\{\mathrm{T.V.}_{x}\,(u^n)\,, \mathrm{T.V.}_{x}\,(v^n)\} \| f - \tilde{f} \|_{\mathrm{Lip}} \Big) \, dy \\
&= \| u^n - v^n \|_{L^1(\mathbb{R}^2)} \\
&\quad + \Delta t \| f - \tilde{f} \|_{\mathrm{Lip}} \int \min\{\mathrm{T.V.}_{x}\,(u^n)\,, \mathrm{T.V.}_{x}\,(v^n)\} \, dy.
\end{aligned}$$

Next we employ the trivial, but useful, inequality

$$a \wedge b + c \wedge d \le (a + c) \wedge (b + d), \quad a, b, c, d \in \mathbb{R}.$$

Thus

$$\begin{aligned}
&\left\| u^{n+1} - v^{n+1} \right\|_{L^1(\mathbb{R}^2)} = \iint \left| u^{n+1} - v^{n+1} \right| \, dx \, dy \\
&\quad \le \int \Big(\int \left| u^{n+1/2} - v^{n+1/2} \right| \, dy \\
&\qquad\qquad + \Delta t \min\left\{ \mathrm{T.V.}_{y} \left(u^{n+1/2} \right), \mathrm{T.V.}_{y} \left(v^{n+1/2} \right) \right\} \| g - \tilde{g} \|_{\mathrm{Lip}} \Big) \, dx \\
&\quad \le \left\| u^{n+1/2} - v^{n+1/2} \right\|_{L^1(\mathbb{R}^2)} \\
&\qquad + \Delta t \| g - \tilde{g} \|_{\mathrm{Lip}} \int \min\left\{ \mathrm{T.V.}_{y} \left(u^{n+1/2} \right), \mathrm{T.V.}_{y} \left(v^{n+1/2} \right) \right\} \, dx \\
&\quad \le \| u^n - v^n \|_{L^1(\mathbb{R}^2)} + \Delta t \max\left\{ \| f - \tilde{f} \|_{\mathrm{Lip}}, \| g - \tilde{g} \|_{\mathrm{Lip}} \right\} \\
&\qquad \times \Bigg(\min\left\{ \int \mathrm{T.V.}_{x}\,(u^n) \, dy, \int \mathrm{T.V.}_{x}\,(v^n) \, dy \right\} \\
&\qquad\qquad + \min\left\{ \int \mathrm{T.V.}_{y}\,(u^n) \, dx, \int \mathrm{T.V.}_{y}\,(v^n) \, dx \right\} \Bigg) \\
&\quad \le \| u^n - v^n \|_{L^1(\mathbb{R}^2)} \\
&\qquad + \Delta t \max\{ \| f - \tilde{f} \|_{\mathrm{Lip}}, \| g - \tilde{g} \|_{\mathrm{Lip}} \} \\
&\qquad\quad \times \min \begin{Bmatrix} \int \mathrm{T.V.}_{x}\,(u^n) \, dy + \int \mathrm{T.V.}_{y}\,(u^n) \, dx, \\ \int \mathrm{T.V.}_{x}\,(v^n) \, dy + \int \mathrm{T.V.}_{y}\,(v^n) \, dx \end{Bmatrix} \\
&\quad = \| u^n - v^n \|_{L^1(\mathbb{R}^2)} \\
&\qquad + \Delta t \max\{ \| f - \tilde{f} \|_{\mathrm{Lip}}, \| g - \tilde{g} \|_{\mathrm{Lip}} \} \min\left\{ \mathrm{T.V.}\,(u^n)\,, \mathrm{T.V.}\,(v^n) \right\},
\end{aligned}$$



which implies

$$\|u^n - v^n\|_{L^1(\mathbb{R}^2)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} + n\Delta t \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \min\{\mathrm{T.V.}(u_0), \mathrm{T.V.}(v_0)\}. \tag{4.21}$$

Consider next $t \in [t_n, t_{n+1/2})$. Then the continuous interpolants defined by (4.7) satisfy

$$\begin{aligned}
\|u_{\Delta t}(t) - v_{\Delta t}(t)\|_{L^1(\mathbb{R}^2)} &= \left\| S^{f,x}_{2(t-t_n)} u^n - S^{\tilde{f},x}_{2(t-t_n)} v^n \right\|_{L^1(\mathbb{R}^2)} \\
&\leq \int \Big[\int |u^n - v^n| \, dx \\
&\qquad + 2(t - t_n) \min\{\mathrm{T.V.}_{\cdot x}(u^n), \mathrm{T.V.}_{\cdot x}(v^n)\} \|f - \tilde{f}\|_{\mathrm{Lip}} \Big] dy \\
&= \|u^n - v^n\|_{L^1(\mathbb{R}^2)} \\
&\quad + 2(t - t_n)\|f - \tilde{f}\|_{\mathrm{Lip}} \int \min\{\mathrm{T.V.}_{\cdot x}(u^n), \mathrm{T.V.}_{\cdot x}(v^n)\} \, dy \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} \\
&\quad + t_n \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \min\{\mathrm{T.V.}(u_0), \mathrm{T.V.}(v_0)\} \\
&\quad + 2(t - t_n) \min\{\mathrm{T.V.}(u_0), \mathrm{T.V.}(v_0)\} \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} \\
&\quad + (t + \Delta t) \min\{\mathrm{T.V.}(u_0), \mathrm{T.V.}(v_0)\} \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\}.
\end{aligned} \tag{4.22}$$

Observe that the above argument also holds mutatis mutandis in the general case of a scalar conservation law in any dimension. We summarize our results in the following theorem.

Theorem 4.3 *Let u_0 be in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$, and let f_j be piecewise twice continuously differentiable functions for $j = 1, \ldots, m$, and set $f = (f_1, \ldots, f_m)$. Then there exists a unique solution $u = u(x_1, \ldots, x_m, t)$ of the initial value problem*

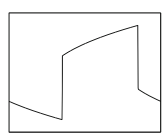
$$u_t + \operatorname{div} f(u) = 0, \quad u|_{t=0} = u_0, \tag{4.23}$$

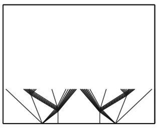
that satisfies the Kružkov entropy condition (4.4). The solution satisfies

$$\begin{aligned}
\|u(t)\|_{L^\infty(\mathbb{R}^m)} &\leq \|u_0\|_{L^\infty(\mathbb{R}^m)}, \\
\mathrm{T.V.}(u(t)) &\leq \mathrm{T.V.}(u_0), \\
\|u(t) - u(s)\|_{L^1(\mathbb{R}^m)} &\leq |t - s| \max_j \{ \|f_j\|_{\mathrm{Lip}} \} \mathrm{T.V.}(u_0).
\end{aligned} \tag{4.24}$$

Furthermore, if v_0 and g share the same properties as u_0 and f, respectively, then the unique weak Kružkov entropy solution of

$$v_t + \operatorname{div} g(v) = 0, \quad v|_{t=0} = v_0, \tag{4.25}$$





satisfies

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^m)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^m)} + t \min\{\text{T.V.}(u_0), \text{T.V.}(v_0)\} \max_j\{ \| f_j - g_j \|_{\text{Lip}} \}. \tag{4.26}$$

If $u_0 \le v_0$ *and* $f = g$*, then also* $u \le v$ *on all of* $\mathbb{R}^m \times [0, \infty)$.

Proof The proof of the Lipschitz continuity in time follows from (4.15). The monotonicity statement at the end follows using the L^1-contractivity (the special case of (4.26) with $f = g$) as in the one-dimensional case by employing the Crandall–Tartar lemma. □

(See also Exercise 4.1.)

4.2 Dimensional Splitting and Front Tracking

It doesn't matter if the cat is black or white. As long as it catches rats, it's a good cat.
— Deng Xiaoping (1904–1997)

In this section we will study the case in which we use front tracking to solve the one-dimensional conservation laws. More precisely, we replace the flux functions f and g (in the two-dimensional case) by piecewise linear continuous interpolations f_δ and g_δ, with the interpolation points spaced a distance δ apart. The aim is to determine the convergence rate toward the solution of the full two-dimensional conservation law as $\delta \to 0$ and $\Delta t \to 0$.

With the front-tracking approximation, the one-dimensional solutions will be piecewise constant if the initial condition is piecewise constant. In order to prevent the number of discontinuities from growing without bound, we will project the one-dimensional solution $S^{f_\delta,x}u$ onto a fixed grid in the (x, y)-plane before applying the operator $S^{g_\delta,y}$.

To be more concrete, let the grid spacing in the x- and y- directions be given by Δx and Δy, respectively, and let I_{ij} denote the grid cell

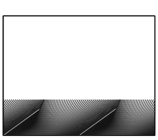
$$I_{ij} = [x_i, x_{i+1}) \times [y_j, y_{j+1}).$$

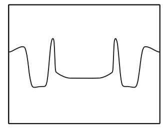
The projection operator π is defined by

$$\pi u(x, y) = \frac{1}{\Delta x \Delta y} \iint_{I_{ij}} u \, dx \, dy \text{ for } (x, y) \in I_{ij}.$$

Let the approximate solution at the discrete times t_l be defined as

$$u^{n+1/2} = \pi \circ S^{f_\delta,x}_{\Delta t} u^n \text{ and } u^{n+1} = \pi \circ S^{g_\delta,y}_{\Delta t} u^{n+1/2},$$





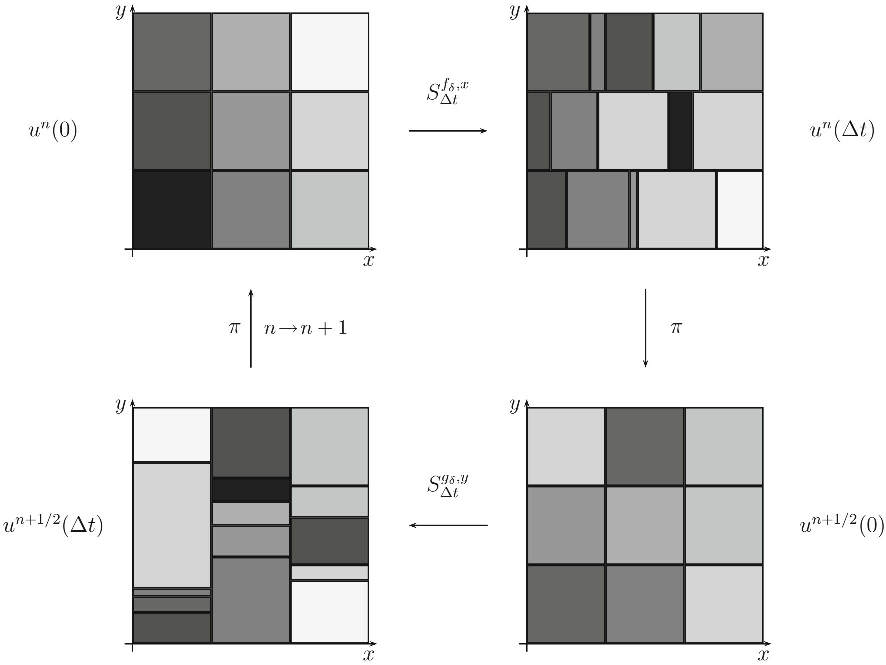


Fig. 4.1 Front tracking and dimensional splitting on a 3×3 grid

for $n = 0, 1, 2, \ldots$, with $u^0 = \pi u_0$. We collect the discretization parameters in $\eta = (\delta, \Delta x, \Delta y, \Delta t)$. In analogy to (4.7), we define u_η as

$$u_\eta(t) = \begin{cases} S^{f_\delta,x}_{2(t-t_n)} u^n & \text{for } t_n \le t < t_{n+1/2}, \\ u^{n+1/2} & \text{for } t = t_{n+1/2}, \\ S^{g_\delta,y}_{2(t-t_{n+1/2})} u^{n+1/2} & \text{for } t_{n+1/2} \le t < t_{n+1}, \\ u^{n+1} & \text{for } t = t_{n+1}. \end{cases} \tag{4.27}$$

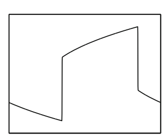
In Fig. 4.1 we illustrate how this works. Starting in the upper left corner, the operator $S^{f_\delta,x}_{\Delta t}$ takes us to the upper right corner; then we apply π and move to the lower right corner. Next, $S^{g_\delta,y}_{\Delta t}$ takes us to the lower left corner, and finally π takes us back to the upper left corner, this time with n incremented by 1.

To prove that u_η converges to the unique solution u as $\eta \to 0$, we essentially mimic the approach we just used to prove Theorem 4.2. First of all we observe that

$$\left\| u_\eta(t) \right\|_{L^\infty(\mathbb{R}^2)} \le \left\| u^0 \right\|_{L^\infty(\mathbb{R}^2)}, \tag{4.28}$$

since $S^{f_\delta,x}$, $S^{g_\delta,y}$, and π all obey a maximum principle. On each rectangle I_{ij} the function u_η is constant for $t = \Delta t$. In a desperate attempt to simplify the notation, we write

$$u^n_{ij} = u_\eta(x, y, n\Delta t) \text{ for } (x, y) \in I_{ij}.$$



Next we go carefully through one full time step in this construction, starting with u_{ij}^n. At each step we define a shorthand notation that we will use in the estimates. When we consider u_{ij}^n as a function of x only, we write

$$u_j^n(0) = u_{ij}^n = u_\eta(\,\cdot\,, j\Delta y, n\Delta t).$$

(The argument "0" on the left-hand side indicates the start of the time variable before we advance time an interval Δt using $S_{\Delta t}^{f_\delta,x}$.) Advancing the solution in time by Δt by applying front tracking in the x-variable produces

$$u_j^n(\Delta t) = \left(S_{\Delta t}^{f_\delta,x} u_j^n\right)(x).$$

(The x-dependence is suppressed in the notation on the left-hand side.) We now apply the projection π, which yields

$$u_{ij}^{n+1/2} = \pi u_j^n(\Delta t).$$

After this sweep in the x-variable, it is time to do the y-direction. Considering $u_{ij}^{n+1/2}$ as a function of y, we write

$$u_i^{n+1/2}(0) = u_{ij}^{n+1/2} = u_\eta\Big(i\Delta x, \cdot, \Big(n + \frac{1}{2}\Big)\Delta t\Big),$$

to which we apply the front-tracking solution operator in the y-direction

$$u_i^{n+1/2}(\Delta t) = \left(S_{\Delta t}^{g_\delta,y} u_i^{n+1/2}\right)(y).$$

(The y-dependence is suppressed in the notation on the left-hand side.) One full time step is completed by a final projection

$$u_{ij}^{n+1} = \pi u_i^{n+1/2}(\Delta t).$$

Using this notation, we first want to prove that the total variation is bounded in the sense that

$$\text{T.V.}\,(u^n) \le \text{T.V.}\,(u_0)\,. \tag{4.29}$$

We will show that

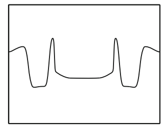
$$\text{T.V.}\left(u^{n+1/2}\right) \le \text{T.V.}\,(u^n)\,; \tag{4.30}$$

an analogous argument gives $\text{T.V.}\left(u^{n+1}\right) \le \text{T.V.}\left(u^{n+1/2}\right)$, from which we conclude that

$$\text{T.V.}\left(u^{n+1}\right) \le \text{T.V.}\,(u^n)\,,$$

and (4.29) follows by induction. By definition,

$$\text{T.V.}\left(u^{n+1/2}\right) = \sum_{i,j}\left(\left|u_{i+1,j}^{n+1/2} - u_{i,j}^{n+1/2}\right|\Delta y + \left|u_{i,j+1}^{n+1/2} - u_{i,j}^{n+1/2}\right|\Delta x\right), \tag{4.31}$$



while

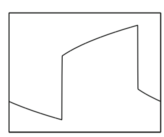
$$\mathrm{T.V.}\,(u^n) = \sum_{i,j} \left(\left| u^n_{i+1,j} - u^n_{i,j} \right| \Delta y + \left| u^n_{i,j+1} - u^n_{i,j} \right| \Delta x \right). \tag{4.32}$$

We first consider

$$\begin{aligned} \sum_i \left| u^{n+1/2}_{i+1,j} - u^{n+1/2}_{i,j} \right| &= \mathrm{T.V.}_{\cdot x} \left(\pi u^n_j(\Delta t) \right) \\ &\le \mathrm{T.V.}_{\cdot x} \left(u^n_j(\Delta t) \right) \le \mathrm{T.V.}_{\cdot x} \left(u^n_j(0) \right) \\ &= \sum_i \left| u^n_{i+1,j} - u^n_{i,j} \right|, \end{aligned} \tag{4.33}$$

where we first used that $\mathrm{T.V.}_{\cdot x}\,(\pi\phi) \le \mathrm{T.V.}_{\cdot x}\,(\phi)$ for step functions ϕ. This follows from the following argument: Let ϕ_c be a continuous function equal to ϕ except close to each jump, where we use a linear interpolation. Then $\mathrm{T.V.}_{\cdot x}\,(\phi) = \mathrm{T.V.}_{\cdot x}\,(\phi_c) \ge \mathrm{T.V.}_{\cdot x}\,(\pi\phi)$, since $\pi\phi$ is just a particular partition of ϕ_c; cf. (A.1). Subsequently we used that $\mathrm{T.V.}\,(v) \le \mathrm{T.V.}\,(v_0)$ for solutions v of one-dimensional conservation laws with initial data v_0. For the second term in the definition of $\mathrm{T.V.}\,\left(u^{n+1/2}\right)$ we obtain (cf. (4.10))

$$\begin{aligned} \sum_{i,j} \left| u^{n+1/2}_{i,j+1} - u^{n+1/2}_{i,j} \right| \Delta x \Delta y &= \sum_{i,j} \int_{I_{ij}} \left| u^{n+1/2}_{i,j+1} - u^{n+1/2}_{i,j} \right| \, dx\, dy \\ &= \sum_{i,j} \int_{I_{ij}} \left| \pi \left(u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right) \right| \, dx\, dy \\ &\le \sum_{i,j} \int_{I_{ij}} \pi \left(\left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \right) \, dx\, dy \\ &= \sum_{i,j} \int_{I_{ij}} \left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \, dx\, dy \\ &= \sum_{i,j} \Delta y \int_{i\Delta x}^{(i+1)\Delta x} \left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \, dx \\ &= \sum_{j} \Delta y \int_{\mathbb{R}} \left| u^n_{j+1}(x,\Delta t) - u^n_j(x,\Delta t) \right| \, dx \\ &\le \sum_{j} \Delta y \int_{\mathbb{R}} \left| u^n_{j+1}(x,0) - u^n_j(x,0) \right| \, dx \\ &= \sum_{i,j} \left| u^n_{i,j+1} - u^n_{i,j} \right| \Delta x \Delta y. \end{aligned} \tag{4.34}$$



The first inequality follows from $|\pi\phi| \leq \pi\,|\phi|$; thereafter, we use $\int_{I_{ij}} \pi\phi = \int_{I_{ij}} \phi$, and finally we use the L^1-contractivity, $\|v - w\|_{L^1(\mathbb{R})} \leq \|v_0 - w_0\|_{L^1(\mathbb{R})}$, of solutions of one-dimensional conservation laws. Multiplying (4.33) by Δy, summing over j, dividing (4.34) by Δx, and finally adding the results gives (4.30).

Finally, we want to show the analogue of Lipschitz continuity in time of the spatial L^1-norm as expressed in (4.12). We want to prove the following result:

$$\begin{aligned}\left\|u_\eta(t_m) - u_\eta(t_n)\right\|_{L^1(\mathbb{R}^2)} &= \sum_{i,j} \left|u_{ij}^m - u_{ij}^n\right| \Delta x \Delta y \\ &\leq \Big(\max\{\,\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\Delta t + 2(\Delta x + \Delta y)\Big) \\ &\quad \times \mathrm{T.V.}\left(u^0\right) |m - n|\,. \end{aligned} \tag{4.35}$$

To prove (4.35), it suffices to show that

$$\sum_{i,j} \left|u_{ij}^{n+1} - u_{ij}^n\right| \Delta x \Delta y \leq \left(\max\{\,\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\Delta t + 2(\Delta x + \Delta y)\right)\mathrm{T.V.}\left(u^0\right). \tag{4.36}$$

We start by writing

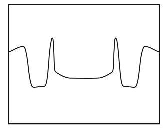
$$\begin{aligned}\left|u_{ij}^{n+1} - u_{ij}^n\right| &\leq \left|u_{ij}^{n+1} - u_i^{n+1/2}(\Delta t)\right| + \left|u_{ij}^{n+1/2} - u_j^n(\Delta t)\right| \\ &\quad + \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right| + \left|u_j^n(\Delta t) - u_j^n(0)\right| \\ &= \left|\pi u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(\Delta t)\right| + \left|\pi u_j^n(\Delta t) - u_j^n(\Delta t)\right| \\ &\quad + \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right| + \left|u_j^n(\Delta t) - u_j^n(0)\right|. \end{aligned}$$

Integrating this inequality over $\mathbb{R}^2$ gives

$$\begin{aligned}\sum_{i,j} \left|u_{ij}^{n+1} - u_{ij}^n\right| \Delta x \Delta y &\leq \iint \left|\pi u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(\Delta t)\right|\, dx\, dy \\ &\quad + \iint \left|\pi u_j^n(\Delta t) - u_j^n(\Delta t)\right|\, dx\, dy \\ &\quad + \iint \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right|\, dx\, dy \\ &\quad + \iint \left|u_j^n(\Delta t) - u_j^n(0)\right|\, dx\, dy. \end{aligned} \tag{4.37}$$

We see that two terms involve the projection operator π. For these terms we prove the estimate

$$\iint |\pi\psi - \psi|\; dx\, dy \leq (\Delta x + \Delta y)\,\mathrm{T.V.}\,(\psi)\,. \tag{4.38}$$



We will prove (4.38) in the one-dimensional case only (See Exercise 4.3). Consider (where $I_i = [x_i, x_{i+1})$)

$$\begin{aligned}
\int |\pi\psi - \psi|\,dx &= \sum_i \int_{I_i} |\pi\psi(x) - \psi(x)|\,dx \\
&= \sum_i \int_{I_i} \left| \frac{1}{\Delta x} \int_{I_i} \psi(y)\,dy - \psi(x) \right| dx \\
&= \frac{1}{\Delta x} \sum_i \int_{I_i} \left| \int_{I_i} (\psi(y) - \psi(x))\,dy \right| dx \\
&\le \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{I_i} |\psi(y) - \psi(x)|\,dy\,dx \\
&= \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{-x+I_i} |\psi(x+\xi) - \psi(x)|\,d\xi\,dx \\
&\le \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{-\Delta x}^{\Delta x} |\psi(x+\xi) - \psi(x)|\,d\xi\,dx \\
&= \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} \int_{\mathbb{R}} |\psi(x+\xi) - \psi(x)|\,dx\,d\xi \\
&\le \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} |\xi| \mathrm{T.V.}\,(\psi)\; d\xi \\
&= \Delta x\, \mathrm{T.V.}\,(\psi)\,. \qquad (4.39)
\end{aligned}$$

For the two remaining terms in (4.37) we obtain, using the Lipschitz continuity in time in the L^1 norm in the x-variable (see Theorem 2.15), that

$$\begin{aligned}
\iint \left| u_j^n(\Delta t) - u_j^n(0) \right| dx\,dy &\le \Delta t\, \|f_\delta\|_{\mathrm{Lip}} \int \mathrm{T.V.}_{\cdot x} \left(u_j^n(0) \right) dy \\
&\le \Delta t\, \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}\,(u^n)\,. \qquad (4.40)
\end{aligned}$$

Combining this result with (4.29), (4.38), we conclude that (4.36), and hence also (4.35), holds.

So far we have obtained the following estimates:

(i) Uniform boundedness,

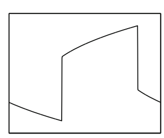
$$\left\| u_\eta(t) \right\|_{L^\infty(\mathbb{R}^2)} \le \left\| u^0 \right\|_{L^\infty(\mathbb{R}^2)}.$$

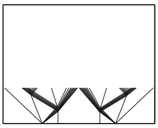
(ii) Uniform bound on the total variation,

$$\mathrm{T.V.}\,(u^n) \le \mathrm{T.V.}\,(u_0)\,.$$

(iii) Lipschitz continuity in time,

$$\begin{aligned}
\left\| u_\eta(t_m) - u_\eta(t_n) \right\|_{L^1(\mathbb{R}^2)} \le &\left(\max\{ \|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}} \} + 2\frac{\Delta x + \Delta y}{\Delta t} \right) \\
&\times \mathrm{T.V.}\left(u^0 \right) |t_m - t_n|\,. \qquad (4.41)
\end{aligned}$$





From Theorem A.11 we conclude that the sequence $\{u_\eta\}$ has a convergent subsequence as $\eta \to 0$, provided that the ratio $\max\{\Delta x, \Delta y\}/\Delta t$ remains bounded. We let u denote its limit. Furthermore, this sequence converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^2))$ for every positive T.

It remains to prove that the limit is indeed an entropy solution of the full two-dimensional conservation law. We first use that $u_j^n(x,t)$ (suppressing the y-dependence) is a solution of the one-dimensional conservation law in the time interval $[t_n, t_{n+1/2}]$. Hence we know that

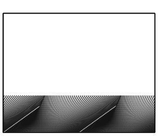
$$\begin{aligned}\int_{\mathbb{R}} \int_{t_n}^{t_{n+1/2}} \left(\frac{1}{2}\left|u_j^n(x,t) - k\right| \phi_t + q^{f_\delta}(u_j^n(x,t),k)\phi_x\right) dt\, dx \\ - \frac{1}{2}\int_{\mathbb{R}} \left|u_j^n(x, t_{n+1/2}-) - k\right| \phi(x, t_{n+1/2})\, dx \\ + \frac{1}{2}\int_{\mathbb{R}} \left|u_j^n(x, t_n+) - k\right| \phi(x, t_n)\, dx \ge 0.\end{aligned}$$

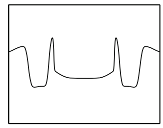
Similarly, we obtain for the y-direction

$$\begin{aligned}\int_{\mathbb{R}} \int_{t_{n+1/2}}^{t_{n+1}} \left(\frac{1}{2}\left|u_i^{n+1/2}(y,t) - k\right| \phi_t + q^{g_\delta}(u_i^{n+1/2}(y,t),k)\phi_y\right) dt\, dy \\ - \frac{1}{2}\int_{\mathbb{R}} \left|u_i^{n+1/2}(y, t_{n+1}-) - k\right| \phi(y, t_{n+1})\, dy \\ + \frac{1}{2}\int_{\mathbb{R}} \left|u_i^{n+1/2}(y, t_{n+1/2}+) - k\right| \phi(y, t_{n+1/2})\, dy \ge 0.\end{aligned}$$

Integrating the first inequality over y and the second over x and adding the results as well as adding over n gives, where $T = N\Delta t$,

$$\begin{aligned}&\iint_{\mathbb{R}^2} \int_0^T \Big(\frac{1}{2}\left|u_\eta - k\right| \phi_t + \sum_n \chi_n q^{f_\delta}(u_\eta,k)\phi_x + \sum_n \tilde{\chi}_n q^{g_\delta}(u_\eta,k)\phi_y\Big)\, dx\, dy\, dt \\ &\qquad - \frac{1}{2}\Big(\iint_{\mathbb{R}^2} \left|u_\eta(x,y,T) - k\right| \phi(x,y,T)\, dx\, dy \\ &\qquad\qquad - \iint_{\mathbb{R}^2} \left|u_\eta(x,y,0) - k\right| \phi(x,y,0)\, dx\, dy\Big) \\ &\ge -\frac{1}{2}\sum_{n=1}^{2N-1} \iint_{\mathbb{R}^2} \Big(\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right|\Big)\phi(x,y,t_{n/2})\, dx\, dy \\ &=: -\frac{1}{2}\sum_{n=1}^{2N-1} I_n,\end{aligned}$$





and as before, χ_n and $\tilde{\chi}_n$ denote the characteristic functions on $\{(x,y,t) \mid t \in [t_n, t_{n+1/2}]\}$ and $\{(x,y,t) \mid t \in [t_{n+1/2}, t_{n+1}]\}$, respectively. Observe that we have obtained the right-hand side by using a projection at each time step. As $n \to \infty$ and $\Delta t \to 0$ while keeping T fixed, we have that $\sum_n \chi_n \overset{*}{\rightharpoonup} \frac{1}{2}$. To estimate the right-hand side we first observe that

$$u_\eta(x,y,t_{n/2}+) - k = \pi\big(u_\eta(x,y,t_{n/2}-) - k\big),$$

and since the absolute value function is convex, Jensen's inequality implies that

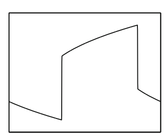
$$\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right| \le 0. \tag{4.42}$$

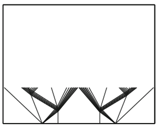
Thus we obtain

$$\begin{aligned}
I_n &= -\iint\limits_{\mathbb{R}^2} \Big(\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right|\Big)\phi(x,y,t_{n/2})\,dx\,dy \\
&= -\sum_{i,j}\iint\limits_{I_{i,j}} \Big(\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right|\Big)\phi(x_i,y_j,t_{n/2})\,dx\,dy \\
&\quad -\sum_{i,j}\iint\limits_{I_{i,j}} \Big(\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right|\Big) \\
&\qquad\qquad \times \big(\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\big)\,dx\,dy \\
&\ge -\sum_{i,j}\iint\limits_{I_{i,j}} \Big(\left|u_\eta(x,y,t_{n/2}+) - k\right| - \left|u_\eta(x,y,t_{n/2}-) - k\right|\Big) \\
&\qquad\qquad \times \big(\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\big)\,dx\,dy \\
&= \tilde{I}_n,
\end{aligned}$$

using (4.42). This implies

$$\begin{aligned}
\left|\tilde{I}_n\right| &\le \sum_{i,j}\iint\limits_{I_{i,j}} \left|u_\eta(x,y,t_{n/2}+) - u_\eta(x,y,t_{n/2}-)\right| \\
&\qquad\qquad \times \left|\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\right|\,dx\,dy \\
&\le (\Delta x + \Delta y)\,\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} \\
&\qquad\qquad \times \sum_{i,j}\iint\limits_{I_{i,j}} \left|u_\eta(x,y,t_{n/2}+) - u_\eta(x,y,t_{n/2}-)\right|\,dx\,dy \\
&\le (\Delta x + \Delta y)\iint\limits_{\mathbb{R}^2} \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}\left|\pi u_\eta(x,y,t_{n/2}-) - u_\eta(x,y,t_{n/2}-)\right|\,dx\,dy \\
&\le (\Delta x + \Delta y)^2\,\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}\,\mathrm{T.V.}\,(u_0)\,,
\end{aligned}$$





since

$$\left|\phi(x,y)-\phi(x_i,y_j)\right| \le \left|(x-x_i,y-y_j)\right| \int_0^1 \left|\nabla\phi(r(x-x_i,y-y_j))\right|\,dr$$
$$\le \left(\Delta x+\Delta y\right)\left\|\nabla\phi\right\|_{L^\infty(\mathbb{R}^2)}, \quad (x,y)\in I_{i,j},$$

where we have used (4.38). Thus

$$\sum_{n=1}^{2N}\left|\tilde{I}_n\right| \le \frac{(\Delta x+\Delta y)^2}{\Delta t}\left\|\nabla\phi\right\|_{L^\infty(\mathbb{R}^2)} \mathrm{T.V.}\,(u_0)\,. \tag{4.43}$$

In order to conclude that u is an entropy solution, we need the right-hand side of (4.43) to vanish as $\Delta x, \Delta y, \Delta t \to 0$; that is, we need to assume that

$$\frac{\Delta x+\Delta y}{\Delta t} \text{ remains bounded}$$

as $\eta \to 0$. Under this assumption,

$$\iint_{\mathbb{R}^2}\int_0^T \left(|u-k|\,\phi_t + q^f(u,k)\phi_x + q^g(u,k)\phi_y\right)\,dt\,dx\,dy$$
$$-\iint_{\mathbb{R}^2} |u(x,y,T)-k|\,\phi(x,y,T)\,dx\,dy$$
$$+\iint_{\mathbb{R}^2} |u(x,y,0)-k|\,\phi(x,y,0)\,dx\,dy \ge 0,$$

which shows that u indeed satisfies the Kružkov entropy condition. We summarize the result.

Theorem 4.4 *Let u_0 be an integrable and bounded function in $L^\infty(\mathbb{R}^m)\cap BV(\mathbb{R}^m)$, and let f_j be piecewise twice continuously differentiable functions for $j=1,\dots,m$. Construct an approximate solution u_η using front tracking by defining*

$$u^0 = \pi u_0, \quad u^{n+j/m} = \pi \circ S_{\Delta t}^{f_{j,\delta},x_j} u^{n+(j-1)/m}, \quad j=1,\dots,m, \quad n\in\mathbb{N},$$

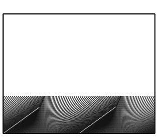
and

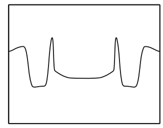
$$u_\eta(x,t) = \begin{cases} S_{m(t-t_{n+(j-1)/m})}^{f_{j,\delta},x_j} u^{n+(j-1)/m}, & \text{for } t\in[t_{n+(j-1)/m}, t_{n+j/m}),\\ u^{n+j/m} & \text{for } t = t_{n+j/m},\end{cases}$$

where $x=(x_1,\dots,x_m)$.

For every sequence $\{\eta\}$, with $\eta=(\Delta x_1,\dots,\Delta x_m,\Delta t,\delta)$, where $\eta\to 0$ and

$$\max_j\left\{\Delta x_j\right\}/\Delta t \text{ remains bounded,}$$





we have that $\{u_\eta\}$ *converges to the unique solution* $u = u(x,t)$ *of the initial value problem*

$$u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = 0, \quad u(x,0) = u_0(x), \tag{4.44}$$

which satisfies the Kružkov entropy condition.

4.3 Convergence Rates

Now I think I'm wrong on account of those damn partial integrations.
I oscillate between right and wrong.
— Letter from Feynman to Welton (1936)

In this section we show how fast front tracking plus dimensional splitting converges to the exact solution. The analysis is based on Kuznetsov's lemma.

We start by generalizing Kuznetsov's lemma, Theorem 3.14, to the present multidimensional setting. Although the argument carries over, we will present the relevant definitions in arbitrary dimension.

Let the class $\mathcal{K}$ consist of maps $u\colon [0,\infty) \to L^1(\mathbb{R}^m) \cap BV(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ such that:

(i) The limits $u(t\pm)$ exist.
(ii) The function u is right continuous, i.e., $u(t+) = u(t)$.
(iii) $\|u(t)\|_{L^\infty(\mathbb{R}^m)} \le \|u(0)\|_{L^\infty(\mathbb{R}^m)}$.
(iv) $\mathrm{T.V.}\,(u(t)) \le \mathrm{T.V.}\,(u(0))$.

Recall the following definition of moduli of continuity in time (cf. (3.54)):

$$\begin{aligned} \nu_t(u,\sigma) &= \sup_{|\tau|\le\sigma} \|u(t+\tau) - u(t)\|_{L^1(\mathbb{R}^m)}, \quad \sigma > 0, \\ \nu(u,\sigma) &= \sup_{0\le t\le T} \nu_t(u,\sigma). \end{aligned}$$

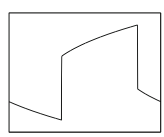
The estimate (3.55) is replaced by

$$\nu(u,\sigma) \le |\sigma|\,\mathrm{T.V.}\,(u_0) \max_j\{\,\|f_j\|_{\mathrm{Lip}}\,\},$$

for a solution u of (4.23).

In several space dimensions, the Kružkov form reads

$$\begin{aligned} \Lambda_T(u,\phi,k) = \iint\limits_{\mathbb{R}^m\times[0,T]} & \Big(|u-k|\,\phi_t + \sum_j q^{f_j}(u,k)\phi_{x_j}\Big)\, dx_1\cdots dx_m \\ & - \int\limits_{\mathbb{R}^m} |u(x,T)-k|\,\phi(x,T)\, dx_1\cdots dx_m\, dt \\ & + \int\limits_{\mathbb{R}^m} |u_0(x)-k|\,\phi(x,0)\, dx_1\cdots dx_m. \end{aligned} \tag{4.45}$$



In this case, we use the test function

$$\Omega(x, x', s, s') = \omega_{\varepsilon_0}(s - s')\omega_\varepsilon(x_1 - x_1') \cdots \omega_\varepsilon(x_m - x_m'), \qquad x = (x_1, \dots, x_m), \quad x' = (x_1', \dots, x_m'). \tag{4.46}$$

Here ω_ε is the standard mollifier defined by

$$\omega_\varepsilon(x_j) = \frac{1}{\varepsilon}\omega\left(\frac{x_j}{\varepsilon}\right)$$

with

$$0 \le \omega \le 1, \quad \operatorname{supp}\omega \subseteq [-1, 1], \quad \omega(-x_j) = \omega(x_j), \quad \int_{-1}^{1} \omega(z)\, dz = 1.$$

When v is the unique solution of the conservation law (4.25), we introduce

$$\Lambda_{\varepsilon,\varepsilon_0}(u, v) = \int_0^T \int_{\mathbb{R}^m} \Lambda_T\left(u, \Omega(\,\cdot\,, x', \,\cdot\,, s'), v(x', s')\right) dx'ds'.$$

Kuznetsov's lemma can be formulated as follows.

Theorem 4.5 *Let u be a function in $\mathcal{K}$, and let v be an entropy solution of* (4.25). *If $0 < \varepsilon_0 < T$ and $\varepsilon > 0$, then*

$$\begin{aligned}\|u(\,\cdot\,, T-) - v(\,\cdot\,, T)\|_{L^1(\mathbb{R}^m)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^m)} &\\ + \mathrm{T.V.}\,(v_0)\left(2\varepsilon + \varepsilon_0 \max_j\{\,\|f_j\|_{\mathrm{Lip}}\,\}\right) &\\ + \nu(u, \varepsilon_0) - \Lambda_{\varepsilon,\varepsilon_0}(u, v), &\end{aligned} \tag{4.47}$$

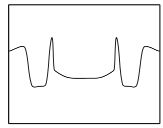
where $u_0 = u(\,\cdot\,, 0)$ and $v_0 = v(\,\cdot\,, 0)$.

The proof of Theorem 3.14 carries over to this setting verbatim.

◇ Example 4.6

Let us first apply this theorem to the case that u is the dimensional splitting approximation, defined with exact solution operators $S_{\Delta t}^{f,x}$ and $S_{\Delta t}^{g,y}$; cf. (4.6). We have established that $\nu(u_{\Delta t}, \varepsilon_0) \le C\varepsilon_0$, where the constant C depends on the total variation of u_0 and the Lipschitz norm of the flux. The inequalities (4.17) and (4.18) imply

$$\begin{aligned} L_T(u_{\Delta t}, k, \varphi) = \int_0^T \iint_{\mathbb{R}^2} & |u_{\Delta t} - k|\,\varphi_t \\ & + 2\chi_n(t) q^f(u_{\Delta t}, k)\varphi_x + 2\tilde{\chi}_n(t) q^g(u_{\Delta t}, k)\varphi_y \, dx\, dy\, dt \\ & - \iint_{\mathbb{R}^2} |u_{\Delta t} - k|\,\varphi\,\Big|_{t=T}\, dx\, dy + \iint_{\mathbb{R}^2} |u_{\Delta t} - k|\,\varphi\,\Big|_{t=0}\, dx\, dy \\ & \ge 0. \end{aligned}$$



Set

$$L_{\varepsilon_0,\varepsilon} = \iiint L_T(u_{\Delta t}, v(x', y', s), \omega_\varepsilon(\,\cdot - x')\omega(\,\cdot - y')\omega_{\varepsilon_0}(\,\cdot - s)\, dx'\, dy'\, ds \geq 0.$$

In the following we always have that $u_{\Delta t} = u_{\Delta t}(x, y, t)$ and $v = v(x', y', s)$, although we sometimes do not indicate that, or indicate only those variables to which we would like to draw the reader's attention. Then

$$\begin{aligned} -\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) &\leq -\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) + L_{\varepsilon_0,\varepsilon} \\ &= \int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} \left(I^x + I^y\right) dx\, dy\, dt\, dx'\, dy'\, ds, \end{aligned}$$

where

$$\begin{aligned} I^x &= (2\chi_n(t) - 1)\, q^f(u_{\Delta t}, v)\omega'_\varepsilon(x - x')\omega_\varepsilon(y - y')\omega_{\varepsilon_0}(t - s), \\ I^y &= (2\tilde{\chi}_n(t) - 1)\, q^g(u_{\Delta t}, v)\omega_\varepsilon(x - x')\omega'_\varepsilon(y - y')\omega_{\varepsilon_0}(t - s). \end{aligned}$$

We shall estimate $\int I^x$; the estimate for I^y is identical. First observe that

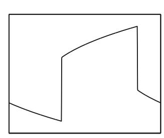
$$2\chi_n(t) - 1 = \begin{cases} 1 & t_n \leq t < t_{n+1/2}, \\ -1 & t_{n+1/2} \leq t < t_{n+1}. \end{cases}$$

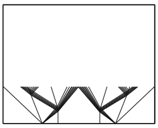
Therefore, if $N\Delta t = T$, then

$$\int_0^T (2\chi_n(t) - 1)\, \psi(t)\, dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \left(\psi(t) - \psi(t + \Delta t/2)\right) dt,$$

for every function ψ. Thus

$$\begin{aligned} &\int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} I^x\, dx\, dy\, dt\, dx'\, dy'\, ds \\ &\quad = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \Big(q^f(u_{\Delta t}(t), v)\omega_{\varepsilon_0}(t - s) \\ &\qquad\quad - q^f(u_{\Delta t}(t + \Delta t/2), v)\omega_{\varepsilon_0}(t + \Delta t/2 - s)\Big) \\ &\qquad\qquad\qquad \times \omega'_\varepsilon(x - x')\omega_\varepsilon(y - y')\, dx\, dy\, dx'\, dy'\, ds\, dt \end{aligned}$$





$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \Big(\omega_{\varepsilon_0}(t-s) - \omega_{\varepsilon_0}(t+\Delta t/2 - s)\Big) \\
&\qquad\qquad \times q^f(u_{\Delta t}(t), v)\omega_\varepsilon'(x-x')\omega_\varepsilon(y-y')\,dx\,dy\,dx'\,dy'\,ds\,dt \\
&\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \omega_{\varepsilon_0}(t+\Delta t/2 - s) \\
&\qquad\qquad\qquad\qquad \times \Big(q^f(u_{\Delta t}(t+\Delta t), v) - q^f(u_{\Delta t}(t), v)\Big) \\
&\qquad\qquad\qquad\qquad \times \omega_\varepsilon'(x-x')\omega_\varepsilon(y-y')\,dx\,dy\,dx'\,dy'\,ds\,dt \\
&=: A + B.
\end{aligned}
$$

Regarding A,

$$
\begin{aligned}
|A| &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} L \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\,dy \int_0^{\Delta t/2} \int_0^T \left|\omega_{\varepsilon_0}'(t-s+\tau)\right|\,ds\,d\tau dt \\
&\le \frac{CT\Delta t}{\varepsilon_0} \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\,dy.
\end{aligned}
$$

Also

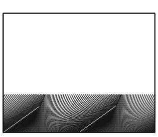
$$
\begin{aligned}
|B| &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \omega_{\varepsilon_0}(t-s+\Delta t/2) \\
&\qquad \times L \iint_{\mathbb{R}^2} |u_{\Delta t}(t+\Delta t/2) - u_{\Delta t}(t)|\,dx\,dy\,|\omega_\varepsilon'(x-x')|\,dx'\,ds\,dt \\
&\le \nu(u_{\Delta t}, \Delta t/2)\frac{C}{\varepsilon} \\
&\le \frac{C\Delta t}{\varepsilon}.
\end{aligned}
$$

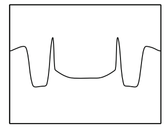
Hence

$$
\begin{aligned}
&\Big|\int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} I^x\,dx\,dy\,dt\,dx'\,dy'\,ds\Big| \\
&\qquad \le \frac{C\Delta t}{\varepsilon_0} \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\,dy + \frac{C\Delta t}{\varepsilon}.
\end{aligned}
$$

We have a similar estimate for the integral of I^y; thus we end up with the estimate

$$
-\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) \le \frac{C\Delta t}{\varepsilon_0}|u_0|_{BV(\mathbb{R}^2)} + \frac{C\Delta t}{\varepsilon}.
$$





Since we have $v(0) = u_{\Delta t}(0) = u_0$, Kuznetsov's lemma yields

$$\|u_{\Delta t}(\cdot, T) - v(\cdot, T)\|_{L^1(\mathbb{R}^2)} \leq C\left(\varepsilon_0 + \varepsilon + \frac{\Delta t}{\varepsilon_0} + \frac{\Delta t}{\varepsilon}\right),$$

which on setting $\varepsilon_0 = \varepsilon = \sqrt{\Delta t}$, yields

$$\|u_{\Delta t}(\cdot, T) - v(\cdot, T)\|_{L^1(\mathbb{R}^2)} \leq C\sqrt{\Delta t}. \tag{4.48}$$

Since this estimate was obtained using the exact solution operator in each direction, there is no hope of obtaining a better estimate using numerical approximations instead of $S_{\Delta t}^{f,g}$. ◇

Next, we use Kuznetsov's lemma to estimate the rate of convergence for the front tracking approximation. This entails using a first-order (in δ) approximation to the exact solution operators, so from the previous example, the best we can hope for is that the error is bounded by $\mathcal{O}(\delta + \sqrt{\Delta t})$.

We want to estimate

$$\left\|S(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)} \leq \|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^m)} + \left\|S_\delta(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)}, \tag{4.49}$$

where $u = S(T)u_0$ and $S_\delta(T)u_0$ denote the exact solutions of the multidimensional conservation law with flux functions f replaced by their piecewise linear and continuous approximations f_δ. The first term can be estimated by

$$\|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^m)} \leq T \max_j \left\{\|f_j - f_{j,\delta}\|_{\mathrm{Lip}}\right\} \mathrm{T.V.}\,(u_0)\,, \tag{4.50}$$

while we apply Kuznetsov's lemma, Theorem 4.5, for the second term. For the function u we choose u_η, the approximate solution using front tracking along each dimension and dimensional splitting, while for v we use the exact solution with piecewise linear continuous flux functions f_δ and g_δ, and u_0 as initial data, that is, $v = v_\delta = S_\delta(T)u_0$. Thus we find, using (4.41), that

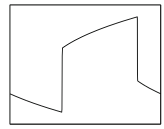
$$\nu(u_\eta, \varepsilon_0) \leq \varepsilon_0\left(C + \mathcal{O}\left(\frac{1}{\Delta t}\max_j\left\{\Delta x_j\right\}\right)\right)\mathrm{T.V.}\,(u_0)\,.$$

Kuznetsov's lemma then reads

$$\begin{aligned}\left\|S_\delta(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)} \leq \left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^m)} &+ \Big[2\varepsilon + \max_j\left\{\|f_{j,\delta}\|_{\mathrm{Lip}}\right\}\varepsilon_0 \\ &+ \varepsilon_0\Big(C + \mathcal{O}\Big(\frac{\max\left\{\Delta x_j\right\}}{\Delta t}\Big)\Big)\Big]\mathrm{T.V.}\,(u_0) \\ &- \Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta),\end{aligned} \tag{4.51}$$

and the name of the game is to estimate $\Lambda_{\varepsilon,\varepsilon_0}$.

To make the estimates more transparent, we start by rewriting $\Lambda_T(u_\eta, \phi, k)$. Since all the complications of several space dimensions are present in two dimensions, we present the argument in two dimensions only, that is, with $m = 2$, and

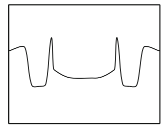


denote the spatial variables by (x, y). All arguments carry over to arbitrary dimensions without any change. By definition we have (in obvious notation, $q^{f_\delta}(u) = \operatorname{sign}(u-k)\,(f_\delta(u) - f_\delta(k))$ and similarly for q^{g_δ})

$$
\begin{aligned}
\Lambda_T(u_\eta, \phi, k) &= \iint \int_0^T \left(\left| u_\eta - k \right| \phi_t + q^{f_\delta}(u_\eta, k)\phi_x + q^{g_\delta}(u_\eta, k)\phi_y \right) \, dt \, dx \, dy \\
&\quad + \iint \left| u_\eta - k \right| \phi|_{t=0+} \, dx \, dy - \iint \left| u_\eta - k \right| \phi|_{t=T-} \, dx \, dy \\
&= \sum_{n=0}^{N-1} \iint \Big(\int_{t_n}^{t_{n+1/2}} + \int_{t_{n+1/2}}^{t_{n+1}} \Big) \Big(\left| u_\eta - k \right| \phi_t \\
&\qquad\qquad + q^{f_\delta}(u_\eta, k)\phi_x + q^{g_\delta}(u_\eta, k)\phi_y \Big) \, dt \, dx \, dy \\
&\quad + \iint \left| u_\eta - k \right| \phi|_{t=0+} \, dx \, dy - \iint \left| u_\eta - k \right| \phi|_{t=T-} \, dx \, dy \\
&= \sum_{n=0}^{N-1} \iint \int_{t_n}^{t_{n+1/2}} \left(\left| u_\eta - k \right| \phi_t + 2q^{f_\delta}(u_\eta, k)\phi_x \right) \, dt \, dx \, dy \\
&\quad + \sum_{n} \iint \int_{t_{n+1/2}}^{t_{n+1}} \left(\left| u_\eta - k \right| \phi_t + 2q^{g_\delta}(u_\eta, k)\phi_y \right) \, dt \, dx \, dy \\
&\quad + \sum_{n=0}^{N-1} \iint \Big(\int_{t_{n+1/2}}^{t_{n+1}} - \int_{t_n}^{t_{n+1/2}} \Big) q^{f_\delta}(u_\eta, k)\phi_x \, dt \, dx \, dy \\
&\quad + \sum_{n=0}^{N-1} \iint \Big(\int_{t_n}^{t_{n+1/2}} - \int_{t_{n+1/2}}^{t_{n+1}} \Big) q^{g_\delta}(u_\eta, k)\phi_y \, dt \, dx \, dy \\
&\quad + \iint \left| u_\eta - k \right| \phi|_{t=0+} \, dx \, dy - \iint \left| u_\eta - k \right| \phi|_{t=T-} \, dx \, dy.
\end{aligned}
$$

We now use that u_η is an exact solution in the x-direction and the y-direction on each strip $[t_n, t_{n+1/2}]$ and $[t_{n+1/2}, t_{n+1}]$, respectively. Thus we can invoke inequalities (4.17) and (4.18), and we conclude that

$$
\begin{aligned}
\Lambda_T(u_\eta, \phi, k) \geq \sum_{n=0}^{N-1} \iint & \Big(\left| u_\eta - k \right| |_{t=t_{n+1/2}-} \phi(t_{n+1/2}) \\
& - \left| u_\eta - k \right| |_{t=t_n+} \phi(t_n) \Big) \, dx \, dy \\
+ \sum_{n=0}^{N-1} \iint & \Big(\left| u_\eta - k \right| |_{t=t_{n+1}-} \phi(t_{n+1}) \\
& - \left| u_\eta - k \right| |_{t=t_{n+1/2}+} \phi(t_{n+1/2}) \Big) \, dx \, dy
\end{aligned}
$$

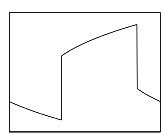


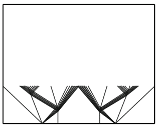
$$
\begin{aligned}
&+\sum_{n=0}^{N-1}\iint\Big(\int_{t_{n+1/2}}^{t_{n+1}}-\int_{t_n}^{t_{n+1/2}}\Big)q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&+\sum_{n=0}^{N-1}\iint\Big(\int_{t_n}^{t_{n+1/2}}-\int_{t_{n+1/2}}^{t_{n+1}}\Big)q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&+\iint\left|u_\eta-k\right|\phi|_{t=0+}\,dx\,dy-\iint\left|u_\eta-k\right|\phi|_{t=T-}\,dx\,dy\\
&=-2\sum_{n=0}^{N-1}\iint\int_{t_n}^{t_{n+1/2}}q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&\quad+\iint\int_0^T q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&\quad-2\sum_{n=0}^{N-1}\iint\int_{t_{n+1/2}}^{t_{n+1}}q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&\quad+\iint\int_0^T q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&\quad+\sum_{n=0}^{N-1}\iint\Big(\left|u_\eta-k\right|\Big|_{t=t_{n+1/2}-}\\
&\qquad\qquad-\left|u_\eta-k\right|\Big|_{t=t_{n+1/2}+}\Big)\phi(t_{n+1/2})\,dx\,dy\\
&\quad+\sum_{n=1}^{N-1}\iint\Big(\left|u_\eta-k\right|\Big|_{t=t_n-}-\left|u_\eta-k\right|\Big|_{t=t_n+}\Big)\phi(t_n)\,dx\,dy\\
&:=-I_1(u_\eta,k)-I_2(u_\eta,k)-I_3(u_\eta,k)-I_4(u_\eta,k). \qquad (4.52)
\end{aligned}
$$

Observe that because we employ the projection operator π between each pair of consecutive times, we solve a conservation law in one dimension; $u^{n+1/2}$ and u^n are in general discontinuous across $t_{n+1/2}$ and t_n, respectively. The terms I_1 and I_2 are due to dimensional splitting, while I_3 and I_4 come from the projections.

Choose now for the constant k the function $v_\delta(x', y', s')$, and for ϕ we use Ω given by (4.46). Integrating over the new variables, we obtain

$$
\begin{aligned}
\Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) &= \iint\int_0^T \Lambda_T(u_\eta, \Omega(\,\cdot\,,x',\,\cdot\,,y',\,\cdot\,,s'), v_\delta(x',y',s'))\,ds'\,dx'\,dy'\\
&\geq -I_1^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_2^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_3^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_4^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta),
\end{aligned}
$$





where $I_j^{\varepsilon,\varepsilon_0}$ are given by

$$I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \iint \int_0^T \iint \Big(2\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} q^{f_\delta}(u_\eta, v_\delta)\Omega_x \, ds - \int_0^T q^{f_\delta}(u_\eta, v_\delta)\Omega_x \, ds \Big) \, dx\, dy\, ds'\, dx'\, dy',$$

$$I_2^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \iint \int_0^T \iint \Big(2\sum_{n=0}^{N-1} \int_{t_{n+1/2}}^{t_{n+1}} q^{g_\delta}(u_\eta, v_\delta)\Omega_y \, ds - \int_0^T q^{g_\delta}(u_\eta, v_\delta)\Omega_y \, ds \Big) \, dx\, dy\, ds'\, dx'\, dy',$$

$$I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=1}^{N-1} \iint \int_0^T \iint \Big(\left|u_\eta - v_\delta\right| |_{s=t_n+} - \left|u_\eta - v_\delta\right| |_{s=t_n-} \Big) \Omega \, dx\, dy\, ds'\, dx'\, dy',$$

$$I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=0}^{N-1} \iint \int_0^T \iint \Big(\left|u_\eta - v_\delta\right| |_{s=t_{n+1/2}+} - \left|u_\eta - v_\delta\right| |_{s=t_{n+1/2}-} \Big) \Omega \, dx\, dy\, ds'\, dx'\, dy'.$$

We will start by estimating $I_1^{\varepsilon,\varepsilon_0}$ and $I_2^{\varepsilon,\varepsilon_0}$.

Lemma 4.7 *We have the following estimate:*

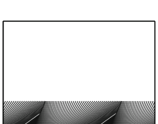
$$\left|I_1^{\varepsilon,\varepsilon_0}\right| + \left|I_2^{\varepsilon,\varepsilon_0}\right| \le T \max\left\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\right\} \mathrm{T.V.}\,(u_0) \times \Big(\frac{\Delta t}{\varepsilon_0} + \frac{1}{\varepsilon}\Big(\{\|f\|_{\mathrm{Lip}} + \|g\|_{\mathrm{Lip}}\}\Delta t + \Delta x + \Delta y \Big)\Big). \quad (4.53)$$

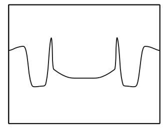
Proof We will detail the estimate for $\left|I_1^{\varepsilon,\varepsilon_0}\right|$. Writing

$$q^{f_\delta}(u_\eta(s), v_\delta(s')) = q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s')) + \left(q^{f_\delta}(u_\eta(s), v_\delta(s')) - q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s')) \right),$$

we rewrite $I_1^{\varepsilon,\varepsilon_0}$ as

$$I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=0}^{N-1} \Big[\left(J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1}) \right) + \left(J_2(t_n, t_{n+1/2}) - J_2(t_{n+1/2}, t_{n+1}) \right) \Big], \quad (4.54)$$





with

$$J_1(\tau_1,\tau_2) = \iint \int_0^T \iint \int_{\tau_1}^{\tau_2} q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(x',y',s')) \times \Omega_x(x,x',y,y',s,s')\, ds\, dx\, dy\, ds'\, dx'\, dy',$$

$$J_2(\tau_1,\tau_2) = \iint \int_0^T \iint \int_{\tau_1}^{\tau_2} \Big(q^{f_\delta}(u_\eta(x,y,s), v_\delta(x',y',s')) - q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(x',y',s')) \Big) \times \Omega_x(x,x',y,y',s,s')\, ds\, dx\, dy\, ds'\, dx'\, dy'.$$

Here we have written out all the variables explicitly; however, in the following we will display only the relevant variables. All spatial integrals are over the real line unless specified otherwise. Rewriting

$$\omega_{\varepsilon_0}(s-s') = \omega_{\varepsilon_0}(t_{n+1/2}-s') + \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s},$$

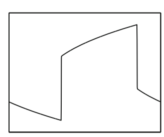
we obtain

$$\begin{aligned} J_1(t_n,t_{n+1/2}) &= \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon \Bigg(\int_{t_n}^{t_{n+1/2}} \omega_{\varepsilon_0}(t_{n+1/2}-s')\, ds \\ &\qquad + \int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds \Bigg) dx\, dy\, ds'\, dx'\, dy' \\ &= \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon \Bigg(\frac{\Delta t}{2}\omega_{\varepsilon_0}(t_{n+1/2}-s') \\ &\qquad + \int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds \Bigg) dx\, dy\, ds'\, dx'\, dy', \end{aligned}$$

where $\Omega^\varepsilon = \omega_\varepsilon(x-x')\omega_\varepsilon(y-y')$ denotes the spatial part of Ω.

If we rewrite $J_1(t_{n+1/2}, t_{n+1})$ in the same way, we obtain

$$\begin{aligned} J_1(t_{n+1/2},t_{n+1}) &= \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon \Bigg(\frac{\Delta t}{2}\omega_{\varepsilon_0}(t_{n+1/2}-s') \\ &\qquad + \int_{t_{n+1/2}}^{t_{n+1}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds \Bigg) dx'\, dy'\, ds'\, dx\, dy, \end{aligned}$$



and hence

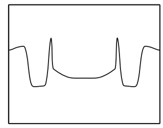
$$
\begin{aligned}
&J_1\big(t_n, t_{n+1/2}\big) - J_1\big(t_{n+1/2}, t_{n+1}\big) \\
&= \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon \Big(\int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s} - s')\, d\bar{s}\, ds \\
&\qquad\qquad - \int_{t_{n+1/2}}^{t_{n+1}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s} - s')\, d\bar{s}\, ds \Big)\, dx\, dy\, ds'\, dx'\, dy'.
\end{aligned}
\tag{4.55}
$$

Now using the Lipschitz continuity of q^{f_δ}, we can replace variation in q^{f_δ} by variation in u, and obtain, using $\iint \omega'_{\varepsilon_0}(x - x')\, dx\, dx' = 0$, that

$$
\begin{aligned}
&\left| \iint q^{f_\delta}(u_\eta(x, y, t_{n+1/2}), v_\delta(s'))\omega'_{\varepsilon_0}(x - x')\, dx\, dx' \right| \\
&= \Big| \iint \omega'_{\varepsilon_0}(x - x')\, dx\, dx' \\
&\qquad \times \left[q^{f_\delta}(u_\eta(x, y, t_{n+1/2}), v_\delta(s')) - q^{f_\delta}(u_\eta(x', y, t_{n+1/2}), v_\delta(s')) \right] \Big| \\
&\le \|f_\delta\|_{\mathrm{Lip}} \iint \left| \omega'_{\varepsilon_0}(x - x') \right| \\
&\qquad\qquad \times \left| u_\eta(x, y, t_{n+1/2}) - u_\eta(x', y, t_{n+1/2}) \right|\, dx\, dx' \\
&= \|f_\delta\|_{\mathrm{Lip}} \iint \left| u_\eta(x' + z, y, t_{n+1/2}) - u_\eta(x', y, t_{n+1/2}) \right| \left| \omega'_{\varepsilon_0}(z) \right|\, dx'\, dz \\
&\le \|f_\delta\|_{\mathrm{Lip}} \int \frac{1}{|z|} \int \left| u_\eta(x' + z, y, t_{n+1/2}) - u_\eta(x', y, t_{n+1/2}) \right|\, dx' \\
&\qquad\qquad \times \left| z\omega'_{\varepsilon_0}(z) \right|\, dz \\
&\le \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}_{\cdot x} \left(u_\eta(t_{n+1/2}) \right) \int \left| z\omega'_{\varepsilon_0}(z) \right|\, dz \\
&\le \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}_{\cdot x} \left(u_\eta(t_{n+1/2}) \right),
\end{aligned}
$$

using that $\int \left| z\omega'_{\varepsilon_0}(z) \right|\, dz = 1$. We combine this with (4.55) to get

$$
\begin{aligned}
&\left| J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1}) \right| \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \iint \mathrm{T.V.}_{\cdot x} \left(u_\eta(t_{n+1/2}) \right) \omega_{\varepsilon_0}(y - y') \\
&\qquad\qquad \times \Big(\int_0^T \int_{t_n}^{t_{n+1/2}} \left| \int_{t_{n+1/2}}^{s} \left| \omega'_{\varepsilon_0}(\bar{s} - s') \right|\, d\bar{s} \right| ds\, ds' \\
&\qquad\qquad\quad + \int_0^T \int_{t_{n+1/2}}^{t_{n+1}} \left| \int_{t_{n+1/2}}^{s} \left| \omega'_{\varepsilon_0}(\bar{s} - s') \right|\, d\bar{s} \right| ds\, ds' \Big)\, dy'\, dy.
\end{aligned}
$$



Inserting the estimate

$$\int_0^T \left|\omega'_{\varepsilon_0}(\bar{s}-s')\right| \, ds' \leq \frac{1}{\varepsilon_0}\int |\omega'(z)| \, dz \leq 2/\varepsilon_0,$$

we obtain

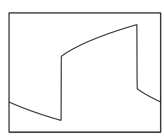
$$\left|J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1})\right| \leq \frac{\|f_\delta\|_{\mathrm{Lip}}(\Delta t)^2}{2\varepsilon_0} \mathrm{T.V.}\left(u_\eta(t_{n+1/2})\right). \quad (4.56)$$

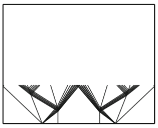
Next we consider the term J_2. We first use the Lipschitz continuity of q^{f_δ}, which yields

$$\begin{aligned}
\left|J_2(t_n, t_{n+1/2})\right| &\leq \|f_\delta\|_{\mathrm{Lip}} \iint \int_0^T \iint \int_{t_n}^{t_{n+1/2}} \left|u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2})\right| \\
&\qquad\qquad \times |\Omega_x| \, ds \, dx' \, dy' \, ds' \, dx \, dy \\
&\leq \frac{\|f_\delta\|_{\mathrm{Lip}}}{\varepsilon} \int_{t_n}^{t_{n+1/2}} \iint \left|u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2})\right| \, ds \, dx \, dy \\
&\leq \frac{\|f_\delta\|_{\mathrm{Lip}}}{\varepsilon} \int_{t_n}^{t_{n+1/2}} \iint \left|u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2}-)\right| \, ds \, dx \, dy \\
&\quad + \frac{\|f_\delta\|_{\mathrm{Lip}}\Delta t}{2\varepsilon} \iint \left|u_\eta(x,y,t_{n+1/2}-) - u_\eta(x,y,t_{n+1/2})\right| \, dx \, dy \\
&\leq \frac{\|f_\delta\|_{\mathrm{Lip}}\Delta t}{\varepsilon} \left(\|f_\delta\|_{\mathrm{Lip}}\Delta t + \Delta x\right) \mathrm{T.V.}\left(u_\eta\left(t_{n+1/2}\right)\right).
\end{aligned}$$

Here we integrated to unity in the variables s' and y', and estimated $\int |\omega'_\varepsilon(x-x')| \, dx'$ by $2/\varepsilon$. Finally, we used the continuity in time of the L^1-norm in the x-direction and estimated the error due to the projection. A similar bound can be obtained for $J_2(t_{n+1/2}, t_{n+1})$, and hence

$$\begin{aligned}
&\left|J_2(t_n, t_{n+1/2}) - J_2(t_{n+1/2}, t_{n+1})\right| \\
&\qquad \leq \left|J_2(t_n, t_{n+1/2})\right| + \left|J_2(t_{n+1/2}, t_{n+1})\right| \\
&\qquad \leq \frac{\|f\|_{\mathrm{Lip}}\Delta t}{\varepsilon} \left(2\|f\|_{\mathrm{Lip}}\Delta t + \Delta x + \Delta y\right) \mathrm{T.V.}\left(u_\eta(t_n)\right), \quad (4.57)
\end{aligned}$$





where we used that T.V. $\left(u_\eta(t_{n+1/2})\right) \le$ T.V. $\left(u_\eta(t_n)\right)$. Inserting estimates (4.56) and (4.57) into (4.54) yields

$$\begin{aligned}\left|I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| &\le \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}\left(u_\eta(0)\right)\\ &\qquad \times \sum_{n=0}^{N-1}\left(\frac{(\Delta t)^2}{2\varepsilon_0} + \frac{\Delta t}{2\varepsilon}(2\|f_\delta\|_{\mathrm{Lip}}\Delta t + \Delta x + \Delta y)\right)\\ &\le T\,\|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}\left(u_\eta(0)\right)\\ &\qquad \times \left(\frac{\Delta t}{2\varepsilon_0} + \frac{1}{2\varepsilon}(2\|f_\delta\|_{\mathrm{Lip}}\Delta t + \Delta x + \Delta y)\right),\end{aligned}$$

where we again used that T.V. $\left(u_\eta\right)$ is nonincreasing. An analogous argument gives the same estimate for $I_2^{\varepsilon,\varepsilon_0}$. Adding the two inequalities, we conclude that (4.53) holds. □

It remains to estimate $I_3^{\varepsilon,\varepsilon_0}$ and $I_4^{\varepsilon,\varepsilon_0}$. We aim at the following result.

Lemma 4.8 *The following estimate holds*:

$$\left|I_3^{\varepsilon,\varepsilon_0}\right| + \left|I_4^{\varepsilon,\varepsilon_0}\right| \le \frac{T(\Delta x + \Delta y)^2}{\Delta t\ \varepsilon}\mathrm{T.V.}\left(u_0\right).$$

Proof We discuss the term $I_3^{\varepsilon,\varepsilon_0}$ only. Recall that

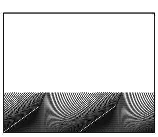
$$\begin{aligned}I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=1}^{N-1} \iint \int_0^T \iint \Big(&\left|u_\eta(x,y,t_n) - v_\delta(x',y',s')\right|\\ &- \left|u_\eta(x,y,t_n-) - v_\delta(x',y',s')\right|\Big)\\ &\times \Omega(x,x',y,y',t_n,s')\,dx'\,dy'\,ds'\,dx\,dy.\end{aligned}$$

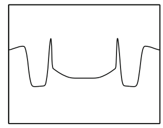
The function $u_\eta(x,y,t_n+)$ is the projection of $u_\eta(x,y,t_n-)$, that is,

$$u_\eta(x,y,t_n+) = \frac{1}{\Delta x \Delta y}\iint_{I_{ij}} u_\eta(\bar{x},\bar{y},t_n-)\,d\bar{x}\,d\bar{y}. \tag{4.58}$$

If we replace $\iint_{\mathbb{R}^2}$ by $\sum_{i,j}\iint_{I_{ij}}$ and use (4.58), we obtain

$$\begin{aligned}&I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\\ &\quad= \sum_{n=1}^{N-1}\iint\int_0^T\sum_{i,j}\iint_{I_{ij}}\left[\left|\frac{1}{\Delta x\Delta y}\iint_{I_{ij}} u_\eta(\bar{x},\bar{y},t_n-)\,d\bar{x}\,d\bar{y} - v_\delta(x',y',s')\right|\right.\\ &\qquad\left. - \left|u_\eta(x,y,t_n-) - v_\delta(x',y',s')\right|\right]\Omega(x,x',y,y',t_n,s')\,dx\,dy\,ds'\,dx'\,dy'\end{aligned}$$

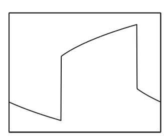




$$
\begin{aligned}
&= \frac{1}{\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(x, x', y, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\big| u_\eta(\bar{x}, \bar{y}, t_n -) - v_\delta(x', y', s') \big| \\
&\qquad - \big| u_\eta(x, y, t_n -) - v_\delta(x', y', s') \big| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy' \\
&= \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(x, x', y, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\big| u_\eta(\bar{x}, \bar{y}, t_n -) - v_\delta(x', y', s') \big| \\
&\qquad - \big| u_\eta(x, y, t_n -) - v_\delta(x', y', s') \big| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy' \\
&\quad + \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\big| u_\eta(x, y, t_n -) - v_\delta(x', y', s') \big| \\
&\qquad - \big| u_\eta(\bar{x}, \bar{y}, t_n -) - v_\delta(x', y', s') \big| \Big) \, dx \, dy \, d\bar{x} \, d\bar{y} \, ds' \, dx' \, dy' \\
&= \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Big(\Omega(x, x', y, y', t_n, s') - \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \Big) \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\big| u_\eta(\bar{x}, \bar{y}, t_n -) - v_\delta(x', y', s') \big| \\
&\qquad - \big| u_\eta(x, y, t_n -) - v_\delta(x', y', s') \big| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy'.
\end{aligned}
$$

Estimating $I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)$ using the inverse triangle inequality, we obtain

$$
\begin{aligned}
&\Big| I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) \Big| \\
&\quad \le \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \big| u_\eta(\bar{x}, \bar{y}, t_n -) - u_\eta(x, y, t_n -) \big| \\
&\qquad \times \left| \Omega(x, x', y, y', t_n, s') - \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \right| \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy'.
\end{aligned}
\tag{4.59}
$$



The next step is to bound the test functions in (4.59) from above. To this end we first consider, for $x, \bar{x} \in (i\Delta x, (i+1)\Delta x)$,

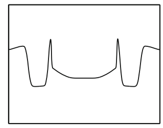
$$\begin{aligned}
\int |\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')|\; dx' &= \int |\omega(z) - \omega(z + (\bar{x}-x)/\varepsilon)|\; dz \\
&= \int \left| \int_z^{z+(\bar{x}-x)/\varepsilon} \omega'(\xi)\, d\xi \right| dz \\
&\le \int \int_z^{z+(\bar{x}-x)/\varepsilon} |\omega'(\xi)|\; d\xi\, dz \\
&\le \int \int_0^{\Delta x/\varepsilon} |\omega'(\alpha+\beta)|\; d\alpha\, d\beta = \frac{2\Delta x}{\varepsilon}.
\end{aligned}$$

Integrating the time variable to unity, we easily see (really, this is easy!) that

$$\begin{aligned}
&\iint \int_0^T |\Omega(x,x',y,y',t_n,s') - \Omega(\bar{x},x',\bar{y},y',t_n,s')|\; ds'\, dx' dy' \\
&= \int_0^T \omega_{\varepsilon_0}(s-s')\, ds' \\
&\qquad \times \iint |\omega_\varepsilon(x-x')\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{x}-x')\omega_\varepsilon(\bar{y}-y')|\; dx' dy' \\
&\le \iint |\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')|\, \omega_\varepsilon(y-y')\, dx' dy' \\
&\quad + \iint |\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{y}-y')|\, \omega_\varepsilon(\bar{x}-x')\, dx' dy' \\
&\le \int |\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')|\; dx' + \int |\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{y}-y')|\; dy' \\
&\le (\Delta x + \Delta y)\frac{2}{\varepsilon}.
\end{aligned} \tag{4.60}$$

Furthermore,

$$\begin{aligned}
\left|u_\eta(\bar{x},\bar{y},t_n-) - u_\eta(x,y,t_n-)\right| &= \left|u_\eta(x,\bar{y},t_n-) - u_\eta(x,y,t_n-)\right| \\
&\le \mathrm{T.V.}_{(j\Delta y,(j+1)\Delta y)}\left(u_\eta(x,\,\cdot\,,t_n-)\right).
\end{aligned} \tag{4.61}$$



Inserting (4.60) and (4.61) into (4.59) yields

$$
\begin{aligned}
&\left|I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \\
&\quad \le \frac{1}{2\Delta x \Delta y}\frac{2(\Delta x+\Delta y)}{\varepsilon} \\
&\qquad \times \sum_{n=1}^{N-1}\sum_{i,j}\iint_{I_{ij}}\iint_{I_{ij}} \mathrm{T.V.}_{\cdot(j\Delta y,(j+1)\Delta y)}\left(u_\eta(x,\,\cdot\,,t_n-)\right)\,d\bar{x}\,d\bar{y}\,dx\,dy \\
&\quad \le \frac{\Delta x+\Delta y}{\varepsilon\Delta x\Delta y}\sum_{n=1}^{N-1}\Delta x(\Delta y)^2\sum_{i,j}\int_{i\Delta x}^{(i+1)\Delta x}\mathrm{T.V.}_{\cdot(j\Delta y,(j+1)\Delta y)}\left(u_\eta(x,\,\cdot\,,t_n-)\right)\,dx \\
&\quad \le \frac{(\Delta x+\Delta y)}{\varepsilon}\Delta y\sum_{n=1}^{N-1}\mathrm{T.V.}\left(u_\eta(t_n-)\right) \\
&\quad \le \frac{(\Delta x+\Delta y)}{\varepsilon}\Delta y\frac{T}{\Delta t}\mathrm{T.V.}\left(u_\eta(0)\right),
\end{aligned}
\tag{4.62}
$$

where in the final step we used that $\mathrm{T.V.}\left(u_\eta(t_n-)\right) \le \mathrm{T.V.}\left(u_\eta(0)\right)$.

The same analysis provides the following estimate for $I_4^{\varepsilon,\varepsilon_0}(v_\delta, u_\eta)$:

$$
\left|I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \le \frac{(\Delta x+\Delta y)}{\varepsilon}\Delta x\frac{T}{\Delta t}\mathrm{T.V.}\left(u_\eta(0)\right). \tag{4.63}
$$

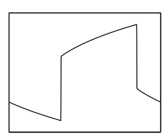
Adding (4.62) and (4.63) proves the lemma. □

We now return to the proof of the estimate of $\Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)$. Combining Lemma 4.7 and Lemma 4.8, we obtain

$$
\begin{aligned}
-\Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) &\le \left|I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_2^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \\
&\le T\Bigg[\left(\frac{\Delta t}{\varepsilon_0} + \frac{1}{\varepsilon}(\{\|f_\delta\|_{\mathrm{Lip}} + \|g_\delta\|_{\mathrm{Lip}}\}\Delta t + \Delta x + \Delta y)\right) \\
&\qquad \times \max\left\{\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\right\} + \frac{(\Delta x+\Delta y)^2}{\Delta t\,\varepsilon}\Bigg]\mathrm{T.V.}\,(u_0) \\
&=: T\,\mathrm{T.V.}\,(u_0)\,\Lambda(\varepsilon,\varepsilon_0,\eta).
\end{aligned}
\tag{4.64}
$$

Returning to (4.49), we combine (4.50), (4.51), as well as (4.64), to obtain

$$
\begin{aligned}
&\left\|S(T)u_0 - u_\eta(T)\right\|_{L^1(\mathbb{R}^2)} \\
&\quad \le \|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^2)} + \left\|S_\delta(T)u_0 - u_\eta(T)\right\|_{L^1(\mathbb{R}^2)} \\
&\quad \le T\max\{\,\|f - f_\delta\|_{\mathrm{Lip}}, \|g - g_\delta\|_{\mathrm{Lip}}\,\}\mathrm{T.V.}\,(u_0) + \left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^2)} \\
&\qquad + \Big(2\varepsilon + \max\{\,\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\varepsilon_0 + \varepsilon_0\Big(C + \mathcal{O}\Big(\frac{\max\{\Delta x, \Delta y\}}{\Delta t}\Big)\Big) \\
&\qquad\qquad + T\,\Lambda(\varepsilon,\varepsilon_0,\eta)\Big)\mathrm{T.V.}\,(u_0).
\end{aligned}
\tag{4.65}
$$



Next we take the minimum over ε and ε_0 on the right-hand side of (4.65). This has the form

$$\min_{\varepsilon,\varepsilon_0}\left(a\,\varepsilon + \frac{b}{\varepsilon} + c\,\varepsilon_0 + \frac{d}{\varepsilon_0}\right) = 2\sqrt{ab} + 2\sqrt{cd}.$$

The minimum is obtained for $\varepsilon = \sqrt{b/a}$ and $\varepsilon_0 = \sqrt{d/c}$. We obtain

$$\begin{aligned}
&\left\| S(T)u_0 - u_\eta(T)\right\|_{L^1(\mathbb{R}^2)} \\
&\quad \le T \max\left\{\|f - f_\delta\|_{\text{Lip}}, \|g - g_\delta\|_{\text{Lip}}\right\} \text{T.V.}\,(u_0) + \left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^2)} \\
&\qquad + \mathcal{O}\left(\left((\Delta x + \Delta y) + \Delta t + \frac{(\Delta x + \Delta y)^2}{\Delta t}\right)^{1/2}\right) \text{T.V.}\,(u_0)\,. \qquad (4.66)
\end{aligned}$$

We may choose the approximation of the initial data such that $\left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^2)} = \mathcal{O}\left(\Delta x + \Delta y\right) \text{T.V.}\,(u_0)$. Furthermore, if the flux functions f and g are piecewise C^2 and Lipschitz continuous, then

$$\|f - f_\delta\|_{\text{Lip}} \le \delta \|f''\|_{L^\infty(\mathbb{R})}.$$

We state the final result in the general case.

Theorem 4.9 *Let u_0 be a function in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ with bounded total variation, and let f_j for $j = 1, \dots, m$ be piecewise C^2 functions that in addition are Lipschitz continuous. Then*

$$\left\|u(T) - u_\eta(T)\right\|_{L^1(\mathbb{R}^m)} \le \mathcal{O}\left(\delta + (\Delta x + \Delta y)^{1/2}\right)$$

as $\eta \to 0$ when

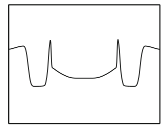
$$\Delta x = K_1 \Delta y = K_2 \Delta t$$

for constants K_1 and K_2.

It is worthwhile to analyze the error terms in the estimate. We are clearly making four approximations with the front-tracking method combined with dimensional splitting. First of all, we are approximating the initial data by step functions. That gives an error of order Δx. Secondly, we are approximating the flux functions by piecewise linear and continuous functions; in this case the error is of order δ. A third source is the intrinsic error in the dimensional splitting, which is of order $(\Delta t)^{1/2}$, and finally, the projection onto the grid gives an error of order $(\Delta x)^{1/2}$.

The advantage of this method over difference methods is the fact that the time step Δt is not bounded by a CFL condition expressed in terms of Δx and Δy. The only relation that must be satisfied is (4.27), which allows for taking large time steps. In practice it is observed that one can choose CFL numbers[4] as high as 10–15 without loss in accuracy. This makes it a very fast method.

[4] In several dimensions the CFL number is defined as $\max_i(\left|f_i'\right| \Delta t/\Delta x_i)$.



4.4 Operator Splitting: Diffusion

The answer, my friend, is blowin' in the wind, the answer is blowin' in the wind.
— Bob Dylan, Blowin' in the Wind (1968)

We show how to use the concept of operator splitting to derive a (weak) solution of the parabolic problem[5] on $\mathbb{R}^m \times [0, T]$,

$$u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = \mu \sum_{j=1}^{m} u_{x_j x_j}, \tag{4.67}$$

by solving

$$u_t + f_j(u)_{x_j} = 0, \quad j = 1, \dots, m, \tag{4.68}$$

and

$$u_t = \mu \Delta u, \tag{4.69}$$

where we employ the notation $\Delta u = \sum_j u_{x_j x_j}$. We augment the equation with initial data $u|_{t=0} = u_0$. Let $S_j(t)u_0$ and $H(t)u_0$ denote the solutions of (4.68) and (4.69), respectively, with initial data u_0. Introducing the heat kernel, we may write

$$\begin{aligned} u(x,t) &= (H(t)u_0)(x,t) \\ &= \int_{\mathbb{R}^m} K(x-y,t)u_0(y)\,dy \\ &= \frac{1}{(4\pi\mu t)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|x-y|^2}{4\mu t}\right) u_0(y)\,dy. \end{aligned}$$

Let Δt be positive and $t_n = n\Delta t$. Define

$$u^0 = u_0, \quad u^{n+1} = (H(\Delta t) S_m(\Delta t) \cdots S_1(\Delta t))\, u^n, \tag{4.70}$$

with the idea that u^n approximates $u(x, t_n)$. We will show that u^n converges to the solution of (4.67) as $\Delta t \to 0$.

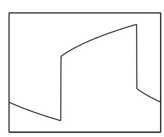
Lemma 4.10 *The following estimates hold*:

$$\|u^n\|_{L^\infty(\mathbb{R}^m)} \le \left\|u^0\right\|_{L^\infty(\mathbb{R}^m)}, \tag{4.71}$$

$$\text{T.V.}\,(u^n) \le \text{T.V.}\left(u^0\right), \tag{4.72}$$

$$\|u^{n_1} - u^{n_2}\|_{L^1_{\text{loc}}(\mathbb{R}^m)} \le C\left(|n_1 - n_2|\,\Delta t\right)^{1/(m+1)}. \tag{4.73}$$

[5] Although we have used the parabolic regularization to motivate the appropriate entropy condition, we have constructed the solution of the multidimensional conservation law independtly, and hence it is logically consistent to use the solution of the conservation law in combination with operator splitting to derive the solution of the parabolic problem. A different approach, where we start with a solution of the parabolic equation and subsequently show that in the limit of vanishing viscosity the solution converges to the solution of the conservation law, is discussed in Appendix B.



Proof Equation (4.71) is obvious, since both the heat equation and the conservation law obey the maximum principle.

We know that the solution of the conservation law has the TVD property (4.72); see (4.24). Thus it remains to show that this property is shared by the solution of the heat equation. To this end, we have

$$\begin{aligned}\Big|H(t)u(x+h) - H(t)u(x)\Big| \\ &= \Big|\int_{\mathbb{R}^m} \big(K(x+h-y,t)u(y) - K(x-y,t)u(y)\big)\,dy\Big| \\ &\le \int_{\mathbb{R}^m} |K(y,t)u(x+h-y) - K(y,t)u(x-y)|\,dy,\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\mathbb{R}^m} \Big|H(t)u(x+h) - H(t)u(x)\Big|\,dx \\ &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |K(y,t)u(x+h-y) - K(y,t)u(x-y)|\,dy\,dx \\ &= \int_{\mathbb{R}^m} K(y,t) \int_{\mathbb{R}^m} |u(x+h-y) - u(x-y)|\,dx\,dy \\ &= \int_{\mathbb{R}^m} K(y,t)\,dy \int_{\mathbb{R}^m} |u(x+h) - u(x)|\,dx \\ &= \int_{\mathbb{R}^m} |u(x+h) - u(x)|\,dx.\end{aligned}$$

Dividing by $|h|$ and letting $h \to 0$, we conclude that

$$\mathrm{T.V.}\,(H(t)u) \le \mathrm{T.V.}\,(u)\,,$$

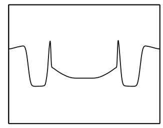
which proves (4.72).

Finally, we consider (4.73). We will first show that the approximate solution obtained by splitting is weakly Lipschitz continuous in time. More precisely, for each ball $\mathcal{B}_r = \{x \mid |x| \le r\}$, we will show that

$$\Big|\int_{\mathcal{B}_r} (u^{n_1} - u^{n_2})\phi\Big| \le C_r\,|n_1 - n_2|\,\Delta t\,\Big(\|\phi\|_\infty + \max_j \|\phi_{x_j}\|_\infty\Big), \tag{4.74}$$

for smooth test functions $\phi = \phi(x)$, where C_r is a constant depending on r. It is enough to study the case $n_2 = n_1 + 1$, and we set $n_1 = n$. Furthermore, we can write

$$\Big|\int (u^{n+1} - u^n)\phi\,dx\Big| \le \Big|\int (H(\Delta t)\tilde{u}^n - \tilde{u}^n)\phi\,dx\Big| + \Big|\int (\tilde{u}^n - u^n)\phi\,dx\Big|, \tag{4.75}$$



where $\tilde{u}^n = (S_m(\Delta t) \cdots S_1(\Delta t))\, u^n$. This shows that it suffices to prove this property for the solutions of the conservation law and the heat equation separately. From Theorem 4.3 we know that the solution of the one-dimensional conservation law satisfies the stronger estimate

$$\|S(t)u - u\|_{L^1(\mathbb{R}^m)} \le C\,|t|\,.$$

This implies that (for simplicity with $m = 2$)

$$\begin{aligned}\|S_2(t)S_1(t)u - u\|_{L^1(\mathbb{R}^2)} &\le \|S_2(t)S_1(t)u - S_1(t)u\|_{L^1(\mathbb{R}^2)} + \|S_1(t)u - u\|_{L^1(\mathbb{R}^2)}\\ &\le C\,|t|\,,\end{aligned}$$

and hence we infer that the last term of (4.75) is of order Δt, that is,

$$\|\tilde{u}^n - u^n\|_{L^1(\mathbb{R}^2)} \le C\,\|\phi\|_{L^\infty(\mathbb{R}^2)}\,|\Delta t|\,.$$

The first term can be estimated as follows (for simplicity of notation we assume $m = 1$). Consider

$$\begin{aligned}\left|\int (H(t)u_0 - u_0)\phi\,dx\right| &= \left|\int\int_0^t u_t\,dt\,\phi\,dx\right| = \left|\int\int_0^t u_{xx}\,dt\,\phi\,dx\right|\\ &\le \int\int_0^t |u_x\phi_x|\;dt\,dx\\ &\le \|\phi_x\|_{L^\infty(\mathbb{R})}\int_0^t\int |u_x|\;dx\,dt\\ &\le \|\phi_x\|_{L^\infty(\mathbb{R})}\int_0^t \mathrm{T.V.}\,(u)\;dt \le \|\phi_x\|_{L^\infty(\mathbb{R})}\mathrm{T.V.}\,(u_0)\;t\,.\end{aligned} \tag{4.76}$$

Thus we conclude that (4.74) holds.

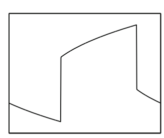
From the TVD property (4.72), we have that

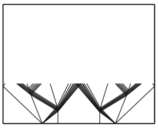
$$\sup_{|\xi|\le\rho}\int |u^n(x+\xi,t) - u^n(x,t)|\;dx \le \rho\,\mathrm{T.V.}\,(u^n)\,. \tag{4.77}$$

Using Kružkov's interpolation lemma (stated and proved right after this proof) we can infer, using (4.74) and (4.77), that

$$\int_{\mathcal{B}_r} |u^{n_1}(x) - u^{n_2}(x)|\;dx \le C_r\left(\varepsilon + \frac{|n_1 - n_2|\,\Delta t}{\varepsilon}\right)$$

for all $\varepsilon \le \rho$. Choosing $\varepsilon = \sqrt{|n_1 - n_2|\,\Delta t}$ proves the result. □





We next state and prove Kružkov's interpolation lemma. It will be convenient to use the multi-index notation. A vector of the form $\alpha = (\alpha_1, \dots, \alpha_m)$, where each component is a nonnegative integer, is called a *multi-index* of order $|\alpha| = \alpha_1 + \dots + \alpha_m$. Given a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.$$

Lemma 4.11 (Kružkov interpolation lemma) *Let $u(x,t)$ be a bounded measurable function defined in the cylinder $\mathcal{B}_{r+\hat{r}} \times [0,T]$, $\hat{r} \geq 0$. For $t \in [0,T]$ and $|\rho| \leq \hat{r}$, assume that u possesses a spatial modulus of continuity*

$$\sup_{|\xi| \leq |\rho|} \int_{\mathcal{B}_r} |u(x+\xi,t) - u(x,t)| \, dx \leq \nu_{r,T,\hat{r}}(|\rho|; u), \tag{4.78}$$

where $\nu_{r,T,\hat{r}}$ does not depend on t. Suppose that for every $\phi \in C_0^\infty(\mathcal{B}_r)$ and $t_1, t_2 \in [0,T]$,

$$\left| \int_{\mathcal{B}_r} \big(u(x,t_2) - u\big(x,t_1\big)\big) \phi(x) \, dx \right| \leq \mathrm{Const}_{r,T} \left(\sum_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^\infty(\mathcal{B}_r)} \right) |t_2 - t_1|, \tag{4.79}$$

where α denotes a multi-index.

Then for t and $t+\tau \in [0,T]$ and for all $\varepsilon \in (0,\hat{r}]$,

$$\int_{\mathcal{B}_r} |u(x,t+\tau) - u(x,t)| \, dx \leq \mathrm{Const}_{r,T} \left(\varepsilon + \nu_{r,T,\hat{r}}(\varepsilon; u) + \frac{|\tau|}{\varepsilon^m} \right). \tag{4.80}$$

Proof Let $\delta \in C_0^\infty$ be a function such that

$$0 \leq \delta(x) \leq 1, \quad \operatorname{supp} \delta \subseteq \mathcal{B}_1, \quad \int \delta(x) \, dx = 1,$$

and define

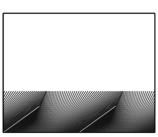
$$\delta_\varepsilon(x) = \frac{1}{\varepsilon^m} \delta\Big(\frac{x}{\varepsilon}\Big).$$

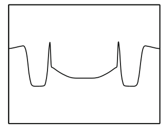
Furthermore, write $f(x) = u(x,t+\tau) - u(x,t)$ (suppressing the time dependence in the notation for f),

$$\sigma(x) = \operatorname{sign}(f(x)) \text{ for } |x| \leq r - \varepsilon, \text{ and } 0 \text{ otherwise},$$

and

$$\sigma_\varepsilon(x) = (\sigma * \delta_\varepsilon)(x) = \int \sigma(x-y) \delta_\varepsilon(y) \, dy.$$





By construction, $\sigma_\varepsilon \in C_0^\infty(\mathbb{R}^m)$ and $\operatorname{supp}\sigma_\varepsilon \subseteq \mathcal{B}_r$. Furthermore, $|\sigma_\varepsilon| \leq 1$ and

$$\begin{aligned}\left|\frac{\partial}{\partial x_j}\sigma_\varepsilon\right| &\leq \frac{1}{\varepsilon^m}\int \left|\frac{\partial}{\partial x_j}\delta(\frac{x-y}{\varepsilon})\right|\sigma(y)\,dy\\ &\leq \frac{1}{\varepsilon^{m+1}}\int \left|\delta_{x_j}(\frac{x-y}{\varepsilon})\right|\sigma(y)\,dy \leq \frac{C}{\varepsilon}.\end{aligned}$$

This easily generalizes to

$$\|D^\alpha\sigma_\varepsilon\|_{L^\infty(\mathbb{R}^m)} \leq \frac{C}{\varepsilon^{|\alpha|}}.$$

Next we have the elementary but important inequality

$$\begin{aligned}\int_{\mathcal{B}_r} |f(x)|\,dx &= \left|\int_{\mathcal{B}_r} |f(x)|\,dx\right|\\ &= \left|\int_{\mathcal{B}_r} (|f(x)| - \sigma_\varepsilon(x)f(x) + \sigma_\varepsilon(x)f(x))\,dx\right|\\ &\leq \left|\int_{\mathcal{B}_r} (|f(x)| - \sigma_\varepsilon(x)f(x))\,dx\right| + \left|\int_{\mathcal{B}_r} \sigma_\varepsilon(x)f(x)\,dx\right|\\ &\leq \int_{\mathcal{B}_r} \big||f(x)| - \sigma_\varepsilon(x)f(x)\big|\,dx + \left|\int_{\mathcal{B}_r} \sigma_\varepsilon(x)f(x)\,dx\right|\\ &=: I_1 + I_2.\end{aligned}$$

We estimate I_1 and I_2 separately. Starting with I_1, we obtain

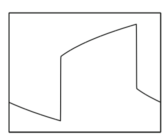
$$\begin{aligned}I_1 &= \int_{\mathcal{B}_r} \big||f(x)| - \sigma_\varepsilon(x)f(x)\big|\,dx\\ &= \int_{\mathcal{B}_r} \left||f(x)|\frac{1}{\varepsilon^m}\int\delta(\frac{x-y}{\varepsilon})\,dy - \frac{1}{\varepsilon^m}\int\delta(\frac{x-y}{\varepsilon})\sigma(y)\,dy\ f(x)\right|\,dx\\ &= \frac{1}{\varepsilon^m}\int\int\delta(\frac{x-y}{\varepsilon})\big||f(x)| - \sigma(y)f(x)\big|\,dy\,dx.\end{aligned}$$

The integrand is integrated over the domain

$$\{(x,y) \mid |x| \leq r,\ |x-y| \leq \varepsilon\}.$$

We further divide this set into two parts: (i) $|y| \geq r-\varepsilon$, and (ii) $|y| \leq r-\varepsilon$; see Fig. 4.2. In case (i) we have

$$\big||f(x)| - \sigma(y)f(x)\big| = |f(x)|,$$



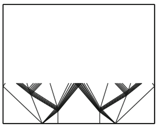
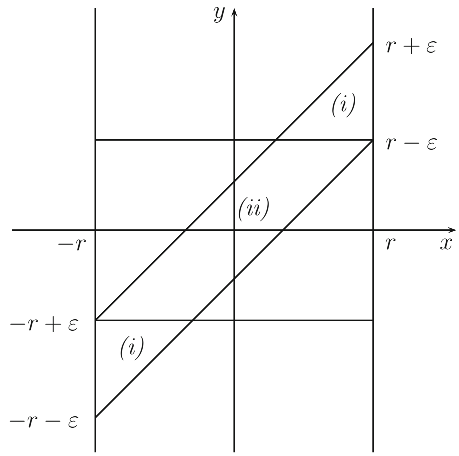


Fig. 4.2 The integration domain



since $\sigma(y) = 0$ whenever $|y| \geq r - \varepsilon$. In case (ii) we have

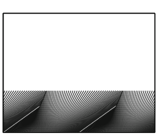
$$\big||f(x)| - \sigma(y) f(x)\big| = \big||f(x)| - \operatorname{sign}(f(y))\, f(x)\big| \leq 2\,|f(x) - f(y)|\,,$$

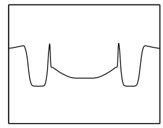
using the elementary inequality

$$\begin{aligned}\big||a| - \operatorname{sign}(b)\, a\big| &= \big||a| - |b| + \operatorname{sign}(b)\,(b-a)\big| \\ &\leq \big||a| - |b|\big| + |\operatorname{sign}(b)\,(b-a)| \\ &\leq 2\,|a-b|\,.\end{aligned}$$

Thus

$$\begin{aligned} I_1 &\leq \frac{2}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{\mathcal{B}_{r-\varepsilon}} \delta(\frac{x-y}{\varepsilon})\,|f(x) - f(y)|\; dy\, dx \\ &\quad + \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{|y| \geq r-\varepsilon} \delta(\frac{x-y}{\varepsilon})\,|f(x)|\; dy\, dx \\ &\leq 2 \int\limits_{\mathcal{B}_r} \int\limits_{\mathcal{B}_1} \delta(z)\,|f(x) - f(x - \varepsilon z)|\; dz\; dx \\ &\quad + \|f\|_\infty \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{|y| \geq r-\varepsilon} \delta(\frac{x-y}{\varepsilon})\, dy\, dx \\ &\leq 2 \int\limits_{\mathcal{B}_1} \delta(z) \sup_{|\xi| \leq \varepsilon} \int\limits_{\mathcal{B}_r} |f(x) - f(x+\xi)|\; dx\, dz \\ &\quad + \|f\|_{L^\infty(\mathbb{R}^m)} \int\limits_{\mathcal{B}_{r+\varepsilon} \setminus \mathcal{B}_{r-\varepsilon}} \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \delta(\frac{x-y}{\varepsilon})\, dx\, dy \\ &\leq 2\nu(\varepsilon; f) + \|f\|_{L^\infty(\mathbb{R}^m)} \operatorname{vol}(\mathcal{B}_{r+\varepsilon} \setminus \mathcal{B}_{r-\varepsilon}) \\ &\leq 2\nu(\varepsilon; f) + \|f\|_{L^\infty(\mathbb{R}^m)} C_r \varepsilon. \end{aligned}$$





Furthermore,

$$\nu(\varepsilon; f) \leq 2\nu(\varepsilon; u).$$

The second term I_2 is estimated by the assumptions of the lemma, namely,

$$I_2 = \left| \int_{\mathcal{B}_r} \sigma_\varepsilon(x) f(x)\, dx \right| \leq \mathrm{Const}_{r,T} \Bigg(\sum_{|\alpha| \leq m} \|D^\alpha \sigma_\varepsilon\|_{L^\infty(\mathcal{B}_r)} \Bigg) |\tau| \leq C\, \frac{|\tau|}{\varepsilon^m}.$$

Combining the two estimates, we conclude that

$$\int_{\mathcal{B}_r} |u(x,t+\tau) - u(x,t)|\; dx \leq C_r \left(\varepsilon + \nu_{r,T,\hat{r}}(\varepsilon; u) + \frac{|\tau|}{\varepsilon^m} \right). \qquad \square$$

Next we need to extend the function u^n to all times. First, define

$$u^{n+j/(m+1)} = S_j u^{n+(j-1)/(m+1)}, \quad j = 1, \dots, m.$$

Now let

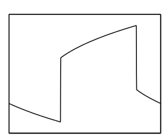
$$u_{\Delta t}(x,t) = \begin{cases} S_j((m+1)(t - t_{n+(j-1)/(m+1)}))u^{n+(j-1)/(m+1)} \\ \qquad \text{for } t \in [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)}), \\ H((m+1)(t - t_{n+m/(m+1)}))u^{n+m/(m+1)} \\ \qquad \text{for } t \in [t_{n+m/(m+1)}, t_{n+1}). \end{cases} \tag{4.81}$$

The estimates in Lemma 4.10 carry over to the function $u_{\Delta t}$. Fix $T > 0$. Applying Theorem A.11, we conclude that there exists a sequence of $\Delta t \to 0$ such that for each $t \in [0,T]$, the function $u_{\Delta t}(t)$ converges to a function $u(t)$, and the convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$. It remains to show that u is a weak solution of (4.67), or

$$\int_{\mathbb{R}^m} \int_0^t (u\phi_t + f(u) \cdot \nabla\phi + \nu u \Delta\phi)\; dt\, dx + \int_{\mathbb{R}^m} u_0 \phi|_{t=0}\, dx = \int_{\mathbb{R}^m} (u\phi)|_{t=T}\, dx \tag{4.82}$$

for all smooth and compactly supported test functions ϕ. We have

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_{t_{n+(j-1)/(m+1)}}^{t_{n+j/(m+1)}} \Big(\frac{1}{m+1} u_{\Delta t}\, \phi_t + f(u_{\Delta t}) \cdot \nabla\phi \Big)\, dt\, dx \\ &\quad = \frac{1}{m+1} \int_{\mathbb{R}^m} \int_0^{\Delta t} \bigg(u^{n+(j-1)/(m+1)}(x,\tilde{t})\, \phi_t \Big(x, \frac{\tilde{t} - t_{n+(j-1)/(m+1)}}{m+1} \Big) \\ &\qquad\qquad + f(u^{n+(j-1)/(m+1)}) \cdot \nabla\phi \Big(x, \frac{\tilde{t} - t_{n+(j-1)/(m+1)}}{m+1} \Big) \bigg)\, d\tilde{t}\, dx \\ &\quad = \frac{1}{m+1} \int_{\mathbb{R}^m} (u_{\Delta t}\phi) \Big|_{t=t_{n+(j-1)/(m+1)}}^{t=t_{n+j/(m+1)}}\; dx, \end{aligned} \tag{4.83}$$



for $j = 1, \dots, m$, where we have used that $u^{n+(j-1)/(m+1)}$ is a solution of the conservation law on the strip $t \in [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)})$. Similarly, we find for the solution of the heat equation that

$$\begin{aligned}\int_{\mathbb{R}^m} \int_{t_{n+m/(m+1)}}^{t_{n+1}} &\left(\frac{1}{m+1} u_{\Delta t}\phi_t + \mu u_{\Delta t}\Delta\phi\right) dt\, dx \\ &= \frac{1}{m+1}\int_{\mathbb{R}^m} \left((u_{\Delta t}\phi)\,|_{t=t_{n+m/(m+1)}} - (u_{\Delta t}\phi)\,|_{t=t_{n+1}}\right) dx.\end{aligned} \tag{4.84}$$

Summing (4.83) for $j = 1, \dots, m$, and adding the result to (4.84), we obtain

$$\begin{aligned}\int_{\mathbb{R}^m}\int_0^t &\left(\frac{1}{m+1} u_{\Delta t}\phi_t + f_{\Delta t}(u_{\Delta t}) \cdot \nabla\phi + \mu\chi_{m+1} u_{\Delta t}\Delta\phi\right) dt\, dx \\ &+ \frac{1}{m+1}\int_{\mathbb{R}^m} u_0\phi|_{t=0}\, dx = \frac{1}{m+1}\int_{\mathbb{R}^m} (u_{\Delta t}\phi)|_{t=T}\, dx,\end{aligned} \tag{4.85}$$

where

$$f_{\Delta t} = (\chi_1 f_1, \dots, \chi_m f_m)$$

and

$$\chi_j = \begin{cases} 1 & \text{for } t \in \cup_n [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)}), \\ 0 & \text{otherwise.} \end{cases}$$

As $\Delta t \to 0$, we have $\chi_j \overset{*}{\rightharpoonup} 1/(m+1)$, which proves (4.82). We summarize the result as follows.

Theorem 4.12 *Let u_0 be a function in $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$, and assume that f_j are piecewise twice continuously differentiable functions for $j = 1, \dots, m$. Define the family of functions $\{u_{\Delta t}\}$ by (4.70) and (4.81). Fix $T > 0$. Then there exists a sequence of $\Delta t \to 0$ such that $\{u_{\Delta t}(t)\}$ converges to a weak solution u of (4.67). The convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$.*

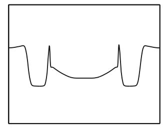
One can prove that a weak solution of (4.67) is indeed a classical solution; see [147]. Hence, by uniqueness of classical solutions, the sequence $\{u_{\Delta t}\}$ converges for *every* sequence $\{\Delta t\}$ tending to zero.

4.5 Operator Splitting: Source

> *Experience must be our only guide; Reason may mislead us.*
> *— J. Dickinson, the Constitutional Convention (1787)*

We will use operator splitting to study the inhomogeneous conservation law

$$u_t + \sum_{j=1}^m f_j(u)_{x_j} = g(x,t,u), \quad u|_{t=0} = u_0, \tag{4.86}$$



where the source term g is assumed to be continuous in (x, t) and Lipschitz continuous in u. In this case the Kružkov entropy condition reads as follows. The bounded function u is a weak entropy solution on $[0, T]$ if it satisfies

$$\int_0^T \int_{\mathbb{R}^m} \Big(|u-k|\,\varphi_t + \operatorname{sign}(u-k) \sum_{j=1}^m \big(f_j(u) - f_j(k)\big)\,\varphi_{x_j} \Big)\, dx_1 \cdots dx_m\, dt$$
$$+ \int_{\mathbb{R}^m} |u_0 - k|\,\varphi|_{t=0}\, dx_1 \cdots dx_m - \int_{\mathbb{R}^m} (|u-k|\,\varphi)|_{t=T}\, dx_1 \cdots dx_m$$
$$\geq - \int_0^T \int_{\mathbb{R}^m} \operatorname{sign}(u-k)\,\varphi g(x,t,u)\, dx_1 \cdots dx_m\, dt, \tag{4.87}$$

for all constants $k \in \mathbb{R}$ and all nonnegative test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0, T])$.

To simplify the presentation we consider only the case with $m = 1$, and where $g = g(u)$. Thus

$$u_t + f(u)_x = g(u). \tag{4.88}$$

The case in which g also depends on (x, t) is treated in Exercise 4.7. Let $S(t)u_0$ and $R(t)u_0$ denote the solutions of

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{4.89}$$

and

$$u_t = g(u), \quad u|_{t=0} = u_0, \tag{4.90}$$

respectively. Define the sequence $\{u^n\}$ by (we still use $t_n = n\Delta t$)

$$u^0 = u_0, \quad u^{n+1} = (S(\Delta t) R(\Delta t))\, u^n$$

for some positive Δt. Furthermore, we need the extension to all times, defined by[6]

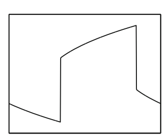
$$u_{\Delta t}(x,t) = \begin{cases} S(2(t-t_n))u^n & \text{for } t \in [t_n, t_{n+1/2}), \\ R\left(2\left(t - t_{n+1/2}\right)\right) u^{n+1/2} & \text{for } t \in [t_{n+1/2}, t_{n+1}), \end{cases} \tag{4.91}$$

with

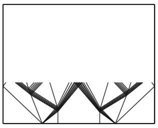
$$u^{n+1/2} = S(\Delta t)u^n, \quad t_{n+1/2} = \Big(n + \frac{1}{2}\Big)\Delta t.$$

For this procedure to be welldefined, we must be sure that the ordinary differential equation (4.90) is welldefined. This is the case if g is uniformly Lipschitz continuous in u, i.e.,

$$|g(u) - g(v)| \leq \|g\|_{\mathrm{Lip}}\, |u - v|\,. \tag{4.92}$$



[6] Essentially replacing the operator H used in operator splitting with respect to diffusion by R in the case of a source.



For convenience, we set $\gamma = \|g\|_{\mathrm{Lip}}$. This assumption also implies that the solution of (4.90) does not "blow up" in finite time, since

$$|g(u)| \le |g(0)| + \gamma\,|u| \le C_g(1 + |u|), \tag{4.93}$$

for some constant C_g. Under this assumption on g we have the following lemma.

Lemma 4.13 *Assume that u_0 is a function in $L^1_{\mathrm{loc}}(\mathbb{R})$, and that u_0 is of bounded variation. Then for $n\Delta t \le T$, the following estimates hold:*

(i) *There is a constant M_1 independent of n and Δt such that*

$$\|u^n\|_{L^\infty(\mathbb{R})} \le M_1. \tag{4.94}$$

(ii) *There is a constant M_2 independent of n and Δt such that*

$$\mathrm{T.V.}\,(u^n) \le M_2. \tag{4.95}$$

(iii) *There is a constant M_3 independent of n and Δt such that for t_1 and t_2, with $0 \le t_1 \le t_2 \le T$, and for each bounded interval $B \subset \mathbb{R}$,*

$$\int_B |u_{\Delta t}(x, t_1) - u_{\Delta t}(x, t_2)|\, dx \le M_3\,|t_1 - t_2|\,. \tag{4.96}$$

Proof We start by proving (i). The solution operator S_t obeys a maximum principle, so that $\left\|u^{n+1/2}\right\|_\infty \le \|u^n\|_\infty$. Multiplying (4.90) by sign (u), we find that

$$|u|_t = \mathrm{sign}\,(u)\, g(u) \le |g(u)| \le C_g(1 + |u|),$$

where we have used (4.93). By Gronwall's inequality (see Exercise 1.10), for a solution of (4.90), we have that

$$|u(t)| \le e^{C_g t}(1 + |u_0|) - 1.$$

This means that

$$\begin{aligned}\left\|u^{n+1}\right\|_{L^\infty(\mathbb{R})} &\le e^{C_g \Delta t}\left(1 + \left\|u^{n+1/2}\right\|_{L^\infty(\mathbb{R})}\right) - 1\\ &\le e^{C_g \Delta t}\left(1 + \|u^n\|_{L^\infty(\mathbb{R})}\right) - 1,\end{aligned}$$

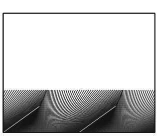
which by induction implies

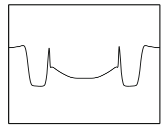
$$\|u^n\|_{L^\infty(\mathbb{R})} \le e^{C_g t_n}\left(1 + \|u_0\|_{L^\infty(\mathbb{R})}\right) - 1.$$

Setting

$$M_1 = e^{C_g T}\left(1 + \|u_0\|_{L^\infty(\mathbb{R})}\right) - 1$$

proves (i).





Next, we prove (ii). The proof is similar to that of the last case, since S_t is TVD, $\text{T.V.}\left(u^{n+1/2}\right) \leq \text{T.V.}\left(u^n\right)$. As before, let u be a solution of (4.90) and let v be another solution with initial data v_0. Then we have $(u-v)_t = g(u) - g(v)$. Setting $w = u - v$, and multiplying by $\operatorname{sign}(w)$, we find that

$$|w|_t = \operatorname{sign}(w)\,(g(u) - g(v)) \leq \gamma\,|w|\,.$$

Then by Gronwall's inequality,

$$|w(t)| \leq e^{\gamma t}\,|w(0)|\,.$$

Hence,

$$\left|u^{n+1}(x) - u^{n+1}(y)\right| \leq e^{\gamma \Delta t}\left|u^{n+1/2}(x) - u^{n+1/2}(y)\right|.$$

This implies that

$$\text{T.V.}\left(u^{n+1}\right) \leq e^{\gamma \Delta t}\,\text{T.V.}\left(u^{n+1/2}\right) \leq e^{\gamma \Delta t}\,\text{T.V.}\left(u^{n}\right).$$

Inductively, we then have that

$$\text{T.V.}\left(u^{n}\right) \leq e^{\gamma t_n}\,\text{T.V.}\left(u_0\right),$$

and setting $M_2 = e^{\gamma T}$ concludes the proof of (ii).

Regarding (iii), we know that

$$\int_B \left|u^{n+1/2}(x) - u^n(x)\right|\,dx \leq C\,\Delta t.$$

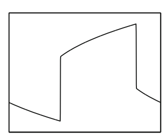
We also have that

$$\begin{aligned}
\int_B \left|u^{n+1}(x) - u^{n+1/2}(x)\right|\,dx &= \int_B \left|\int_0^{\Delta t} g\left(u_{\Delta t}(x, t - t_n)\right)\,dt\right|\,dx \\
&\leq \int_B \int_0^{\Delta t} \left|g\left(u_{\Delta t}(x, t - t_n)\right)\right|\,dt\,dx \\
&\leq C_g \int_0^{\Delta t} \int_B (1 + M_1)\,dx\,dt \\
&= |B|\,C_g(1 + M_1)\Delta t,
\end{aligned}$$

where $|B|$ denotes the length of B. Setting $M_3 = C + |B|\,C_g(1 + M_1)$ shows that

$$\int_B \left|u^{n+1}(x) - u^n(x)\right| \leq M_3 \Delta t,$$

which implies (iii). □



Fix $T > 0$. Theorem A.11 implies the existence of a sequence $\Delta t \to 0$ such that for each $t \in [0, T]$, the function $u_{\Delta t}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to a bounded function of bounded variation $u(t)$. The convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$. It remains to show that u solves (4.88) in the sense of (4.87).

Using that $u_{\Delta t}$ is an entropy solution of the conservation law without source term (4.89) in the interval $[t_n, t_{n+1/2}]$, we obtain[7]

$$2\int_{t_n}^{t_{n+1/2}}\int \left(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t + \operatorname{sign}(u_{\Delta t}-k)\,(f(u_{\Delta t})-f(k))\varphi_x\right) dx\,dt \\ + \int \left(\left|u_{\Delta t}-k\right|\varphi\right)\Big|_{t=t_{n+1/2}}^{t=t_n} dx \ge 0. \tag{4.97}$$

Regarding solutions of (4.90), since $k_t = 0$ for every constant k, we find that

$$\left|u-k\right|_t = \operatorname{sign}(u-k)\,(u-k)_t = \operatorname{sign}(u-k)\,g(u).$$

Multiplying this by a test function $\phi(t)$ and integrating over $s \in [0,t]$, we find after a partial integration that

$$\int_0^t \left(\left|u-k\right|\phi_s + \operatorname{sign}(u-k)\,g(u)\phi\right) ds + u\phi\big|_{s=0}^{s=t} = 0.$$

Since $u_{\Delta t}$ is a solution of the ordinary differential equation (4.90) on the interval $[t_{n+1/2}, t_{n+1}]$ (with time running "twice as fast"; see (4.91)), we find that

$$2\int_{t_n}^{t_{n+1/2}}\int \left(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t + \operatorname{sign}(u_{\Delta t}-k)\,g(u_{\Delta t})\varphi\right) dx\,dt \\ + \int \left(\left|u_{\Delta t}-k\right|\varphi\right)\Big|_{t=t_{n+1}}^{t=t_{n+1/2}} dx = 0.$$

Adding this and (4.97), and summing over n, we obtain

$$2\int_0^T\int \left(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t + \chi_{\Delta t}\operatorname{sign}(u_{\Delta t}-k)\,(f(u_{\Delta t})-f(k))\varphi_x \right. \\ \left. + \tilde{\chi}_{\Delta t}\operatorname{sign}(u_{\Delta t}-k)\,g(u_{\Delta t})\varphi\right) dx\,dt \\ - \int \left(\left|u_{\Delta t}-k\right|\varphi\right)\big|_{t=0}^{t=T} dx \ge 0,$$

where $\chi_{\Delta t}$ and $\tilde{\chi}_{\Delta t}$ denote the characteristic functions of the sets $\cup_n [t_n, t_{n+1/2})$ and $\cup_n [t_{n+1/2}, t_{n+1})$, respectively. We have that $\chi_{\Delta t} \overset{*}{\rightharpoonup} \frac{1}{2}$ and $\tilde{\chi}_{\Delta t} \overset{*}{\rightharpoonup} \frac{1}{2}$, and hence we conclude that (4.87) holds in the limit as $\Delta t \to 0$.

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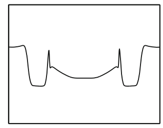
[7] The constants 2 and $\frac{1}{2}$ come from the fact that time is running "twice as fast" in the solution operators S and R in (4.91) (cf. also (4.16)–(4.17)).



Theorem 4.14 *Let $f(u)$ be piecewise twice continuously differentiable, and assume that $g = g(u)$ satisfies the bound (4.92). Let u_0 be a bounded function of bounded variation. Then the initial value problem*

$$u_t + f(u)_x = g(u), \qquad u(x,0) = u_0(x) \tag{4.98}$$

has a weak entropy solution, which can be constructed as the limit of the sequence $\{u_{\Delta t}\}$ defined by (4.91).

4.6 Notes

Dimensional splitting for hyperbolic equations was first introduced by Bagrinovskiĭ and Godunov [7] in 1957. Crandall and Majda made a comprehensive and systematic study of dimensional splitting (or the fractional steps method) in [52]. In [53] they used dimensional splitting to prove convergence of monotone schemes as well as the Lax–Wendroff scheme and the Glimm scheme, i.e., the random choice method. A more general introduction to operator splitting can be found in [91].

There are also methods for multidimensional conservation laws that are intrinsically multidimensional. However, we have here decided to use dimensional splitting as our technique because it is conceptually simple and allows us to take advantage of the one-dimensional analysis.

Another natural approach to the study of multidimensional equations based on the front-tracking concept is first to make the standard front-tracking approximation: Replace the initial data by a piecewise constant function, and replace flux functions by piecewise linear and continuous functions. That gives rise to truly two-dimensional Riemann problems at each grid point $(i\Delta x, j\Delta y)$. However, that approach has turned out to be rather cumbersome even for a single Riemann problem and piecewise linear and continuous flux functions f and g. See Risebro [159].

The one-dimensional front-tracking approach combined with dimensional splitting was first introduced in Holden and Risebro [93]. The theorem on the convergence rate of dimensional splitting was proved independently by Teng [178] and Karlsen [105, 106]. Our presentation here follows Haugse, Lie, and Karlsen [133]. Sect. 4.4, using operator splitting to solve the parabolic regularization, is taken from Karlsen and Risebro [108]. The Kružkov interpolation lemma, Lemma 4.11, is taken from [117]; see also [108].

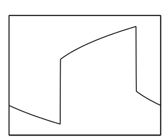
The presentation in Sect. 4.5 can be found in Holden and Risebro [95], where also the case with a stochastic source is treated. The convergence rate in the case of operator splitting applied to a conservation law with a source term is discussed in Langseth, Tveito, and Winther [123].

4.7 Exercises

4.1 Consider the initial value problem

$$u_t + f(u)_x + g(u)_y = 0, \quad u|_{t=0} = u_0,$$

where f, g are piecewise twice continuously differentiable functions, and u_0 is a bounded integrable function with finite total variation.



(a) Show that the solution u is Lipschitz continuous in time; that is,

$$\|u(t) - u(s)\|_{L^1(\mathbb{R}^2)} \le |t - s| \left(\|f\|_{\text{Lip}} \vee \|g\|_{\text{Lip}}\right) \text{T.V.}\,(u_0)\,.$$

(b) Let v_0 be another function with the same properties as u_0. Show that if $u_0 \le v_0$, then also $u \le v$ almost everywhere, where v is the solution with initial data v_0.

4.2 Consider the initial value problem

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{4.99}$$

where f is a piecewise twice continuously differentiable function and u_0 is a bounded, integrable function with finite total variation. Write

$$f = f_1 + f_2$$

and let $S_j(t)u_0$ denote the solution of

$$u_t + f_j(u)_x = 0, \quad u|_{t=0} = u_0.$$

Prove that operator splitting converges to the solution of (4.99). Determine the convergence rate.

4.3 Prove (4.38), that is, that

$$\iint |\pi\psi - \psi|\, dx\, dy \le (\Delta x + \Delta y)\,\text{T.V.}\,(\psi)\,,$$

for all functions ψ of bounded variation.

4.4 Consider the heat equation in $\mathbb{R}^m$,

$$u_t = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}, \qquad u(x,0) = u_0(x). \tag{4.100}$$

Let H_t^i denote the solution operator for the heat equation in the ith direction, i.e., we write the solution of

$$u_t = \frac{\partial^2 u}{\partial x_i^2}, \quad u(x,0) = u_0(x),$$

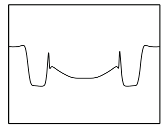
as $H_t^i u_0$. Define

$$u^n(x) = \left[H_{\Delta t}^m \circ \cdots \circ H_{\Delta t}^1\right]^n u_0(x),$$
$$u^{n+j/m}(x) = H_{\Delta t}^j \circ H_{\Delta t}^{j-1} \circ \cdots \circ H_{\Delta t}^1 u^n(x),$$

for $j = 1, \ldots, m$, and $n \ge 0$.
For t in the interval $[t_n + ((j-1)/m)\Delta t, t_n + (j/m)\Delta t]$ define

$$u_{\Delta t}(x,t) = H_{m(t-t_{n+(j-1)/m})}^j u^{n+(j-1)/m}(x).$$

If the initial function $u_0(x)$ is bounded and of bounded variation, show that $\{u_{\Delta t}\}$ converges in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^m))$ to a weak solution of (4.100).



4.5 We consider the viscous conservation law in one space dimension,

$$u_t + f(u)_x = u_{xx}, \quad u(x,0) = u_0(x), \tag{4.101}$$

where f satisfies the "usual" assumptions and u_0 is in $L^1 \cap BV$. Consider the following scheme based on operator splitting:

$$\begin{aligned} U_j^{n+1/2} &= \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \lambda\left(f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right), \\ U_j^{n+1} &= U_j^{n+1/2} + \mu\left(U_{j+1}^{n+1/2} - 2U_j^{n+1/2} + U_{j-1}^{n+1/2}\right), \end{aligned}$$

for $n \geq 0$, where $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/\Delta x^2$. Set

$$U_j^0 = \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_0(x)\,dx.$$

We see that we use the Lax–Friedrichs scheme for the conservation law and an explicit difference scheme for the heat equation. Let

$$u_{\Delta t}(x,t) = U_j^n$$

for $\left(j - \frac{1}{2}\right)\Delta x \leq x < \left(j + \frac{1}{2}\right)\Delta x$ and $n\Delta t < t \leq (n+1)\Delta t$.

(a) Show that this gives a monotone and consistent scheme, provided that a CFL condition holds.
(b) Show that there is a sequence of Δt's such that $u_{\Delta t}$ converges to a weak solution of (4.101) as $\Delta t \to 0$.

(a) Assume that u, f, and g are in $L^1([0,T])$, and that g is nonnegative, while f is strictly positive and nondecreasing. Assume that

$$u(t) \leq f(t) + \int_0^t g(s)u(s)\,ds, \quad t \in [0,T].$$

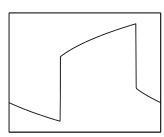
Show that

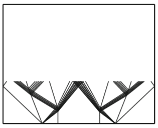
$$u(t) \leq f(t) \exp\Big(\int_0^t g(s)\,ds\Big), \quad t \in [0,T].$$

4.6 Assume that u and v are entropy solutions of

$$\begin{aligned} u_t + f(u)_x &= g(u), \quad u(x,0) = u_0(x), \\ v_t + f(v)_x &= g(v), \quad v(x,0) = v_0(x), \end{aligned}$$

where u_0 and v_0 are in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and f and g satisfy the assumptions of Theorem 4.14.





(a) Use the entropy formulation (4.87) and mimic the arguments used to prove (2.60) to show that for every nonnegative test function ψ,

$$\begin{aligned}\iint \big(|u(x,t) - v(x,t)|\, \psi_t + q(u,v)\psi_x \big)\, dt\, dx \\ - \int |u(x,T) - v(x,T)|\, \psi(x,T)\, dx \\ + \int |u_0(x) - v_0(x)|\, \psi(x,0)\, dx \\ \geq \iint \operatorname{sign}(u-v)\, (g(u) - g(v))\psi\, dt\, dx.\end{aligned}$$

(b) Define $\psi(x,t)$ by (2.61), and set

$$h(t) = \int |u(x,t) - v(x,t)|\, \psi(x,t)\, dx.$$

Show that

$$h(T) \leq h(0) + \gamma \int_0^T h(t)\, dt,$$

where γ denotes the Lipschitz constant of g. Use the previous exercise to conclude that

$$h(T) \leq h(0) \left(1 + \gamma T e^{\gamma T}\right).$$

(c) Show that

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} \left(1 + \gamma t e^{\gamma t}\right),$$

and hence that entropy solutions of (4.98) are unique. Note that this implies that $\{u_{\Delta t}\}$ defined by (4.91) converges to the entropy solution for *every* sequence $\{\Delta t\}$ such that $\Delta t \to 0$.

4.7 We consider the case that the source depends on (x,t). For $u_0 \in L^1_{\text{loc}} \cap BV$, let u be an entropy solution of

$$u_t + f(u)_x = g(x,t,u), \quad u(x,0) = u_0(x), \tag{4.102}$$

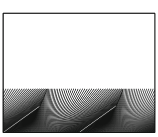
where g is bounded for each fixed u and continuous in t, and satisfies

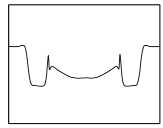
$$\begin{aligned}|g(x,t,u) - g(x,t,v)| \leq \gamma\, |u - v|, \\ \text{T.V.}\,(g(\,\cdot\,,t,u)) \leq b(t),\end{aligned}$$

where the constant γ is independent of x and t, for all u and v and for a bounded function $b(t)$ in $L^1([0,T])$. We let S_t be as before, and let $R(x,t,s)u_0$ denote the solution of

$$u'(t) = g(x,t,u), \quad u(s) = u_0,$$

for $t > s$.





(a) Define an operator splitting approximation $u_{\Delta t}$ using S_t and $R(x, t, s)$.
(b) Show that there is a sequence of Δt's such that $u_{\Delta t}$ converges in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ to a function of bounded variation u.
(c) Show that u is an entropy solution of (4.102).

4.8 Show that if the initial data u_0 of the heat equation $u_t = \Delta u$ is smooth, that is, $u_0 \in C_0^\infty$, then

$$\|u(t) - u_0\|_{L^1} \leq C\, t.$$

Compare this result with (4.76).