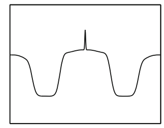


Chapter 1

Introduction

I have no objection to the use of the term "Burgers' equation"
for the nonlinear heat equation
(provided it is not written "Burger's equation").
— Letter from Burgers to Batchelor (*1968*)

Hyperbolic conservation laws are partial differential equations of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0.$$

If we write $f = (f_1, \dots, f_m)$, $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, and introduce initial data u_0 at $t = 0$, the Cauchy problem for hyperbolic conservation laws reads

$$\frac{\partial u(x,t)}{\partial t} + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f_j\,(u(x,t)) = 0, \quad u|_{t=0} = u_0. \tag{1.1}$$

In applications, t normally denotes the time variable, while x describes the spatial variation in m space dimensions. The unknown function u (as well as each f_j) can be a vector, in which case we say that we have a system of equations, or u and each f_j can be a scalar. This book covers the theory of scalar conservation laws in several space dimensions as well as the theory of systems of hyperbolic conservation laws in one space dimension. In the present chapter we study the one-dimensional scalar case to highlight some of the fundamental issues in the theory of conservation laws.

We use subscripts to denote partial derivatives, i.e., $u_t(x,t) = \partial u(x,t)/\partial t$. Hence we may write (1.1) when $m = 1$ as

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0. \tag{1.2}$$

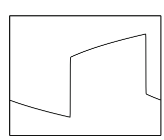
If we formally integrate equation (1.2) between two points x_1 and x_2, we obtain

$$\int_{x_1}^{x_2} u_t\,dx = -\int_{x_1}^{x_2} f(u)_x\,dx = f\,(u\,(x_1,t)) - f\,(u\,(x_2,t))\,.$$

Assuming that u is sufficiently regular to allow us to take the derivative outside the integral, we get

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f\,(u\,(x_1,t)) - f\,(u\,(x_2,t))\,. \tag{1.3}$$

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_1



This equation expresses conservation of the quantity measured by u in the sense that the rate of change in the amount of u between x_1 and x_2 is given by the difference in $f(u)$ evaluated at these points.[1] Therefore, it is natural to interpret $f(u)$ as the *flux density* of u. Often, $f(u)$ is referred to as the *flux function*.

Consider a fluid with density $\rho = \rho(x,t)$ and velocity v. Assume that there are no sources or sinks, so that amount of fluid is conserved. For a given and fixed bounded domain $D \subset \mathbb{R}^m$, conservation of fluid implies

$$\frac{d}{dt}\int_D \rho(x,t)\,dx = -\int_{\partial D} (\rho v)\cdot n\,dS_x, \tag{1.4}$$

where n is the outward unit normal at the boundary ∂D of D. If we interchange the time derivative and the integral on the left-hand side of the equation, and apply the divergence theorem on the right-hand side, we obtain

$$\int_D \rho(x,t)_t\,dx = -\int_D \operatorname{div}(\rho v)\,dx \tag{1.5}$$

which we rewrite as

$$\int_D \big(\rho_t + \operatorname{div}(\rho v)\big)\,dx = 0. \tag{1.6}$$

Since the domain D was arbitrary, we obtain the hyperbolic conservation law

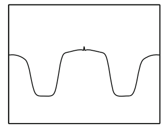
$$\rho_t + \operatorname{div}(\rho v) = 0. \tag{1.7}$$

The above derivation is very fundamental, and only two assumptions are made. First of all, we make the physical assumption of conservation, and secondly, we assume sufficient smoothness of the functions to perform the necessary mathematical manipulations. The latter aspect will a recurring theme throughout the book.

As a simple example of a conservation law, consider a one-dimensional medium consisting of noninteracting particles, or material points, identified by their coordinates y along a line. Let $\phi(y,t)$ denote the position of material point y at time t. The velocity and the acceleration of y at time t are given by $\phi_t(y,t)$ and $\phi_{tt}(y,t)$, respectively. Assume that for each t, $\phi(\,\cdot\,,t)$ is strictly increasing, so that two distinct material points cannot occupy the same position at the same time. Then the function $\phi(\,\cdot\,,t)$ has an inverse $\psi(\,\cdot\,,t)$, so that $y = \psi(\phi(y,t),t)$ for all t. Hence $x = \phi(y,t)$ is equivalent to $y = \psi(x,t)$. Now let u denote the velocity of the material point occupying position x at time t, i.e., $u(x,t) = \phi_t(\psi(x,t),t)$, or equivalently, $u(\phi(y,t),t) = \phi_t(y,t)$. Then the acceleration of material point y at time t is

$$\begin{aligned}\phi_{tt}(y,t) &= u_t(\phi(y,t),t) + u_x(\phi(y,t),t)\phi_t(y,t)\\ &= u_t(x,t) + u_x(x,t)u(x,t).\end{aligned}$$

[1] In physics one normally describes conservation of a quantity in integral form, that is, one starts with (1.3). The differential equation (1.2) then follows under additional regularity conditions on u.



If the material particles are noninteracting, so that they exert no force on each other, and there is no external force acting on them, then Newton's second law requires the acceleration to be zero, giving

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{1.8}$$

The last equation, (1.8), is a conservation law; it expresses that u is conserved with a flux density given by $u^2/2$. This equation is often referred to as the *Burgers equation without viscosity,*[2] and is in some sense the simplest nonlinear conservation law.

Burgers's equation, and indeed any conservation law, is an example of a *quasilinear* equation, meaning that the highest derivatives occur linearly. A general inhomogeneous quasilinear equation for functions of two variables x and t can be written

$$a(x,t,u)u_t + b(x,t,u)u_x = c(x,t,u). \tag{1.9}$$

If the coefficients a and b are independent of u, i.e., $a = a(x,t)$, $b = b(x,t)$, we say that the equation is *semilinear*, while the equation is *linear* if, in addition, the same applies to c, i.e., $c = c(x,t)$.

We may consider the solution as the surface $S = \{(t,x,u(x,t)) \in \mathbb{R}^3 \mid (t,x) \in \mathbb{R}^2\}$ in $\mathbb{R}^3$. Let Γ be a given curve in $\mathbb{R}^3$ (which one may think of as the initial data if t is constant) parameterized by $(t(y), x(y), z(y))$ for y in some interval. We want to construct the surface $S \subset \mathbb{R}^3$ parameterized by $(t, x, u(x,t))$ such that $u = u(x,t)$ satisfies (1.9) and $\Gamma \subset S$. It turns out to be advantageous to consider the surface S parameterized by new variables (s, y), thus $t = t(s,y)$, $x = x(s,y)$, $z = z(s,y)$, in such a way that $u(x,t) = z(s,y)$. We solve the system of ordinary differential equations

$$\frac{\partial t}{\partial s} = a, \quad \frac{\partial x}{\partial s} = b, \quad \frac{\partial z}{\partial s} = c, \tag{1.10}$$

with

$$t(s_0, y) = t(y), \quad x(s_0, y) = x(y), \quad z(s_0, y) = z(y). \tag{1.11}$$

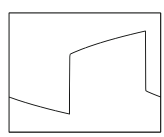
In this way we obtain the parameterized surface $S = \{(t(s,y), x(s,y), z(s,y) \mid (s,y) \in \mathbb{R}^2\}$. Assume that we can invert the relations $x = x(s,y)$, $t = t(s,y)$ and write $s = s(x,t)$, $y = y(x,t)$. Then

$$u(x,t) = z(s(x,t), y(x,t)) \tag{1.12}$$

satisfies both (1.9) and the condition $\Gamma \subset S$. Namely, we have

$$c = \frac{\partial z}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s} = u_x b + u_t a. \tag{1.13}$$

[2] Henceforth we will adhere to common practice and call it the inviscid Burgers equation.



However, there are many pitfalls in the above construction: the solution (1.10) may only be local, and we may not be able to invert the solution of the differential equation to express (s, y) as functions of (x, t). These problems are intrinsic to equations of this type and will be discussed at length.

Equation (1.10) is called the *characteristic equation*, and its solutions are called *characteristics*. This can sometimes be used to find explicit solutions of conservation laws. In the homogeneous case, that is, when $c = 0$, the solution u is constant along characteristics, namely,

$$\frac{d}{ds}u(x(s,y),t(s,y)) = u_x x_s + u_t t_s = u_x b + u_t a = 0. \tag{1.14}$$

◇ Example 1.1
Consider the (quasi)linear equation

$$u_t - x u_x = -2u, \quad u(x,0) = x,$$

with associated characteristic equations

$$\frac{\partial t}{\partial s} = 1, \quad \frac{\partial x}{\partial s} = -x, \quad \frac{\partial z}{\partial s} = -2z.$$

The general solution of the characteristic equations reads

$$t = t_0 + s, \quad x = x_0 e^{-s}, \quad z = z_0 e^{-2s}.$$

Parameterizing the initial data for $s = 0$ by $t = 0$, $x = y$, and $z = y$, we obtain

$$t = s, \quad x = y e^{-s}, \quad z = y e^{-2s},$$

which can be inverted to yield

$$u = u(x,t) = z(s,y) = x e^{-t}. \qquad \diamond$$

◇ Example 1.2
Consider the (quasi)linear equation

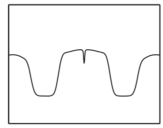
$$x u_t - t^2 u_x = 0. \tag{1.15}$$

Its associated characteristic equation is

$$\frac{\partial t}{\partial s} = x, \quad \frac{\partial x}{\partial s} = -t^2.$$

This has solutions given implicitly by $x^2/2 + t^3/3$ equals a constant, since after all, $\partial(x^2/2 + t^3/3)/\partial s = 0$, so the solution of (1.15) is any function φ of $x^2/2 + t^3/3$, i.e., $u(x,t) = \varphi(x^2/2 + t^3/3)$. For example, if we wish to solve the initial value problem (1.15) with $u(x,0) = \sin|x|$, then $u(x,0) = \varphi(x^2/2) = \sin|x|$. Consequently, $\varphi(\zeta) = \sin\sqrt{2\zeta}$ with $\zeta \geq 0$, and the solution u is given by

$$u(x,t) = \sin\sqrt{x^2 + 2t^3/3}, \qquad t \geq 0. \qquad \diamond$$



◇ Example 1.3 (Burgers's equation)
If we apply this technique to Burgers's equation(1.8) with initial data $u(x, 0) = u_0(x)$, we get that

$$\frac{\partial t}{\partial s} = 1, \qquad \frac{\partial x}{\partial s} = z, \quad \text{and} \quad \frac{\partial z}{\partial s} = 0$$

with initial conditions $t(0, y) = 0$, $x(0, y) = y$, and $z(0, y) = u_0(y)$. We cannot solve these equations without knowing more about u_0, but since u (or z) is constant along characteristics, cf. (1.14), we see that the characteristics are straight lines. In other words, the value of z is transported along characteristics, so that

$$t(s, y) = s, \quad x(s, y) = y + sz = y + su_0(\eta), \quad z(s, y) = u_0(y).$$

We may write this as

$$x = y + u_0(y)t. \tag{1.16}$$

If we solve this equation in terms of $y = y(x, t)$, we can use y to obtain $u(x, t) = z(s, y) = u_0(y(x, t))$, yielding the implicit relation

$$u(x, t) = u_0(x - u(x, t)t). \tag{1.17}$$

Given a point (x, t), one can in principle determine the solution $u = u(x, t)$ from equation (1.17). By differentiating equation (1.16) we find that

$$\frac{\partial x}{\partial y} = 1 + tu_0'(y). \tag{1.18}$$

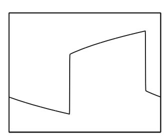
Thus a solution certainly exists for all $t > 0$ if $u_0' > 0$, since x is a strictly increasing function of η in that case. On the other hand, if $u_0'(\tilde{x}) < 0$ for some $\tilde{x}$, then a solution cannot be found for $t > t^* = -1/u_0'(\tilde{x})$. For example, if $u_0(x) = -\arctan(x)$, there is no smooth solution for $t > 1$.

What actually happens when a smooth solution cannot be defined? From (1.18) we see that for $t > t^*$, there are several y that satisfy (1.16) for each x, since x is no longer a strictly increasing function of y. In some sense, we can say that the solution u is multivalued at such points. To illustrate this, consider the surface in (t, x, u)-space parameterized by s and y,

$$(s, y + su_0(y), u_0(\eta)).$$

Let us assume that the initial data are given by $u_0(x) = -\arctan(x)$ and $t_0 = 0$. For each fixed t, the curve traced out by the surface is the graph of a (multivalued) function of x. In Fig. 1.1 we see how the multivaluedness starts at $t = 1$ when the surface "folds over," and that for $t > 1$ there are some x that have three associated u values. To continue the solution beyond $t = 1$ we have to choose among these three u values. In any case, it is impossible to continue the solution and at the same time keep it continuous. ◇

Now we have seen that no matter how smooth the initial function is, we cannot expect to be able to define classical solutions of nonlinear conservation laws for



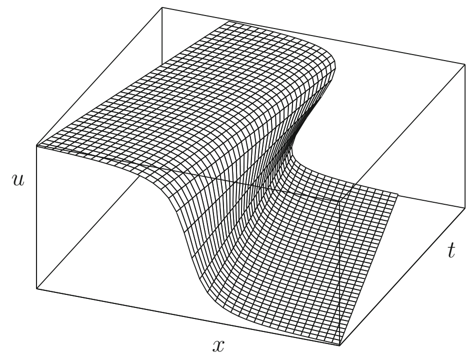


Fig. 1.1 A multivalued solution

all time. In this case we have to extend the concept of solution in order to allow discontinuities.

The standard way of extending the admissible set of solutions to partial differential equations is to look for *weak solutions* rather than so-called *classical solutions*, by introducing distribution theory. Classical solutions are sufficiently differentiable functions such that the differential equation is satisfied for all values of the independent arguments. However, there is no unique definition of weak solutions. In the context of hyperbolic conservation laws we do not need the full machinery of distribution theory, and our solutions will be functions that may be nondifferentiable.

In this book we use the following standard notation: $C^i(U)$ is the set of i times continuously differentiable functions on a set $U \subseteq \mathbb{R}^n$, and $C_0^i(U)$ denotes the set of such functions that have compact support in U. Then $C^\infty(U) = \bigcap_{i=0}^\infty C^i(U)$, and similarly for C_0^∞. Where there is no ambiguity, we sometimes omit the set U and write only C^0, etc.

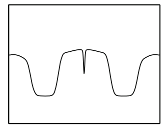
If we have a classical solution to (1.2), we can multiply the equation by a function $\varphi = \varphi(x,t) \in C_0^\infty(\mathbb{R} \times [0,\infty))$, called a *test function*, and integrate by parts to get

$$
\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}} \left(u_t\varphi + f(u)_x\varphi\right) dx\, dt \\
&= -\int_0^\infty \int_{\mathbb{R}} \left(u\varphi_t + f(u)\varphi_x\right) dx\, dt - \int_{\mathbb{R}} u(x,0)\varphi(x,0)\, dx.
\end{aligned}
$$

Observe that the boundary terms at $t = \infty$ and at $x = \pm\infty$ vanish, since φ has compact support, and that the final expression incorporates the initial data. Now we *define* a weak solution of (1.2) to be a measurable function $u(x,t)$ such that

$$
\int_0^\infty \int_{\mathbb{R}} \left(u\varphi_t + f(u)\varphi_x\right) dxdt + \int_{\mathbb{R}} u_0\varphi(x,0)\, dx = 0 \tag{1.19}
$$

holds for all $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$. We see that the weak solution u is no longer required to be differentiable, and that a classical solution is also a weak solution. We will spend considerable time in understanding the constraints that the equation (1.19) puts on u.



We employ the usual notation that for $p \in [0, \infty)$, $L^p(U)$ denotes the set of all measurable functions $F: U \to \mathbb{R}$ such that the integral

$$\int_U |F|^p \, dx$$

is finite. The set $L^p(U)$ is equipped with the norm

$$\|F\|_p = \|F\|_{L^p} = \|F\|_{L^p(U)} = \left(\int_U |F|^p dx \right)^{1/p}.$$

If $p = \infty$, $L^\infty(U)$ denotes the set of all measurable functions F such that

$$\operatorname{ess\,sup}_U |F|$$

is finite. The space $L^\infty(U)$ has the norm $\|F\|_\infty = \operatorname{ess\,sup}_U |F|$. As is well-known, the spaces $L^p(U)$ are Banach spaces for $p \in [1, \infty]$, and $L^2(U)$ is a Hilbert space. In addition, we will frequently use the spaces

$$L^p_{\mathrm{loc}}(U) = \{f: U \to \mathbb{R} \mid f \in L^p(K) \text{ for every compact set } K \subseteq U\}.$$

So what kind of discontinuities are compatible with (1.19)? If we assume that u is constant outside some finite interval, the remarks below (1.2) imply that

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) \, dx = 0.$$

Hence, the total amount of u is independent of time, or equivalently, the area below the graph of $u(\,\cdot\,, t)$ is constant.

◇ Example 1.4 (Burgers's equation (cont'd.))
We now wish to determine a discontinuous function such that the graph of the function lies on the surface given earlier with $u(x, 0) = -\arctan x$. Furthermore, the area under the graph of the function should be equal to the area between the x-axis and the surface. In Fig. 1.2 we see a section of the surface making up the solution for $t = 3$. The curve is parameterized by x_0, and explicitly given by $u = -\arctan(x_0)$, $x = x_0 - 3\arctan(x_0)$.

The function u is shown by a thick line, and the surface is shown by a dotted line. A function $u(x)$ that has the correct integral, $\int u \, dx = \int u_0 \, dx$, is easily

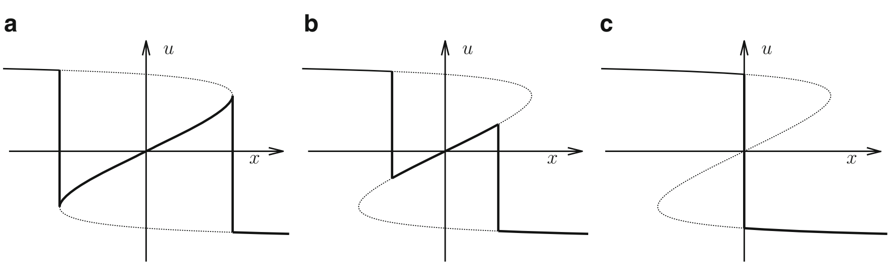
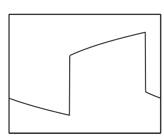


Fig. 1.2 Different solutions with u conserved



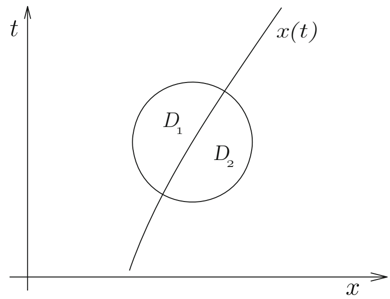


Fig. 1.3 An isolated discontinuity

found by making any cut from the upper fold to the middle fold at some negative x_c with $x_c \geq -\sqrt{2}$, and then making a cut from the middle part to the lower part at $-x_c$. We see that in all cases, the area below the thick line is the same as the area bounded by the curve $(x(x_0), u(x_0))$. Consequently, conservation of u is not sufficient to determine a unique weak solution. ◇

Let us examine what kind of discontinuities are compatible with (1.19) in the general case. Assume that we have an isolated discontinuity that moves along a smooth curve $\Gamma: x = x(t)$. The discontinuity being isolated means that the function $u(x,t)$ is differentiable in a sufficiently small neighborhood of $x(t)$ and satisfies equation (1.2) classically on each side of $x(t)$. We also assume that u is uniformly bounded in a neighborhood of the discontinuity.

Now we choose a neighborhood D around the point $(x(t), t)$ and a test function $\phi(x,t)$ whose support lies entirely inside the neighborhood. The situation is as depicted in Fig. 1.3. The neighborhood consists of two parts D_1 and D_2, and is chosen so small that u is differentiable everywhere inside D except on $x(t)$. Let D_i^ε denote the set of points

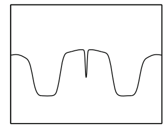
$$D_i^\varepsilon = \left\{(x,t) \in D_i \mid \mathrm{dist}\big((x,t), (x(t),t)\big) > \varepsilon\right\}.$$

The function u is bounded, and hence

$$0 = \int\limits_D (u\phi_t + f(u)\phi_x)\, dx\, dt = \lim_{\varepsilon \to 0} \int\limits_{D_1^\varepsilon \cup D_2^\varepsilon} (u\phi_t + f(u)\phi_x)\, dx\, dt. \tag{1.20}$$

Since u is a classical solution inside each D_i^ε, we can use Green's theorem and obtain

$$\begin{aligned}
\int\limits_{D_i^\varepsilon} (u\phi_t + f(u)\phi_x)\, dx\, dt &= \int\limits_{D_i^\varepsilon} (u\phi_t + f(u)\phi_x + (u_t + f(u)_x)\phi)\, dx\, dt \\
&= \int\limits_{D_i^\varepsilon} ((u\phi)_t + (f(u)\phi)_x)\, dx\, dt \\
&= \int\limits_{D_i^\varepsilon} (\partial_x, \partial_t) \cdot \big(f(u)\phi, u\phi\big)\, dx\, dt \\
&= \int\limits_{\partial D_i^\varepsilon} \phi\,(f(u), u) \cdot n_i\, ds.
\end{aligned} \tag{1.21}$$



Here n_i is the outward unit normal at ∂D_i^ε. But ϕ is zero everywhere on ∂D_i^ε except in the vicinity of $x(t)$. Let Γ_i^ε denote this part of ∂D_i^ε. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_i^\varepsilon} \phi\,(f(u),u)\cdot n_i\,ds &= \int_I \phi\left(-u_l x'(t) + f_l\right)\,dt\\ &= -\int_I \phi\left(-u_r x'(t) + f_r\right)\,dt\end{aligned}$$

for some suitable time interval I. Here u_l denotes the limit of $u(x,t)$ as $x \to x(t)-$, and u_r the limit as x approaches $x(t)$ from the right, i.e., $u_r = \lim_{x\to x(t)+} u(x,t)$. Similarly, $f_l = f(u_l)$ and $f_r = f(u_r)$. The reason for the difference in sign is that according to Green's theorem, we must integrate along the boundary counterclockwise. Therefore, the positive sign holds for $i = 1$, and the negative for $i = 2$. Using (1.20) we obtain (slightly abusing notation by writing $u(t) = u(x(t),t)$, etc.)

$$\int_I \phi\left[-\left(u_l(t) - u_r(t)\right)x'(t) + \left(f_l(t) - f_r(t)\right)\right]\,dt = 0.$$

Since this is to hold for all test functions ϕ, we must have

$$s\,(u_r - u_l) = f_r - f_l, \tag{1.22}$$

where $s = x'(t)$. This equality is called the *Rankine–Hugoniot condition* or the *jump condition*, and it expresses conservation of u across jump discontinuities. It is common in the theory of conservation laws to introduce a notation for the jump in a quantity. Write

$$[\![a]\!] = a_r - a_l \tag{1.23}$$

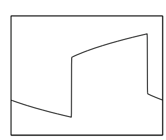
for the jump in any quantity a. With this notation the Rankine–Hugoniot relation takes the form

$$s\,[\![u]\!] = [\![f]\!]. \tag{1.24}$$

◇ Example 1.5 (Burgers's equation (cont'd.))
For Burgers's equation we see that the shock speed must satisfy

$$s = \frac{[\![u^2/2]\!]}{[\![u]\!]} = \frac{\left(u_r^2 - u_l^2\right)}{2\,(u_r - u_l)} = \frac{1}{2}\,(u_l + u_r)\,.$$

Consequently, the left shock in parts **a** and **b** in Fig. 1.2 above will have greater speed than the right shock, and will, eventually, collide. Therefore, solutions of type **a** or **b** cannot be isolated discontinuities moving along two trajectories starting at $t = 1$. Type **c** yields a stationary shock. ◇



◇ Example 1.6 (Traffic flow)

I am ill at these numbers.
— *W. Shakespeare, Hamlet* (*1603*)

Rather than continue to develop the theory, we shall now consider an example of a conservation law in some detail. We will try to motivate how a conservation law can model the flow of cars on a crowded highway.

Consider a road consisting of a single lane, with traffic in one direction only. The road is parameterized by a single coordinate x, and we assume that the traffic moves in the direction of increasing x.

Suppose we position ourselves at a point x on the road and observe the number of cars $N = N(x,t,h)$ in the interval $[x, x+h]$. If some car is located at the boundary of this interval, we account for that by allowing N to take any real value. If the traffic is dense, and if h is large compared with the average length of a car, but at the same time small compared with the length of our road, we can introduce the density given by

$$\rho(x,t) = \lim_{h\to 0} \frac{N(x,t,h)}{h}.$$

Then $N(x,t,h) = \int_x^{x+h} \rho(y,t)\,dy$.

Let now the position of some vehicle be given by $r(t)$, and its velocity by $v(r(t),t)$. Considering the interval $[a,b]$, we wish to determine how the number of cars changes in this interval. Since we have assumed that there are no entries or exits on our road, this number can change only as cars are entering the interval from the left endpoint, or leaving the interval at the right endpoint. The rate of cars passing a point x at some time t is given by

$$v(x,t)\rho(x,t).$$

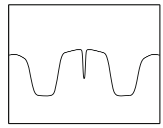
Consequently,

$$-\left(v(b,t)\rho(b,t) - v(a,t)\rho(a,t)\right) = \frac{d}{dt}\int_a^b \rho(y,t)\,dy.$$

Comparing this with (1.3) and (1.2), we see that the density satisfies the conservation law

$$\rho_t + (\rho v)_x = 0. \tag{1.25}$$

In the simplest case we assume that the velocity v is given as a function of the density ρ only. This may be a good approximation if the road is uniform and does not contain any sharp bends or similar obstacles that force the cars to slow down. It is also reasonable to assume that there is some maximal speed $v_{\max}$ that any car can attain. When traffic is light, a car will drive at this maximum speed, and as the road gets more crowded, the cars will have to slow down, until they come to a complete standstill as the traffic stands bumper to bumper. Hence, we assume that the velocity



v is a monotone decreasing function of ρ such that $v(0) = v_{\max}$ and $v(\rho_{\max}) = 0$. The simplest such function is a linear function, resulting in a flux function given by

$$f(\rho) = v\rho = \rho v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{1.26}$$

For convenience we normalize by introducing $u = \rho/\rho_{\max}$ and $\tilde{x} = v_{\max}x$. The resulting normalized conservation law reads

$$u_t + (u(1-u))_x = 0. \tag{1.27}$$

Setting $\tilde{u} = \frac{1}{2} - u$, we recover Burgers's equation, but this time with a new interpretation of the solution.

Let us solve an initial value problem explicitly by the method of characteristics described earlier. We wish to solve (1.27), with initial function $u_0(x)$ given by

$$u_0(x) = u(x,0) = \begin{cases} \frac{3}{4} & \text{for } x \le -a, \\ \frac{1}{2} - x/(4a) & \text{for } -a < x < a, \\ \frac{1}{4} & \text{for } a \le x. \end{cases}$$

The characteristics satisfy $t'(\xi) = 1$ and $x'(\xi) = 1 - 2u(x(\xi), t(\xi))$. The solution of these equations is given by $x = x(t)$, where

$$x(t) = \begin{cases} x_0 - t/2 & \text{for } x_0 < -a, \\ x_0 + x_0 t/(2a) & \text{for } -a \le x_0 \le a, \\ x_0 + t/2 & \text{for } a < x_0. \end{cases}$$

Inserting this into the solution $u(x,t) = u_0(x_0(x,t))$, we find that

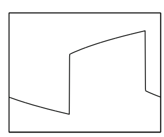
$$u(x,t) = \begin{cases} \frac{3}{4} & \text{for } x \le -a - t/2, \\ \frac{1}{2} - x/(4a+2t) & \text{for } -a - t/2 < x < a + t/2, \\ \frac{1}{4} & \text{for } a + t/2 \le x. \end{cases}$$

This solution models a situation in which the traffic density initially is small for positive x, and high for negative x. If we let a tend to zero, the solution reads

$$u(x,t) = \begin{cases} \frac{3}{4} & \text{for } x \le -t/2, \\ \frac{1}{2} - x/(2t) & \text{for } -t/2 < x < t/2, \\ \frac{1}{4} & \text{for } t/2 \le x. \end{cases}$$

As the reader may check directly, this is also a classical solution everywhere except at $x = \pm t/2$. It takes discontinuous initial values:

$$u(x,0) = \begin{cases} \frac{3}{4} & \text{for } x < 0, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{1.28}$$



This initial function may model the situation when a traffic light turns green at $t = 0$. The density of cars facing the traffic light is high, while on the other side of the light there is a small constant density.

Initial value problems of the kind (1.28), where the initial function consists of two constant values, are called *Riemann problems*. We will discuss Riemann problems at great length in this book.

If we simply insert $u_l = \frac{3}{4}$ and $u_r = \frac{1}{4}$ in the Rankine–Hugoniot condition (1.22), we find another weak solution to this initial value problem. These left and right values give $s = 0$, so the solution found here is simply $u_2(x,t) = u_0(x)$. A priori, this solution is no better or worse than the solution computed earlier. But when we examine the situation the equation is supposed to model, the second solution u_2 is unsatisfactory, since it describes a situation in which the traffic light is green, but the density of cars facing the traffic light does not decrease!

In the first solution the density decreased. Examining the model a little more closely, we find, perhaps from experience of traffic jams, that the allowable discontinuities are those in which the density is increasing. This corresponds to the situation in which there is a traffic jam ahead, and we suddenly have to slow down when we approach it.

When we emerge from a traffic jam, we experience a gradual decrease in the density of cars around us, not a sudden jump from a bumper to bumper situation to a relatively empty road.

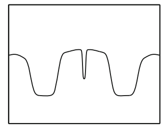
We have now formulated a condition, in addition to the Rankine–Hugoniot condition, that allows us to reduce the number of weak solutions to our conservation law. This condition says that *every weak solution u has to increase across discontinuities.* Such conditions are often called *entropy conditions*. This terminology comes from gas dynamics, where similar conditions state that the physical entropy has to increase across any discontinuity.

Let us consider the opposite initial value problem, namely,

$$u_0(x) = \begin{cases} \frac{1}{4} & \text{for } x < 0, \\ \frac{3}{4} & \text{for } x \geq 0. \end{cases}$$

Now the characteristics starting at negative x_0 are given by $x(t) = x_0 + t/2$, and the characteristics starting on the positive half-line are given by $x(t) = x_0 - t/2$. We see that these characteristics immediately will run into each other, and therefore the solution is multivalued for every positive time t. Thus there is no hope of finding a continuous solution to this initial value problem for any time interval $(0, \delta)$, no matter how small δ is. When inserting the initial values $u_l = \frac{1}{4}$ and $u_r = \frac{3}{4}$ into the Rankine–Hugoniot condition, we see that the initial function is already a weak solution. This time, the solution increases across the discontinuity, and therefore satisfies our entropy condition. Thus, an admissible solution is given by $u(x,t) = u_0(x)$.

Now we shall attempt to solve a more complicated problem in some detail. Assume that we have a road with a uniform density of cars initially. At $t = 0$ a traffic light placed at $x = 0$ changes from green to red. It remains red for some time interval Δt, then turns green again and stays green thereafter. We assume that the initial uniform density is given by $u = \frac{1}{2}$, and we wish to determine the traffic density for $t > 0$.



When the traffic light initially turns red, the situation for the cars to the left of the traffic light is the same as when the cars stand bumper to bumper to the right of the traffic light. So in order to determine the situation for t in the interval $[0, \Delta t)$, we must solve the Riemann problem with the initial function

$$u_0^l(x) = \begin{cases} \frac{1}{2} & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{1.29}$$

For the cars to the right of the traffic light, the situation is similar to the situation in which the traffic abruptly stopped at $t = 0$ behind the car located at $x = 0$. Therefore, to determine the density for $x > 0$ we have to solve the Riemann problem given by

$$u_0^r(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{2} & \text{for } x \geq 0. \end{cases} \tag{1.30}$$

Returning to (1.29), here u is increasing over the initial discontinuity, so we can try to insert this into the Rankine–Hugoniot condition. This gives

$$s = \frac{f_r - f_l}{u_r - u_l} = \frac{\frac{1}{4} - 0}{\frac{1}{2} - 1} = -\frac{1}{2}.$$

Therefore, an admissible solution for $x < 0$ and t in the interval $[0, \Delta t)$ is given by

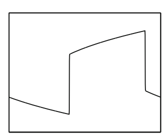
$$u^l(x,t) = \begin{cases} \frac{1}{2} & \text{for } x < -t/2, \\ 1 & \text{for } x \geq -t/2. \end{cases}$$

This is indeed close to what we experience when we encounter a traffic light. We see the discontinuity approaching as the brake lights come on in front of us, and the discontinuity has passed us when we have come to a halt. Note that although each car moves only in the positive direction, the discontinuity moves to the left.

In general, we have to deal with three different speeds when we study conservation laws: the particle speed, in our case the speed of each car; the characteristic speed; and the speed of a discontinuity. These three speeds are not equal if the conservation law is nonlinear. In our case, the speed of each car is nonnegative, but both the characteristic speed and the speed of a discontinuity may take both positive and negative values. Note that the speed of an admissible discontinuity is less than the characteristic speed to the left of the discontinuity, and larger than the characteristic speed to the right. This is a general feature of admissible discontinuities.

It remains to determine the density for positive x. The initial function given by (1.30) also has a positive jump discontinuity, so we obtain an admissible solution if we insert it into the Rankine–Hugoniot condition. Then we obtain $s = \frac{1}{2}$, so the solution for positive x is

$$u^r(x,t) = \begin{cases} 0 & \text{for } x < t/2, \\ \frac{1}{2} & \text{for } x \geq t/2. \end{cases}$$



Piecing together u^l and u^r, we find that the density u in the time interval $[0, \Delta t)$ reads

$$u(x,t) = \begin{cases} \frac{1}{2} & \text{for } x \le -t/2, \\ 1 & \text{for } -t/2 < x \le 0, \\ 0 & \text{for } 0 < x \le t/2, \\ \frac{1}{2} & \text{for } t/2 < x, \end{cases} \qquad t \in [0, \Delta t).$$

What happens for $t > \Delta t$? To find out, we have to solve the Riemann problem

$$u(x, \Delta t) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \ge 0. \end{cases}$$

Now the initial discontinuity is not acceptable according to our entropy condition, so we have to look for some other solution. We can try to mimic the example above in which we started with a nonincreasing initial function that was linear on some small interval $(-a, a)$. Therefore, let $v(x, t)$ be the solution of the initial value problem

$$v_t + (v(1-v))_x = 0,$$
$$v(x, 0) = v_0(x) = \begin{cases} 1 & \text{for } x < -a, \\ \frac{1}{2} - x/(2a) & \text{for } -a \le x < a, \\ 0 & \text{for } a \le x. \end{cases}$$

As in the above example, we find that the characteristics are not overlapping, and they fill out the positive half-plane exactly. The solution is given by $v(x,t) = v_0\left(x_0(x,t)\right)$:

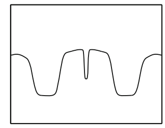
$$v(x,t) = \begin{cases} 1 & \text{for } x < -a - t, \\ \frac{1}{2} - x/(2a + 2t) & \text{for } -a - t \le x < a + t, \\ 0 & \text{for } a + t \le x. \end{cases}$$

Letting $a \to 0$, we obtain the solution to the Riemann problem with a left value 1 and a right value 0. For simplicity we also denote this function by $v(x, t)$.

This type of solution can be depicted as a "fan" of characteristics emanating from the origin, and it is called a *centered rarefaction wave*, or sometimes just a *rarefaction wave*. The origin of this terminology lies in gas dynamics.

We see that the rarefaction wave, which is centered at $(0, \Delta t)$, does not immediately influence the solution away from the origin. The leftmost part of the wave moves with a speed -1, and the front of the wave moves with speed 1. So for some time after Δt, the density is obtained by piecing together three solutions, $u^l(x,t)$, $v(x, t - \Delta t)$, and $u^r(x,t)$.

The rarefaction wave will of course catch up with the discontinuities in the solutions u^l and u^r. Since the speeds of the discontinuities are $\mp\frac{1}{2}$, and the speeds of the rear and the front of the rarefaction wave are ∓ 1, and the rarefaction wave starts at $(0, \Delta t)$, we conclude that this will happen at $(\mp\Delta t, 2\Delta t)$.



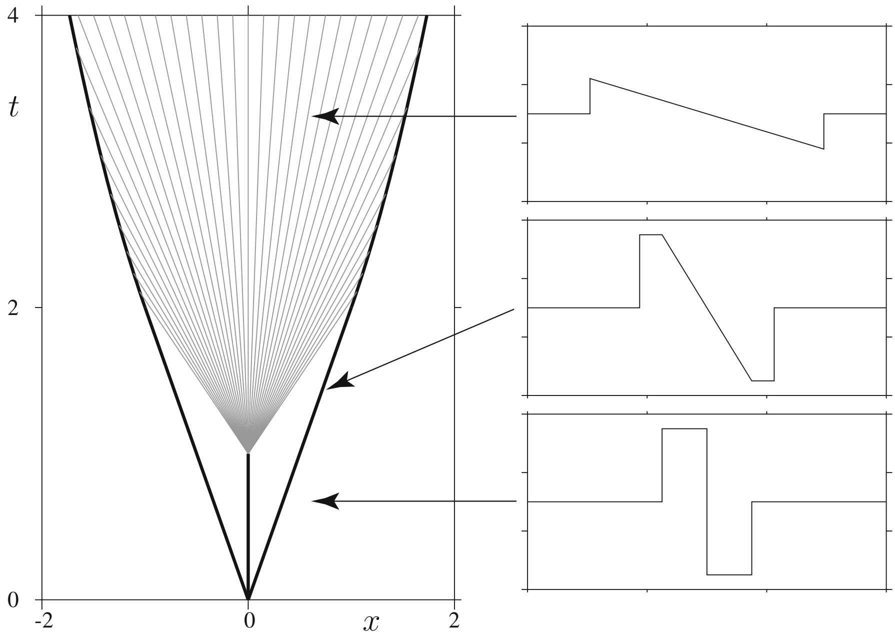


Fig. 1.4 A traffic light on a single road. *To the left* we show the solution in (x, t), and *to the right* the solution $u(x, t)$ at three different times t

It remains to compute the solution for $t > 2\Delta t$. Let us start with examining what happens for positive x. Since the u values that are transported along the characteristics in the rarefaction wave are less than $\frac{1}{2}$, we can construct an admissible discontinuity using the Rankine–Hugoniot condition (1.22). Define a function that has a discontinuity moving along a path $x(t)$. The value to the right of the discontinuity is $\frac{1}{2}$, and the value to the left is determined by $v(x, t - \Delta t)$. Inserting this into (1.22), we get

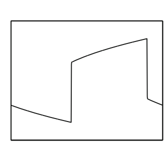
$$x'(t) = s = \frac{\frac{1}{4} - \left(\frac{1}{2} + \frac{x}{2(t-\Delta t)}\right)\left(\frac{1}{2} - \frac{x}{2(t-\Delta t)}\right)}{\frac{1}{2} - \left(\frac{1}{2} - \frac{x}{2(t-\Delta t)}\right)} = \frac{x}{2(t - \Delta t)}.$$

Since $x(2\Delta t) = \Delta t$, this differential equation has solution

$$x_+(t) = \sqrt{\Delta t(t - \Delta t)}.$$

The situation is similar for negative x. Here, we use the fact that the u values in the left part of the rarefaction fan are larger than $\frac{1}{2}$. This gives a discontinuity with a left value $\frac{1}{2}$ and right values taken from the rarefaction wave. The path of this discontinuity is found to be $x_-(t) = -x_+(t)$.

Now we have indeed found a solution that is valid for all positive time. This function has the property that it is a classical solution at all points x and t where it is differentiable, and it satisfies both the Rankine–Hugoniot condition and the entropy condition at points of discontinuity. We show this weak solution in Fig. 1.4,



both in the (x,t)-plane, where we show characteristics and discontinuities, and u as a function of x for various times. The characteristics are shown as gray lines, and the discontinuities as thicker black lines. This concludes our example. Note that we have been able to find the solution to a complicated initial value problem by piecing together solutions from Riemann problems. This is indeed the main idea behind front tracking, and a theme to which we shall give considerable attention in this book. ◇

1.1 Linear Equations

I don't make unconventional stories;
I don't make nonlinear stories.
I like linear storytelling a lot.
— Steven Spielberg

We now make a pause in the exposition of nonlinear hyperbolic conservation laws and take a brief look at linear transport equations. Many of the methods and concepts introduced later in the book are much simpler if the equations are linear.

Let $u \in \mathbb{R}$ be an unknown scalar function of $x \in \mathbb{R}$ and $t \in [0,\infty)$ satisfying the Cauchy problem

$$\begin{cases} u_t + a\,u_x = 0, & x \in \mathbb{R},\quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.31}$$

where a is a given (positive) constant, and u_0 is a known function. Recall the theory of characteristics. Since this case is particularly simple, we can use t as a parameter, and we will here use (t, x_0) rather than (s, y) as parameters. Thus the characteristics $x = \xi(t; x_0)$ are defined as

$$\frac{d}{dt}\xi(t;x_0) = a, \quad \xi(0;x_0) = x_0,$$

with solution

$$\xi(t;x_0) = at + x_0.$$

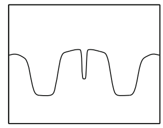
We know that $\frac{d}{dt}u\left(\xi(t;x_0),t\right) = 0$, and thus $u(\xi(t;x_0),t) = u(\xi(0;x_0),0) = u(x_0,0) = u_0(x_0)$. We can use the solution of ξ to write

$$u(at + x_0, t) = u_0(x_0).$$

If we set $x = at + x_0$, i.e., $x_0 = x - at$, we get the solution formula

$$u(x,t) = u_0(x - at).$$

Thus (1.31) expresses that the initial function u_0 is transported with a constant velocity a.



The same reasoning works if now $a = a(x,t)$, where the map $x \mapsto a(x,t)$ is Lipschitz continuous for all t. In this case let $u = u(x,t)$ satisfy the Cauchy problem

$$\begin{cases} u_t + a(x,t)u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.32}$$

First we observe that this equation is not conservative, and the interpretation of $a(x,t)u$ is *not* the flux of u across a point. Now let $\xi(t;x_0)$ denote the unique solution of the ordinary differential equation

$$\frac{d}{dt}\xi(t;x_0) = a(\xi(t;x_0),t), \quad \xi(0;x_0) = x_0. \tag{1.33}$$

By the chain rule we also now find that

$$\frac{d}{dt}u\left(\xi(t;x_0),t\right) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{d}{dt}\xi(t;x_0) = u_t(\xi,t) + a(\xi,t)u_x(\xi,t) = 0.$$

Therefore $u(\xi(t;x_0),t) = u_0(x_0)$. In order to get a solution formula, we must solve $x = \xi(t;x_0)$ in terms of x_0, or equivalently, find a function $\zeta(\tau;x)$ that solves the backward characteristic equation,

$$\frac{d}{d\tau}\zeta(\tau;x) = -a(\zeta(\tau;x),t-\tau), \quad \zeta(0;x) = x. \tag{1.34}$$

Then

$$\frac{d}{d\tau}u(\zeta(\tau;x),t-\tau) = 0,$$

which means that $u(x,t) = u(\zeta(0;x),t) = u(\zeta(t;x),0) = u_0(\zeta(t;x))$.

◇ Example 1.7
Let us study the simple example with $a(x,t) = x$. Thus

$$u_t + xu_x = 0, \quad u(x,0) = u_0(x).$$

Then the characteristic equation is

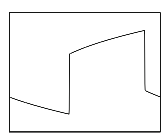
$$\frac{d}{dt}\xi = \xi, \quad \xi(0) = x_0,$$

with solution

$$\xi(t;x_0) = x_0e^t.$$

Solving $\xi(t;x_0) = x$ in terms of x_0 gives $x_0 = xe^{-t}$, and thus

$$u(x,t) = u_0\left(xe^{-t}\right).$$ ◇



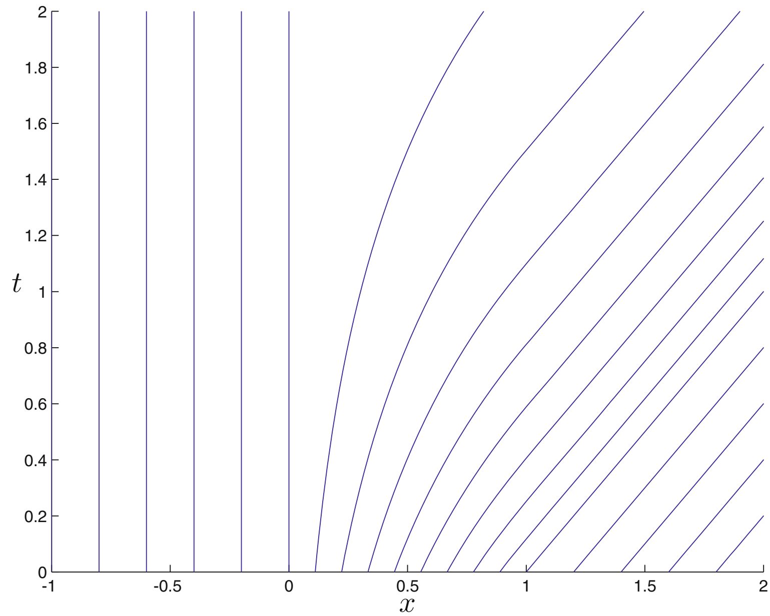


Fig. 1.5 Characteristics in the (x, t)-plane for (1.35)

◇ Example 1.8
Let us look at another example:

$$a(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x \le 1, \\ 1 & 1 < x. \end{cases} \tag{1.35}$$

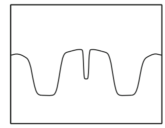
In this case the characteristics are straight lines $\xi(t; x_0) = x_0$ if $x_0 \le 0$, and $\xi(t; x_0) = x_0 + t$ if $x_0 \ge 1$. Finally, whenever $0 < x_0 < 1$, the characteristics are given by

$$\xi(t; x_0) = \begin{cases} x_0 e^t & t \le -\ln(x_0), \\ 1 + t + \ln(x_0) & t > -\ln(x_0). \end{cases}$$

See Fig. 1.5 for a picture of this. In this case a is increasing in x, and therefore the characteristics are no closer than they were initially. Since u is constant along characteristics, this means that

$$\max_x |u_x(x, t)| \le \max_x |u_0'(x)| \, .$$

If a is decreasing, such a bound cannot be found, as the next example shows. ◇



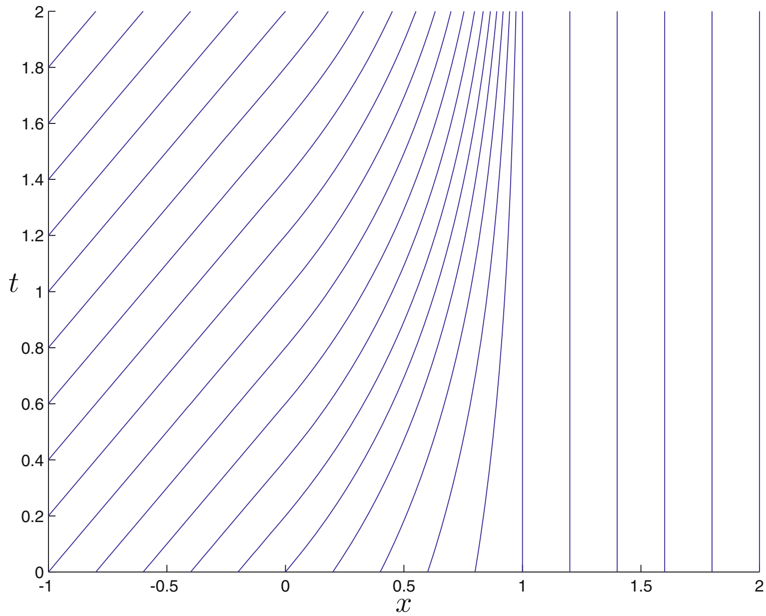


Fig. 1.6 The characteristics for (1.36)

◇ Example 1.9

Let now

$$a(x) = \begin{cases} 1 & x < 0, \\ 1 - x & 0 \leq x \leq 1, \\ 0 & 1 < x. \end{cases} \tag{1.36}$$

In this case the characteristics are given by

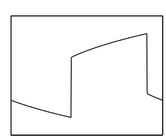
$$\xi(t; x_0) = \begin{cases} \begin{cases} x_0 + t, & t < -x_0, \\ 1 - e^{-(t+x_0)} & t \geq -x_0, \end{cases} & x_0 < 0, \\ 1 - (1 - x_0)e^{-t} & 0 \leq x_0 < 1, \\ x_0 & 1 \leq x_0. \end{cases}$$

See Fig. 1.6 for an illustration of these characteristics. Let now x_0 be in the interval $(0, 1)$, and assume that u_0 is continuously differentiable. Since u is constant along characteristics, $u(\cdot, t)$ is also continuously differentiable for all $t > 0$. Thus

$$u_0'(x_0) = \frac{\partial}{\partial x_0} u(\xi(t; x_0), t) = u_x(\xi(t; x_0), t) \frac{\partial \xi}{\partial x_0},$$

which, when $x_0 \in (0, 1)$, implies that $u_x(x, t) = u_0'(x_0)e^t$ for $x = \xi(t; x_0)$. From this we see that the only bound on the derivative that we can hope for is of the type

$$\max_x |u_x(x, t)| \leq e^t \max_x |u_0'(x)| . \qquad \diamond$$



Numerics (I)

If we (pretend that we) do not have the characteristics, and still want to know the solution, we can try to approximate it by some numerical method.

To this end we introduce approximations to the first spatial derivative

$$\begin{aligned} D_-u(x) &= \frac{u(x)-u(x-\Delta x)}{\Delta x}, \\ D_+u(x) &= \frac{u(x+\Delta x)-u(x)}{\Delta x}, \quad \text{and} \\ D_0u(x) &= \frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}, \end{aligned}$$

where Δx is a small positive number. When we deal with numerical approximations, we shall always use the notation $u_j(t)$ to indicate an approximation to $u(j\,\Delta x, t)$ for some integer j. We also use the notation

$$x_j = j\,\Delta x, \quad x_{j\pm 1/2} = \Big(j \pm \frac{1}{2}\Big)\Delta x = x_j \pm \frac{\Delta x}{2}.$$

Now consider the case in which a is a positive constant. As a semidiscrete numerical scheme for (1.31) we propose to let u_j solve the (infinite) system of ordinary differential equations

$$u_j'(t) + aD_-u_j(t) = 0, \;\; u_j(0) = u_0(x_j). \tag{1.37}$$

We need to define an approximation to $u(x,t)$ for every x and t, and we do this by linear interpolation:

$$u_{\Delta x}(x,t) = u_j(t) + \big(x-x_j\big)\, D_-u_{j+1}(t), \quad \text{for } x \in [x_j, x_{j+1}). \tag{1.38}$$

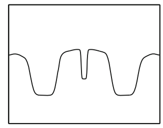
We want to show that (**a**) $u_{\Delta x}$ converges to some function u as $\Delta x \to 0$, and (**b**) the limit u solves the equation.

If u_0 is continuously differentiable, we know that a solution to (1.31) exists (and we can find it by the method of characteristics). Since the equation is linear, we can easily study the error $e_{\Delta x}(x,t) = u(x,t) - u_{\Delta x}(x,t)$. In the calculation that follows, we use the following properties:

$$D_+u_j - D_-u_j = \Delta x D_+D_-u_j \text{ and } D_-u_{j+1} = D_+u_j.$$

Inserting the error term $e_{\Delta x}$ into the equation, we obtain for $x \in (x_j, x_{j+1})$,

$$\begin{aligned} \frac{\partial}{\partial t}e_{\Delta x} + a\frac{\partial}{\partial x}e_{\Delta x} &= -\frac{\partial}{\partial t}u_{\Delta x} - a\frac{\partial}{\partial x}u_{\Delta x} \\ &= -\frac{d}{dt}\big[u_j(t) + \big(x-x_j\big)\, D_-u_{j+1}(t)\big] - aD_-u_{j+1}(t) \\ &= -u_j'(t) - \big(x-x_j\big)\, D_-u_{j+1}'(t) - aD_+u_j(t) \\ &= aD_-u_j(t) - aD_+u_j(t) + a\big(x-x_j\big)\, D_-D_-u_{j+1}(t) \\ &= -a\Delta x D_+D_-u_j(t) + a\big(x-x_j\big)\, D_+D_-u_j(t) \\ &= a\big(\big(x-x_j\big) - \Delta x\big)\, D_+D_-u_j(t). \end{aligned}$$



Next let $f_{\Delta x}$ be defined by

$$f_{\Delta x}(x,t) = a\left(\left(x - x_j\right) - \Delta x\right) D_+ D_- u_j(t) \ \text{ for } x \in [x_j, x_{j+1}),$$

so that

$$(e_{\Delta x})_t + a\,(e_{\Delta x})_x = f_{\Delta x}. \tag{1.39}$$

Using the method of characteristics on this equation gives (see Exercise 1.3)

$$e_{\Delta x}(x,t) = e_{\Delta x}(x - at, 0) + \int_0^t f_{\Delta x}(x - a(t-s), s)\,ds. \tag{1.40}$$

(Here we tacitly assume uniqueness of the solution.) Hence we get the bound

$$|e_{\Delta x}(x,t)| \le \sup_x |e_{\Delta x}(x,0)| + t\,\|f_{\Delta x}\|_{L^\infty(\mathbb{R}\times[0,t])}. \tag{1.41}$$

In trying to bound $f_{\Delta x}$, note first that

$$|f_{\Delta x}(x,t)| \le \Delta x\, a\,\left|D_- D_+ u_j(t)\right|,$$

so $f_{\Delta x}$ tends to zero with Δx if $D_- D_+ u_j$ is bounded. Writing $w_j = D_- D_+ u_j$ and applying $D_- D_+$ to (1.37), we get

$$w_j'(t) + a D_-\left(w_j\right) = 0, \quad w_j(0) = D_- D_+ u_0(x).$$

Now it is time to use the fact that $a > 0$. To bound w_j, observe that if $w_j \le w_{j-1}$, then $D_- w_j \le 0$. Hence, if $w_j(t) \le w_{j-1}(t)$, then

$$\frac{d}{dt} w_j(t) = -a D_- w_j(t) \ge 0.$$

Similarly, if for some t, $w_j(t) \ge w_{j-1}(t)$, then $w_j'(t) \le 0$. This means that

$$\inf_x u_0''(x) \le \inf_k D_- D_+ u_k(0) \le w_j(t) \le \sup_k D_- D_+ u_k(0) \le \sup_x u_0''(x).$$

Thus w_j is bounded if u_0' is Lipschitz continuous. Note that it is the choice of the difference scheme (1.37) (choosing D_- instead of D_+ or D) that allows us to conclude that we have a bounded approximation. It remains to study $e_{\Delta x}(x,0)$. For $x \in [x_j, x_{j+1})$,

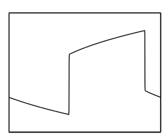
$$\begin{aligned} |e_{\Delta x}(x,0)| &= \left|u_0(x) - u_0(x_j) - \frac{x - x_j}{\Delta x}\left(u_0(x_{j+1}) - u_0(x_j)\right)\right| \\ &\le 2\Delta x \max_{x\in[x_j,x_{j+1}]} |u_0'(x)|. \end{aligned}$$

Then we have proved the bound

$$|u_{\Delta x}(x,t) - u(x,t)| \le \Delta x\left(2\,\|u_0'\|_{L^\infty(\mathbb{R})} + t\,a\,\|u_0''\|_{L^\infty(\mathbb{R})}\right), \tag{1.42}$$

for all x and $t > 0$.

Strictly speaking, in order for this argument to be valid, we have implicitly assumed in (1.40) that equation (1.39) has *only* the solution (1.40). This brings us to another topic.



Entropy Solutions (I)

You should call it entropy . . . [since] . . . no one knows what entropy really is, so in a debate you will always have the advantage.[3]
— John von Neumann

Without much extra effort, we can generalize slightly, and we want to ensure that the equation

$$u_t + a(x,t)u_x = f(x,t) \tag{1.43}$$

has only one differentiable solution. If we let the characteristic curves be defined by (1.33), a solution is given by (see Exercise 1.3)

$$u\left(\xi(t;x_0),t\right) = u_0(x_0) + \int_0^t f(\xi(s;x_0),s)\,ds.$$

In terms of the inverse characteristic ζ defined by (1.34) this formula reads (see Exercise 1.3)

$$u(x,t) = u_0\left(\zeta(t;x)\right) + \int_0^t f\left(\zeta(\tau;x),t-\tau\right)\,d\tau.$$

If u_0 is differentiable and f is bounded, this formula gives a differentiable function $u(x,t)$.

Now we can turn to the uniqueness question. Since (1.43) is linear, to prove uniqueness means to show that the equation with $f = 0$ and $u_0 = 0$ has only the zero solution. Therefore, we consider

$$u_t + a(x,t)u_x = 0.$$

Now let $\eta(u)$ be a differentiable function, and multiply the above by $\eta'(u)$ to get

$$0 = \frac{\partial}{\partial t}\eta(u) + a\frac{\partial}{\partial x}\eta(u) = \eta(u)_t + (a\eta(u))_x - a_x\eta(u).$$

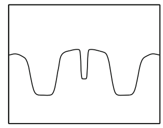
Assume that $\eta(0) = 0$ and $\eta(u) > 0$ for $u \neq 0$, and that $|a_x(x,t)| < C$ for all x and t. If $\eta(u(\cdot\,,t))$ is integrable, then we can integrate this to get

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(u(x,t))\,dx = \int_{\mathbb{R}} a_x(x,t)\eta(u(x,t))\,dx \le C\int_{\mathbb{R}} \eta(u(x,t))\,dx.$$

By Gronwall's inequality (see Exercise 1.10),

$$\int_{\mathbb{R}} \eta(u(x,t))\,dx \le e^{Ct}\int_{\mathbb{R}} \eta(u_0(x))\,dx.$$

[3] In a discussion with Claude Shannon about Shannon's new concept called "entropy."



If $u_0 = 0$, then $\eta(u_0) = 0$, and we must have $u(x,t) = 0$ as well. We have shown that if $\eta(u_0)$ is integrable for some differentiable function η with $\eta(0) = 0$ and $\eta(u) > 0$ for $u \neq 0$, and a_x is bounded, then (1.43) has only one differentiable solution.

Frequently, the model (1.43) (with f identically zero) is obtained by the limit of a physically more realistic model,

$$u_t^\varepsilon + a(x,t)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \tag{1.44}$$

as ε becomes small. You can think of u^ε as the temperature in a long rod moving with speed a. In this case ε is proportional to the heat conductivity of the rod. Equation (1.44) has more regular solutions than the initial data u_0 (see Appendix B). If we multiply this equation by $\eta'(u^\varepsilon)$, where $\eta \in C^2(\mathbb{R})$ is a *convex* function, we get

$$\eta\left(u^\varepsilon\right)_t + a\,\eta\left(u^\varepsilon\right)_x = \varepsilon\left(\eta'\left(u^\varepsilon\right)u_x^\varepsilon\right)_x - \varepsilon\eta''\left(u^\varepsilon\right)\left(u_x^\varepsilon\right)^2.$$

The function η is often called an *entropy*. The term with $(u_x^\varepsilon)^2$ is problematic when $\varepsilon \to 0$, since the derivative will not be square integrable in this limit. For linear equations the integrability of this term depends on the integrability of this term initially. However, for nonlinear equations, we have seen that jumps can form independently of the smoothness of the initial data, and the limit of u_x^ε will in general not be square integrable.

The key to circumventing this problem is to use the convexity of η, that is, $\eta''(u) \geq 0$, and hence $\varepsilon\eta''\left(u^\varepsilon\right)\left(u_x^\varepsilon\right)^2$ is nonnegative, to replace this term by the appropriate inequality. Thus we get that

$$\eta\left(u^\varepsilon\right)_t + \left(a\eta\left(u^\varepsilon\right)\right)_x - a_x\eta\left(u^\varepsilon\right) \leq \varepsilon\left(\eta'\left(u^\varepsilon\right)u_x^\varepsilon\right)_x. \tag{1.45}$$

Now the right-hand side of (1.45) converges to zero weakly.[4] We define an entropy solution to be the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ of solutions to (1.44) as $\varepsilon \to 0$. Formally, an entropy solution to (1.43) should satisfy (reintroducing the function f)

$$\eta(u)_t + (a\eta(u))_x - a_x\eta(u) \leq \eta'(u)f(x,t), \tag{1.46}$$

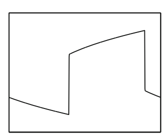
for all convex functions $\eta \in C^2(\mathbb{R})$. We shall see later that this is sufficient to establish uniqueness even if u is not assumed to be differentiable.

Numerics (II)

Let us for the moment return to the transport equation

$$u_t + a(x,t)u_x = 0. \tag{1.47}$$

[4] That is, $\varepsilon \iint_{\mathbb{R}\times[0,\infty)} \varphi_x \eta'(u^\varepsilon)u_x^\varepsilon \, dx\, dt \to 0$ as $\varepsilon \to 0$ for any test function φ.



We want to construct a fully discrete scheme for this equation, and the simplest such scheme is the explicit Euler scheme,

$$D_+^t u_j^n + a_j^n D_- u_j^n = 0, \quad n \geq 0, \tag{1.48}$$

and $u_j^0 = u_0(x_j)$. Here D_+^t denotes the discrete forward time difference

$$D_+^t u(t) = \frac{u(t+\Delta t) - u(t)}{\Delta t},$$

and u_j^n is an approximation of $u(x_j, t_n)$, with $t_n = n\Delta t$, $n \geq 0$. Furthermore, a_j^n denotes some approximation of $a(x_j, t_n)$, to be determined later. We can rewrite (1.48) as

$$u_j^{n+1} = u_j^n - a_j^n \lambda \left(u_j^n - u_{j-1}^n \right),$$

where $\lambda = \Delta t / \Delta x$.[5]

Let us first return to the case that a is constant. We can then use von Neumann stability analysis. Assume that the scheme produces approximations that converge to a bounded solution for almost all x and t; in particular, assume that u_j^n is bounded independently of Δx and Δt. Consider the periodic case. We make the ansatz that $u_j^n = \alpha^n e^{ij\Delta x}$ with $i = \sqrt{-1}$ (the equation is linear, so we might as well expand the solution in a Fourier series). Inserting this into the equation for u_j^{n+1}, we get

$$\begin{aligned} \alpha^{n+1} e^{ij\Delta x} &= \alpha^n e^{ij\Delta x} - \lambda a \left(\alpha^n e^{ij\Delta x} - \alpha^n e^{i(j-1)\Delta x} \right) \\ &= \alpha^n e^{ij\Delta x} \big(1 - \lambda a (1 - e^{-i\Delta x}) \big), \end{aligned}$$

so that

$$\alpha = 1 - \lambda a \big(1 - \cos(\Delta x) + i \sin(\Delta x) \big).$$

If $|\alpha| \leq 1$, then the sup-norm estimate will hold also for the solution generated by the scheme. In this case the scheme is called *von Neumann stable*.

We calculate

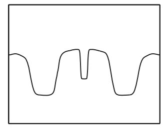
$$\begin{aligned} |\alpha|^2 &= 1 + 2\lambda^2 a^2 - 2\lambda a (1 + (1 - \lambda a) \cos(\Delta x)) \\ &= 1 - 2a\lambda \, (1 - a\lambda) \, (1 - \cos(\Delta x)) \, . \end{aligned}$$

This is less than or equal to 1 if and only if $a\lambda(1 - a\lambda) \geq 0$. Thus we require

$$0 \leq \lambda a \leq 1. \tag{1.49}$$

This relationship between the spatial and temporal discretization (as measured by λ) and the wave speed given by a is the simplest example of the celebrated *CFL condition*, named after Courant–Friedrichs–Lewy. We will return to the CFL condition repeatedly throughout the book.

[5] Unless otherwise is stated, you can safely assume that this is the definition of λ.



Returning to the scheme for the transport equation with variable and nonnegative speed, we say that the scheme will be von Neumann stable if

$$\lambda \max_{(x,t)} a(x,t) \le 1. \tag{1.50}$$

Consider now the scheme (1.48) with

$$a_j^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} a(x_j, t)\, dt.$$

We wish to establish the convergence of u_j^n. To this end, set

$$e_j^n = u(x_j, t_n) - u_j^n,$$

where u is the unique solution to (1.47). Inserting this into the scheme, we find that

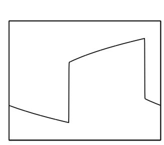
$$\begin{aligned}
D_+^t e_j^n + a_j^n D_- e_j^n &= D_+^t u(x_j, t_n) + a_j^n D_- u(x_j, t_n) \\
&= \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} u_t(x_j, t)\, dt + \frac{a_j^n}{\Delta x} \int_{x_{j-1}}^{x_j} u_x(x, t_n)\, dx \\
&= \frac{1}{\Delta x \Delta t} \int_{t_n}^{t_{n+1}} \int_{x_{j-1}}^{x_j} \big(u_t(x_j, t) + a(x_j, t) u_x(x, t_n)\big)\, dx\, dt \\
&= \frac{1}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} a(x_j, t) \big(u_x(x, t_n) - u_x(x_j, t)\big)\; dt\, dx \\
&= \frac{1}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} a(x_j, t) \Big(\int_{x_j}^{x} u_{xx}(z, t_n)\, dz - \int_{t_n}^{t} u_{xt}(x_j, s)\, ds \Big) dt\, dx \\
&=: R_j^n.
\end{aligned}$$

Assuming now that u_{xx} and u_{tx} are bounded, which they will be if we consider a finite time interval $[0, T]$, choose M such that $\max\{\|u_{xx}\|_{L^\infty}, \|u_{tx}\|_{L^\infty}, \|a\|_{L^\infty}\} \le M$. Then we get the bound

$$\begin{aligned}
\left| R_j^n \right| &\le \frac{M^2}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} \big((x_j - x) + (t - t_n) \big)\, dt\, dx \\
&= \frac{M^2}{2} (\Delta x + \Delta t).
\end{aligned}$$

Therefore the error will satisfy the inequality

$$e_j^{n+1} \le \underbrace{e_j^n \left(1 - \lambda a_j^n\right) + \lambda a_j^n e_{j-1}^n}_{\Upsilon} + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$



If $\|a\|_{L^\infty}\lambda < 1$ (recall the CFL condition), then Υ is a convex combination of e_j^n and e_{j-1}^n, which is less than or equal to $\max\left\{e_j^n, e_{j-1}^n\right\}$. Taking the supremum over j, first on the right, and then on the left, we get

$$\sup_j \left\{e_j^{n+1}\right\} \leq \sup_j \left\{e_j^n\right\} + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

We also have that

$$e_j^{n+1} \geq \underbrace{e_j^n \left(1 - \lambda a_j^n\right) + \lambda a_j^n e_{j-1}^n}_{\Upsilon} - \Delta t \frac{M^2}{2} (\Delta x + \Delta t),$$

which implies that

$$\inf_j \left\{e_j^{n+1}\right\} \geq \inf_j \left\{e_j^n\right\} - \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

With $\bar{e}^n = \sup_j \left|e_j^n\right|$, the above means that

$$\bar{e}^{n+1} \leq \bar{e}^n + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

Inductively, we then find that

$$\bar{e}^n \leq \bar{e}^0 + t_n \frac{M^2}{2} (\Delta x + \Delta t) = t_n \frac{M^2}{2} (\Delta x + \Delta t),$$

since $e_j^0 = 0$ by definition. Hence, the approximation defined by (1.48) converges to the unique solution if u is twice differentiable with bounded second derivatives.

We have seen that if $x \mapsto a(x,t)$ is decreasing on some interval, the best bounds for u_{xx} and u_{xt} are likely to be of the form Ce^{Ct}, which means that the "constant" M is likely to be large if we want to study the solution for large (or even moderate) times.

Similarly, if $a(x,t) < 0$, the scheme

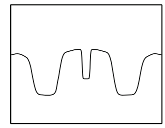
$$D_+^t u_j^n + a_j^n D_+ u_j^n = 0$$

will give a convergent sequence.

Entropy Solutions (II)

Consider the Cauchy problem

$$\begin{cases} u_t + a(x,t)u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.51}$$



where a is a continuously differentiable function (in this section not assumed to be nonnegative). Recall that an entropy solution is defined as the limit of the singularly perturbed equation (1.44). For every positive ε, u^ε satisfies (1.45), implying that the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ should satisfy (1.46) with f identically zero. Multiplying the inequality (1.46) by a nonnegative test function ψ, and integrating by parts, we find that

$$\int_0^\infty\int_{\mathbb{R}} \left(\eta(u)\psi_t + a\eta(u)\psi_x + a_x\eta(u)\psi\right) dx\,dt + \int_{\mathbb{R}} \eta(u_0(x))\psi(x,0)\,dx \ge 0 \tag{1.52}$$

should hold for all nonnegative test functions $\psi \in C_0^\infty(\mathbb{R}\times[0,\infty))$, and for all convex η. If $u(x,t)$ is a function in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))$ that satisfies (1.52) for all convex entropies η, then u is called a weak entropy solution to (1.51). The point of this is that we no longer require u to be differentiable, or even continuous. Therefore, showing that approximations converge to an entropy solution should be much easier than showing that the limit is a classical solution.

We are going to show that there is only one entropy solution. Again, since the equation is linear, it suffices to show that $u_0 = 0$ (in $L^1(\mathbb{R})$) implies $u(\cdot,t) = 0$ (in $L^1(\mathbb{R})$).

To do this, we specify a particular test function. Let ω be a C^∞ function such that

$$0 \le \omega(\sigma) \le 1, \quad \operatorname{supp}\omega \subseteq [-1,1], \quad \omega(-\sigma) = \omega(\sigma), \quad \int_{-1}^{1} \omega(\sigma)\,d\sigma = 1.$$

Now define

$$\omega_\varepsilon(\sigma) = \frac{1}{\varepsilon}\omega\left(\frac{\sigma}{\varepsilon}\right). \tag{1.53}$$

Let $x_1 < x_2$, and introduce

$$\varphi_\varepsilon(x,t) = \int_{x_1+Lt}^{x_2-Lt} \omega_\varepsilon(x-y)\,dy,$$

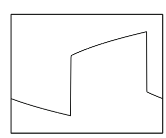
where L is a constant such that $L > \|a\|_{L^\infty(\Omega)}$ and $\Omega = \mathbb{R}\times[0,\infty)$. We fix a T such that $T < (x_2-x_1)/(2L)$, and consider $t < T$. Observe that $\varphi_\varepsilon(\cdot,t)$ is an approximation to the characteristic function for the interval (x_1+Lt, x_2-Lt).

Next introduce

$$h_\varepsilon(t) = 1 - \int_0^t \omega_\varepsilon(s-T)\,ds.$$

This is an approximation to the characteristic function of the interval $(-\infty, T]$. Finally, we choose the test function

$$\psi_\varepsilon(x,t) = h_\varepsilon(t)\varphi_\varepsilon(x,t) \in C_0^\infty(\Omega).$$



Inserting this into the entropy inequality (1.52), we get

$$\begin{aligned} &\iint_{\Omega} \eta(u)\varphi_\varepsilon h'_\varepsilon(t)\,dx\,dt \\ &+ \iint_{\Omega} h_\varepsilon(t)\eta(u)\left(\frac{\partial}{\partial t}\varphi_\varepsilon(x,t) + a(x,t)\frac{\partial}{\partial x}\varphi_\varepsilon(x,t)\right)\,dx\,dt \\ &+ \iint_{\Omega} a_x \eta(u) h_\varepsilon \varphi_\varepsilon\,dx\,dt + \int_{\mathbb{R}} \eta(u_0)\varphi_\varepsilon(x,0)\,dx \ge 0. \end{aligned} \tag{1.54}$$

We treat the second integral first, and calculate

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_\varepsilon(x,t) &= -L\left(\omega_\varepsilon(x-x_2+Lt) + \omega_\varepsilon(x-x_1-Lt)\right), \\ \frac{\partial}{\partial x}\varphi_\varepsilon(x,t) &= -\omega_\varepsilon(x-x_2+Lt) + \omega_\varepsilon(x-x_1-Lt). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_\varepsilon + a\,\frac{\partial}{\partial x}\varphi_\varepsilon &= (-L+a)\omega_\varepsilon(x-x_2+Lt) + (-L-a)\omega_\varepsilon(x-x_1-Lt) \\ &\le (|a|-L)\big(\omega_\varepsilon(x-x_2+Lt) + \omega_\varepsilon(x-x_1-Lt)\big) \le 0, \end{aligned}$$

since L is chosen to be larger than $|a|$. Hence, if $\eta(u) \ge 0$, then the second integral in (1.54) is nonpositive. Thus we have

$$\begin{aligned} \iint_{\Omega} \eta(u)\varphi_\varepsilon h'_\varepsilon(t)\,dx\,dt + \iint_{\Omega} a_x\eta(u)h_\varepsilon\varphi_\varepsilon\,dx\,dt \\ + \int_{\mathbb{R}} \eta(u_0)\varphi_\varepsilon(x,0)\,dx \ge 0. \end{aligned} \tag{1.55}$$

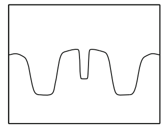
Let us for the moment proceed formally. The function h_ε approximates the characteristic function $\chi_{(-\infty,T]}$, which has derivative $-\delta_T$, a negative Dirac delta function at T. Similarly, φ_ε approximates the characteristic function $\chi_{(x_1+Lt,x_2-Lt)}$, with derivative $L(\delta_{x_1+Lt} - \delta_{x_2-Lt})$. From (1.54) we formally obtain by sending $\varepsilon \to 0$, that

$$-\int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx + \int_0^T \int_{x_1+Lt}^{x_2-Lt} a_x(x,t)\eta(u(x,t))\,dx\,dt + \int_{x_1}^{x_2} \eta(u(x,0))\,dx \ge 0, \tag{1.56}$$

and this is what we intend to prove next.

The first integral in (1.54) reads

$$-\iint_{\Omega} \eta(u)\varphi_\varepsilon(x,t)\omega_\varepsilon(t-T)\,dx\,dt = -\int_0^\infty f_\varepsilon(t)\omega_\varepsilon(t-T)\,dt,$$



where

$$f_\varepsilon(t) = \int_{\mathbb{R}} \varphi_\varepsilon(x,t)\eta(u(x,t))\,dx.$$

Keeping t fixed, we obtain

$$f_\varepsilon(t) \to \int_{x_1+Lt}^{x_2-Lt} \eta(u(x,t))\,dx = f_0(t) \quad \text{as } \varepsilon \to 0,$$

the limit being uniform in t for $t \in [0,T]$. If $t \mapsto u(\cdot\,,t)$ is continuous as a map from $[0,\infty)$ with values in $L^1(\mathbb{R})$, then f_ε and f_0 are continuous in t. In that case,

$$\begin{aligned}\int_0^\infty f_\varepsilon(t)\omega_\varepsilon(t-T)\,dt &= \int_0^\infty \big(f_\varepsilon(t) - f_0(t)\big)\omega_\varepsilon(t-T)\,dt + \int_0^\infty f_0(t)\omega_\varepsilon(t-T)\,dt \\ &\to f_0(T) \quad \text{as } \varepsilon \to 0,\end{aligned}$$

since

$$\begin{aligned}\left|\int_0^\infty \big(f_\varepsilon(t) - f_0(t)\big)\omega_\varepsilon(t-T)\,dt\right| &\le \|f_\varepsilon - f_0\|_{L^\infty} \int_0^\infty \omega_\varepsilon(t-T)\,dt \\ &= \|f_\varepsilon - f_0\|_{L^\infty} \to 0.\end{aligned}$$

In order to ensure that $t \mapsto u(\cdot\,,t)$ is continuous as a map from $[0,\infty)$ to $L^1(\mathbb{R})$, we *define* an entropy solution to have this property; see Definition 1.10 below. We have that

$$h_\varepsilon(t)\varphi_\varepsilon(x,t) \to \chi_{\Pi_T}(x,t) \quad \text{in } L^1(\Omega_T),$$

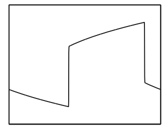
where $\Pi_T = \{(x,t) \mid 0 \le t \le T,\ x_1 + Lt \le x \le x_2 - Lt\}$ and $\Omega_T = \mathbb{R} \times [0,T]$. By sending $\varepsilon \to 0$ in (1.54), we then find that (cf. (1.56))

$$\int_{x_1}^{x_2} \eta(u(x,0))\,dx + \int_0^T \int_{x_1+Lt}^{x_2-Lt} a_x(x,t)\eta(u(x,t))\,dx\,dt \ge \int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx, \tag{1.57}$$

which implies that

$$f_0(T) \le f_0(0) + \|a_x\|_{L^\infty(\Omega_T)} \int_0^T f_0(t)\,dt,$$

assuming that η is positive.



Gronwall's inequality then implies

$$f_0(T) \le f_0(0)e^{\|a_x\|_{L^\infty(\Omega_T)}T},$$

or, writing it out explicitly,

$$\int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx \le \int_{x_1}^{x_2} \eta(u_0(x))\,dx\, e^{\|a_x\|_{L^\infty(\Omega_T)}T},$$

for every nonnegative convex function η. Observe that this proves the finite speed of propagation.

Choosing $\eta(u) = |u|^p$ for $1 \le p < \infty$, assuming $\eta(u)$ to be integrable, and sending x_1 to $-\infty$ and x_2 to ∞, we get

$$\|u(\cdot,T)\|_{L^p(\mathbb{R})} \le \|u_0\|_{L^p(\mathbb{R})}\, e^{\|a_x\|_{L^\infty(\Omega_T)}T/p}, \quad 1 \le p < \infty. \tag{1.58}$$

Next, we can let $p \to \infty$, assuming $\eta(u)$ to be integrable for all $1 \le p < \infty$, to get

$$\|u(\cdot,T)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}\,. \tag{1.59}$$

In order to formalize the preceding argument, we introduce the following definition.

Definition 1.10 A function $u = u(x,t)$ in $C([0,\infty); L^1(\mathbb{R}))$ is called a weak entropy solution to the problem

$$\begin{cases} u_t + a(x,t)u_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

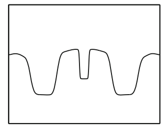
if for all nonnegative and convex functions $\eta(u)$ and all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$, the inequality

$$\int_0^\infty \int_{\mathbb{R}} \big(\eta(u)\varphi_t + a\,\eta(u)\varphi_x + a_x\eta(u)\varphi\big)\,dx\,dt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x,0)\,dx \ge 0$$

holds.

Theorem 1.11 *Assume that $a = a(x,t)$ is such that a_x is bounded. Then the problem (1.32) has at most one entropy solution $u = u(x,t)$, and the bounds (1.58) and (1.59) hold.*

Remark 1.12 From the proof of this theorem (applying (1.58) for $p = 1$), we see that if we define an entropy solution to satisfy the entropy condition only for $\eta(u) = |u|$, then we get uniqueness in $C([0,\infty), L^1(\mathbb{R}))$.



Numerics (III)

We now reconsider the transport equation

$$\begin{cases} u_t + a(x,t)u_x = 0, & t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.60}$$

and the corresponding difference scheme

$$D_+^t u_j^n + a_j^n D_- u_j^n = 0,$$

with

$$a_j^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} a(x_j, t)\, dt, \qquad u_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\, dx,$$

where as before, we assume that $a(x,t) \geq 0$. In order to have an approximation defined for all x and t, we define

$$u_{\Delta x}(x,t) = u_j^n \ \text{ for } (x,t) \in I_{j-1/2}^n := [x_{j-1}, x_j) \times [t_n, t_{n+1}),$$

where $t_n = n\Delta t$. We wish to show that $u_{\Delta x}$ converges to an entropy solution (the only one!) of (1.60). Now we do not use the linearity, and first prove that $\{u_{\Delta x}\}_{\Delta x > 0}$ has a convergent subsequence.

First we recall that the scheme can be written

$$u_j^{n+1} = \left(1 - a_j^n \lambda\right) u_j^n + a_j^n \lambda u_{j-1}^n.$$

We aim to use Theorem A.11 to prove compactness. First we show that the approximation is uniformly bounded. This is easy, since u_j^{n+1} is a *convex* combination of u_j^n and u_{j-1}^n, so new maxima or minima are not introduced. Thus

$$\|u_{\Delta x}(\,\cdot\,, t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}.$$

Therefore, the first condition of Theorem A.11 is satisfied.

To show that the second condition holds, recall, or consult Appendix A, that the total variation of a function $u\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$\mathrm{T.V.}\,(u) = \sup_{\{x_i\}} \sum_i |u(x_i) - u(x_{i-1})|,$$

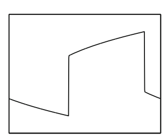
where the supremum is taken over all finite partitions $\{x_i\}$ such that $x_i < x_{i+1}$. This is a seminorm, and we also write $|u|_{BV} := \mathrm{T.V.}\,(u)$.

We have to estimate the total variation of $u_{\Delta x}$. For $t \in [t_n, t_{n+1})$ this is given by

$$|u_{\Delta x}(\,\cdot\,, t)|_{BV} = \sum_j \left| u_j^n - u_{j-1}^n \right|.$$

We also have that

$$\begin{aligned} u_j^{n+1} - u_{j-1}^{n+1} &= (1 - a_j^n \lambda) u_j^n + a_j^n \lambda u_{j-1}^n - (1 - a_{j-1}^n \lambda) u_{j-1}^n - a_{j-1}^n \lambda u_{j-2}^n \\ &= (1 - a_j^n \lambda)(u_j^n - u_{j-1}^n) + a_{j-1}^n \lambda (u_{j-1}^n - u_{j-2}^n). \end{aligned}$$



By the CFL condition $0 \le a_j^n \lambda \le 1$ for all n and j, we infer

$$\left|u_j^{n+1} - u_{j-1}^{n+1}\right| \le (1 - a_j^n \lambda) \left|u_j^n - u_{j-1}^n\right| + \lambda a_{j-1}^n \left|u_{j-1}^n - u_{j-2}^n\right|.$$

Therefore

$$\begin{aligned}\sum_j \left|u_j^{n+1} - u_{j-1}^{n+1}\right| &\le \sum_j (1 - a_j^n \lambda) \left|u_j^n - u_{j-1}^n\right| + \sum_j \lambda a_{j-1}^n \left|u_{j-1}^n - u_{j-2}^n\right| \\ &= \sum_j \left|u_j^n - u_{j-1}^n\right| - \sum_j \lambda a_j^n \left|u_j^n - u_{j-1}^n\right| + \sum_j \lambda a_j^n \left|u_j^n - u_{j-1}^n\right| \\ &= \sum_j \left|u_j^n - u_{j-1}^n\right|.\end{aligned}$$

Hence

$$|u_{\Delta x}(\cdot, t)|_{BV} \le |u_{\Delta x}(\cdot, 0)|_{BV} \le |u_0|_{BV}.$$

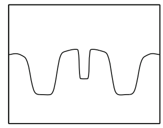
This shows that the second condition of Theorem A.11 is satisfied; see Remark A.12.

To show that the third condition holds, i.e., the continuity of the L^1-norm in time, we assume that $s \in [t_n, t_{n+1})$, and that t is such that $t - s \le \Delta t$. Then

$$\begin{aligned}\int_{\mathbb{R}} |u_{\Delta x}(x,t) - u_{\Delta x}(x,s)|\, dx &\le \Delta x \sum_j \left|u_j^{n+1} - u_j^n\right| \\ &= \Delta x \sum_j a_j^n \lambda \left|u_j^n - u_{j-1}^n\right| \\ &\le \Delta t\, \|a\|_{L^\infty(\Omega)} \sum_j \left|u_j^n - u_{j-1}^n\right| \\ &\le \Delta t\, \|a\|_{L^\infty(\Omega)} |u_0|_{BV}.\end{aligned}$$

If $s \in [t_n, t_{n+1})$ and $t \in [t_{n+k}, t_{n+k+1})$, we have

$$\begin{aligned}\int_{\mathbb{R}} |u_{\Delta x}(x,t) - u_{\Delta x}(x,s)|\, dx &= \Delta x \sum_j \left|u_j^{n+k} - u_j^n\right| \\ &\le \sum_{m=n}^{n+k-1} \Delta x \sum_j \left|u_j^{m+1} - u_j^m\right| \\ &= \sum_{m=n}^{n+k-1} \Delta x \sum_j a_j^m \lambda \left|u_j^m - u_{j-1}^m\right| \\ &\le \sum_{m=n}^{n+k-1} \Delta t\, \|a\|_{L^\infty(\Omega)} \sum_j \left|u_j^m - u_{j-1}^m\right| \\ &\le k \Delta t\, \|a\|_{L^\infty(\Omega)} |u_0|_{BV} \\ &\le (t - s + \Delta t)\, \|a\|_{L^\infty(\Omega)} |u_0|_{BV}.\end{aligned}$$



Hence, also the third condition of Theorem A.11 is fulfilled, and we have the convergence (of a subsequence) $u_{\Delta x} \to u$ as $\Delta x \to 0$. It remains to prove that u is the entropy solution.

To do this, start by observing that

$$\begin{aligned}\eta\left(u_j^{n+1}\right) &= \eta\left((1-a_j^n\lambda)u_j^n + a_j^n\lambda u_{j-1}^n\right)\\ &\le (1-a_j^n\lambda)\eta\left(u_j^n\right) + a_j^n\lambda\eta\left(u_{j-1}^n\right),\end{aligned}$$

since η is assumed to be a convex function. This can be rearranged as

$$D_+^t\eta_j^n + a_j^n D_-\eta_j^n \le 0,$$

where $\eta_j^n = \eta(u_j^n)$, and as

$$D_+^t\eta_j^n + D_-\left(a_j^n\eta_j^n\right) - \eta_{j-1}^n D_- a_j^n \le 0. \tag{1.61}$$

The operators D_-, D_+, and D_+^t satisfy the following "summation by parts" formulas:

$$\begin{aligned}\sum_j a_j D_- b_j &= -\sum_j b_j D_+ a_j, \quad \text{if } a_{\pm\infty} = 0 \text{ or } b_{\pm\infty} = 0,\\ \sum_{n=0}^{\infty} a^n D_+^t b^n &= -\frac{1}{\Delta t}a^0 b^0 - \sum_{n=1}^{\infty} b^n D_-^t a^n \quad \text{if } a^\infty = 0 \text{ or } b^\infty = 0.\end{aligned}$$

Let φ be a nonnegative test function in $C_0^\infty(\Omega)$ and set

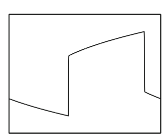
$$\varphi_j^n = \frac{1}{|I_{j-1/2}^n|}\iint\limits_{I_{j-1/2}^n} \varphi(x,t)\,dx\,dt.$$

We multiply (1.61) by $\Delta t\,\Delta x\,\varphi_j^n$ and sum over $n \ge 0$ and $j \in \mathbb{Z}$, using the summation by parts formulas above, to get

$$\begin{aligned}&\Delta x\,\Delta t\sum_{n=1}^{\infty}\sum_j \eta(u_j^n)D_-^t\varphi_j^n\\ &\quad + \Delta x\,\Delta t\sum_{n=0}^{\infty}\sum_j\left(a_j^n\eta(u_j^n)D_+\varphi_j^n + \eta(u_{j-1}^n)D_- a_j^n\varphi_j^n\right) + \Delta x\sum_j \eta(u_j^0)\varphi_j^0 \ge 0.\end{aligned}$$

Call the left-hand side of the above inequality $B_{\Delta x}$, and set

$$A_{\Delta x} = \iint\limits_{\Omega}\left(\eta(u_{\Delta x})\varphi_t + a\eta(u_{\Delta x})\varphi_x + a_x\eta(u_{\Delta x})\varphi\right)dx\,dt + \int\limits_{\mathbb{R}}\eta(u_0)\varphi(x,0)\,dx.$$



Then we have

$$A_{\Delta x} = B_{\Delta x} + (A_{\Delta x} - B_{\Delta x}) \geq A_{\Delta x} - B_{\Delta x}.$$

We find that

$$A_{\Delta x} - B_{\Delta x} = \sum_{n=1}^{\infty} \sum_{j} \iint\limits_{I_{j-1/2}^n} \eta_j^n \left(\varphi_t - D_-^t \varphi_j^n\right) \, dx \, dt \tag{1.62a}$$

$$+ \sum_{j} \iint\limits_{I_{j-1/2}^0} \eta_j^0 \varphi_t \, dx \, dt \tag{1.62b}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^n} \eta_j^n a \left(\varphi_x - D_+ \varphi_j^n\right) \, dx dt \tag{1.62c}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^n} \eta_j^n D_+ \varphi_j^n \left(a - a_j^n\right) \, dx \, dt \tag{1.62d}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^n} \left(\eta_j^n - \eta_{j-1}^n\right) a_x \varphi \, dx \, dt \tag{1.62e}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^n} a_x \eta_{j-1}^n \left(\varphi - \varphi_j^n\right) \, dx \, dt \tag{1.62f}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^n} \eta_{j-1}^n \left(a_x - D_- a_j^n\right) \varphi_j^n \, dx \, dt \tag{1.62g}$$

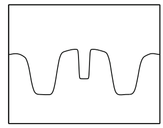
$$+ \sum_{j} \int\limits_{I_{j-1/2}} \left(\eta(u_0) - \eta_j^0\right) \varphi(x, 0) \, dx \tag{1.62h}$$

$$+ \sum_{j} \int\limits_{I_{j-1/2}} \eta_j^0 \left(\varphi(x, 0) - \varphi_j^0\right) \, dx. \tag{1.62i}$$

Here $I_{j-1/2} = [x_{j-1}, x_j)$. To show that the limit u is an entropy solution, we must show that all the terms (1.62a)–(1.62i) vanish when Δx becomes small. A small but useful device is contained in the following remark.

Remark 1.13 For a continuously differentiable function ϕ we have

$$\begin{aligned}
|\varphi(x, t) - \varphi(y, s)| &= \left| \int_0^1 \frac{d}{d\sigma} \varphi(\sigma(x, t) + (1 - \sigma)(y, s)) d\sigma \right| \\
&= \left| \int_0^1 \nabla \varphi(\sigma(x, t) + (1 - \sigma)(y, s)) \cdot (x - y, t - s) d\sigma \right| \\
&\leq |x - y| \, \|\varphi_x\|_{L^\infty} + |t - s| \, \|\varphi_t\|_{L^\infty} .
\end{aligned}$$



We start with the last term (1.62i). Now

$$
\begin{aligned}
\int_{I_{j-1/2}} \eta_j^0 \left(\varphi(x,0) - \varphi_j^0\right) dx
&= \frac{\eta_j^0}{\Delta x \Delta t} \int_{I_{j-1/2}} \iint_{I_{j-1/2}^0} \varphi(x,0) - \varphi(y,t)\, dy\, dt\, dx \\
&= \frac{\eta_j^0}{\Delta x \Delta t} \int_{I_{j-1/2}} \iint_{I_{j-1/2}^0} \Big(\int_y^x \varphi_x(z,0)\, dz + \int_0^t \varphi_t(y,s)\, ds \Big)\, dy\, dt\, dx.
\end{aligned}
$$

Therefore,

$$
|(1.62\text{i})| \le \|\eta(u_0)\|_{L^1(\mathbb{R})} \left(\|\varphi_x\|_{L^\infty(\Omega)} \Delta x + \|\varphi_t\|_{L^\infty(\Omega)} \Delta t \right),
$$

where we used the convexity of η. Next, we consider the term (1.62h): Since η is convex, we have

$$
|\eta(b) - \eta(a)| \le \max\left\{ |\eta'(a)|, |\eta'(b)| \right\} |b - a| .
$$

Furthermore, if both x and y are in $I_{j-1/2}$, then

$$
|u_0(x) - u_0(y)| \le |u_0|_{BV(I_{j-1/2})} .
$$

Using this and choosing $C = \|\eta'(u_0)\|_{L^\infty}$ yields

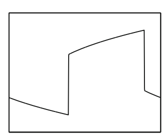
$$
\begin{aligned}
\Big| \int_{I_{j-1/2}} \left(\eta(u_0) - \eta(u_j^0) \right) \varphi(x,0)\, dx \Big|
&\le C \|\varphi\|_{L^\infty(\Omega)} \int_{I_{j-1/2}} \left| u_0(x) - u_j^0 \right| dx \\
&\le C \|\varphi\|_{L^\infty(\Omega)} \int_{I_{j-1/2}} \frac{1}{\Delta x} \int_{I_{j-1/2}} |u_0(x) - u_0(y)|\, dx\, dy \\
&\le C \|\varphi\|_{L^\infty(\Omega)} \Delta x\, |u_0|_{BV(I_{j-1/2})} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(1.62\text{h})| &\le C \|\varphi\|_{L^\infty(\Omega)} \Delta x \sum_j |u_0|_{BV(I_{j-1/2})} \\
&\le C \|\varphi\|_{L^\infty(\Omega)} \Delta x\, |u_0|_{BV} .
\end{aligned}
$$

Next, we consider (1.62g). First observe that

$$
D_- a_j^n = D_- \Big(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} a(x_j, t)\, dt \Big) = \frac{1}{\Delta x \Delta t} \iint_{I_{j-1/2}^n} a_x(x,t)\, dx\, dt.
$$



Therefore,

$$\eta^n_{j-1} \iint\limits_{I^n_{j-1/2}} a_x(x,t) - D_- a^n_j \, dx\, dt = \eta^n_{j-1}\Big(\iint\limits_{I^n_{j-1/2}} a_x(x,t)\, dx\, dt - \Delta x \Delta t a^n_j \Big) \\ = 0,$$

and (1.62g) = 0. We continue with the term (1.62f), namely

$$\begin{aligned} &\iint\limits_{I^n_{j-1/2}} |a_x|\, \eta^n_{j-1} \left|\varphi - \varphi^n_j\right| \, dx\, dt \\ &\quad\le \|a_x\|_{L^\infty(\Omega)}\, \eta^n_{j-1} \frac{1}{\Delta x \Delta t} \iint\limits_{I^n_j} \iint\limits_{I^n_j} |\varphi(x,t) - \varphi(y,s)| \, dy\, ds\, dx\, dt \\ &\quad\le \|a_x\|_{L^\infty(\Omega)}\, \eta^n_{j-1} \Delta x \Delta t \left(\Delta x \, \|\varphi_x\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_t\|_{L^\infty(\Omega)} \right). \end{aligned}$$

Recall that the test function φ has compact support, contained in $\{t < T\}$. Furthermore, using the scheme for η^n_j, cf. (1.61), it is straightforward to show that

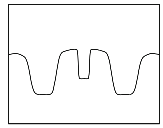
$$\Delta x \sum_j \eta^n_j \le e^{C t_n} \Delta x \sum_j \eta^0_j \le e^{C t_n} \, \|\eta(u_0)\|_{L^1(\mathbb{R})},$$

where C is a bound on $D_- a^n_j$. Therefore,

$$\begin{aligned} \Big| \sum_{j,n} \iint\limits_{I^n_{j-1/2}} a_x \eta^n_{j-1} \left(\varphi - \varphi^n_j \right) \, dx\, dt \Big| &\le C_T \Delta x \sum_{j,n} \eta^n_j \Delta t (\Delta x + \Delta t) \\ &\le C_T \Delta x \sum_{n,j} \eta^0_j \Delta t (\Delta x + \Delta t) \\ &\le C_T T \, \|\eta(u_0)\|_{L^1(\mathbb{R})} \, (\Delta x + \Delta t), \end{aligned}$$

since the sum in n is only over those n such that $t_n = n\Delta t \le T$. Regarding (1.62e), and setting $M > \|u_0\|_{L^\infty(\mathbb{R})}$, we have that

$$\begin{aligned} &\Big| \sum_{j,n} \iint\limits_{I^n_{j-1/2}} \left(\eta^n_j - \eta^n_{j-1} \right) a_x \varphi \, dx\, dt \Big| \\ &\quad\le \|\eta'\|_{L^\infty((-M,M))} \, \|a_x\|_{L^\infty(\Omega)} \, \|\varphi\|_{L^\infty(\Omega)} \, \Delta x \Delta t \sum_{j,n} \left| u^n_j - u^n_{j-1} \right| \\ &\quad\le C \Delta x T \, |u_0|_{BV}. \end{aligned}$$



Next, we turn to (1.62d):

$$
\begin{aligned}
|(1.62\mathrm{d})| &\le \|\varphi_x\|_{L^\infty(\Omega)} \sum_{j,n} \eta_j^n \iint\limits_{I_{j-1/2}^n} \left|a(x,t) - a(x_j,t)\right| \, dxdt \\
&\le \|\varphi_x\|_{L^\infty(\Omega)} \|a_x\|_{L^\infty(\Omega)} \Delta x \sum_{j,n} \eta_j^n \Delta x \Delta t \\
&\le \|\varphi_x\|_{L^\infty(\Omega)} \|a_x\|_{L^\infty(\Omega)} C_T T \Delta x \, \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

We can use the same type of argument to estimate (1.62c):

$$
\begin{aligned}
|(1.62\mathrm{c})| &\le \|a\|_{L^\infty(\Omega)} \left(\Delta x \, \|\varphi_{xx}\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_{xt}\|_{L^\infty(\Omega)}\right) \sum_{n,j} \eta_j^n \Delta x \Delta t \\
&\le \|a\|_{L^\infty(\Omega)} \left(\Delta x \, \|\varphi_{xx}\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_{xt}\|_{L^\infty(\Omega)}\right) C_T T \, \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

Similarly, we show that

$$
|(1.62\mathrm{b})| \le C_{\Delta t} \Delta t \, \|\eta(u_0)\|_{L^1(\mathbb{R})} \, \|\varphi_t\|_{L^\infty(\Omega)} .
$$

Now the end is in sight. We estimate the right-hand side of (1.62a). This will be less than

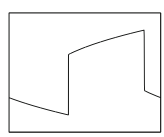
$$
\begin{aligned}
&\sum_{j,n\ge 1} \eta_j^n \iint\limits_{I_{j-1/2}^n} \left|\varphi_t - D_+^t \varphi_j^n\right| \, dx \, dt \\
&\qquad \le \left(\Delta x \, \|\varphi_{xt}\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_{tt}\|_{L^\infty(\Omega)}\right) \sum_{j,n\ge 1} \eta_j^n \Delta x \Delta t \\
&\qquad \le \left(\Delta x \, \|\varphi_{xt}\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_{tt}\|_{L^\infty(\Omega)}\right) C_T T \, \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

To sum up, what we have shown is that for every test function $\varphi(x,t)$,

$$
\begin{aligned}
&\iint\limits_{\Omega} \left(\eta(u)\varphi_t + a\eta(u)\varphi_x + a_x \eta(u)\right) dx \, dt + \int\limits_{\mathbb{R}} \eta(u_0)\varphi(x,0) \, dx \\
&\qquad = \lim_{\Delta x \to 0} A_{\Delta x} \\
&\qquad \ge \lim_{\Delta x \to 0} \left(A_{\Delta x} - B_{\Delta x}\right) = 0,
\end{aligned}
$$

if a_x is (locally) continuous and $u_0 \in BV(\mathbb{R})$. Hence the scheme (1.60) produces a subsequence that converges to the unique weak solution. Since the limit is the unique entropy solution, every subsequence will produce a further subsequence that converges to the *same* limit, and thus the *whole* sequence converges!

If u_0'' is bounded, we have seen that the scheme (1.60) converges at a rate $\mathcal{O}(\Delta x)$ to the entropy solution. The significance of the above computations is that we have the convergence to the unique entropy solution even if u_0 is assumed to be only in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$. However, in this case we have not shown any convergence rate.



Systems of Equations

I have a different way of thinking. I think synergistically.
I'm not linear in thinking, I'm not very logical.
— Imelda Marcos

Now we generalize, and let $u\colon \mathbb{R} \times [0, \infty) \to \mathbb{R}^n$ be a solution of the linear system

$$\begin{cases} u_t + Au_x = 0, & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.63}$$

where A is an $n \times n$ matrix with *real and distinct eigenvalues* $\{\lambda_i\}_{i=1}^n$. We order these such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

If this holds, then the system is said to be *strictly hyperbolic*. The matrix A will also have n linearly independent right eigenvectors $r_1, \dots, r_n$ such that

$$Ar_i = \lambda_i r_i.$$

Similarly, it has n independent left eigenvectors $l_1, \dots, l_n$ such that

$$l_i A = \lambda_i l_i.$$

We assume r_i to be column vectors and l_i to be row vectors. However, we will not enforce this strictly, and will write, e.g., $l_i \cdot r_k$. For $k \neq m$, l_k and r_m are orthogonal, since

$$\lambda_m r_m \cdot l_k = Ar_m \cdot l_k = r_m \cdot l_k A = \lambda_k r_m \cdot l_k.$$

Let

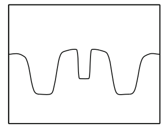
$$L = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}, \quad R = \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix}.$$

Normalize the eigenvectors so that $l_k \cdot r_i = \delta_{ki}$, i.e., $L = R^{-1}$, or $LR = I$. Then

$$LAR = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

We can multiply (1.63) by L from the left to get

$$Lu_t + LAu_x = 0,$$



and defining w by $u = Rw$, we find that

$$w_t + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} w_x = 0. \tag{1.64}$$

This is n decoupled equations, one for each component of $w = (w_1, \dots, w_n)$,

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \quad \text{for } i = 1, \dots, n.$$

The initial data transforms into

$$w_0 = L u_0 = (l_1 \cdot u_0, \dots, l_n \cdot u_0),$$

and hence we obtain the solution

$$w_i(x,t) = l_i \cdot u_0(x - \lambda_i t).$$

Transforming back into the original variables, we obtain

$$u(x,t) = \sum_{i=1}^{n} w_i(x,t) r_i = \sum_{i=1}^{n} \left[l_i \cdot u_0(x - \lambda_i t) \right] r_i. \tag{1.65}$$

◇ Example 1.14 (The linear wave equation)
Now consider the linear wave equation; $\alpha\colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is a solution of

$$\begin{cases} \alpha_{tt} - c^2 \alpha_{xx} = 0, & x \in \mathbb{R}, \quad t > 0, \\ \alpha(x,0) = \alpha_0(x), \quad \alpha_t(x,0) = \beta_0(x), & \end{cases}$$

where c is a positive constant. Defining

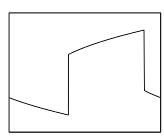
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_t \\ \alpha_x \end{pmatrix}$$

implies that

$$\begin{aligned} \frac{\partial u_1}{\partial t} - c^2 \frac{\partial u_2}{\partial x} &= 0, \\ \frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial x} &= 0, \end{aligned} \qquad \text{or} \qquad u_t + \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} u_x = 0.$$

The matrix

$$A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$



has eigenvalues and eigenvectors

$$\lambda_1 = -c, \;\; r_1 = \begin{pmatrix} c \\ 1 \end{pmatrix}, \quad \lambda_2 = c, \;\; r_2 = \begin{pmatrix} -c \\ 1 \end{pmatrix}.$$

Thus

$$R = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix}, \quad L = R^{-1} = \frac{1}{2c}\begin{pmatrix} 1 & c \\ -1 & c \end{pmatrix}.$$

Hence we find that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{2c}\begin{pmatrix} u_1 + cu_2 \\ -u_1 + cu_2 \end{pmatrix}.$$

Writing the solution in terms of α_x and α_t, we find that

$$\begin{aligned} \alpha_t(x,t) + c\alpha_x(x,t) &= \beta_0(x+ct) + c\alpha_0'(x+ct), \\ -\alpha_t(x,t) + c\alpha_x(x,t) &= -\beta_0(x-ct) + c\alpha_0'(x-ct). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_x(x,t) &= \frac{1}{2}\left(\alpha_0'(x+ct) + \alpha_0'(x-ct)\right) + \frac{1}{2c}\left(\beta_0(x+ct) - \beta_0(x-ct)\right), \\ \alpha_t(x,t) &= \frac{1}{2}\left(\beta_0(x+ct) + \beta_0(x-ct)\right) + \frac{c}{2}\left(\alpha_0'(x+ct) - \alpha_0'(x-ct)\right). \end{aligned}$$

To find α, we can integrate the last equation in t,

$$\alpha(x,t) = \frac{1}{2}\left(\alpha_0(x+ct) + \alpha_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta_0(y)\,dy,$$

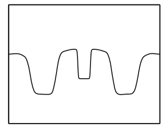
after a change of variables in the integral involving β_0. This is the famous d'Alembert formula for the solution of the linear wave equation in one dimension. ◇

Next, we discuss the notion of entropy solutions. The meaning of an entropy solution to an equation written in characteristic variables is that for some convex function $\hat{\eta}(u)$, the entropy solution should satisfy

$$\hat{\eta}(u)_t + \hat{q}(u)_x \le 0,$$

in the weak sense. Here the entropy flux $\hat{q}$ should satisfy

$$\nabla_w \hat{q}(u) = \nabla_u \hat{\eta}(u)\Lambda, \quad \text{i.e.,} \;\; \frac{\partial \hat{q}}{\partial u_i} = \lambda_i \frac{\partial \hat{\eta}}{\partial u_i} \;\text{for}\; i = 1,\dots,n.$$



An entropy solution to (1.63) is the limit (if such a limit exists) of the parabolic reqularization

$$u_t^\varepsilon + Au_x^\varepsilon = \varepsilon u_{xx}^\varepsilon$$

as $\varepsilon \to 0$. To check whether we have a convex entropy $\eta\colon \mathbb{R}^n \to \mathbb{R}$, we take the inner product of the above with $\nabla\eta(u^\varepsilon)$ to get

$$\eta\left(u^\varepsilon\right)_t + \nabla\eta\left(u^\varepsilon\right)\cdot A\, u_x^\varepsilon \le \varepsilon\left(\nabla\eta\left(u^\varepsilon\right)\cdot u_x^\varepsilon\right)_x,$$

by the convexity of η. Observe that the convexity is used to get rid of a term containing $(u_x^\varepsilon)^2$, which may not be wellbehaved (for nonlinear equations) in the limit $\varepsilon \to 0$, and we obtain an inequality rather than an equality. We want to write the second term on the left as the x derivative of some function $q(u^\varepsilon)$. Using

$$q\left(u^\varepsilon\right)_x = \nabla q\left(u^\varepsilon\right)\cdot u_x^\varepsilon,$$

we see that if this is so, then

$$\frac{\partial q}{\partial u_j} = \sum_i a_{ij}\frac{\partial\eta}{\partial u_i} \quad \text{for } j = 1,\ldots,n. \tag{1.66}$$

This is n equations in the two unknowns η and q. Thus we cannot expect any solution if $n > 2$. The right-hand side of (1.66) is given, and hence we are looking for a potential q with a given gradient. This problem has a solution if

$$\frac{\partial^2 q}{\partial u_k\partial u_j} = \frac{\partial^2 q}{\partial u_j\partial u_k},$$

or

$$\sum_i a_{ik}\frac{\partial^2\eta}{\partial u_i\partial u_j} = \sum_i a_{ij}\frac{\partial^2\eta}{\partial u_i\partial u_k} \quad \text{for } 1 \le j,k \le n.$$

If we wish to find an entropy flux for the entropy $\eta(u) = |u|^2/2$, note that

$$\frac{\partial^2\eta}{\partial u_i\partial u_k} = \delta_{ik}.$$

Thus we can find an entropy flux if $a_{jk} = a_{kj}$ for $1 \le j,k \le n$; in other words, A must be symmetric. In this case the entropy flux q reads

$$q(u) = \sum_{i,j} a_{ij}u_iu_j - \frac{1}{2}\sum_i a_{ii}u_i^2.$$

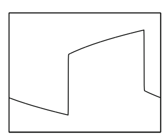
Hence, an entropy (using the entropy $\eta(u) = |u|^2/2$) solution satisfies

$$|u|_t^2 + q(u)_x \le 0 \quad \text{weakly.}$$

This means that

$$\|u(\,\cdot\,,t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})},$$

and thus there is at most one entropy solution to (1.63) if A is symmetric.



The Riemann Problem for Linear Systems

Returning to the general case, recall that the solution $u = u(x,t)$ to (1.63) is given by (1.65), namely

$$u(x,t) = \sum_{i=1}^{n} [l_i \cdot u_0(x - \lambda_i t)]\, r_i. \tag{1.67}$$

Now we shall look at a type of initial value problem where u_0 is given by two constant values, namely

$$u_0(x) = \begin{cases} u_{\text{left}} & x < 0, \\ u_{\text{right}} & x \geq 0, \end{cases} \tag{1.68}$$

where u_{left} and u_{right} are two constant vectors. This type of initial value problem is called a *Riemann problem*, (cf. (1.28)) which will a problem of considerable interest throughout the book.

For a single equation ($n = 1$), the weak solution to this Riemann problem reads

$$u(x,t) = u_0(x - \lambda_1 t) = \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_{\text{right}} & x \geq \lambda_1 t. \end{cases}$$

Note that u is not continuous. Nevertheless, it is the unique entropy solution in the sense of Definition 1.10 to (1.63) with initial data (1.68) (see Exercise 1.4).

For two equations ($n = 2$), we write

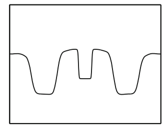
$$u_{\text{left}} = \sum_{i=1}^{2} [l_i \cdot u_{\text{left}}]\, r_i, \qquad u_{\text{right}} = \sum_{i=1}^{2} \left[l_i \cdot u_{\text{right}}\right] r_i.$$

We can find the solution of each component separately. Namely, using (1.67) for initial data (1.68), we obtain

$$[l_1 \cdot u(x,t)] = \begin{cases} l_1 \cdot u_{\text{left}} & x < \lambda_1 t, \\ l_1 \cdot u_{\text{right}} & x \geq \lambda_1 t, \end{cases} \quad [l_2 \cdot u(x,t)] = \begin{cases} l_2 \cdot u_{\text{left}} & x < \lambda_2 t, \\ l_2 \cdot u_{\text{right}} & x \geq \lambda_2 t. \end{cases}$$

Combining these we see that

$$\begin{aligned} u(x,t) &= [l_1 \cdot u(x,t)]\, r_1 + [l_2 \cdot u(x,t)]\, r_2 \\ &= \begin{cases} [l_1 \cdot u_{\text{left}}]\, r_1 + [l_2 \cdot u_{\text{left}}]\, r_2 & x < \lambda_1 t, \\ \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2 & t\lambda_1 < x \leq t\lambda_2, \\ \left[l_1 \cdot u_{\text{right}}\right] r_1 + \left[l_2 \cdot u_{\text{right}}\right] r_2 & x \geq \lambda_1 t, \end{cases} \\ &= \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_{\text{middle}} & t\lambda_1 < x \leq t\lambda_2, \\ u_{\text{right}} & x \geq \lambda_1 t, \end{cases} \end{aligned}$$



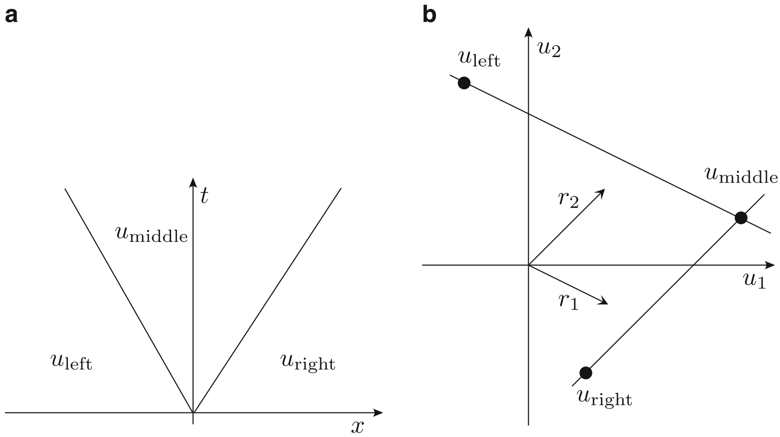


Fig. 1.7 The solution of the Riemann problem. **a** In (x, t)-space. **b** In phase space

with $u_{\text{middle}} = \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2$. Observe the structure of the different states:

$$\begin{aligned}
u_{\text{left}} &= [l_1 \cdot u_{\text{left}}]\, r_1 + [l_2 \cdot u_{\text{left}}]\, r_2, \\
u_{\text{middle}} &= \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2, \\
u_{\text{right}} &= \left[l_1 \cdot u_{\text{right}}\right] r_1 + \left[l_2 \cdot u_{\text{right}}\right] r_2.
\end{aligned}$$

We can also view the solution in *phase space*, that is, in the (u_1, u_2)-plane. We see that for every u_{left} and u_{right}, we have the solution $u(x,t) = u_{\text{left}}$ for $x < \lambda_1 t$ and $u(x,t) = u_{\text{right}}$ for $x \geq \lambda_2 t$. In the middle, $u(x,t) = u_{\text{middle}}$ for $\lambda_1 t \leq x < \lambda_2 t$. The middle value u_{middle} is on the intersection of the line through u_{left} parallel to r_1 and the line through u_{right} parallel to r_2. See Fig. 1.7. In the general, nonlinear, case, the straight lines connecting u_{left}, u_m, and u_{right} will be replaced by arcs, not necessarily straight. However, the same structure prevails, at least locally.

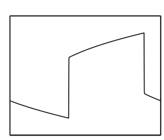
Now we can find the solution to the Riemann problem for any n, namely

$$u(x,t) = \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_i & \lambda_i t \leq x < \lambda_{i+1} t, \quad i = 1, \dots, n-1, \\ u_{\text{right}} & x \geq \lambda_n t, \end{cases}$$

where

$$u_i = \sum_{j=1}^{i} \left[l_j \cdot u_{\text{right}}\right] r_j + \sum_{j=i+1}^{n} \left[l_j \cdot u_{\text{left}}\right] r_j.$$

Observe that this solution can also be viewed in phase space as the path from $u_0 = u_{\text{left}}$ to $u_n = u_{\text{right}}$ obtained by going from u_{i-1} to u_i on a line parallel to r_i for $i = 1, \dots, n$. This viewpoint will be important when we consider nonlinear equations, where the straight lines will be replaced by arcs. Locally, the structure will remain unaltered.



Numerics for Linear Systems with Constant Coefficients

If $\lambda_i > 0$, then we know that the scheme

$$D_+^t w_{i,j}^m + \lambda_i D_- w_{i,j}^m = 0$$

will produce a sequence of functions $\{w_{i,\Delta x}\}$ that converges to the unique entropy solution of

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0.$$

Similarly, if $\lambda_i < 0$, the scheme

$$D_+^t w_{i,j}^m + \lambda_i D_+ w_{i,j}^m = 0$$

will give a convergent sequence. Both of these schemes will be convergent only if $\Delta t \leq \Delta x \, |\lambda_i|$, which is the CFL condition. In eigenvector coordinates, with

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \qquad w_j^m \approx w(j\Delta x, m\Delta t),$$

the resulting scheme for w reads

$$D_+^t w_j^m + \Lambda_+ D_- w_j^m + \Lambda_- D_+ w_j^n = 0, \tag{1.69}$$

where

$$\Lambda_- = \begin{pmatrix} \lambda_1 \wedge 0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \wedge 0 \end{pmatrix} \text{ and } \Lambda_+ = \begin{pmatrix} \lambda_1 \vee 0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \vee 0 \end{pmatrix},$$

and we have introduced the notation

$$a \vee b = \max\{a, b\} \text{ and } a \wedge b = \min\{a, b\}.$$

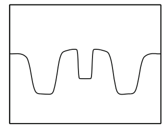
Observe that $\Lambda = \Lambda_+ + \Lambda_-$. If the CFL condition

$$\frac{\Delta t}{\Delta x} \max_i |\lambda_i| = \frac{\Delta t}{\Delta x} \max\{|\lambda_1|, |\lambda_n|\} \leq 1$$

holds, then the scheme (1.69) will produce a convergent sequence, and the limit w will be the unique entropy solution to

$$w_t + \Lambda w_x = 0. \tag{1.70}$$

By defining $u = Rw$, we obtain a solution of (1.63).



We can also make the same transformation on the discrete level. Multiplying (1.69) by L from the left and using that $u = Rw$ yields

$$D_+^t u_j^m + A_+ D_- u_j^m + A_- D_+ u_j^m = 0, \tag{1.71}$$

where

$$A_\pm = R\Lambda_\pm L,$$

and this finite difference scheme will converge directly to u.

1.2 Notes

Never any knowledge was delivered in the same order it was invented.[6]
— Sir Francis Bacon (1561–1626)

The simplest nontrivial conservation law, the inviscid Burgers equation, has been extensively analyzed. Burgers introduced the "nonlinear diffusion equation"

$$u_t + \frac{1}{2}(u^2)_x = u_{xx}, \tag{1.72}$$

which is currently called (the viscous) Burgers's equation, in 1940 [37] (see also [38]) as a model of turbulence. Burgers's equation is linearized, and thereby solved, by the Cole–Hopf transformation [46, 98]. Both the equation and the Cole–Hopf transformation were, however, known already in 1906; see Forsyth [66, p. 100]. See also Bateman [14]. The early history of hyperbolic conservation laws is presented in [56, pp. XV–XXX]. A source of some of the early papers is [104].

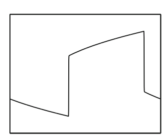
The most common elementary example of application of scalar conservation laws is the model of traffic flow called "traffic hydrodynamics" that was introduced independently by Lighthill and Whitham [134] and Richards [155]. A modern treatment can be found in Haberman [81]. Example 1.6 presents some of the fundamentals, and serves as a nontechnical introduction to the lack of uniqueness for weak solutions. Extensions to traffic flow on networks exist; see [94] and [68].

The jump condition, or the Rankine–Hugoniot condition, was derived heuristically from the conservation principle independently by Rankine in 1870 [152] and Hugoniot in 1886 [101–103]. Our presentation of the Rankine–Hugoniot condition is taken from Smoller [169].

The notion of "Riemann problem" is fundamental in the theory of conservation laws. It was introduced by Riemann in 1859 [156, 157] in the context of gas dynamics. He studied the situation in which one initially has two gases with different (constant) pressures and densities separated by a thin membrane in a one-dimensional cylindrical tube. See [97] and [56, pp. XV–XXX] for a historical discussion.

The final section of this chapter contains a detailed description of the one-dimensional linear case, both in the scalar case and in the case of systems. This allows us to introduce some of the methods in a simpler case. Here existence of

[6] in *Valerius Terminus*: *Of the Interpretation of Nature*, c. 1603.



solutions is shown using appropriate finite difference schemes, in contrast to the front-tracking method used in the text proper.

There are by now several books on various aspects of hyperbolic conservation laws, starting with the classical book by Courant and Friedrichs [51]. Nice treatments with emphasis on the mathematical theory can be found in books by Lax [126, 127], Chorin and Marsden [42], Roždestvenskiĭ and Janenko [164], Smoller [169], Rhee, Aris, and Amundson [153, 154], Málek et al. [141], Hörmander [99], Liu [137], Serre [167, 168], Benzoni-Gavage and Serre [15], Bressan [24, 27], Dafermos [56], Lu [139], LeFloch [129], Perthame [150], Zheng [192]. The books by Bouchut [19], Godlewski and Raviart [78, 79], LeVeque [130, 131], Kröner [116], Toro [180], Thomas [179], and Trangenstein [183] focus more on the numerical theory.

1.3 Exercises

1.1 Determine characteristics for the following quasilinear equations:

$$\begin{aligned} u_t + \sin(x)u_x &= u, \\ \sin(t)u_t + \cos(x)u_x &= 0, \\ u_t + \sin(u)u_x &= u, \\ \sin(u)u_t + \cos(u)u_x &= 0. \end{aligned}$$

1.2 Use characteristics to solve the following initial value problems:

(a) $uu_x + xu_y = 0$, $u(0,s) = 2s$ for $s > 0$.
(b) $e^y u_x + uu_y + u^2 = 0$, $u(x,0) = 1/x$ for $x > 0$.
(c) $xu_y - yu_x = u$, $u(x,0) = h(x)$ for $x > 0$.
(d) $(x+1)^2 u_x + (y-1)^2 u_y = (x+y)u$, $u(x,0) = -1-x$.
(e) $u_x + 2xu_y = x + xu$, $u(1,y) = e^y - 1$.
(f) $u_x + 2xu_y = x + xu$, $u(0,y) = y^2 - 1$.
(g) $xuu_x + u_y = 2y$, $u(x,0) = x$.

1.3 (a) Use characteristics to show that

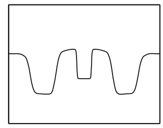
$$u_t + au_x = f(x,t), \quad u|_{t=0} = u_0,$$

with a a constant, has solution

$$u(x,t) = u_0(x - at) + \int_0^t f(x - a(t-s), s)\, ds.$$

(b) Show that

$$u\left(\xi(t;x_0), t\right) = u_0(x_0) + \int_0^t f(\xi(s;x_0), s)\, ds$$



holds if u is the solution of

$$u_t + a(x,t)u_x = f(x,t), \quad u|_{t=0} = u_0, \tag{1.73}$$

where ξ satisfies

$$\frac{d}{dt}\xi(t;x_0) = a(\xi(t;x_0),t), \;\; \xi(0;x_0) = x_0.$$

(c) Show that

$$u(x,t) = u_0\left(\zeta(t;x)\right) + \int_0^t f\left(\zeta(\tau;x), t-\tau\right)\, d\tau$$

holds if u is the solution of (1.73) and

$$\frac{d}{d\tau}\zeta(\tau;x) = -a(\zeta(\tau;x), t-\tau), \;\; \zeta(0;x) = x.$$

1.4 Show that

$$u(x,t) = \begin{cases} u_{\text{left}} & x < at, \\ u_{\text{right}} & x \geq at, \end{cases}$$

is the entropy solution in the sense of Definition 1.10 for the equation $u_t + au_x = 0$ (where a is constant) and $u|_{t=0}(x) = u_{\text{left}}\chi_{x<0} + u_{\text{right}}\chi_{x\geq 0}$.

1.5 Find the shock condition (i.e., the Rankine–Hugoniot condition) for one-dimensional systems, i.e., the unknown u is a vector $u = (u_1, \ldots, u_n)$ for some $n > 1$, and also $f(u) = (f_1(u), \ldots, f_n(u))$.

1.6 Consider a scalar conservation law in two space dimensions,

$$u_t + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0,$$

where the flux functions f and g are continuously differentiable. Now the unknown u is a function of x, y, and t. Determine the Rankine–Hugoniot condition across a jump discontinuity in u, assuming that u jumps across a regular surface in (x, y, t). Try to generalize your answer to a conservation law in n space dimensions.

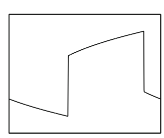
1.7 We shall consider a linearization of Burgers's equation. Let

$$u_0(x) = \begin{cases} 1 & \text{for } x < -1, \\ -x & \text{for } -1 \leq x \leq 1, \\ -1 & \text{for } 1 < x. \end{cases}$$

(a) First determine the maximum time that the solution of the initial value problem

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = u_0(x),$$

will remain continuous. Find the solution for t less than this time.



(b) Then find the solution v of the linearized problem

$$v_t + u_0(x)v_x = 0, \quad v(x,0) = u_0(x).$$

Determine the solution also in the case $v(x,0) = u_0(\alpha x)$, where α is nonnegative.

(c) Next, we shall determine a procedure for finding u by solving a sequence of linearized equations. Fix $n \in \mathbb{N}$. For t in the interval $(m/n, (m+1)/n]$ and $m \geq 0$, let v_n solve

$$(v_n)_t + v_n\,(x, m/n)\,(v_n)_x = 0,$$

and set $v_n(x,0) = u_0(x)$. Then show that

$$v_n\left(x, \frac{m}{n}\right) = u_0\left(\alpha_{m,n} x\right)$$

and find a recurrence relation (in m) satisfied by $\alpha_{m,n}$.

(d) Assume that

$$\lim_{n\to\infty} \alpha_{m,n} = \bar{\alpha}(t),$$

for some continuously differentiable $\bar{\alpha}(t)$, where $t = m/n < 1$. Show that $\bar{\alpha}(t) = 1/(1-t)$, and thus $v_n(x) \to u(x)$ for $t < 1$. What happens for $t \geq 1$?

1.8 (a) Solve the initial value problem for Burgers's equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{1.74}$$

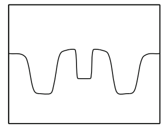
(b) Then find the solution where the initial data are

$$u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

(c) If we multiply Burgers's equation by u, we formally find that u satisfies

$$\frac{1}{2}\left(u^2\right)_t + \frac{1}{3}\left(u^3\right)_x = 0, \quad u(x,0) = u_0(x). \tag{1.75}$$

Are the solutions to (1.74) you found in parts **a** and **b** weak solutions to (1.75)? If not, then find the corresponding weak solutions to (1.75). *Warning: This shows that manipulations valid for smooth solutions are not necessarily so for weak solutions.*



1.9 ([169, p. 250]) Show that

$$u(x,t) = \begin{cases} 1 & \text{for } x \le (1-\alpha)t/2, \\ -\alpha & \text{for } (1-\alpha)t/2 < x \le 0, \\ \alpha & \text{for } 0 < x \le (\alpha-1)t/2, \\ -1 & \text{for } x \ge (\alpha-1)t/2 \end{cases}$$

is a weak solution of

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad u(x,0) = \begin{cases} 1 & \text{for } x \le 0, \\ -1 & \text{for } x > 0, \end{cases}$$

for all $\alpha \ge 1$. *Warning: Thus we see that weak solutions are not necessarily unique.*

1.10 We outline a proof of some *Gronwall inequalities*.

(a) Assume that u satisfies

$$u'(t) \le \gamma u(t).$$

Show that $u(t) \le e^{\gamma t} u(0)$.

(b) Assume now that u satisfies

$$u'(t) \le C(1 + u(t)).$$

Show that $u(t) \le e^{Ct}(1 + u(0)) - 1$.

(c) Assume that u satisfies

$$u'(t) \le c(t)u(t) + d(t),$$

for $0 \le t \le T$, where $c(t)$ and $d(t)$ are in $L^1([0,T])$. Show that

$$u(t) \le u(0) + \int_0^t d(s) \exp\Big(\int_s^t c(\tilde{s})\, d\tilde{s} \Big)\, ds$$

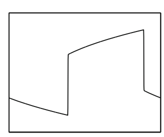
for $t \le T$.

(d) Assume that u is in $L^1([0,T])$ and that for $t \in [0,T]$,

$$u(t) \le C_1 \int_0^t u(s)\, ds + C_2.$$

Show that

$$u(t) \le C_2 e^{C_1 t}.$$



1.11 Consider the semidiscrete difference scheme (1.37). The goal of this exercise is to prove that a unique solution exists for all $t > 0$.

(a) Let $\eta(u)$ be a smooth function. Show that

$$D_-\eta(u_j) = \eta'(u_j)D_-u_j - \frac{\Delta x}{2}\eta''(u_{j-1/2})\left(D_-u_j\right)^2,$$

where $u_{j-1/2}$ is some value between u_j and u_{j-1}.

(b) Assume now that $\eta'' \geq 0$. Show that

$$\frac{d}{dt}\sum_j \eta(u_j) \leq \sup_j \left|D_+a_j\right| \sum_j \eta(u_j).$$

Note that in particular, this holds for $\eta(u) = u^2$.

(c) Show that for fixed Δx, and $u \in l_2$, the function $F : l_2 \to l_2$ defined by $F_j(u) = aD_-u_j$ is Lipschitz continuous.
If we view $u(t)_j = u_j(t)$, then the difference scheme (1.37) reads $u' = -F(u)$. Since we know that the solution is bounded in l_2, we cannot have a blowup, and the solution exists for all time.

1.12 Consider the fully discrete scheme (1.48). Show that

$$\sum_j \eta(u_j^{n+1}) \leq \sum_j \eta(u_j^n) + \Delta t \sum_j \eta(u_j^n)D_+a_j^n.$$

Use this to show that

$$\Delta x \sum_j \eta(u_j^n) \leq e^{Ct_n} \left\|\eta(u_0)\right\|_{L^1(\mathbb{R})},$$

where C is a bound on a_x.

1.13 The linear *variational* wave equation reads

$$\begin{aligned} \alpha_{tt} + c(x)\left(c(x)\alpha_x\right)_x = 0, \qquad t > 0,\ x \in \mathbb{R}, \\ \alpha(x,0) = \alpha_0(x),\ \ \alpha_t(x,0) = \beta_0(x), \end{aligned} \tag{1.76}$$

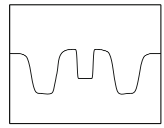
where c is a positive Lipschitz continuous function, and α_0 and β_0 are suitable initial data.

(a) Set $u = \alpha_t + c\alpha_x$ and $v = \alpha_t - c\alpha_x$. Find the equations satisfied by u and v.

(b) Find the solutions of these equations in terms of the characteristics.

(c) Formulate a difference scheme to approximate $u(x,t)$ and $v(x,t)$, and give suitable conditions on your scheme and the initial data (here you have a large choice) that guarantee the convergence of the scheme.

(d) Test your scheme with $c(x) = \sqrt{1 + \sin^2(x)}$, $\alpha_0(x) = \max\{0, 1 - |x|\}$, $\beta_0 = 0$, and periodic boundary conditions in $[-\pi, \pi]$.



1.14 Consider the transport equation

$$u_t + a(x,t)u_x = 0, \ \ t > 0, \ x \in \mathbb{R},$$
$$u(x,0) = u_0(x).$$

We know that the (unique) solution can be written in terms of the backward characteristics, $u(x,t) = u_0(\zeta(t;x))$, where ζ solves

$$\frac{d}{d\tau}\zeta(\tau;x) = -a(\zeta(\tau;x), t-\tau), \ \zeta(0;x) = x.$$

We want to use this numerically. Write a routine that given t, u_0, and a, calculates an approximation to $u(x,t)$ using a numerical method to find $\zeta(t;x)$. Test the routine for the initial function $u_0(x) = \sin(x)$, and for a given by (1.35) and (1.36), as well as for the example $a(x,t) = x^2 \sin(t)$.