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Helge Holden
Nils Henrik Risebro

Front Tracking for Hyperbolic Conservation Laws

Second Edition

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Helge Holden · Nils Henrik Risebro

Front Tracking for Hyperbolic Conservation Laws

2nd Edition

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In memory of Raphael, who started it all

Preface to the Second Edition

On general grounds I deprecate prefaces.[1]
— Winston Churchill

In this edition we have added the following new material: In Chapt. 1 we have added a section on linear equations, which allows us to present some of the material in the book in the simpler linear setting. In Chapt. 2 we have made some changes in the presentation of Kružkov's fundamental doubling of variables method. In Chapt. 3 on finite difference methods the focus has been changed to finite volume methods. A section on higher-order schemes has been added. The section on measure-valued solutions has been rewritten. The main existence theorem in Chapt. 4, Theorem 4.3, now resembles the one-dimensional case. The presentation of the solution of the Riemann problem for systems in Chapt. 5 has been supplemented by the complete solution of the Riemann problem for the 3×3 Euler equations of gas dynamics. The solution of the Cauchy problem for systems in Chapt. 6 has been rewritten and simplified. We have added a new chapter, Chapt. 8, on one-dimensional scalar conservation laws where the flux function depends explicitly on space in a discontinuous manner.

In addition, we have corrected mistakes that we have discovered. Furthermore, we have polished the presentation in several places, and new exercises have been added. We are grateful to those who have given us feedback, in particular G.M. Coclite, U. Skre Fjordholm, F. Gossler, K. Grunert, H. Hanche-Olsen, Espen R. Jakobsen, Qifan Li, S. May, A. Nordli, X. Raynaud, M. Rejske, O. Sete, K. Varholm, and F. Weber. The extensive help from Olivier Buffet in setting up the flip cartoons is much appreciated. We are very grateful to David Kramer for careful copyediting.

[1] in *The Story of the Malakand Field Force: An Episode of Frontier War* (*1898*).

Preface to the First Edition

Все счастливые семьи похожи друг на друга,
каждая несчастливая семья несчастлива по-своему.[2]
— Лев Толстой, Анна Каренина (1875)

While it is not strictly speaking true that all linear partial differential equations are the same, the theory that encompasses these equations can be considered well developed (and these are the happy families). Large classes of linear partial differential equations can be studied using linear functional analysis, which was developed in part as a tool to investigate important linear differential equations.

In contrast to the well-understood (and well-studied) classes of linear partial differential equations, each nonlinear equation presents its own particular difficulties. Nevertheless, over the last forty years some rather general classes of nonlinear partial differential equations have been studied and at least partly understood. These include the theory of viscosity solutions for Hamilton–Jacobi equations, the theory of Korteweg–de Vries equations, as well as the theory of hyperbolic conservation laws.

The purpose of this book is to present the modern theory of hyperbolic conservation laws in a largely self-contained manner. In contrast to the modern theory of linear partial differential equations, the mathematician interested in nonlinear hyperbolic conservation laws does not have to cover a large body of general theory to understand the results. Therefore, to follow the presentation in this book (with some minor exceptions), the reader does not have to be familiar with many complicated function spaces, nor does he or she have to know much theory of linear partial differential equations.

The methods used in this book are almost exclusively constructive, and largely based on the front-tracking construction. We feel that this gives the reader an intuitive feeling for the nonlinear phenomena that are described by conservation laws. In addition, front tracking is a viable numerical tool, and our book is also suitable for practical scientists interested in computations.

We focus on scalar conservation laws in several space dimensions and systems of hyperbolic conservation laws in one space dimension. In the scalar case we first discuss the one-dimensional case before we consider its multidimensional generalization. Multidimensional systems will not be treated. For multidimensional

[2] All happy families resemble one another, but every unhappy family is unhappy in its own way (Leo Tolstoy, *Anna Karenina*).

equations we combine front tracking with the method of dimensional splitting. We have included a chapter on standard difference methods that provides a brief introduction to the fundamentals of difference methods for conservation laws.

This book has grown out of courses we have given over some years: full-semester courses at the Norwegian University of Science and Technology, the University of Oslo, and Eidgenössische Technische Hochschule Zürich (ETH), as well as shorter courses at Universität Kaiserslautern, S.I.S.S.A., Trieste, and Helsinki University of Technology.

We have taught this material for graduate and advanced undergraduate students. A solid background in real analysis and integration theory is an advantage, but key results concerning compactness and functions of bounded variation are proved in Appendix A.

Our main audience consists of students and researchers interested in analytical properties as well as numerical techniques for hyperbolic conservation laws.

We have benefited from the kind advice and careful proofreading of various versions of this manuscript by several friends and colleagues, among them Petter I. Gustafson, Runar Holdahl, Helge Kristian Jenssen, Kenneth H. Karlsen, Odd Kolbjørnsen, Kjetil Magnus Larsen, Knut-Andreas Lie, Achim Schroll. Special thanks are due to Harald Hanche-Olsen, who has helped us on several occasions with both mathematical and TEX-nical issues.

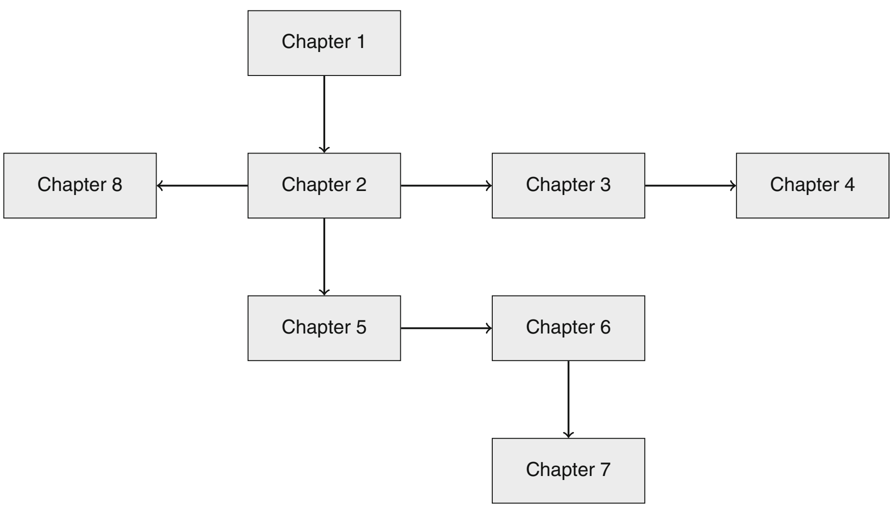
Our research has been supported in part by the BeMatA program of the Research Council of Norway.

A list of corrections can be found at

http://www.math.ntnu.no/~holden/FrontBook/

Whenever you find an error, please send us an email about it.

The logical interdependence of the material in this book is depicted in the diagram below. The main line, Chapts. 1, 2, 5–7, has most of the emphasis on the theory for systems of conservation laws in one space dimension. Another possible track is Chapts. 1–4, with emphasis on numerical methods and theory for scalar equations in one and several space dimensions. Chapt. 8, on the theory for one-dimensional scalar conservation laws with spatially depending flux function, requires only Chapts. 1 and 2.



Dependencies among the chapters

Flip Cartoons[3]

Well, the silent pictures were the purest form of cinema.
— Alfred Hitchcock

We have included four flip cartoons in the book: At the bottom of the odd-numbered pages (starting from the back) you see the solution of the equation

$$u_t + \frac{1}{3}(u^3)_x = 0, \quad u|_{t=0} = \cos(\pi x),$$

using a second-order finite difference method, more specifically, the Lax–Wendroff method with minmod limiter; see (3.43). On the bottom of the even-numbered pages (starting from the front) you see the fronts in the (x, t)-plane for the same problem; see (2.44).

At at top of the odd-numbered pages (starting from the back) you see the solution of the Euler equations (5.150) with $\gamma = 1.4$. The initial data are

$$p(x,0) = \begin{cases} 3 & \text{for } |x| \leq 0.5, \\ 1 & \text{otherwise,} \end{cases} \qquad \rho(x,0) = \begin{cases} 2.5 & \text{for } |x| \leq 0.25, \\ 1 & \text{otherwise,} \end{cases} \qquad v(x,0) = 0,$$

and the data are extended periodically outside the interval $(-1, 1)$. The pressure p is displayed for $t \in [0, 1]$, and the solution is obtained using the Godunov method with a Roe approximate Riemann solver. We use $\Delta x = 1/250$. On the bottom of the even-numbered pages (starting from the front) you see the fronts in the (x, t)-plane for the same problem; see (6.9).

We do not want now and we shall never want the human voice with our films.
— D.W. Griffiths (1875–1948), movie pioneer

As for readers of the eBook, we refer to Springer's web site where one can watch the flip cartoons.

Maybe eBooks are going to take over, one day, but not until those whizzkids in Silicon Valley invent a way to bend the corners, fold the spine, yellow the pages, add a coffee ring or two and allow the plastic tablet to fall open at a favorite page.
— R.T. Davies, in foreword to D. Adams's The Hitchhiker's Guide to the Galaxy

[3] Assistance from Olivier Buffet is much appreciated.

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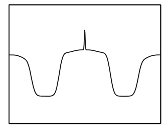


Chapter 1

Introduction

I have no objection to the use of the term "Burgers' equation" for the nonlinear heat equation (provided it is not written "Burger's equation").
— Letter from Burgers to Batchelor (1968)

Hyperbolic conservation laws are partial differential equations of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0.$$

If we write $f = (f_1, \dots, f_m)$, $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, and introduce initial data u_0 at $t = 0$, the Cauchy problem for hyperbolic conservation laws reads

$$\frac{\partial u(x,t)}{\partial t} + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f_j\,(u(x,t)) = 0, \quad u|_{t=0} = u_0. \tag{1.1}$$

In applications, t normally denotes the time variable, while x describes the spatial variation in m space dimensions. The unknown function u (as well as each f_j) can be a vector, in which case we say that we have a system of equations, or u and each f_j can be a scalar. This book covers the theory of scalar conservation laws in several space dimensions as well as the theory of systems of hyperbolic conservation laws in one space dimension. In the present chapter we study the one-dimensional scalar case to highlight some of the fundamental issues in the theory of conservation laws.

We use subscripts to denote partial derivatives, i.e., $u_t(x,t) = \partial u(x,t)/\partial t$. Hence we may write (1.1) when $m = 1$ as

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0. \tag{1.2}$$

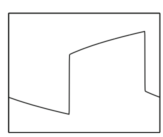
If we formally integrate equation (1.2) between two points x_1 and x_2, we obtain

$$\int_{x_1}^{x_2} u_t\,dx = -\int_{x_1}^{x_2} f(u)_x\,dx = f\,(u\,(x_1,t)) - f\,(u\,(x_2,t))\,.$$

Assuming that u is sufficiently regular to allow us to take the derivative outside the integral, we get

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f\,(u\,(x_1,t)) - f\,(u\,(x_2,t))\,. \tag{1.3}$$

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_1



This equation expresses conservation of the quantity measured by u in the sense that the rate of change in the amount of u between x_1 and x_2 is given by the difference in $f(u)$ evaluated at these points.[1] Therefore, it is natural to interpret $f(u)$ as the *flux density* of u. Often, $f(u)$ is referred to as the *flux function*.

Consider a fluid with density $\rho = \rho(x,t)$ and velocity v. Assume that there are no sources or sinks, so that amount of fluid is conserved. For a given and fixed bounded domain $D \subset \mathbb{R}^m$, conservation of fluid implies

$$\frac{d}{dt}\int_D \rho(x,t)\,dx = -\int_{\partial D} (\rho v)\cdot n\,dS_x, \tag{1.4}$$

where n is the outward unit normal at the boundary ∂D of D. If we interchange the time derivative and the integral on the left-hand side of the equation, and apply the divergence theorem on the right-hand side, we obtain

$$\int_D \rho(x,t)_t\,dx = -\int_D \operatorname{div}(\rho v)\,dx \tag{1.5}$$

which we rewrite as

$$\int_D \big(\rho_t + \operatorname{div}(\rho v)\big)\,dx = 0. \tag{1.6}$$

Since the domain D was arbitrary, we obtain the hyperbolic conservation law

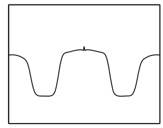
$$\rho_t + \operatorname{div}(\rho v) = 0. \tag{1.7}$$

The above derivation is very fundamental, and only two assumptions are made. First of all, we make the physical assumption of conservation, and secondly, we assume sufficient smoothness of the functions to perform the necessary mathematical manipulations. The latter aspect will a recurring theme throughout the book.

As a simple example of a conservation law, consider a one-dimensional medium consisting of noninteracting particles, or material points, identified by their coordinates y along a line. Let $\phi(y,t)$ denote the position of material point y at time t. The velocity and the acceleration of y at time t are given by $\phi_t(y,t)$ and $\phi_{tt}(y,t)$, respectively. Assume that for each t, $\phi(\,\cdot\,,t)$ is strictly increasing, so that two distinct material points cannot occupy the same position at the same time. Then the function $\phi(\,\cdot\,,t)$ has an inverse $\psi(\,\cdot\,,t)$, so that $y = \psi(\phi(y,t),t)$ for all t. Hence $x = \phi(y,t)$ is equivalent to $y = \psi(x,t)$. Now let u denote the velocity of the material point occupying position x at time t, i.e., $u(x,t) = \phi_t(\psi(x,t),t)$, or equivalently, $u(\phi(y,t),t) = \phi_t(y,t)$. Then the acceleration of material point y at time t is

$$\begin{aligned}\phi_{tt}(y,t) &= u_t(\phi(y,t),t) + u_x(\phi(y,t),t)\phi_t(y,t)\\ &= u_t(x,t) + u_x(x,t)u(x,t).\end{aligned}$$

[1] In physics one normally describes conservation of a quantity in integral form, that is, one starts with (1.3). The differential equation (1.2) then follows under additional regularity conditions on u.



If the material particles are noninteracting, so that they exert no force on each other, and there is no external force acting on them, then Newton's second law requires the acceleration to be zero, giving

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{1.8}$$

The last equation, (1.8), is a conservation law; it expresses that u is conserved with a flux density given by $u^2/2$. This equation is often referred to as the *Burgers equation without viscosity,*[2] and is in some sense the simplest nonlinear conservation law.

Burgers's equation, and indeed any conservation law, is an example of a *quasilinear* equation, meaning that the highest derivatives occur linearly. A general inhomogeneous quasilinear equation for functions of two variables x and t can be written

$$a(x,t,u)u_t + b(x,t,u)u_x = c(x,t,u). \tag{1.9}$$

If the coefficients a and b are independent of u, i.e., $a = a(x,t)$, $b = b(x,t)$, we say that the equation is *semilinear*, while the equation is *linear* if, in addition, the same applies to c, i.e., $c = c(x,t)$.

We may consider the solution as the surface $S = \{(t,x,u(x,t)) \in \mathbb{R}^3 \mid (t,x) \in \mathbb{R}^2\}$ in $\mathbb{R}^3$. Let Γ be a given curve in $\mathbb{R}^3$ (which one may think of as the initial data if t is constant) parameterized by $(t(y), x(y), z(y))$ for y in some interval. We want to construct the surface $S \subset \mathbb{R}^3$ parameterized by $(t,x,u(x,t))$ such that $u = u(x,t)$ satisfies (1.9) and $\Gamma \subset S$. It turns out to be advantageous to consider the surface S parameterized by new variables (s,y), thus $t = t(s,y)$, $x = x(s,y)$, $z = z(s,y)$, in such a way that $u(x,t) = z(s,y)$. We solve the system of ordinary differential equations

$$\frac{\partial t}{\partial s} = a, \quad \frac{\partial x}{\partial s} = b, \quad \frac{\partial z}{\partial s} = c, \tag{1.10}$$

with

$$t(s_0, y) = t(y), \quad x(s_0, y) = x(y), \quad z(s_0, y) = z(y). \tag{1.11}$$

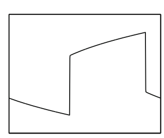
In this way we obtain the parameterized surface $S = \{(t(s,y), x(s,y), z(s,y) \mid (s,y) \in \mathbb{R}^2\}$. Assume that we can invert the relations $x = x(s,y)$, $t = t(s,y)$ and write $s = s(x,t)$, $y = y(x,t)$. Then

$$u(x,t) = z(s(x,t), y(x,t)) \tag{1.12}$$

satisfies both (1.9) and the condition $\Gamma \subset S$. Namely, we have

$$c = \frac{\partial z}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s} = u_x b + u_t a. \tag{1.13}$$

[2] Henceforth we will adhere to common practice and call it the inviscid Burgers equation.



However, there are many pitfalls in the above construction: the solution (1.10) may only be local, and we may not be able to invert the solution of the differential equation to express (s, y) as functions of (x, t). These problems are intrinsic to equations of this type and will be discussed at length.

Equation (1.10) is called the *characteristic equation*, and its solutions are called *characteristics*. This can sometimes be used to find explicit solutions of conservation laws. In the homogeneous case, that is, when $c = 0$, the solution u is constant along characteristics, namely,

$$\frac{d}{ds}u(x(s, y), t(s, y)) = u_x x_s + u_t t_s = u_x b + u_t a = 0. \tag{1.14}$$

◇ Example 1.1
Consider the (quasi)linear equation

$$u_t - xu_x = -2u, \quad u(x, 0) = x,$$

with associated characteristic equations

$$\frac{\partial t}{\partial s} = 1, \quad \frac{\partial x}{\partial s} = -x, \quad \frac{\partial z}{\partial s} = -2z.$$

The general solution of the characteristic equations reads

$$t = t_0 + s, \quad x = x_0 e^{-s}, \quad z = z_0 e^{-2s}.$$

Parameterizing the initial data for $s = 0$ by $t = 0$, $x = y$, and $z = y$, we obtain

$$t = s, \quad x = ye^{-s}, \quad z = ye^{-2s},$$

which can be inverted to yield

$$u = u(x, t) = z(s, y) = xe^{-t}. \qquad \diamond$$

◇ Example 1.2
Consider the (quasi)linear equation

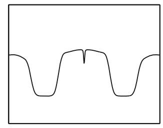
$$xu_t - t^2 u_x = 0. \tag{1.15}$$

Its associated characteristic equation is

$$\frac{\partial t}{\partial s} = x, \quad \frac{\partial x}{\partial s} = -t^2.$$

This has solutions given implicitly by $x^2/2 + t^3/3$ equals a constant, since after all, $\partial(x^2/2 + t^3/3)/\partial s = 0$, so the solution of (1.15) is any function φ of $x^2/2 + t^3/3$, i.e., $u(x, t) = \varphi(x^2/2 + t^3/3)$. For example, if we wish to solve the initial value problem (1.15) with $u(x, 0) = \sin|x|$, then $u(x, 0) = \varphi(x^2/2) = \sin|x|$. Consequently, $\varphi(\zeta) = \sin\sqrt{2\zeta}$ with $\zeta \geq 0$, and the solution u is given by

$$u(x, t) = \sin\sqrt{x^2 + 2t^3/3}, \qquad t \geq 0. \qquad \diamond$$



◇ Example 1.3 (Burgers's equation)
If we apply this technique to Burgers's equation(1.8) with initial data $u(x,0) = u_0(x)$, we get that

$$\frac{\partial t}{\partial s} = 1, \qquad \frac{\partial x}{\partial s} = z, \quad \text{and} \quad \frac{\partial z}{\partial s} = 0$$

with initial conditions $t(0,y) = 0$, $x(0,y) = y$, and $z(0,y) = u_0(y)$. We cannot solve these equations without knowing more about u_0, but since u (or z) is constant along characteristics, cf. (1.14), we see that the characteristics are straight lines. In other words, the value of z is transported along characteristics, so that

$$t(s,y) = s, \quad x(s,y) = y + sz = y + su_0(\eta), \quad z(s,y) = u_0(y).$$

We may write this as

$$x = y + u_0(y)t. \tag{1.16}$$

If we solve this equation in terms of $y = y(x,t)$, we can use y to obtain $u(x,t) = z(s,y) = u_0(y(x,t))$, yielding the implicit relation

$$u(x,t) = u_0(x - u(x,t)t). \tag{1.17}$$

Given a point (x,t), one can in principle determine the solution $u = u(x,t)$ from equation (1.17). By differentiating equation (1.16) we find that

$$\frac{\partial x}{\partial y} = 1 + tu_0'(y). \tag{1.18}$$

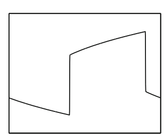
Thus a solution certainly exists for all $t > 0$ if $u_0' > 0$, since x is a strictly increasing function of η in that case. On the other hand, if $u_0'(\tilde{x}) < 0$ for some $\tilde{x}$, then a solution cannot be found for $t > t^* = -1/u_0'(\tilde{x})$. For example, if $u_0(x) = -\arctan(x)$, there is no smooth solution for $t > 1$.

What actually happens when a smooth solution cannot be defined? From (1.18) we see that for $t > t^*$, there are several y that satisfy (1.16) for each x, since x is no longer a strictly increasing function of y. In some sense, we can say that the solution u is multivalued at such points. To illustrate this, consider the surface in (t,x,u)-space parameterized by s and y,

$$(s, y + su_0(y), u_0(\eta)).$$

Let us assume that the initial data are given by $u_0(x) = -\arctan(x)$ and $t_0 = 0$. For each fixed t, the curve traced out by the surface is the graph of a (multivalued) function of x. In Fig. 1.1 we see how the multivaluedness starts at $t = 1$ when the surface "folds over," and that for $t > 1$ there are some x that have three associated u values. To continue the solution beyond $t = 1$ we have to choose among these three u values. In any case, it is impossible to continue the solution and at the same time keep it continuous. ◇

Now we have seen that no matter how smooth the initial function is, we cannot expect to be able to define classical solutions of nonlinear conservation laws for



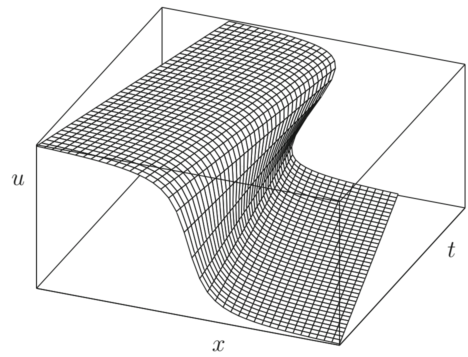


Fig. 1.1 A multivalued solution

all time. In this case we have to extend the concept of solution in order to allow discontinuities.

The standard way of extending the admissible set of solutions to partial differential equations is to look for *weak solutions* rather than so-called *classical solutions*, by introducing distribution theory. Classical solutions are sufficiently differentiable functions such that the differential equation is satisfied for all values of the independent arguments. However, there is no unique definition of weak solutions. In the context of hyperbolic conservation laws we do not need the full machinery of distribution theory, and our solutions will be functions that may be nondifferentiable.

In this book we use the following standard notation: $C^i(U)$ is the set of i times continuously differentiable functions on a set $U \subseteq \mathbb{R}^n$, and $C_0^i(U)$ denotes the set of such functions that have compact support in U. Then $C^\infty(U) = \bigcap_{i=0}^\infty C^i(U)$, and similarly for C_0^∞. Where there is no ambiguity, we sometimes omit the set U and write only C^0, etc.

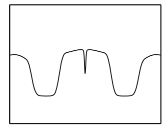
If we have a classical solution to (1.2), we can multiply the equation by a function $\varphi = \varphi(x,t) \in C_0^\infty(\mathbb{R} \times [0,\infty))$, called a *test function*, and integrate by parts to get

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} \left(u_t \varphi + f(u)_x \varphi\right) dx\, dt \\ &= -\int_0^\infty \int_{\mathbb{R}} \left(u \varphi_t + f(u) \varphi_x\right) dx\, dt - \int_{\mathbb{R}} u(x,0)\varphi(x,0)\, dx. \end{aligned}$$

Observe that the boundary terms at $t = \infty$ and at $x = \pm\infty$ vanish, since φ has compact support, and that the final expression incorporates the initial data. Now we *define* a weak solution of (1.2) to be a measurable function $u(x,t)$ such that

$$\int_0^\infty \int_{\mathbb{R}} \left(u \varphi_t + f(u) \varphi_x\right) dx dt + \int_{\mathbb{R}} u_0 \varphi(x,0)\, dx = 0 \tag{1.19}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$. We see that the weak solution u is no longer required to be differentiable, and that a classical solution is also a weak solution. We will spend considerable time in understanding the constraints that the equation (1.19) puts on u.



We employ the usual notation that for $p \in [0, \infty)$, $L^p(U)$ denotes the set of all measurable functions $F : U \to \mathbb{R}$ such that the integral

$$\int_U |F|^p \, dx$$

is finite. The set $L^p(U)$ is equipped with the norm

$$\|F\|_p = \|F\|_{L^p} = \|F\|_{L^p(U)} = \left(\int_U |F|^p dx \right)^{1/p}.$$

If $p = \infty$, $L^\infty(U)$ denotes the set of all measurable functions F such that

$$\operatorname{ess\,sup}_U |F|$$

is finite. The space $L^\infty(U)$ has the norm $\|F\|_\infty = \operatorname{ess\,sup}_U |F|$. As is well-known, the spaces $L^p(U)$ are Banach spaces for $p \in [1, \infty]$, and $L^2(U)$ is a Hilbert space. In addition, we will frequently use the spaces

$$L^p_{\mathrm{loc}}(U) = \{ f : U \to \mathbb{R} \mid f \in L^p(K) \text{ for every compact set } K \subseteq U \}.$$

So what kind of discontinuities are compatible with (1.19)? If we assume that u is constant outside some finite interval, the remarks below (1.2) imply that

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) \, dx = 0.$$

Hence, the total amount of u is independent of time, or equivalently, the area below the graph of $u(\,\cdot\,, t)$ is constant.

◇ Example 1.4 (Burgers's equation (cont'd.))
We now wish to determine a discontinuous function such that the graph of the function lies on the surface given earlier with $u(x,0) = -\arctan x$. Furthermore, the area under the graph of the function should be equal to the area between the x-axis and the surface. In Fig. 1.2 we see a section of the surface making up the solution for $t = 3$. The curve is parameterized by x_0, and explicitly given by $u = -\arctan(x_0)$, $x = x_0 - 3\arctan(x_0)$.

The function u is shown by a thick line, and the surface is shown by a dotted line. A function $u(x)$ that has the correct integral, $\int u \, dx = \int u_0 \, dx$, is easily

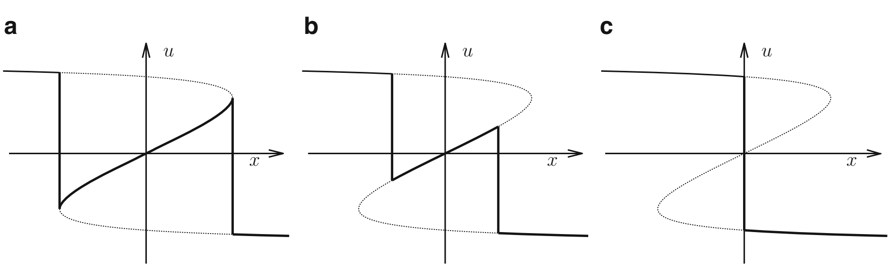
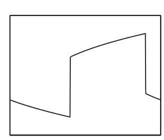


Fig. 1.2 Different solutions with u conserved



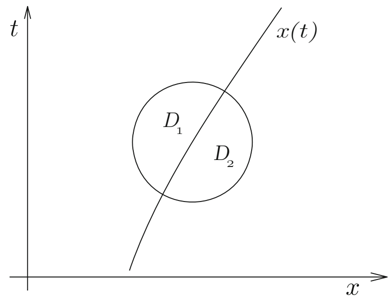


Fig. 1.3 An isolated discontinuity

found by making any cut from the upper fold to the middle fold at some negative x_c with $x_c \geq -\sqrt{2}$, and then making a cut from the middle part to the lower part at $-x_c$. We see that in all cases, the area below the thick line is the same as the area bounded by the curve $(x(x_0), u(x_0))$. Consequently, conservation of u is not sufficient to determine a unique weak solution. $\diamond$

Let us examine what kind of discontinuities are compatible with (1.19) in the general case. Assume that we have an isolated discontinuity that moves along a smooth curve $\Gamma\colon x = x(t)$. The discontinuity being isolated means that the function $u(x,t)$ is differentiable in a sufficiently small neighborhood of $x(t)$ and satisfies equation (1.2) classically on each side of $x(t)$. We also assume that u is uniformly bounded in a neighborhood of the discontinuity.

Now we choose a neighborhood D around the point $(x(t),t)$ and a test function $\phi(x,t)$ whose support lies entirely inside the neighborhood. The situation is as depicted in Fig. 1.3. The neighborhood consists of two parts D_1 and D_2, and is chosen so small that u is differentiable everywhere inside D except on $x(t)$. Let D_i^ε denote the set of points

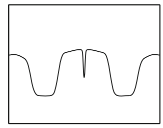
$$D_i^\varepsilon = \left\{(x,t) \in D_i \mid \operatorname{dist}\big((x,t),(x(t),t)\big) > \varepsilon\right\}.$$

The function u is bounded, and hence

$$0 = \int\limits_D (u\phi_t + f(u)\phi_x)\, dx\, dt = \lim_{\varepsilon\to 0} \int\limits_{D_1^\varepsilon \cup D_2^\varepsilon} (u\phi_t + f(u)\phi_x)\, dx\, dt. \tag{1.20}$$

Since u is a classical solution inside each D_i^ε, we can use Green's theorem and obtain

$$\begin{aligned}
\int\limits_{D_i^\varepsilon} (u\phi_t + f(u)\phi_x)\, dx\, dt &= \int\limits_{D_i^\varepsilon} (u\phi_t + f(u)\phi_x + (u_t + f(u)_x)\phi)\, dx\, dt \\
&= \int\limits_{D_i^\varepsilon} ((u\phi)_t + (f(u)\phi)_x)\, dx\, dt \\
&= \int\limits_{D_i^\varepsilon} (\partial_x, \partial_t)\cdot \big(f(u)\phi, u\phi\big)\, dx\, dt \\
&= \int\limits_{\partial D_i^\varepsilon} \phi\,(f(u),u)\cdot n_i\, ds.
\end{aligned} \tag{1.21}$$



Here n_i is the outward unit normal at ∂D_i^ε. But ϕ is zero everywhere on ∂D_i^ε except in the vicinity of $x(t)$. Let Γ_i^ε denote this part of ∂D_i^ε. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_i^\varepsilon} \phi\,(f(u),u)\cdot n_i\,ds &= \int_I \phi\left(-u_l x'(t)+f_l\right)\,dt\\ &= -\int_I \phi\left(-u_r x'(t)+f_r\right)\,dt\end{aligned}$$

for some suitable time interval I. Here u_l denotes the limit of $u(x,t)$ as $x\to x(t)-$, and u_r the limit as x approaches $x(t)$ from the right, i.e., $u_r=\lim_{x\to x(t)+}u(x,t)$. Similarly, $f_l=f(u_l)$ and $f_r=f(u_r)$. The reason for the difference in sign is that according to Green's theorem, we must integrate along the boundary counterclockwise. Therefore, the positive sign holds for $i=1$, and the negative for $i=2$. Using (1.20) we obtain (slightly abusing notation by writing $u(t)=u(x(t),t)$, etc.)

$$\int_I \phi\left[-\left(u_l(t)-u_r(t)\right)x'(t)+\left(f_l(t)-f_r(t)\right)\right]\,dt=0.$$

Since this is to hold for all test functions ϕ, we must have

$$s\,(u_r-u_l)=f_r-f_l, \tag{1.22}$$

where $s=x'(t)$. This equality is called the *Rankine–Hugoniot condition* or the *jump condition*, and it expresses conservation of u across jump discontinuities. It is common in the theory of conservation laws to introduce a notation for the jump in a quantity. Write

$$[\![a]\!]=a_r-a_l \tag{1.23}$$

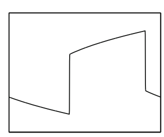
for the jump in any quantity a. With this notation the Rankine–Hugoniot relation takes the form

$$s\,[\![u]\!]=[\![f]\!]. \tag{1.24}$$

◇ Example 1.5 (Burgers's equation (cont'd.))
For Burgers's equation we see that the shock speed must satisfy

$$s=\frac{[\![u^2/2]\!]}{[\![u]\!]}=\frac{\left(u_r^2-u_l^2\right)}{2\,(u_r-u_l)}=\frac{1}{2}\,(u_l+u_r)\,.$$

Consequently, the left shock in parts **a** and **b** in Fig. 1.2 above will have greater speed than the right shock, and will, eventually, collide. Therefore, solutions of type **a** or **b** cannot be isolated discontinuities moving along two trajectories starting at $t=1$. Type **c** yields a stationary shock. ◇



◇ Example 1.6 (Traffic flow)

I am ill at these numbers.
— W. Shakespeare, Hamlet (1603)

Rather than continue to develop the theory, we shall now consider an example of a conservation law in some detail. We will try to motivate how a conservation law can model the flow of cars on a crowded highway.

Consider a road consisting of a single lane, with traffic in one direction only. The road is parameterized by a single coordinate x, and we assume that the traffic moves in the direction of increasing x.

Suppose we position ourselves at a point x on the road and observe the number of cars $N = N(x, t, h)$ in the interval $[x, x + h]$. If some car is located at the boundary of this interval, we account for that by allowing N to take any real value. If the traffic is dense, and if h is large compared with the average length of a car, but at the same time small compared with the length of our road, we can introduce the density given by

$$\rho(x, t) = \lim_{h \to 0} \frac{N(x, t, h)}{h}.$$

Then $N(x, t, h) = \int_x^{x+h} \rho(y, t)\, dy$.

Let now the position of some vehicle be given by $r(t)$, and its velocity by $v(r(t), t)$. Considering the interval $[a, b]$, we wish to determine how the number of cars changes in this interval. Since we have assumed that there are no entries or exits on our road, this number can change only as cars are entering the interval from the left endpoint, or leaving the interval at the right endpoint. The rate of cars passing a point x at some time t is given by

$$v(x, t)\rho(x, t).$$

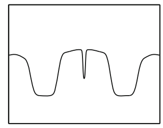
Consequently,

$$-\left(v(b, t)\rho(b, t) - v(a, t)\rho(a, t)\right) = \frac{d}{dt} \int_a^b \rho(y, t)\, dy.$$

Comparing this with (1.3) and (1.2), we see that the density satisfies the conservation law

$$\rho_t + (\rho v)_x = 0. \tag{1.25}$$

In the simplest case we assume that the velocity v is given as a function of the density ρ only. This may be a good approximation if the road is uniform and does not contain any sharp bends or similar obstacles that force the cars to slow down. It is also reasonable to assume that there is some maximal speed $v_{\max}$ that any car can attain. When traffic is light, a car will drive at this maximum speed, and as the road gets more crowded, the cars will have to slow down, until they come to a complete standstill as the traffic stands bumper to bumper. Hence, we assume that the velocity



v is a monotone decreasing function of ρ such that $v(0) = v_{\max}$ and $v(\rho_{\max}) = 0$. The simplest such function is a linear function, resulting in a flux function given by

$$f(\rho) = v\rho = \rho v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{1.26}$$

For convenience we normalize by introducing $u = \rho/\rho_{\max}$ and $\tilde{x} = v_{\max}x$. The resulting normalized conservation law reads

$$u_t + (u(1-u))_x = 0. \tag{1.27}$$

Setting $\tilde{u} = \frac{1}{2} - u$, we recover Burgers's equation, but this time with a new interpretation of the solution.

Let us solve an initial value problem explicitly by the method of characteristics described earlier. We wish to solve (1.27), with initial function $u_0(x)$ given by

$$u_0(x) = u(x,0) = \begin{cases} \frac{3}{4} & \text{for } x \le -a, \\ \frac{1}{2} - x/(4a) & \text{for } -a < x < a, \\ \frac{1}{4} & \text{for } a \le x. \end{cases}$$

The characteristics satisfy $t'(\xi) = 1$ and $x'(\xi) = 1 - 2u(x(\xi), t(\xi))$. The solution of these equations is given by $x = x(t)$, where

$$x(t) = \begin{cases} x_0 - t/2 & \text{for } x_0 < -a, \\ x_0 + x_0 t/(2a) & \text{for } -a \le x_0 \le a, \\ x_0 + t/2 & \text{for } a < x_0. \end{cases}$$

Inserting this into the solution $u(x,t) = u_0(x_0(x,t))$, we find that

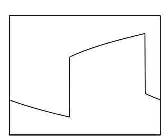
$$u(x,t) = \begin{cases} \frac{3}{4} & \text{for } x \le -a - t/2, \\ \frac{1}{2} - x/(4a+2t) & \text{for } -a - t/2 < x < a + t/2, \\ \frac{1}{4} & \text{for } a + t/2 \le x. \end{cases}$$

This solution models a situation in which the traffic density initially is small for positive x, and high for negative x. If we let a tend to zero, the solution reads

$$u(x,t) = \begin{cases} \frac{3}{4} & \text{for } x \le -t/2, \\ \frac{1}{2} - x/(2t) & \text{for } -t/2 < x < t/2, \\ \frac{1}{4} & \text{for } t/2 \le x. \end{cases}$$

As the reader may check directly, this is also a classical solution everywhere except at $x = \pm t/2$. It takes discontinuous initial values:

$$u(x,0) = \begin{cases} \frac{3}{4} & \text{for } x < 0, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{1.28}$$



This initial function may model the situation when a traffic light turns green at $t = 0$. The density of cars facing the traffic light is high, while on the other side of the light there is a small constant density.

Initial value problems of the kind (1.28), where the initial function consists of two constant values, are called *Riemann problems*. We will discuss Riemann problems at great length in this book.

If we simply insert $u_l = \frac{3}{4}$ and $u_r = \frac{1}{4}$ in the Rankine–Hugoniot condition (1.22), we find another weak solution to this initial value problem. These left and right values give $s = 0$, so the solution found here is simply $u_2(x,t) = u_0(x)$. A priori, this solution is no better or worse than the solution computed earlier. But when we examine the situation the equation is supposed to model, the second solution u_2 is unsatisfactory, since it describes a situation in which the traffic light is green, but the density of cars facing the traffic light does not decrease!

In the first solution the density decreased. Examining the model a little more closely, we find, perhaps from experience of traffic jams, that the allowable discontinuities are those in which the density is increasing. This corresponds to the situation in which there is a traffic jam ahead, and we suddenly have to slow down when we approach it.

When we emerge from a traffic jam, we experience a gradual decrease in the density of cars around us, not a sudden jump from a bumper to bumper situation to a relatively empty road.

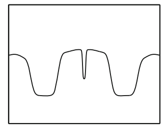
We have now formulated a condition, in addition to the Rankine–Hugoniot condition, that allows us to reduce the number of weak solutions to our conservation law. This condition says that *every weak solution u has to increase across discontinuities.* Such conditions are often called *entropy conditions*. This terminology comes from gas dynamics, where similar conditions state that the physical entropy has to increase across any discontinuity.

Let us consider the opposite initial value problem, namely,

$$u_0(x) = \begin{cases} \frac{1}{4} & \text{for } x < 0, \\ \frac{3}{4} & \text{for } x \geq 0. \end{cases}$$

Now the characteristics starting at negative x_0 are given by $x(t) = x_0 + t/2$, and the characteristics starting on the positive half-line are given by $x(t) = x_0 - t/2$. We see that these characteristics immediately will run into each other, and therefore the solution is multivalued for every positive time t. Thus there is no hope of finding a continuous solution to this initial value problem for any time interval $(0, \delta)$, no matter how small δ is. When inserting the initial values $u_l = \frac{1}{4}$ and $u_r = \frac{3}{4}$ into the Rankine–Hugoniot condition, we see that the initial function is already a weak solution. This time, the solution increases across the discontinuity, and therefore satisfies our entropy condition. Thus, an admissible solution is given by $u(x,t) = u_0(x)$.

Now we shall attempt to solve a more complicated problem in some detail. Assume that we have a road with a uniform density of cars initially. At $t = 0$ a traffic light placed at $x = 0$ changes from green to red. It remains red for some time interval Δt, then turns green again and stays green thereafter. We assume that the initial uniform density is given by $u = \frac{1}{2}$, and we wish to determine the traffic density for $t > 0$.



When the traffic light initially turns red, the situation for the cars to the left of the traffic light is the same as when the cars stand bumper to bumper to the right of the traffic light. So in order to determine the situation for t in the interval $[0, \Delta t)$, we must solve the Riemann problem with the initial function

$$u_0^l(x) = \begin{cases} \frac{1}{2} & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{1.29}$$

For the cars to the right of the traffic light, the situation is similar to the situation in which the traffic abruptly stopped at $t = 0$ behind the car located at $x = 0$. Therefore, to determine the density for $x > 0$ we have to solve the Riemann problem given by

$$u_0^r(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{2} & \text{for } x \geq 0. \end{cases} \tag{1.30}$$

Returning to (1.29), here u is increasing over the initial discontinuity, so we can try to insert this into the Rankine–Hugoniot condition. This gives

$$s = \frac{f_r - f_l}{u_r - u_l} = \frac{\frac{1}{4} - 0}{\frac{1}{2} - 1} = -\frac{1}{2}.$$

Therefore, an admissible solution for $x < 0$ and t in the interval $[0, \Delta t)$ is given by

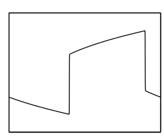
$$u^l(x,t) = \begin{cases} \frac{1}{2} & \text{for } x < -t/2, \\ 1 & \text{for } x \geq -t/2. \end{cases}$$

This is indeed close to what we experience when we encounter a traffic light. We see the discontinuity approaching as the brake lights come on in front of us, and the discontinuity has passed us when we have come to a halt. Note that although each car moves only in the positive direction, the discontinuity moves to the left.

In general, we have to deal with three different speeds when we study conservation laws: the particle speed, in our case the speed of each car; the characteristic speed; and the speed of a discontinuity. These three speeds are not equal if the conservation law is nonlinear. In our case, the speed of each car is nonnegative, but both the characteristic speed and the speed of a discontinuity may take both positive and negative values. Note that the speed of an admissible discontinuity is less than the characteristic speed to the left of the discontinuity, and larger than the characteristic speed to the right. This is a general feature of admissible discontinuities.

It remains to determine the density for positive x. The initial function given by (1.30) also has a positive jump discontinuity, so we obtain an admissible solution if we insert it into the Rankine–Hugoniot condition. Then we obtain $s = \frac{1}{2}$, so the solution for positive x is

$$u^r(x,t) = \begin{cases} 0 & \text{for } x < t/2, \\ \frac{1}{2} & \text{for } x \geq t/2. \end{cases}$$



Piecing together u^l and u^r, we find that the density u in the time interval $[0, \Delta t)$ reads

$$u(x,t) = \begin{cases} \frac{1}{2} & \text{for } x \leq -t/2, \\ 1 & \text{for } -t/2 < x \leq 0, \\ 0 & \text{for } 0 < x \leq t/2, \\ \frac{1}{2} & \text{for } t/2 < x, \end{cases} \qquad t \in [0, \Delta t).$$

What happens for $t > \Delta t$? To find out, we have to solve the Riemann problem

$$u(x, \Delta t) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

Now the initial discontinuity is not acceptable according to our entropy condition, so we have to look for some other solution. We can try to mimic the example above in which we started with a nonincreasing initial function that was linear on some small interval $(-a, a)$. Therefore, let $v(x, t)$ be the solution of the initial value problem

$$v_t + (v(1-v))_x = 0,$$
$$v(x,0) = v_0(x) = \begin{cases} 1 & \text{for } x < -a, \\ \frac{1}{2} - x/(2a) & \text{for } -a \leq x < a, \\ 0 & \text{for } a \leq x. \end{cases}$$

As in the above example, we find that the characteristics are not overlapping, and they fill out the positive half-plane exactly. The solution is given by $v(x,t) = v_0\left(x_0(x,t)\right)$:

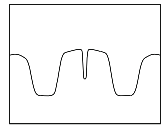
$$v(x,t) = \begin{cases} 1 & \text{for } x < -a - t, \\ \frac{1}{2} - x/(2a + 2t) & \text{for } -a - t \leq x < a + t, \\ 0 & \text{for } a + t \leq x. \end{cases}$$

Letting $a \to 0$, we obtain the solution to the Riemann problem with a left value 1 and a right value 0. For simplicity we also denote this function by $v(x, t)$.

This type of solution can be depicted as a "fan" of characteristics emanating from the origin, and it is called a *centered rarefaction wave*, or sometimes just a *rarefaction wave*. The origin of this terminology lies in gas dynamics.

We see that the rarefaction wave, which is centered at $(0, \Delta t)$, does not immediately influence the solution away from the origin. The leftmost part of the wave moves with a speed -1, and the front of the wave moves with speed 1. So for some time after Δt, the density is obtained by piecing together three solutions, $u^l(x,t)$, $v(x, t - \Delta t)$, and $u^r(x,t)$.

The rarefaction wave will of course catch up with the discontinuities in the solutions u^l and u^r. Since the speeds of the discontinuities are $\mp\frac{1}{2}$, and the speeds of the rear and the front of the rarefaction wave are ∓ 1, and the rarefaction wave starts at $(0, \Delta t)$, we conclude that this will happen at $(\mp\Delta t, 2\Delta t)$.



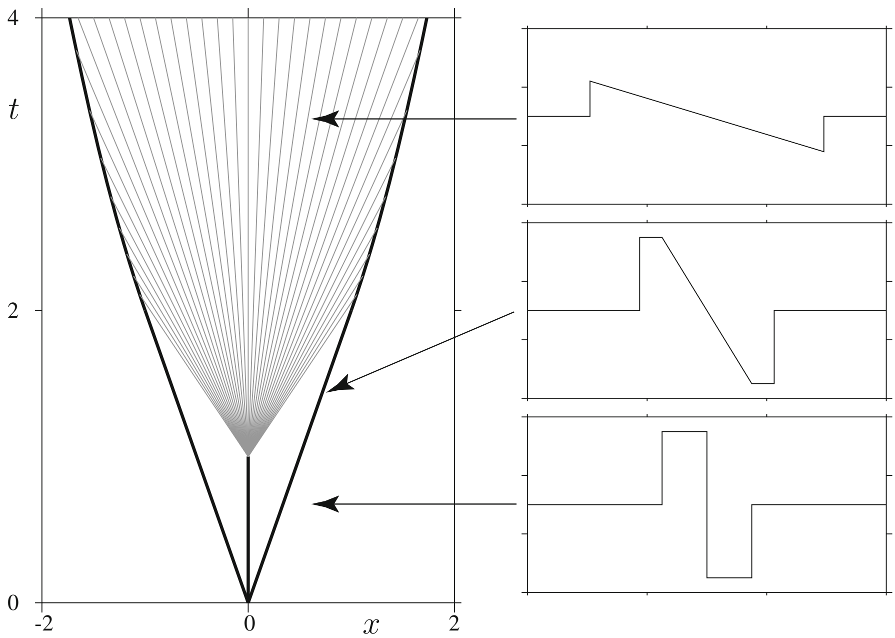


Fig. 1.4 A traffic light on a single road. *To the left* we show the solution in (x, t), and *to the right* the solution $u(x, t)$ at three different times t

It remains to compute the solution for $t > 2\Delta t$. Let us start with examining what happens for positive x. Since the u values that are transported along the characteristics in the rarefaction wave are less than $\frac{1}{2}$, we can construct an admissible discontinuity using the Rankine–Hugoniot condition (1.22). Define a function that has a discontinuity moving along a path $x(t)$. The value to the right of the discontinuity is $\frac{1}{2}$, and the value to the left is determined by $v(x, t - \Delta t)$. Inserting this into (1.22), we get

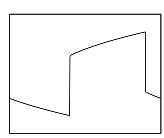
$$x'(t) = s = \frac{\frac{1}{4} - \left(\frac{1}{2} + \frac{x}{2(t-\Delta t)}\right)\left(\frac{1}{2} - \frac{x}{2(t-\Delta t)}\right)}{\frac{1}{2} - \left(\frac{1}{2} - \frac{x}{2(t-\Delta t)}\right)} = \frac{x}{2(t - \Delta t)}.$$

Since $x(2\Delta t) = \Delta t$, this differential equation has solution

$$x_+(t) = \sqrt{\Delta t(t - \Delta t)}.$$

The situation is similar for negative x. Here, we use the fact that the u values in the left part of the rarefaction fan are larger than $\frac{1}{2}$. This gives a discontinuity with a left value $\frac{1}{2}$ and right values taken from the rarefaction wave. The path of this discontinuity is found to be $x_-(t) = -x_+(t)$.

Now we have indeed found a solution that is valid for all positive time. This function has the property that it is a classical solution at all points x and t where it is differentiable, and it satisfies both the Rankine–Hugoniot condition and the entropy condition at points of discontinuity. We show this weak solution in Fig. 1.4,



both in the (x, t)-plane, where we show characteristics and discontinuities, and u as a function of x for various times. The characteristics are shown as gray lines, and the discontinuities as thicker black lines. This concludes our example. Note that we have been able to find the solution to a complicated initial value problem by piecing together solutions from Riemann problems. This is indeed the main idea behind front tracking, and a theme to which we shall give considerable attention in this book. ◇

1.1 Linear Equations

I don't make unconventional stories;
I don't make nonlinear stories.
I like linear storytelling a lot.
— Steven Spielberg

We now make a pause in the exposition of nonlinear hyperbolic conservation laws and take a brief look at linear transport equations. Many of the methods and concepts introduced later in the book are much simpler if the equations are linear.

Let $u \in \mathbb{R}$ be an unknown scalar function of $x \in \mathbb{R}$ and $t \in [0, \infty)$ satisfying the Cauchy problem

$$\begin{cases} u_t + a\, u_x = 0, & x \in \mathbb{R},\ \ t > 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.31}$$

where a is a given (positive) constant, and u_0 is a known function. Recall the theory of characteristics. Since this case is particularly simple, we can use t as a parameter, and we will here use (t, x_0) rather than (s, y) as parameters. Thus the characteristics $x = \xi(t; x_0)$ are defined as

$$\frac{d}{dt}\xi(t; x_0) = a, \quad \xi(0; x_0) = x_0,$$

with solution

$$\xi(t; x_0) = at + x_0.$$

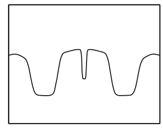
We know that $\frac{d}{dt}u\left(\xi(t; x_0), t\right) = 0$, and thus $u(\xi(t; x_0), t) = u(\xi(0; x_0), 0) = u(x_0, 0) = u_0(x_0)$. We can use the solution of ξ to write

$$u(at + x_0, t) = u_0(x_0).$$

If we set $x = at + x_0$, i.e., $x_0 = x - at$, we get the solution formula

$$u(x, t) = u_0(x - at).$$

Thus (1.31) expresses that the initial function u_0 is transported with a constant velocity a.



The same reasoning works if now $a = a(x, t)$, where the map $x \mapsto a(x, t)$ is Lipschitz continuous for all t. In this case let $u = u(x, t)$ satisfy the Cauchy problem

$$\begin{cases} u_t + a(x, t)u_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.32}$$

First we observe that this equation is not conservative, and the interpretation of $a(x, t)u$ is *not* the flux of u across a point. Now let $\xi(t; x_0)$ denote the unique solution of the ordinary differential equation

$$\frac{d}{dt}\xi(t; x_0) = a(\xi(t; x_0), t), \quad \xi(0; x_0) = x_0. \tag{1.33}$$

By the chain rule we also now find that

$$\frac{d}{dt}u\left(\xi(t; x_0), t\right) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{d}{dt}\xi(t; x_0) = u_t(\xi, t) + a(\xi, t)u_x(\xi, t) = 0.$$

Therefore $u(\xi(t; x_0), t) = u_0(x_0)$. In order to get a solution formula, we must solve $x = \xi(t; x_0)$ in terms of x_0, or equivalently, find a function $\zeta(\tau; x)$ that solves the backward characteristic equation,

$$\frac{d}{d\tau}\zeta(\tau; x) = -a(\zeta(\tau; x), t - \tau), \quad \zeta(0; x) = x. \tag{1.34}$$

Then

$$\frac{d}{d\tau}u(\zeta(\tau; x), t - \tau) = 0,$$

which means that $u(x, t) = u(\zeta(0; x), t) = u(\zeta(t; x), 0) = u_0(\zeta(t; x))$.

◇ Example 1.7
Let us study the simple example with $a(x, t) = x$. Thus

$$u_t + xu_x = 0, \quad u(x, 0) = u_0(x).$$

Then the characteristic equation is

$$\frac{d}{dt}\xi = \xi, \quad \xi(0) = x_0,$$

with solution

$$\xi(t; x_0) = x_0 e^t.$$

Solving $\xi(t; x_0) = x$ in terms of x_0 gives $x_0 = xe^{-t}$, and thus

$$u(x, t) = u_0\left(xe^{-t}\right). \qquad \diamond$$

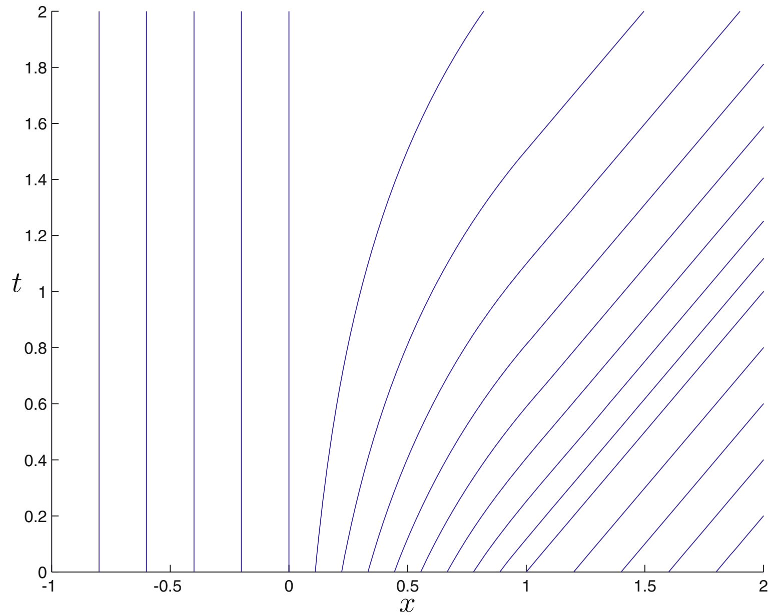


Fig. 1.5 Characteristics in the (x, t)-plane for (1.35)

◇ Example 1.8
Let us look at another example:

$$a(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x \le 1, \\ 1 & 1 < x. \end{cases} \tag{1.35}$$

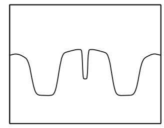
In this case the characteristics are straight lines $\xi(t; x_0) = x_0$ if $x_0 \le 0$, and $\xi(t; x_0) = x_0 + t$ if $x_0 \ge 1$. Finally, whenever $0 < x_0 < 1$, the characteristics are given by

$$\xi(t; x_0) = \begin{cases} x_0 e^t & t \le -\ln(x_0), \\ 1 + t + \ln(x_0) & t > -\ln(x_0). \end{cases}$$

See Fig. 1.5 for a picture of this. In this case a is increasing in x, and therefore the characteristics are no closer than they were initially. Since u is constant along characteristics, this means that

$$\max_x |u_x(x, t)| \le \max_x |u_0'(x)|.$$

If a is decreasing, such a bound cannot be found, as the next example shows. ◇



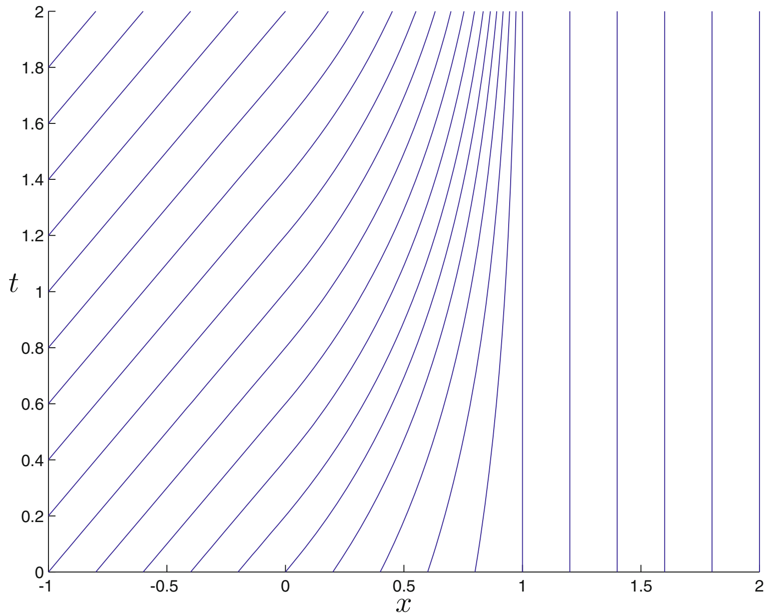


Fig. 1.6 The characteristics for (1.36)

◇ Example 1.9

Let now

$$a(x) = \begin{cases} 1 & x < 0, \\ 1 - x & 0 \le x \le 1, \\ 0 & 1 < x. \end{cases} \tag{1.36}$$

In this case the characteristics are given by

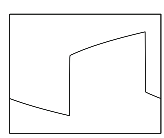
$$\xi(t; x_0) = \begin{cases} \begin{cases} x_0 + t, & t < -x_0, \\ 1 - e^{-(t+x_0)} & t \ge -x_0, \end{cases} & x_0 < 0, \\ 1 - (1 - x_0)e^{-t} & 0 \le x_0 < 1, \\ x_0 & 1 \le x_0. \end{cases}$$

See Fig. 1.6 for an illustration of these characteristics. Let now x_0 be in the interval $(0, 1)$, and assume that u_0 is continuously differentiable. Since u is constant along characteristics, $u(\cdot, t)$ is also continuously differentiable for all $t > 0$. Thus

$$u_0'(x_0) = \frac{\partial}{\partial x_0} u(\xi(t; x_0), t) = u_x(\xi(t; x_0), t) \frac{\partial \xi}{\partial x_0},$$

which, when $x_0 \in (0, 1)$, implies that $u_x(x, t) = u_0'(x_0)e^t$ for $x = \xi(t; x_0)$. From this we see that the only bound on the derivative that we can hope for is of the type

$$\max_x |u_x(x, t)| \le e^t \max_x |u_0'(x)| . \qquad \diamond$$



Numerics (I)

If we (pretend that we) do not have the characteristics, and still want to know the solution, we can try to approximate it by some numerical method.

To this end we introduce approximations to the first spatial derivative

$$\begin{aligned} D_- u(x) &= \frac{u(x) - u(x - \Delta x)}{\Delta x}, \\ D_+ u(x) &= \frac{u(x + \Delta x) - u(x)}{\Delta x}, \quad \text{and} \\ D_0 u(x) &= \frac{u(x + \Delta x) - u(x - \Delta x)}{2\Delta x}, \end{aligned}$$

where Δx is a small positive number. When we deal with numerical approximations, we shall always use the notation $u_j(t)$ to indicate an approximation to $u(j\,\Delta x, t)$ for some integer j. We also use the notation

$$x_j = j\,\Delta x, \quad x_{j\pm 1/2} = \left(j \pm \frac{1}{2}\right)\Delta x = x_j \pm \frac{\Delta x}{2}.$$

Now consider the case in which a is a positive constant. As a semidiscrete numerical scheme for (1.31) we propose to let u_j solve the (infinite) system of ordinary differential equations

$$u_j'(t) + a D_- u_j(t) = 0, \quad u_j(0) = u_0(x_j). \tag{1.37}$$

We need to define an approximation to $u(x,t)$ for every x and t, and we do this by linear interpolation:

$$u_{\Delta x}(x,t) = u_j(t) + \left(x - x_j\right) D_- u_{j+1}(t), \quad \text{for } x \in [x_j, x_{j+1}). \tag{1.38}$$

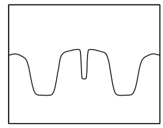
We want to show that (**a**) $u_{\Delta x}$ converges to some function u as $\Delta x \to 0$, and (**b**) the limit u solves the equation.

If u_0 is continuously differentiable, we know that a solution to (1.31) exists (and we can find it by the method of characteristics). Since the equation is linear, we can easily study the error $e_{\Delta x}(x,t) = u(x,t) - u_{\Delta x}(x,t)$. In the calculation that follows, we use the following properties:

$$D_+ u_j - D_- u_j = \Delta x D_+ D_- u_j \text{ and } D_- u_{j+1} = D_+ u_j.$$

Inserting the error term $e_{\Delta x}$ into the equation, we obtain for $x \in (x_j, x_{j+1})$,

$$\begin{aligned} \frac{\partial}{\partial t} e_{\Delta x} + a \frac{\partial}{\partial x} e_{\Delta x} &= -\frac{\partial}{\partial t} u_{\Delta x} - a \frac{\partial}{\partial x} u_{\Delta x} \\ &= -\frac{d}{dt}\left[u_j(t) + \left(x - x_j\right) D_- u_{j+1}(t)\right] - a D_- u_{j+1}(t) \\ &= -u_j'(t) - \left(x - x_j\right) D_- u_{j+1}'(t) - a D_+ u_j(t) \\ &= a D_- u_j(t) - a D_+ u_j(t) + a\left(x - x_j\right) D_- D_- u_{j+1}(t) \\ &= -a \Delta x D_+ D_- u_j(t) + a\left(x - x_j\right) D_+ D_- u_j(t) \\ &= a\left(\left(x - x_j\right) - \Delta x\right) D_+ D_- u_j(t). \end{aligned}$$



Next let $f_{\Delta x}$ be defined by

$$f_{\Delta x}(x,t) = a\left(\left(x - x_j\right) - \Delta x\right) D_+ D_- u_j(t) \quad \text{for } x \in [x_j, x_{j+1}),$$

so that

$$(e_{\Delta x})_t + a\,(e_{\Delta x})_x = f_{\Delta x}. \tag{1.39}$$

Using the method of characteristics on this equation gives (see Exercise 1.3)

$$e_{\Delta x}(x,t) = e_{\Delta x}(x - at, 0) + \int_0^t f_{\Delta x}(x - a(t-s), s)\, ds. \tag{1.40}$$

(Here we tacitly assume uniqueness of the solution.) Hence we get the bound

$$|e_{\Delta x}(x,t)| \le \sup_x |e_{\Delta x}(x,0)| + t\, \|f_{\Delta x}\|_{L^\infty(\mathbb{R}\times[0,t])}. \tag{1.41}$$

In trying to bound $f_{\Delta x}$, note first that

$$|f_{\Delta x}(x,t)| \le \Delta x\, a\, \left|D_- D_+ u_j(t)\right|,$$

so $f_{\Delta x}$ tends to zero with Δx if $D_- D_+ u_j$ is bounded. Writing $w_j = D_- D_+ u_j$ and applying $D_- D_+$ to (1.37), we get

$$w_j'(t) + a D_-\left(w_j\right) = 0, \quad w_j(0) = D_- D_+ u_0(x).$$

Now it is time to use the fact that $a > 0$. To bound w_j, observe that if $w_j \le w_{j-1}$, then $D_- w_j \le 0$. Hence, if $w_j(t) \le w_{j-1}(t)$, then

$$\frac{d}{dt} w_j(t) = -a D_- w_j(t) \ge 0.$$

Similarly, if for some t, $w_j(t) \ge w_{j-1}(t)$, then $w_j'(t) \le 0$. This means that

$$\inf_x u_0''(x) \le \inf_k D_- D_+ u_k(0) \le w_j(t) \le \sup_k D_- D_+ u_k(0) \le \sup_x u_0''(x).$$

Thus w_j is bounded if u_0' is Lipschitz continuous. Note that it is the choice of the difference scheme (1.37) (choosing D_- instead of D_+ or D) that allows us to conclude that we have a bounded approximation. It remains to study $e_{\Delta x}(x,0)$. For $x \in [x_j, x_{j+1})$,

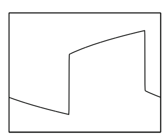
$$\begin{aligned} |e_{\Delta x}(x,0)| &= \left| u_0(x) - u_0(x_j) - \frac{x - x_j}{\Delta x}\left(u_0(x_{j+1}) - u_0(x_j)\right)\right| \\ &\le 2\Delta x \max_{x\in[x_j,x_{j+1}]} |u_0'(x)|. \end{aligned}$$

Then we have proved the bound

$$|u_{\Delta x}(x,t) - u(x,t)| \le \Delta x\left(2\,\|u_0'\|_{L^\infty(\mathbb{R})} + t\,a\,\|u_0''\|_{L^\infty(\mathbb{R})}\right), \tag{1.42}$$

for all x and $t > 0$.

Strictly speaking, in order for this argument to be valid, we have implicitly assumed in (1.40) that equation (1.39) has *only* the solution (1.40). This brings us to another topic.



Entropy Solutions (I)

You should call it entropy . . . [since] . . . no one knows what entropy really is, so in a debate you will always have the advantage.[3]
— John von Neumann

Without much extra effort, we can generalize slightly, and we want to ensure that the equation

$$u_t + a(x,t)u_x = f(x,t) \tag{1.43}$$

has only one differentiable solution. If we let the characteristic curves be defined by (1.33), a solution is given by (see Exercise 1.3)

$$u\left(\xi(t;x_0),t\right) = u_0(x_0) + \int_0^t f(\xi(s;x_0),s)\,ds.$$

In terms of the inverse characteristic ζ defined by (1.34) this formula reads (see Exercise 1.3)

$$u(x,t) = u_0\left(\zeta(t;x)\right) + \int_0^t f\left(\zeta(\tau;x),t-\tau\right)\,d\tau.$$

If u_0 is differentiable and f is bounded, this formula gives a differentiable function $u(x,t)$.

Now we can turn to the uniqueness question. Since (1.43) is linear, to prove uniqueness means to show that the equation with $f = 0$ and $u_0 = 0$ has only the zero solution. Therefore, we consider

$$u_t + a(x,t)u_x = 0.$$

Now let $\eta(u)$ be a differentiable function, and multiply the above by $\eta'(u)$ to get

$$0 = \frac{\partial}{\partial t}\eta(u) + a\frac{\partial}{\partial x}\eta(u) = \eta(u)_t + (a\eta(u))_x - a_x\eta(u).$$

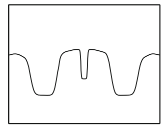
Assume that $\eta(0) = 0$ and $\eta(u) > 0$ for $u \neq 0$, and that $|a_x(x,t)| < C$ for all x and t. If $\eta(u(\cdot,t))$ is integrable, then we can integrate this to get

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(u(x,t))\,dx = \int_{\mathbb{R}} a_x(x,t)\eta(u(x,t))\,dx \le C\int_{\mathbb{R}} \eta(u(x,t))\,dx.$$

By Gronwall's inequality (see Exercise 1.10),

$$\int_{\mathbb{R}} \eta(u(x,t))\,dx \le e^{Ct}\int_{\mathbb{R}} \eta(u_0(x))\,dx.$$

[3] In a discussion with Claude Shannon about Shannon's new concept called "entropy."



If $u_0 = 0$, then $\eta(u_0) = 0$, and we must have $u(x,t) = 0$ as well. We have shown that if $\eta(u_0)$ is integrable for some differentiable function η with $\eta(0) = 0$ and $\eta(u) > 0$ for $u \neq 0$, and a_x is bounded, then (1.43) has only one differentiable solution.

Frequently, the model (1.43) (with f identically zero) is obtained by the limit of a physically more realistic model,

$$u_t^\varepsilon + a(x,t)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \tag{1.44}$$

as ε becomes small. You can think of u^ε as the temperature in a long rod moving with speed a. In this case ε is proportional to the heat conductivity of the rod. Equation (1.44) has more regular solutions than the initial data u_0 (see Appendix B). If we multiply this equation by $\eta'(u^\varepsilon)$, where $\eta \in C^2(\mathbb{R})$ is a *convex* function, we get

$$\eta\left(u^\varepsilon\right)_t + a\,\eta\left(u^\varepsilon\right)_x = \varepsilon\left(\eta'\left(u^\varepsilon\right)u_x^\varepsilon\right)_x - \varepsilon\eta''\left(u^\varepsilon\right)\left(u_x^\varepsilon\right)^2.$$

The function η is often called an *entropy*. The term with $(u_x^\varepsilon)^2$ is problematic when $\varepsilon \to 0$, since the derivative will not be square integrable in this limit. For linear equations the integrability of this term depends on the integrability of this term initially. However, for nonlinear equations, we have seen that jumps can form independently of the smoothness of the initial data, and the limit of u_x^ε will in general not be square integrable.

The key to circumventing this problem is to use the convexity of η, that is, $\eta''(u) \geq 0$, and hence $\varepsilon\eta''\left(u^\varepsilon\right)\left(u_x^\varepsilon\right)^2$ is nonnegative, to replace this term by the appropriate inequality. Thus we get that

$$\eta\left(u^\varepsilon\right)_t + \left(a\eta\left(u^\varepsilon\right)\right)_x - a_x\eta\left(u^\varepsilon\right) \leq \varepsilon\left(\eta'\left(u^\varepsilon\right)u_x^\varepsilon\right)_x. \tag{1.45}$$

Now the right-hand side of (1.45) converges to zero weakly.[4] We define an entropy solution to be the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ of solutions to (1.44) as $\varepsilon \to 0$. Formally, an entropy solution to (1.43) should satisfy (reintroducing the function f)

$$\eta(u)_t + (a\eta(u))_x - a_x\eta(u) \leq \eta'(u)f(x,t), \tag{1.46}$$

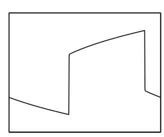
for all convex functions $\eta \in C^2(\mathbb{R})$. We shall see later that this is sufficient to establish uniqueness even if u is not assumed to be differentiable.

Numerics (II)

Let us for the moment return to the transport equation

$$u_t + a(x,t)u_x = 0. \tag{1.47}$$

[4] That is, $\varepsilon \iint_{\mathbb{R}\times[0,\infty)} \varphi_x \eta'(u^\varepsilon)u_x^\varepsilon \, dx\, dt \to 0$ as $\varepsilon \to 0$ for any test function φ.



We want to construct a fully discrete scheme for this equation, and the simplest such scheme is the explicit Euler scheme,

$$D_+^t u_j^n + a_j^n D_- u_j^n = 0, \quad n \geq 0, \tag{1.48}$$

and $u_j^0 = u_0(x_j)$. Here D_+^t denotes the discrete forward time difference

$$D_+^t u(t) = \frac{u(t + \Delta t) - u(t)}{\Delta t},$$

and u_j^n is an approximation of $u(x_j, t_n)$, with $t_n = n\Delta t$, $n \geq 0$. Furthermore, a_j^n denotes some approximation of $a(x_j, t_n)$, to be determined later. We can rewrite (1.48) as

$$u_j^{n+1} = u_j^n - a_j^n \lambda \left(u_j^n - u_{j-1}^n \right),$$

where $\lambda = \Delta t / \Delta x$.[5]

Let us first return to the case that a is constant. We can then use von Neumann stability analysis. Assume that the scheme produces approximations that converge to a bounded solution for almost all x and t; in particular, assume that u_j^n is bounded independently of Δx and Δt. Consider the periodic case. We make the ansatz that $u_j^n = \alpha^n e^{ij\Delta x}$ with $i = \sqrt{-1}$ (the equation is linear, so we might as well expand the solution in a Fourier series). Inserting this into the equation for u_j^{n+1}, we get

$$\begin{aligned} \alpha^{n+1} e^{ij\Delta x} &= \alpha^n e^{ij\Delta x} - \lambda a \left(\alpha^n e^{ij\Delta x} - \alpha^n e^{i(j-1)\Delta x} \right) \\ &= \alpha^n e^{ij\Delta x} \big(1 - \lambda a (1 - e^{-i\Delta x}) \big), \end{aligned}$$

so that

$$\alpha = 1 - \lambda a \big(1 - \cos(\Delta x) + i \sin(\Delta x) \big).$$

If $|\alpha| \leq 1$, then the sup-norm estimate will hold also for the solution generated by the scheme. In this case the scheme is called *von Neumann stable*.

We calculate

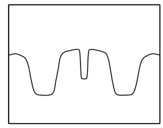
$$\begin{aligned} |\alpha|^2 &= 1 + 2\lambda^2 a^2 - 2\lambda a (1 + (1 - \lambda a) \cos(\Delta x)) \\ &= 1 - 2a\lambda \, (1 - a\lambda) \, (1 - \cos(\Delta x)) \, . \end{aligned}$$

This is less than or equal to 1 if and only if $a\lambda(1 - a\lambda) \geq 0$. Thus we require

$$0 \leq \lambda a \ \leq 1. \tag{1.49}$$

This relationship between the spatial and temporal discretization (as measured by λ) and the wave speed given by a is the simplest example of the celebrated *CFL condition*, named after Courant–Friedrichs–Lewy. We will return to the CFL condition repeatedly throughout the book.

[5] Unless otherwise is stated, you can safely assume that this is the definition of λ.



Returning to the scheme for the transport equation with variable and nonnegative speed, we say that the scheme will be von Neumann stable if

$$\lambda \max_{(x,t)} a(x,t) \le 1. \tag{1.50}$$

Consider now the scheme (1.48) with

$$a_j^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} a(x_j, t)\, dt.$$

We wish to establish the convergence of u_j^n. To this end, set

$$e_j^n = u(x_j, t_n) - u_j^n,$$

where u is the unique solution to (1.47). Inserting this into the scheme, we find that

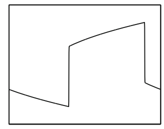
$$\begin{aligned}
D_+^t e_j^n + a_j^n D_- e_j^n &= D_+^t u(x_j, t_n) + a_j^n D_- u(x_j, t_n) \\
&= \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} u_t(x_j, t)\, dt + \frac{a_j^n}{\Delta x} \int_{x_{j-1}}^{x_j} u_x(x, t_n)\, dx \\
&= \frac{1}{\Delta x \Delta t} \int_{t_n}^{t_{n+1}} \int_{x_{j-1}}^{x_j} \big(u_t(x_j, t) + a(x_j, t) u_x(x, t_n)\big)\, dx\, dt \\
&= \frac{1}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} a(x_j, t) \big(u_x(x, t_n) - u_x(x_j, t)\big)\, dt\, dx \\
&= \frac{1}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} a(x_j, t) \Big(\int_{x_j}^{x} u_{xx}(z, t_n)\, dz - \int_{t_n}^{t} u_{xt}(x_j, s)\, ds\Big) dt\, dx \\
&=: R_j^n.
\end{aligned}$$

Assuming now that u_{xx} and u_{tx} are bounded, which they will be if we consider a finite time interval $[0, T]$, choose M such that $\max\{\|u_{xx}\|_{L^\infty}, \|u_{tx}\|_{L^\infty}, \|a\|_{L^\infty}\} \le M$. Then we get the bound

$$\begin{aligned}
\left| R_j^n \right| &\le \frac{M^2}{\Delta x \Delta t} \int_{x_{j-1}}^{x_j} \int_{t_n}^{t_{n+1}} \big((x_j - x) + (t - t_n) \big)\, dt\, dx \\
&= \frac{M^2}{2} (\Delta x + \Delta t).
\end{aligned}$$

Therefore the error will satisfy the inequality

$$e_j^{n+1} \le \underbrace{e_j^n \left(1 - \lambda a_j^n\right) + \lambda a_j^n e_{j-1}^n}_{\Upsilon} + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$



If $\|a\|_{L^\infty}\lambda < 1$ (recall the CFL condition), then Υ is a convex combination of e_j^n and e_{j-1}^n, which is less than or equal to $\max\left\{e_j^n, e_{j-1}^n\right\}$. Taking the supremum over j, first on the right, and then on the left, we get

$$\sup_j \left\{e_j^{n+1}\right\} \le \sup_j \left\{e_j^n\right\} + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

We also have that

$$e_j^{n+1} \ge \underbrace{e_j^n \left(1 - \lambda a_j^n\right) + \lambda a_j^n e_{j-1}^n}_{\Upsilon} - \Delta t \frac{M^2}{2} (\Delta x + \Delta t),$$

which implies that

$$\inf_j \left\{e_j^{n+1}\right\} \ge \inf_j \left\{e_j^n\right\} - \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

With $\bar{e}^n = \sup_j \left|e_j^n\right|$, the above means that

$$\bar{e}^{n+1} \le \bar{e}^n + \Delta t \frac{M^2}{2} (\Delta x + \Delta t).$$

Inductively, we then find that

$$\bar{e}^n \le \bar{e}^0 + t_n \frac{M^2}{2} (\Delta x + \Delta t) = t_n \frac{M^2}{2} (\Delta x + \Delta t),$$

since $e_j^0 = 0$ by definition. Hence, the approximation defined by (1.48) converges to the unique solution if u is twice differentiable with bounded second derivatives.

We have seen that if $x \mapsto a(x,t)$ is decreasing on some interval, the best bounds for u_{xx} and u_{xt} are likely to be of the form Ce^{Ct}, which means that the "constant" M is likely to be large if we want to study the solution for large (or even moderate) times.

Similarly, if $a(x,t) < 0$, the scheme

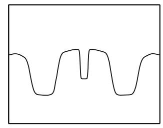
$$D_+^t u_j^n + a_j^n D_+ u_j^n = 0$$

will give a convergent sequence.

Entropy Solutions (II)

Consider the Cauchy problem

$$\begin{cases} u_t + a(x,t) u_x = 0, & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.51}$$



where a is a continuously differentiable function (in this section not assumed to be nonnegative). Recall that an entropy solution is defined as the limit of the singularly perturbed equation (1.44). For every positive ε, u^ε satisfies (1.45), implying that the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ should satisfy (1.46) with f identically zero. Multiplying the inequality (1.46) by a nonnegative test function ψ, and integrating by parts, we find that

$$\int_0^\infty\int_{\mathbb{R}} \big(\eta(u)\psi_t + a\eta(u)\psi_x + a_x\eta(u)\psi\big)\,dx\,dt + \int_{\mathbb{R}} \eta(u_0(x))\psi(x,0)\,dx \ge 0 \qquad (1.52)$$

should hold for all nonnegative test functions $\psi \in C_0^\infty(\mathbb{R}\times[0,\infty))$, and for all convex η. If $u(x,t)$ is a function in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))$ that satisfies (1.52) for all convex entropies η, then u is called a weak entropy solution to (1.51). The point of this is that we no longer require u to be differentiable, or even continuous. Therefore, showing that approximations converge to an entropy solution should be much easier than showing that the limit is a classical solution.

We are going to show that there is only one entropy solution. Again, since the equation is linear, it suffices to show that $u_0 = 0$ (in $L^1(\mathbb{R})$) implies $u(\,\cdot\,,t) = 0$ (in $L^1(\mathbb{R})$).

To do this, we specify a particular test function. Let ω be a C^∞ function such that

$$0 \le \omega(\sigma) \le 1, \quad \operatorname{supp}\omega \subseteq [-1,1], \quad \omega(-\sigma) = \omega(\sigma), \quad \int_{-1}^{1} \omega(\sigma)\,d\sigma = 1.$$

Now define

$$\omega_\varepsilon(\sigma) = \frac{1}{\varepsilon}\omega\left(\frac{\sigma}{\varepsilon}\right). \qquad (1.53)$$

Let $x_1 < x_2$, and introduce

$$\varphi_\varepsilon(x,t) = \int_{x_1+Lt}^{x_2-Lt} \omega_\varepsilon(x-y)\,dy,$$

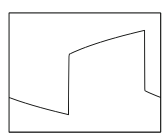
where L is a constant such that $L > \|a\|_{L^\infty(\Omega)}$ and $\Omega = \mathbb{R}\times[0,\infty)$. We fix a T such that $T < (x_2 - x_1)/(2L)$, and consider $t < T$. Observe that $\varphi_\varepsilon(\,\cdot\,,t)$ is an approximation to the characteristic function for the interval $(x_1 + Lt, x_2 - Lt)$.

Next introduce

$$h_\varepsilon(t) = 1 - \int_0^t \omega_\varepsilon(s-T)\,ds.$$

This is an approximation to the characteristic function of the interval $(-\infty, T]$. Finally, we choose the test function

$$\psi_\varepsilon(x,t) = h_\varepsilon(t)\varphi_\varepsilon(x,t) \in C_0^\infty(\Omega).$$



Inserting this into the entropy inequality (1.52), we get

$$\begin{aligned}
&\iint_{\Omega} \eta(u)\varphi_\varepsilon h'_\varepsilon(t)\,dx\,dt \\
&+ \iint_{\Omega} h_\varepsilon(t)\eta(u)\left(\frac{\partial}{\partial t}\varphi_\varepsilon(x,t) + a(x,t)\frac{\partial}{\partial x}\varphi_\varepsilon(x,t)\right)dx\,dt \\
&+ \iint_{\Omega} a_x\eta(u)h_\varepsilon\varphi_\varepsilon\,dx\,dt + \int_{\mathbb{R}} \eta(u_0)\varphi_\varepsilon(x,0)\,dx \ge 0.
\end{aligned} \tag{1.54}$$

We treat the second integral first, and calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_\varepsilon(x,t) &= -L\left(\omega_\varepsilon(x - x_2 + Lt) + \omega_\varepsilon(x - x_1 - Lt)\right), \\
\frac{\partial}{\partial x}\varphi_\varepsilon(x,t) &= -\omega_\varepsilon(x - x_2 + Lt) + \omega_\varepsilon(x - x_1 - Lt).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_\varepsilon + a\,\frac{\partial}{\partial x}\varphi_\varepsilon &= (-L + a)\omega_\varepsilon(x - x_2 + Lt) + (-L - a)\omega_\varepsilon(x - x_1 - Lt) \\
&\le (|a| - L)\big(\omega_\varepsilon(x - x_2 + Lt) + \omega_\varepsilon(x - x_1 - Lt)\big) \le 0,
\end{aligned}$$

since L is chosen to be larger than $|a|$. Hence, if $\eta(u) \ge 0$, then the second integral in (1.54) is nonpositive. Thus we have

$$\begin{aligned}
\iint_{\Omega} \eta(u)\varphi_\varepsilon h'_\varepsilon(t)\,dx\,dt &+ \iint_{\Omega} a_x\eta(u)h_\varepsilon\varphi_\varepsilon\,dx\,dt \\
&+ \int_{\mathbb{R}} \eta(u_0)\varphi_\varepsilon(x,0)\,dx \ge 0.
\end{aligned} \tag{1.55}$$

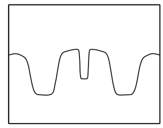
Let us for the moment proceed formally. The function h_ε approximates the characteristic function $\chi_{(-\infty,T]}$, which has derivative $-\delta_T$, a negative Dirac delta function at T. Similarly, φ_ε approximates the characteristic function $\chi_{(x_1+Lt,x_2-Lt)}$, with derivative $L(\delta_{x_1+Lt} - \delta_{x_2-Lt})$. From (1.54) we formally obtain by sending $\varepsilon \to 0$, that

$$-\int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx + \int_0^T \int_{x_1+Lt}^{x_2-Lt} a_x(x,t)\eta(u(x,t))\,dx\,dt + \int_{x_1}^{x_2} \eta(u(x,0))\,dx \ge 0, \tag{1.56}$$

and this is what we intend to prove next.

The first integral in (1.54) reads

$$-\iint_{\Omega} \eta(u)\varphi_\varepsilon(x,t)\omega_\varepsilon(t-T)\,dx\,dt = -\int_0^\infty f_\varepsilon(t)\omega_\varepsilon(t-T)\,dt,$$



where

$$f_\varepsilon(t) = \int_{\mathbb{R}} \varphi_\varepsilon(x,t)\eta(u(x,t))\,dx.$$

Keeping t fixed, we obtain

$$f_\varepsilon(t) \to \int_{x_1+Lt}^{x_2-Lt} \eta(u(x,t))\,dx = f_0(t) \quad \text{as } \varepsilon \to 0,$$

the limit being uniform in t for $t \in [0,T]$. If $t \mapsto u(\cdot\,,t)$ is continuous as a map from $[0,\infty)$ with values in $L^1(\mathbb{R})$, then f_ε and f_0 are continuous in t. In that case,

$$\begin{aligned}\int_0^\infty f_\varepsilon(t)\omega_\varepsilon(t-T)\,dt &= \int_0^\infty \big(f_\varepsilon(t)-f_0(t)\big)\omega_\varepsilon(t-T)\,dt + \int_0^\infty f_0(t)\omega_\varepsilon(t-T)\,dt \\ &\to f_0(T) \quad \text{as } \varepsilon \to 0,\end{aligned}$$

since

$$\begin{aligned}\left|\int_0^\infty \big(f_\varepsilon(t)-f_0(t)\big)\omega_\varepsilon(t-T)\,dt\right| &\le \|f_\varepsilon - f_0\|_{L^\infty}\int_0^\infty \omega_\varepsilon(t-T)\,dt \\ &= \|f_\varepsilon - f_0\|_{L^\infty} \to 0.\end{aligned}$$

In order to ensure that $t \mapsto u(\cdot\,,t)$ is continuous as a map from $[0,\infty)$ to $L^1(\mathbb{R})$, we *define* an entropy solution to have this property; see Definition 1.10 below. We have that

$$h_\varepsilon(t)\varphi_\varepsilon(x,t) \to \chi_{\Pi_T}(x,t) \quad \text{in } L^1(\Omega_T),$$

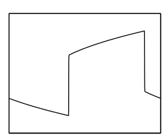
where $\Pi_T = \{(x,t) \mid 0 \le t \le T,\ x_1 + Lt \le x \le x_2 - Lt\}$ and $\Omega_T = \mathbb{R}\times[0,T]$. By sending $\varepsilon \to 0$ in (1.54), we then find that (cf. (1.56))

$$\int_{x_1}^{x_2} \eta(u(x,0))\,dx + \int_0^T \int_{x_1+Lt}^{x_2-Lt} a_x(x,t)\eta(u(x,t))\,dx\,dt \ge \int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx, \tag{1.57}$$

which implies that

$$f_0(T) \le f_0(0) + \|a_x\|_{L^\infty(\Omega_T)}\int_0^T f_0(t)\,dt,$$

assuming that η is positive.



Gronwall's inequality then implies

$$f_0(T) \le f_0(0) e^{\|a_x\|_{L^\infty(\Omega_T)} T},$$

or, writing it out explicitly,

$$\int_{x_1+LT}^{x_2-LT} \eta(u(x,T))\,dx \le \int_{x_1}^{x_2} \eta(u_0(x))\,dx\, e^{\|a_x\|_{L^\infty(\Omega_T)} T},$$

for every nonnegative convex function η. Observe that this proves the finite speed of propagation.

Choosing $\eta(u) = |u|^p$ for $1 \le p < \infty$, assuming $\eta(u)$ to be integrable, and sending x_1 to $-\infty$ and x_2 to ∞, we get

$$\|u(\,\cdot\,,T)\|_{L^p(\mathbb{R})} \le \|u_0\|_{L^p(\mathbb{R})}\, e^{\|a_x\|_{L^\infty(\Omega_T)} T/p}, \quad 1 \le p < \infty. \tag{1.58}$$

Next, we can let $p \to \infty$, assuming $\eta(u)$ to be integrable for all $1 \le p < \infty$, to get

$$\|u(\,\cdot\,,T)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}\,. \tag{1.59}$$

In order to formalize the preceding argument, we introduce the following definition.

Definition 1.10 A function $u = u(x,t)$ in $C([0,\infty); L^1(\mathbb{R}))$ is called a weak entropy solution to the problem

$$\begin{cases} u_t + a(x,t)u_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

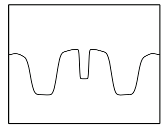
if for all nonnegative and convex functions $\eta(u)$ and all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$, the inequality

$$\int_0^\infty \int_{\mathbb{R}} \big(\eta(u)\varphi_t + a\,\eta(u)\varphi_x + a_x \eta(u)\varphi\big)\,dx\,dt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x,0)\,dx \ge 0$$

holds.

Theorem 1.11 *Assume that $a = a(x,t)$ is such that a_x is bounded. Then the problem (1.32) has at most one entropy solution $u = u(x,t)$, and the bounds (1.58) and (1.59) hold.*

Remark 1.12 From the proof of this theorem (applying (1.58) for $p = 1$), we see that if we define an entropy solution to satisfy the entropy condition only for $\eta(u) = |u|$, then we get uniqueness in $C([0,\infty), L^1(\mathbb{R}))$.



Numerics (III)

We now reconsider the transport equation

$$\begin{cases} u_t + a(x,t)u_x = 0, & t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.60}$$

and the corresponding difference scheme

$$D_+^t u_j^n + a_j^n D_- u_j^n = 0,$$

with

$$a_j^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a(x_j,t)\,dt, \qquad u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx,$$

where as before, we assume that $a(x,t) \ge 0$. In order to have an approximation defined for all x and t, we define

$$u_{\Delta x}(x,t) = u_j^n \ \text{ for } (x,t) \in I_{j-1/2}^n := [x_{j-1}, x_j) \times [t_n, t_{n+1}),$$

where $t_n = n\Delta t$. We wish to show that $u_{\Delta x}$ converges to an entropy solution (the only one!) of (1.60). Now we do not use the linearity, and first prove that $\{u_{\Delta x}\}_{\Delta x > 0}$ has a convergent subsequence.

First we recall that the scheme can be written

$$u_j^{n+1} = \left(1 - a_j^n \lambda\right) u_j^n + a_j^n \lambda u_{j-1}^n.$$

We aim to use Theorem A.11 to prove compactness. First we show that the approximation is uniformly bounded. This is easy, since u_j^{n+1} is a *convex* combination of u_j^n and u_{j-1}^n, so new maxima or minima are not introduced. Thus

$$\|u_{\Delta x}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}.$$

Therefore, the first condition of Theorem A.11 is satisfied.

To show that the second condition holds, recall, or consult Appendix A, that the total variation of a function $u\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$\text{T.V.}\,(u) = \sup_{\{x_i\}} \sum_i |u(x_i) - u(x_{i-1})|,$$

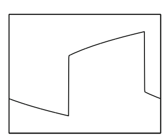
where the supremum is taken over all finite partitions $\{x_i\}$ such that $x_i < x_{i+1}$. This is a seminorm, and we also write $|u|_{BV} := \text{T.V.}\,(u)$.

We have to estimate the total variation of $u_{\Delta x}$. For $t \in [t_n, t_{n+1})$ this is given by

$$|u_{\Delta x}(\cdot,t)|_{BV} = \sum_j \left| u_j^n - u_{j-1}^n \right|.$$

We also have that

$$\begin{aligned} u_j^{n+1} - u_{j-1}^{n+1} &= (1 - a_j^n\lambda)u_j^n + a_j^n\lambda u_{j-1}^n - (1 - a_{j-1}^n\lambda)u_{j-1}^n - a_{j-1}^n\lambda u_{j-2}^n \\ &= (1 - a_j^n\lambda)(u_j^n - u_{j-1}^n) + a_{j-1}^n\lambda(u_{j-1}^n - u_{j-2}^n). \end{aligned}$$



By the CFL condition $0 \leq a_j^n \lambda \leq 1$ for all n and j, we infer

$$\left|u_j^{n+1} - u_{j-1}^{n+1}\right| \leq (1 - a_j^n \lambda) \left|u_j^n - u_{j-1}^n\right| + \lambda a_{j-1}^n \left|u_{j-1}^n - u_{j-2}^n\right|.$$

Therefore

$$\begin{aligned}
\sum_j \left|u_j^{n+1} - u_{j-1}^{n+1}\right| &\leq \sum_j (1 - a_j^n \lambda) \left|u_j^n - u_{j-1}^n\right| + \sum_j \lambda a_{j-1}^n \left|u_{j-1}^n - u_{j-2}^n\right| \\
&= \sum_j \left|u_j^n - u_{j-1}^n\right| - \sum_j \lambda a_j^n \left|u_j^n - u_{j-1}^n\right| + \sum_j \lambda a_j^n \left|u_j^n - u_{j-1}^n\right| \\
&= \sum_j \left|u_j^n - u_{j-1}^n\right|.
\end{aligned}$$

Hence

$$\left|u_{\Delta x}(\,\cdot\,, t)\right|_{BV} \leq \left|u_{\Delta x}(\,\cdot\,, 0)\right|_{BV} \leq |u_0|_{BV}.$$

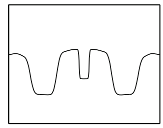
This shows that the second condition of Theorem A.11 is satisfied; see Remark A.12.

To show that the third condition holds, i.e., the continuity of the L^1-norm in time, we assume that $s \in [t_n, t_{n+1})$, and that t is such that $t - s \leq \Delta t$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \left|u_{\Delta x}(x, t) - u_{\Delta x}(x, s)\right| \, dx &\leq \Delta x \sum_j \left|u_j^{n+1} - u_j^n\right| \\
&= \Delta x \sum_j a_j^n \lambda \left|u_j^n - u_{j-1}^n\right| \\
&\leq \Delta t \, \|a\|_{L^\infty(\Omega)} \sum_j \left|u_j^n - u_{j-1}^n\right| \\
&\leq \Delta t \, \|a\|_{L^\infty(\Omega)} |u_0|_{BV}.
\end{aligned}$$

If $s \in [t_n, t_{n+1})$ and $t \in [t_{n+k}, t_{n+k+1})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left|u_{\Delta x}(x, t) - u_{\Delta x}(x, s)\right| \, dx &= \Delta x \sum_j \left|u_j^{n+k} - u_j^n\right| \\
&\leq \sum_{m=n}^{n+k-1} \Delta x \sum_j \left|u_j^{m+1} - u_j^m\right| \\
&= \sum_{m=n}^{n+k-1} \Delta x \sum_j a_j^m \lambda \left|u_j^m - u_{j-1}^m\right| \\
&\leq \sum_{m=n}^{n+k-1} \Delta t \, \|a\|_{L^\infty(\Omega)} \sum_j \left|u_j^m - u_{j-1}^m\right| \\
&\leq k \Delta t \, \|a\|_{L^\infty(\Omega)} |u_0|_{BV} \\
&\leq (t - s + \Delta t) \, \|a\|_{L^\infty(\Omega)} |u_0|_{BV}.
\end{aligned}$$



Hence, also the third condition of Theorem A.11 is fulfilled, and we have the convergence (of a subsequence) $u_{\Delta x} \to u$ as $\Delta x \to 0$. It remains to prove that u is the entropy solution.

To do this, start by observing that

$$\begin{aligned}\eta\left(u_j^{n+1}\right) &= \eta\left((1-a_j^n\lambda)u_j^n + a_j^n\lambda u_{j-1}^n\right)\\ &\le (1-a_j^n\lambda)\eta\left(u_j^n\right) + a_j^n\lambda\eta\left(u_{j-1}^n\right),\end{aligned}$$

since η is assumed to be a convex function. This can be rearranged as

$$D_+^t\eta_j^n + a_j^n D_-\eta_j^n \le 0,$$

where $\eta_j^n = \eta(u_j^n)$, and as

$$D_+^t\eta_j^n + D_-\left(a_j^n\eta_j^n\right) - \eta_{j-1}^n D_- a_j^n \le 0. \tag{1.61}$$

The operators D_-, D_+, and D_+^t satisfy the following "summation by parts" formulas:

$$\begin{aligned}\sum_j a_j D_- b_j &= -\sum_j b_j D_+ a_j, \quad \text{if } a_{\pm\infty} = 0 \text{ or } b_{\pm\infty} = 0,\\ \sum_{n=0}^{\infty} a^n D_+^t b^n &= -\frac{1}{\Delta t}a^0 b^0 - \sum_{n=1}^{\infty} b^n D_-^t a^n \quad \text{if } a^\infty = 0 \text{ or } b^\infty = 0.\end{aligned}$$

Let φ be a nonnegative test function in $C_0^\infty(\Omega)$ and set

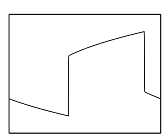
$$\varphi_j^n = \frac{1}{|I_{j-1/2}^n|}\iint\limits_{I_{j-1/2}^n} \varphi(x,t)\,dx\,dt.$$

We multiply (1.61) by $\Delta t\,\Delta x\,\varphi_j^n$ and sum over $n \ge 0$ and $j \in \mathbb{Z}$, using the summation by parts formulas above, to get

$$\begin{aligned}&\Delta x\Delta t\sum_{n=1}^{\infty}\sum_j \eta(u_j^n)D_-^t\varphi_j^n\\ &\quad + \Delta x\Delta t\sum_{n=0}^{\infty}\sum_j\left(a_j^n\eta(u_j^n)D_+\varphi_j^n + \eta(u_{j-1}^n)D_-a_j^n\varphi_j^n\right) + \Delta x\sum_j \eta(u_j^0)\varphi_j^0 \ge 0.\end{aligned}$$

Call the left-hand side of the above inequality $B_{\Delta x}$, and set

$$A_{\Delta x} = \iint\limits_{\Omega}\left(\eta(u_{\Delta x})\varphi_t + a\eta(u_{\Delta x})\varphi_x + a_x\eta(u_{\Delta x})\varphi\right)dx\,dt + \int\limits_{\mathbb{R}}\eta(u_0)\varphi(x,0)\,dx.$$



Then we have

$$A_{\Delta x} = B_{\Delta x} + (A_{\Delta x} - B_{\Delta x}) \geq A_{\Delta x} - B_{\Delta x}.$$

We find that

$$A_{\Delta x} - B_{\Delta x} = \sum_{n=1}^{\infty} \sum_{j} \iint\limits_{I_{j-1/2}^{n}} \eta_j^n \left(\varphi_t - D_-^t \varphi_j^n \right) \, dx \, dt \tag{1.62a}$$

$$+ \sum_{j} \iint\limits_{I_{j-1/2}^{0}} \eta_j^0 \varphi_t \, dx \, dt \tag{1.62b}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^{n}} \eta_j^n a \left(\varphi_x - D_+ \varphi_j^n \right) \, dx dt \tag{1.62c}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^{n}} \eta_j^n D_+ \varphi_j^n \left(a - a_j^n \right) \, dx \, dt \tag{1.62d}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^{n}} \left(\eta_j^n - \eta_{j-1}^n \right) a_x \varphi \, dx \, dt \tag{1.62e}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^{n}} a_x \eta_{j-1}^n \left(\varphi - \varphi_j^n \right) \, dx \, dt \tag{1.62f}$$

$$+ \sum_{j,n} \iint\limits_{I_{j-1/2}^{n}} \eta_{j-1}^n \left(a_x - D_- a_j^n \right) \varphi_j^n \, dx \, dt \tag{1.62g}$$

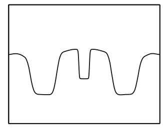
$$+ \sum_{j} \int\limits_{I_{j-1/2}} \left(\eta(u_0) - \eta_j^0 \right) \varphi(x, 0) \, dx \tag{1.62h}$$

$$+ \sum_{j} \int\limits_{I_{j-1/2}} \eta_j^0 \left(\varphi(x, 0) - \varphi_j^0 \right) \, dx. \tag{1.62i}$$

Here $I_{j-1/2} = [x_{j-1}, x_j)$. To show that the limit u is an entropy solution, we must show that all the terms (1.62a)–(1.62i) vanish when Δx becomes small. A small but useful device is contained in the following remark.

Remark 1.13 For a continuously differentiable function ϕ we have

$$\begin{aligned} |\varphi(x,t) - \varphi(y,s)| &= \left| \int_0^1 \frac{d}{d\sigma} \varphi(\sigma(x,t) + (1-\sigma)(y,s)) d\sigma \right| \\ &= \left| \int_0^1 \nabla\varphi(\sigma(x,t) + (1-\sigma)(y,s)) \cdot (x-y, t-s) d\sigma \right| \\ &\leq |x-y| \, \|\varphi_x\|_{L^\infty} + |t-s| \, \|\varphi_t\|_{L^\infty} . \end{aligned}$$



We start with the last term (1.62i). Now

$$\begin{aligned}
&\int\limits_{I_{j-1/2}} \eta_j^0 \left(\varphi(x,0) - \varphi_j^0\right)\, dx \\
&\qquad = \frac{\eta_j^0}{\Delta x \Delta t} \int\limits_{I_{j-1/2}} \iint\limits_{I_{j-1/2}^0} \varphi(x,0) - \varphi(y,t)\, dy\, dt\, dx \\
&\qquad = \frac{\eta_j^0}{\Delta x \Delta t} \int\limits_{I_{j-1/2}} \iint\limits_{I_{j-1/2}^0} \Big(\int_y^x \varphi_x(z,0)\, dz + \int_0^t \varphi_t(y,s)\, ds \Big)\, dy\, dt\, dx.
\end{aligned}$$

Therefore,

$$|(1.62i)| \le \|\eta(u_0)\|_{L^1(\mathbb{R})} \left(\|\varphi_x\|_{L^\infty(\Omega)}\, \Delta x + \|\varphi_t\|_{L^\infty(\Omega)}\, \Delta t \right),$$

where we used the convexity of η. Next, we consider the term (1.62h): Since η is convex, we have

$$|\eta(b) - \eta(a)| \le \max\left\{ |\eta'(a)|, |\eta'(b)| \right\} |b - a|.$$

Furthermore, if both x and y are in $I_{j-1/2}$, then

$$|u_0(x) - u_0(y)| \le |u_0|_{BV(I_{j-1/2})}.$$

Using this and choosing $C = \|\eta'(u_0)\|_{L^\infty}$ yields

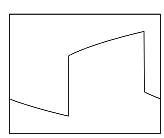
$$\begin{aligned}
&\Big| \int\limits_{I_{j-1/2}} \left(\eta(u_0) - \eta(u_j^0)\right) \varphi(x,0)\, dx \Big| \\
&\qquad\qquad \le C\, \|\varphi\|_{L^\infty(\Omega)} \int\limits_{I_{j-1/2}} \left| u_0(x) - u_j^0 \right|\, dx \\
&\qquad\qquad \le C\, \|\varphi\|_{L^\infty(\Omega)} \int\limits_{I_{j-1/2}} \frac{1}{\Delta x} \int\limits_{I_{j-1/2}} |u_0(x) - u_0(y)|\, dx\, dy \\
&\qquad\qquad \le C\, \|\varphi\|_{L^\infty(\Omega)}\, \Delta x\, |u_0|_{BV(I_{j-1/2})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(1.62h)| &\le C\, \|\varphi\|_{L^\infty(\Omega)}\, \Delta x \sum_j |u_0|_{BV(I_{j-1/2})} \\
&\le C\, \|\varphi\|_{L^\infty(\Omega)}\, \Delta x\, |u_0|_{BV}.
\end{aligned}$$

Next, we consider (1.62g). First observe that

$$D_- a_j^n = D_- \Big(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} a(x_j, t)\, dt \Big) = \frac{1}{\Delta x \Delta t} \iint\limits_{I_{j-1/2}^n} a_x(x,t)\, dx\, dt.$$



Therefore,

$$\eta_{j-1}^n \iint\limits_{I_{j-1/2}^n} a_x(x,t) - D_- a_j^n \, dx\, dt = \eta_{j-1}^n \Big(\iint\limits_{I_{j-1/2}^n} a_x(x,t)\, dx\, dt - \Delta x \Delta t a_j^n \Big)$$
$$= 0,$$

and (1.62g) $= 0$. We continue with the term (1.62f), namely

$$\begin{aligned}
&\iint\limits_{I_{j-1/2}^n} |a_x|\, \eta_{j-1}^n \left|\varphi - \varphi_j^n\right| \, dx\, dt \\
&\qquad \le \|a_x\|_{L^\infty(\Omega)}\, \eta_{j-1}^n \frac{1}{\Delta x \Delta t} \iint\limits_{I_j^n} \iint\limits_{I_j^n} |\varphi(x,t) - \varphi(y,s)| \, dy\, ds\, dx\, dt \\
&\qquad \le \|a_x\|_{L^\infty(\Omega)}\, \eta_{j-1}^n \Delta x \Delta t \left(\Delta x \, \|\varphi_x\|_{L^\infty(\Omega)} + \Delta t \, \|\varphi_t\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

Recall that the test function φ has compact support, contained in $\{t < T\}$. Furthermore, using the scheme for η_j^n, cf. (1.61), it is straightforward to show that

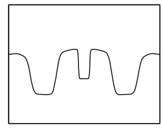
$$\Delta x \sum_j \eta_j^n \le e^{C t_n} \Delta x \sum_j \eta_j^0 \le e^{C t_n} \, \|\eta(u_0)\|_{L^1(\mathbb{R})},$$

where C is a bound on $D_- a_j^n$. Therefore,

$$\begin{aligned}
\Big|\sum_{j,n} \iint\limits_{I_{j-1/2}^n} a_x \eta_{j-1}^n \left(\varphi - \varphi_j^n\right) \, dx\, dt\Big| &\le C_T \Delta x \sum_{j,n} \eta_j^n \Delta t (\Delta x + \Delta t) \\
&\le C_T \Delta x \sum_{n,j} \eta_j^0 \Delta t (\Delta x + \Delta t) \\
&\le C_T T \, \|\eta(u_0)\|_{L^1(\mathbb{R})} \, (\Delta x + \Delta t),
\end{aligned}$$

since the sum in n is only over those n such that $t_n = n\Delta t \le T$. Regarding (1.62e), and setting $M > \|u_0\|_{L^\infty(\mathbb{R})}$, we have that

$$\begin{aligned}
&\Big|\sum_{j,n} \iint\limits_{I_{j-1/2}^n} \left(\eta_j^n - \eta_{j-1}^n\right) a_x \varphi \, dx\, dt\Big| \\
&\qquad \le \|\eta'\|_{L^\infty((-M,M))} \, \|a_x\|_{L^\infty(\Omega)} \, \|\varphi\|_{L^\infty(\Omega)} \, \Delta x \Delta t \sum_{j,n} \left|u_j^n - u_{j-1}^n\right| \\
&\qquad \le C \Delta x T \, |u_0|_{BV}.
\end{aligned}$$



Next, we turn to (1.62d):

$$
\begin{aligned}
|(1.62\mathrm{d})| &\le \|\varphi_x\|_{L^\infty(\Omega)} \sum_{j,n} \eta_j^n \iint\limits_{I_{j-1/2}^n} \left|a(x,t) - a(x_j,t)\right| \, dxdt \\
&\le \|\varphi_x\|_{L^\infty(\Omega)} \|a_x\|_{L^\infty(\Omega)} \Delta x \sum_{j,n} \eta_j^n \Delta x \Delta t \\
&\le \|\varphi_x\|_{L^\infty(\Omega)} \|a_x\|_{L^\infty(\Omega)} C_T T \Delta x \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

We can use the same type of argument to estimate (1.62c):

$$
\begin{aligned}
|(1.62\mathrm{c})| &\le \|a\|_{L^\infty(\Omega)} \left(\Delta x \|\varphi_{xx}\|_{L^\infty(\Omega)} + \Delta t \|\varphi_{xt}\|_{L^\infty(\Omega)}\right) \sum_{n,j} \eta_j^n \Delta x \Delta t \\
&\le \|a\|_{L^\infty(\Omega)} \left(\Delta x \|\varphi_{xx}\|_{L^\infty(\Omega)} + \Delta t \|\varphi_{xt}\|_{L^\infty(\Omega)}\right) C_T T \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

Similarly, we show that

$$
|(1.62\mathrm{b})| \le C_{\Delta t} \Delta t \|\eta(u_0)\|_{L^1(\mathbb{R})} \|\varphi_t\|_{L^\infty(\Omega)} .
$$

Now the end is in sight. We estimate the right-hand side of (1.62a). This will be less than

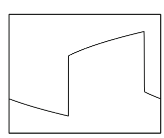
$$
\begin{aligned}
&\sum_{j,n\ge 1} \eta_j^n \iint\limits_{I_{j-1/2}^n} \left|\varphi_t - D_+^t \varphi_j^n\right| \, dx\, dt \\
&\qquad \le \left(\Delta x \|\varphi_{xt}\|_{L^\infty(\Omega)} + \Delta t \|\varphi_{tt}\|_{L^\infty(\Omega)}\right) \sum_{j,n\ge 1} \eta_j^n \Delta x \Delta t \\
&\qquad \le \left(\Delta x \|\varphi_{xt}\|_{L^\infty(\Omega)} + \Delta t \|\varphi_{tt}\|_{L^\infty(\Omega)}\right) C_T T \|\eta(u_0)\|_{L^1(\mathbb{R})} .
\end{aligned}
$$

To sum up, what we have shown is that for every test function $\varphi(x,t)$,

$$
\begin{aligned}
&\iint\limits_{\Omega} \left(\eta(u)\varphi_t + a\eta(u)\varphi_x + a_x\eta(u)\right) dx\, dt + \int\limits_{\mathbb{R}} \eta(u_0)\varphi(x,0)\, dx \\
&\qquad = \lim_{\Delta x \to 0} A_{\Delta x} \\
&\qquad \ge \lim_{\Delta x \to 0} \left(A_{\Delta x} - B_{\Delta x}\right) = 0,
\end{aligned}
$$

if a_x is (locally) continuous and $u_0 \in BV(\mathbb{R})$. Hence the scheme (1.60) produces a subsequence that converges to the unique weak solution. Since the limit is the unique entropy solution, every subsequence will produce a further subsequence that converges to the *same* limit, and thus the *whole* sequence converges!

If u_0'' is bounded, we have seen that the scheme (1.60) converges at a rate $\mathcal{O}(\Delta x)$ to the entropy solution. The significance of the above computations is that we have the convergence to the unique entropy solution even if u_0 is assumed to be only in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$. However, in this case we have not shown any convergence rate.



Systems of Equations

I have a different way of thinking. I think synergistically.
I'm not linear in thinking, I'm not very logical.
— Imelda Marcos

Now we generalize, and let $u\colon \mathbb{R} \times [0,\infty) \to \mathbb{R}^n$ be a solution of the linear system

$$\begin{cases} u_t + Au_x = 0, & x \in \mathbb{R},\ \ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.63}$$

where A is an $n \times n$ matrix with *real and distinct eigenvalues* $\{\lambda_i\}_{i=1}^n$. We order these such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

If this holds, then the system is said to be *strictly hyperbolic*. The matrix A will also have n linearly independent right eigenvectors $r_1, \dots, r_n$ such that

$$Ar_i = \lambda_i r_i.$$

Similarly, it has n independent left eigenvectors $l_1, \dots, l_n$ such that

$$l_i A = \lambda_i l_i.$$

We assume r_i to be column vectors and l_i to be row vectors. However, we will not enforce this strictly, and will write, e.g., $l_i \cdot r_k$. For $k \neq m$, l_k and r_m are orthogonal, since

$$\lambda_m r_m \cdot l_k = Ar_m \cdot l_k = r_m \cdot l_k A = \lambda_k r_m \cdot l_k.$$

Let

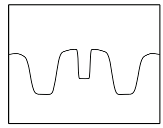
$$L = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}, \quad R = \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix}.$$

Normalize the eigenvectors so that $l_k \cdot r_i = \delta_{ki}$, i.e., $L = R^{-1}$, or $LR = I$. Then

$$LAR = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

We can multiply (1.63) by L from the left to get

$$Lu_t + LAu_x = 0,$$



and defining w by $u = Rw$, we find that

$$w_t + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} w_x = 0. \tag{1.64}$$

This is n decoupled equations, one for each component of $w = (w_1, \dots, w_n)$,

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \quad \text{for } i = 1, \dots, n.$$

The initial data transforms into

$$w_0 = L u_0 = (l_1 \cdot u_0, \dots, l_n \cdot u_0),$$

and hence we obtain the solution

$$w_i(x,t) = l_i \cdot u_0(x - \lambda_i t).$$

Transforming back into the original variables, we obtain

$$u(x,t) = \sum_{i=1}^{n} w_i(x,t) r_i = \sum_{i=1}^{n} \left[l_i \cdot u_0(x - \lambda_i t) \right] r_i. \tag{1.65}$$

◇ Example 1.14 (The linear wave equation)
Now consider the linear wave equation; $\alpha\colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is a solution of

$$\begin{cases} \alpha_{tt} - c^2 \alpha_{xx} = 0, & x \in \mathbb{R}, \quad t > 0, \\ \alpha(x,0) = \alpha_0(x), \quad \alpha_t(x,0) = \beta_0(x), & \end{cases}$$

where c is a positive constant. Defining

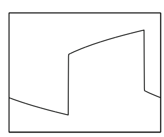
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_t \\ \alpha_x \end{pmatrix}$$

implies that

$$\begin{aligned} \frac{\partial u_1}{\partial t} - c^2 \frac{\partial u_2}{\partial x} &= 0, \\ \frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial x} &= 0, \end{aligned} \qquad \text{or} \qquad u_t + \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} u_x = 0.$$

The matrix

$$A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$



has eigenvalues and eigenvectors

$$\lambda_1 = -c, \ \ r_1 = \begin{pmatrix} c \\ 1 \end{pmatrix}, \quad \lambda_2 = c, \ \ r_2 = \begin{pmatrix} -c \\ 1 \end{pmatrix}.$$

Thus

$$R = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix}, \quad L = R^{-1} = \frac{1}{2c} \begin{pmatrix} 1 & c \\ -1 & c \end{pmatrix}.$$

Hence we find that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{2c} \begin{pmatrix} u_1 + cu_2 \\ -u_1 + cu_2 \end{pmatrix}.$$

Writing the solution in terms of α_x and α_t, we find that

$$\begin{aligned} \alpha_t(x,t) + c\alpha_x(x,t) &= \beta_0(x+ct) + c\alpha_0'(x+ct), \\ -\alpha_t(x,t) + c\alpha_x(x,t) &= -\beta_0(x-ct) + c\alpha_0'(x-ct). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_x(x,t) &= \frac{1}{2}\left(\alpha_0'(x+ct) + \alpha_0'(x-ct)\right) + \frac{1}{2c}\left(\beta_0(x+ct) - \beta_0(x-ct)\right), \\ \alpha_t(x,t) &= \frac{1}{2}\left(\beta_0(x+ct) + \beta_0(x-ct)\right) + \frac{c}{2}\left(\alpha_0'(x+ct) - \alpha_0'(x-ct)\right). \end{aligned}$$

To find α, we can integrate the last equation in t,

$$\alpha(x,t) = \frac{1}{2}\left(\alpha_0(x+ct) + \alpha_0(x-ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \beta_0(y)\,dy,$$

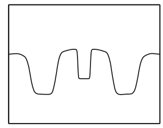
after a change of variables in the integral involving β_0. This is the famous d'Alembert formula for the solution of the linear wave equation in one dimension. $\diamond$

Next, we discuss the notion of entropy solutions. The meaning of an entropy solution to an equation written in characteristic variables is that for some convex function $\hat{\eta}(u)$, the entropy solution should satisfy

$$\hat{\eta}(u)_t + \hat{q}(u)_x \le 0,$$

in the weak sense. Here the entropy flux $\hat{q}$ should satisfy

$$\nabla_w \hat{q}(u) = \nabla_u \hat{\eta}(u) \Lambda, \quad \text{i.e.,} \quad \frac{\partial \hat{q}}{\partial u_i} = \lambda_i \frac{\partial \hat{\eta}}{\partial u_i} \text{ for } i = 1, \dots, n.$$



An entropy solution to (1.63) is the limit (if such a limit exists) of the parabolic reqularization

$$u_t^\varepsilon + A u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon$$

as $\varepsilon \to 0$. To check whether we have a convex entropy $\eta\colon \mathbb{R}^n \to \mathbb{R}$, we take the inner product of the above with $\nabla\eta(u^\varepsilon)$ to get

$$\eta\left(u^\varepsilon\right)_t + \nabla\eta\left(u^\varepsilon\right) \cdot A\, u_x^\varepsilon \leq \varepsilon \left(\nabla\eta\left(u^\varepsilon\right) \cdot u_x^\varepsilon\right)_x ,$$

by the convexity of η. Observe that the convexity is used to get rid of a term containing $(u_x^\varepsilon)^2$, which may not be wellbehaved (for nonlinear equations) in the limit $\varepsilon \to 0$, and we obtain an inequality rather than an equality. We want to write the second term on the left as the x derivative of some function $q(u^\varepsilon)$. Using

$$q\left(u^\varepsilon\right)_x = \nabla q\left(u^\varepsilon\right) \cdot u_x^\varepsilon,$$

we see that if this is so, then

$$\frac{\partial q}{\partial u_j} = \sum_i a_{ij} \frac{\partial \eta}{\partial u_i} \quad \text{for } j = 1, \ldots, n. \tag{1.66}$$

This is n equations in the two unknowns η and q. Thus we cannot expect any solution if $n > 2$. The right-hand side of (1.66) is given, and hence we are looking for a potential q with a given gradient. This problem has a solution if

$$\frac{\partial^2 q}{\partial u_k \partial u_j} = \frac{\partial^2 q}{\partial u_j \partial u_k},$$

or

$$\sum_i a_{ik} \frac{\partial^2 \eta}{\partial u_i \partial u_j} = \sum_i a_{ij} \frac{\partial^2 \eta}{\partial u_i \partial u_k} \quad \text{for } 1 \leq j, k \leq n.$$

If we wish to find an entropy flux for the entropy $\eta(u) = |u|^2/2$, note that

$$\frac{\partial^2 \eta}{\partial u_i \partial u_k} = \delta_{ik}.$$

Thus we can find an entropy flux if $a_{jk} = a_{kj}$ for $1 \leq j, k \leq n$; in other words, A must be symmetric. In this case the entropy flux q reads

$$q(u) = \sum_{i,j} a_{ij} u_i u_j - \frac{1}{2} \sum_i a_{ii} u_i^2.$$

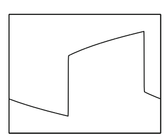
Hence, an entropy (using the entropy $\eta(u) = |u|^2/2$) solution satisfies

$$|u|_t^2 + q(u)_x \leq 0 \quad \text{weakly.}$$

This means that

$$\|u(\,\cdot\,, t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})} ,$$

and thus there is at most one entropy solution to (1.63) if A is symmetric.



The Riemann Problem for Linear Systems

Returning to the general case, recall that the solution $u = u(x,t)$ to (1.63) is given by (1.65), namely

$$u(x,t) = \sum_{i=1}^{n} [l_i \cdot u_0(x - \lambda_i t)]\, r_i. \tag{1.67}$$

Now we shall look at a type of initial value problem where u_0 is given by two constant values, namely

$$u_0(x) = \begin{cases} u_{\text{left}} & x < 0, \\ u_{\text{right}} & x \geq 0, \end{cases} \tag{1.68}$$

where u_{left} and u_{right} are two constant vectors. This type of initial value problem is called a *Riemann problem*, (cf. (1.28)) which will a problem of considerable interest throughout the book.

For a single equation ($n = 1$), the weak solution to this Riemann problem reads

$$u(x,t) = u_0(x - \lambda_1 t) = \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_{\text{right}} & x \geq \lambda_1 t. \end{cases}$$

Note that u is not continuous. Nevertheless, it is the unique entropy solution in the sense of Definition 1.10 to (1.63) with initial data (1.68) (see Exercise 1.4).

For two equations ($n = 2$), we write

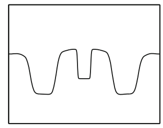
$$u_{\text{left}} = \sum_{i=1}^{2} [l_i \cdot u_{\text{left}}]\, r_i, \qquad u_{\text{right}} = \sum_{i=1}^{2} \left[l_i \cdot u_{\text{right}}\right] r_i.$$

We can find the solution of each component separately. Namely, using (1.67) for initial data (1.68), we obtain

$$[l_1 \cdot u(x,t)] = \begin{cases} l_1 \cdot u_{\text{left}} & x < \lambda_1 t, \\ l_1 \cdot u_{\text{right}} & x \geq \lambda_1 t, \end{cases} \quad [l_2 \cdot u(x,t)] = \begin{cases} l_2 \cdot u_{\text{left}} & x < \lambda_2 t, \\ l_2 \cdot u_{\text{right}} & x \geq \lambda_2 t. \end{cases}$$

Combining these we see that

$$\begin{aligned} u(x,t) &= [l_1 \cdot u(x,t)]\, r_1 + [l_2 \cdot u(x,t)]\, r_2 \\ &= \begin{cases} [l_1 \cdot u_{\text{left}}]\, r_1 + [l_2 \cdot u_{\text{left}}]\, r_2 & x < \lambda_1 t, \\ \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2 & t\lambda_1 < x \leq t\lambda_2, \\ \left[l_1 \cdot u_{\text{right}}\right] r_1 + \left[l_2 \cdot u_{\text{right}}\right] r_2 & x \geq \lambda_1 t, \end{cases} \\ &= \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_{\text{middle}} & t\lambda_1 < x \leq t\lambda_2, \\ u_{\text{right}} & x \geq \lambda_1 t, \end{cases} \end{aligned}$$



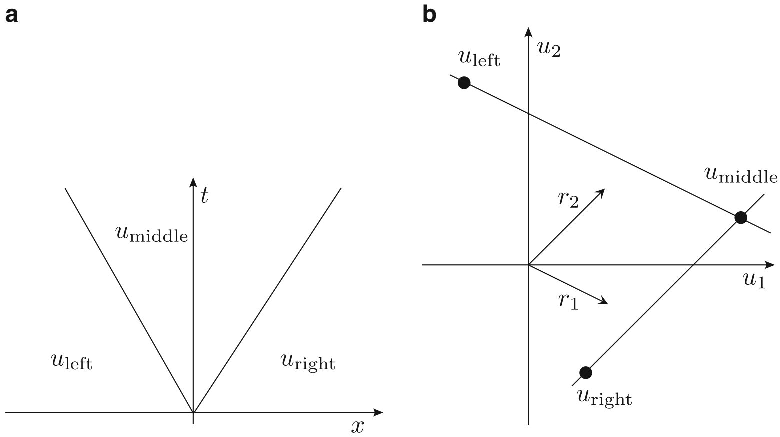


Fig. 1.7 The solution of the Riemann problem. **a** In (x,t)-space. **b** In phase space

with $u_{\text{middle}} = \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2$. Observe the structure of the different states:

$$\begin{aligned} u_{\text{left}} &= [l_1 \cdot u_{\text{left}}]\, r_1 + [l_2 \cdot u_{\text{left}}]\, r_2, \\ u_{\text{middle}} &= \left[l_1 \cdot u_{\text{right}}\right] r_1 + [l_2 \cdot u_{\text{left}}]\, r_2, \\ u_{\text{right}} &= \left[l_1 \cdot u_{\text{right}}\right] r_1 + \left[l_2 \cdot u_{\text{right}}\right] r_2. \end{aligned}$$

We can also view the solution in *phase space*, that is, in the (u_1, u_2)-plane. We see that for every u_{left} and u_{right}, we have the solution $u(x,t) = u_{\text{left}}$ for $x < \lambda_1 t$ and $u(x,t) = u_{\text{right}}$ for $x \geq \lambda_2 t$. In the middle, $u(x,t) = u_{\text{middle}}$ for $\lambda_1 t \leq x < \lambda_2 t$. The middle value u_{middle} is on the intersection of the line through u_{left} parallel to r_1 and the line through u_{right} parallel to r_2. See Fig. 1.7. In the general, nonlinear, case, the straight lines connecting u_{left}, u_m, and u_{right} will be replaced by arcs, not necessarily straight. However, the same structure prevails, at least locally.

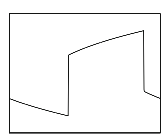
Now we can find the solution to the Riemann problem for any n, namely

$$u(x,t) = \begin{cases} u_{\text{left}} & x < \lambda_1 t, \\ u_i & \lambda_i t \leq x < \lambda_{i+1} t, \quad i = 1, \ldots, n-1, \\ u_{\text{right}} & x \geq \lambda_n t, \end{cases}$$

where

$$u_i = \sum_{j=1}^{i} \left[l_j \cdot u_{\text{right}}\right] r_j + \sum_{j=i+1}^{n} \left[l_j \cdot u_{\text{left}}\right] r_j.$$

Observe that this solution can also be viewed in phase space as the path from $u_0 = u_{\text{left}}$ to $u_n = u_{\text{right}}$ obtained by going from u_{i-1} to u_i on a line parallel to r_i for $i = 1, \ldots, n$. This viewpoint will be important when we consider nonlinear equations, where the straight lines will be replaced by arcs. Locally, the structure will remain unaltered.



Numerics for Linear Systems with Constant Coefficients

If $\lambda_i > 0$, then we know that the scheme

$$D_+^t w_{i,j}^m + \lambda_i D_- w_{i,j}^m = 0$$

will produce a sequence of functions $\{w_{i,\Delta x}\}$ that converges to the unique entropy solution of

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0.$$

Similarly, if $\lambda_i < 0$, the scheme

$$D_+^t w_{i,j}^m + \lambda_i D_+ w_{i,j}^m = 0$$

will give a convergent sequence. Both of these schemes will be convergent only if $\Delta t \leq \Delta x \, |\lambda_i|$, which is the CFL condition. In eigenvector coordinates, with

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \qquad w_j^m \approx w(j\Delta x, m\Delta t),$$

the resulting scheme for w reads

$$D_+^t w_j^m + \Lambda_+ D_- w_j^m + \Lambda_- D_+ w_j^n = 0, \tag{1.69}$$

where

$$\Lambda_- = \begin{pmatrix} \lambda_1 \wedge 0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \wedge 0 \end{pmatrix} \text{ and } \Lambda_+ = \begin{pmatrix} \lambda_1 \vee 0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \vee 0 \end{pmatrix},$$

and we have introduced the notation

$$a \vee b = \max\{a, b\} \text{ and } a \wedge b = \min\{a, b\}.$$

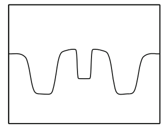
Observe that $\Lambda = \Lambda_+ + \Lambda_-$. If the CFL condition

$$\frac{\Delta t}{\Delta x} \max_i |\lambda_i| = \frac{\Delta t}{\Delta x} \max\{|\lambda_1|, |\lambda_n|\} \leq 1$$

holds, then the scheme (1.69) will produce a convergent sequence, and the limit w will be the unique entropy solution to

$$w_t + \Lambda w_x = 0. \tag{1.70}$$

By defining $u = Rw$, we obtain a solution of (1.63).



We can also make the same transformation on the discrete level. Multiplying (1.69) by L from the left and using that $u = Rw$ yields

$$D_+^t u_j^m + A_+ D_- u_j^m + A_- D_+ u_j^m = 0, \tag{1.71}$$

where

$$A_\pm = R\Lambda_\pm L,$$

and this finite difference scheme will converge directly to u.

1.2 Notes

Never any knowledge was delivered in the same order it was invented.[6]
— Sir Francis Bacon (1561–1626)

The simplest nontrivial conservation law, the inviscid Burgers equation, has been extensively analyzed. Burgers introduced the "nonlinear diffusion equation"

$$u_t + \frac{1}{2}(u^2)_x = u_{xx}, \tag{1.72}$$

which is currently called (the viscous) Burgers's equation, in 1940 [37] (see also [38]) as a model of turbulence. Burgers's equation is linearized, and thereby solved, by the Cole–Hopf transformation [46, 98]. Both the equation and the Cole–Hopf transformation were, however, known already in 1906; see Forsyth [66, p. 100]. See also Bateman [14]. The early history of hyperbolic conservation laws is presented in [56, pp. XV–XXX]. A source of some of the early papers is [104].

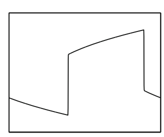
The most common elementary example of application of scalar conservation laws is the model of traffic flow called "traffic hydrodynamics" that was introduced independently by Lighthill and Whitham [134] and Richards [155]. A modern treatment can be found in Haberman [81]. Example 1.6 presents some of the fundamentals, and serves as a nontechnical introduction to the lack of uniqueness for weak solutions. Extensions to traffic flow on networks exist; see [94] and [68].

The jump condition, or the Rankine–Hugoniot condition, was derived heuristically from the conservation principle independently by Rankine in 1870 [152] and Hugoniot in 1886 [101–103]. Our presentation of the Rankine–Hugoniot condition is taken from Smoller [169].

The notion of "Riemann problem" is fundamental in the theory of conservation laws. It was introduced by Riemann in 1859 [156, 157] in the context of gas dynamics. He studied the situation in which one initially has two gases with different (constant) pressures and densities separated by a thin membrane in a one-dimensional cylindrical tube. See [97] and [56, pp. XV–XXX] for a historical discussion.

The final section of this chapter contains a detailed description of the one-dimensional linear case, both in the scalar case and in the case of systems. This allows us to introduce some of the methods in a simpler case. Here existence of

[6] in *Valerius Terminus*: *Of the Interpretation of Nature*, c. 1603.



solutions is shown using appropriate finite difference schemes, in contrast to the front-tracking method used in the text proper.

There are by now several books on various aspects of hyperbolic conservation laws, starting with the classical book by Courant and Friedrichs [51]. Nice treatments with emphasis on the mathematical theory can be found in books by Lax [126, 127], Chorin and Marsden [42], Roždestvenskiĭ and Janenko [164], Smoller [169], Rhee, Aris, and Amundson [153, 154], Málek et al. [141], Hörmander [99], Liu [137], Serre [167, 168], Benzoni-Gavage and Serre [15], Bressan [24, 27], Dafermos [56], Lu [139], LeFloch [129], Perthame [150], Zheng [192]. The books by Bouchut [19], Godlewski and Raviart [78, 79], LeVeque [130, 131], Kröner [116], Toro [180], Thomas [179], and Trangenstein [183] focus more on the numerical theory.

1.3 Exercises

1.1 Determine characteristics for the following quasilinear equations:

$$\begin{aligned} u_t + \sin(x)u_x &= u, \\ \sin(t)u_t + \cos(x)u_x &= 0, \\ u_t + \sin(u)u_x &= u, \\ \sin(u)u_t + \cos(u)u_x &= 0. \end{aligned}$$

1.2 Use characteristics to solve the following initial value problems:

(a) $uu_x + xu_y = 0$, $u(0, s) = 2s$ for $s > 0$.
(b) $e^y u_x + uu_y + u^2 = 0$, $u(x, 0) = 1/x$ for $x > 0$.
(c) $xu_y - yu_x = u$, $u(x, 0) = h(x)$ for $x > 0$.
(d) $(x + 1)^2 u_x + (y - 1)^2 u_y = (x + y)u$, $u(x, 0) = -1 - x$.
(e) $u_x + 2xu_y = x + xu$, $u(1, y) = e^y - 1$.
(f) $u_x + 2xu_y = x + xu$, $u(0, y) = y^2 - 1$.
(g) $xuu_x + u_y = 2y$, $u(x, 0) = x$.

1.3 (a) Use characteristics to show that

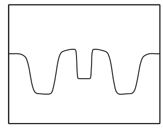
$$u_t + au_x = f(x, t), \quad u|_{t=0} = u_0,$$

with a a constant, has solution

$$u(x, t) = u_0(x - at) + \int_0^t f(x - a(t - s), s)\, ds.$$

(b) Show that

$$u\left(\xi(t; x_0), t\right) = u_0(x_0) + \int_0^t f(\xi(s; x_0), s)\, ds$$



holds if u is the solution of

$$u_t + a(x,t)u_x = f(x,t), \quad u|_{t=0} = u_0, \tag{1.73}$$

where ξ satisfies

$$\frac{d}{dt}\xi(t;x_0) = a(\xi(t;x_0),t), \;\; \xi(0;x_0) = x_0.$$

(c) Show that

$$u(x,t) = u_0\left(\zeta(t;x)\right) + \int_0^t f\left(\zeta(\tau;x), t-\tau\right)\,d\tau$$

holds if u is the solution of (1.73) and

$$\frac{d}{d\tau}\zeta(\tau;x) = -a(\zeta(\tau;x), t-\tau), \;\; \zeta(0;x) = x.$$

1.4 Show that

$$u(x,t) = \begin{cases} u_{\text{left}} & x < at, \\ u_{\text{right}} & x \geq at, \end{cases}$$

is the entropy solution in the sense of Definition 1.10 for the equation $u_t + au_x = 0$ (where a is constant) and $u|_{t=0}(x) = u_{\text{left}}\chi_{x<0} + u_{\text{right}}\chi_{x\geq 0}$.

1.5 Find the shock condition (i.e., the Rankine–Hugoniot condition) for one-dimensional systems, i.e., the unknown u is a vector $u = (u_1, \ldots, u_n)$ for some $n > 1$, and also $f(u) = (f_1(u), \ldots, f_n(u))$.

1.6 Consider a scalar conservation law in two space dimensions,

$$u_t + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0,$$

where the flux functions f and g are continuously differentiable. Now the unknown u is a function of x, y, and t. Determine the Rankine–Hugoniot condition across a jump discontinuity in u, assuming that u jumps across a regular surface in (x, y, t). Try to generalize your answer to a conservation law in n space dimensions.

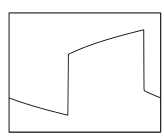
1.7 We shall consider a linearization of Burgers's equation. Let

$$u_0(x) = \begin{cases} 1 & \text{for } x < -1, \\ -x & \text{for } -1 \leq x \leq 1, \\ -1 & \text{for } 1 < x. \end{cases}$$

(a) First determine the maximum time that the solution of the initial value problem

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = u_0(x),$$

will remain continuous. Find the solution for t less than this time.



(b) Then find the solution v of the linearized problem

$$v_t + u_0(x)v_x = 0, \quad v(x,0) = u_0(x).$$

Determine the solution also in the case $v(x,0) = u_0(\alpha x)$, where α is nonnegative.

(c) Next, we shall determine a procedure for finding u by solving a sequence of linearized equations. Fix $n \in \mathbb{N}$. For t in the interval $(m/n, (m+1)/n]$ and $m \geq 0$, let v_n solve

$$(v_n)_t + v_n\,(x, m/n)\,(v_n)_x = 0,$$

and set $v_n(x,0) = u_0(x)$. Then show that

$$v_n\left(x, \frac{m}{n}\right) = u_0\left(\alpha_{m,n} x\right)$$

and find a recurrence relation (in m) satisfied by $\alpha_{m,n}$.

(d) Assume that

$$\lim_{n\to\infty} \alpha_{m,n} = \bar{\alpha}(t),$$

for some continuously differentiable $\bar{\alpha}(t)$, where $t = m/n < 1$. Show that $\bar{\alpha}(t) = 1/(1-t)$, and thus $v_n(x) \to u(x)$ for $t < 1$. What happens for $t \geq 1$?

1.8 (a) Solve the initial value problem for Burgers's equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{1.74}$$

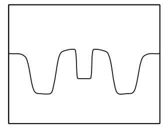
(b) Then find the solution where the initial data are

$$u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

(c) If we multiply Burgers's equation by u, we formally find that u satisfies

$$\frac{1}{2}\left(u^2\right)_t + \frac{1}{3}\left(u^3\right)_x = 0, \quad u(x,0) = u_0(x). \tag{1.75}$$

Are the solutions to (1.74) you found in parts **a** and **b** weak solutions to (1.75)? If not, then find the corresponding weak solutions to (1.75). *Warning: This shows that manipulations valid for smooth solutions are not necessarily so for weak solutions.*



1.9 ([169, p. 250]) Show that

$$u(x,t) = \begin{cases} 1 & \text{for } x \le (1-\alpha)t/2, \\ -\alpha & \text{for } (1-\alpha)t/2 < x \le 0, \\ \alpha & \text{for } 0 < x \le (\alpha-1)t/2, \\ -1 & \text{for } x \ge (\alpha-1)t/2 \end{cases}$$

is a weak solution of

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad u(x,0) = \begin{cases} 1 & \text{for } x \le 0, \\ -1 & \text{for } x > 0, \end{cases}$$

for all $\alpha \ge 1$. *Warning: Thus we see that weak solutions are not necessarily unique.*

1.10 We outline a proof of some *Gronwall inequalities*.

(a) Assume that u satisfies

$$u'(t) \le \gamma u(t).$$

Show that $u(t) \le e^{\gamma t} u(0)$.

(b) Assume now that u satisfies

$$u'(t) \le C(1 + u(t)).$$

Show that $u(t) \le e^{Ct}(1 + u(0)) - 1$.

(c) Assume that u satisfies

$$u'(t) \le c(t)u(t) + d(t),$$

for $0 \le t \le T$, where $c(t)$ and $d(t)$ are in $L^1([0,T])$. Show that

$$u(t) \le u(0) + \int_0^t d(s) \exp\Big(\int_s^t c(\tilde{s})\, d\tilde{s}\Big)\, ds$$

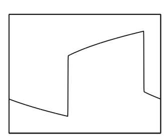
for $t \le T$.

(d) Assume that u is in $L^1([0,T])$ and that for $t \in [0,T]$,

$$u(t) \le C_1 \int_0^t u(s)\, ds + C_2.$$

Show that

$$u(t) \le C_2 e^{C_1 t}.$$



1.11 Consider the semidiscrete difference scheme (1.37). The goal of this exercise is to prove that a unique solution exists for all $t > 0$.

(a) Let $\eta(u)$ be a smooth function. Show that

$$D_-\eta(u_j) = \eta'(u_j)D_-u_j - \frac{\Delta x}{2}\eta''(u_{j-1/2})\left(D_-u_j\right)^2,$$

where $u_{j-1/2}$ is some value between u_j and u_{j-1}.

(b) Assume now that $\eta'' \geq 0$. Show that

$$\frac{d}{dt}\sum_j \eta(u_j) \leq \sup_j \left|D_+a_j\right| \sum_j \eta(u_j).$$

Note that in particular, this holds for $\eta(u) = u^2$.

(c) Show that for fixed Δx, and $u \in l_2$, the function $F : l_2 \to l_2$ defined by $F_j(u) = aD_-u_j$ is Lipschitz continuous.
If we view $u(t)_j = u_j(t)$, then the difference scheme (1.37) reads $u' = -F(u)$. Since we know that the solution is bounded in l_2, we cannot have a blowup, and the solution exists for all time.

1.12 Consider the fully discrete scheme (1.48). Show that

$$\sum_j \eta(u_j^{n+1}) \leq \sum_j \eta(u_j^n) + \Delta t \sum_j \eta(u_j^n) D_+a_j^n.$$

Use this to show that

$$\Delta x \sum_j \eta(u_j^n) \leq e^{Ct_n} \left\|\eta(u_0)\right\|_{L^1(\mathbb{R})},$$

where C is a bound on a_x.

1.13 The linear *variational* wave equation reads

$$\begin{aligned} \alpha_{tt} + c(x)\left(c(x)\alpha_x\right)_x = 0, \qquad t > 0,\ x \in \mathbb{R}, \\ \alpha(x,0) = \alpha_0(x),\ \ \alpha_t(x,0) = \beta_0(x), \end{aligned} \tag{1.76}$$

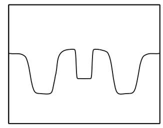
where c is a positive Lipschitz continuous function, and α_0 and β_0 are suitable initial data.

(a) Set $u = \alpha_t + c\alpha_x$ and $v = \alpha_t - c\alpha_x$. Find the equations satisfied by u and v.

(b) Find the solutions of these equations in terms of the characteristics.

(c) Formulate a difference scheme to approximate $u(x,t)$ and $v(x,t)$, and give suitable conditions on your scheme and the initial data (here you have a large choice) that guarantee the convergence of the scheme.

(d) Test your scheme with $c(x) = \sqrt{1 + \sin^2(x)}$, $\alpha_0(x) = \max\{0, 1 - |x|\}$, $\beta_0 = 0$, and periodic boundary conditions in $[-\pi, \pi]$.



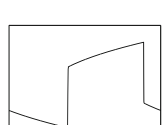
1.14 Consider the transport equation

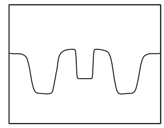
$$u_t + a(x,t)u_x = 0, \quad t > 0, \ x \in \mathbb{R},$$
$$u(x,0) = u_0(x).$$

We know that the (unique) solution can be written in terms of the backward characteristics, $u(x,t) = u_0(\zeta(t;x))$, where ζ solves

$$\frac{d}{d\tau}\zeta(\tau;x) = -a(\zeta(\tau;x), t-\tau), \ \zeta(0;x) = x.$$

We want to use this numerically. Write a routine that given t, u_0, and a, calculates an approximation to $u(x,t)$ using a numerical method to find $\zeta(t;x)$. Test the routine for the initial function $u_0(x) = \sin(x)$, and for a given by (1.35) and (1.36), as well as for the example $a(x,t) = x^2 \sin(t)$.





Chapter 2

Scalar Conservation Laws

It is a capital mistake to theorise before one has data.
Insensibly one begins to twist facts to suit theories,
instead of theories to suit facts.
— Sherlock Holmes, A Scandal in Bohemia (1891)

In this chapter we consider the Cauchy problem for a scalar conservation law. Our goal is to show that subject to certain conditions, there exists a unique solution to the general initial value problem. Our method will be completely constructive, and we shall exhibit a procedure by which this solution can be constructed. This procedure is, of course, front tracking. The basic ingredient in the front-tracking algorithm is the solution of the Riemann problem.

Already in the example on traffic flow, we observed that conservation laws may have several weak solutions, and that some principle is needed to pick out the correct ones. The problem of lack of uniqueness for weak solutions is intrinsic in the theory of conservation laws. There are by now several different approaches to this problem, and they are commonly referred to as "entropy conditions."

Thus the solution of Riemann problems requires some mechanism to choose one of possibly several weak solutions. Therefore, before we turn to front tracking, we will discuss entropy conditions.

2.1 Entropy Conditions

We study the conservation law[1]

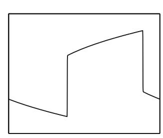
$$u_t + f(u)_x = 0, \tag{2.1}$$

whose solutions $u = u(x,t)$ are to be understood in the distributional sense; see (1.19). We will not state any continuity properties of f, but tacitly assume that f is sufficiently smooth for all subsequent formulas to make sense.

One of the most common entropy conditions is so-called viscous regularization, where the scalar conservation law $u_t + f(u)_x = 0$ is replaced by $u_t + f(u)_x = \epsilon u_{xx}$ with ϵ positive. The idea is that the physical problem has some diffusion, and that the conservation law represents a limit model when the diffusion is small. Based

[1] The analysis up to and including (2.7) could have been carried out for systems on the line as well.

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_2



on this, one is looking for solutions of the conservation law that are limits of the regularized equation when $\epsilon \to 0$.

Therefore, we are interested in the *viscous regularization* of the conservation law (2.1),

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \tag{2.2}$$

as $\varepsilon \to 0$. In order for this equation to be well posed, ε must be nonnegative. Equations such as (2.2) are called viscous, because the right-hand side u^ε_{xx} models the effect of viscosity or diffusion. We then demand that the distributional solutions of (2.1) be limits of solutions of the more fundamental equation (2.2) as the viscous term disappears.

This has some interesting consequences. Assume that (2.1) has a solution consisting of constant states on each side of a discontinuity moving with a speed s, i.e.,

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st. \end{cases} \tag{2.3}$$

We say that $u(x,t)$ satisfies a *traveling wave entropy condition* if $u(x,t)$ is the pointwise limit almost everywhere of some $u^\varepsilon(x,t) = U((x-st)/\varepsilon)$ as $\varepsilon \to 0$, where u^ε solves (2.2) in the classical sense.

Inserting $U((x-st)/\varepsilon)$ into (2.2), we obtain

$$-s\dot{U} + \frac{df(U)}{d\xi} = \ddot{U}. \tag{2.4}$$

Here $U = U(\xi)$, $\xi = (x-st)/\varepsilon$, and $\dot{U}$ denotes the derivative of U with respect to ξ. This equation can be integrated once, yielding

$$\dot{U} = -sU + f(U) + A, \tag{2.5}$$

where A is a constant of integration. We see that as $\varepsilon \to 0$, ξ tends to plus or minus infinity, depending on whether $x - st$ is positive or negative.

If u should be the limit of u^ε, we must have that

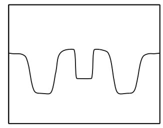
$$\lim_{\varepsilon\to 0} u^\varepsilon = \lim_{\varepsilon\to 0} U(\xi) = \begin{Bmatrix} u_l & \text{for } x < st, \\ u_r & \text{for } x > st, \end{Bmatrix} = \begin{cases} \lim_{\xi\to-\infty} U(\xi), \\ \lim_{\xi\to+\infty} U(\xi). \end{cases}$$

From the differential equation (2.5) we see that $\lim_{\xi\to\pm\infty} \dot{U}(\xi)$ exists and equals $-su_{l,r} + f(u_{l,r}) + A$. We get a contradiction unless this limit vanishes, and therefore,

$$\lim_{\xi\to\pm\infty} \dot{U}(\xi) = 0.$$

Inserting this into (2.5), we obtain (recall that $f_r = f(u_r)$, etc.)

$$A = su_l - f_l = su_r - f_r, \tag{2.6}$$



which again gives us the Rankine–Hugoniot condition

$$s\,(u_r - u_l) = f_r - f_l.$$

Summing up, the traveling wave U must satisfy the following boundary value problem:

$$\dot{U} = -s\,(U - u_l) + (f\,(U) - f_l)\,, \quad U(\pm\infty) = \begin{cases} u_r, \\ u_l. \end{cases} \tag{2.7}$$

Using the Rankine–Hugoniot condition, we see that both u_l and u_r are fixed points for this equation. What we want is an orbit of (2.7) going from u_l to u_r. If the triplet (s, u_l, u_r) has such an orbit, we say that the discontinuous solution (2.3) satisfies a traveling wave entropy condition, or that the discontinuity has a *viscous profile*. (For the analysis so far in this section we were not restricted to the scalar case, and could as well have worked with the case of systems in which u is a vector in $\mathbb{R}^n$ and $f(u)$ is some function $\mathbb{R}^n \to \mathbb{R}^n$.)

From now on we say that an isolated discontinuity satisfies the traveling wave entropy condition if (2.7) holds locally across the discontinuity.

Let us examine this in more detail. First we assume that $u_l < u_r$. Observe that $\dot{U}$ can never be zero. Assuming otherwise, namely that $\dot{U}(\xi_0) = 0$ for some ξ_0, would result in the constant $U(\xi_0)$ being the unique solution, which contradicts that $U(-\infty) = u_l < u_r = U(\infty)$. Thus $\dot{U}(\xi) > 0$ for all ξ, and hence

$$f_l + s\,(u - u_l) < f(u), \tag{2.8}$$

for all $u \in (u_l, u_r)$. Recall that according to the Rankine–Hugoniot conditions, $s = (f_l - f_r)\,/(u_l - u_r)$, which means that the graph of $f(u)$ has to lie *above* the straight line segment joining the points (u_l, f_l) and (u_r, f_r). On the other hand, if the graph of $f(u)$ is above the straight line, then (2.8) is satisfied, and we can find a solution of (2.7). Similarly, if $u_l > u_r$, $\dot{U}$ must be negative in the whole interval (u_r, u_l). Consequently, the graph of $f(u)$ must be *below* the straight line.

By combining the two cases we conclude that the viscous profile or *traveling wave entropy condition* is equivalent to

$$s\,|k - u_l| < \operatorname{sign}(k - u_l)\,(f(k) - f\,(u_l))\,, \tag{2.9}$$

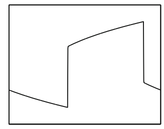
for all k strictly between u_l and u_r when the Rankine–Hugoniot condition $s\,[\![u]\!] = [\![f]\!]$ holds. Note that an identical inequality holds with u_l replaced by u_r. This is equivalent to the *Oleĭnik entropy condition*

$$\frac{f(k) - f_r}{k - u_r} < s < \frac{f(k) - f_l}{k - u_l}, \tag{2.10}$$

to be valid for all k strictly between u_l and u_r.

Furthermore, we claim that the traveling wave entropy condition is equivalent to the condition that

$$s\,[\![\,|u - k|\,]\!] \geq [\![\,\operatorname{sign}(u - k)\,(f(u) - f(k))\,]\!] \tag{2.11}$$



(recall that $[\![a]\!] = a_r - a_l$ for every quantity a) is satisfied for *all* k. To show this, we first assume that (2.11) holds. Consider first the case $u_l < u_r$, and choose k to be between u_l and u_r. We obtain from (2.11) that

$$s\left((u_r - k) + (u_l - k)\right) \geq \left(f_r - f(k)\right) + \left(f_l - f(k)\right),$$

or

$$f(k) \geq \bar{f} - s(\bar{u} - k). \tag{2.12}$$

Here, $\bar{f}$ denotes $(f_l + f_r)/2$, and similarly, $\bar{u} = (u_l + u_r)/2$. The right-hand side is a straight line connecting (u_l, f_l) and (u_r, f_r) (here we have to use the Rankine–Hugoniot condition). Thus, the graph of $f(u)$ must lie above the straight line segment between (u_l, f_l) and (u_r, f_r). Similarly, if $u_r < u_l$, we find that the graph has to lie below the line segment. Hence (2.11) implies (2.9).

Assume next that (2.9) holds across an isolated discontinuity, with limits u_l and u_r that are such that the Rankine–Hugoniot equality holds. Then

$$s\,[\![\,|u - k|\,]\!] = [\![\,\operatorname{sign}(u - k)\left(f(u) - f(k)\right)]\!] \tag{2.13}$$

for every constant k *not* between u_l and u_r. For constants k between u_l and u_r, we have seen that if $f(k) \geq \bar{f} - s(\bar{u} - k)$ for $u_l < u_r$, i.e., the viscous profile entropy condition holds, then

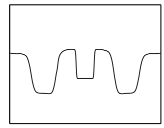
$$s\,[\![\,|u - k|\,]\!] \geq [\![\,\operatorname{sign}(u - k)\left(f(u) - f(k)\right)]\!]\,. \tag{2.14}$$

In the same way one can show that (2.14) holds whenever $u_l > u_r$. Thus we conclude that (2.11) will be satisfied.

The inequality (2.9) motivates another entropy condition, the *Kružkov entropy condition*. This condition is often more convenient to work with, since it combines the definition of a weak solution with that of the entropy condition.

Choose a smooth convex function $\eta = \eta(u)$ and a nonnegative test function ϕ in $C_0^\infty(\mathbb{R} \times (0, \infty))$. (Such a test function will be supported away from the x-axis, and thus we get no contribution from the initial data.) Then we obtain

$$\begin{aligned}
0 &= \iint \left(u_t^\varepsilon + f(u^\varepsilon)_x - \epsilon u_{xx}^\varepsilon\right) \eta'(u^\varepsilon)\phi \, dx\, dt \\
&= \iint \eta(u^\varepsilon)_t \phi \, dx\, dt + \iint q'(u^\varepsilon) u_x^\varepsilon \phi \, dx\, dt \\
&\quad - \varepsilon \iint \left(\eta(u^\varepsilon)_{xx} - \eta''(u^\varepsilon)(u_x^\varepsilon)^2\right) \phi \, dx\, dt \\
&= -\iint \eta(u^\varepsilon)\phi_t \, dx\, dt - \iint q(u^\varepsilon)\phi_x \, dx\, dt \\
&\quad - \varepsilon \iint \eta(u^\varepsilon)\phi_{xx} \, dx\, dt + \varepsilon \iint \eta''(u^\varepsilon)(u_x^\varepsilon)^2 \phi \, dx\, dt \\
&\geq -\iint \left(\eta(u^\varepsilon)\phi_t + q(u^\varepsilon)\phi_x + \varepsilon \eta(u^\varepsilon)\phi_{xx}\right) dx\, dt,
\end{aligned} \tag{2.15}$$



where we first introduced q such that

$$q'(u) = f'(u)\eta'(u) \tag{2.16}$$

and subsequently used the convexity of η, i.e., $\eta'' \geq 0$, to remove the term $\varepsilon \iint \eta''(u^\varepsilon)(u_x^\varepsilon)^2\phi\, dx\, dt$. This term is problematic, since $(u_x^\varepsilon)^2$ in general will not be integrable in the limit as $\varepsilon \to 0$. If this is to hold as $\varepsilon \to 0$, we need

$$\iint \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt \geq 0, \tag{2.17}$$

and we say that the Kružkov entropy condition holds if (2.17) is valid for all convex functions η and all nonnegative test functions ϕ. However, we will soon see that we can simplify this further.

Consider now the case with

$$\eta(u) = \big((u-k)^2 + \delta^2\big)^{1/2}, \quad \delta > 0,$$

for some constant k. By taking $\delta \to 0$ we can extend the analysis to the case

$$\eta(u) = |u - k|\,. \tag{2.18}$$

In this case, we find that

$$q(u) = \mathrm{sign}\,(u-k)\,(f(u) - f(k))\,.$$

Remark 2.1 Consider a fixed bounded weak solution u and a nonnegative test function ϕ, and define the linear functional

$$\Lambda(\eta) = \iint \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt. \tag{2.19}$$

(The function q depends linearly on η; cf. (2.16).) Assume that the Kružkov entropy condition holds, that is, $\Lambda(\eta) \geq 0$ with η convex. Introduce

$$\eta_i(u) = \alpha_i\,|u - k_i|\,, \quad k_i \in \mathbb{R},\, \alpha_i \geq 0.$$

Clearly,

$$\Lambda\Big(\sum_i \eta_i\Big) \geq 0.$$

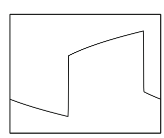
Since u is a weak solution, we have

$$\Lambda(\alpha u + \beta) = 0, \quad \alpha, \beta \in \mathbb{R},$$

and hence the convex piecewise linear function

$$\eta(u) = \alpha u + \beta + \sum_i \eta_i(u) \tag{2.20}$$

satisfies $\Lambda(\eta) \geq 0$. On the other hand, any convex piecewise linear function η can be written in the form (2.20). This can be proved by induction on the number of



breakpoints for η, where by breakpoints we mean those points where η' is discontinuous. The induction step goes as follows. Consider a breakpoint for η, which we without loss of generality can assume is at the origin. Near the origin we may write η as

$$\eta(u) = \begin{cases} \sigma_1 u & \text{for } u \leq 0, \\ \sigma_2 u & \text{for } u > 0, \end{cases}$$

for $|u|$ small. Since η is convex, $\sigma_1 < \sigma_2$. Then the function

$$\tilde{\eta}(u) = \eta(u) - \frac{1}{2}(\sigma_2 - \sigma_1)\,|u| - \frac{1}{2}(\sigma_1 + \sigma_2)u \tag{2.21}$$

is a convex piecewise linear function with one breakpoint fewer than η for which one can use the induction hypothesis. Hence we infer that $\Lambda(\eta) \geq 0$ for all convex, piecewise linear functions η. Consider now any convex function η. By sampling points, we can approximate η with convex, piecewise linear functions η_j such that $\eta_j \to \eta$ in L^∞. Thus we find that

$$\Lambda(\eta) \geq 0.$$

We conclude that if $\Lambda(\eta) \geq 0$ for the *Kružkov function* $\eta(u) = |u-k|$ for all $k \in \mathbb{R}$, then this inequality holds for all convex functions.

We say that a function is a *Kružkov entropy solution* to (2.1) if the inequality

$$\iint \left(|u-k|\,\phi_t + \operatorname{sign}(u-k)\,(f(u) - f(k))\,\phi_x\right)\,dx\,dt \geq 0 \tag{2.22}$$

holds for all constants $k \in \mathbb{R}$ and all nonnegative test functions ϕ in $C_0^\infty(\mathbb{R} \times (0,\infty))$.

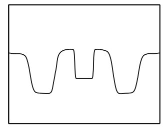
If we instead consider solutions on a time interval $[0,T]$, and thus use nonnegative test functions $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$, we find that the appropriate definition is that u is a Kružkov entropy solution on $\mathbb{R} \times [0,T]$ if

$$\begin{aligned} &\iint_0^T \left[\,|u-k|\,\phi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\,\phi_x\,\right]\,dx\,dt \\ &\quad - \int |u(x,T)-k|\,\phi(x,T)\,dx + \int |u_0(x)-k|\,\phi(x,0)\,dx \geq 0 \end{aligned} \tag{2.23}$$

holds for all $k \in \mathbb{R}$ and for all nonnegative test functions ϕ in $C_0^\infty(\mathbb{R} \times [0,T])$.

Next, let us analyze the consequences of definition (2.22). If we assume that u is bounded, and set $k \leq -\|u\|_\infty$, (2.22) gives

$$0 \leq \iint \left((u-k)\phi_t + (f(u)-f(k))\,\phi_x\right)\,dx\,dt = \iint \left(u\phi_t + f(u)\phi_x\right)\,dx\,dt.$$



Similarly, setting $k \geq \|u\|_\infty$ gives

$$0 \geq \iint \left(u\phi_t + f(u)\phi_x\right)\, dx\, dt.$$

These two inequalities now imply that

$$\iint \left(u\phi_t + f(u)\phi_x\right)\, dx\, dt = 0 \tag{2.24}$$

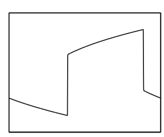
for all nonnegative ϕ. By considering test functions ϕ in $C_0^\infty(\mathbb{R} \times (0,\infty))$ of the form $\phi_+ - \phi_-$, with $\phi_\pm \in C_0^\infty(\mathbb{R} \times (0,\infty))$ nonnegative, which are dense C_0^∞, equation (2.24) implies the usual definition (1.19) of a weak solution. Thus, we find that a Kružkov entropy solution is also a weak solution. In particular, jump discontinuities satisfy the Rankine–Hugoniot condition.

We will now study the relationship between the Kružkov entropy condition and the traveling wave condition. First assume that u is a classical solution away from isolated jump discontinuities along piecewise smooth curves, and that it satisfies the Kružkov entropy condition (2.22). By applying to (2.22) the argument (cf. (1.21)) used to derive the Rankine–Hugoniot condition in a neighborhood of a jump discontinuity, we obtain precisely the inequality (2.11). Thus the traveling wave entropy condition holds.

On the other hand, consider the situation in which we have a smooth solution except for jump discontinuities on isolated curves where the traveling wave condition (2.11) holds. For smooth solutions we have directly that $\eta(u)_t + q(u)_x = 0$. For simplicity we assume that there is exactly one curve Γ where u has a jump discontinuity. We write $\mathbb{R} \times (0,\infty) = D_- \cup \Gamma \cup D_+$, where $D_\pm$ are on either side of the curve Γ. Thus (cf. (1.20)–(1.22))

$$\begin{aligned}
0 &= \Big(\iint_{D_-} + \iint_{D_+}\Big)\big(\eta(u)_t + q(u)_x\big)\phi\, dx\, dt \\
&= \Big(\iint_{D_-} + \iint_{D_+}\Big)\big((\eta(u)\phi)_t + (q(u)\phi)_x\big)\, dx\, dt \\
&\quad - \Big(\iint_{D_-} + \iint_{D_+}\Big)\big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt \\
&= \Big(\int_{\partial D_-} + \int_{\partial D_+}\Big)\phi\big(q(u), \eta(u)\big)\cdot n\, ds - \iint \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt \\
&= \int_\Gamma \phi\big(-[\![q]\!] + s\,[\![\eta]\!]\big) ds - \iint \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt \\
&\geq -\iint \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt
\end{aligned}$$

using (2.11). Here $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$ is the usual nonnegative test function. Thus the Kružkov entropy condition is satisfied.



Hence, for sufficiently regular solutions, these two entropy conditions are equivalent. We shall later see that Kružkov's entropy condition implies that for sufficiently regular initial data, the solution indeed posseses the necessary regularity; consequently, these two entropy conditions "pick" the same solutions. We will therefore in the following use whichever entropy condition is more convenient to work with.

2.2 The Riemann Problem

With my two algorithms one can solve all problems—without error, if God will!
— Al-Khwarizmi (c. 780–c. 850)

For conservation laws, the Riemann problem is the initial value problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0. \end{cases} \tag{2.25}$$

Assume temporarily that $f \in C^2$ with finitely many inflection points. We have seen examples of Riemann problems and their solutions in the previous chapter, in the context of traffic flow. Since both the equation and the initial data are invariant under the transformation $x \mapsto kx$ and $t \mapsto kt$, it is reasonable to look for solutions of the form $u = u(x,t) = w(x/t)$. We set $z = x/t$ and insert this into (2.25) to obtain

$$-\frac{x}{t^2}w' + \frac{1}{t}f'(w)w' = 0, \text{ or } z = f'(w). \tag{2.26}$$

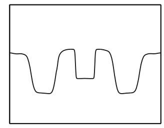
If f' is strictly monotone, we can simply invert this relation to obtain the solution $w = (f')^{-1}(z)$. In the general case we have to replace f' by a monotone function on the interval between u_l and u_r. In the example of traffic flow, we saw that it was important whether $u_l < u_r$ or vice versa. Assume first that $u_l < u_r$. Now we claim that the solution of (2.26) is given by

$$u(x,t) = w(z) = \begin{cases} u_l & \text{for } x \leq f_\smile'(u_l)t, \\ \left(f_\smile'\right)^{-1}(x/t) & \text{for } f_\smile'(u_l)t \leq x \leq f_\smile'(u_r)t, \\ u_r & \text{for } x \geq f_\smile'(u_r)t, \end{cases} \tag{2.27}$$

for $u_l < u_r$. Here $f_\smile$ denotes the *lower convex envelope* of f in the interval $[u_l, u_r]$, and $\left((f_\smile)'\right)^{-1}$, or, to be less pedantic, $(f_\smile')^{-1}$, denotes the inverse of its derivative. The lower convex envelope is defined to be the largest convex function that is less than or equal to f in the interval $[u_l, u_r]$, i.e.,

$$f_\smile(u) = \sup\left\{g(u) \mid g \leq f \text{ and } g \text{ convex on } [u_l, u_r]\right\}. \tag{2.28}$$

To picture the envelope of f, we can imagine the graph of f cut out from a board so that the lower boundary of the board has the shape of the graph. An elastic rubber band stretched from $(u_l, f(u_l))$ to $(u_r, f(u_r))$ will then follow the graph of $f_\smile$.



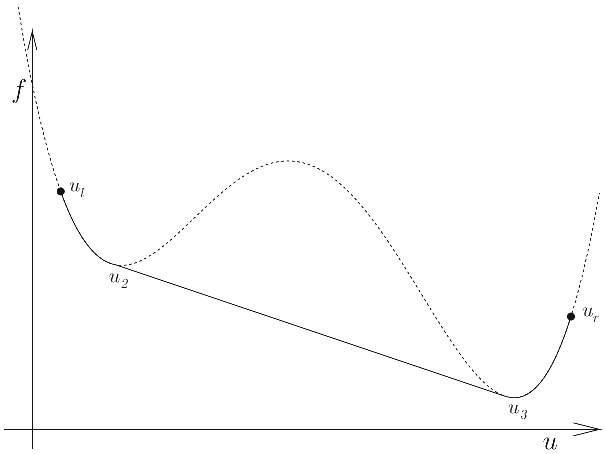


Fig. 2.1 In a series of figures we will illustrate the solution of an explicit Riemann problem. We start by giving the flux function f, two states u_l and u_r (with $u_l < u_r$), and the convex envelope of f relative to the interval $[u_l, u_r]$

Note that $f_\smile$ depends on the interval $[u_l, u_r]$, and thus is a highly nonlocal function of f.

Since $f_\smile'' \ge 0$, we have that $f_\smile'$ is nondecreasing, and hence we can form its inverse, denoted by $(f_\smile')^{-1}$, permitting jump discontinuities where $f_\smile'$ is constant. Hence formula (2.27) at least makes sense. In Fig. 2.1 we see a flux function and the envelope between two points u_l and u_r.

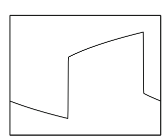
If $f \in C^2$ with finitely many inflection points, there will be a finite number of intervals with endpoints $u_l = u_1 < u_2 < \cdots < u_n = u_r$ such that $f_\smile = f$ on every other interval. That is, if $f_\smile(u) = f(u)$ for $u \in [u_i, u_{i+1}]$, then $f_\smile(u) < f(u)$ for $u \in (u_{i+1}, u_{i+2}) \cup (u_{i-1}, u_i)$. In this case the solution $u(\,\cdot\,, t)$ consists of finitely many intervals where u is a regular solution given by $u(x,t) = (f')^{-1}(x/t)$ separated by jump discontinuities at points x such that $x = f'(u_j)t = t(f(u_{j+1}) - f(u_j))/(u_{j+1} - u_j) = f'(u_{j+1})t$ that clearly satisfy the Rankine–Hugoniot relation. Furthermore, we see that the traveling wave entropy condition (2.9) is satisfied as the graph of f is above the segment connecting the left and right states. In Fig. 2.1 we have three intervals, where $f_\smile < f$ on the middle interval.

To show that (2.27) defines a Kružkov entropy solution, we shall need some notation. For $i = 1, \dots, n$ set $\sigma_i = f_\smile'(u_i)$ and define $\sigma_0 = -\infty$, $\sigma_{n+1} = \infty$. By discarding identical σ_i's and relabeling if necessary, we can and will assume that $\sigma_0 < \sigma_1 < \cdots < \sigma_{n+1}$. Then for $i = 2, \dots, n$ define (see Fig. 2.2)

$$v_i(x,t) = \left(f_\smile'\right)^{-1}\left(\frac{x}{t}\right), \quad \sigma_{i-1} \le \frac{x}{t} \le \sigma_i$$

and set $v_1(x,t) = u_l$ for $x \le \sigma_1 t$ and $v_{n+1}(x,t) = u_r$ for $x \ge \sigma_n t$. Let Ω_i denote the set

$$\Omega_i = \{(x,t) \mid 0 \le t \le T, \quad \sigma_{i-1} t < x < \sigma_i t\}$$



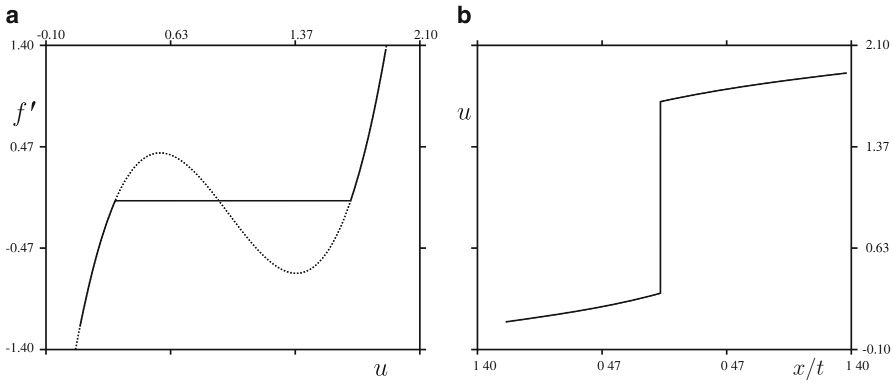


Fig. 2.2 The function $f'_{\smile}$ (**a**) and its inverse (**b**)

for $i = 1, \dots, n+1$. Using this notation, u defined by (2.27) can be written

$$u(x,t) = \sum_{i=1}^{n+1} \chi_{\Omega_i}(x,t) v_i(x,t), \tag{2.29}$$

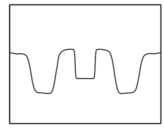
where χ_{Ω_i} denotes the characteristic function of the set Ω_i. For $i = 1, \dots, n$ we then define

$$\underline{u}_i = \lim_{x \to \sigma_i t-} u(x,t) \quad \text{and} \quad \bar{u}_i = \lim_{x \to \sigma_i t+} u(x,t).$$

The values $\underline{u}_i$ and $\bar{u}_i$ are the left and right limits of the discontinuities of u.

With this notation at hand, we show that u defined by (2.29) is a Kružkov entropy solution of the initial value problem (2.25) in the sense of (2.23). Observe that each shock by construction satisfies the traveling wave entropy condition as given by (2.8). Note that u is continuously differentiable in each Ω_i. First we use Green's theorem (similarly as in proving the Rankine–Hugoniot relation (1.21)) on each Ω_i to show that

$$\begin{aligned}
\iint_0^T \left(\eta\varphi_t + q\varphi_x\right) dx\,dt &= \sum_{i=1}^{n+1} \iint_{\Omega_i} \left(\eta_i\varphi_t + q_i\varphi_x\right) dx\,dt \\
&= \sum_{i=1}^{n+1} \iint_{\Omega_i} \left((\eta_i\varphi)_t + (q_i\varphi)_x\right) dx\,dt \\
&= \sum_{i=1}^{n+1} \int_{\partial\Omega_i} \varphi\left(-\eta_i\,dx + q_i\,dt\right) \\
&= \int \left(\eta(x,T)\varphi(x,T) - \eta(x,0)\varphi(x,0)\right) dx \\
&\quad + \sum_{i=1}^{n} \int_0^T \varphi(\sigma_i t, t)\left[\sigma_i\left(\bar{\eta}_i - \underline{\eta}_i\right) - \left(\bar{q}_i - \underline{q}_i\right)\right] dt.
\end{aligned}$$



Here

$$\begin{aligned} \eta &= \eta(u,k) = |u-k|, \\ \eta_i &= \eta(v_i(x,t),k), \quad \bar{\eta}_i = \eta(\bar{v}_i,k), \quad \underline{\eta}_i = \eta(\underline{v}_i,k), \\ q &= q(u,k) = \operatorname{sign}(u-k)\,(f(u)-f(k)), \\ q_i &= q(v_i(x,t),k), \quad \bar{q}_i = q(\bar{v}_i,k), \quad \text{and} \quad \underline{q}_i = q(\underline{v}_i,k). \end{aligned}$$

By construction, the traveling wave entropy condition (2.9) is satisfied. We have shown in Sect. 2.1 that this implies that (2.11) holds. Thus

$$\sigma_i\left(\bar{\eta}_i - \underline{\eta}_i\right) - \left(\bar{q}_i - \underline{q}_i\right) \geq 0,$$

for all constants k. Hence

$$\iint_0^T \left(\eta\varphi_t + q\varphi_x\right) dx\,dt + \int \left(\eta(x,0)\varphi(x,0) - \eta(x,T)\varphi(x,T)\right) dx \geq 0, \quad (2.30)$$

i.e., u satisfies (2.23). Now we have found a Kružkov entropy-satisfying solution to the Riemann problem if $u_l < u_r$.

If $u_l > u_r$, we can transform the problem to the case discussed above by sending $x \mapsto -x$. Then we obtain the Riemann problem

$$u_t - f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_r & \text{for } x < 0, \\ u_l & \text{for } x \geq 0. \end{cases}$$

In order to solve this, we have to take the lower convex envelope of $-f$ from u_r to u_l. But this envelope is exactly the negative of the *upper concave envelope* from u_l to u_r. The upper concave envelope is defined to be

$$f_{\frown}(u) = \inf\left\{ g(u) \;\middle|\; g \geq f \text{ and } g \text{ concave on } [u_r, u_l] \right\}. \quad (2.31)$$

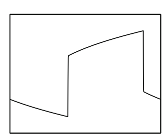
In this case the weak solution is given by

$$u(x,t) = w(z) = \begin{cases} u_l & \text{for } x \leq f'_{\frown}(u_l)t, \\ \left(f'_{\frown}\right)^{-1}(z) & \text{for } f'_{\frown}(u_l)t \leq x \leq f'_{\frown}(u_r)t, \\ u_r & \text{for } x \geq f'_{\frown}(u_r)t, \end{cases} \quad (2.32)$$

for $u_l > u_r$, where $z = x/t$.

This construction of the solution is valid as long as the envelope consists of a finite number of intervals where $f_{\smile,\frown} \neq f$, alternating with intervals where the envelope and the function coincide. We will later extend the solution to the case in which f is a piecewise, twice continuously differentiable function.

We have now proved a theorem about the solution of the Riemann problem for scalar conservation laws.



Theorem 2.2 *The initial value problem*

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0, \end{cases}$$

with a flux function $f(u)$ such that $f_{\smile,\frown} \neq f$ on finitely many intervals, alternating with intervals where they coincide, has a weak solution given by equation (2.27) if $u_l < u_r$, or by (2.32) if $u_r < u_l$. This solution satisfies the Kružkov entropy condition (2.23).

The solution $u(x,t)$ given by (2.27) and (2.32) consists of a finite number of discontinuities separated by "wedges" (i.e., intervals (z_i, z_{i+1})) inside which u is a classical solution. A discontinuity that satisfies the entropy condition is called a *shock wave* or simply a *shock*, and the continuous parts of the solution of the Riemann problem are called *rarefaction waves*. This terminology, as well as the term "entropy condition," comes from gas dynamics. Thus we may say that the solution of a Riemann problem consists of a finite sequence of rarefaction waves alternating with shocks.

◇ Example 2.3 (Traffic flow (cont'd.))
In the conservation law model of traffic flow, we saw in Example 1.6 that the flux function was given as

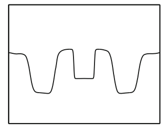
$$f(u) = u(1-u).$$

This is a concave function. Consequently, every upper envelope will be the function f itself, whereas a lower envelope will be the straight line segment between its endpoints. Every Riemann problem with $u_l > u_r$ will be solved by a rarefaction wave, and if $u_l < u_r$, the solution will consist of a single shock. This is, of course, in accordance with our earlier results, and perhaps also with our experience. ◇

The solution of a Riemann problem is frequently depicted in (x,t)-space as a collection of rays emanating from the origin. The slope of these rays is the reciprocal of $f'(u)$ for rarefaction waves, and if the ray illustrates a shock, the reciprocal of $[\![f]\!] \,/\, [\![u]\!]$. In Fig. 2.3 we illustrate the solution of the previous Riemann problem in this way; broken lines indicate rarefaction waves, and the solid line the shock. Note that Theorem 2.2 does *not* require the flux function f to be differentiable. Assume now that the flux function is a *polygon*, i.e., that f is *continuous* and *piecewise linear* on a finite number of intervals. Thus f' will then be a step function taking a finite number of values. The discontinuity points of f' will hereinafter be referred to as *breakpoints*.

Making this approximation is reasonable in many applications, since the precise form of the flux function is often the result of some measurements. These measurements are taken for a discrete set of u values, and a piecewise linear flux function is the result of a linear interpolation between these values.

Both upper concave and lower convex envelopes will also be piecewise linear functions with a finite number of breakpoints. This means that $f'_{\smile}$ and $f'_{\frown}$ will be step functions, as will their inverses. Furthermore, the inverses of the derivatives will take their values among the breakpoints of $f_{\smile}$ (or $f_{\frown}$), and therefore also of f. If the initial states in a Riemann problem are breakpoints, then the entire solution will take values in the set of breakpoints.



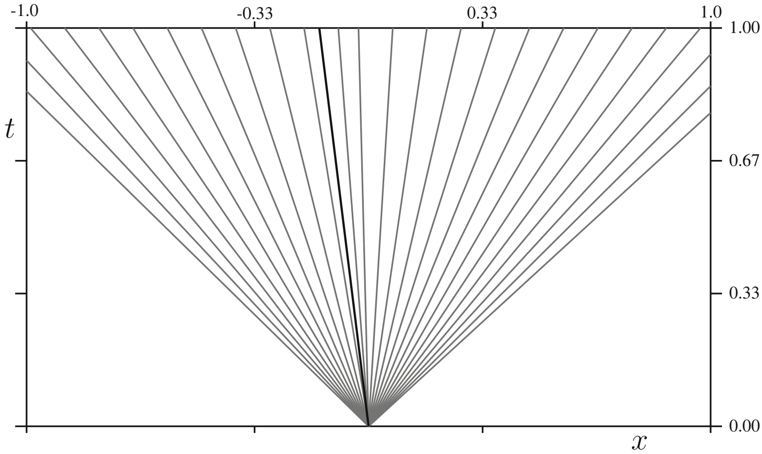


Fig. 2.3 The solution of a Riemann problem, shown in (x, t)-space

If we assume that $u_l < u_r$, and label the breakpoints $u_l = u_0 < u_1 < \cdots < u_n = u_r$, then $f_{\smile}$ will have breakpoints in some subset of this, say $u_l < u_{i_1} < \cdots < u_{i_k} < u_r$. The solution will be a step function in $z = x/t$, monotonically nondecreasing between u_l and u_r. The discontinuities will be located at z_{i_k}, given by

$$z_{i_k} = \frac{f\left(u_{i_{k-1}}\right) - f\left(u_{i_k}\right)}{u_{i_{k-1}} - u_{i_k}}.$$

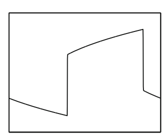
Thus the following corollary of Theorem 2.2 holds.

Corollary 2.4 *Assume that f is a continuous piecewise linear function $f\colon [-K, K] \to \mathbb{R}$ for some constant K. Denote the breakpoints of f by $-K = u_0 < u_1 < \cdots < u_{n-1} < u_n = K$. Then the Riemann problem*

$$u_t + f(u)_x = 0, \qquad u(x, 0) = \begin{cases} u_j & \text{for } x < 0, \\ u_k & \text{for } x \geq 0, \end{cases} \tag{2.33}$$

has a piecewise constant (in $z = x/t$) solution. If $u_j < u_k$, let $u_j = v_1 < \cdots < v_m = u_k$ denote the breakpoints of $f_{\smile}$, and if $u_j > u_k$, let $u_k = v_m < \cdots < v_1 = u_j$ denote the breakpoints of $f_{\frown}$. The weak solution of the Riemann problem is then given by

$$u(x, t) = \begin{cases} v_1 & \text{for } x \leq s_1 t, \\ v_2 & \text{for } s_1 t < x \leq s_2 t, \\ \vdots & \\ v_i & \text{for } s_{i-1} t < x \leq s_i t, \\ \vdots & \\ v_m & \text{for } s_{m-1} t < x. \end{cases} \tag{2.34}$$



Here, the speeds s_i are computed from the derivative of the envelope, that is,

$$s_i = \frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}.$$

For a fixed time t, the solution is monotone in the x variable. Furthermore,

$$\|u(\cdot,t) - u_0\|_{L^1} \le t \|f\|_{\mathrm{Lip}} \left|u_j - u_k\right|. \tag{2.35}$$

Proof It remains to prove (2.35). With the given notation,

$$\begin{aligned}
\|u(\cdot,t) - u_0\|_{L^1} &= \sum_{\substack{j \\ s_j \le 0}} (v_{j+1} - v_j)(-s_j)t + \sum_{\substack{j \\ s_j > 0}} (v_{j+1} - v_j) s_j t \\
&\le \max_j \left|s_j\right| \left|v_m - v_1\right| \, t \\
&\le t \, \|f\|_{\mathrm{Lip}} \left|u_j - u_k\right|.
\end{aligned}$$ □

Note that this solution is an admissible solution in the sense that it satisfies the Kružkov entropy condition. The viscous profile entropy condition is somewhat degenerate in this case. Across discontinuities over which $f(u)$ differs from the envelope, it is satisfied. But across those discontinuities over which the envelope and the flux function coincide, the right-hand side of the defining ordinary differential equation (2.7) collapses to zero. The conservation law is called *linearly degenerate* in each such interval (v_i, v_{i+1}). Nevertheless, these discontinuities are also limits of the viscous regularization, as can be seen by changing to Lagrangian coordinates $x \mapsto x - s_i t$; see Exercise 2.4.

With this we conclude our discussion of the Riemann problem, and in the next section we shall see how the solutions of Riemann problems may be used as a building block to solve more general initial value problems.

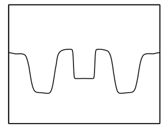
2.3 Front Tracking

This algorithm is admittedly complicated,
but no simpler mechanism seems to do nearly as much.
— D.E. Knuth, The TEXbook (1984)

We begin this section with an example that illustrates the ideas of front tracking for scalar conservation laws, as well as some of the properties of solutions.

◇ Example 2.5

In this example we shall study a piecewise linear approximation of Burgers's equation, $u_t + (u^2/2)_x = 0$. This means that we study a conservation law with a flux function that is piecewise linear and agrees with $u^2/2$ at its breakpoints. To be specific, we choose intervals of unit length. We shall be interested in the flux function



only in the interval $[-1, 2]$, where we define it to be

$$f(u) = \begin{cases} -u/2 & \text{for } u \in [-1, 0], \\ u/2 & \text{for } u \in [0, 1], \\ 3u/2 - 1 & \text{for } u \in [1, 2]. \end{cases} \tag{2.36}$$

This flux function has two breakpoints, and is convex.

We wish to solve the initial value problem

$$u_t + f(u)_x = 0, \qquad u_0(x) = \begin{cases} 2 & \text{for } x \le x_1, \\ -1 & \text{for } x_1 < x \le x_2, \\ 1 & \text{for } x_2 < x, \end{cases} \tag{2.37}$$

with f given by (2.36). Initially, the solution must consist of the solutions of the two Riemann problems located at x_1 and x_2. This is so, since the waves from these solutions move with a finite speed and will not interact until some positive time.

This feature, sometimes called *finite speed of propagation*, characterizes hyperbolic, as opposed to elliptic or parabolic, partial differential equations. It implies that if we change the initial condition locally around some point, it will not immediately influence the solution "far away." Recalling the almost universally accepted assumption that nothing moves faster than the speed of light, one can say that hyperbolic equations are more fundamental than the other types of partial differential equations.

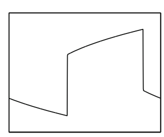
Returning to our example, we must then solve the two initial Riemann problems. We commence with the one at x_1. Since f is convex, and $u_l = 2 > -1 = u_r$, the solution will consist of a single shock wave with speed $s_1 = \frac{1}{2}$ given from the Rankine–Hugoniot condition of this Riemann problem. For small t and x near x_1 the solution reads

$$u(x,t) = \begin{cases} 2 & \text{for } x < s_1 t + x_1, \\ -1 & \text{for } x \ge s_1 t + x_1. \end{cases} \tag{2.38}$$

The other Riemann problem has $u_l = -1$ and $u_r = 1$, so we must use the lower convex envelope, which in this case coincides with the flux function f. The flux function has two linear segments and one breakpoint $u = 0$ in the interval $(-1, 1)$. Hence, the solution will consist of two discontinuities moving apart. The speeds of the discontinuities are computed from $f'(u)$, or equivalently from the Rankine–Hugoniot condition, since f is linearly degenerate over each discontinuity. This gives $s_2 = -\frac{1}{2}$ and $s_3 = \frac{1}{2}$. The solution equals

$$u(x,t) = \begin{cases} -1 & \text{for } x < s_2 t + x_2, \\ 0 & \text{for } s_2 t + x_2 \le x < s_3 t + x_2, \\ 1 & \text{for } s_3 t + x_2 \le x, \end{cases} \tag{2.39}$$

for small t and x near x_2.



It remains to connect the two solutions (2.38) and (2.39). This is easily done for sufficiently small t:

$$u(x,t) = \begin{cases} 2 & \text{for } x < x_1 + s_1 t, \\ -1 & \text{for } x_1 + s_1 t \le x \le x_2 + s_2 t, \\ 0 & \text{for } x_2 + s_2 t \le x < x_2 + s_3 t, \\ 1 & \text{for } x_2 + s_3 t \le x. \end{cases} \tag{2.40}$$

The problem now is that the shock wave located at $x_1(t) = x_1 + t/2$ will collide with the discontinuity $x_2(t) = x_2 - t/2$. Then equation (2.40) is no longer valid, since the middle interval has collapsed. This will happen at time $t = t_1 = (x_2 - x_1)$ and position $x = x_4 = (x_1 + x_2)/2$.

To continue the solution, we must solve the interaction between the shock and the discontinuity. Again, using finite speed of propagation, we have that the solution away from (x_4, t_1) will not be directly influenced by the behavior here. Consider now the solution at time t_1 and in a vicinity of x_4. Here u takes two constant values, 2 for $x < x_4$ and 0 for $x > x_4$. Therefore, the interaction of the shock wave $x_1(t)$ and the discontinuity $x_2(t)$ is determined by solving the Riemann problem with $u_l = 2$ and $u_r = 0$.

Again, this Riemann problem is solved by a single shock, since the flux function is convex and $u_l > u_r$. The speed of this shock is $s_4 = 1$. Thus, for t larger than $x_2 - x_1$, the solution consists of a shock located at $x_4(t)$ and a discontinuity located at $x_3(t)$. The locations are given by

$$\begin{aligned} x_4(t) &= \frac{1}{2}(x_1 + x_2) + 1\,(t - (x_2 - x_1)) = t + \frac{1}{2}(3x_1 - x_2), \\ x_3(t) &= x_2 + \frac{1}{2}t. \end{aligned}$$

We can then write the solution $u(x,t)$ as

$$u(x,t) = 2 + [\![u\,(x_4(t))]\!]\, H\,(x - x_4(t)) + [\![u\,(x_3(t))]\!]\, H\,(x - x_3(t)), \tag{2.41}$$

where H is the Heaviside function.

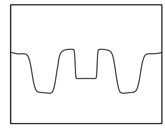
Indeed, every function $u(x,t)$ that is piecewise constant in x with discontinuities located at $x_j(t)$ can be written in the form

$$u(x,t) = u_l + \sum_j [\![u\left(x_j(t)\right)]\!]\, H\left(x - x_j(t)\right), \tag{2.42}$$

where u_l now denotes the value of u to the left of the leftmost discontinuity.

Since the speed of $x_4(t)$ is greater than the speed of $x_3(t)$, these two discontinuities will collide. This will happen at $t = t_2 = 3\,(x_2 - x_1)$ and $x = x_5 = (5x_2 - 3x_1)/2$. In order to resolve the interaction of these two discontinuities, we have to solve the Riemann problem with $u_l = 2$ and $u_r = 1$.

In the interval $[1, 2]$, $f(u)$ is linear, and hence the solution of the Riemann problem will consist of a single discontinuity moving with speed $s_5 = \frac{3}{2}$. Therefore, for



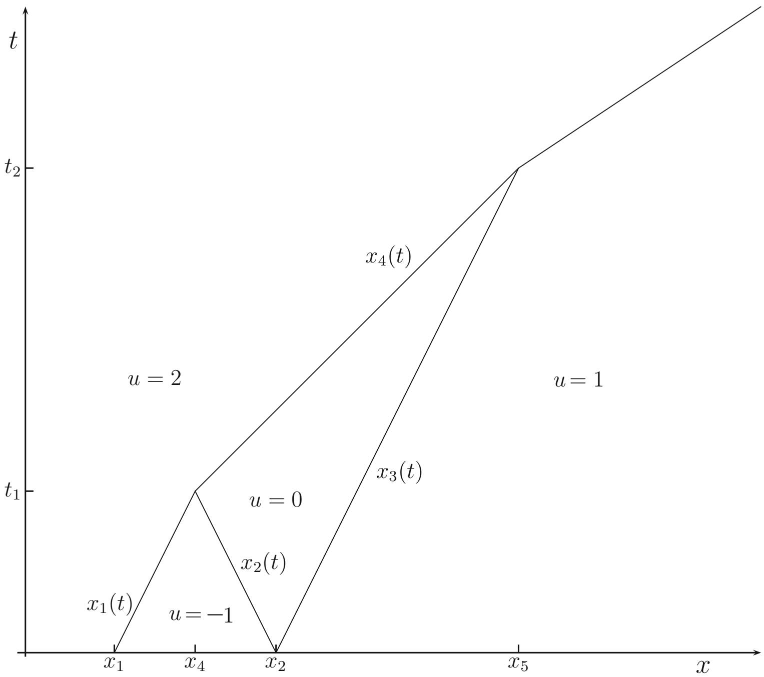


Fig. 2.4 The solution of (2.37) with the piecewise linear continuous flux function (2.36)

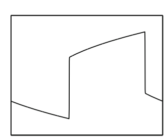
$t > t_2$ the solution is defined as

$$u(x,t) = \begin{cases} 2 & \text{for } x < 3t/2 + 3x_1 - 2x_2, \\ 1 & \text{for } x \geq 3t/2 + 3x_1 - 2x_2. \end{cases} \tag{2.43}$$

Since the solution now consists of a single moving discontinuity, there will be no further interactions, and we have found the solution for all positive t. Figure 2.4 depicts this solution in the (x,t)-plane; the discontinuities are shown as solid lines. We call the method that we have used to obtain the solution *front tracking*. Front tracking consists in *tracking* all discontinuities in the solution, whether they represent shocks or not. Hereinafter, if the flux function is continuous and piecewise linear, all discontinuities in the solution will be referred to as *fronts*.

Notice that if the flux function is continuous and piecewise linear, the Rankine–Hugoniot condition can be used to calculate the speed of any front. So from a computational point of view, all discontinuities are equivalent. ◇

With this example in mind we can define a general front-tracking algorithm for scalar conservation laws. Loosely speaking, front tracking consists in making a step-function approximation to the initial data, and a piecewise linear approximation to the flux function. The approximate initial function will define a series of Riemann problems, one at each step. One can solve each Riemann problem, and since the solutions have finite speed of propagation, they will be independent of each other



until waves from neighboring solutions interact. Front tracking should then resolve this interaction in order to propagate the solution to larger times.

By considering flux functions that are continuous and piecewise linear, we are providing a method for resolving interactions.

Front tracking in a box (scalar case)

(i) We are given a scalar one-dimensional conservation law

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0. \tag{2.44}$$

(ii) Approximate f by a continuous piecewise linear flux function f^δ.

(iii) Approximate initial data u_0 by a piecewise constant function u_0^η.

(iv) Solve the initial value problem

$$u_t + f^\delta(u)_x = 0, \quad u|_{t=0} = u_0^\eta$$

exactly. Denote the solution by $u_{\delta,\eta}$.

(v) As f^δ and u_0^η approach f and u_0, respectively, the approximate solution $u_{\delta,\eta}$ will converge to u, the solution of (2.44).

We have seen that the solution of a Riemann problem always is a monotone function taking values between u_l and u_r. Another way of stating this is to say that the solution of a Riemann problem obeys a *maximum principle*. This means that if we solve a collection of Riemann problems, the solutions (all of them) will remain between the minimum and the maximum of the left and right states.

Therefore, fix a large positive number M and let $u_i = i\delta$, for $-M \leq i\delta \leq M$. In this section we shall assume, unless otherwise stated, that the flux function $f(u)$ is continuous and piecewise linear, with breakpoints u_i.

We assume that u_0 is some piecewise constant function taking values in the set $\{u_i\}$ with a finite number of discontinuities, and we wish to solve the initial value problem

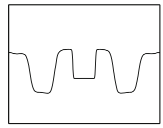
$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{2.45}$$

As remarked above, the solution will initially consist of a number of noninteracting solutions of Riemann problems. Each solution will be a piecewise constant function with discontinuities traveling at constant speed. Hence, at some later time $t_1 > 0$, two discontinuities from neighboring Riemann problems will interact.

At $t = t_1$ we can proceed by considering the initial value problem with solution $v(x,t)$:

$$v_t + f(v)_x = 0, \qquad v(x,t_1) = u(x,t_1).$$

Since the solutions of the initial Riemann problems will take values among the breakpoints of f, i.e., $\{u_i\}$, the initial data $u(x,t_1)$ is the same type of function as



$u_0(x)$. Consequently, we can proceed as we did initially, by solving the Riemann problems at the discontinuities of $u(x, t_1)$. However, except for the Riemann problem at the interaction point, these Riemann problems have all been solved initially, and their solution merely consists in continuing the discontinuities at their present speed. The Riemann problem at the interaction point has to be solved, giving a new fan of discontinuities. In this fashion the solution can be calculated up to the next interaction at t_2, say. Note that what we calculate in this way is *not* an approximation to the entropy weak solution of (2.45), but the exact solution.

It is clear that we can continue this process for any number of interactions occurring at times t_n, where $0 < t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n \leq \cdots$. However, we cannot a priori be sure that $\lim t_n = \infty$, or in other words, that we can calculate the solution up to any predetermined time. One might envisage that the number of discontinuities grows for each interaction, and that this number increases without bound at some finite time. The next lemma assures us that this does not happen.

Lemma 2.6 *For each fixed δ, and for each piecewise constant function u_0 taking values in the set $\{u_i\}$, there is only a finite number of interactions between discontinuities of the weak solution to* (2.45) *for t in the interval* $[0, \infty)$.

Remark 2.7 In particular, this means that we can calculate the solution by front tracking up to infinite time using only a *finite* number of operations. In connection with front tracking used as a numerical method, this property is called *hyperfast*. In the rest of this book we call a discontinuity in a front-tracking solution a *front*. Thus a front can represent either a shock or a discontinuity over which the flux function is linearly degenerate.

Proof (of Lemma 2.6) Let $N(t)$ denote the total number of fronts in the front-tracking solution $u(x, t)$ at time t.

If a front represents a jump from u_l to u_r, we say that the front contains l linear segments if the flux function has $l - 1$ breakpoints between u_l and u_r. We use the notation $[\![u]\!]$ to denote the jump in u across a front. In this notation, $l = |[\![u]\!]| / \delta$.

Let $L(t)$ be the total number of linear segments present in all fronts of $u(x, t)$ at time t. Thus, if we number the fronts from left to right, and the ith front contains l_i linear segments, then

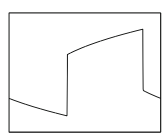
$$L(t) = \sum_i l_i = \frac{1}{\delta} \sum_i \left| [\![u]\!]_i \right| .$$

Let Q denote the number of linear segments in the piecewise linear flux function $f(u)$ for u in the interval $[-M, M]$. Now we claim that the functional

$$T(t) = QL(t) + N(t)$$

is strictly decreasing for each collision of fronts. Since $T(t)$ takes only integer values, this means that we can have at most $T(0)$ collisions.

It remains to prove that $T(t)$ is strictly decreasing for each collision. Assume that a front separating values u_l and u_m collides from the left with a front separating u_m and u_r. We will first show that T is decreasing if u_m is between u_l and u_r.



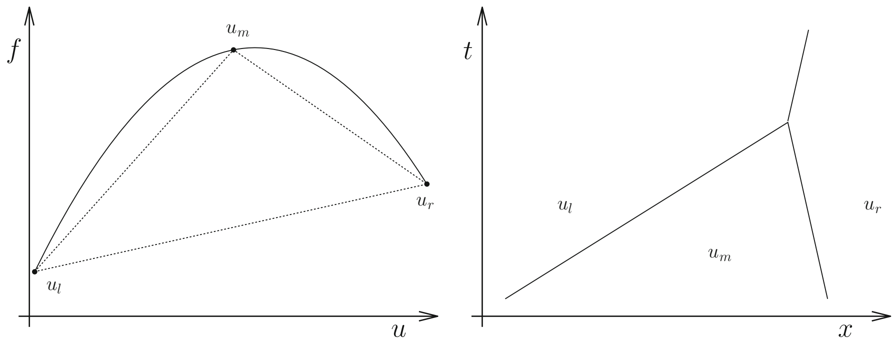


Fig. 2.5 An interaction of fronts where $u_l < u_m < u_r$

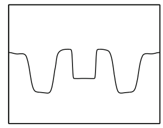
We assume that $u_l < u_m < u_r$. If $u_r < u_m < u_l$, the situation is analogous, and the statement can be proved with the same arguments. The situation is as depicted in Fig. 2.5. Since a single front connects u_l with u_m, the graph of the flux function cannot cross the straight line segment connecting the points $(u_l, f(u_l))$ and $(u_m, f(u_m))$. The entropy condition also implies that the graph of the flux function must be above this segment. The same holds for the front on the right separating u_m and u_r. As the two fronts are colliding, the speed of the left front must be larger than the speed of the right front. This means that the slope of the segment from $(u_l, f(u_l))$ to $(u_m, f(u_m))$ is greater than the slope of the segment from $(u_m, f(u_m))$ to $(u_r, f(u_r))$. Therefore, the lower convex envelope from u_l to u_r consists of the line from $(u_l, f(u_l))$ to $(u_r, f(u_r))$. Accordingly, the solution of the Riemann problem consists of a single front separating u_l and u_r. See Fig. 2.5. Consequently, L does not change at the interaction, and N decreases by one. Thus, when u_m is between u_r and u_l, T decreases.

It remains to show that T also decreases if u_m is not between u_l and u_r. We will do this for the case $u_m < u_l < u_r$. The other cases are similar, and can be proved by analogous arguments.

Since the Riemann problem with a left state u_l and right state u_m is solved by a single discontinuity, the graph of the flux function cannot lie above the straight line segment connecting the points $(u_l, f(u_l))$ and $(u_m, f(u_m))$. Similarly, the graph of the flux function must lie entirely above the straight line segment connecting $(u_m, f(u_m))$ and $(u_r, f(u_r))$. Also, the slope of the latter segment must be smaller than that of the former, since the fronts are colliding. This means that the Riemann problem with left state u_l and right state u_r defined at the collision of the fronts will have a solution consisting of fronts with speed smaller than or equal to the speed of the right colliding front. See Fig. 2.6.

The maximal number of fronts resulting from the collision is $|u_l - u_r|/\delta$. This is strictly less than Q. Hence N increases by at most $Q - 1$. At the same time, L decreases by at least one. Consequently, T must decrease by at least one. This concludes the proof of Lemma 2.6. $\square$

As a corollary of Lemma 2.6, we infer that for a piecewise constant initial function with a finite number of discontinuities, and for a continuous and piecewise



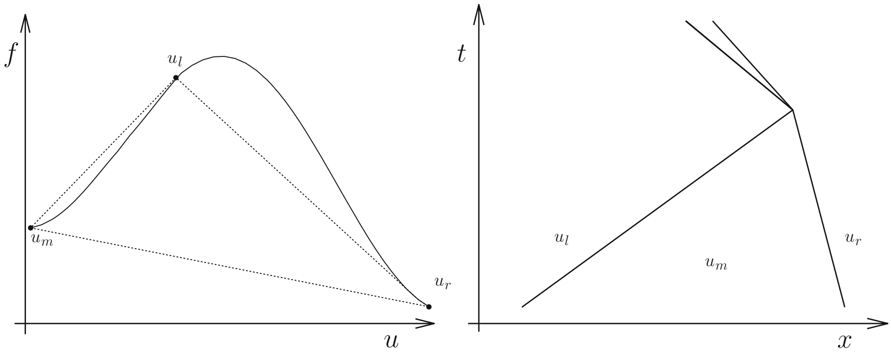


Fig. 2.6 An interaction of fronts where $u_m < u_l < u_r$

linear flux function with a finite number of breakpoints, the initial value problem has a weak solution satisfying the Kružkov entropy condition (2.22), as well as the viscous entropy condition for every discontinuity. Before we state the precise result, it is convenient to introduce the notion of *total variation*. For a piecewise constant function $u = u(x)$ with finitely many jumps, its total variation is the sum of the absolute values of its jumps, that is,

$$\mathrm{T.V.}\,(u) = \sum_i \left| [\![u]\!]_i \right| .$$

This notation can and will be generalized to arbitrary functions; see Appendix A. It is not difficult to prove the following slight generalization of what we have already shown:

Corollary 2.8 *Let $f(u)$ be a continuous and piecewise linear function with a finite number of breakpoints for u in the interval $[-M, M]$, where M is some constant. Assume that u_0 is a piecewise constant function with a finite number of discontinuities, $u_0\colon \mathbb{R} \to [-M, M]$. Then the initial value problem*

$$u_t + f(u)_x = 0, \qquad u|_{t=0} = u_0, \tag{2.46}$$

has a weak solution $u = u(x,t)$. The function $u = u(x,t)$ is a piecewise constant function of x for each t, and $u(x,t)$ takes values in the finite set $\{u_0(x)\} \cup \{\text{the breakpoints of } f\}$. Furthermore, there is only a finite number of interactions between the fronts of u. The function u also satisfies the Kružkov entropy condition (2.23)*. In addition,*

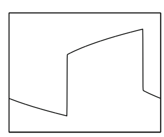
$$\mathrm{T.V.}\,(u(\,\cdot\,,t)) \leq \mathrm{T.V.}\,(u_0)\,. \tag{2.47}$$

Finally, we have

$$\|u(\,\cdot\,,t) - u_0\|_{L^1} \leq t\,\|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,, \tag{2.48}$$

and, more generally,

$$\|u(\,\cdot\,,t) - u(\,\cdot\,,s)\|_{L^1} \leq \|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,|t-s|\,. \tag{2.49}$$



Proof For the proof that the solution satisfies the Kružkov entropy condition, see Exercise 2.21. It remains to prove (2.47), (2.48), and (2.49). Regarding (2.47), recall that at each time two or more fronts interact, the solution of the resulting Riemann problem is always monotone, and hence no new extrema are introduced. Thus the total variation cannot increase, which proves (2.47).

To prove (2.48), we observe that Corollary 2.4 yields (cf. (2.35)) that

$$\|u(\cdot,t) - u_0\|_{L^1} \le t \,\|f\|_{\mathrm{Lip}} \mathrm{T.V.}\,(u_0)$$

for all $t < t_1$, the first collision time. We use the same argument for all $t \in (t_1, t_2)$, where t_2 is the second collision time, to conclude that

$$\begin{aligned}
\|u(\cdot,t) - u_0\|_{L^1} &\le \|u(\cdot,t) - u(\cdot,t_1)\|_{L^1} + \|u(\cdot,t_1) - u_0\|_{L^1} \\
&\le (t - t_1)\|f\|_{\mathrm{Lip}} \mathrm{T.V.}\,(u(\cdot,t_1)) + t_1\|f\|_{\mathrm{Lip}} \mathrm{T.V.}\,(u_0) \\
&\le t\,\|f\|_{\mathrm{Lip}} \mathrm{T.V.}\,(u_0)\,.
\end{aligned}$$

Repeating this argument for all collision times, we conclude that (2.48) holds. The estimate (2.49) follows by considering the previous result with initial data $u(\cdot,s)$ when $s < t$. □

This is all well and good, but we could wish for more. For instance, is this solution the only one? And piecewise linear flux functions and piecewise constant initial functions seem more like an approximation than what we would expect to see in "real life." So what happens when the piecewise constant initial function and the piecewise linear flux function converge to general initial data and flux functions, respectively?

It turns out that these two questions are connected and can be answered by elegant, but indirect, analysis starting from the Kružkov formulation (2.22).

2.4 Existence and Uniqueness

Det var en ustyrtelig mængde lag!
Kommer ikke kærnen snart for en dag?[2]
— Henrik Ibsen, Peer Gynt (1867)

By a clever choice of the test function ϕ, we shall use the Kružkov formulation to show stability with respect to the initial value function, and thereby uniqueness.

The approach used in this section is also very useful in estimating the error in numerical methods. We shall return to this in a later chapter.

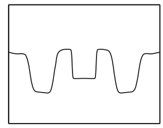
Let therefore $u = u(x,t)$ and $v = v(x,t)$ be two weak solutions to

$$u_t + f(u)_x = 0,$$

with initial data

$$u|_{t=0} = u_0, \qquad v|_{t=0} = v_0,$$

[2] So many layers I've peeled and peeled! Will the kernel never be revealed?



respectively, satisfying the Kružkov entropy condition. Equivalently,

$$\iint \left(|u-k|\,\phi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\phi_x \right)\,dx\,dt \\ + \int |u_0 - k|\,\phi|_{t=0}\,dx \ge 0 \tag{2.50}$$

for every nonnegative test function ϕ with compact support (and similarly for the function v). Throughout the calculations we will assume that both u and v are bounded and integrable, thus

$$u, v \in L^1 \cap L^\infty(\mathbb{R} \times (0,\infty)). \tag{2.51}$$

We assume that f is *Lipschitz continuous*, that is, that there is a constant L such that

$$\|f\|_{\mathrm{Lip}} := \sup_{u \neq v} \left| \frac{f(u)-f(v)}{u-v} \right| \le L, \tag{2.52}$$

and we denote by $\|f\|_{\mathrm{Lip}}$ the Lipschitz constant, or seminorm,[3] of f.

If ϕ is compactly supported in $t > 0$, then (2.50) reads

$$\iint \left(|u-k|\,\phi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\phi_x \right)\,dx\,dt \ge 0. \tag{2.53}$$

For simplicity we shall in this section use the notation

$$q(u,k) = \operatorname{sign}(u-k)\,(f(u)-f(k)).$$

For functions of two variables we define the Lipschitz constant by

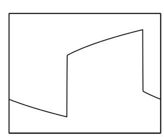
$$\|q\|_{\mathrm{Lip}} = \sup_{(u_1,v_1)\neq(u_2,v_2)} \frac{|q(u_1,v_1)-q(u_2,v_2)|}{|u_1-u_2|+|v_1-v_2|}.$$

Since $q_u(u,k) = \operatorname{sign}(u-k)\,f'(u)$ and $q_k(u,k) = -\operatorname{sign}(u-k)\,f'(k)$, it follows that if $\|f\|_{\mathrm{Lip}} \le L$, then also $\|q\|_{\mathrm{Lip}} \le L$.

Now we introduce the famous *Kružkov doubling of variables method*. To that end, let $\phi = \phi(x,t,y,s)$ be a nonnegative test function in both (x,t) and (y,s) with compact support in $t > 0$ and $s > 0$. Using that both u and v satisfy (2.53), we can set $k = v(y,s)$ in the equation for u, and set $k = u(x,t)$ in the equation for $v = v(y,s)$. We integrate the equation for $u(x,t)$ with respect to y and s, and the equation for $v(y,s)$ with respect to x and t, and add the two resulting equations. We then obtain

$$\iiiint \big(|u(x,t)-v(y,s)|\,(\phi_t+\phi_s) + q(u,v)(\phi_x+\phi_y) \big)\,dx\,dt\,dy\,ds \ge 0. \tag{2.54}$$

[3] $\|f\|_{\mathrm{Lip}}$ is *not* a norm; after all, constants k have $\|k\|_{\mathrm{Lip}} = 0$.



Now we temporarily leave the topic of conservation laws in order to establish some facts about "approximate δ distributions," or mollifiers. This is a sequence of smooth functions ω_ε such that the corresponding distributions tend to the δ_0 distribution, i.e., $\omega_\varepsilon \to \delta_0$ as $\varepsilon \to 0$. There are several ways of defining these distributions. Recall the following (cf. (1.53)): Let $\omega(\sigma)$ be a C^∞ function such that

$$0 \le \omega(\sigma) \le 1, \quad \operatorname{supp}\omega \subseteq [-1,1], \quad \omega(-\sigma) = \omega(\sigma), \quad \int_{-1}^{1} \omega(\sigma)\,d\sigma = 1.$$

Now define

$$\omega_\varepsilon(\sigma) = \frac{1}{\varepsilon}\omega\left(\frac{\sigma}{\varepsilon}\right). \tag{2.55}$$

It is not hard to verify that ω_ε has the necessary properties such that as a distribution, $\lim_{\varepsilon\to 0}\omega_\varepsilon = \delta_0$.

We will need the following result:

Lemma 2.9 *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be locally Lipschitz continuous and let $\Psi \in C_0^\infty(\mathbb{R}^2)$. Assume that $u, v \in L^1 \cap L^\infty(\mathbb{R}\times(0,\infty))$. Then*

$$\begin{aligned}\iiiint F(u(x,t), v(y,s))\Psi\Big(\frac{1}{2}(x+y), \frac{1}{2}(t+s)\Big)\omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,dx\,dt\,dy\,ds \\ \underset{\varepsilon,\varepsilon_0\downarrow 0}{\to} \iint F(u(x,t), v(x,t))\Psi(x,t)\,dx\,dt.\end{aligned} \tag{2.56}$$

Proof To ease the notation, we drop the time variation and want to show that

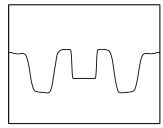
$$\begin{aligned}\iint F(u(x), v(y))\Psi\Big(\frac{1}{2}(x+y)\Big)\omega_\varepsilon(x-y)\,dx\,dy \\ \underset{\varepsilon\downarrow 0}{\to} \int F(u(x), v(x))\Psi(x)\,dx.\end{aligned} \tag{2.57}$$

Observe first that

$$\int F(u(x), v(x))\Psi(x)\,dx = \iint F(u(x), v(x))\Psi(x)\omega_\varepsilon(x-y)\,dx\,dy.$$

Next we obtain

$$\begin{aligned}&\iint F(u(x), v(y))\Psi\Big(\frac{1}{2}(x+y)\Big)\omega_\varepsilon(x-y)\,dx\,dy \\ &\quad - \iint F(u(x), v(x))\Psi(x)\omega_\varepsilon(x-y)\,dx\,dy \\ &= \iint \big(F(u(x), v(y)) - F(u(x), v(x))\big)\Psi\Big(\frac{1}{2}(x+y)\Big)\omega_\varepsilon(x-y)\,dx\,dy \\ &\quad + \iint F(u(x), v(x))\Big(\Psi\big(\frac{1}{2}(x+y)\big) - \Psi(x)\Big)\omega_\varepsilon(x-y)\,dx\,dy.\end{aligned}$$



We estimate the two terms separately. We obtain

$$\begin{aligned}
&\left|\iint \big(F(u(x),v(y)) - F(u(x),v(x))\big)\Psi\Big(\frac{1}{2}(x+y)\Big)\omega_\varepsilon(x-y)\,dx\,dy\right| \\
&\quad\le \|\Psi\|_{L^\infty}\, C \iint |v(x)-v(y)|\,\omega_\varepsilon(x-y)\,dx\,dy \\
&\quad\le \|\Psi\|_{L^\infty}\, C \iint |v(y+z)-v(y)|\,\omega_\varepsilon(z)\,dz\,dy \\
&\quad\le \|\Psi\|_{L^\infty}\, C \sup_{|z|\le\varepsilon} \|v(\,\cdot\,+z)-v\|_{L^1} \int \omega_\varepsilon(z)\,dz \\
&\quad= \|\Psi\|_{L^\infty}\, C \sup_{|z|\le\varepsilon} \|v(\,\cdot\,+z)-v\|_{L^1}\,,
\end{aligned}$$

using that u is bounded and the Lipschitz continuity of F. Since the L^1-norm is continuous with respect to translations (see Exercise 2.20), we conclude that this term vanishes as $\varepsilon \downarrow 0$. As for the second term, we use a similar approach:

$$\begin{aligned}
&\left|\iint F(u(x),v(x))\Big(\Psi\big(\frac{1}{2}(x+y)\big) - \Psi(x)\Big)\omega_\varepsilon(x-y)\,dxdy\right| \\
&\quad\le \|F(u,v)\|_{L^\infty} \iint \left|\Psi\Big(\frac{1}{2}z+y\Big) - \Psi(z+y)\right|\omega_\varepsilon(z)dzdy \\
&\quad\le \|F(u,v)\|_{L^\infty} \sup_{|z|\le\varepsilon} \left\|\Psi\Big(\,\cdot\,+\frac{1}{2}z\Big) - \Psi\right\|_{L^1} \int \omega_\varepsilon(z)dz \\
&\quad= \|F(u,v)\|_{L^\infty} \sup_{|z|\le\varepsilon} \left\|\Psi\Big(\,\cdot\,+\frac{1}{2}z\Big) - \Psi\right\|_{L^1}\,,
\end{aligned}$$

which again vanishes as $\varepsilon \downarrow 0$. □

Returning now to conservation laws and (2.54), we must make a smart choice of a test function $\phi(x,t,y,s)$. Let $\psi(x,t)$ be a test function that has support in $t > 0$, and define

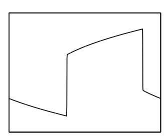
$$\phi(x,t,y,s) = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_{\varepsilon_0}(t-s)\omega_\varepsilon(x-y),$$

where ε_0 and ε are (small) positive numbers. In this case,[4]

$$\phi_t + \phi_s = \frac{\partial\psi}{\partial t}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_{\varepsilon_0}(t-s)\omega_\varepsilon(x-y)$$

and[5]

$$\phi_x + \phi_y = \frac{\partial\psi}{\partial x}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_{\varepsilon_0}(t-s)\omega_\varepsilon(x-y).$$



[4] Beware! Here $\frac{\partial\psi}{\partial t}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$ means the partial derivative of ψ with respect to the second variable, and this derivative is evaluated at $\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$.

[5] As in the previous equation, $\frac{\partial\psi}{\partial x}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$ means the partial derivative of ψ with respect to the first variable, and this derivative is evaluated at $\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$.

Observe that the derivatives of ω_{ε_0} and ω_ε, the approximate δ distributions, cancel. Now apply Lemma 2.9 to $F(u,v) = |u-v|$, $\Psi = \partial\psi/\partial t$, and $F(u,v) = q(u,v)$, $\Psi = \partial\psi/\partial x$, respectively. Then, as ε_0 and ε tend to zero, (2.54) and Lemma 2.9 give

$$\iint \left(|u(x,t) - v(x,t)|\, \psi_t + q(u,v)\psi_x \right)\, dt\, dx \geq 0 \tag{2.58}$$

for any two weak solutions u and v and any nonnegative test function ψ with support in $t > \epsilon$.

If we considered (2.23) in the strip $t \in [0,T]$ and test functions whose support included 0 and T, the Kružkov formulation would imply

$$\begin{aligned} \iiiint &\left(|u(x,t) - v(y,s)|\, (\phi_t + \phi_s) + q(u,v)\left(\phi_x + \phi_y\right)\right) dx\, dt\, dy\, ds \\ &- \iiint |u(x,T) - v(y,s)|\, \phi(x,T,y,s)\, dx\, dy\, ds \\ &- \iiint |u(x,t) - v(y,T)|\, \phi(x,t,y,T)\, dx\, dy\, dt \\ &+ \iiint |u_0(x) - v(y,s)|\, \phi(x,0,y,s)\, dx\, dy\, ds \\ &+ \iiint |u(x,t) - v_0(y)|\, \phi(x,t,y,0)\, dx\, dy\, dt \geq 0. \end{aligned} \tag{2.59}$$

We can make the same choice of test function as before. Since we are integrating over only half the support of the test functions, we get a factor $\frac{1}{2}$ in front of each of the boundary terms for $t = 0$ and $t = T$. Thus we end up with

$$\begin{aligned} &\iint \left(|u(x,t) - v(x,t)|\, \psi_t + q(u,v)\psi_x \right)\, dt\, dx \\ &\quad - \int |u(x,T) - v(x,T)|\, \psi(x,T)\, dx + \int |u_0(x) - v_0(x)|\, \psi(x,0)\, dx \geq 0. \end{aligned} \tag{2.60}$$

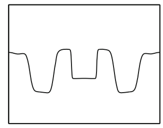
In order to exploit (2.60), we define ψ as

$$\psi(x,t) = \left(\chi_{[-M+Lt+\varepsilon, M-Lt-\varepsilon]} * \omega_\varepsilon\right)(x), \tag{2.61}$$

for $t \in [0,T]$. Here L denotes the Lipschitz constant of f, $\chi_{[a,b]}$ the characteristic function of the interval $[a,b]$, and $*$ the convolution product. We make the constant M so large that $M - Lt - \varepsilon > -M + Lt + 3\varepsilon$ for $t < T$. In order to make ψ an admissible test function, we modify it to go smoothly to zero for $t > T$.

We can compute for $t < T$,

$$\begin{aligned} \psi_t &= \frac{d}{dt} \int\limits_{-M+Lt+\varepsilon}^{M-Lt-\varepsilon} \omega_\varepsilon(x-y)\, dy \\ &= -L\left(\omega_\varepsilon\left(x - M + Lt + \varepsilon\right) + \omega_\varepsilon\left(x + M - Lt - \varepsilon\right)\right) \leq 0 \end{aligned} \tag{2.62}$$



and

$$\psi_x = -\left(\omega_\varepsilon\left(x - M + Lt + \varepsilon\right) - \omega_\varepsilon\left(x + M - Lt - \varepsilon\right)\right). \tag{2.63}$$

With our choice of M, the two functions on the right-hand side of (2.63) have nonoverlapping support. Therefore,

$$0 = \psi_t + L\left|\psi_x\right| \geq \psi_t + \frac{q(u,v)}{|u-v|}\psi_x,$$

and hence[6]

$$|u-v|\,\psi_t + q(u,v)\psi_x \leq 0.$$

Using this in (2.60) and letting ε go to zero, we find that

$$\int_{-M+Lt}^{M-Lt} |u(x,t) - v(x,t)|\; dx \leq \int_{-M}^{M} |u_0(x) - v_0(x)|\; dx. \tag{2.64}$$

By letting $M \to \infty$, we find that

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \tag{2.65}$$

in this case. Thus we have proved the following result.

Proposition 2.10 *Assume that f is Lipschitz continuous, and let $u, v \in L^1 \cap L^\infty(\mathbb{R} \times (0,\infty))$ be weak solutions of the initial value problems*

$$\begin{aligned} u_t + f(u)_x &= 0, \quad u|_{t=0} = u_0, \\ v_t + f(v)_x &= 0, \quad v|_{t=0} = v_0, \end{aligned}$$

respectively, satisfying the Kružkov entropy condition. Then

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}. \tag{2.66}$$

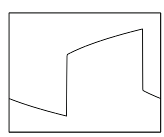
In particular, if $u_0 = v_0$, then $u = v$.

In other words, we have shown, starting from the Kružkov formulation of the entropy condition, that the initial value problem is stable in L^1, assuming the existence of solutions.

The idea is now to obtain the existence of solutions using front tracking; for Riemann initial data and continuous piecewise linear flux functions, we already have existence from Corollary 2.8. For given initial data and flux function, we show that the solution can be obtained by approximating with front-tracking solutions.

Now that we have shown stability with respect to the initial data, we proceed to study how the solution varies with the flux function. We start by studying two

[6] Recall that we had the same property in the linear case; see (1.54).



Riemann problems with the same initial data, but with different flux functions. Let u and v be the weak solutions of

$$u_t + f(u)_x = 0, \quad v_t + g(v)_x = 0 \tag{2.67}$$

with common initial data

$$u(x,0) = v(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0. \end{cases}$$

We assume that both f and g are continuous and piecewise linear with the same breakpoints. The solutions u and v of (2.67) will be piecewise constant functions of x/t that are equal outside a finite interval in x/t. More concretely, we have that

$$u(x,t) = v(x,t) = u_l \text{ if } x < \sigma_m t, \text{ and } u(x,t) = v(x,t) = u_r \text{ if } x > \sigma_M t.$$

We have to estimate the difference in L^1 between the two solutions.

Lemma 2.11 *The following inequality holds*:

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} \le t\,\|f - g\|_{\mathrm{Lip}}\,|u_l - u_r|\,, \tag{2.68}$$

where the Lipschitz seminorm is taken over over all u between u_l and u_r.

Proof Assume that $u_l \le u_r$; the case $u_l \ge u_r$ is similar. Consider first the case in which f and g both are convex. Without loss of generality we may assume that f and g have common breakpoints $u_l = w_1 < w_2 < \cdots < w_n = u_r$, and let the speeds be denoted by

$$f'|_{(w_j,w_{j+1})} = s_j \text{ and } g'|_{(w_j,w_{j+1})} = \tilde{s}_j\,.$$

Then

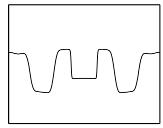
$$\int_{u_l}^{u_r} |f'(u) - g'(u)|\,du = \sum_{j=1}^{n-1} \left|s_j - \tilde{s}_j\right| (w_{j+1} - w_j).$$

Let σ_j be an ordering, that is, $\sigma_j < \sigma_{j+1}$, of all the speeds $\{s_j, \tilde{s}_j\}$. Then we may write

$$\begin{aligned} u(x,t)|_{x\in(\sigma_j t,\sigma_{j+1} t)} &= u_{j+1}, \\ v(x,t)|_{x\in(\sigma_j t,\sigma_{j+1} t)} &= v_{j+1}, \end{aligned}$$

where both u_{j+1} and v_{j+1} are from the set of all possible breakpoints, namely $\{w_1, w_2, \ldots, w_n\}$, and $u_j \le u_{j+1}$ and $v_j \le v_{j+1}$. Thus

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} = t \sum_{j=1}^{m} \left|u_{j+1} - v_{j+1}\right| (\sigma_{j+1} - \sigma_j).$$



However, we see that

$$\begin{aligned}
&\sum_{j=1}^{n-1} \left|s_j - \tilde{s}_j\right| (w_{j+1} - w_j) \\
&\quad = \sum_{j=1}^{n-1} \sum_{k=1}^{m} (\sigma_{k+1} - \sigma_k)(w_{j+1} - w_j)[\sigma_{k+1}, \sigma_k \text{ between } s_j \text{ and } \tilde{s}_j] \\
&\quad = \sum_{j=1}^{n-1} \sum_{k=1}^{m} (\sigma_{k+1} - \sigma_k)(w_{j+1} - w_j)[w_{j+1}, w_j \text{ between } u_{k+1} \text{ and } v_{k+1}] \\
&\quad = \sum_{k=1}^{m} |u_{k+1} - v_{k+1}| \, (\sigma_{k+1} - \sigma_k),
\end{aligned} \tag{2.69}$$

where we have introduced the Iverson bracket notation

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases} \tag{2.70}$$

Remark 2.12 The equality (2.69) simply says that

$$\begin{aligned}
\int_{u_1}^{u_2} |F(u) - G(u)| \, du &= \text{the area between } F \text{ and } G \\
&= \int_{\xi_1}^{\xi_2} \left|F^{-1}(\xi) - G^{-1}(\xi)\right| d\xi,
\end{aligned}$$

where $F, G\colon [u_1, u_2] \to [\xi_1, \xi_2]$ are two nondecreasing functions such that $F(u_j) = G(u_j) = \xi_j$ for $j = 1, 2$. In the present case, the functions F and G are piecewise constant with finitely many jumps. Thus, a certain amount of care is needed in defining the inverse functions.

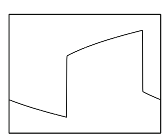
Thus we see that

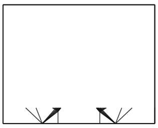
$$\begin{aligned}
\|u(\,\cdot\,, t) - v(\,\cdot\,, t)\|_{L^1} &= t \int_{u_l}^{u_r} |f'(u) - g'(u)| \; du \\
&\le t \, \|f - g\|_{\mathrm{Lip}} \, |u_l - u_r| \, .
\end{aligned} \tag{2.71}$$

The case in which f and g are not necessarily convex is more involved. We will show that

$$\int_{u_l}^{u_r} |f'_{\smile}(u) - g'_{\smile}(u)| \; du \le \int_{u_l}^{u_r} |f'(u) - g'(u)| \; du \tag{2.72}$$

when the convex envelopes are taken on the interval $[u_l, u_r]$, which together with (2.71) implies the lemma. To this end, we use the following general lemma:





Lemma 2.13 (Crandall–Tartar) *Let D be a subset of $L^1(\Omega)$, where Ω is some measure space. Assume that if ϕ and ψ are in D, then also $\phi \vee \psi = \max\{\phi, \psi\}$ is in D. Assume furthermore that there is a map $T\colon D \to L^1(\Omega)$ such that*

$$\int_\Omega T(\phi) = \int_\Omega \phi, \quad \phi \in D.$$

Then the following statements, valid for all $\phi, \psi \in D$, are equivalent:

(i) *If $\phi \le \psi$, then $T(\phi) \le T(\psi)$.*
(ii) $\int_\Omega (T(\phi) - T(\psi))^+ \le \int_\Omega (\phi - \psi)^+$, *where $\phi^+ = \phi \vee 0$.*
(iii) $\int_\Omega |T(\phi) - T(\psi)| \le \int_\Omega |\phi - \psi|$.

Proof (of Lemma 2.13) For completeness we include a proof of this lemma. Assume (i). Then $T(\phi \vee \psi) - T(\phi) \ge 0$, which trivially implies $T(\phi) - T(\psi) \le T(\phi \vee \psi) - T(\psi)$, and thus $(T(\phi) - T(\psi))^+ \le T(\phi \vee \psi) - T(\psi)$. Furthermore,

$$\int_\Omega (T(\phi) - T(\psi))^+ \le \int_\Omega (T(\phi \vee \psi) - T(\psi)) = \int_\Omega (\phi \vee \psi - \psi) = \int_\Omega (\phi - \psi)^+,$$

proving (ii). Assume now (ii). Then

$$\begin{aligned}\int_\Omega |T(\phi) - T(\psi)| &= \int_\Omega (T(\phi) - T(\psi))^+ + \int_\Omega (T(\psi) - T(\phi))^+ \\ &\le \int_\Omega (\phi - \psi)^+ + \int_\Omega (\psi - \phi)^+ \\ &= \int_\Omega |\phi - \psi|,\end{aligned}$$

which is (iii). It remains to prove that (iii) implies (ii). Let $\phi \le \psi$. For real numbers we have $x^+ = (|x| + x)/2$. This implies

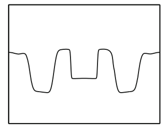
$$\begin{aligned}\int_\Omega (T(\phi) - T(\psi))^+ &= \frac{1}{2}\int_\Omega |T(\phi) - T(\psi)| + \frac{1}{2}\int_\Omega (T(\phi) - T(\psi)) \\ &\le \frac{1}{2}\int_\Omega |\phi - \psi| + \frac{1}{2}\int_\Omega (\phi - \psi) = 0. \qquad \square\end{aligned}$$

We use this lemma to prove (2.72), that is,

$$\int_{u_l}^{u_r} |f_\smile'(u) - g_\smile'(u)|\, du \le \int_{u_l}^{u_r} |f'(u) - g'(u)|\, dx. \tag{2.73}$$

In our context, we let D be the set of all piecewise constant functions on $[u_l, u_r]$. For any piecewise linear and continuous function f, its derivative f' is in D, and we define

$$T(f') = (f_\smile)',$$



where the convex envelope is taken on the full interval $[u_l, u_r]$. Then

$$\begin{aligned}\int_{u_l}^{u_r} T(f')\,du &= \int_{u_l}^{u_r} (f_{\smile})'(u)\,du = f_{\smile}(u_r) - f_{\smile}(u_l)\\ &= f(u_r) - f(u_l) = \int_{u_l}^{u_r} f'(u)\,du.\end{aligned}$$

To prove (2.73), it suffices to prove that (i) holds, that is,

$$f' \le g' \text{ implies } T(f') \le T(g')$$

for another piecewise linear and continuous flux function g. Assume otherwise, i.e., $f'_{\smile}(u) > g'_{\smile}(u)$ for some $u \in (u_l, u_r)$. Recall that both $f'_{\smile}$ and $g'_{\smile}$ are piecewise constant, and thus we set

$$u_1 = \inf_u f'_{\smile}(u) > g'_{\smile}(u) \;\text{ and }\; u_2 = \sup_u f'_{\smile}(u) > g'_{\smile}(u).$$

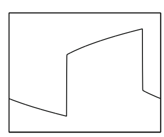
We have that $f'_{\smile}(u_1-) \le f'_{\smile}(u_1+) > g'_{\smile}(u_1+)$, and since u_1 is the smallest such value, we must have that $f'_{\smile}(u_1-) < f'_{\smile}(u_1+)$. Therefore $f(u_1) = f_{\smile}(u_1)$. Similarly we deduce that $g_{\smile}(u_2) = g(u_2)$. Using this yields

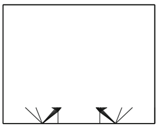
$$\begin{aligned}\int_{u_1}^{u_2} f'(u)\,du \le \int_{u_1}^{u_2} g'(u)\,du &= g(u_2) - g(u_1)\\ &\le g_{\smile}(u_2) - g_{\smile}(u_1), \quad \text{since } g_{\smile}(u) \le g(u),\\ &= \int_{u_1}^{u_2} g'_{\smile}(u)\,du < \int_{u_1}^{u_2} f'_{\smile}(u)\,du\\ &= f_{\smile}(u_2) - f_{\smile}(u_1)\\ &\le f(u_2) - f(u_1) = \int_{u_1}^{u_2} f'(u)\,du,\end{aligned}$$

which is a contradiction. Thus (2.73) follows. Hence, from (2.71) we get

$$\begin{aligned}\int_{\mathbb{R}} |u(x,t) - v(x,t)|\,dx &\le t \int_{u_l}^{u_r} |f'(u) - g'(u)|\,du\\ &\le t\,\|f - g\|_{\mathrm{Lip}}\,|u_r - u_l|. \qquad \square\end{aligned}$$

Next we consider an arbitrary piecewise constant function $u_0(x)$ with a finite number of discontinuities, and let u and v be the solutions of the initial value problem (2.67), but $u(x,0) = v(x,0) = u_0(x)$. By Lemma 2.11 applied at each of the





jumps in the initial data, the following inequality holds for all t until the first front collision:

$$\|u(t)-v(t)\|_{L^1} \le t\,\|f-g\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,. \tag{2.74}$$

This estimate holds until the first interaction of fronts for either u or v. Let t_1 be this first collision time, and let w be the weak solution constructed by front tracking of

$$w_t + f(w)_x = 0, \qquad w\,(x,t_1) = v\,(x,t_1)\,.$$

Then for $t_1 < t < t_2$, with t_2 denoting the next time two fronts of either v or u interact,

$$\begin{aligned}\|u(t)-v(t)\|_{L^1} &\le \|u(t)-w(t)\|_{L^1} + \|w(t)-v(t)\|_{L^1}\\ &\le \|u\,(t_1)-w\,(t_1)\|_{L^1} + (t-t_1)\,\|f-g\|_{\mathrm{Lip}}\,\mathrm{T.V.}\,(v\,(t_1))\,.\end{aligned} \tag{2.75}$$

However,

$$\|u\,(t_1)-w\,(t_1)\|_{L^1} = \|u\,(t_1)-v\,(t_1)\|_{L^1} \le t_1\,\|f-g\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,. \tag{2.76}$$

Recall from Corollary 2.8 that front-tracking solutions have the property that the total variation is nonincreasing. When this and (2.76) are used in (2.75), we obtain

$$\|u(t)-v(t)\|_{L^1} \le t\,\|f-g\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,. \tag{2.77}$$

This now holds for $t_1 < t < t_2$, but we can repeat the above argument inductively for every collision time t_i. Consequently, (2.77) holds for all positive t.

Now we are ready to prove the convergence of the front-tracking approximations as the piecewise linear flux function and the piecewise constant initial data converge.

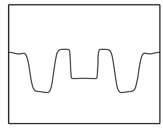
Let u_0 be a bounded function in $L^1(\mathbb{R})\cap BV(\mathbb{R})$ such that $u_0(x)\in[-M,M]$ for some positive constant M. Set $\delta_n = M/2^n$, and $u_{j,n} = j\delta_n$ for $j = -2^n,\dots,2^n$. Let f be a piecewise twice continuously differentiable function and define the piecewise linear interpolation

$$f_n(u) = f(u_{j,n}) + \frac{1}{\delta_n}\left(u-u_{j,n}\right)\left(f(u_{j+1,n})-f(u_{j,n})\right), \quad \text{for } u\in(u_{j,n},u_{j+1,n}]. \tag{2.78}$$

We assume that the possible points where f is not a twice continuously differentiable function are contained in the set of points $u_{j,n}$ for all sufficiently large n.[7] Define the approximate initial data $u_{0,n}$ to be a piecewise constant function taking values in the set $\left\{u_{j,n}\right\}_{j=-2^n}^{2^n}$ such that $\|u_{0,n}-u_0\|_{L^1}\to 0$ as $n\to\infty$. Now let u_n be the front-tracking solution to

$$(u_n)_t + f_n(u_n)_x = 0, \quad u_n(x,0) = u_{0,n}(x).$$

[7] Otherwise, we would have to replace the regular discretization by an irregular one containing the points where the second derivative does not exist.



We shall prove that the sequence $\{u_n(\,\cdot\,,t)\}$ is a Cauchy sequence in $L^1(\mathbb{R})$. Let $n_2 > n_1$, and let w solve

$$w_t + f_{n_2}(w)_x = 0, \;\; w(x,0) = u_{n_1,0}(x).$$

Then

$$\begin{aligned}
&\|u_{n_2}(\,\cdot\,,t) - u_{n_1}(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \\
&\quad \le \|u_{n_2}(\,\cdot\,,t) - w(\,\cdot\,,t)\|_{L^1(\mathbb{R})} + \|w(\,\cdot\,,t) - u_{n_1}(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \\
&\quad \le \|u_{0,n_2} - u_{0,n_1}\|_{L^1(\mathbb{R})} + t\,\mathrm{T.V.}\,(u_0)\,\|f_{n_1} - f_{n_2}\|_{\mathrm{Lip}(-M,M)}\,.
\end{aligned}$$

We have the following estimate:

$$\|f_{n_1} - f_{n_2}\|_{\mathrm{Lip}(-M,M)} \le C\delta_{n_1}\,\|f''\|_{L^\infty(-M,M)}$$

for some constant C (see Exercise 2.13).

Using this, we see that $\{u_n(\,\cdot\,,t)\}_{n\ge 0}$ is a Cauchy sequence, and thus strongly convergent to some $u(\,\cdot\,,t)$ in $L^1(\mathbb{R})$. Furthermore, recall from Corollary 2.8 that for $s \le t$,

$$\|u_n(\,\cdot\,,t) - u_n(\,\cdot\,,s)\|_{L^1(\mathbb{R})} \le \|f_n\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0)\,(t-s).$$

Since

$$\|f_n\|_{\mathrm{Lip}} \le \|f_n - f_m\|_{\mathrm{Lip}} + \|f_m\|_{\mathrm{Lip}} \le \delta_m\,\|f''\|_{L^\infty} + \|f_m\|_{\mathrm{Lip}},$$

we see that the limit u is in $C([0,\infty);L^1(\mathbb{R}))$.

It remains to show that u is an entropy solution. To this end, let η be a convex entropy and $q_n = \int^u f_n'\eta'\,du$, the corresponding entropy flux. Since u_n is the unique entropy solution taking the initial value $u_{n,0}$, we have

$$\int_0^\infty \int_{\mathbb{R}} \big(\eta(u_n)\varphi_t + q_n(u_n)\varphi_x\big)\,dx\,dt + \int_{\mathbb{R}} \eta(u_{n,0})\varphi(x,0)\,dx \ge 0.$$

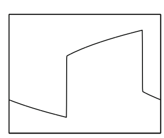
Since $u_n \to u$, and $q_n(u) \to q(u)$ in $C(-M,M)$, we have that $\eta(u_n) \to \eta(u)$, $q_n(u_n) \to q(u)$, and $\eta(u_{n,0})$ to $\eta(u_0)$ in L^1. Thus the limit u is the unique entropy solution.

If v_n is the front-tracking solution to

$$(v_n)_t + g_n(v_n)_x = 0, \;\; v_n(x,0) = v_{n,0}(x),$$

where g_n is a piecewise linear interpolation to the twice differentiable function g, we can compare the difference between u_n and v_n as follows: Let w be the solution of the initial value problem

$$w_t + f_n(w)_x = 0, \;\; w(x,0) = v_{n,0}(x).$$



Then we obtain

$$\begin{aligned}\|u_n(\cdot,t)-v_n(\cdot,t)\|_{L^1(\mathbb{R})} &\le \|u_n(\cdot,t)-w(\cdot,t)\|_{L^1}+\|w(\cdot,t)-v_n(\cdot,t)\|_{L^1}\\ &\le \|u_{0,n}-v_{0,n}\|_{L^1}+t\|f_n-g_n\|_{\mathrm{Lip}}\mathrm{T.V.}\,(v_0)\end{aligned}$$

by combining Proposition 2.10 and (2.77). By interchanging the roles of flux functions f and g and the initial data $u_{0,n}$ and $v_{0,n}$ in the definition of w, we infer

$$\begin{aligned}&\|u_n(\cdot,t)-v_n(\cdot,t)\|_{L^1(\mathbb{R})}\\ &\qquad\le \|u_{0,n}-v_{0,n}\|_{L^1(\mathbb{R})}+t\,\|f_n-g_n\|_{\mathrm{Lip}}\min\{\mathrm{T.V.}\,(u_0)\,,\mathrm{T.V.}\,(v_0)\}. \quad (2.79)\end{aligned}$$

Thus we have proved the following theorem.

Theorem 2.14 *Let u_0 be a function of bounded variation that is also in L^1, and let $f(u)$ be a piecewise twice continuously differentiable function. Then there exists a unique weak solution $u=u(x,t)$ to the initial value problem*

$$u_t+f(u)_x=0, \quad u(x,0)=u_0(x),$$

which also satisfies the Kružkov entropy condition (2.50). Furthermore, if v_0 is another function in $BV\cap L^1(\mathbb{R})$, $g(v)$ is a piecewise twice continuously differentiable function, and v is the unique weak Kružkov entropy solution to

$$v_t+g(v)_x=0, \quad v(x,0)=v_0(x),$$

then

$$\begin{aligned}\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbb{R})} &\le \|u_0-v_0\|_{L^1(\mathbb{R})}\\ &\qquad+t\min\{\mathrm{T.V.}\,(u_0)\,,\mathrm{T.V.}\,(v_0)\}\,\|f-g\|_{\mathrm{Lip}}. \quad (2.80)\end{aligned}$$

We end this section by summarizing some of the fundamental properties of solutions of scalar conservation laws in one dimension.

Theorem 2.15 *Let u_0 be an integrable function of bounded variation, and let $f(u)$ be a piecewise twice continuously differentiable function. Then the unique weak entropy solution $u=u(x,t)$ to the initial value problem*

$$u_t+f(u)_x=0, \quad u(x,0)=u_0(x), \tag{2.81}$$

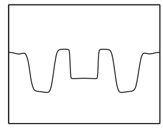
satisfies the following properties for all $t\in[0,\infty)$:

(i) *Maximum principle*:

$$\|u(\cdot,t)\|_{L^\infty}\le\|u_0\|_{L^\infty}.$$

(ii) *Total variation diminishing (TVD)*:

$$\mathrm{T.V.}\,(u(\cdot,t))\le\mathrm{T.V.}\,(u_0)\,.$$



(iii) *L^1-contractive*: *If v_0 is a function in $BV \cap L^1(\mathbb{R})$ and $v = v(x,t)$ denotes the entropy solution with v_0 as initial data, then*

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

(iv) *Monotonicity preservation*:

$$u_0 \textit{ monotone implies } u(\,\cdot\,,t) \textit{ monotone.}$$

(v) *Monotonicity*: *Let v_0 be a function in $BV \cap L^1(\mathbb{R})$, and let $v = v(x,t)$ denote the entropy solution with v_0 as initial data. Then*

$$u_0 \le v_0 \textit{ implies } u(\,\cdot\,,t) \le v(\,\cdot\,,t).$$

(vi) *Lipschitz continuity in time*:

$$\|u(\,\cdot\,,t) - u(\,\cdot\,,s)\|_{L^1(\mathbb{R})} \le \|f\|_{\mathrm{Lip}} \,\mathrm{T.V.}\,(u_0)\,|t-s|\,,$$

for all $s, t \in [0,\infty)$.

Proof The maximum principle and the monotonicity preservation properties are all easily seen to be true for the front-tracking approximation by checking the solution of isolated Riemann problems, and the properties carry over in the limit.

Monotonicity holds by the Crandall–Tartar lemma, Lemma 2.13, applied with the solution operator $u_0 \mapsto u(x,t)$ as the operator T and with the L^1 contraction property.

The fact that the total variation is nonincreasing follows using Theorem 2.14 (with $g = f$ and $v_0 = u_0(\,\cdot\, + h)$) and

$$\begin{aligned}\mathrm{T.V.}\,(u(\,\cdot\,,t)) &= \lim_{h\to 0}\frac{1}{h}\int |u(x+h,t) - u(x,t)|\;dx\\ &\le \lim_{h\to 0}\frac{1}{h}\int |u_0(x+h) - u_0(x)|\;dx = \mathrm{T.V.}\,(u_0)\,.\end{aligned}$$

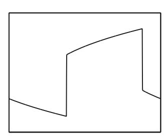
The L^1-contractivity is a special case of (2.80). Finally, to prove the Lipschitz continuity in time of the spatial L^1-norm, we first observe that by translation invariance in time it suffices to prove the result for $s = 0$. Thus

$$\|u(\,\cdot\,,t) - u_0\|_{L^1} \le t\,\|f\|_{\mathrm{Lip}}\,\mathrm{T.V.}\,(u_0)\,,$$

for all $t \in [0,\infty)$. Consider a step-function approximation $u_{0,n}$ to u_0 and a polygonal approximation f_n to f. From Corollary 2.8 we see that

$$\|u_n(\,\cdot\,,t) - u_{0,n}\|_{L^1} \le t\,\|f_n\|_{\mathrm{Lip}}\,\mathrm{T.V.}\,(u_{0,n})\,, \tag{2.82}$$

for all $t \in [0,\infty)$. From Theorem 2.14 we know that $u_n(t)$ converges to $u(t)$, the solution of (2.81). By taking the limit in (2.82), the result follows. For an alternative argument, see Theorem 7.10. □



2.5 Notes

Ofte er det jo sådan, at når man kigger det nye efter i sømmene, såer det bare sømmene, der er nye.[8]
— Kaj Munk, En Digters Vej og andre Artikler (*1948*)

The “viscous regularization” as well as the weak formulation of the scalar conservation law were studied in detail by Hopf [98] in the case of Burgers’s equation where $f(u) = u^2/2$. Hopf’s paper initiated the rigorous analysis of conservation laws. Oleĭnik, [145], [146] gave a systematic analysis of the scalar case, proving existence and uniqueness of solutions using finite differences. See also the survey by Gel’fand [69].

Kružkov’s approach, which combines the notion of weak solution and uniqueness into one equation, (2.22), was introduced in [118], in which he studied general scalar conservation laws in many dimensions with explicit time and space dependence in flux functions and a source term.

The solution of the Riemann problem when the flux function f has one or more inflection points was given by Gel’fand [69], Cho-Chun [41], and Ballou [11].

It is quite natural to approximate a flux function by a continuous and piecewise linear function. This method is frequently referred to as “Dafermos’s method” [55]. Dafermos used it to derive existence of solutions of scalar conservation laws. Prior to that, a similar approach was studied numerically by Barker [13]. Further numerical work based on Dafermos’s paper can be found in Hedstrom [86, 87] and Swartz and Wendroff [172]. Applications of front tracking to hyperbolic conservation laws on a half-line appeared in [175].

Unaware of this earlier development, Holden, Holden, and Høegh-Krohn rediscovered the method [90], [89] and proved L^1-stability, the extension to nonconvex flux functions, and that the method in fact can be used as a numerical method. We here use the name “front-tracking method” as a common name for this approach and an analogous method that works for systems of hyperbolic conservation laws. We combine the front-tracking method with Kružkov’s ingenious method of “doubling the variables”; Kružkov’s method shows stability (and thereby uniqueness) of the solution, and we use front tracking to construct the solution.

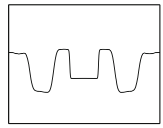
The original argument in [89] followed a direct but more cumbersome analysis. An alternative approach to show convergence of the front-tracking approximation is first to establish boundedness of the approximation in both L^∞ and total variation, and then to use Helly’s theorem to deduce convergence. Subsequently one has to show that the limit is a Kružkov entropy solution, and finally, invoke Kuznetsov’s theory to conclude stability in the sense of Theorem 2.14. We will use this argument in Chapts. 3 and 4.

Lemma 2.13 is due to Crandall and Tartar; see [54] and [52].

The L^1-contractivity of solutions of scalar conservation laws is due to Volpert [186]; see also Keyfitz [84, 151]. We simplify the presentation by assuming solutions of bounded variation.

The uniqueness result, Theorem 2.14, was first proved by Lucier [140], using an approach due to Kutznetsov [120]. Our presentation here is different in that we

[8] It is said that to tell if a cloth is new, you should examine the seams; yet what you thought you knew, may not be what it seems, for it might be the seams that are all that is new.



avoid Kutznetsov's theory; see Sect. 3.3. For an alternative proof of (2.65) we refer to Málek et al. [141, pp. 92 ff].

The term "front tracking" is also used to denote other approaches to hyperbolic conservation laws. Glimm and coworkers [73–76] have used a front-tracking method as a computational tool. In their approach, the discontinuities, or shocks, are introduced as independent computational objects and moved separately according to their own dynamics. Away from the shocks, traditional numerical methods can be employed. This method yields sharp fronts. The name "front tracking" is also used in connection with level set methods, in particular in connection with Hamilton–Jacobi equations; see, e.g., [149]. Here one considers the dynamics of interfaces or fronts. These methods are distinct from those treated in this book.

A different approach to hyperbolic conservation laws is based on the so-called kinetic formulation. The approach allows for a complete existence theory in the scalar case for initial data that are only assumed to be integrable. See Perthame [150] for an extensive presentation of this theory.

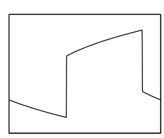
2.6 Exercises

Our problems are manmade; therefore they may be solved by man.
— John F. Kennedy (1963)

2.1 Let
$$f(u) = \frac{u^2}{u^2 + (1-u)^2}.$$
Find the solution of the Riemann problem for the scalar conservation law $u_t + f(u)_x = 0$, where $u_l = 0$ and $u_r = 1$. This equation is an example of the so-called Buckley–Leverett equation and represents a simple model of two-phase fluid flow in a porous medium. In this case u is a number between 0 and 1 and denotes the saturation of one of the phases.

2.2 In Example 1.6 one uses a linear velocity model, i.e., the velocity depends linearly on the density. Other models have been analyzed [68, Sect. 3.1.2] (v_0, $v_{\max}$, and $\rho_{\max}$ are constants):
$$\begin{aligned}
v(\rho) &= v_0 \ln\left(\frac{\rho_{\max}}{\rho}\right) \quad \text{(the Greenberg model)},\\
v(\rho) &= v_{\max} \exp\left(-\frac{\rho}{\rho_{\max}}\right) \quad \text{(the Underwood model)},\\
v(\rho) &= v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\max}}\right)^n\right) \quad n \in \mathbb{N}, \quad \text{(the Greenshield model)},\\
v(\rho) &= v_0\left(\frac{1}{\rho} - \frac{1}{\rho_{\max}}\right) \quad \text{(the California model)}.
\end{aligned}$$
Solve the Riemann problem with these velocity functions.

2.3 Consider the following initial value problem for Burgers's equation:
$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} -1 & \text{for } x < 0,\\ 1 & \text{for } x \geq 0.\end{cases}$$



(a) Show that $u(x,t) = u(x,0)$ is a weak solution.
(b) Let

$$u_0^\varepsilon(x) = \begin{cases} -1 & \text{for } x < -\varepsilon, \\ x/\varepsilon & \text{for } -\varepsilon \le x \le \varepsilon, \\ 1 & \text{for } \varepsilon < x. \end{cases}$$

Find the solution $u^\varepsilon(x,t)$ of Burgers's equation if $u(x,0) = u_0^\varepsilon(x)$.
(c) Find $\bar{u}(x,t) = \lim_{\varepsilon\downarrow 0} u^\varepsilon(x,t)$.
(d) Since $\bar{u}(x,0) = u_0(x)$, why do we not have $\bar{u} = u$?

2.4 For $\varepsilon > 0$, consider the linear viscous regularization

$$u_t^\varepsilon + a u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, \qquad u^\varepsilon(x,0) = u_0(x) = \begin{cases} u_l, & \text{for } x \le 0, \\ u_r, & \text{for } x > 0, \end{cases}$$

where a is a constant. Show that

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(x,t) = \begin{cases} u_l, & \text{for } x < at, \\ u_r, & \text{for } x > at, \end{cases}$$

and thus that $u^\varepsilon \to u_0(x-at)$ in $L^1(\mathbb{R}\times[0,T])$.

2.5 This exercise outlines another way to prove monotonicity. If u and v are entropy solutions, then we have

$$\iint \left[(u-v)\psi_t + (f(u)-f(v))\psi_x\right]\,dx\,dt - \int (u-v)\psi\,\Big|_0^T\,dx = 0.$$

Set $\Phi(\sigma) = |\sigma| + \sigma$, and use (2.60) to conclude that

$$\iint \big(\Phi(u-v)\psi_t + \Psi(u,v)\psi_x\big)\,dx\,dt - \int \Phi(u-v)\psi\,\Big|_0^T\,dx \ge 0 \qquad (2.83)$$

for a Lipschitz continuous Ψ. Choose a suitable test function ψ to show that (2.83) implies the monotonicity property.

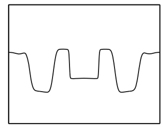
2.6 Let $c(x)$ be a continuous and locally bounded function. Consider the conservation law with "coefficient" c,

$$u_t + c(x)f(u)_x = 0, \quad u(x,0) = u_0(x). \qquad (2.84)$$

(a) Define the characteristics for (2.84).
(b) What is the Rankine–Hugoniot condition in this case?
(c) Set $f(u) = u^2/2$, $c(x) = 1 + x^2$, and

$$u_0(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x \ge 0. \end{cases}$$

Find the solution of (2.84) in this case.
(d) Formulate a front-tracking algorithm for the general case of (2.84).
(e) What is the entropy condition for (2.84)?



2.7 Consider the conservation law where the x dependency is "inside the derivation,"

$$u_t + (c(x) f(u))_x = 0. \tag{2.85}$$

The coefficient c is assumed to be continuously differentiable.

(a) Define the characteristics for (2.85).
(b) What is the entropy condition for this problem?
(c) Modify the proof of Proposition 2.10 to show that if u and v are entropy solutions of (2.85) with initial data u_0 and v_0, respectively, then

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})}\,.$$

2.8 Let η and q be an entropy/entropy flux pair as in (2.17). Assume that u is a piecewise continuous solution (in the distributional sense) of

$$\eta(u)_t + q(u)_x \le 0.$$

Show that across any discontinuity u satisfies

$$\sigma\,(\eta_l - \eta_r) - (q_l - q_r) \ge 0,$$

where σ is the speed of the discontinuity, and $q_{l,r}$ and $\eta_{l,r}$ are the values to the left and right of the discontinuity.

2.9 Consider the initial value problem for (the inviscid) Burgers's equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0, \quad u(x,0) = u_0(x),$$

and assume that the entropy solution is bounded. Set $\eta = \frac{1}{2}u^2$, and find the corresponding entropy flux $q(u)$. Then choose a test function $\psi(x,t)$ to show that

$$\|u(\,\cdot\,,t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}\,.$$

If v is another bounded entropy solution of Burgers's equation with initial data v_0, do we have $\|u - v\|_{L^2(\mathbb{R})} \le \|u_0 - v_0\|_{L^2(\mathbb{R})}$?

2.10 Define the positive and negative part of a number $x \in \mathbb{R}$ by $x^\pm = \frac{1}{2}(|x| \pm x)$. Show that

$$\left\|(u(\,\cdot\,,t) - v(\,\cdot\,,t))^\pm\right\|_1 \le \left\|(u_0 - v_0)^\pm\right\|_1,$$

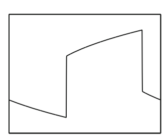
where u and v are weak entropy solutions of the equation $u_t + f(u)_x = 0$ with initial data u_0 and v_0, respectively.

2.11 Consider the scalar conservation law with a zeroth-order term

$$u_t + f(u)_x = g(u), \tag{2.86}$$

where $g(u)$ is a locally bounded and Lipschitz continuous function.

(a) Determine the Rankine–Hugoniot relation for (2.86).
(b) Find the entropy condition for (2.86).



2.12 The initial value problem

$$v_t + H(v_x) = 0, \quad v(x,0) = v_0(x), \tag{2.87}$$

is called a Hamilton–Jacobi equation. One is interested in solving (2.87) for $t > 0$, and the initial function v_0 is assumed to be bounded and uniformly continuous. Since the differentiation is inside the nonlinearity, we cannot define solutions in the distributional sense as for conservation laws. A *viscosity solution* of (2.87) is a bounded and uniformly continuous function v such that for all test functions φ, the following hold:

$$\text{subsolution} \begin{cases} \text{if } v - \varphi \text{ has a local maximum at } (x,t), \text{ then} \\ \varphi(x,t)_t + H(\varphi(x,t)_x) \le 0, \end{cases}$$

$$\text{supsolution} \begin{cases} \text{if } v - \varphi \text{ has a local minimum at } (x,t), \text{ then} \\ \varphi(x,t)_t + H(\varphi(x,t)_x) \ge 0. \end{cases}$$

If we set $p = v_x$, then formally p satisfies the conservation law

$$p_t + H(p)_x = 0, \quad p(x,0) = \partial_x v_0(x). \tag{2.88}$$

Assume that

$$v_0(x) = v_0(0) + \begin{cases} p_l x & \text{for } x \le 0, \\ p_r x & \text{for } x > 0, \end{cases}$$

where p_l and p_r are constants. Let p be an entropy solution of (2.88) and set

$$v(x,t) = v_0(0) + xp(x,t) - tH(p(x,t)).$$

Show that v defined in this way is a viscosity solution of (2.87).

2.13 Let f be piecewise C^2. Show that if we define the continuous, piecewise linear interpolation f_δ by $f_\delta(j\delta) = f(j\delta)$, then we have

$$\|f - f_\delta\|_{\text{Lip}} \le C_1 \delta \, \|f''\|_\infty,$$

where C_1 is a constant that equals one plus the number of points where f is not twice continuously differentiable. Use this to show that

$$\|f_{n_1} - f_{n_2}\|_{\text{Lip}(-M,M)} \le 2C_1 \delta_{n_1} \, \|f''\|_{L^\infty(-M,M)},$$

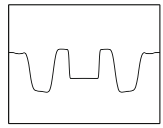
where f_n is defined by (2.78).

2.14 (a) Let f be a continuous function on $[a,b]$. Show that

$$f_\smile(u) = f^{**}(u), \quad u \in [a,b],$$

where $f^{**} = (f^*)^*$, and f^* denotes the Legendre transform

$$f^*(u) = \max_{v \in [a,b]} \big(uv - f(v)\big), \quad u \in [a,b].$$



(b) Let $u(\xi) = (f'_{\smile})^{-1}(\xi)$. Show that

$$u(\xi) = \frac{d}{d\xi} f^*(\xi).$$

This provides an alternative formula for the solution of the Riemann problem in Sect. 2.2.

2.15 Find the unique weak entropy solution of the initial value problem (cf. Exercise 2.11)

$$u_t + \left(\frac{1}{2}u^2\right)_x = -u,$$
$$u|_{t=0} = \begin{cases} 1 & \text{for } x \leq -\frac{1}{2}, \\ -2x & \text{for } -\frac{1}{2} < x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

2.16 Find the weak entropy solution of the initial value problem

$$u_t + (e^u)_x = 0, \quad u(x,0) = \begin{cases} 2 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

2.17 Find the weak entropy solution of the initial value problem

$$u_t + \left(u^3\right)_x = 0$$

with initial data

(a) $u(x,0) = \begin{cases} 1 & \text{for } x < 2, \\ 0 & \text{for } x \geq 2, \end{cases}$

(b) $u(x,0) = \begin{cases} 0 & \text{for } x < 2, \\ 1 & \text{for } x \geq 2. \end{cases}$

2.18 Find the weak entropy solution of the initial value problem

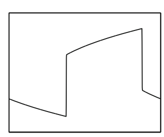
$$u_t + \frac{1}{2}\left(u^2\right)_x = 0$$

with initial data

$$u(x,0) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

2.19 Redo Example 2.5 with the same flux function but initial data

$$u_0(x) = \begin{cases} -1 & \text{for } x \leq x_1, \\ 1 & \text{for } x_1 < x < x_2, \\ -1 & \text{for } x \geq x_2. \end{cases}$$

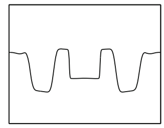


2.20 Show that the L^1 norm is continuous with respect to translations, that is,

$$\|\phi(\cdot\,+x)-\phi\|_{L^1(\mathbb{R})} \underset{|x|\to 0}{\to} 0,$$

for every $\phi \in L^1(\mathbb{R})$. (The same result is true if $L^1(\mathbb{R})$ is replaced by $L^p(M)$ for any Lebesgue measurable set $M \subseteq \mathbb{R}$ and $p \in [1,\infty)$.)

2.21 Show that the solution constructed in Corollary 2.8 satisfies the Kružkov entropy condition.



Chapter 3

A Short Course in Difference Methods

Computation will cure what ails you.
— Clifford Truesdell, The Computer, Ruin of Science and Threat to Mankind, 1980/1982

Although front tracking can be thought of as a numerical method, and has indeed been shown to be excellent for one-dimensional conservation laws, it is not part of the standard repertoire of numerical methods for conservation laws. Traditionally, difference methods have been central to the development of the theory of conservation laws, and the study of such methods is very important in applications.

This chapter is intended to give a brief introduction to difference methods for conservation laws. The emphasis throughout will be on methods and general results rather than on particular examples. Although difference methods and the concepts we discuss can be formulated for systems, we will exclusively concentrate on scalar equations. This is partly because we want to keep this chapter introductory, and partly due to the lack of general results for difference methods applied to systems of conservation laws.

3.1 Conservative Methods

We are interested in numerical methods for the scalar conservation law in one dimension. (We will study multidimensional problems in Chapter 4.) Thus we consider

$$u_t + f(u)_x = 0, \qquad u|_{t=0} = u_0. \tag{3.1}$$

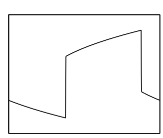
A difference method is created by replacing the derivatives by finite differences, e.g.,

$$\frac{\Delta u}{\Delta t} + \frac{\Delta f(u)}{\Delta x} = 0. \tag{3.2}$$

Here Δt and Δx are small positive numbers. We shall use the notation

$$u_j^n \approx u\left(j\Delta x, n\Delta t\right) \quad \text{and} \quad u^n = \left(u_{-K}^n, \dots, u_j^n, \dots, u_K^n\right),$$

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_3



where u_j^n now is our numerical approximation to the solution u of (3.1) at the point $(j\Delta x, n\Delta t)$. Normally, since we are interested in the initial value problem (3.1), we know the initial approximation

$$u_j^0, \quad -K \le j \le K,$$

and we want to use (3.2) to calculate u^n for $n \in \mathbb{N}$. We will not say much about boundary conditions in this book. Often one assumes that the initial data is periodic, i.e.,

$$u_{-K+j}^0 = u_{K+j}^0, \quad \text{for } 0 \le j \le 2K,$$

which gives $u_{-K+j}^n = u_{K+j}^n$. Another commonly used device is to assume that $\partial_x f(u) = 0$ at the boundary of the computational domain. For a numerical scheme this means that

$$f\left(u_{-K-j}^n\right) = f\left(u_{-K}^n\right) \quad \text{and} \quad f\left(u_{K+j}^n\right) = f\left(u_K^n\right) \quad \text{for } j > 0.$$

For nonlinear equations, *explicit* methods are most common. These can be written

$$u^{n+1} = G\left(u^n, \dots, u^{n-l}\right) \tag{3.3}$$

for some function G. We see that u^{n+1} can depend on the previous $l+1$ approximations $u^n, \dots, u^{n-l}$. The simplest methods are those with $l = 0$, where $u^{n+1} = G(u^n)$, and we shall restrict ourselves to such methods in this presentation.

◇ Example 3.1 (A nonconservative method)
Consider Burgers's equation written in nonconservative form (writing uu_x instead of $\frac{1}{2}(u^2)_x$)

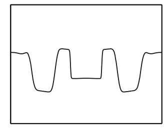
$$u_t + uu_x = 0.$$

Based on the linear transport equation, if $u_j^n > 0$, a natural discretization of this would be

$$u_j^{n+1} = u_j^n - \lambda u_j^n\left(u_j^n - u_{j-1}^n\right), \tag{3.4}$$

with $\lambda = \Delta t/\Delta x$. Since it is based on the nonconservative formulation, we do not automatically have conservation of u. Indeed,

$$\begin{aligned}
\Delta x \sum_j u_j^{n+1} &= \Delta x \sum_j u_j^n - \lambda \Delta x \sum_j u_j^n \left(u_j^n - u_{j-1}^n\right), \\
&= \Delta x \sum_j u_j^n - \frac{1}{2}\lambda \Delta x \sum_j \left(\left(u_j^n\right)^2 - \left(u_{j-1}^n\right)^2 + \left(u_j^n - u_{j-1}^n\right)^2 \right) \\
&= \Delta x \sum_j u_j^n - \frac{1}{2}\lambda \Delta x \sum_j \left(u_j^n - u_{j-1}^n\right)^2.
\end{aligned}$$



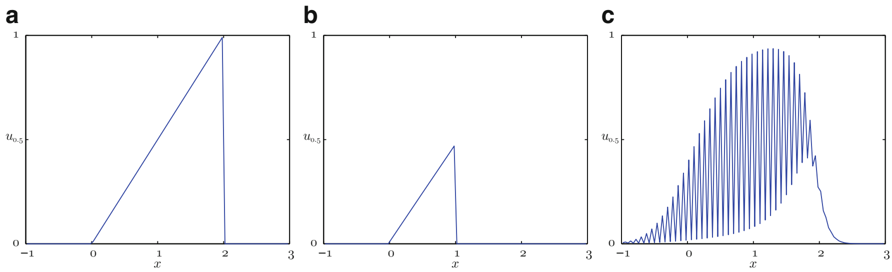


Fig. 3.1 **a** The entropy solution; **b** the scheme (3.4); **c** the scheme (3.5)

This in itself might not seem so bad, since it may happen that $\Delta x \sum_j (u_j^n - u_{j-1}^n)^2$ vanishes as $\Delta x \to 0$. However, let us examine what happens in a specific case. Let the initial data be given by

$$u_0(x) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The entropy solution to Burgers's equation consists of a rarefaction wave, centered at $x = 0$, and a shock with left value $u = 1$ and right value $u = 0$, starting from $x = 1$ and moving to the right with speed $1/2$. At $t = 2$ the rarefaction wave will catch up with the shock. Thus at $t = 2$ the entropy solution reads

$$u(x, 2) = \begin{cases} \frac{x}{2} & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

We use $u_j^0 = u_0(j\Delta x)$ as initial data for the scheme. Then we have that for every j such that $j\Delta x > 1$, $u_j^n = 0$ for all $n \ge 0$. So if $N\Delta t = 2$, then $u_j^N = 0$, and clearly $u_j^N \not\approx u(j\Delta x, 2)$ for $1 \le j\Delta x \le 2$. This method simply fails to "detect" the moving shock.

We might think that the situation would be better if we used a (second-order) approximation to u_x instead, resulting in the scheme

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\lambda}{2} u_j^n \left(u_{j+1}^n - u_{j-1}^n\right). \tag{3.5}$$

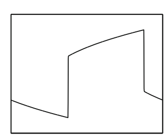
In practice, this scheme computes something that moves to the right, but the rarefaction part of the solution is not well approximated. In Fig. 3.1 we show how these two nonconservative schemes work on this example. Henceforth, we will not discuss nonconservative schemes. ◇

We call a difference method *conservative* if it can be written in the form

$$u_j^{n+1} = u_j^n - \lambda\left(F\left(u_{j-p}^n, \dots, u_{j+q}^n\right) - F\left(u_{j-1-p}^n, \dots, u_{j-1+q}^n\right)\right), \tag{3.6}$$

where

$$\lambda = \frac{\Delta t}{\Delta x}.$$



The function F is referred to as the *numerical flux*. For brevity, we shall often use the notation

$$G_j(u) = G\left(u_{j-1-p}, \dots, u_{j+q}\right),$$
$$F_{j+1/2}(u) = F\left(u_{j-p}, \dots, u_{j+q}\right),$$

so that (3.6) reads

$$u_j^{n+1} = G_j(u^n) = u_j^n - \lambda\left(F_{j+1/2}(u^n) - F_{j-1/2}(u^n)\right). \tag{3.7}$$

The above equation has a nice formal explanation. Set $x_j = j\Delta x$ and $x_{j+1/2} = x_j + \Delta x/2$ for $j \in \mathbb{Z}$. Likewise, set $t_n = n\Delta t$ for $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Define the interval $I_j = [x_{j-1/2}, x_{j+1/2})$ and the cell $I_j^n = I_j \times [t_n, t_{n+1})$. If we integrate the conservation law

$$u_t + f(u)_x = 0$$

over the cell I_j^n, we obtain

$$\int_{I_j} u(x, t_{n+1})\, dx = \int_{I_j} u(x, t_n)\, dx$$
$$+ \Big(\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt\Big).$$

Now defining u_j^n as the average of $u(x, t_n)$ in I_j, i.e.,

$$u_j^n = \frac{1}{\Delta x} \int_{I_j} u(x, t_n)\, dx,$$

we obtain the exact expression

$$u_j^{n+1} = u_j^n - \lambda\Big(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\, dt\Big).$$

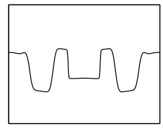
Comparing this with (3.7), we see that it is reasonable that the numerical flux $F_{j+1/2}$ approximates the average flux through the line segment $x_{j+1/2} \times [t_n, t_{n+1}]$. Thus

$$F_{j+1/2}(u^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\, dt.$$

With this interpretation of $F_{j+1/2}^n = F_{j+1/2}(u^n)$, equation (3.7) states that the change in the amount of u inside the "volume" I_j equals (approximately) the influx minus the outflux. Methods that can be written on the form (3.7) are often called *finite volume methods*.

If $u(x, t_n)$ is the piecewise constant function

$$u(x, t_n) = u_j^n \text{ for } x \in I_j,$$



we can solve the conservation law exactly for $0 \leq t - t_n \leq \Delta x / (2 \max_u |f'(u)|)$. This is true because the initial data is a series of Riemann problems, whose solutions will not interact in this short time interval. We also see that $f(u(x_{j+1/2}, t))$ is independent of t, and depends only on u_j^n and u_{j+1}^n. So if we set $v = w(x/t)$ to be the entropy solution to

$$v_t + f(v)_x = 0, \quad v(x,0) = \begin{cases} u_j^n & x < 0, \\ u_{j+1}^n & x > 0, \end{cases}$$

then

$$F_{j+1/2}^n = f(w(0)). \tag{3.8}$$

This method is called the *Godunov method*. In general, it is well defined (see Exercise 3.5) for

$$\Delta t \max |f'(u)| \leq \Delta x. \tag{3.9}$$

This last condition is called the Courant–Friedrichs–Lewy (CFL) condition.

If $f'(u) \geq 0$ for all u, then $v(0) = u_j^n$, and the Godunov method simplifies to

$$u_j^{n+1} = u_j^n - \lambda \left(f(u_j^n) - f(u_{j-1}^n) \right). \tag{3.10}$$

This is called the *upwind method*.

Conservative methods have the property that $\int u\, dx$ is conserved, since

$$\sum_{j=-K}^{K} u_j^{n+1} \Delta x = \sum_{j=-K}^{K} u_j^n \Delta x - \Delta t \left(F_{K+1/2}^n - F_{-K-1/2}^n \right).$$

If we set u_j^0 equal to the average of u_0 over the jth grid cell, i.e.,

$$u_j^0 = \frac{1}{\Delta x} \int\limits_{I_j} u_0(x)\, dx,$$

and for the moment assume that $F_{-K-1/2}^n = F_{K+1/2}^n$, then

$$\int u^n(x)\, dx = \int u_0(x)\, dx. \tag{3.11}$$

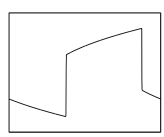
A conservative method is said to be *consistent* if

$$F(c, \ldots, c) = f(c), \tag{3.12}$$

and in addition, we demand that F be Lipschitz continuous in all its variables, that is,

$$\left| F(a_{j-p}, \ldots, a_{j+q}) - F(b_{j-p}, \ldots, b_{j+q}) \right| \leq L \sum_{i=-p}^{q} \left| a_{j+i} - b_{j+i} \right|, \tag{3.13}$$

for some constant L.



◇ Example 3.2 (Some conservative methods)
We have already seen that the Godunov method (and in particular the upwind method) is an example of a conservative finite volume method.

Another prominent examples is the *Lax–Friedrichs scheme*, usually written

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{1}{2}\lambda\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right). \tag{3.14}$$

This can be written in conservative form by defining

$$F_{j+1/2}^n = \frac{1}{2\lambda}\left(u_j^n - u_{j+1}^n\right) + \frac{1}{2}\left(f\left(u_j^n\right) + f\left(u_{j+1}^n\right)\right).$$

Some methods, so-called *two-step methods*, use iterates of the flux function. One such method is the *Richtmyer two-step Lax–Wendroff scheme*:

$$F_{j+1/2}^n = f\left(\frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{\lambda}{2}\left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right)\right)\right). \tag{3.15}$$

Another two-step method is the *MacCormack scheme*:

$$F_{j+1/2}^n = \frac{1}{2}\left(f\left(u_j^n - \lambda(f\left(u_{j+1}^n\right) - f\left(u_j^n\right))\right) + f\left(u_j^n\right)\right). \tag{3.16}$$

The Lax–Friedrichs and Godunov schemes are both of first order in the sense that the local truncation error is of order one. (We shall return to this concept below.) On the other hand, both the Lax–Wendroff and MacCormack methods are of second order. In general, higher-order methods are good for smooth solutions, but they also produce solutions that oscillate in the vicinity of discontinuities. See Sect. 3.2. Lower-order methods have "enough diffusion" to prevent oscillations. Therefore, one often uses *hybrid methods*. These methods usually consist of a linear combination of a lower- and a higher-order method. The numerical flux is then given by

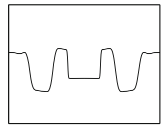
$$F_{j+1/2}^n = \theta_{j+1/2}(u^n)F_{L,j+1/2}^n + \left(1 - \theta_{j+1/2}(u^n)\right)F_{H,j+1/2}^n, \tag{3.17}$$

where F_L denotes a lower-order numerical flux, and F_H a higher-order numerical flux. The function $\theta_{j+1/2}$ is close to zero where u^n is smooth, and close to one near discontinuities. Needless to say, choosing appropriate θ's is a discipline in its own right. We have implemented a method (called *fluxlim* in Fig. 3.2) that is a combination of the (second-order) MacCormack method and the (first-order) Lax–Friedrichs scheme, and this scheme is compared with the "pure" methods in this figure. We somewhat arbitrarily used

$$\theta_{j+1/2} = 1 - \frac{1}{1 + \left|D_+D_-u_j^n\right|},$$

where $D_\pm$ are the forward and backward divided differences,

$$D_\pm u_j = \pm\frac{u_{j\pm1} - u_j}{\Delta x},$$



so that D_+D_- is an approximation to the second derivative of u with respect to x, namely

$$D_+D_-u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}.$$

Another approach is to try to generalize Godunov's method by replacing the piecewise constant data u^n by a smoother function. The simplest such replacement is by a piecewise linear function. To obtain a proper generalization, one should then solve a generalized "Riemann problem" with linear initial data to the left and right. While this is difficult to do exactly, one can use approximations instead. One such approximation leads to the following method:

$$F_{j+1/2} = \frac{1}{2}\left(g_j + g_{j+1}\right) - \frac{1}{2\lambda}\Delta_+ u_j^n.$$

Here $\Delta_\pm u_j^n = \pm(u_{j\pm1}^n - u_j^n) = \Delta x\, D_\pm u_j^n$, and

$$g_j = f(u_j^{n+1/2}) + \frac{1}{2\lambda}\tilde{u}_j,$$

where

$$\tilde{u}_j = \operatorname{minmod}\left(\Delta_- u_j^n, \Delta_+ u_j^n\right),$$
$$u_j^{n+1/2} = u_j^n - \frac{\lambda}{2} f'\left(u_j^n\right)\tilde{u}_j,$$

and

$$\operatorname{minmod}(a, b) := \frac{1}{2}\left(\operatorname{sign}(a) + \operatorname{sign}(b)\right)\min\{|a|, |b|\}.$$

This method is labeled *slopelim* in the figures. Now we show how these methods perform on two test examples. In both examples the flux function is given by (see Exercise 2.1)

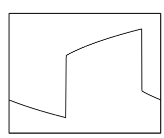
$$f(u) = \frac{u^2}{u^2 + (1-u)^2}. \tag{3.18}$$

The example is motivated by applications in oil recovery, where one often encounters flux functions that have a shape similar to that of f, that is, $f' \geq 0$ and $f''(u) = 0$ at a single point u. The model is called the *Buckley–Leverett* equation. The first example uses initial data

$$u_0(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 0 & \text{for } x > 0. \end{cases} \tag{3.19}$$

In Fig. 3.2 we show the computed solution at time $t = 1$ for all methods, using 30 grid points in the interval $[-0.1, 1.6]$, and $\Delta x = 1.7/29$, $\Delta t = 0.5\Delta x$. The second example uses initial data

$$u_0(x) = \begin{cases} 1 & \text{for } x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases} \tag{3.20}$$



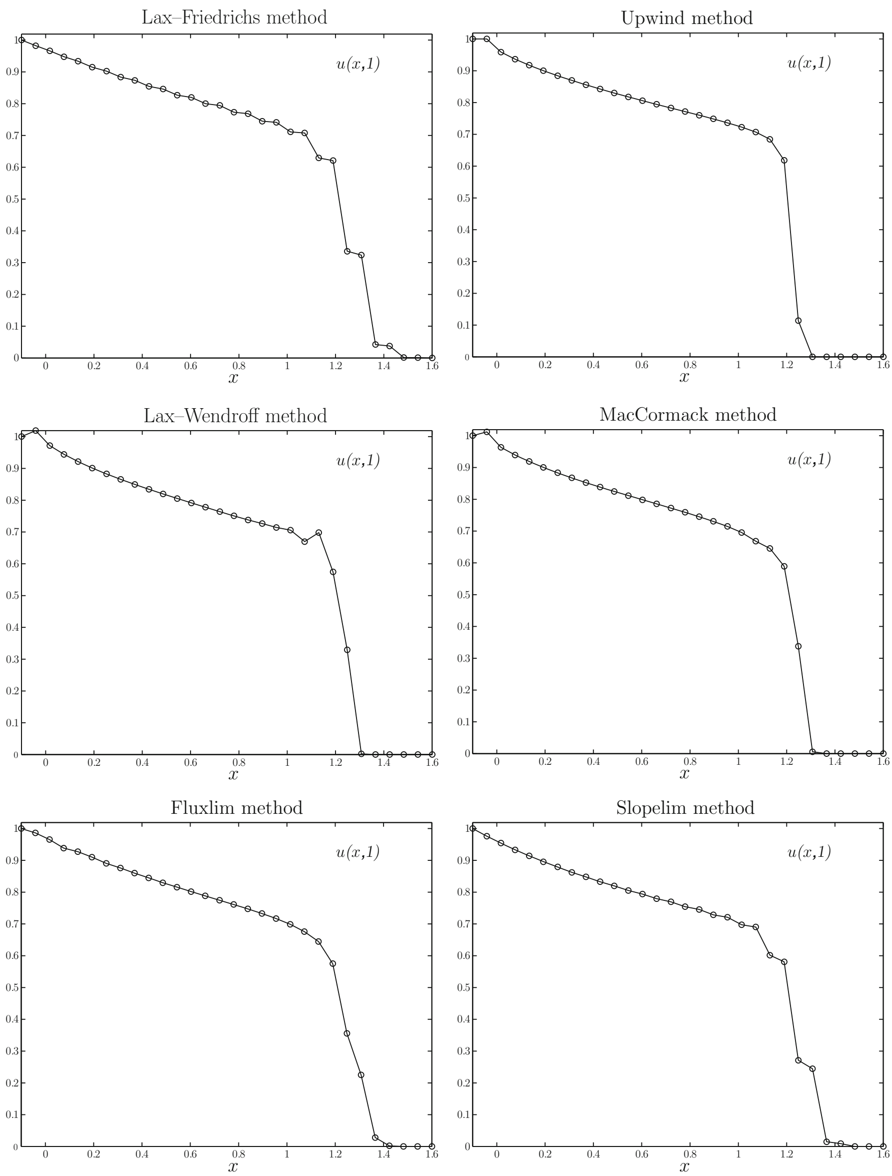
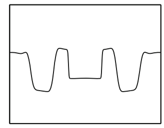


Fig. 3.2 Computed solutions at time $t = 1$ for flux function (3.18) and initial data (3.19)

and 30 grid points in the interval $[-0.1, 2.6]$, $\Delta x = 2.7/29$, $\Delta t = 0.5\Delta x$. In Fig. 3.3 we also show a reference solution computed by the upwind method using 500 grid points. The most notable feature of the plots in Fig. 3.3 is the solutions computed by the second-order methods. We shall show that if a sequence of solutions produced by a consistent conservative method converges, then the limit is a weak solution. The exact solution to both these problems can be calculated by the method of characteristics. $\diamond$



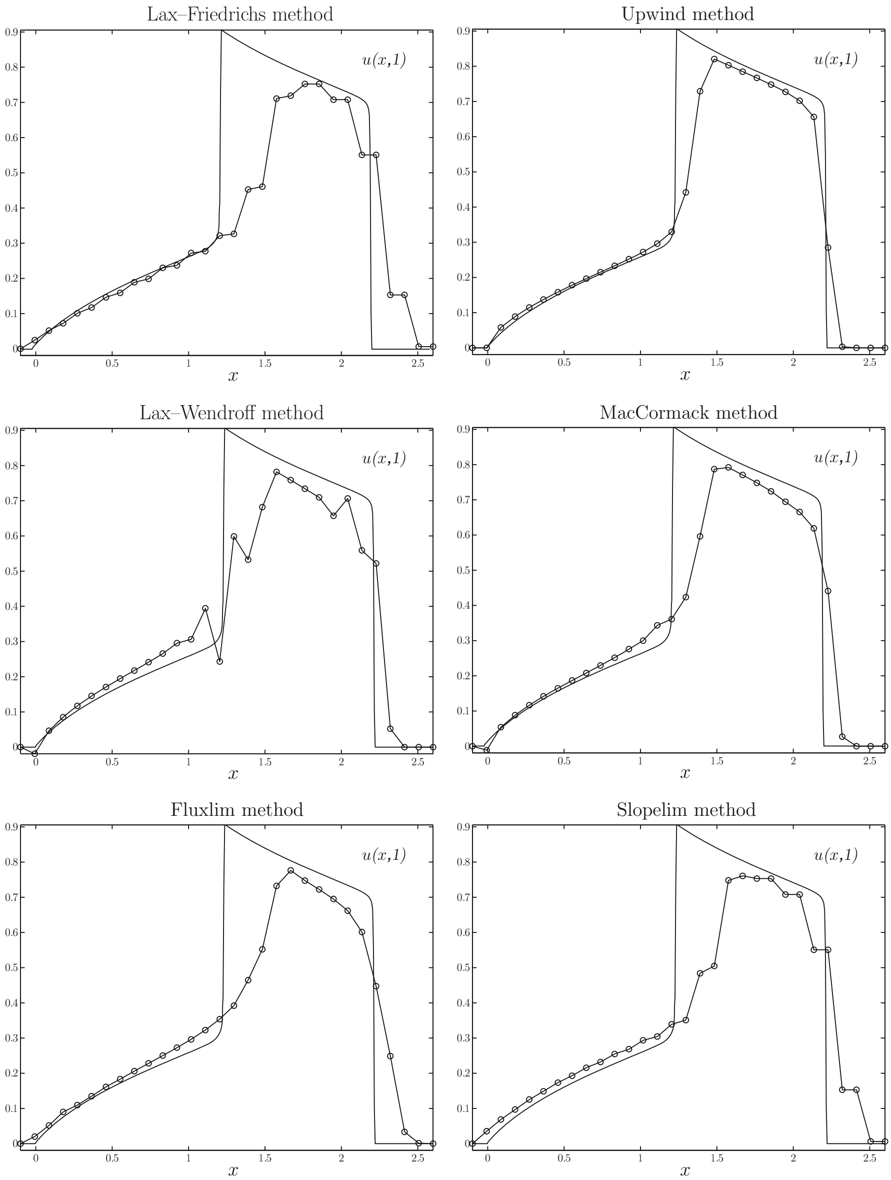
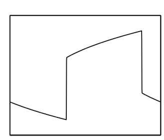


Fig. 3.3 Computed solutions at time $t = 1$ for flux function (3.18) and initial data (3.20)

The *local truncation error* of a numerical method $L_{\Delta t}$ is defined as

$$L_{\Delta t}(x) = \frac{1}{\Delta t}\left(S(\Delta t)u - S_N(\Delta t)u\right)(x), \tag{3.21}$$

where $S(t)$ is the solution operator associated with (3.1), that is, $u = S(t)u_0$ denotes the solution at time t, and $S_N(t)$ is the solution operator associated with the



numerical method, i.e.,

$$S_N(\Delta t)u(x) = u(x) - \lambda \left(F_{j+1/2}(u) - F_{j-1/2}(u)\right).$$

Assuming that we have a *smooth* solution of the conservation law, allowing us to expand all relevant quantities in Taylor series, we say that the method is of kth order if

$$|L_{\Delta t}(x)| = \mathcal{O}\left(\Delta t^k\right)$$

as $\Delta t \to 0$. To compute $L_{\Delta t}(x)$ one uses a Taylor expansion of the exact solution $u(x,t)$ near x. We know that u may have discontinuities, so it does not necessarily have a Taylor expansion. Therefore, the concept of truncation error is formal. However, if $u(x,t)$ is smooth near (x,t), then one would expect that a higher-order method would approximate u better than a lower-order method near (x,t).

◇ Example 3.3 (Local truncation error)
Consider the upwind method. Then

$$S_N(\Delta t)u(x) = u(x) - \frac{\Delta t}{\Delta x}\left(f(u(x)) - f(u(x-\Delta x))\right).$$

We verify that the upwind method is of first order:

$$\begin{aligned}
L_{\Delta t}(x) &= \frac{1}{\Delta t}\Big(u(x,t+\Delta t) - u(x,t) + \frac{\Delta t}{\Delta x}(f(u(x,t)) - f(u(x-\Delta x,t)))\Big) \\
&= \frac{1}{\Delta t}\Big(u + \Delta t\, u_t + \frac{(\Delta t)^2}{2}u_{tt} + \cdots - u \\
&\qquad + \frac{\Delta t}{\Delta x}\big(f(u) - f(u) - (-\Delta x) f(u)_x - \frac{1}{2}(-\Delta x)^2 f(u)_{xx} + \cdots\big)\Big) \\
&= u_t + f(u)_x + \frac{1}{\Delta t}\Big(\frac{(\Delta t)^2}{2}u_{tt} - \frac{\Delta t\,\Delta x}{2} f(u)_{xx} + \cdots\Big) \\
&= \frac{\Delta x}{2}\left(\lambda\, u_{tt} - f(u)_{xx}\right) + \mathcal{O}\left((\Delta t)^2\right).
\end{aligned}$$

Since u is a smooth solution of (3.1), we find that

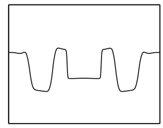
$$u_{tt} = \left((f'(u))^2 u_x\right)_x,$$

and inserting this into the previous equation, we obtain

$$L_{\Delta t} = \frac{\Delta t}{2\lambda}\frac{\partial}{\partial x}\left(f'(u)\left(\lambda f'(u) - 1\right)u_x\right) + \mathcal{O}\left((\Delta t)^2\right). \tag{3.22}$$

Hence, the upwind method is of first order. This means that Godunov's scheme is also of first order. Similarly, computations based on the Lax–Friedrichs scheme yield

$$L_{\Delta t} = \frac{\Delta t}{2\lambda^2}\frac{\partial}{\partial x}\left(\left((\lambda f'(u))^2 - 1\right)u_x\right) + \mathcal{O}\left(\Delta t^2\right). \tag{3.23}$$



Consequently, the Lax–Friedrichs scheme is indeed of first order. From the above computations it also emerges that the Lax–Friedrichs scheme is *second-order* accurate when applied to the equation (see Exercise 3.6)

$$u_t + f(u)_x = \frac{\Delta t}{2\lambda^2}\left(\left(1-(\lambda f'(u))^2\right) u_x\right)_x. \tag{3.24}$$

This is called the *model equation* for the Lax–Friedrichs scheme. In order for this to be well posed, the coefficient of u_{xx} on the right-hand side must be nonnegative, that is,

$$|\lambda f'(u)| \le 1. \tag{3.25}$$

This is a stability restriction on λ, and it is the Courant–Friedrichs–Lewy (CFL) condition that we encountered in (3.9); see also (1.50).

The model equation for the upwind method is

$$u_t + f(u)_x = \frac{\Delta t}{2\lambda}\left(f'(u)\left(1-\lambda f'(u)\right) u_x\right)_x. \tag{3.26}$$

In order for this equation to be well posed, we must have $f'(u) \ge 0$ and $\lambda f'(u) \le 1$. ◇

From the above examples, we see that first-order methods have model equations with a diffusive term. Similarly, one finds that second-order methods have model equations with a dispersive right-hand side. Therefore, the oscillations observed in the computations were to be expected.

From now on we let the function $u_{\Delta t}$ be defined by

$$u_{\Delta t}(x,t) = u_j^n, \quad \text{for } (x,t) \in I_j^n. \tag{3.27}$$

Observe that

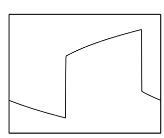
$$\int_{\mathbb{R}} u_{\Delta t}(x,t)\, dx = \Delta x \sum_j u_j^n, \qquad \text{for } t_n \le t < t_{n+1}.$$

We briefly mentioned in Example 3.2 the fact that if $u_{\Delta t}$ converges, then the limit is a weak solution. Precisely, we have the well-known Lax–Wendroff theorem.

Theorem 3.4 (Lax–Wendroff theorem) *Let $u_{\Delta t}$ be computed from a conservative and consistent method. Assume that* $\mathrm{T.V.}_x(u_{\Delta t})$ *is uniformly bounded in Δt. Consider a subsequence $u_{\Delta t_k}$ such that $\Delta t_k \to 0$, and assume that $u_{\Delta t_k}$ converges in L^1_{loc} as $\Delta t_k \to 0$. Then the limit is a weak solution to* (3.1)*.*

Proof The proof uses summation by parts. Let $\varphi(x,t)$ be a test function. For simplicity we write $\varphi_j^n = \varphi(x_j, t_n)$. By the definition of u_j^{n+1},

$$\sum_{n=0}^{N}\sum_{j=-\infty}^{\infty} \varphi_j^n\left(u_j^{n+1}-u_j^n\right) = -\frac{\Delta t}{\Delta x}\sum_{n=0}^{N}\sum_{j=-\infty}^{\infty} \varphi_j^n\left(F_{j+1/2}^n - F_{j-1/2}^n\right),$$



where we choose $T = N\Delta t$ such that $\varphi = 0$ for $t \geq T$. After a summation by parts we get

$$-\sum_{j=-\infty}^{\infty} \varphi_j^0 u_j^0 - \sum_{j=-\infty}^{\infty} \sum_{n=1}^{N} \left(\varphi_j^n - \varphi_j^{n-1}\right) u_j^n$$
$$- \frac{\Delta t}{\Delta x} \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left(\varphi_{j-1}^n - \varphi_j^n\right) F_{j+1/2}^n = 0.$$

Rearranging, we find that

$$\Delta t \Delta x \sum_{n=1}^{N} \sum_{j=-\infty}^{\infty} \left(\frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t}\right) u_j^n + \Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left(\frac{\varphi_{j-1}^n - \varphi_j^n}{\Delta x}\right) F_{j+1/2}^n$$
$$= -\Delta x \sum_{j=-\infty}^{\infty} \varphi\left(x_j, 0\right) u_j^0. \tag{3.28}$$

This almost looks like a Riemann sum for the weak formulation of (3.1). Thus

$$\Delta x \sum_{j=-\infty}^{\infty} \varphi\left(x_j, 0\right) u_j^0 \to \int_0^{\infty} \varphi(x, 0) u_0(x)\, dx$$

as $\Delta x \to 0$, and

$$\Delta t \Delta x \sum_{n=1}^{N} \sum_{j=-\infty}^{\infty} \left(\frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t}\right) u_j^n \to \int_0^{T} \int_{-\infty}^{\infty} \varphi_t(x, t) u(x, t)\, dx\, dt$$

as $\Delta x, \Delta t \to 0$.

Since

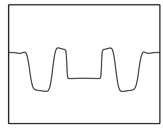
$$\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left(\frac{\varphi_{j-1}^n - \varphi_j^n}{\Delta x}\right) f(u_j^n) \to \int_0^{T} \int_{-\infty}^{\infty} \varphi_x(x, t) f(u(x, t))\, dx\, dt \tag{3.29}$$

as $\Delta x, \Delta t \to 0$, it remains to show that

$$\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left| F_{j+1/2}^n - f(u_j^n) \right| \tag{3.30}$$

tends to zero as $\Delta t \to 0$ in order to conclude that the limit is a weak solution. Using consistency, (3.12), we find that (3.30) equals

$$\Delta t \Delta x \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left| F\left(u_{j-p}^n, \ldots, u_{j+q}^n\right) - F\left(u_j^n, \ldots, u_j^n\right) \right|,$$



which by the Lipschitz continuity of F is less than

$$\begin{aligned}
&\Delta t \Delta x L \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=-p}^{q} \left| u_{j+k}^n - u_j^n \right| \\
&\qquad \le \frac{1}{2}(q(q+1) + p(p+1)) \Delta t \Delta x L \sum_{n=0}^{N} \sum_{j=-\infty}^{\infty} \left| u_{j+1}^n - u_j^n \right| \\
&\qquad \le (q^2 + p^2) \Delta x \, L \, \mathrm{T.V.}\,(u_{\Delta t}) \; T,
\end{aligned}$$

where L is the Lipschitz constant of F. Using the uniform boundedness of the total variation of $u_{\Delta x}$, we infer that (3.30) is small for small Δx, and the limit is a weak solution. □

We proved in Theorem 2.15 that the solution of a scalar conservation law in one dimension possesses several properties. The corresponding properties for conservative and consistent numerical schemes read as follows:

Definition 3.5 Let $u_{\Delta t}$ be computed from a conservative and consistent method.

(i) A method is said to be *total variation bounded (TVB)*, or *total variation stable*,[1] if the total variation of u^n is uniformly bounded, independently of Δx and Δt.

(ii) Assume that u_0 has finite total variation. We say that a numerical method is *total variation diminishing (TVD)* if $\mathrm{T.V.}\left(u^{n+1}\right) \le \mathrm{T.V.}\,(u^n)$ for all $n \in \mathbb{N}_0$.

(iii) A method is called *monotonicity preserving* if the initial data being monotone implies that u^n is monotone for all $n \in \mathbb{N}$.

(iv) Assume that $u_0 \in L^1(\mathbb{R})$. Let $v_{\Delta t}$ be another solution with initial data $v_0 \in L^1(\mathbb{R})$. A numerical method is called L^1*-contractive* if

$$\|u_{\Delta t}(t) - v_{\Delta t}(t)\|_{L^1} \le \|u_{\Delta t}(0) - v_{\Delta t}(0)\|_{L^1}$$

for all $t \ge 0$. Alternatively, we can of course write this as

$$\sum_j \left| u_j^{n+1} - v_j^{n+1} \right| \le \sum_j \left| u_j^n - v_j^n \right|, \quad n \in \mathbb{N}_0.$$

(v) A method is said to be *monotone* if for initial data u^0 and v^0, we have

$$u_j^0 \le v_j^0, \quad j \in \mathbb{Z} \quad \Rightarrow \quad v_j^n \le v_j^n, \quad j \in \mathbb{Z},\ n \in \mathbb{N}.$$

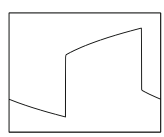
The above notions are strongly interrelated, as the next theorem shows.

Theorem 3.6 *For conservative and consistent methods the following hold*:

(i) *Assume initial data to be integrable. In that case, every monotone method is* L^1*-contractive.*

(ii) *Every* L^1*-contractive method is TVD.*

(iii) *Every TVD method is monotonicity preserving.*



[1] This definition is slightly different from the standard definition of T.V. stable methods.

Proof (i) We apply the Crandall–Tartar lemma, Lemma 2.13, with $\Omega = \mathbb{R}$, and D equal to the set of all functions in L^1 that are piecewise constant on the grid I_j, $j \in \mathbb{Z}$, and we define $T(u^0) = u^n$. Since the method is conservative (cf. (3.11)), we have that

$$\sum_j u_j^n = \sum_j u_j^0, \quad \text{or} \quad \int T(u^0)\,dx = \int u^n\,dx = \int u^0\,dx.$$

Lemma 2.13 immediately implies that (for $t \in [t_n, t_{n+1})$)

$$\begin{aligned}\|u_{\Delta t}(t) - v_{\Delta t}(t)\|_{L^1} &= \Delta x \sum_j \left|v_j^n - v_j^n\right| \le \Delta x \sum_j \left|u_j^0 - v_j^0\right| \\ &= \|u_{\Delta t}(0) - v_{\Delta t}(0)\|_{L^1}.\end{aligned}$$

(ii) Assume now that the method is L^1-contractive, i.e.,

$$\sum_j \left|u_j^{n+1} - v_j^{n+1}\right| \le \sum_j \left|u_j^n - v_j^n\right|.$$

Let v^n be the numerical solution with initial data

$$v_j^0 = u_{j+1}^0.$$

Then by the translation invariance induced by (3.6), we have $v_i^n = u_{i+1}^n$ for all n. Furthermore,

$$\begin{aligned}\text{T.V.}\left(u_j^{n+1}\right) &= \sum_j \left|u_{j+1}^{n+1} - u_j^{n+1}\right| = \sum_j \left|u_j^{n+1} - v_j^{n+1}\right| \\ &\le \sum_j \left|u_j^n - v_j^n\right| = \text{T.V.}\left(u_j^n\right).\end{aligned}$$

(iii) Consider now a TVD method, and assume that we have monotone initial data. Since $\text{T.V.}\left(u^0\right)$ is finite by assumption, the limits

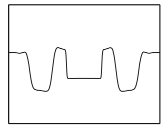
$$u_L = \lim_{j\to-\infty} u_j^0 \quad \text{and} \quad u_R = \lim_{j\to\infty} u_j^0$$

exist. Then $\text{T.V.}\left(u^0\right) = |u_R - u_L|$. If u^1 were not monotone, then $\text{T.V.}\left(u^1\right) > |u_R - u_L| = \text{T.V.}\left(u^0\right)$, which is a contradiction. □

We can summarize the above theorem as follows:

$$\text{monotone} \Rightarrow L^1\text{-contractive} \Rightarrow \text{TVD} \Rightarrow \text{monotonicity preserving}.$$

Monotonicity is relatively easy to check for explicit methods, e.g., by calculating the partial derivatives $\partial G/\partial u^i$ in (3.3).



◇ Example 3.7 (Lax–Friedrichs scheme)
Recall from Example 3.2 that the Lax–Friedrichs scheme is given by

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{1}{2}\lambda\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right).$$

Computing partial derivatives, we obtain, assuming the flux function f to be continuously differentiable,

$$\frac{\partial u_j^{n+1}}{\partial u_k^n} = \begin{cases} (1-\lambda f'(u_k^n))/2 & \text{for } k = j+1, \\ (1+\lambda f'(u_k^n))/2 & \text{for } k = j-1, \\ 0 & \text{otherwise,} \end{cases}$$

and hence we see that the Lax–Friedrichs scheme is monotone as long as the CFL condition

$$\lambda \left| f'(u) \right| \le 1$$

is fulfilled. See also Exercise 3.7. ◇

Theorem 3.8 *Fix $T > 0$. Assume that f is Lipschitz continuous. Let $u_0 \in L^1(\mathbb{R})$ have bounded variation. Assume that $u_{\Delta t}$ is computed with a method that is conservative, consistent, total variation bounded, and uniformly bounded, that is,*

$$\mathrm{T.V.}\,(u_{\Delta t}) \le M \text{ and } \|u_{\Delta t}\|_\infty \le M,$$

where M is independent of Δx and Δt.

Then $\{u_{\Delta t}(t)\}$ has a subsequence that converges for all $t \in [0,T]$ to a weak solution $u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, the limit is in $C\left([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\right)$.

Proof We intend to apply Theorem A.11. It remains to show that

$$\int_a^b \left|u_{\Delta t}(x,t) - u_{\Delta t}(x,s)\right| \, dx \le C\,|t-s| + \nu(\Delta t), \text{ as } \Delta t \to 0, \quad s,t \in [0,T],$$

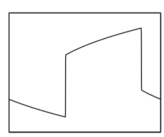
for some nonnegative continuous function ν with $\nu(0) = 0$.

The Lipschitz continuity of the flux function implies, for fixed Δt,

$$\begin{aligned} \left|u_j^{n+1} - u_j^n\right| &= \lambda \left|F_{j+1/2}^n - F_{j-1/2}^n\right| \\ &= \lambda \left|F(u_{j-p}^n, \dots, u_{j+q}^n) - F(u_{j-p-1}^n, \dots, u_{j+q-1}^n)\right| \\ &\le \lambda L \left(\left|u_{j-p}^n - u_{j-p-1}^n\right| + \dots + \left|u_{j+q}^n - u_{j+q-1}^n\right|\right), \end{aligned}$$

from which we conclude that

$$\begin{aligned} \|u_{\Delta t}(\,\cdot\,, t_{n+1}) - u_{\Delta t}(\,\cdot\,, t_n)\|_{L^1} &= \sum_j \left|u_j^{n+1} - u_j^n\right| \Delta x \\ &\le L(p+q+1)\mathrm{T.V.}\,(u^n)\,\Delta t \\ &\le L(p+q+1)M\,\Delta t, \end{aligned}$$



where L is the Lipschitz constant of F. More generally,

$$\begin{aligned}\|u_{\Delta t}(\,\cdot\,,t_m)-u_{\Delta t}(\,\cdot\,,t_n)\|_{L^1} &\le L(p+q+1)M\,|n-m|\,\Delta t\\ &= L(p+q+1)M\,|t_n-t_m|\,.\end{aligned}$$

Now let $\tau_1,\tau_2\in[0,T]$, and choose $\tilde t_1,\tilde t_2\in\{n\Delta t\mid 0\le n\le T/\Delta t\}$ such that

$$0\le\tau_j-\tilde t_j<\Delta t \text{ for } j=1,2.$$

By construction $u_{\Delta t}(\tau_j)=u_{\Delta t}(\tilde t_j)$, and hence

$$\begin{aligned}&\|u_{\Delta t}(\,\cdot\,,\tau_{L^1})-u_{\Delta t}(\,\cdot\,,\tau_2)\|_{L^1}\\ &\qquad\le\left\|u_{\Delta t}(\,\cdot\,,\tau_1)-u_{\Delta t}(\,\cdot\,,\tilde t_1)\right\|_{L^1}+\left\|u_{\Delta t}(\,\cdot\,,\tilde t_1)-u_{\Delta t}(\,\cdot\,,\tilde t_2)\right\|_{L^1}\\ &\qquad\quad+\left\|u_{\Delta t}(\,\cdot\,,\tilde t_2)-u_{\Delta t}(\,\cdot\,,\tau_2)\right\|_{L^1}\\ &\qquad\le(p+q+1)L\,M\left|\tilde t_1-\tilde t_2\right|\\ &\qquad\le(p+q+1)L\,M\,|\tau_1-\tau_2|+\mathcal{O}(\Delta t)\,.\end{aligned}$$

Observe that this estimate is uniform in $\tau_1,\tau_2\in[0,T]$. We conclude that

$$u_{\Delta t}\to u \text{ in } C([0,T];L^1([a,b]))$$

for a sequence $\Delta t\to 0$. The Lax–Wendroff theorem then says that this limit is a weak solution. □

At this point, the reader should review the concept of a Kružkov entropy condition; see Sect. 2.1. A function u is a Kružkov entropy solution of

$$u_t+f(u)_x=0$$

if it satisfies

$$\eta(u)_t+q(u)_x\le 0 \tag{3.31}$$

in the sense of distributions, where

$$\eta(u)=|u-k|\,,\quad q(u)=\operatorname{sign}(u-k)\,(f(u)-f(k)),$$

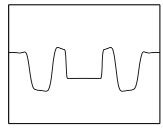
for all $k\in\mathbb{R}$.

The analogue of the Kružkov entropy pair for difference schemes reads as follows. We still employ $\eta(u)=|u-k|$. Write

$$a\vee b=\max\{a,b\}\quad\text{ and }\quad a\wedge b=\min\{a,b\},$$

and observe the trivial identity

$$|a-b|=a\vee b-a\wedge b.$$



Then we define the *numerical entropy flux* Q by

$$Q_{j+1/2}(u) = F_{j+1/2}(u \vee k) - F_{j+1/2}(u \wedge k), \tag{3.32}$$

or more explicitly,

$$Q\big(u_{j-p}, \dots, u_{j+q}\big) = F\big(u_{j-p} \vee k, \dots, u_{j+q} \vee k\big) - F\big(u_{j-p} \wedge k, \dots, u_{j+q} \wedge k\big).$$

Note that Q is consistent with the Kružkov entropy flux, i.e.,

$$Q\,(c, \dots, c) = \operatorname{sign}\,(c-k)\,(f(c) - f(k)).$$

Returning to monotone difference schemes, we have the following result.

Theorem 3.9 *Fix $T > 0$. Assume that f is Lipschitz continuous. Let $u_0 \in L^1(\mathbb{R})$ have bounded variation. Assume that $u_{\Delta t}$ is computed with a method that is conservative, consistent, and monotone.*

For every sequence $\Delta t_k \to 0$, the family $\{u_{\Delta t_k}(t)\}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to the Kružkov entropy solution $u(t)$ for all $t \in [0, T]$. Furthermore, the limit is in $C\big([0, T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$.

Proof Consider a sequence $\Delta t_k \to 0$. Theorem 3.8 allows us to conclude that $u_{\Delta t_k}$ has a subsequence that converges in $C([0, T]; L^1([a, b]))$ to a weak solution. It remains to show that the limit satisfies a discrete Kružkov form. First we find, using (3.7) and (3.32), that

$$G(u^n \vee k) - G(u^n \wedge k) = |u^n - k| - \lambda\big(Q^n_{j+1/2} - Q^n_{j-1/2}\big).$$

Using that $u^{n+1}_j = G_j(u^n)$, cf. (3.3), and the consistency of the scheme, see (3.12), which implies $k = G(k, \dots, k) = G(k)$, we conclude from the monotonicity of the scheme that

$$\begin{aligned} G_j(u^n \vee k) &\geq G_j(u^n) \vee G(k) = G_j(u^n) \vee k, \\ -G_j(u^n \wedge k) &\geq -\big(G_j(u^n) \wedge G(k)\big) = -\big(G_j(u^n) \wedge k\big). \end{aligned}$$

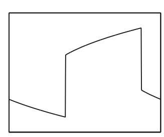
Therefore,

$$\left|u^{n+1}_j - k\right| - \left|u^n_j - k\right| + \lambda\big(Q^n_{j+1/2} - Q^n_{j-1/2}\big) \leq 0. \tag{3.33}$$

Applying the technique used in proving the Lax–Wendroff theorem to (3.33) shows that the limit u satisfies

$$\begin{aligned} &\iint \big(|u-k|\,\varphi_t + \operatorname{sign}\,(u-k)\,\big(f(u) - f(k)\big)\varphi_x\big)\,dx\,dt \\ &\quad + \int_{\mathbb{R}} |u_0 - k|\,\varphi(x, 0)\,dx - \int_{\mathbb{R}} (|u-k|\,\varphi)|_{t=T}\,dx \geq 0, \end{aligned}$$

for every nonnegative test function $\varphi \in C^\infty_0(\mathbb{R} \times [0, T])$ and for every $k \in \mathbb{R}$.



Suppose there is another subsequence for which $u_{\Delta t}$ does not converge to the entropy solution. Then by the above argument, this subsequence has another subsequence for which the limit is the unique entropy solution. The uniqueness of the limit gives a contradiction, and we conclude that for all sequences $\Delta t_k \to 0$, the sequence $\{u_{\Delta t_k}(t)\}$ converges to the unique entropy solution $u(t)$. □

Note that the above theorem offers a constructive proof of the existence of weak entropy solutions to scalar conservation laws. The fact that monotone schemes converge to the entropy solution provides an alternative to the front-tracking method discussed in Chapt. 2.

Now we shall examine the local truncation error of a general conservative, consistent, and monotone method. Since this can be written

$$\begin{aligned} u_j^{n+1} &= G_j\left(u^n\right) = G\left(u_{j-p-1}^n, \dots, u_{j+q}^n\right) \\ &= u_j^n - \lambda\left(F\left(u_{j-p}^n, \dots, u_{j+q}^n\right) - F\left(u_{j-p-1}^n, \dots, u_{j+q-1}^n\right)\right), \end{aligned}$$

we write

$$G = G(\alpha_0, \dots, \alpha_{p+q+1}) \quad \text{and} \quad F = F(\alpha_1, \dots, \alpha_{p+q+1}).$$

We assume that F, and hence G, is three times continuously differentiable with respect to all arguments, and write the derivatives with respect to the ith argument as

$$\partial_i G(\alpha_0, \dots, \alpha_{p+q+1}) \quad \text{and} \quad \partial_i F(\alpha_1, \dots, \alpha_{p+q+1}).$$

We set $\partial_i F = 0$ if $i = 0$. Throughout this calculation, we assume that the jth slot of G contains u_j^n, so that $G(\alpha_0, \dots, \alpha_{p+q+1}) = u_j - \lambda(\cdots)$. By consistency we have that

$$G(u, \dots, u) = u \quad \text{and} \quad F(u, \dots, u) = f(u).$$

Using this, we find that

$$\sum_{i=1}^{p+q+1} \partial_i F(u, \dots, u) = f'(u), \tag{3.34}$$

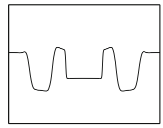
$$\partial_i G = \delta_{i,j} - \lambda\left(\partial_{i-1} F - \partial_i F\right), \tag{3.35}$$

and

$$\partial_{i,k}^2 G = -\lambda\left(\partial_{i-1,k-1}^2 F - \partial_{i,k}^2 F\right). \tag{3.36}$$

Therefore,

$$\sum_{i=0}^{p+q+1} \partial_i G(u, \dots, u) = \sum_{i=0}^{p+q+1} \delta_{i,j} = 1. \tag{3.37}$$



Furthermore,

$$\begin{aligned}\sum_{i=0}^{p+q+1}(i-j)\partial_i G(u,\dots,u) &= \sum_{i=0}^{p+q+1}\Big[(i-j)\delta_{i,j}\\ &\qquad -\lambda(i-j)\left(\partial_{i-1}F(u,\dots,u)-\partial_i F(u,\dots,u)\right)\Big]\\ &= -\lambda\sum_{i=0}^{p+q+1}((i+1)-i))\partial_i F(u,\dots,u)\\ &= -\lambda f'(u). \end{aligned}\tag{3.38}$$

We also find that

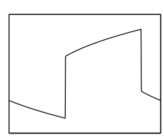
$$\begin{aligned}&\sum_{i,k=0}^{p+q+1}(i-k)^2\partial_{i,k}^2 G(u,\dots,u)\\ &\qquad = -\lambda\sum_{i,k=0}^{p+q+1}(i-k)^2\left(\partial_{i-1,k-1}^2F(u,\dots,u)-\partial_{i,k}^2F(u,\dots,u)\right)\\ &\qquad = -\lambda\sum_{i,k=0}^{p+q+1}\left(((i+1)-(k+1))^2-(i-k)^2\right)\partial_{i,k}^2F(u,\dots,u)\\ &\qquad = 0. \end{aligned}\tag{3.39}$$

Having established this, we now let $u = u(x,t)$ be a smooth solution of the conservation law (3.1). We are interested in applying G to $u(x,t)$, i.e., in calculating

$$G(u(x-(p+1)\Delta x,t)\dots,u(x,t),\dots,u(x+q\Delta x,t)).$$

Set $u_i = u(x+(i-(p+1))\Delta x,t)$ for $i=0,\dots,p+q+1$. Then we find that

$$\begin{aligned}&G(u_0,\dots,u_{p+q+1})\\ &\quad = G(u_j,\dots,u_j)+\sum_{i=0}^{p+q+1}\partial_i G(u_j,\dots,u_j)\left(u_i-u_j\right)\\ &\qquad + \frac{1}{2}\sum_{i,k=0}^{p+q+1}\partial_{i,k}^2 G(u_j,\dots,u_j)\left(u_i-u_j\right)\left(u_k-u_j\right)+\mathcal{O}\left(\Delta x^3\right)\\ &\quad = u(x,t)+u_x(x,t)\Delta x\sum_{i=0}^{p+q+1}(i-j)\partial_i G(u_j,\dots,u_j)\\ &\qquad + \frac{1}{2}u_{xx}(x,t)\Delta x^2\sum_{i=0}^{p+q+1}(i-j)^2\partial_i G(u_j,\dots,u_j)\\ &\qquad + \frac{1}{2}u_x^2(x,t)\Delta x^2\sum_{i,k=0}^{p+q+1}(i-j)(k-j)\partial_{i,k}^2 G(u_j,\dots,u_j)+\mathcal{O}\left(\Delta x^3\right)\end{aligned}$$



$$
\begin{aligned}
&= u(x,t) + u_x(x,t)\Delta x \sum_{i=0}^{p+q+1} (i-j)\partial_i G(u_j,\dots,u_j) \\
&\quad + \frac{1}{2}\Delta x^2 \sum_{i=0}^{p+q+1} (i-j)^2 \left[\partial_i G(u_j,\dots,u_j) u_x(x,t)\right]_x \\
&\quad - \frac{1}{2}\Delta x^2 u_x^2(x,t) \sum_{i,k}^{p+q+1} \left((i-j)^2 - (i-j)(k-j)\right) \partial_{i,k}^2 G(u_j,\dots,u_j) \\
&\quad + \mathcal{O}\left(\Delta x^3\right).
\end{aligned}
$$

Next we observe, since $\partial_{i,k}^2 G = \partial_{k,i}^2 G$ and using (3.39), that

$$
\begin{aligned}
0 &= \sum_{i,k} (i-k)^2 \partial_{i,k}^2 G = \sum_{i,k} ((i-j)-(k-j))^2 \partial_{i,k}^2 G \\
&= \sum_{i,k} ((i-j)^2 - 2(i-j)(k-j)) \partial_{i,k}^2 G + \sum_{i,k} (k-j)^2 \partial_{k,i}^2 G \\
&= 2 \sum_{i,k} ((i-j)^2 - (i-j)(k-j)) \partial_{i,k}^2 G.
\end{aligned}
$$

Consequently, the penultimate term in the Taylor expansion of G above is zero, and we have that

$$
\begin{aligned}
G(u(x-(p+1)\Delta x,t),\dots,u(x+q\Delta x,t)) &= u(x,t) - \Delta t f(u(x,t))_x \\
+ \frac{\Delta x^2}{2} \sum_i (i-j)^2 \left[\partial_i G(u(x,t),\dots,u(x,t)) u_x\right]_x &+ \mathcal{O}\left(\Delta x^3\right).
\end{aligned} \tag{3.40}
$$

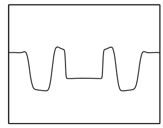
Since u is a smooth solution of (3.1), we have already established that

$$
u(x,t+\Delta t) = u(x,t) - \Delta t f(u)_x + \frac{\Delta t^2}{2} \left[\left(f'(u)\right)^2 u_x\right]_x + \mathcal{O}\left(\Delta t^3\right).
$$

Hence, we compute the local truncation error as

$$
\begin{aligned}
L_{\Delta t} &= -\frac{\Delta t}{2\lambda^2} \left[\left(\sum_{i=1}^{p+q+1} (i-j)^2 \partial_i G(u,\dots,u) - \lambda^2 (f'(u))^2\right) u_x\right]_x \\
&=: -\frac{\Delta t}{2\lambda^2} \left[\beta(u) u_x\right]_x + \mathcal{O}\left(\Delta t^2\right).
\end{aligned} \tag{3.41}
$$

Thus if $\beta > 0$, then the method is of first order. What we have done so far is valid for every conservative and consistent method where the numerical flux function is three times continuously differentiable. Next, we use that $\partial_i G \geq 0$, so that $\sqrt{\partial_i G}$



is well defined. This means that

$$
\begin{aligned}
|-\lambda f'(u)| &= \left| \sum_{i=0}^{p+q+1} (i-j)\partial_i G(u,\dots,u) \right| \\
&= \sum_{i=0}^{p+q+1} |i-j| \sqrt{\partial_i G(u,\dots,u)} \sqrt{\partial_i G(u,\dots,u)}.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality and (3.37), we find that

$$
\begin{aligned}
\lambda^2 \left(f'(u)\right)^2 &\le \sum_{i=0}^{p+q+1} (i-j)^2 \partial_i G(u,\dots,u) \sum_{i=0}^{p+q+1} \partial_i G(u,\dots,u) \\
&= \sum_{i=0}^{p+q+1} (i-j)^2 \partial_i G(u,\dots,u).
\end{aligned}
$$

Thus, $\beta(u) \ge 0$. Furthermore, the inequality is strict if more than one term in the sum on the right-hand side is different from zero. If $\partial_i G(u,\dots,u) = 0$ except for $i = k$ for some k, then $G(u_0,\dots,u_{p+q+1}) = u_k$ by (3.37). Hence the scheme is a linear translation, and by consistency, $f(u) = cu$, where $c = (j-k)\lambda$. Therefore, monotone methods for nonlinear conservation laws are at most first-order accurate. This is indeed their main drawback. To recapitulate, we have proved the following theorem:

Theorem 3.10 *Assume that the numerical flux F is three times continuously differentiable, and that the corresponding scheme is monotone. Then the method is at most first-order accurate.*

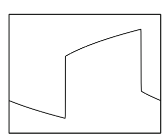
3.2 Higher-Order Schemes

We want to derive a second-order difference approximation to the solution of a conservation law

$$
u_t + f(u)_x = 0.
$$

In order to derive scheme that is second-order accurate, the local truncation error must be third-order accurate. For a smooth solution we have

$$
\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + \Delta t u_t(x,t) + \frac{\Delta t^2}{2} u_{tt}(x,t) + \mathcal{O}\left(\Delta t^3\right) \\
&= u(x,t) - \Delta t f(u(x,t))_x - \frac{\Delta t^2}{2} f(u(x,t))_{xt} + \mathcal{O}\left(\Delta t^3\right) \\
&= u - \Delta t f(u)_x + \frac{\Delta t^2}{2} \left(f'(u) f(u)_x\right)_x + \mathcal{O}\left(\Delta t^3\right).
\end{aligned}
$$



For a difference scheme we have $\Delta x = \mathcal{O}(\Delta t)$, so if the resulting scheme is of second order, the difference approximation to $f(u)_x$ must be second-order accurate, and the approximation to $(f' f_x)_x$ can be first-order accurate. We can use the following (where we write $D_0(g(x)) = (g(x+\Delta x) - g(x-\Delta x))/(2\Delta x)$) relations:

$$\begin{aligned}
f(u(x,t))_x &= D_0 f(u(x,t)) + \mathcal{O}\left(\Delta x^2\right) \\
&= \frac{f(u(x+\Delta x,t)) - f(u(x-\Delta x,t))}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right), \\
(f'(u(x,t)) f(u(x,t))_x)_x &= \frac{1}{\Delta x}\left(f'\left(u\left(x+\frac{\Delta x}{2},t\right)\right) \frac{f(u(x+\Delta x,t)) - f(u(x,t))}{\Delta x} \right. \\
&\qquad \left. - f'\left(u\left(x-\frac{\Delta x}{2},t\right)\right) \frac{f(u(x,t)) - f(u(x-\Delta x,t))}{\Delta x} \right) \\
&\qquad + \mathcal{O}\left(\Delta x^2\right), \\
f'\left(u\left(x \pm \frac{\Delta x}{2},t\right)\right) &= \frac{f(u(x\pm\Delta x,t)) - f(u(x,t))}{u(x\pm\Delta x,t) - u(x,t)} + \mathcal{O}\left(\Delta u^2\right).
\end{aligned}$$

This leads to the scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda^2}{2}\left(\nu_{j+1/2}^2 \Delta_+ u_j^n - \nu_{j-1/2}^2 \Delta_- u_j^n\right), \tag{3.42}$$

where

$$\lambda = \frac{\Delta t}{\Delta x}, \quad f_j^n = f(u_j^n), \quad \Delta_\pm v_j = \pm(v_{j\pm 1} - v_j), \quad \nu_{j+1/2} = \frac{\Delta_+ f_j^n}{\Delta_+ u_j^n}.$$

The scheme (3.42) is called the Lax–Wendroff scheme, and by construction it is of second order. We can see that it is conservative with a two-point numerical flux function given by $F_{j+1/2} = F(u_j, u_{j+1})$, where

$$F(u,v) = \frac{1}{2}\left(f(v) + f(u) - \lambda \nu^2(u,v)(v-u)\right), \quad \nu(u,v) = \frac{f(v) - f(u)}{v - u}.$$

◇ Example 3.11

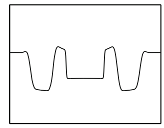
We test this second-order scheme on the equation

$$u_t + u_x = 0$$

with two sets of periodic initial data

$$u^1(x,0) = \sin^2(\pi x), \quad u^2(x,0) = \begin{cases} 1 & x \in [0.3,\ 0.7], \\ 0 & x \in [0,1] \setminus [0.3,\ 0.7], \end{cases}$$

and u^2 extended periodically. By periodicity, we know that $u^i(x,k) = u^i(x,0)$ for $k \in \mathbb{N}$. In Fig. 3.4 we have plotted the numerical solution at $t = 3$ with initial data u^1 and u^2 and $\Delta x = 1/30$. Note that for the smooth solution the method gives very



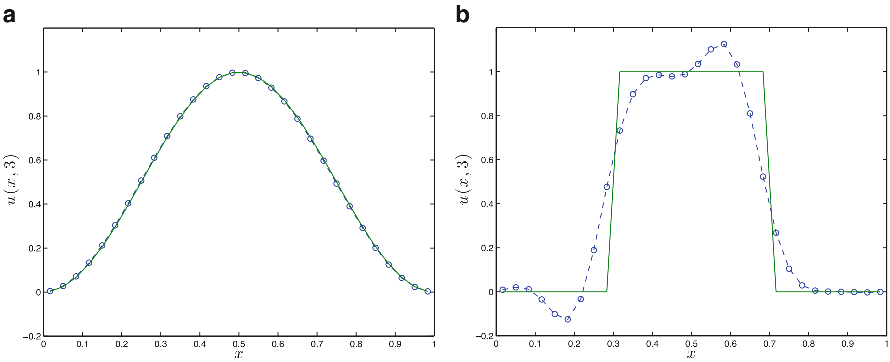


Fig. 3.4 **a** The numerical solution with initial values u^1. **b** The numerical solution with initial value u^2. We use $\Delta x = 1/30$

accurate results, and the errors are indeed of second order. For the discontinuous solution, the errors seem large, and we also see the prominent oscillations trailing the discontinuity. ◇

For simplicity we will for the moment assume that $f' \geq 0$, so that the upwind method is monotone (and hence TVD). If f is not monotone, then the upwind flux below should be replaced by a numerical flux giving a monotone method.

The Lax–Wendroff numerical flux function can be rearranged to read

$$\begin{aligned} F^n_{j+1/2} &= f(u^n_j) - \frac{1}{2}\nu_{j+1/2}\left(\lambda\nu_{j+1/2} - 1\right)\Delta_+ u^n_j \\ &= \text{upwind} + \text{second-order correction.} \end{aligned}$$

We would like to modify the Lax–Wendroff method so that it is locally of second order where the solution is smooth, and first order and monotone near discontinuities. Hence, we would like to turn off the second-order correction near discontinuities. One way of doing this is to observe that the oscillations occur near discontinuities (this is the Gibbs phenomenon), and use oscillations as an indicator of when the second-order term should be turned off. As an important side effect, this is likely to make the resulting method TVD.

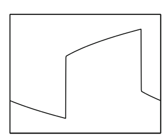
To this end let r_j (whose exact form will be specified later) be some "indicator of oscillations" near x_j. We assume that if there are oscillations, then $r_j < 0$. Let $\varphi(r)$ be a continuous function that is zero if $r < 0$.

Now we modify the numerical flux for the Lax–Wendroff method to read

$$F^n_{j+1/2} = f^n_j - \frac{1}{2}\varphi(r_j)\nu_{j+1/2}\left(\lambda\nu_{j+1/2} - 1\right)\Delta_+ u^n_j. \tag{3.43}$$

If we set

$$\alpha_{j+1/2} = \frac{1}{2}\nu_{j+1/2}(1 - \lambda\nu_{j+1/2}), \tag{3.44}$$



the modified scheme reads

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \lambda\Delta_- f_j^n - \lambda\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)\\
&= u_j^n - \lambda\nu_{j-1/2}\Delta_- u_j^n - \lambda\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)\\
&= u_j^n - \lambda\left(\nu_{j-1/2} + \lambda\frac{\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)}{\Delta_- u_j^n}\right)\Delta_- u_j^n\\
&= u_j^n - A_{j-1/2}\Delta_- u_j^n,
\end{aligned}$$

where we have defined

$$A_{j-1/2} = \nu_{j-1/2} + \lambda\frac{\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)}{\Delta_- u_j^n}.$$

At this point the following lemma is convenient.

Lemma 3.12 (Harten's lemma) *Let v_j be given by*

$$v_j = u_j - A_{j-1/2}\Delta_- u_j + B_{j+1/2}\Delta_+ u_j,$$

where $\Delta_\pm u_j = \pm(u_{j\pm 1} - u_j)$.

(i) *If $A_{j+1/2}$ and $B_{j+1/2}$ are nonnegative for all j, and $A_{j+1/2} + B_{j+1/2} \le 1$ for all j, then*

$$\mathrm{T.V.}\,(v) \le \mathrm{T.V.}\,(u)\,.$$

(ii) *If $A_{j+1/2}$ and $B_{j+1/2}$ are nonnegative for all j, and $A_{j-1/2} + B_{j+1/2} \le 1$ for all j, then*

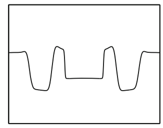
$$\min_k u_k \le v_j \le \max_k u_k, \quad j \in \mathbb{Z}.$$

Proof (i) We have

$$\begin{aligned}
\Delta_+ v_j &= u_{j+1} - u_j - A_{j+1/2}\Delta_+ u_j + B_{j+3/2}\Delta_+ u_{j+1}\\
&\quad + A_{j-1/2}\Delta_- u_j - B_{j+1/2}\Delta_+ u_j\\
&= \left(1 - A_{j+1/2} - B_{j+1/2}\right)\Delta_+ u_j + A_{j-1/2}\Delta_- u_j + B_{j+3/2}\Delta_+ u_{j+3/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_j \left|\Delta_+ v_j\right| &\le \sum_j \left(1 - A_{j+1/2} - B_{j+1/2}\right)\left|\Delta_+ u_j\right|\\
&\qquad + \sum_j A_{j-1/2}\left|\Delta_- u_j\right| + \sum_j B_{j+3/2}\left|\Delta_+ u_{j+3/2}\right|\\
&= \sum_j \left|\Delta_+ u_j\right|.
\end{aligned}$$



(ii) We may write

$$v_j = A_{j-1/2}u_{j-1/2} + (1 - A_{j-1/2} - B_{j+1/2})u_j + B_{j+1/2}u_{j+1},$$

from which the statement follows. $\square$

Returning to the scheme (3.43), we introduce

$$\alpha_{j+1/2} = \frac{1}{2}\nu_{j+1/2}\left(1 - \lambda\nu_{j+1/2}\right).$$

Hence, we get the scheme

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \lambda\left(f_j^n - f_{j-1}^n\right) \\
&\quad - \lambda\Big(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n - \varphi(r_{j-1})\alpha_{j-1/2}\Delta_- u_j^n\Big) \\
&= u_j^n - \lambda\nu_{j-1/2}\Delta_- u_j^n - \lambda\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right) \\
&= u_j^n - \lambda\left[\nu_{j-1/2} + \frac{\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)}{\Delta_- u_j^n}\right]\Delta_- u_j^n \\
&= u_j^n - A_{j-1/2}\Delta_- u_j^n.
\end{aligned}$$

We want to choose φ and r such that we can use the above lemma, with $B_{j+1/2} = 0$, to conclude that the scheme is TVD. Note that $\lambda \max_u f'(u) \le 1$ by the CFL condition and thus $\alpha_{j+1/2} \ge 0$ and $\lambda\alpha_{j+1/2} \le 1$.

We define

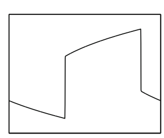
$$r_j = \frac{\alpha_{j-1/2}\Delta_- u_j}{\alpha_{j+1/2}\Delta_+ u_j}. \tag{3.45}$$

To see that this can be used as an "indicator of oscillations," note that since we have assumed that $f' \ge 0$, we have $v_{j+1/2} \ge 0$ for all j, and by the CFL condition, $\lambda v_{j+1/2} \le 1$ for all j. Hence $\alpha_{j+1/2} = \frac{1}{2}\nu_{j+1/2}(1 - \lambda v_{j+1/2}) \ge 0$ for all j. We say that "oscillations" are present at x_j if u_j is a local maximum or minimum. If so, then $\operatorname{sign}\left(\Delta_- u_j\right) \ne \operatorname{sign}\left(\Delta_+ u_j\right)$, and consequently, $r_j \le 0$. We also calculate

$$\begin{aligned}
\frac{\Delta_-\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n\right)}{\Delta_- u_j^n} &= \frac{1}{\Delta_- u_j^n}\left(\varphi(r_j)\alpha_{j+1/2}\Delta_+ u_j^n - \varphi(r_{j-1})\alpha_{j-1/2}\Delta_- u_j^n\right) \\
&= \alpha_{j-1/2}\left(\frac{\varphi(r_j)}{r_j} - \varphi(r_{j-1})\right).
\end{aligned}$$

Hence

$$A_{j+1/2} = \lambda\Big(\nu_{j+1/2} + \alpha_{j+1/2}\Big(\frac{\varphi(r_{j+1})}{r_{j+1}} - \varphi(r_j)\Big)\Big).$$



Let us assume that

$$\max\left\{\frac{\varphi(r)}{r},\varphi(r)\right\}\le 2,\quad\text{or}\ \ 0\le\varphi(r)\le\max\{0,\min\{2r,2\}\}. \tag{3.46}$$

If this assumption holds, then

$$\left|\frac{\varphi(r)}{r}-\varphi(s)\right|\le 2\ \ \text{for all } r \text{ and } s.$$

This means that

$$\begin{aligned}
A_{j+1/2}&\le\lambda\left(\nu_{j+1/2}+2\alpha_{j+1/2}\right)\\
&=\lambda\left(\nu_{j+1/2}+\nu_{j+1/2}\left(1-\lambda\nu_{j+1/2}\right)\right)\\
&=\lambda\left(2\nu_{j+1/2}-\lambda\nu_{j+1/2}^2\right)\\
&=1-(1-\lambda\nu_{j+1/2})^2\\
&\le 1.
\end{aligned}$$

For the other bound,

$$\begin{aligned}
A_{j+1/2}&\ge\lambda\left(\nu_{j+1/2}-2\alpha_{j+1/2}\right)\\
&=\lambda\left(\nu_{j+1/2}-\nu_{j+1/2}(1-\lambda\nu_{j+1/2})\right)\\
&=\left(\lambda\nu_{j+1/2}\right)^2\ge 0.
\end{aligned}$$

Summing up, we have proved the following result.

Lemma 3.13 *Assume $f'\ge 0$. Let r_j be defined by (3.45), and assume $\lambda>0$ is such that the CFL condition $\lambda\max_u f'(u)\le 1$ holds. Assume further that the function φ is such that $\varphi(r)$ vanishes for $r\le 0$ and satisfies (3.46). Then the finite volume scheme with numerical flux function (3.43) is TVD.*

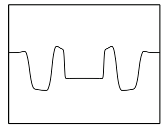
If we choose $\varphi(r)=r$, we get another scheme, called the Beam–Warming (BW) scheme. The Beam–Warming scheme is also of second order, but not TVD. The Lax–Wendroff (LW) scheme is obtained by choosing $\varphi(r)=1$.

If (for the moment) we do not care about TVD, we can define a family of second-order schemes by linear interpolation between the Beam–Warming and the Lax–Wendroff schemes. This interpolation can be done locally, meaning that we choose φ as

$$\varphi(r)=(1-\theta(r))\varphi_{\mathrm{LW}}(r)+\theta(r)\varphi_{\mathrm{BW}}(r).$$

The scheme reads

$$u_j^{n+1}=u_j^n-\lambda\Delta_-f_j^n+\lambda\Delta_-\varphi(r_j)\alpha_{j+1/2}\Delta_+u_j^n. \tag{3.47}$$



If now $u_j^n = u(x_j, t_n)$ is the exact solution, then we can calculate

$$\begin{aligned}
u_j^{n+1} &= u_j - \lambda\Delta f_j + \lambda\Delta_-\left(\left((1-\theta(r_j)) + \theta(r_j)r_j\right)\alpha_{j+1/2}\Delta_+ u_j\right) \\
&= \left(1-\theta(r_j)\right)\left(u_j - \lambda\Delta_- f_j + \lambda\Delta_-\left(\alpha_{j+1/2}\Delta_+ u_j\right)\right) \\
&\quad + \theta(r_j)\left(u_j - \lambda\Delta_- f_j + \lambda\Delta_-\left(r_j\alpha_{j+1/2}\Delta_+ u_j\right)\right) \\
&\quad + \lambda\alpha_{j-1/2}\left(r_{j-1} - 1\right)\Delta_- u_j\,\Delta_-\theta(r_j).
\end{aligned}$$

This means that

$$\begin{aligned}
u(x_j, t+\Delta t) - u_j^{n+1} = (1-\theta(r_j)) &\quad (\text{“LW truncation error”}) \\
+ \theta(r_j) &\quad (\text{“BW truncation error”}) \\
+ \lambda\,\underbrace{\alpha_{j-1/2}\left(r_{j-1}-1\right)\Delta_- u_j\,\Delta_-\theta(r_j)}_{I}&.
\end{aligned}$$

If $I = \mathcal{O}\left(\Delta t^3\right)$, then the combination of the LW and the BW schemes is of second order. By the CFL condition, $0 \le \lambda\alpha_{j-1/2} \le 1$. Furthermore, since u is an exact smooth solution, $\alpha_{j+1/2}\Delta_+ u \approx \Delta x\, f'(u)(1-\lambda f'(u))u_x$, or more precisely

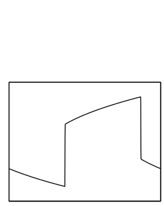
$$\alpha_{j+1/2}\frac{\Delta_+ u_j}{\Delta x} = f'(u)(1-\lambda f'(u))u_x \,\big|_{x=x_{j+1/2}} + \mathcal{O}\left(\Delta x^2\right).$$

Recall the definition of r_j, equation (3.45), and set $h(x) = f'(u(x,t))(1 - \lambda f'(u(x,t)))u_x(x,t)$. With this notation we get

$$\begin{aligned}
\left|\alpha_{j-1/2}(r_{j-1}-1)\Delta_- u_j\right| &= \left|\Delta_-\left(\alpha_{j-1/2}\Delta_+ u_{j-1}\right)\right| \\
&= \Delta x\left|h(x_{j-1/2}) - h(x_{j-3/2}) + \mathcal{O}\left(\Delta x^2\right)\right| \\
&\le \Delta x^2 \max_{(x,t)}|h'(x)| + \mathcal{O}\left(\Delta x^3\right).
\end{aligned}$$

Therefore, to show that $I = \mathcal{O}\left(\Delta t^3\right)$, it suffices to show that $\Delta_-\theta_j = \mathcal{O}(\Delta t)$. Since θ is a smooth function with values in [0, 1], we get

$$\begin{aligned}
\left|\Delta_-\theta(r_j)\right| &= \left|\theta(r_j) - \theta(r_{j-1})\right| \\
&\le C\left|r_j - r_{j-1}\right| \\
&\le C\left|\frac{\alpha_{j-1/2}\Delta_- u_j}{\alpha_{j+1/2}\Delta_+ u_j} - \frac{\alpha_{j-3/2}\Delta_- u_{j-1}}{\alpha_{j-1/2}\Delta_- u_j}\right| \\
&= C\left|\frac{h_{j-1/2} + \mathcal{O}\left(\Delta x^2\right)}{h_{j+1/2} + \mathcal{O}\left(\Delta x^2\right)} - \frac{h_{j-3/2} + \mathcal{O}\left(\Delta x^2\right)}{h_{j-1/2} + \mathcal{O}\left(\Delta x^2\right)}\right| \\
&= C\left|\frac{h_{j-1/2}^2 - h_{j+1/2}h_{j-3/2} + \mathcal{O}\left(\Delta x^2\right)}{h_{j+1/2}h_{j-1/2} + \mathcal{O}\left(\Delta x^2\right)}\right| \\
&\le C\,\frac{\Delta x \max_{(x,t)}|h'(x)| + \mathcal{O}\left(\Delta x^2\right)}{h_{j+1/2}h_{j-3/2} + \mathcal{O}\left(\Delta x^2\right)} \\
&= \mathcal{O}(\Delta t).
\end{aligned}$$



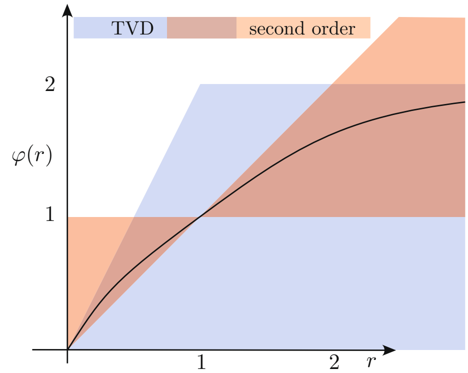


Fig. 3.5 The graph of the limiter must lie in both the TVD region and the second-order region. The graph shown is a possible limiter

Thus we have shown that if θ is a Lipschitz continuous function, the resulting scheme is of second order.

Returning to φ, we have shown that the scheme (3.47) is of second order if φ is Lipschitz continuous and

$$\min\{1, r\} \leq \varphi(r) \leq \max\{1, r\}. \tag{3.48}$$

If φ satisfies both (3.46) and (3.48), then the resulting scheme (3.47) is TVD, and second-order accurate away from local extrema. The scheme also produces a convergent sequence of approximations, and the limit is a weak solution (prove this!).

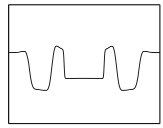
The function φ is called a limiter; a list of popular limiters follows. It is clear that the graph of a limiter must lie in the shaded region in Fig. 3.5.

$$\begin{aligned}
\varphi(r) &= \max\{0, \min\{r, 1\}\}, && \text{minmod} \\
\varphi(r) &= \max\{0, \min\{2r, 1\}, \min\{r, 2\}\}, && \text{superbee,} \\
\varphi(r) &= \frac{|r| + r}{1 + r}, && \text{van Leer} \\
\varphi(r) &= \frac{r^2 + r}{1 + r^2}, && \text{van Albada} \\
\varphi(r) &= \max\{0, \min\{r, \beta\}\}, \quad 1 \leq \beta \leq 2, && \text{Chakarvarthy \& Osher}
\end{aligned}$$

In Fig. 3.6 we show the approximate solutions to

$$u_t + u_x = 0, \quad u(x, 0) = \begin{cases} 1 & x \in [0, 3, \ 0.7], \\ 0 & x \in [0, 1] \setminus [0, 3, \ 0.7], \end{cases}$$

and for $x \notin [0, 1]$ we extend $u(x, 0)$ periodically. The figure shows approximate solutions at $t = 0$ as well as the exact solution. To the left we see that both the Lax–Wendroff and the Beam–Warming schemes have pronounced oscillations, but the linear combination of the two schemes, in this case using the van Leer limiter, does not. This solution is also superior to the solution found by the upwind method. Since these methods limit the contribution of the higher-order numerical flux function, they are often called *flux-limiter methods*.



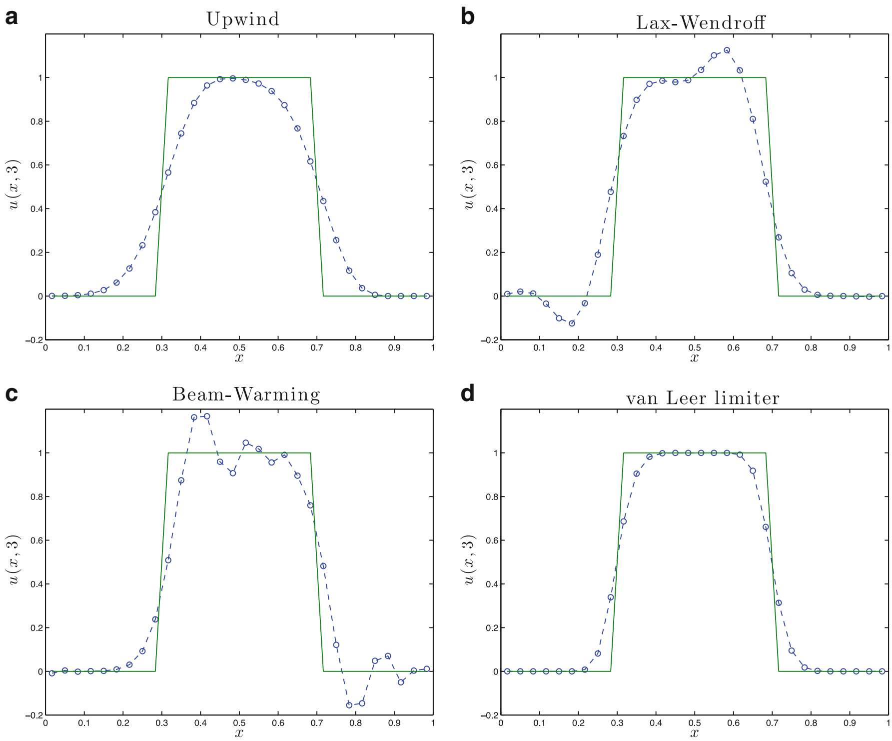


Fig. 3.6 The approximate solutions found by the upwind method (**a**), the Lax–Wendroff method (**b**), the Beam–Warming method (**c**), and the TVD method using the van Leer limiter (**d**). All computations used $\Delta x = 1/30$

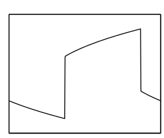
Semidiscrete Higher-Order Methods

Let us now consider semidiscrete higher-order methods, where we do not (initially) discretize time, only space. Based on the finite volume approach, such methods can be written

$$u_j'(t) = -\frac{1}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right), \tag{3.49}$$

where $u_j(t)$ is some approximation to the average of u in the cell $(x_{j-1/2}, x_{j+1/2}]$. If the right-hand side of the above is a second-order approximation to $-f(u)_x$ for smooth functions $u(x)$, then the method is said to be second-order accurate. To get second-order accuracy in time as well, one could use a second-order Runge–Kutta method to integrate (3.49) numerically. One such example is Heun's method:

$$\begin{aligned}
\tilde{u}_j^n &= u_j^n - \lambda\left(F_{j+1/2} - F_{j-1/2}\right),\\
u_j^{n+1} &= u_j^n - \frac{\lambda}{2}\left(\tilde{F}_{j+1/2} - \tilde{F}_{j-1/2}\right) - \frac{\lambda}{2}\left(F_{j+1/2} - F_{j-1/2}\right).
\end{aligned}$$



The simplest way of achieving second-order accuracy is by choosing

$$F_{j+1/2} = f\left(\frac{u_{j+1} + u_j}{2}\right). \tag{3.50}$$

This, however, gives a nonviable method if we combine it with a first-order Euler method in time. This combination is not stable[2]. To see this, set $f(u) = u$. With the Euler method it gives

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(u_{j+1} - u_{j-1}\right).$$

Making the *ansatz* $u_j^n = \mu_n e^{ij\,\Delta x}$ (here $i = \sqrt{-1}$) yields

$$\mu_{n+1} = \mu_n\left(1 + i\lambda \sin(\Delta x)\right).$$

Therefore, $|\mu_{n+1}| = |\mu_n| \sqrt{1 + \lambda^2 \sin^2(\Delta x)}$, or

$$|\mu_n| = |\mu_0| \left(1 + \lambda^2 \sin^2(\Delta x)\right)^{n/2}.$$

This is unconditionally unstable. Also using the second-order Heun's method with (3.50) gives an unstable method (see Exercise 3.8). Thus the choice (3.50) is of second order, but useless.

In order to overcome this, we define values to the left and right of a cell edge $u^L_{j+1/2}$ and $u^R_{j-1/2}$ by

$$\begin{aligned} u^L_{j+1/2} &= u_j + \frac{1}{2}\Delta_- u_j, \\ u^R_{j-1/2} &= u_j - \frac{1}{2}\Delta_+ u_j. \end{aligned} \tag{3.51}$$

Then we can use any two-point monotone first-order numerical flux $F(u,v)$ to define a second-order approximation

$$f(u(x))_x = \frac{1}{\Delta x}\left(F\left(u^L_{j+1/2}, u^R_{j+1/2}\right) - F\left(u^L_{j-1/2}, u^R_{j-1/2}\right)\right) + \mathcal{O}\left(\Delta x^2\right). \tag{3.52}$$

Even if we use Heun's method for time integration, the extrapolation values (3.51) do not give a TVD method. This is to be expected, since the method is formally second-order accurate. We illustrate this in Fig. 3.7 for the linear equation $u_t + u_x = 0$ with smooth and discontinuous initial values. We used the upwind first-order numerical flux $F(u,v) = f(u) = u$. From Fig. 3.7 we see that for smooth initial data, the approximation is "reasonably close" to the correct function, whereas for discontinuous initial data, the approximation bears little relation to the exact solution.

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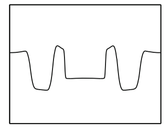
[2] Often called von Neumann stability.



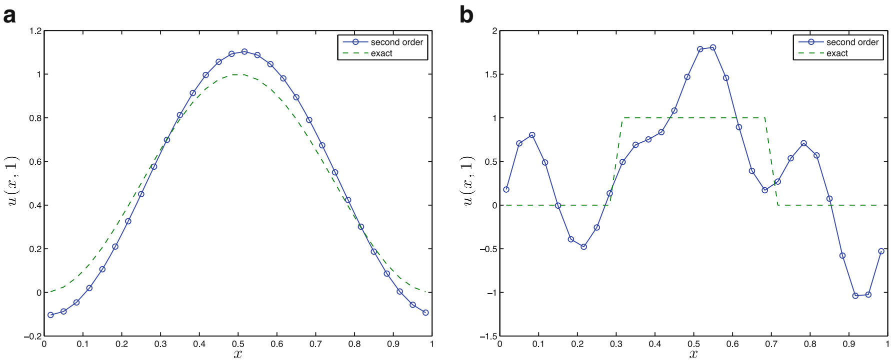


Fig. 3.7 Using the extrapolation (3.51). **a** $u(x,1)$ with smooth initial data. **b** $u(x,1)$ with discontinuous intitial data

These results suggest that the method will be improved if we use some kind of limiter to define the extrapolated values $u^{L,R}_{j+1/2}$. To this end, set $\varphi_j = \varphi(r_j)$, where r_j is to be defined, and redefine the extrapolations as

$$\begin{aligned} u^L_{j+1/2} &= u_j + \frac{1}{2}\varphi_j \Delta_- u_j, \\ u^R_{j-1/2} &= u_j - \frac{1}{2}\varphi_j \Delta_+ u_j. \end{aligned} \tag{3.53}$$

For simplicity, we now assume that $f' \geq 0$, and that the numerical flux function is the upwind flux, i.e., $F(u,v) = f(u)$. In this case the resulting scheme is

$$u^{n+1}_j = u^n_j - \lambda\left(f(u^L_{j+1/2}) - f(u^L_{j-1/2})\right).$$

We aim to define r_j and find conditions on φ such that the above scheme is TVD but retains the formal second order away from oscillations. In order to use Lemma 3.12, we rewrite the scheme as

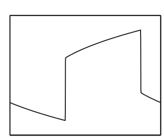
$$u^{n+1}_j = u^n_j - \lambda \frac{\Delta_- f(u^L_{j+1/2})}{\Delta_- u^n_j} \Delta_- u^n_j,$$

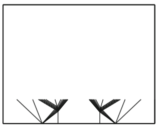
where we have used a first-order Euler method for the integration in time. This will of course destroy the formal second-order accuracy, but it is convenient for analysis. With

$$A_{j-1/2} = \lambda \frac{\Delta_- f(u^L_{j+1/2})}{\Delta_- u^n_j}$$

the scheme will be TVD if $0 \leq A_{j-1/2} \leq 1$. Dropping the superscript n, we calculate

$$\begin{aligned} A_{j-1/2} &= \lambda f'(\bar{u}_j) \frac{u_j + \frac{1}{2}\varphi_j \Delta_- u_j - u_{j-1} - \frac{1}{2}\varphi_{j-1}\Delta_- u_{j-1}}{\Delta_- u_j} \\ &= \lambda f'(\bar{u}_j)\left(\left(1 + \frac{1}{2}\varphi_j\right) - \frac{1}{2}\varphi_{j-1}\frac{\Delta_- u_{j-1}}{\Delta_- u_j}\right), \end{aligned}$$





where $\bar{u}_j$ is some value between $u^L_{j-1/2}$ and $u^L_{j+1/2}$. If we now choose

$$r_j = \frac{\Delta_+ u_j}{\Delta_- u_j},$$

this can be rewritten as

$$A_{j-1/2} = \lambda f'\left(\bar{u}_j\right)\left(1 - \frac{1}{2}\left(\frac{\varphi(r_{j-1})}{r_{j-1}} - \varphi(r_j)\right)\right).$$

We now demand that the scheme satisfy the CFL condition

$$\lambda \max_u f'(u) \leq \frac{1}{2}.$$

In this case $0 \leq A_{j-1/2} \leq 1$ if

$$0 \leq \left(1 - \frac{1}{2}\left(\frac{\varphi(r_{j-1})}{r_{j-1}} - \varphi(r_j)\right)\right) \leq 2,$$

which can be rewritten

$$-2 \leq \frac{\varphi_{j-1}}{r_{j-1}} - \varphi_j \leq 2.$$

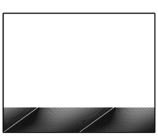
This is the case if

$$0 \leq \varphi(r) \leq \min\left\{2r, 2\right\},$$

which gives the same TVD-region as for the flux-limiter schemes; see Fig. 3.5.

The scheme with $\phi(r) \equiv 1$ is not TVD, but of second order, and the choice $\phi(r) = r$ gives the (useless) second-order scheme with numerical flux (3.50). It follows as before that every smooth (in r) convex combination of these two schemes will also be of second order. Therefore, we get the same second-order region as in Fig. 3.5. Hence we have the same choice of limiter functions as before. Each choice will give a formally second-order scheme away from local extrema. This method is called *MUSCL* (monotone upstream centered scheme for conservation laws).

If Fig. 3.8 we show how the above schemes perform on the model equation $u_t + u_x = 0$ with smooth and discontinuous initial data. The MUSCL method does not perform as well as the flux limiter method, but a clear difference can be seen between the first-order upstream method and the high-resolution methods (MUSCL and flux limiter). For both the high-resolution methods, the computations in Fig. 3.8 use the van Leer limiter. The perceptive reader may have noticed that the flux-limiter method is further from the exact solution than the methods shown in Fig. 3.6. This is because we choose to use the same timestep for all the methods, this being limited by the MUSCL method. Thus, the upwind and flux limiter methods will also have a time step $\Delta t \leq \lambda \Delta x$, with $\lambda = 0.49$.



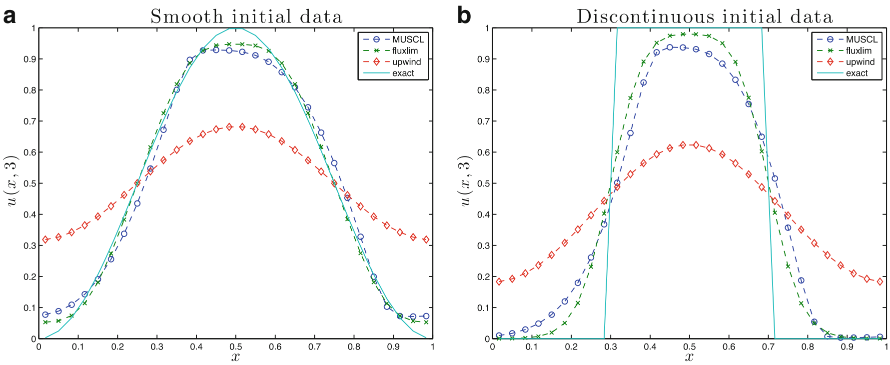
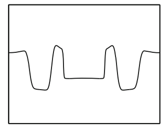


Fig. 3.8 A comparison of the first-order monotone upstream method and high-resolution methods for smooth (**a**) and discontinuous initial data (**b**)



3.3 Error Estimates

Let others bring order to chaos. I would bring chaos to order instead.
— Kurt Vonnegut, Breakfast of Champions (1973)

The concept of local error estimates is based on formal computations, and such estimates indicate how the method performs in regions where the solution is smooth. Since the convergence of the methods discussed was in L^1, it is reasonable to ask how far the approximated solution is from the true solution in this space.

In this section we will consider functions u that are maps $t \mapsto u(t)$ from $[0,\infty)$ to $L^1_{\mathrm{loc}} \cap BV(\mathbb{R})$ such that the one-sided limits $u(t\pm)$ exist in L^1_{loc}, and for definiteness we assume that this map is right continuous. Furthermore, we assume that

$$\|u(t)\|_\infty \le \|u(0)\|_\infty, \quad \mathrm{T.V.}\,(u(t)) \le \mathrm{T.V.}\,(u(0))\,.$$

We denote this class of functions by $\mathcal{K}$. From Theorem 2.15 we know that solutions of scalar conservation laws are in the class $\mathcal{K}$.

It is convenient to introduce *moduli of continuity in time* (see Appendix A)

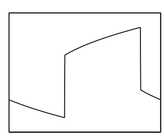
$$\begin{aligned} \nu_t(u,\sigma) &= \sup_{|\tau|\le\sigma} \|u(t+\tau) - u(t)\|_{L^1}, \quad \sigma > 0, \\ \nu(u,\sigma) &= \sup_{0\le t\le T} \nu_t(u,\sigma). \end{aligned} \tag{3.54}$$

From Theorem 2.15 we have that

$$\nu(u,\sigma) \le |\sigma|\,\|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(u_0) \tag{3.55}$$

for weak solutions of conservation laws.

Now let $u(x,t)$ be any function in $\mathcal{K}$, not necessarily a solution of (3.1). In order to measure how far u is from being a solution of (3.1) we insert u in the Kružkov



form (cf. (2.23))

$$\Lambda_T(u,\phi,k) = \int_0^T \int \left(|u-k|\,\phi_t + q(u,k)\phi_x\right)\,dx\,ds \qquad (3.56)$$
$$- \int |u(x,T)-k|\,\phi(x,T)\,dx + \int |u_0(x)-k|\,\phi(x,0)\,dx.$$

If u is a solution, then $\Lambda_T \geq 0$ for all constants k and all nonnegative test functions ϕ. We shall now use the special test function

$$\Omega(x,x',s,s') = \omega_{\varepsilon_0}(s-s')\omega_\varepsilon(x-x'),$$

where

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right)$$

and $\omega(x)$ is an even C^∞ function satisfying

$$0 \leq \omega \leq 1, \quad \omega(x) = 0 \quad \text{for } |x| > 1, \quad \int \omega(x)\,dx = 1.$$

Let $v(x',s')$ be the unique weak solution of (3.1), and define

$$\Lambda_{\varepsilon,\varepsilon_0}(u,v) = \int_0^T \int \Lambda_T\left(u, \Omega(\,\cdot\,,x',\,\cdot\,,s'), v(x',s')\right)\,dx'ds'.$$

The comparison result reads as follows.

Theorem 3.14 (Kuznetsov's lemma) *Let $u(\,\cdot\,,t)$ be a function in $\mathcal{K}$, and v a solution of (3.1). If $0 < \varepsilon_0 < T$ and $\varepsilon > 0$, then*

$$\|u(\,\cdot\,,T-) - v(\,\cdot\,,T)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} + \text{T.V.}\,(v_0)\left(2\varepsilon + \varepsilon_0\|f\|_{\text{Lip}}\right)$$
$$+ \nu(u,\varepsilon_0) - \Lambda_{\varepsilon,\varepsilon_0}(u,v), \qquad (3.57)$$

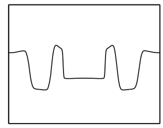
where $u_0 = u(\,\cdot\,,0)$ and $v_0 = v(\,\cdot\,,0)$.

Proof We use special properties of the test function Ω, namely that

$$\Omega(x,x',s,s') = \Omega(x',x,s,s') = \Omega(x,x',s',s) = \Omega(x',x,s',s) \qquad (3.58)$$

and

$$\Omega_x = -\Omega_{x'}, \quad \text{and} \quad \Omega_s = -\Omega_{s'}. \qquad (3.59)$$



Using (3.58) and (3.59), we find that

$$\begin{aligned}\Lambda_{\varepsilon,\varepsilon_0}(u,v) = -\Lambda_{\varepsilon,\varepsilon_0}(v,u) - \int_0^T \iint \Omega(x,x',s,T)\big(&|u(x,T)-v(x',s)| \\ &+ |v(x',T)-u(x,s)|\big)\,dx\,dx'\,ds \\ + \int_0^T \iint \Omega(x,x',s,0)\big(|v_0(x')-u(x,s)|& \\ + |u_0(x)-v(x',s)|\big)\,dx\,dx'\,ds& \\ := -\Lambda_{\varepsilon,\varepsilon_0}(v,u) - A\ + B.&\end{aligned}$$

Since v is a weak solution, $\Lambda_{\varepsilon,\varepsilon_0}(v,u) \geq 0$, and hence

$$A \leq B - \Lambda_{\varepsilon,\varepsilon_0}(u,v).$$

Therefore, we would like to obtain a lower bound on A and an upper bound on B, the lower bound on A involving $\|u(T)-v(T)\|_{L^1}$ and the upper bound on B involving $\|u_0 - v_0\|_{L^1}$. We start with the lower bound on A.

Let ρ_ε be defined by

$$\rho_\varepsilon(u,v) = \iint \omega_\varepsilon(x-x')\,|u(x)-v(x')|\;dx\,dx'. \tag{3.60}$$

Then

$$A = \int_0^T \omega_{\varepsilon_0}(T-s)\,(\rho_\varepsilon(u(T),v(s)) + \rho_\varepsilon(u(s),v(T)))\;ds.$$

Now by a use of the triangle inequality,

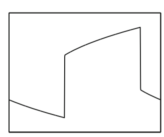
$$\begin{aligned}\|u(x,T)-v(x',s)\| &+ |u(x,s)-v(x',T)| \\ &\geq |u(x,T)-v(x,T)| + |u(x,T)-v(x,T)| \\ &\quad - |v(x,T)-v(x',T)| - |u(x,T)-u(x,s)| \\ &\quad - |v(x',T)-v(x',s)| - |v(x,T)-v(x',T)|\,.\end{aligned}$$

Hence

$$\begin{aligned}\rho_\varepsilon(u(T),v(s)) + \rho_\varepsilon(u(s),v(T)) \geq 2\|u(T)-v(T)\|_{L^1} - 2\rho_\varepsilon(v(T),v(T))& \\ - \|u(T)-u(s)\|_{L^1} - \|v(T)-v(s)\|_{L^1}.&\end{aligned}$$

Regarding the upper estimate on B, we similarly have that

$$B = \int_0^T \omega_{\varepsilon_0}(s)\,[\rho_\varepsilon(u_0,v(s)) + \rho_\varepsilon(u(s),v_0)]\;ds,$$



and we also obtain

$$\begin{aligned}\rho_\varepsilon(u_0, v(s)) + \rho_\varepsilon(u(s), v_0) \le 2\|u_0 - v_0\|_{L^1} + 2\rho_\varepsilon(v_0, v_0) \\ + \|u_0 - u(s)\|_{L^1} + \|v_0 - v(s)\|_{L^1}.\end{aligned}$$

Since v is a solution, it satisfies the TVD property, and hence

$$\begin{aligned}\rho_\varepsilon(v(T), v(T)) &= \int \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(z)\,|v(x+z,T) - v(x,T)|\;dz\,dx \\ &\le \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(z) \sup_{|z|\le\varepsilon} \left(\int |v(x+z,T) - v(x,T)|\;dx \right) dz \\ &= |\varepsilon| \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(z) \mathrm{T.V.}\,(v(T))\;dz \le |\varepsilon|\,\mathrm{T.V.}\,(v_0)\,,\end{aligned}$$

using (A.10). By the properties of ω,

$$\int_0^T \omega_\varepsilon(T-s)\,ds = \int_0^T \omega_\varepsilon(s)\,ds = \frac{1}{2}.$$

Applying (3.55), we obtain (recall that $\varepsilon_0 < T$)

$$\begin{aligned}&\int_0^T \omega_{\varepsilon_0}(T-s)\|v(T) - v(s)\|_{L^1}\,ds \\ &\qquad \le \int_0^T \omega_{\varepsilon_0}(T-s)\,(T-s)\|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(v_0)\;ds \\ &\qquad \le \frac{1}{2}\varepsilon_0\|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(v_0)\end{aligned}$$

and

$$\int_0^T \omega_{\varepsilon_0}(s)\|v_0 - v(s)\|_{L^1}\,ds \le \frac{1}{2}\varepsilon_0\|f\|_{\mathrm{Lip}}\mathrm{T.V.}\,(v_0)\,.$$

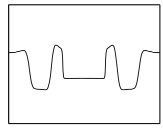
Similarly,

$$\int_0^T \omega_{\varepsilon_0}(T-s)\|u(T) - u(s)\|_{L^1}\,ds \le \frac{1}{2}\nu\,(u, \varepsilon_0)$$

and

$$\int_0^T \omega_{\varepsilon_0}(s)\|u_0 - u(s)\|_{L^1}\,ds \le \frac{1}{2}\nu\,(u, \varepsilon_0)\,.$$

If we collect all the above bounds, we should obtain the statement of the theorem. □



Observe that in the special case that u is a solution of the conservation law (3.1), we know that $\Lambda_{\varepsilon,\varepsilon_0}(u,v) \geq 0$, and hence we obtain, as $\varepsilon, \varepsilon_0 \to 0$, the familiar stability result

$$\|u(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

We shall now show in three cases how Kuznetsov's lemma can be used to give estimates on how fast a method converges to the entropy solution of (3.1).

◇ Example 3.15 (The smoothing method)
While not a proper numerical method, the smoothing method provides an example of how the result of Kuznetsov may be used. The smoothing method is a (semi)numerical method approximating the solution of (3.1) as follows: Let $\omega_\delta(x)$ be a standard mollifier with support in $[-\delta,\delta]$, and let $t_n = n\Delta t$. Set $u^0 = u_0 * \omega_\delta$. For $0 \leq t < \Delta t$ define u^1 to be the solution of (3.1) with initial data u^0. If Δt is small enough, u^1 remains differentiable for $t < \Delta t$. In the interval $[(n-1)\Delta t, n\Delta t)$, we define u^n to be the solution of (3.1), with $u^n\,(x,(n-1)\Delta t) = u^{n-1}(\,\cdot\,,t_n-) * \omega_\delta$. The advantage of doing this is that u^n will remain differentiable in x for all times, and the solution in the strips $[t_n, t_{n+1})$ can be found by, e.g., the method of characteristics. To show that u^n is differentiable, we calculate

$$\begin{aligned} |u_x^n(x,t_{n-1})| &= \left| \int u_x^{n-1}(y,t_{n-1})\omega_\delta(x-y)\,dy \right| \\ &\leq \frac{1}{\delta}\mathrm{T.V.}\left(u^{n-1}(t_{n-1})\right) \leq \frac{\mathrm{T.V.}\,(u_0)}{\delta}. \end{aligned}$$

Let $\mu(t) = \max_x |u_x(x,t)|$. Using that u is a classical solution of (3.1), we find by differentiating (3.1) with respect to x that

$$u_{xt} + f'(u)u_{xx} + f''(u)u_x^2 = 0.$$

Write

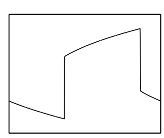
$$\mu(t) = u_x(x_0(t),t),$$

where $x_0(t)$ is the location of the maximum of $|u_x|$. Then

$$\begin{aligned} \mu'(t) &= u_{xx}(x_0(t),t)x_0'(t) + u_{xt}(x_0(t),t) \\ &\leq u_{xt}(x_0(t),t) = -f''(u)\big(u_x(x_0(t),t)\big)^2 \\ &\leq c\mu(t)^2, \end{aligned}$$

since $u_{xx} = 0$ at an extremum of u_x. Thus

$$\mu'(t) \leq c\mu^2(t), \tag{3.61}$$



where $c = \|f''\|_\infty$. The idea is now that (3.61) has a blowup at some finite time, and we choose Δt less than this time. We shall be needing a precise relation between Δt and δ and must therefore investigate (3.61) further. Solving (3.61) we obtain

$$\mu(t) \le \frac{\mu(t_n)}{1 - c\mu(t_n)(t - t_n)} \le \frac{\text{T.V.}(u_0)}{\delta - c\text{T.V.}(u_0)\,\Delta t}.$$

So if

$$\Delta t < \frac{\delta}{c\text{T.V.}(u_0)}, \tag{3.62}$$

the method is well defined. Choosing $\Delta t = \delta/(2c\text{T.V.}(u_0))$ will do.

Since u is an exact solution in the strips $[t_n, t_{n+1})$, we have

$$\int_{t_n}^{t_{n+1}} \int \left(|u - k|\,\phi_t + q(u,k)\phi_x\right)\,dx\,dt \\ + \int \Big(|u(x, t_n+) - k|\,\phi(x, t_n) - |u(x, t_{n+1}-) - k|\,\phi(x, t_{n+1})\Big)\,dx \ge 0.$$

Summing these inequalities and setting $k = v(y, s)$, where v is an exact solution of (3.1), we obtain

$$\Lambda_T(u, \Omega, v(y,s)) \ge -\sum_{n=0}^{N-1} \int \Omega(x, y, t_n, s)\Big(|u(x, t_n+) - v(y,s)| \\ - |u(x, t_n-) - v(y,s)|\Big)\,dx,$$

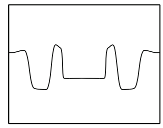
where we use the test function $\Omega(x, y, t, s) = \omega_{\varepsilon_0}(t - s)\omega_\varepsilon(x - y)$. Integrating this over y and s, and letting ε_0 tend to zero, we get

$$\liminf_{\varepsilon_0 \to 0} \Lambda_{\varepsilon,\varepsilon_0}(u, v) \ge -\sum_{n=0}^{N-1} \left(\rho_\varepsilon(u(t_n+), v(t_n)) - \rho_\varepsilon(u(t_n-), v(t_n))\right).$$

Using this in Kuznetsov's lemma, and letting $\varepsilon_0 \to 0$, we obtain

$$\|u(T) - v(T)\|_1 \le \left\|u_0 - u^0\right\|_1 + 2\varepsilon\,\text{T.V.}(u_0) \\ + \sum_{n=0}^{N-1} \left(\rho_\varepsilon(u(t_n+), v(t_n)) - \rho_\varepsilon(u(t_n-), v(t_n))\right), \tag{3.63}$$

where we have used that $\lim_{\varepsilon_0 \to 0} \nu_t(u, \varepsilon_0) = 0$, which holds because u is a solution of the conservation law in each strip $[t_n, t_{n+1})$.



To obtain a more explicit bound on the difference of u and v, we investigate $\rho_\varepsilon(\omega_\delta * u, v) - \rho_\varepsilon(u, v)$, where ρ_ε is defined by (3.60),

$$
\begin{aligned}
\rho_\varepsilon(u * \omega_\delta, v) - \rho_\varepsilon(u, v) &\le \iiint\limits_{|z|\le 1} \omega_\varepsilon(x-y)\omega(z)\Big(|u(x+\delta z) - v(y)| \\
&\qquad\qquad - |u(x) - v(y)|\Big)\, dx\, dy\, dz \\
&= \frac{1}{2}\iiint\limits_{|z|\le 1} \left(\omega_\varepsilon(x-y) - \omega_\varepsilon(x+\delta z - y)\right)\omega(z) \\
&\qquad \times \left(|u(x+\delta z) - v(y)| - |u(x) - v(y)|\right)\, dx\, dy\, dz,
\end{aligned}
$$

which follows after writing $\iiint = \frac{1}{2}\iiint + \frac{1}{2}\iiint$ and making the substitution $x \mapsto x - \delta z$, $z \mapsto -z$ in one of these integrals. Therefore,

$$
\begin{aligned}
\rho_\varepsilon(u * \omega_\delta, v) - \rho_\varepsilon(u, v) &\le \frac{1}{2}\iiint\limits_{|z|\le 1} |\omega_\varepsilon(y+\delta z) - \omega_\varepsilon(y)| \\
&\qquad \times \omega(z)\,|u(x+\delta z) - u(x)|\, dx\, dy\, dz \\
&\le \frac{1}{2}\,\mathrm{T.V.}\,(\omega_\varepsilon)\ \mathrm{T.V.}\,(u)\ \delta^2 \\
&\le \mathrm{T.V.}\,(u)\,\frac{\delta^2}{\varepsilon},
\end{aligned}
$$

by the triangle inequality and a further substitution $y \mapsto x - y$. Since $N = T/\Delta t$, the last term in (3.63) is less than

$$
N\,\mathrm{T.V.}\,(u_0)\,\frac{\delta^2}{\varepsilon} \le (\mathrm{T.V.}\,(u_0))^2\ 2cT\frac{\delta}{\varepsilon},
$$

using (3.62). Furthermore, we have that

$$
\left\|u^0 - u_0\right\|_1 \le \delta\mathrm{T.V.}\,(u_0)\,.
$$

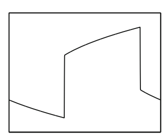
Letting $K = \mathrm{T.V.}\,(u_0)\,c$, we find that

$$
\|u(T) - v(T)\|_1 \le 2\mathrm{T.V.}\,(u_0)\left[\delta + \varepsilon + \frac{KT\delta}{\varepsilon}\right],
$$

using (3.63). Minimizing with respect to ε, we find that

$$
\|u(T) - v(T)\|_1 \le 2\mathrm{T.V.}\,(u_0)\left(\delta + 2\sqrt{KT\delta}\right). \tag{3.64}
$$

So, we have shown that the smoothing method is of order $\frac{1}{2}$ in the smoothing coefficient δ. ◇



◇ Example 3.16 (The method of vanishing viscosity)
Another (semi)numerical method for (3.1) is the method of vanishing viscosity. Here we approximate the solution of (3.1) by the solution of

$$u_t + f(u)_x = \delta u_{xx}, \quad \delta > 0, \tag{3.65}$$

using the same initial data. Let u^δ denote the solution of (3.65). Due to the dissipative term on the right-hand side, the solution of (3.65) remains a classical (twice differentiable) solution for all $t > 0$. Furthermore, the solution operator for (3.65) is TVD. Hence a numerical method for (3.65) will (presumably) not experience the same difficulties as a numerical method for (3.1). If (η, q) is a convex entropy pair, we have, using the differentiability of the solution, that

$$\eta(u)_t + q(u)_x = \delta\eta'(u)u_{xx} = \delta\left(\eta(u)_{xx} - \eta''(u)u_x^2\right).$$

Multiplying by a nonnegative test function φ and integrating by parts, we get

$$\iint (\eta(u)\varphi_t + q(u)\varphi_x)\,dx\,dt \geq \delta \iint \eta(u)_x\varphi_x\,dx\,dt,$$

where we have used the convexity of η. Applying this with $\eta = \left|u^\delta - u\right|$ and $q = F(u^\delta, u)$, we can bound $\lim_{\varepsilon_0\to 0} \Lambda_{\varepsilon,\varepsilon_0}(u^\delta, u)$ as follows:

$$\begin{aligned}
-\lim_{\varepsilon_0\to 0} \Lambda_{\varepsilon,\varepsilon_0}(u^\delta, u) &\leq \delta \int_0^T \iint \left|\frac{\partial\omega_\varepsilon(x-y)}{\partial x}\right| \frac{\partial\left|u^\delta(x,t) - u(y,t)\right|}{\partial x}\,dx\,dy\,dt \\
&\leq \delta \int_0^T \iint \left|\frac{\partial\omega_\varepsilon(x-y)}{\partial x}\right| \left|\frac{\partial u^\delta(x,t)}{\partial x}\right|\,dx\,dy\,dt \\
&\leq 2\mathrm{T.V.}\left(u^\delta\right) T\frac{\delta}{\varepsilon} \\
&\leq 2T\,\mathrm{T.V.}\left(u_0\right)\,\frac{\delta}{\varepsilon}.
\end{aligned}$$

Now letting $\varepsilon_0 \to 0$ in (3.57), we obtain

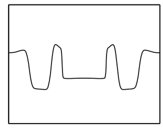
$$\left\|u^\delta(T) - u(T)\right\|_1 \leq \min_\varepsilon\left(2\varepsilon + \frac{2T\delta}{\varepsilon}\right)\mathrm{T.V.}\left(u_0\right) = 2\mathrm{T.V.}\left(u_0\right)\sqrt{T\delta}.$$

So the method of vanishing viscosity also has order $\frac{1}{2}$. ◇

◇ Example 3.17 (Monotone schemes)
We will here show that monotone schemes converge in L^1 to the solution of (3.1) at a rate of $(\Delta t)^{1/2}$. In particular, this applies to the Lax–Friedrichs scheme.

Let $u_{\Delta t}$ be defined by (3.27), where u_j^n is defined by (3.6), that is,

$$u_j^{n+1} = u_j^n - \lambda\left(F_{j+1/2}^n - F_{j-1/2}^n\right), \tag{3.66}$$



where $F^n_{j+1/2} = F\left(u^n_{j-p}, \dots, u^n_{j+p'}\right)$, for a scheme that is assumed to be monotone; cf. Definition 3.5. In the following we use the notation

$$\eta^n_j = \left|u^n_j - k\right|, \quad q^n_j = f\left(u^n_j \vee k\right) - f\left(u^n_j \wedge k\right).$$

We find that

$$\begin{aligned}
-\Lambda_T(u_{\Delta t}, \phi, k) &= -\sum_j \sum_{n=0}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \left(\eta^n_j \phi_t(x,s) + q^n_j \phi_x(x,s)\right) ds\, dx \\
&\qquad - \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \eta^0_j \phi(x,0)\, dx + \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \eta^N_j \phi(x,T)\, dx \\
&= -\sum_j \Bigg[\sum_{n=0}^{N-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta^n_j \left(\phi(x,t_{n+1}) - \phi(x,t_n)\right) dx \\
&\qquad + \int_{x_{j-1/2}}^{x_{j+1/2}} \eta^0_j \phi(x,0)\, dx - \int_{x_{j-1/2}}^{x_{j+1/2}} \eta^N_j \phi(x,T)\, dx \\
&\qquad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} q^n_j \left(\phi(x_{j+1/2}, s) - \phi(x_{j-1/2}, s)\right) ds \Bigg] \\
&= \sum_j \sum_{n=0}^{N-1} \Bigg((\eta^{n+1}_j - \eta^n_j) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \\
&\qquad + (q^n_j - q^n_{j-1}) \int_{t_n}^{t_{n+1}} \phi(x_{j-1/2}, s)\, ds \Bigg)
\end{aligned}$$

by a summation by parts. Recall that we define the numerical entropy flux by

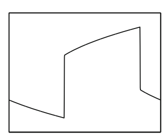
$$Q^n_{j+1/2} = F(u^n_{j-p} \vee k, \dots, u^n_{j+p'} \vee k) - F(u^n_{j-p} \wedge k, \dots, u^n_{j+p'} \wedge k).$$

Monotonicity of the scheme implies, cf. (3.33), that

$$\eta^{n+1}_j - \eta^n_j + \lambda(Q^n_{j+1/2} - Q^n_{j-1/2}) \le 0.$$

For a nonnegative test function ϕ we obtain

$$\begin{aligned}
-\Lambda_T(u_{\Delta t}, \phi, k) \le \sum_j \sum_{n=0}^{N-1} \Bigg(&-\lambda(Q^n_{j+1/2} - Q^n_{j-1/2}) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \\
&+ (q^n_j - q^n_{j-1}) \int_{t_n}^{t_{n+1}} \phi(x_j, s)\, ds \Bigg)
\end{aligned}$$



$$
\begin{aligned}
&= \sum_j \sum_{n=0}^{N-1} \Bigg[\lambda \left(\left(q_j^n - Q_{j+1/2}^n \right) - \left(q_{j-1}^n - Q_{j-1/2}^n \right) \right) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \\
&\qquad + (q_j^n - q_{j-1}^n) \Big(\int_{t_n}^{t_{n+1}} \phi(x_{j-1/2}, s)\, ds - \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \Big) \Bigg] \\
&= \sum_j \sum_{n=0}^{N-1} \Bigg[\lambda \left(Q_{j+1/2}^n - q_j^n \right) \Big(\int_{x_{j+1/2}}^{x_{j+3/2}} \phi(x, t_{n+1})\, dx - \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \Big) \\
&\qquad + \left(q_j^n - q_{j-1}^n \right) \Big(\int_{t_n}^{t_{n+1}} \phi(x_{j-1/2}, s)\, ds - \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \Big) \Bigg] \\
&= \sum_j \sum_{n=0}^{N-1} \Bigg[\lambda \left(Q_{j+1/2}^n - q_j^n \right) \Big(\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x + \Delta x, t_{n+1}) - \phi(x, t_{n+1})\, dx \Big) \\
&\qquad + (q_j^n - q_{j-1}^n) \Big(\int_{t_n}^{t_{n+1}} \phi(x_{j-1/2}, s)\, ds - \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x, t_{n+1})\, dx \Big) \Bigg].
\end{aligned}
$$

We also have that

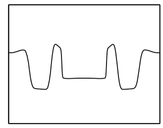
$$
\left| q_j^n - Q_{j+1/2}^n \right| \le \|f\|_{\mathrm{Lip}} \sum_{m=-p}^{p'} \left| u_{j+m}^n - u_j^n \right|
$$

and

$$
\left| q_j^n - q_{j-1}^n \right| \le \|f\|_{\mathrm{Lip}} \left| u_j^n - u_{j-1}^n \right|,
$$

which implies that

$$
\begin{aligned}
-\Lambda_T(u_{\Delta t}, \phi, k) \le \|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \Bigg[&\Big(\sum_{m=-p}^{p'} \left| u_{j+m}^n - u_j^n \right| \Big) \\
&\times \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} \left| \phi(x + \Delta x, t_{n+1}) - \phi(x, t_{n+1}) \right| \, dx \\
&+ \left| u_j^n - u_{j-1}^n \right| \\
&\times \Big| \int_{t_n}^{t_{n+1}} \phi(x_{j-1/2}, s)\, ds - \lambda \int_{x_j}^{x_{j+1}} \phi(x, t_{n+1})\, dx \Big| \Bigg].
\end{aligned}
$$



Next, we subtract $\phi(x_{j-1/2}, t_{n+1})$ from the integrand in each of the latter two integrals. Since $\Delta t = \lambda \Delta x$, the extra terms cancel, and we obtain

$$
\begin{aligned}
-\Lambda_T(u_{\Delta t}, \phi, k) \le \|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \Bigg[\Big(\sum_{m=-p}^{p'} \left| u^n_{j+m} - u^n_j \right| \Big) \\
\times \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} |\phi(x + \Delta x, t_{n+1}) - \phi(x, t_{n+1})| \, dx \\
+ \left| u^n_j - u^n_{j-1} \right| \Big(\int_{t_n}^{t_{n+1}} \left| \phi(x_{j-1/2}, t) - \phi(x_{j-1/2}, t_{n+1}) \right| dt \\
+ \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} \left| \phi(x, t_{n+1}) - \phi(x_{j-1/2}, t_{n+1}) \right| dx \Big) \Bigg].
\end{aligned}
\tag{3.67}
$$

Let $v = v(y, s)$ denote the unique entropy solution of (3.1), and let $k = v(y, s)$. Then

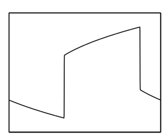
$$
-\Lambda_{\varepsilon_0, \varepsilon}(u, v) = -\int_0^T \int_{\mathbb{R}} \Lambda_T \left(u, v(y, s), \omega_{\varepsilon_0}(\cdot - s) \omega_\varepsilon(\cdot - x) \right) dy \, ds.
$$

Thus to estimate $-\Lambda_{\varepsilon_0, \varepsilon}(u, v)$ we must integrate the terms on the right-hand side of (3.67) in (y, s). To this end,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \int_{x_{j-1/2}}^{x_{j+1/2}} \omega_{\varepsilon_0}(t_{n+1} - s) \, |\omega_\varepsilon(x + \Delta x - y) - \omega_\varepsilon(x - y)| \, dx \, dy \, ds \\
&\qquad = \int_{\mathbb{R}} \int_{x_{j-1/2}}^{x_{j+1/2}} |\omega_\varepsilon(x + \Delta x - y) - \omega_\varepsilon(x - y)| \, dx \, dy \\
&\qquad \le \Delta x^2 \, |\omega_\varepsilon|_{BV} \\
&\qquad \le \frac{2 \Delta x^2}{\varepsilon}.
\end{aligned}
$$

Recalling that $\lambda = \Delta t / \Delta x$, we get

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \Bigg[\Big(\sum_{m=-p}^{p'} \left| u^n_{j+m} - u^n_j \right| \Big) \\
&\qquad\qquad \times \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} |\phi(x + \Delta x, t_{n+1}) - \phi(x, t_{n+1})| \, dx \, dy \, ds \\
&\quad \le \|f\|_{\mathrm{Lip}} \frac{1}{2} (p(p-1) + p'(p'-1)) \sum_{n=0}^{N-1} \sum_j \left| u^n_j - u^n_{j-1} \right| \frac{2 \Delta x^2}{\varepsilon} \lambda \\
&\quad \le C T \frac{\Delta x}{\varepsilon}.
\end{aligned}
\tag{3.68}
$$



We also have that

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} & \omega_\varepsilon(x_{j-1/2}-y)\left|\omega_{\varepsilon_0}(t-s)-\omega_{\varepsilon_0}(t_{n+1}-s)\right| \, dt\, dy\, ds \\
&= \int_0^T \int_{t_n}^{t_{n+1}} \left|\omega_{\varepsilon_0}(t-s)-\omega_{\varepsilon_0}(t_{n+1}-s)\right| \, dt\, ds \\
&\le \int_{t_n}^{t_{n+1}} \int_t^{t_{n+1}} \int_0^T \left|\omega'_{\varepsilon_0}(\tau-s)\right| \, ds\, d\tau\, dt \\
&\le \frac{C\,\Delta t^2}{\varepsilon_0}.
\end{aligned}
$$

Therefore,

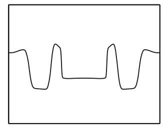
$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \left|u_j^n - u_{j-1}^n\right| \int_{t_n}^{t_{n+1}} & \left|\phi(x_{j-1/2},t)-\phi(x_{j-1/2},t_{n+1})\right| \, dt\, dy\, ds \\
&\le \|f\|_{\mathrm{Lip}} \sum_j \left|u_j^0 - u_{j-1}^0\right| \sum_{n=0}^{N-1} \frac{C\,\Delta t^2}{\varepsilon_0} \\
&\le C\,T \frac{\Delta t}{\varepsilon_0}. \qquad (3.69)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} & \omega_{\varepsilon_0}(t_{n+1}-s)\left|\omega_\varepsilon(x-y)-\omega_\varepsilon(x_{j-1/2}-y)\right| \, dx\, dy\, ds \\
&\le \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x} \int_{\mathbb{R}} \left|\omega'_\varepsilon(z-y)\right| \, dy\, dz\, dx \\
&\le \frac{C\,\Delta x \Delta t}{\varepsilon_0},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \int_0^T \int_{\mathbb{R}} \left|u_j^n - u_{j-1}^n\right| \lambda \int_{x_{j-1/2}}^{x_{j+1/2}} & \left|\phi(x,t_{n+1})-\phi(x_{j-1/2},t_{n+1})\right| \, dx dy ds \\
&\le \|f\|_{\mathrm{Lip}} \sum_j \sum_{n=0}^{N-1} \left|u_j^0 - u_{j-1}^0\right| \lambda \frac{C\,\Delta x \Delta t}{\varepsilon_0} \\
&\le C\,T \frac{\Delta t}{\varepsilon_0}. \qquad (3.70)
\end{aligned}
$$



Collecting the estimates (3.68)–(3.70), we obtain

$$-\Lambda_{\varepsilon_0,\varepsilon}(u,v) \le CT\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0}\right), \tag{3.71}$$

where the constant C depends only on f, F, and $|u_0|_{BV}$. Regarding the term $\nu(u,\varepsilon_0)$, we have that $t \mapsto u_{\Delta t}(x,\,\cdot\,)$ is "almost" L^1 Lipschitz continuous, so

$$\nu(u_{\Delta t},\varepsilon_0) \le C\left(\max\{\varepsilon_0,\Delta t\} + \Delta t\right).$$

The entropy solution v is of uniformly bounded variation in x for each t. Therefore, we conclude that

$$\begin{aligned}\|u_{\Delta t}(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1} &\le \|u_{\Delta t}(\,\cdot\,,0) - v_0\|_1 \\ &\quad + CT\left(\max\{\varepsilon_0,\Delta t\} + \varepsilon_0 + \varepsilon + \frac{\Delta t}{\varepsilon_0} + \frac{\Delta x}{\varepsilon}\right).\end{aligned}$$

Choosing

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} v_0(y)\,dy,$$

we have that $\|u_{\Delta t}(\,\cdot\,,0) - v_0\|_1 \le \Delta x\,|v_0|_{BV}$. Then we can choose $\varepsilon = \sqrt{\Delta x}$ and $\varepsilon_0 = \sqrt{\Delta t}$ to find that

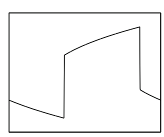
$$\|u_{\Delta t}(\,\cdot\,,T) - v(\,\cdot\,,T)\|_1 \le C\sqrt{\Delta t}, \tag{3.72}$$

where C depends on T, $|v_0|_{BV}$, f, and F. ◇

If one uses Kuznetsov's lemma to estimate the error of a scheme, one must estimate the modulus of continuity $\tilde{\nu}_t\,(u,\varepsilon_0)$ and the term $\Lambda_{\varepsilon,\varepsilon_0}(u,v)$. In other words, one must obtain regularity estimates on the *approximation* u. Therefore, this approach gives a posteriori error estimates, and perhaps the proper use for this approach should be in adaptive methods, in which it would provide error control and govern mesh refinement. However, despite this weakness, Kuznetsov's theory is still actively used.

3.4 A Priori Error Estimates

We shall now describe an application of a variation of Kuznetsov's approach in which we obtain an error estimate for the method of vanishing viscosity without using the regularity properties of the viscous approximation. Of course, this application only motivates the approach, since regularity of the solutions of parabolic equations is not difficult to obtain elsewhere. Nevertheless, it is interesting in its own right, since many difference methods have (3.73) as their model equation. We first state the result.



Theorem 3.18 *Let $v(x,t)$ be a solution of (3.1) with initial value v_0, and let u solve the equation*

$$u_t + f(u)_x = (\delta(u)u_x)_x\,, \qquad u(x,0) = u_0(x), \tag{3.73}$$

in the classical sense, with $\delta(u) > 0$. Then

$$\|u(T) - v(T)\|_{L^1(\mathbb{R})} \leq 2\|u_0 - v_0\|_{L^1(\mathbb{R})} + 4\mathrm{T.V.}\,(v_0)\,\sqrt{8T\,\|\delta\|_v},$$

where

$$\|\delta\|_v = \sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}}} \tilde{\delta}\,(v(x-,t), v(x+,t))$$

and

$$\tilde{\delta}(a,b) = \frac{1}{b-a}\int_a^b \delta(c)\,dc.$$

This result is not surprising, and in some sense is weaker than the corresponding result found using Kuznetsov's lemma. The new element here is that the proof does *not* rely on any smoothness properties of the function u, and is therefore also considerably more complicated than the proof using Kuznetsov's lemma.

Proof The proof consists in choosing new Λ's, and using a special form of the test function φ. Let ω^∞ be defined as

$$\omega^\infty(x) = \begin{cases} \frac{1}{2} & \text{for } |x| \leq 1,\\ 0 & \text{otherwise.}\end{cases}$$

We will consider a family of smooth functions ω such that $\omega \to \omega^\infty$. To keep the notation simple we will not add another parameter to the functions ω, but rather write $\omega \to \omega^\infty$ when we approach the limit. Let

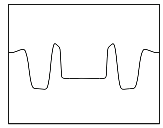
$$\varphi(x,y,t,s) = \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)$$

with $\omega_\alpha(x) = (1/\alpha)\,\omega(x/\alpha)$ as usual. In this notation,

$$\omega_\varepsilon^\infty(x) = \begin{cases} 1/(2\varepsilon) & \text{for } |x| \leq \varepsilon,\\ 0 & \text{otherwise.}\end{cases}$$

In the following we will use the entropy pair

$$\eta(u,k) = |u-k| \quad \text{and} \quad q(u,k) = \mathrm{sign}\,(u-k)\,(f(u) - f(k)),$$



and except where explicitly stated, we always let $u = u(y,s)$ and $v = v(x,t)$. Let $\eta_\sigma(u,k)$ and $q_\sigma(u,k)$ be smooth approximations to η and q such that

$$\eta_\sigma(u) \to \eta(u) \quad \text{as } \sigma \to 0, \qquad q_\sigma(u,k) = \int \eta'_\sigma(z-k)(f(z)-f(k))\,dz.$$

For a test function φ define

$$\Lambda_T^\sigma(u,k) = \int_0^T \int \eta'_\sigma(u-k)\Big(u_s + f(u)_y - \big(\delta(u)u_y\big)_y\Big)\varphi\,dy\,ds$$

(which is clearly zero because of (3.73)) and

$$\Lambda_{\varepsilon,\varepsilon_0}^\sigma(u,v) = \int_0^T \int \Lambda_T^\sigma(u, v(x,t))\,dx\,dt.$$

Note that since u satisfies (3.73), $\Lambda_{\varepsilon,\varepsilon_0}^\sigma = 0$ for every v. We now split $\Lambda_{\varepsilon,\varepsilon_0}^\sigma$ into two parts. Writing (cf. (2.15))

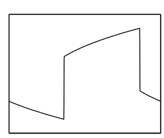
$$\begin{aligned}
\big(u_s + &f(u)_x - (\delta(u)u_y)_y\big)\eta'_\sigma(u-k) \\
&= \eta(u-k)_s + ((f(u)-f(k))'\eta'_\sigma(u-k)u_y - (\delta(u)u_y)_y\eta'_\sigma(u-k) \\
&= \eta_\sigma(u-k)_s + q_\sigma(u,k)_u u_y - (\delta(u)u_y)_y\eta'_\sigma(u-k) \\
&= \eta_\sigma(u-k)_s + q_\sigma(u,k)_y - (\delta(u)\eta_\sigma(u-k)_y)_y + \eta''_\sigma(u-k)\delta(u)(u_y)^2 \\
&= \eta_\sigma(u-k)_s + (q_\sigma(u,k) - \delta(u)\eta_\sigma(u-k)_y)_y + \eta''(u-k)\delta(u)(u_y)^2,
\end{aligned}$$

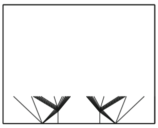
we may introduce

$$\begin{aligned}
\Lambda_1^\sigma(u,v) &= \int_0^T\!\!\int\!\!\int_0^T\!\!\int \eta''_\sigma(u-v)\delta(u)\big(u_y\big)^2\varphi\,dy\,ds\,dx\,dt, \\
\Lambda_2^\sigma(u,v) &= \int_0^T\!\!\int\!\!\int_0^T\!\!\int \Big(\eta_\sigma(u-v)_s + \big(q_\sigma(u,v) - \delta(u)\eta_\sigma(u-v)_y\big)_y\Big)\varphi\,dy\,ds\,dx\,dt,
\end{aligned}$$

such that $\Lambda_{\varepsilon,\varepsilon_0}^\sigma = \Lambda_1^\sigma + \Lambda_2^\sigma$. Note that if $\delta(u) > 0$, we always have $\Lambda_1^\sigma \geq 0$, and hence $\Lambda_2^\sigma \leq 0$. Then we have that

$$\Lambda_2 := \limsup_{\sigma\to 0} \Lambda_2^\sigma \leq 0.$$





To estimate Λ_2, we integrate by parts:

$$\begin{aligned}
\Lambda_2(u,v) &= \int_0^T\!\!\iint_0^T\!\!\int \big(-\eta(u-v)\varphi_s - q(u,v)\varphi_y + V(u,v)\varphi_{yy}\big)\,dy\,ds\,dx\,dt \\
&\quad + \int_0^T\!\!\iint \eta(u(T)-v)\varphi|_{s=T}\,dy\,dx\,dt - \int_0^T\!\!\iint \eta(u_0-v)\varphi|_{s=0}\,dy\,dx\,dt \\
&= \int_0^T\!\!\iint_0^T\!\!\int \big(\eta(u-v)\varphi_t + F(u,v)\varphi_x - V(u,v)\varphi_{xy}\big)\,dy\,ds\,dx\,dt \\
&\quad + \int_0^T\!\!\iint \eta(u(T)-v)\varphi|_{s=T}\,dy\,dx\,dt - \int_0^T\!\!\iint \eta(u_0-v)\varphi|_{s=0}\,dy\,dx\,dt,
\end{aligned}$$

where

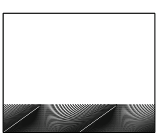
$$V(u,v) = \int_u^v \delta(s)\eta'(s-v)\,ds.$$

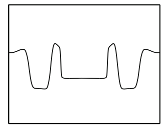
Now define (the "dual of Λ_2")

$$\begin{aligned}
\Lambda_2^* := &-\int_0^T\!\!\iint_0^T\!\!\int \big(\eta(u-v)\varphi_t + q(u,v)\varphi_x - V(u,v)\varphi_{xy}\big)\,dy\,ds\,dx\,dt \\
&- \int_0^T\!\!\iint \eta(u-v(T))\varphi\,\Big|_{t=0}^{t=T}\,dx\,dy\,ds.
\end{aligned}$$

Then we can write

$$\begin{aligned}
\Lambda_2 = -\Lambda_2^* &\underbrace{+ \int_0^T\!\!\iint \big(\eta(u(T)-v)\varphi\big)|_{s=T}\,dy\,dx\,dt}_{\Phi_1} \\
&\underbrace{- \int_0^T\!\!\iint \big(\eta(u_0-v)\varphi\big)|_{s=0}\,dy\,dx\,dt}_{\Phi_2} \\
&\underbrace{+ \int_0^T\!\!\iint \big(\eta(u-v(T))\varphi\big)|_{t=T}\,dx\,dy\,ds}_{\Phi_3} \\
&\underbrace{- \int_0^T\!\!\iint \big(\eta(u_0-v_0)\varphi\big)|_{t=0}\,dx\,dy\,ds}_{\Phi_4} \\
=:& -\Lambda_2^* + \Phi.
\end{aligned}$$





We will need later that

$$\Phi = \Lambda_2^* + \Lambda_2 \le \Lambda_2^*. \tag{3.74}$$

Let

$$\Omega_{\varepsilon_0}(t) = \int_0^t \omega_{\varepsilon_0}(s)\,ds$$

and

$$e(t) = \|u(t) - v(t)\|_{L^1} = \int \eta(u(x,t) - v(x,t))\,dx.$$

To continue estimating, we need the following proposition.

Proposition 3.19

$$\begin{aligned}\Phi \ge\, &\Omega_{\varepsilon_0}(T)e(T) - \Omega_{\varepsilon_0}(T)e(0) + \int_0^T \omega_{\varepsilon_0}(T-t)e(t)\,dt - \int_0^T \omega_{\varepsilon_0}(t)e(t)\,dt \\ &- 4\Omega_{\varepsilon_0}(T)\left(\varepsilon_0\|f\|_{\mathrm{Lip}} + \varepsilon\right)\mathrm{T.V.}\,(v_0)\,.\end{aligned}$$

Proof (of Proposition 3.19) We start by estimating Φ_1. First note that

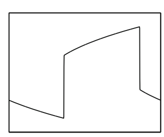
$$\begin{aligned}\eta(u(y,T) - v(x,t)) &= |u(y,T) - v(x,t)| \\ &\ge |u(y,T) - v(y,T)| \\ &\quad - |v(y,T) - v(y,t)| - |v(y,t) - v(x,t)| \\ &= \eta(u(y,T) - v(y,T)) \\ &\quad - |v(y,T) - v(y,t)| - |v(y,t) - v(x,t)|\,.\end{aligned}$$

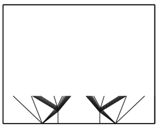
Thus

$$\begin{aligned}\Phi_1 \ge &\int_0^T \iint \eta(u(y,T) - v(y,T))\varphi|_{s=T}\,dy\,dx\,dt \\ &- \int_0^T \iint |v(y,T) - v(y,t)|\,\varphi|_{s=T}\,dy\,dx\,dt \\ &- \int_0^T \iint |v(y,t) - v(x,t)|\,\varphi|_{s=T}\,dy\,dx\,dt \\ \ge\, &\Omega_{\varepsilon_0}(T)e(T) - \Omega_{\varepsilon_0}(T)\left(\varepsilon_0\|f\|_{\mathrm{Lip}} + \varepsilon\right)\mathrm{T.V.}\,(v_0)\,.\end{aligned}$$

Here we have used that v is an exact solution. The estimate for Φ_2 is similar, yielding

$$\Phi_2 \ge -\Omega_{\varepsilon_0}(T)e(0) - \Omega_{\varepsilon_0}(T)\left(\varepsilon_0\|f\|_{\mathrm{Lip}} + \varepsilon\right)\mathrm{T.V.}\,(v_0)\,.$$





To estimate Φ_3 we proceed in the same manner:

$$\eta(u(y,s) - v(x,T)) \geq \eta(u(y,s) - v(y,s)) - |v(y,s) - v(x,s)| \\ - |v(x,s) - v(x,T)|\,.$$

This gives

$$\Phi_3 \geq \int_0^T \omega_{\varepsilon_0}(T-t)e(t)\,dt - \Omega_{\varepsilon_0}(T)\left(\varepsilon_0\|f\|_{\mathrm{Lip}} + \varepsilon\right)\mathrm{T.V.}\,(v_0)\,,$$

while by the same reasoning, the estimate for Φ_4 reads

$$\Phi_4 \geq -\int_0^T \omega_{\varepsilon_0}(t)e(t)\,dt - \Omega_{\varepsilon_0}(T)\left(\|f\|_{\mathrm{Lip}}\varepsilon_0 + \varepsilon\right)\mathrm{T.V.}\,(v_0)\,.$$

The proof of Proposition 3.19 is complete. □

To proceed further, we shall need the following Gronwall-type lemma:

Lemma 3.20 *Let θ be a nonnegative function that satisfies*

$$\Omega^\infty_{\varepsilon_0}(\tau)\theta(\tau) + \int_0^\tau \omega^\infty_{\varepsilon_0}(\tau - t)\theta(t)\,dt \leq C\,\Omega^\infty_{\varepsilon_0}(\tau) + \int_0^\tau \omega^\infty_{\varepsilon_0}(t)\theta(t)\,dt, \tag{3.75}$$

for all $\tau \in [0,T]$ and some constant C. Then

$$\theta(\tau) \leq 2C.$$

Proof (of Lemma 3.20) If $\tau \leq \varepsilon_0$, then for $t \in [0,\tau]$, $\omega^\infty_{\varepsilon_0}(t) = \omega^\infty_{\varepsilon_0}(\tau - t) = 1/(2\varepsilon_0)$. In this case (3.75) immediately simplifies to $\theta(t) \leq C$.

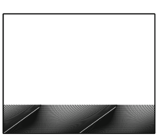
For $\tau > \varepsilon_0$, we can write (3.75) as

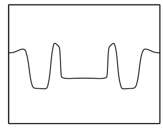
$$\theta(\tau) \leq C + \frac{1}{\Omega^\infty_{\varepsilon_0}(\tau)} \int_0^{\varepsilon_0} \left(\omega^\infty_{\varepsilon_0}(t) - \omega^\infty_{\varepsilon_0}(\tau - t)\right)\theta(t)\,dt.$$

For $t \in [0,\varepsilon_0]$ we have $\theta(t) \leq C$, and this implies

$$\theta(\tau) \leq C\left(1 + \frac{1}{\Omega^\infty_{\varepsilon_0}(\tau)} \int_0^{\varepsilon_0} \left(\omega^\infty_{\varepsilon_0}(t) - \omega^\infty_{\varepsilon_0}(\tau - t)\right)\,dt\right) \leq 2C.$$

This concludes the proof of the lemma. □





Now we can continue the estimate of $e(T)$.

Proposition 3.21 *We have that*

$$e(T) \leq 2e(0) + 8\left(\varepsilon + \varepsilon_0 \|f\|_{\mathrm{Lip}}\right) \mathrm{T.V.}\,(v_0) + 2 \lim_{\omega \to \omega^\infty} \sup_{t \in [0,T]} \frac{\Lambda_2^*(u,v)}{\Omega_{\varepsilon_0}^\infty(t)}.$$

Proof (of Proposition 3.21) Starting with the inequality (3.74), using the estimate for Φ from Proposition 3.19, we have, after passing to the limit $\omega \to \omega^\infty$, that

$$\begin{aligned} \Omega_{\varepsilon_0}^\infty(T)e(T) + \int_0^T \omega_{\varepsilon_0}^\infty(T-t)e(t)\,dt &\leq \Omega_{\varepsilon_0}^\infty(t)e(0) + \int_0^T \omega_{\varepsilon_0}^\infty(t)e(t)\,dt \\ &\quad + 4\Omega_{\varepsilon_0}^\infty(t)\left(\varepsilon + \varepsilon_0 \|f\|_{\mathrm{Lip}}\right) \mathrm{T.V.}\,(v_0) \\ &\quad + \Omega_{\varepsilon_0}^\infty(T) \lim_{\omega \to \omega^\infty} \sup_{t \in [0,T]} \frac{\Lambda_2^*(u,v)}{\Omega_{\varepsilon_0}^\infty(t)}. \end{aligned}$$

We apply Lemma 3.20 with

$$C = 4\left(\varepsilon + \varepsilon_0 \|f\|_{\mathrm{Lip}}\right) \mathrm{T.V.}\,(v_0) + \lim_{\omega \to \omega^\infty} \sup_{t \in [0,T]} \frac{\Lambda_2^*(u,v)}{\Omega_{\varepsilon_0}^\infty(t)} + e(0)$$

to complete the proof. □

To finish the proof of the theorem, it remains only to estimate

$$\lim_{\omega \to \omega^\infty} \sup_{t \in [0,T]} \frac{\Lambda_2^*(u,v)}{\Omega(t)}.$$

We will use the following inequality:

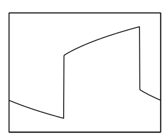
$$\left| \frac{V\left(u, v^+\right) - V\left(u, v^-\right)}{v^+ - v^-} \right| \leq \frac{1}{v^+ - v^-} \int_{v^-}^{v^+} \delta(s)\,ds. \tag{3.76}$$

Since v is an entropy solution to (3.1), we have that

$$\Lambda_2^* \leq -\int_0^T \int \int_0^T \int V(u,v)\varphi_{xy}\,dy\,ds\,dx\,dt. \tag{3.77}$$

Since v is of bounded variation, it suffices to study the case that v is differentiable except on a countable number of curves $x = x(t)$. We shall bound Λ_2^* in the case that we have one such curve; the generalization to more than one is straightforward. Integrating (3.77) by parts, we obtain

$$\Lambda_2^* \leq \int_0^T \int \Psi(y,s)\,dy\,ds, \tag{3.78}$$



where Ψ is given by

$$\Psi(y,s) = \int_0^T \Big(\int_{-\infty}^{x(t)} V(u,v)_v \, v_x \varphi_y \, dx \\ + \frac{[\![V]\!]}{[\![v]\!]} [\![v]\!] \, \varphi_y|_{x=x(t)} + \int_{x(t)}^{\infty} V(u,v)_v \, v_x \varphi_y \, dx \Big) \, dt.$$

As before, $[\![a]\!]$ denotes the jump in a, i.e., $[\![a]\!] = a(x(t)+,t) - a(x(t)-,t)$. Using (3.76), we obtain

$$|\Psi(y,s)| \le \|\delta\|_v \int_0^T \Big(\int_{-\infty}^{x(t)} |v_x| \, |\varphi_y| \, dx \\ + |[\![v]\!]| \, \big|\varphi_y|_{x=x(t)}\big| + \int_{x(t)}^{\infty} |v_x| \, |\varphi_y| \, dx \Big) \, dt. \tag{3.79}$$

Let D be given by

$$D(x,t) = \int_0^T \int |\varphi_y| \, dy \, ds.$$

A simple calculation shows that

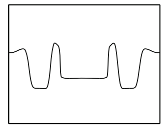
$$D(x,t) = \frac{1}{\varepsilon} \int_0^T \omega_{\varepsilon_0}(t-s) \, ds \int |\omega'(y)| \, dy \le \frac{1}{\varepsilon} \int_0^T \omega_{\varepsilon_0}(t-s) \, ds.$$

Consequently,

$$\begin{aligned} \int_0^T \sup_x D(x,t) \, dt &\le \frac{1}{\varepsilon} \int_0^T \int_0^T \omega_{\varepsilon_0}(t-s) \, ds \, dt \\ &= \frac{2}{\varepsilon} \int_0^T (T-t) \omega_{\varepsilon_0}(t) \, dt \\ &\le \frac{2T \Omega(T)}{\varepsilon}. \end{aligned}$$

Inserting this in (3.79), and the result in (3.78), we find that

$$\Lambda_2^*(u,v,T) \le \frac{2}{\varepsilon} T \, \mathrm{T.V.}\,(v_0) \, \|\delta\|_v \, \Omega(T).$$



Summing up, we have now shown that

$$e(T) \leq 2e(0) + 8\left(\varepsilon + \varepsilon_0\|f\|_{\mathrm{Lip}}\right)\mathrm{T.V.}\,(v_0) + \frac{4}{\varepsilon}T\,\mathrm{T.V.}\,(v_0)\,\|\delta\|_v.$$

We can set ε_0 to zero, and minimize over ε, obtaining

$$\|u(T) - v(T)\|_{L^1} \leq 2\|u_0 - v_0\|_{L^1} + 4\mathrm{T.V.}\,(v_0)\,\sqrt{8T\|\delta\|_v}.$$

The theorem is proved. □

The main idea behind this approach to getting a priori error estimates is to choose the "Kuznetsov-type" form $\Lambda_{\varepsilon,\varepsilon_0}$ such that

$$\Lambda_{\varepsilon,\varepsilon_0}(u, v) = 0$$

for every function v, and then write $\Lambda_{\varepsilon,\varepsilon_0}$ as the sum of a nonnegative and a nonpositive part. Given a numerical scheme, the task is then to prove a discrete analogue of the previous theorem.

3.5 Measure-Valued Solutions

You try so hard, but you don't understand . . .
— Bob Dylan, Ballad of a Thin Man (1965)

Monotone methods are at most first-order accurate. Consequently, one must work harder to show that higher-order methods converge to the entropy solution. While this is possible in one space dimension, i.e., in the above setting, it is much more difficult in several space dimensions. One useful tool to aid the analysis of higher-order methods is the concept of *measure-valued solutions*. This is a rather complicated concept, which requires a solid background in analysis beyond this book. Therefore, the presentation in this section is brief, and is intended to give the reader a first flavor, and an idea of what this method can accomplish.

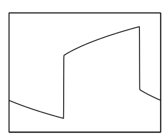
The Young Measure

Consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ that is uniformly bounded in $L^\infty(\mathbb{R}\times[0,\infty))$. This is typically the result of a numerical method, where one has L^∞ bounds, but no uniform bounds on the total variation. Passing to a subsequence, we can still infer that the weak-star limit

$$u_n \overset{*}{\rightharpoonup} u,$$

exists, which means that for all $\varphi \in L^1(\mathbb{R}\times[0,\infty))$,

$$\iint\limits_{\Omega} u_n\varphi\,dx\,dt \to \iint\limits_{\Omega} u\varphi\,dx\,dt,$$



with $\Omega = \mathbb{R} \times [0, \infty)$. In order to show that the limit u is a weak solution to the conservation law, we must study

$$\iint\limits_{\Omega} \big(u_n \varphi_t + f(u_n)\varphi_x\big)\, dx\, dt.$$

The first term in this equation has a limit $\iint u\varphi_t \, dx\, dt$, but the second term is more complicated, as the next example shows.

◇ Example 3.22

Let $u_n = \sin(nx)$ and $f(u) = u^2$, and φ a smooth function in $L^1(\mathbb{R})$. Then

$$\left| \int \sin(nx)\varphi(x)\, dx \right| \le \frac{1}{n} \left| \int \cos(nx)\varphi'(x)\, dx \right| \le \frac{C}{n} \to 0 \text{ as } n \to \infty.$$

On the other hand, $f(u_n) = \sin^2(nx) = (1 - \cos(2nx))/2$, and hence a similar estimate shows that

$$\left| \int (f(u_n) - \frac{1}{2})\varphi(x)\, dx \right| \le \frac{C}{n} \to 0 \text{ as } n \to \infty.$$

Thus we conclude that

$$u_n \overset{*}{\rightharpoonup} 0, \quad f(u_n) \overset{*}{\rightharpoonup} \frac{1}{2} \ne 0 = f(0). \qquad \diamond$$

The Young measure is one method for studying the weak limits of nonlinear functions of a weak-star convergent sequence.

In order to define it, we first define the function

$$\chi(\lambda, u) = \begin{cases} 1 & 0 \le \lambda \le u, \\ -1 & u \le \lambda \le 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.80}$$

It is easily verified that for every differentiable function f,

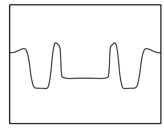
$$\int\limits_{-\infty}^{\infty} f'(\lambda)\chi(\lambda, u)\, d\lambda = f(u) - f(0). \tag{3.81}$$

Furthermore, let $g(\lambda)$ be a function such that

$$u = \int\limits_{\mathbb{R}} g(\lambda)\, d\lambda, \qquad \operatorname{sign}(\lambda)\, g(\lambda) = |g(\lambda)| \le 1. \tag{3.82}$$

Define $m(\lambda)$ by

$$m(\lambda) = \int\limits_{-\infty}^{\lambda} \big(\chi(\xi, u) - g(\xi)\big)\, d\xi.$$



Then $\lim_{\lambda\to-\infty} m(\lambda) = 0$, and

$$\lim_{\lambda\to\infty} m(\lambda) = \int_{-\infty}^{\infty} \chi(\xi, u) - g(\xi)\, d\xi = u - u = 0.$$

Furthermore, by (3.82), we have that m is nondecreasing in the interval $(-\infty, u)$ and nonincreasing in the interval (u, ∞). Hence $m(\lambda)$ is nonnegative. For every twice differentiable convex function $S(\lambda)$ we have

$$\int_{\mathbb{R}} S'(\lambda)\left(\chi(\lambda, u) - g(\lambda)\right) d\lambda = -\int_{\mathbb{R}} S''(\lambda) m(\lambda)\, d\lambda \le 0.$$

Thus, for a strictly convex function S, the function $\chi(\,\cdot\,, u)$ is the unique minimizer of the problem: Find $g \in L^1(\mathbb{R})$ such that (3.82) holds and

$$\int_{\mathbb{R}} S'(\lambda) g(\lambda)\, d\lambda \qquad \text{is minimized.} \tag{3.83}$$

If $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ is uniformly bounded, then $\{\chi(\,\cdot\,, u_n)\}_{n\in\mathbb{N}} \subset L^\infty(\mathbb{R} \times \Omega)$ is also uniformly bounded. Thus it has (modulo subsequences) a weak-star limit, which we call $f(\lambda, x, t)$. The next lemma gives some properties of this limit.

Lemma 3.23 *Let $f(\lambda, x, t)$ denote the weak-star limit of $\chi(\lambda, u_n)$. Then f is in $L^\infty(\mathbb{R} \times \Omega)$ and satisfies*

$$\int_{\mathbb{R}} f(\lambda, x, t)\, d\lambda = u(x, t) \tag{3.84}$$

for almost all (x, t). Furthermore,

$$|f(\lambda, x, t)| = \operatorname{sign}(\lambda)\, f(\lambda, x, t), \tag{3.85}$$

$$\frac{\partial}{\partial\lambda} f(\lambda, x, t) = \delta(\lambda) - \nu_{(x,t)}(\lambda), \tag{3.86}$$

where $\delta(\lambda)$ is the Dirac measure, and $\nu_{(x,t)}(\lambda)$ is a nonnegative measure in (λ, x, t) such that

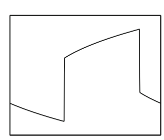
$$\int_{\mathbb{R}} \nu_{(x,t)}(\lambda)\, d\lambda = 1 \tag{3.87}$$

for almost all (x, t).

Remark 3.24 The derivative in (3.86) is to be interpreted in the distributional sense, i.e., (3.86) means that

$$\begin{aligned} -\int_{\mathbb{R}} f(\lambda, x, t)\varphi'(\lambda)\, d\lambda &= \int_{\mathbb{R}} \frac{\partial}{\partial\lambda} f(\lambda, x, t)\varphi(\lambda)\, d\lambda \\ &= \int_{\mathbb{R}} \left(\delta(\lambda) - \nu_{(x,t)}(\lambda)\right)\varphi(\lambda)\, d\lambda, \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.



Proof The first equality, (3.84) follows from the observation

$$u_n(x,t) = \int_{\mathbb{R}} \chi(\lambda, u_n(x,t))\, d\lambda.$$

To prove (3.85) we choose a test function of the form $\varphi(x,t)\psi(\lambda)$, where the ψ has support in $(0,\infty)$ and $\varphi \geq 0$. By definition of the weak-star limit,

$$\begin{aligned}
&\iint_{\Omega} \int_{\mathbb{R}} f(\lambda,x,t)\psi(\lambda)\varphi(x,t)\, d\lambda\, dx\, dt \\
&\qquad\qquad = \lim_{n\to\infty} \iint_{\Omega} \int_{\mathbb{R}} \chi(\lambda, u_n(x,t))\psi(\lambda)\varphi(x,t)\, d\lambda\, dx\, dt \geq 0.
\end{aligned}$$

Thus $f \geq 0$ for $\lambda \geq 0$, and one similarly shows that $f \leq 0$ if $\lambda \leq 0$.

To prove (3.86), by Remark 3.24 we have that for all test functions $\varphi(\lambda,x,t)$,

$$\begin{aligned}
&\iint_{\Omega} \int_{\mathbb{R}} \frac{\partial}{\partial\lambda}\chi(\lambda,u_n)\varphi(\lambda,x,t)\, d\lambda\, dx\, dt \\
&\qquad = -\iint_{\Omega} \int_{\mathbb{R}} \chi(\lambda,u_n)\frac{\partial}{\partial\lambda}\varphi(\lambda,x,t)\, d\lambda\, dx\, dt \\
&\qquad = \iint_{\Omega} \big(\varphi(0,x,t) - \varphi(u_n,x,t)\big)\, dx\, dt \\
&\qquad = \iint_{\Omega} \int_{\mathbb{R}} \Big(\delta(\lambda)\varphi(\lambda,x,t) - \delta_{u_n}(\lambda)\varphi(\lambda,x,t)\Big)\, d\lambda\, dx\, dt,
\end{aligned}$$

where δ_{u_n} is the Dirac mass centered at u_n. Thus we define

$$\nu_{n,(x,t)}(\lambda) = \delta_{u_n}(\lambda),$$

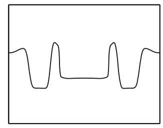
so that

$$\frac{\partial}{\partial\lambda}\chi(\lambda, u_n(x,t)) = \delta(\lambda) - \nu_{n,(x,t)}(\lambda).$$

The measure $\nu_{n,(x,t)}$ is a probability measure in the first variable, in the sense that it is nonnegative and has unit total mass. Thus we have that there exists a nonnegative measure $\nu_{(x,t)}$ such that

$$\int_{\mathbb{R}} \nu_{n,(x,t)}(\lambda)\psi(\lambda)\, d\lambda \to \int_{\mathbb{R}} \psi(\lambda)\, \nu_{(x,t)}(\lambda)\, d\lambda,$$

for all continuous functions ψ. In order to conclude, we must prove (3.87). Choose a test function of the form $\psi(\lambda)\varphi(x,t)$, where ψ has compact support and $\psi \equiv 1$



for $|\lambda| \leq \|u_n\|_\infty$. Then

$$\begin{aligned}
0 &= -\iint\limits_{\Omega} \int\limits_{\mathbb{R}} \chi(\lambda, u_n)\psi'(\lambda)\varphi(x,t)\, d\lambda\, dx\, dt \\
&= \iint\limits_{\Omega} \Big(1 - \int\limits_{\mathbb{R}} \nu_{n,(x,t)}(\lambda)\, d\lambda\Big)\varphi(x,t)\, dx\, dt \\
&\to \iint\limits_{\Omega} \Big(1 - \int\limits_{\mathbb{R}} \nu_{(x,t)}(\lambda)\, d\lambda\Big)\varphi(x,t)\, dx\, dt \quad \text{as } n \to \infty.
\end{aligned}$$

Thus (3.87) holds. □

If now $u_n \overset{*}{\rightharpoonup} u$ in L^∞, then we have

$$u_n(x,t) = \int\limits_{\mathbb{R}} \chi(\lambda, u_n(x,t))\, d\lambda \to \int\limits_{\mathbb{R}} f(\lambda, x, t)\, d\lambda = u(x,t).$$

Similarly, for every function $S(u)$ with S' bounded and $S(0) = 0$,

$$S(u_n) = \int\limits_{\mathbb{R}} S'(\lambda)\chi(\lambda, u_n)\, d\lambda = \int\limits_{\mathbb{R}} S(\lambda)\nu_{n,(x,t)}(\lambda)\, d\lambda.$$

Therefore, if $\bar{S}(x,t)$ denotes the weak-star limit of $S(u_n)$, then

$$\bar{S}(x,t) = \int\limits_{\mathbb{R}} S'(\lambda) f(\lambda, x, t)\, d\lambda = \int\limits_{\mathbb{R}} S(\lambda)\nu_{(x,t)}(\lambda)\, d\lambda. \tag{3.88}$$

The limit measure $\nu_{(x,t)}$ is called the Young measure associated with the sequence $\{u_n\}$. If S is strictly convex, then using (3.83), we obtain

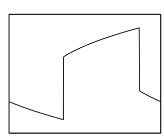
$$\bar{S}(x,t) = \int\limits_{\mathbb{R}} S'(\lambda) f(\lambda, x, t)\, d\lambda \leq \int\limits_{\mathbb{R}} S'(\lambda)\chi(\lambda, u)\, d\lambda = S(u),$$

with equality if and only if $f(\lambda, x, t) = \chi(\lambda, u(x,t))$. Hence, $u_n \to u$ strongly, if and only if $\nu_{(x,t)}(\lambda) = \delta_u(\lambda)$.

We have proved the following theorem:

Theorem 3.25 (Young's theorem) *Let $\{u_n\}$ be a sequence of functions from $\Omega = \mathbb{R} \times [0,\infty)$ with values in $[-K, K]$. Then there exists a family of probability measures $\{\nu_{(x,t)}(\lambda)\}_{(x,t)\in\Omega}$, depending weak-star measurably on (x,t), such that for every continuously differentiable function $S\colon [-K, K] \to \mathbb{R}$ with S' bounded and $S(0) = 0$, we have*

$$S(u_n) \overset{*}{\rightharpoonup} \bar{S} \text{ in } L^\infty(\Omega) \text{ as } n \to \infty,$$



where

$$\bar{S}(x,t) = \int_{\mathbb{R}} S(\lambda)\, d\nu_{(x,t)}(\lambda) \text{ for a.e. } (x,t) \in \Omega,$$

and where the exceptional set possibly depends on S. *Furthermore,*

$$\operatorname{supp} \nu_{(x,t)} \subset [-K, K] \text{ for a.e. } (x,t) \in \Omega.$$

We also have that $u_n \to u$ *strongly in* $L^1_{\mathrm{loc}}(\Omega)$ *if and only if* $\nu_{(x,t)}(\lambda) = \delta_{u(x,t)}(\lambda)$.

◇ Example 3.26

Let us compute the Young measure associated with the sequence $\{\sin(nx)\}$. In this case the weak limit of $\chi(\lambda, \sin(nx))$ will be independent of x. If $\lambda > 0$, then

$$\int_a^b \chi(\lambda, \sin(nx))\, dx = \frac{\operatorname{meas}\{x \in [a,b] \mid \sin(nx) > \lambda\}}{b-a},$$

and similarly, if $\lambda < 0$, then

$$\int_a^b \chi(\lambda, \sin(nx))\, dx = -\frac{\operatorname{meas}\{x \in [a,b] \mid \sin(nx) < \lambda\}}{b-a}.$$

We have $\chi(\lambda, \sin(nx)) \overset{*}{\rightharpoonup} f(\lambda)$, where

$$f(\lambda) = \frac{1}{2\pi} \begin{cases} 2(\frac{\pi}{2} - \sin^{-1}(\lambda)) & 0 < \lambda \le 1, \\ -2(\frac{\pi}{2} + \sin^{-1}(\lambda)) & -1 \le \lambda \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

This can be rewritten

$$f(\lambda) = \chi_{[-1,1]}(\lambda) \left(\frac{1}{2} \operatorname{sign}(\lambda) - \frac{1}{\pi} \sin^{-1}(\lambda) \right).$$

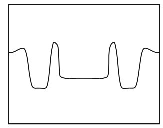
Thus from (3.86),

$$f'(\lambda) = \delta(\lambda) - \nu_x(\lambda) = \delta(\lambda) - \chi_{[-1,1]}(\lambda) \frac{1}{\pi\sqrt{1-\lambda^2}},$$

and we see that

$$\nu_x(\lambda) = \frac{\chi_{[-1,1]}(\lambda)}{\pi\sqrt{1-\lambda^2}}.$$ ◇

Theorem 3.25 is indeed the main reason why measure-valued solutions are easier to obtain than weak solutions, since for every bounded sequence of approximations to a solution of a conservation law we can associate (at least) one probability measure $\nu_{(x,t)}$ representing the weak-star limits of the sequence. Thus we avoid having to show that the method is TVD stable and use Helly's theorem to be able to work with the limit of the sequence. The measures associated with weakly convergent sequences are frequently called Young measures.



Intuitively, when we are in the situation that we have no knowledge of eventual oscillations in u_ε as $\varepsilon \to 0$, the Young measure $\nu_{(x,t)}(E)$ can be thought of as the probability that the "limit" at the point (x,t) takes a value in the set E. To be a bit more precise, define

$$\nu^{\varepsilon,r}_{(x,t)}(E) = \frac{1}{r^2} \operatorname{meas}\Big\{(y,s) \;\Big|\; |x-y|\,, |t-s| \le r \quad \text{and} \quad u_\varepsilon(y,s) \in E\Big\}.$$

Then for small r, $\nu^{\varepsilon,r}_{(x,t)}(E)$ is the probability that u^ε takes values in E near x. It can be shown that

$$\nu_{(x,t)} = \lim_{r\to 0} \lim_{\varepsilon\to 0} \nu^{\varepsilon,r}_{(x,t)};$$

see [10].

Measure-Valued Solutions

Now we can define measure-valued solutions. We use the notation

$$\langle \nu_{(x,t)}, g\rangle = \int_{\mathbb{R}} g(\lambda)\, d\nu_{(x,t)}(\lambda).$$

A probability measure $\nu_{(x,t)}$ is a measure-valued solution to (3.1) if

$$\big\langle \nu_{(x,t)}, \mathrm{Id}\big\rangle_t + \big\langle \nu_{(x,t)}, f\big\rangle_x = 0$$

in the distributional sense, where Id is the identity map, $\mathrm{Id}(\lambda) = \lambda$. As with weak solutions, we call a measure-valued solution compatible with the entropy pair (η, q) (recall that $q' = \eta' f'$) if

$$\big\langle \nu_{(x,t)}, \eta\big\rangle_t + \big\langle \nu_{(x,t)}, q\big\rangle_x \le 0 \tag{3.89}$$

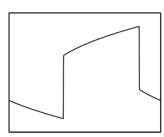
in the distributional sense. If (3.89) holds for *all* convex η, we call $\nu_{(x,t)}$ a measure-valued entropy solution. Clearly, weak entropy solutions are also measure-valued solutions, as we can see by setting

$$\nu_{(x,t)} = \delta_{u(x,t)}$$

for a weak entropy solution u. But measure-valued solutions are more general than weak solutions, since for every two measure-valued solutions $\nu_{(x,t)}$ and $\mu_{(x,t)}$ and $\theta \in [0,1]$, the convex combination

$$\theta\nu_{(x,t)} + (1-\theta)\mu_{(x,t)} \tag{3.90}$$

is also a measure-valued solution. It is not clear, however, what are the initial data satisfied by the measure-valued solution defined by (3.90). We would like our



measure-valued solutions initially to be Dirac masses, i.e., $\nu_{(x,0)} = \delta_{u_0(x)}$. Concretely, we shall assume the following:

$$\lim_{T\downarrow 0} \frac{1}{T} \int_0^T \int_{-A}^{A} \left\langle \nu_{(x,t)}, |\mathrm{Id} - u_0(x)| \right\rangle \, dx \, dt = 0 \tag{3.91}$$

for every A. For every Young measure $\nu_{(x,t)}$ we have the following lemma.

Lemma 3.27 *Let $\nu_{(x,t)}$ be a Young measure with* $\operatorname{supp} \nu_{(x,t)} \subset [-K, K]$*, and let ω_ε be a standard mollifier in x and t. Then:*

(i) *there exists a Young measure $\nu^{\varepsilon}_{(x,t)}$ defined by*

$$\begin{aligned} \left\langle \nu^{\varepsilon}_{(x,t)}, g \right\rangle &= \left\langle \nu_{(x,t)}, g \right\rangle * \omega_\varepsilon \\ &= \iint \omega_\varepsilon(x-y)\omega_\varepsilon(t-s) \left\langle \nu_{(y,s)}, g \right\rangle \, dy \, ds. \end{aligned} \tag{3.92}$$

(ii) *For all $(x, t) \in \mathbb{R} \times [0, T]$ there exist bounded measures $\partial_x \nu^{\varepsilon}_{(x,t)}$ and $\partial_t \nu^{\varepsilon}_{(x,t)}$, defined by*

$$\begin{aligned} \left\langle \partial_t \nu^{\varepsilon}_{(x,t)}, g \right\rangle &= \partial_t \left\langle \nu^{\varepsilon}_{(x,t)}, g \right\rangle, \\ \left\langle \partial_x \nu^{\varepsilon}_{(x,t)}, g \right\rangle &= \partial_x \left\langle \nu^{\varepsilon}_{(x,t)}, g \right\rangle. \end{aligned} \tag{3.93}$$

Proof Clearly, the right-hand side of (3.92) is a bounded linear functional on $C_0(\mathbb{R})$, the set of compactly supported continuous functions, and hence the Riesz representation theorem guarantees the existence of $\nu^{\varepsilon}_{(x,t)}$. To show that $\|\nu^{\varepsilon}_{(x,t)}\|_{\mathcal{M}(\mathbb{R})} = 1$, where $\mathcal{M}(\mathbb{R})$ is the set of all Radon measures, we let $\{\psi_n\}$ be a sequence of test functions such that

$$\left\langle \nu_{(x,t)}, \psi_n \right\rangle \to 1, \quad \text{as } n \to \infty.$$

Then for all $1 > \kappa > 0$ we can find an N such that

$$\left\langle \nu_{(x,t)}, \psi_n \right\rangle > 1 - \kappa,$$

for $n \geq N$. Thus, for such n,

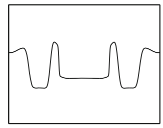
$$\left\langle \nu^{\varepsilon}_{(x,t)}, \psi_n \right\rangle \geq 1 - \kappa,$$

and therefore $\|\nu^{\varepsilon}_{(x,t)}\|_{\mathcal{M}(\mathbb{R})} \geq 1$. The opposite inequality is immediate, since

$$\left| \left\langle \nu^{\varepsilon}_{(x,t)}, \psi \right\rangle \right| \leq \left| \left\langle \nu_{(x,t)}, \psi \right\rangle \right|$$

for all test functions ψ. Therefore, $\nu^{\varepsilon}_{(x,t)}$ is a probability measure. Similarly, the existence of $\partial_x \nu^{\varepsilon}_{(x,t)}$ and $\partial_t \nu^{\varepsilon}_{(x,t)}$ follows by the Riesz representation theorem. Since $\nu_{(x,t)}$ is bounded, the boundedness of $\partial_x \nu^{\varepsilon}_{(x,t)}$ and $\partial_t \nu^{\varepsilon}_{(x,t)}$ follows for each fixed $\varepsilon > 0$. □

Now that we have established the existence of the "smooth approximation" to a Young measure, we can use this to prove the following lemma.



Lemma 3.28 *Assume that f is a Lipschitz continuous function and that $\nu_{(x,t)}(\lambda)$ and $\sigma_{(x,t)}(\mu)$ are measure-valued solutions with support in $[-K, K]$. Then*

$$\partial_t \left\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \right\rangle + \partial_x \left\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, q(\lambda, \mu) \right\rangle \leq 0, \tag{3.94}$$

in the distributional sense, where

$$q(\lambda, \mu) = \operatorname{sign}(\lambda - \mu)\,(f(\lambda) - f(\mu)),$$

and $\nu_{(x,t)} \otimes \sigma_{(x,t)}$ denotes the product measure $d\nu_{(x,t)} d\sigma_{(x,t)}$ on $\mathbb{R} \times \mathbb{R}$.

Proof If $\nu^\varepsilon_{(x,t)}$ and $\sigma^\varepsilon_{(x,t)}$ are defined by (3.92), and $\varphi \in C_0^\infty(\mathbb{R} \times [0, T])$, then we have that

$$\begin{aligned}
\iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu_{(x,t)}, g \right\rangle \partial_t \left(\varphi * \omega_\varepsilon\right)\, dx\, dt &= \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu^\varepsilon_{(x,t)}, g \right\rangle \partial_t \varphi\, dx\, dt \\
&= - \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \partial_t \nu^\varepsilon_{(x,t)}, g \right\rangle \varphi\, dx\, dt,
\end{aligned}$$

and similarly,

$$\iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu_{(x,t)}, g \right\rangle \partial_x \left(\varphi * \omega_\varepsilon\right)\, dx\, dt = - \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \partial_x \nu^\varepsilon_{(x,t)}, g \right\rangle \varphi\, dx\, dt,$$

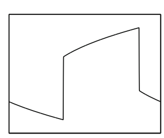
and analogous identities also hold for $\sigma_{(x,t)}$. Therefore,

$$\left\langle \partial_t \nu^\varepsilon_{(x,t)}, |\lambda - \mu| \right\rangle + \left\langle \partial_x \nu^\varepsilon_{(x,t)}, q(\lambda, \mu) \right\rangle \leq 0, \tag{3.95}$$

$$\left\langle \partial_t \sigma^\varepsilon_{(x,t)}, |\lambda - \mu| \right\rangle + \left\langle \partial_x \sigma^\varepsilon_{(x,t)}, q(\lambda, \mu) \right\rangle \leq 0. \tag{3.96}$$

Next, we observe that for every continuous function g,

$$\begin{aligned}
\partial_t \left\langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, g(\lambda, \mu) \right\rangle &= \int\limits_{\mathbb{R}} \partial_t \left(\int\limits_{\mathbb{R}} g(\lambda, \mu)\, d\nu^\varepsilon_{(x,t)}(\lambda) \right) d\sigma^\varepsilon_{(x,t)}(\mu) \\
&\quad + \int\limits_{\mathbb{R}} \partial_t \left(\int\limits_{\mathbb{R}} g(\lambda, \mu)\, d\sigma^\varepsilon_{(x,t)}(\mu) \right) d\nu^\varepsilon_{(x,t)}(\lambda) \\
&= \int\limits_{\mathbb{R}} \left\langle \partial_t \nu^\varepsilon_{(x,t)}, g(\lambda, \mu) \right\rangle d\sigma^\varepsilon_{(x,t)}(\mu) \\
&\quad + \int\limits_{\mathbb{R}} \left\langle \partial_t \sigma^\varepsilon_{(x,t)}, g(\lambda, \mu) \right\rangle d\nu^\varepsilon_{(x,t)}(\lambda),
\end{aligned}$$



and an analogous equality holds for

$$\partial_x \left\langle \nu^{\varepsilon}_{(x,t)} \otimes \sigma^{\varepsilon}_{(x,t)}, g(\lambda,\mu) \right\rangle.$$

Therefore, we find that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}\times[0,T]} \left[\left\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu| \right\rangle \varphi_t + \left\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle \varphi_x(x,t) \right] dx\,dt \\
&\quad = -\iint\limits_{\mathbb{R}\times[0,T]} \Bigl(\int\limits_{\mathbb{R}} \left\langle \partial_t \nu^{\varepsilon_1}_{(x,t)}, |\lambda-\mu| \right\rangle + \left\langle \partial_x \nu^{\varepsilon_1}_{(x,t)}, q(\lambda,\mu) \right\rangle d\sigma^{\varepsilon_2}_{(x,t)}(\mu) \Bigr) \varphi\, dx\,dt \\
&\qquad - \iint\limits_{\mathbb{R}\times[0,T]} \Bigl(\int\limits_{\mathbb{R}} \left\langle \partial_t \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu| \right\rangle + \left\langle \partial_x \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle d\nu^{\varepsilon_1}_{(x,t)}(\lambda) \Bigr) \varphi\, dx\,dt \\
&\quad \geq 0,
\end{aligned}$$

for every nonnegative test function φ. Now we would like to conclude the proof by sending ε_1 and ε_2 to zero. Consider the second term:

$$\begin{aligned}
I^{\varepsilon_1,\varepsilon_2} &= \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle \varphi_x(x,t)\, dx\,dt \\
&= \iint\limits_{\mathbb{R}\times[0,T]} \iiint \left\langle \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle d\nu_{(y,s)} \\
&\qquad\qquad \times \omega_{\varepsilon_1}(x-y)\omega_{\varepsilon_1}(t-s)\varphi_x(x,t)\, dy\,ds\,dx\,dt.
\end{aligned}$$

Since

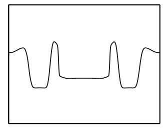
$$\begin{aligned}
&\iiint \left\langle \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle d\nu_{(y,s)} \omega_{\varepsilon_1}(x-y)\omega_{\varepsilon_1}(t-s)\varphi_x(x,t)\, dy\,ds \\
&\qquad \to \int \left\langle \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle d\nu_{(x,t)} \varphi_x(x,t) < \infty
\end{aligned}$$

for almost all (x,t) as $\varepsilon_1 \to 0$, we can use the Lebesgue dominated convergence theorem to conclude that

$$\lim_{\varepsilon_1\to 0} I^{\varepsilon_1,\varepsilon_2} = \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \right\rangle \varphi_x(x,t)\, dx\,dt.$$

We can apply this argument once more for ε_2, obtaining

$$\lim_{\varepsilon_2\to 0} \lim_{\varepsilon_1\to 0} I^{\varepsilon_1,\varepsilon_2} = \iint\limits_{\mathbb{R}\times[0,T]} \left\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, q(\lambda,\mu) \right\rangle \varphi_x(x,t)\, dx\,dt. \tag{3.97}$$



Similarly, we obtain

$$\begin{aligned}\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}&\iint\limits_{\mathbb{R}\times[0,T]}\left\langle \nu^{\varepsilon_1}_{(x,t)}\otimes\sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\right\rangle \varphi_t(x,t)\,dx\,dt\\&=\iint\limits_{\mathbb{R}\times[0,T]}\left\langle \nu_{(x,t)}\otimes\sigma_{(x,t)}, |\lambda-\mu|\right\rangle \varphi_t(x,t)\,dx\,dt.\end{aligned}\tag{3.98}$$

This concludes the proof of the lemma. □

Let $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ be the sequences associated with $\nu_{(x,t)}$ and $\sigma_{(x,t)}$, respectively, and assume that for $t\le T$, the support of $u_\varepsilon(\,\cdot\,,t)$ and $v_\varepsilon(\,\cdot\,,t)$ is contained in a finite interval I. Then both $u_\varepsilon(\,\cdot\,,t)$ and $v_\varepsilon(\,\cdot\,,t)$ are in $L^1(\mathbb{R})$ uniformly in ε. This means that both

$$\left\langle \nu_{(x,t)}, |\lambda|\right\rangle \quad\text{and}\quad \left\langle \sigma_{(x,t)}, |\lambda|\right\rangle$$

are in $L^1(\mathbb{R})$ for almost all t. Using this observation and the preceding lemma, Lemma 3.28, we can continue. Define a smooth approximation to the characteristic function of $[t_1,t_2]$ by

$$\phi_\varepsilon(t)=\int_0^t \left(\omega_\varepsilon(s-t_1)-\omega_\varepsilon(s-t_2)\right)ds,$$

where $t_2>t_1>0$ and ω_ε is the usual mollifier. Also define

$$\psi_n(x)=\begin{cases}1 & \text{for } |x|\le n,\\ 2(1-x/(2n)) & \text{for } n<|x|\le 2n,\\ 0 & \text{otherwise,}\end{cases}$$

and set $\psi_{\varepsilon,n}=\psi_n*\omega_\varepsilon(x)$. Hence

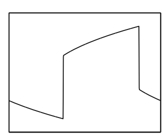
$$\varphi(x,t)=\phi_\varepsilon(t)\psi_{\varepsilon,n}(x)$$

is an admissible test function. Furthermore, $\left|\psi'_{\varepsilon,n}\right|\le 1/n$, and $\phi_\varepsilon(t)$ tends to the characteristic function of the interval $[t_1,t_2]$ as $\varepsilon\to 0$. Therefore,

$$\begin{aligned}-\lim_{\varepsilon\to 0}\iint\limits_{\mathbb{R}\times[0,T]}&\Big[\left\langle \nu_{(x,t)}\otimes\sigma_{(x,t)}, |\lambda-\mu|\right\rangle\varphi_t\\&\qquad+\left\langle \nu_{(x,t)}\otimes\sigma_{(x,t)}, q(\lambda,\mu)\right\rangle\varphi_x\Big]\,dx\,dt\le 0.\end{aligned}$$

Set

$$A_n(t)=\int_{\mathbb{R}}\left\langle \nu_{(x,t)}\otimes\sigma_{(x,t)}, |\lambda-\mu|\right\rangle\psi_n(x)\,dx.$$



Using this definition, we find that

$$A_n(t_2) - A_n(t_1) \le \int_{t_1}^{t_2} \int_{\mathbb{R}} \left\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \right\rangle |\psi_n'(x)| \, dx \, dt. \tag{3.99}$$

The right-hand side of this is bounded by

$$\|f\|_{\mathrm{Lip}} \frac{1}{n} \left(\left\| \left\langle \nu_{(x,t)}, |\lambda| \right\rangle \right\|_{L^1(\mathbb{R})} + \left\| \left\langle \sigma_{(x,t)}, |\mu| \right\rangle \right\|_{L^1(\mathbb{R})} \right) \to 0$$

as $n \to \infty$. Since $\nu_{(x,t)}$ and $\sigma_{(x,t)}$ are probability measures, for almost all t, the set

$$\left\{ x \mid \langle \nu_{(x,t)}, 1 \rangle \ne 1 \text{ and } \langle \sigma_{(x,t)}, 1 \rangle \ne 1 \right\}$$

has zero Lebesgue measure. Therefore, for almost all t,

$$\begin{aligned} A_n(t) &\le \int_{\mathbb{R}} \left\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - u_0(x)| + |\mu - u_0(x)| \right\rangle dx \\ &= \int_{\mathbb{R}} \left\langle \nu_{(x,t)}, |\lambda - u_0(x)| \right\rangle dx + \int_{\mathbb{R}} \left\langle \sigma_{(x,t)}, |\mu - u_0(x)| \right\rangle dx. \end{aligned}$$

Integrating (3.99) with respect to t_1 from 0 to T, then dividing by T and sending T to 0, using (3.91), and finally sending $n \to \infty$, we find that

$$\iint_{\mathbb{R}\times\mathbb{R}} |\lambda - \mu| \, d\nu_{(x,t)} \, d\sigma_{(x,t)} = 0, \quad \text{for } (x,t) \notin E, \tag{3.100}$$

where the Lebesgue measure of the (exceptional) set E is zero. Suppose now that for $(x,t) \notin E$ there is a $\bar{\lambda}$ in the support of $\nu_{(x,t)}$ and a $\bar{\mu}$ in the support of $\sigma_{(x,t)}$ and $\bar{\lambda} \ne \bar{\mu}$. Then we can find positive functions g and h such that

$$0 \le g \le 1, \quad 0 \le h \le 1,$$

and

$$\bar{\lambda} \in \mathrm{supp}(g), \quad \bar{\mu} \in \mathrm{supp}(h), \quad \mathrm{supp}(g) \cap \mathrm{supp}(h) = \emptyset.$$

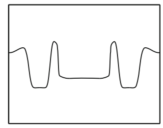
Furthermore,

$$\left\langle \nu_{(x,t)}, g \right\rangle > 0 \quad \text{and} \quad \left\langle \sigma_{(x,t)}, h \right\rangle > 0.$$

Thus

$$\begin{aligned} 0 &< \iint_{\mathbb{R}\times\mathbb{R}} g(\lambda) h(\mu) \, d\nu_{(x,t)} \, d\sigma_{(x,t)} \\ &\le \sup_{\lambda,\mu} \left| \frac{g(\lambda)h(\mu)}{\lambda - \mu} \right| \iint_{\mathbb{R}\times\mathbb{R}} |\lambda - \mu| \, d\nu_{(x,t)} \, d\sigma_{(x,t)} = 0. \end{aligned}$$

This contradiction shows that both $\nu_{(x,t)}$ and $\sigma_{(x,t)}$ are unit point measures with support at a common point. Precisely, we have proved the following theorem:



Theorem 3.29 *Let* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(i) *Suppose that* $\nu_{(x,t)}$ *is a measure-valued entropy solution to the conservation law*

$$u_t + f(u)_x = 0$$

such that $\nu_{(x,t)}$ *satisfies the initial condition* (3.91), *and that* $\langle \nu_{(x,t)}, |\lambda| \rangle$ *is in* $L^\infty([0,T]; L^1(\mathbb{R}))$. *Then there exists a function* $u \in L^\infty([0,T]; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times [0,T])$ *such that*

$$\nu_{(x,t)} = \delta_{u(x,t)}, \quad \textit{for almost all } (x,t).$$

(ii) *Assume that* $\sigma_{(x,t)}$ *is (another) measure-valued entropy solution satisfying the same regularity assumptions as* $\nu_{(x,t)}$. *Then*

$$\nu_{(x,t)} = \sigma_{(x,t)} = \delta_{u(x,t)}, \quad \textit{for almost all } (x,t).$$

In order to avoid checking (3.91) directly, we can use the following lemma.

Lemma 3.30 *Let* $\nu_{(x,t)}$ *be a probability measure, and assume that for all test functions* $\varphi(x)$ *we have*

$$\lim_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int \left\langle \nu_{(x,t)}, \mathrm{Id} \right\rangle \varphi(x)\, dx\, dt = \int u_0(x) \varphi(x)\, dx, \tag{3.101}$$

and that for all nonnegative $\varphi(x)$ *and for at least one strictly convex continuous function* η,

$$\limsup_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int \left\langle \nu_{(x,t)}, \eta \right\rangle \varphi(x)\, dx\, dt \le \int \eta\left(u_0(x)\right) \varphi(x)\, dx. \tag{3.102}$$

Then (3.91) *holds.*

Proof We shall prove

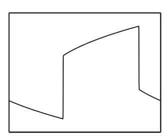
$$\lim_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{-A}^{A} \langle \nu_{(x,t)}, \left(\mathrm{Id} - u_0(x)\right)^+ \rangle\, dx\, dt = 0, \tag{3.103}$$

from which the desired result will follow from (3.101) and the identity

$$|\lambda - u_0(x)| = 2\left(\lambda - u_0(x)\right)^+ - \left(\lambda - u_0(x)\right),$$

where $a^+ = \max\{a, 0\}$ denotes the positive part of a. To get started, we write η'_+ for the right-hand derivative of η. It exists by virtue of the convexity of η; moreover,

$$\eta(\lambda) \ge \eta(y) + \eta'_+(y)(\lambda - y)$$



for all λ. Whenever $\varepsilon > 0$, write

$$\zeta(y,\varepsilon) = \frac{\eta(y+\varepsilon) - \eta(y)}{\varepsilon} - \eta'_+(y).$$

Since η is *strictly* convex, $\zeta(y,\varepsilon) > 0$, and this quantity is an increasing function of ε. In particular, if $\lambda > y + \varepsilon$, then $\zeta(y, \lambda - y) > \zeta(y,\varepsilon)$, or

$$\eta(\lambda) > \eta(y) + \eta'_+(y)(\lambda - y) + \zeta(y,\varepsilon)(\lambda - y).$$

In *every* case, then,

$$\eta(\lambda) > \eta(y) + \eta'_+(y)(\lambda - y) + \zeta(y,\varepsilon)\big((\lambda - y)^+ - \varepsilon\big). \tag{3.104}$$

On the other hand, whenever $y < \lambda < y + \varepsilon$, then $\zeta(y, \lambda - y) > \zeta(y,\varepsilon)$, so

$$\eta(\lambda) < \eta(y) + \eta'_+(y)(\lambda - y) + \varepsilon\zeta(y,\varepsilon) \qquad (y \le \lambda < y + \varepsilon). \tag{3.105}$$

Let us now assume that $\varphi \ge 0$ is such that

$$\varphi(x) \ne 0 \Rightarrow y \le u_0(x) < y + \varepsilon. \tag{3.106}$$

We use (3.104) on the left-hand side and (3.105) on the right-hand side of (3.102), and get

$$\begin{aligned}\limsup_{\tau\to 0+} \frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} \langle \nu_{(x,t)}, \big[\eta(y) + \eta'_+(y)(\mathrm{Id} - y) \\ + \zeta(y,\varepsilon)\big((\mathrm{Id} - y)^+ - \varepsilon\big)\big]\rangle\varphi(x)\,dx\,dt \\ \le \int_{\mathbb{R}} \big[\eta(y) + \eta'_+(y)\big(u(x_0) - y\big) + \varepsilon\zeta(y,\varepsilon)\big]\varphi(x)\,dx.\end{aligned}$$

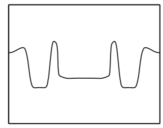
Here, thanks to (3.101) and the fact that $\nu_{(x,t)}$ is a probability measure, all the terms not involving $\zeta(y,\varepsilon)$ cancel, and then we can divide by $\zeta(y,\varepsilon) \ne 0$ to arrive at

$$\limsup_{\tau\to 0+} \frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} \langle \nu_{(x,t)}, (\mathrm{Id} - y)^+\rangle\varphi(x)\,dx\,dt \le 2\varepsilon\int_{\mathbb{R}} \varphi(x)\,dx.$$

Now, remembering (3.106), we see that whenever $\varphi(x) \ne 0$ we have $(\lambda - y)^+ \le (\lambda - u_0(x))^+ + \varepsilon$, so the above implies

$$\limsup_{\tau\to 0+} \frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} \langle \nu_{(x,t)}, \big(\mathrm{Id} - u_0(x)\big)^+\rangle\varphi(x)\,dx\,dt \le 3\varepsilon\int_{\mathbb{R}} \varphi(x)\,dx$$

whenever (3.106) holds.



It remains only to divide up the common support $[-M, M]$ of all the measures $\nu_{(x,t)}$, writing $y_i = -M + i\varepsilon$ for $i = 0, 1, \dots, N-1$, where $\varepsilon = 2M/N$. Let φ_i be the characteristic function of $[-A, A] \cap u_0^{-1}([y_i, y_i + \varepsilon))$, and add together the above inequalities, one for each i, to arrive at

$$\limsup_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{-A}^{A} \langle \nu_{(x,t)}, \left(\mathrm{Id} - u_0(x)\right)^+ \rangle \varphi(x)\, dx\, dt \leq 3\varepsilon \int_{-A}^{A} \varphi(x)\, dx.$$

Since ε can be made arbitrarily small, (3.103) follows, and the proof is complete. □

Remark 3.31 We *cannot* conclude that[3]

$$\lim_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}} \langle \nu_{(x,t)}, |\,\mathrm{Id} - u_0(x)| \rangle \, dx\, dt = 0 \tag{3.107}$$

from the present assumptions. Here is an example to show this.

Let $\nu_{(x,t)} = \mu_{\gamma(x,t)}$, where $\mu_\beta = \frac{1}{2}(\delta_{-\beta} + \delta_\beta)$ and γ is a continuous, nonnegative function with $\gamma(x, 0) = 0$. Let $u_0(x) = 0$ and $\eta(y) = y^2$.

Then (3.101) holds trivially, and (3.102) becomes

$$\limsup_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}} \gamma(x,t)^2 \varphi(x)\, dx\, dt = 0,$$

which is also true due to the stated assumptions on γ.

The desired conclusion (3.107), however, is now

$$\limsup_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}} \gamma(x,t)\, dx\, dt = 0.$$

But the simple choice

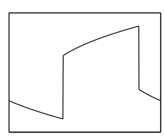
$$\gamma(x,t) = t e^{-(xt)^2}$$

yields

$$\limsup_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}} \gamma(x,t)\, dx\, dt = \sqrt{\pi}.$$

We shall now describe a framework that allows one to prove convergence of a sequence of approximations without proving that the method is TV stable. Unfortunately, the application of this method to concrete examples, while not very

[3] Where the integral over the compact interval $[-A, A]$ in (3.91) has been replaced by an integral over the entire real line.



difficult, involves quite large calculations, and will be omitted here. Readers are encouraged to try their hands at it themselves.

We give one application of these concepts. The setting is as follows. Let u^n be computed from a conservative and consistent scheme, and assume uniform boundedness of u^n. Young's theorem states that there exists a family of probability measures $\nu_{(x,t)}$ such that $g(u^n) \overset{*}{\rightharpoonup} \langle \nu_{(x,t)}, g\rangle$ for Lipschitz continuous functions g. We assume that the CFL condition, $\lambda \sup_u |f'(u)| \le 1$, is satisfied. The next theorem states conditions, strictly weaker than TVD, for which we prove that the limit measure $\nu_{(x,t)}$ is a measure-valued solution of the scalar conservation law.

Theorem 3.32 *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume that the sequence $\{u^n\}$ is the result of a conservative, consistent method, and define $u_{\Delta t}$ as in (3.27). Assume that $u_{\Delta t}$ is uniformly bounded in $L^\infty(\mathbb{R}\times[0,T])$, $T = n\Delta t$. Let $\Delta t_n \to 0$ be a sequence such that $u_{\Delta t_n} \overset{*}{\rightharpoonup} u$, and let $\nu_{(x,t)}$ be the Young measure associated with $u_{\Delta t_n}$, and assume that u_j^n satisfies the estimate*

$$(\Delta x)^\beta \sum_{n=0}^{N} \sum_j \left| u_{j+1}^n - u_j^n \right| \Delta t \le C(T), \tag{3.108}$$

for some $\beta \in [0,1)$ and some constant $C(T)$. Then $\nu_{(x,t)}$ is a measure-valued solution to (3.1).

Furthermore, let (η, q) be a strictly convex entropy pair, and let Q be a numerical entropy flux consistent with q. Write $\eta_j^n = \eta(u_j^n)$ and $Q_{j+1/2}^n = Q(u^n)_{j+1/2}$. Assume that

$$\frac{1}{\Delta t}\left(\eta_j^{n+1} - \eta_j^n\right) + \frac{1}{\Delta x}\left(Q_{j+1/2}^n - Q_{j-1/2}^n\right) \le R_j^n \tag{3.109}$$

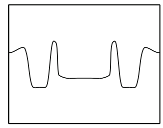
for all n and j, where R_j^n satisfies,

$$\lim_{\Delta t \to 0} \sum_{n=0}^{N} \sum_j \varphi_j^n R_j^n \Delta x \Delta t = 0 \tag{3.110}$$

for all nonnegative $\varphi \in C_0^1$ where $\varphi_j^n = \varphi(j\Delta x, n\Delta t)$. Then $\nu_{(x,t)}$ is a measure-valued solution compatible with (η, q), and the initial data is assumed in the sense of (3.101), (3.102). If (3.109) and (3.110) hold for all entropy pairs (η, q), then $\nu_{(x,t)}$ is a measure-valued entropy solution to (3.1).

Remark 3.33 For $\beta = 0$, (3.108) is the standard TV estimate, while for $\beta > 0$, (3.108) is genuinely weaker than a TV estimate.

Proof We start by proving the first statement in the theorem, assuming (3.108). As before, we obtain (3.28) by rearranging. For simplicity, we now write $F_{j+1/2}^n =$



$F(u^n)_{j+1/2}$, $f_j^n = f(u_j^n)$, and observe that $F_{j+1/2}^n = f_j^n + \left(F_{j+1/2}^n - f_j^n\right)$, getting

$$\iint \left(u_{\Delta t} D_+^t \varphi_j^n + f(u_{\Delta t}) D_+ \varphi_j^n\right) dx\, dt \\ = \sum_{j,n} D_+\varphi_j^n \left(F_{j+1/2}^n - f_j^n\right) \Delta t\, \Delta x. \tag{3.111}$$

Here we use the notation

$$u_{\Delta t} = u_j^n \quad \text{for} \quad (x,t) \in [j\Delta x, (j+1)\Delta x) \times [n\Delta t, (n+1)\Delta t),$$

and

$$D_+^t \varphi_j^n = \frac{1}{\Delta t}\left(\varphi_j^{n+1} - \varphi_j^n\right),$$
$$D_+ \varphi_j^n = \frac{1}{\Delta x}\left(\varphi_{j+1}^n - \varphi_j^n\right).$$

The first term on the left-hand side in (3.111) reads

$$\iint u_{\Delta t} D_+^t \varphi_j^n \, dx\, dt = \iint \left\langle \nu_{(x,t)}, \mathrm{Id}\right\rangle \varphi_t \, dx\, dt + \iint \left(u_{\Delta t} - \left\langle \nu_{(x,t)}, \mathrm{Id}\right\rangle\right) \varphi_t \, dx\, dt \\ + \iint u_{\Delta t} \left(D_+^t \varphi_j^n - \varphi_t\right) dx\, dt. \tag{3.112}$$

The third term on the right-hand side of (3.112) clearly tends to zero as Δt goes to zero. Furthermore, by definition of the Young measure $\nu_{(x,t)}$, the second term tends to zero as well. Thus the left-hand side of (3.112) approaches $\iint \langle \nu_{(x,t)}, \mathrm{Id}\rangle \varphi_t \, dx\, dt$.

One can use a similar argument for the second term on the left-hand side of (3.111) to show that the (whole) left-hand side of (3.111) tends to

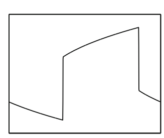
$$\iint \left(\left\langle \nu_{(x,t)}, \mathrm{Id}\right\rangle \varphi_t + \left\langle \nu_{(x,t)}, f\right\rangle \varphi_x\right) dx\, dt \tag{3.113}$$

as $\Delta t \to 0$. We now study the right-hand side of (3.111). Mimicking the proof of the Lax–Wendroff theorem, we have

$$\left|F_{j+1/2}^n - f_j^n\right| \le C \sum_{k=-p}^{q} \left|u_{j+k}^n - u_j^n\right|.$$

Therefore,

$$\left|\sum_{j,n} D_+\varphi_j^n \left(F_{j+1/2}^n - f_j^n\right) \Delta t\, \Delta x\right| \\ \le C \left\|\varphi\right\|_{\mathrm{Lip}} (p+q+1) \sum_{n=0}^{N} \sum_j \left|u_{j+1}^n - u_j^n\right| \Delta t\, \Delta x \\ \le C \left\|\varphi\right\|_{\mathrm{Lip}} (p+q+1) (\Delta x)^{1-\beta}, \tag{3.114}$$



using the assumption (3.108). Thus the right-hand side of (3.114), and hence also of (3.111), tends to zero. Since the left-hand side of (3.111) tends to (3.113), we conclude that $\nu_{(x,t)}$ is a measure-valued solution. Using similar calculations, and (3.110), one shows that $\nu_{(x,t)}$ is also an entropy measure-valued solution.

It remains to show consistency with the initial condition, i.e., (3.101) and (3.102). Let $\varphi(x)$ be a test function, and we use the notation $\varphi(j\,\Delta x) = \varphi_j$. From the definition of u_j^{n+1}, after a summation by parts, we have that

$$\sum_j \varphi_j \left(u_j^{n+1} - u_j^n \right) \Delta x = \Delta t \sum_j F_{j+1/2}^n D_+ \varphi_j \, \Delta x \leq \mathcal{O}\left(1\right) \Delta t,$$

since u_j^n is bounded. Recall that $\varphi = \varphi(x)$, we get

$$\left| \sum_j \varphi_j \left(u_j^n - u_j^0 \right) \Delta x \right| \leq \mathcal{O}\left(1\right) n \Delta t. \tag{3.115}$$

Let $t_1 = n_1 \Delta t$ and $t_2 = n_2 \Delta t$. Then (3.115) yields

$$\left| \frac{1}{\left(n_2 + 1 - n_1\right) \Delta t} \sum_{n=n_1}^{n_2} \sum_j \varphi_j \left(u_j^n - u_j^0 \right) \Delta x \Delta t \right| \leq \mathcal{O}\left(1\right) t_2,$$

which implies that the Young measure $\nu_{(x,t)}$ satisfies

$$\left| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \varphi(x) \left\langle \nu_{(x,t)}, \mathrm{Id} \right\rangle dx\,dt - \int \varphi(x) u_0(x)\,dx \right| \leq \mathcal{O}\left(1\right) t_2. \tag{3.116}$$

We let $t_1 \to 0$ and set $t_2 = \tau$ in (3.116), obtaining

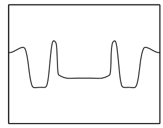
$$\left| \frac{1}{\tau} \int_0^{\tau} \int \varphi(x) \left\langle \nu_{(x,t)}, \mathrm{Id} \right\rangle dx\,dt - \int \varphi(x) u_0(x)\,dx \right| \leq \mathcal{O}\left(1\right) \tau, \tag{3.117}$$

which proves (3.101). Now for (3.102). We have that there exists a strictly convex entropy η for which (3.109) holds. Now let $\varphi(x)$ be a nonnegative test function. Using (3.109), and proceeding as before, we obtain

$$\left| \sum_j \left(\eta_j^n - \eta_j^0 \right) \varphi_j \, \Delta x \right| \leq \mathcal{O}\left(1\right) n \Delta t + \sum_{l=0}^{n} \sum_j R_j^l \varphi_j \, \Delta t \, \Delta x.$$

Using this estimate and the assumption on R_j^l, (3.110), we can use the same arguments as in proving (3.117) to prove (3.102). The proof of the theorem is complete. □

A trivial application of this approach is found by considering monotone schemes. Here we have seen that (3.108) holds for $\beta = 0$, and (3.109) for $R_j^n = 0$. The theorem then gives the convergence of these schemes without using Helly's theorem. However, in this case the application does not give the existence of a solution, since we must have this in order to use DiPerna's theorem. The main usefulness of the method is for schemes in several space dimensions, where TV bounds are more difficult to obtain.



3.6 Notes

The Lax–Friedrichs scheme was introduced by Lax in 1954; see [124]. Godunov discussed what has later become the Godunov scheme in 1959 as a method to study gas dynamics; see [80]. The CFL condition was introduced in the seminal paper [50]; see also [57].

The Lax–Wendroff theorem, Theorem 3.4, was first proved in [128]. Theorem 3.8 was proved by Oleĭnik in her fundamental paper [145]; see also [169]. Several of the key results concerning monotone schemes are due to Crandall and Majda [53], [52]. Theorem 3.10 is due to Harten, Hyman, and Lax; see [84]. Harten's lemma, Lemma 3.12, can be found in [83]. See also [148].

The error analysis is based on the fundamental analysis by Kuznetsov, [119], where one also can find a short discussion of the examples we have analyzed, namely the smoothing method, the method of vanishing viscosity, as well as monotone schemes. Our presentation of the a priori estimates follows the approach due to Cockburn and Gremaud; see [44] and [45], where also applications to numerical methods are given.

The concept of measure-valued solutions is due to DiPerna, and the key results can be found in [62], while Lemma 3.30 is to be found in [61]. Our presentation of the Young measure follows the exposition of Perthame, [150]. For further information regarding the functional-analytic framework, see, e.g., [34] and references therein. The proof of Lemma 3.30 and Remark 3.31 are due to H. Hanche-Olsen. Our presentation of the uniqueness of measure-valued solutions, Theorem 3.29, is taken mainly from Szepessy, [173]. Theorem 3.32 is due to Coquel and LeFloch, [48]; see also [49], where several extensions are discussed. For numerical schemes that satisfy the criteria in Theorem 3.32, see [49] and [65].

3.7 Exercises

3.1 Consider the difference scheme (3.4). Show that if u^0 is given by

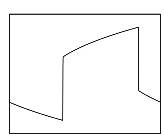
$$u_j^0 = \begin{cases} 0 & \text{for } j < 0, \\ 1 & \text{for } j \geq 0, \end{cases}$$

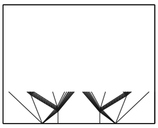
then $u^n = u^0$ for all n, thus indicating the solution $u(x,t) = \chi_{[0,\infty)}$. Determine the weak entropy solution.

3.2 Show that the Lax–Wendroff and the MacCormack methods are of second order.

3.3 The Engquist–Osher (or generalized upwind) method, see [63], is a conservative difference scheme with a numerical flux defined as follows:

$$F_{j+1/2}(u) = f^{\mathrm{EO}}\left(u_j, u_{j+1}\right), \quad \text{where}$$
$$f^{\mathrm{EO}}(u,v) = \int_0^u \max\{f'(s),0\}\,ds + \int_0^v \min\{f'(s),0\}\,ds + f(0).$$





(a) Show that this method is consistent and monotone.
(b) Find the order of the scheme.
(c) Show that the Engquist–Osher flux f^{EO} can be written

$$f^{\mathrm{EO}}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \int_u^v |f'(s)|\, ds \right).$$

(d) If $f(u) = u^2/2$, show that the numerical flux can be written

$$f^{\mathrm{EO}}(u,v) = \frac{1}{2}\left(\max\{u,0\}^2 + \min\{v,0\}^2 \right).$$

Generalize this simple expression to the case that $f''(u) \neq 0$ and $\lim_{|u|\to\infty} |f(u)| = \infty$.

3.4 Why does the method

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right)$$

not give a viable difference scheme?

3.5 In the derivation of the Godunov scheme it is assumed that $\Delta t \max_u |f'(u)| \leq \frac{1}{2}\Delta x$, yet it is stated that the method is well defined if the CFL condition $\Delta t \max_u |f'(u)| \leq \Delta x$ is satisfied; see (3.9). Please explain.

3.6 Show that (3.24) is the model equation for the Lax–Friedrichs scheme.

3.7 Show that the Lax–Friedrichs scheme is monotone also in the case that the flux function is assumed only to be Lipschitz continuous.

3.8 Show that Heun's method is unstable.

3.9 We study a nonconservative method for Burgers's equation. Assume that $u_j^0 \in [0,1]$ for all j. Then the characteristic speed is nonnegative, and we define

$$u_j^{n+1} = u_j^n - \lambda u_j^{n+1}\left(u_j^n - u_{j-1}^n\right), \quad n \geq 0, \tag{3.118}$$

where $\lambda = \Delta t/\Delta x$.

(a) Show that this yields a monotone method, provided that a CFL condition holds.
(b) Show that this method is consistent and determine the truncation error.

3.10 Assume that $f'(u) > 0$ and that $f''(u) \geq 2c > 0$ for all u in the range of u_0. We use the upwind method to generate approximate solutions to

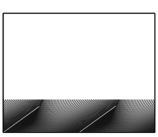
$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x); \tag{3.119}$$

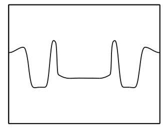
i.e., we set

$$u_j^{n+1} = u_j^n - \lambda\left(f(u_j^n) - f(u_{j-1}^n)\right).$$

Set

$$v_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x}.$$





(a) Show that

$$v_j^{n+1} = \left(1 - \lambda f'(u_{j-1}^n)\right) v_j^n + \lambda f'(u_{j-1}^n) v_{j-1}^n \\ - \frac{\Delta t}{2}\left(f''(\eta_{j-1/2}) \left(v_j^n\right)^2 + f''(\eta_{j-3/2}) \left(v_{j-1}^n\right)^2 \right),$$

where $\eta_{j-1/2}$ is between u_j^n and u_{j-1}^n.

(b) Next, assume inductively that

$$v_j^n \le \frac{1}{(n+2)c\Delta t}, \quad \text{for all } j,$$

and set $\hat{v}^n = \max\{\max_j v_j^n, 0\}$. Then show that

$$\hat{v}^{n+1} \le \hat{v}^n - c\Delta t\, (\hat{v}^n)^2 .$$

(c) Use this to show that

$$\hat{v}^n \le \frac{\hat{v}^0}{1 + \hat{v}^0 c n \Delta t}.$$

(d) Show that this implies that

$$u_i^n - u_j^n \le \Delta x (i - j) \frac{\hat{v}^0}{1 + \hat{v}^0 c n \Delta t},$$

for $i \ge j$.

(e) Let u be the entropy solution of (3.119), and assume that

$$0 \le \max_x u_0'(x) = M < \infty.$$

Show that for almost every x, y, and t we have that

$$\frac{u(x,t) - u(y,t)}{x - y} \le \frac{M}{1 + cMt}. \tag{3.120}$$

This is the Oleĭnik entropy condition for convex scalar conservation laws.

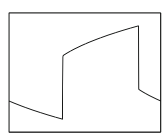
3.11 Assume that f is as in the previous exercise, and that u_0 is periodic with period p.

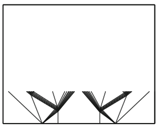
(a) Use uniqueness of the entropy solution to (3.119) to show that the entropy solution $u(x,t)$ is also periodic in x with period p.

(b) Then use the Oleĭnik entropy condition (3.120) to deduce that

$$\sup_x u(x,t) - \inf_x u(x,t) \le \frac{Mp}{1 + cMt}.$$

Thus $\lim_{t\to\infty} u(x,t) = \bar{u}$ for some constant $\bar{u}$.





(c) Use conservation to show that

$$\bar{u} = \frac{1}{p}\int_0^p u_0(x)\,dx.$$

3.12 Let $u_n\colon [0,1) \to [-1,1]$ be defined as

$$u_n(x) = \begin{cases} 1 & x \in [2k/2n, (2k+1)/2n), \\ -1 & x \in [(2k+1)/2n, (2k+2)/2n), \end{cases} \qquad \text{for } k = 0, \dots, n-1,$$

for $n \in \mathbb{N}$. Find the weak limit of u_n as $n \to \infty$, and the associated Young measure.

3.13 We shall consider a scalar conservation law with a "fractal" function as the initial data. Define the set of piecewise linear functions

$$\mathcal{D} = \{\phi(x) = Ax + B \mid x \in [a,b], A, B \in \mathbb{R}\},$$

and the map

$$F(\phi) = \begin{cases} 2D(x-a) + \phi(a) & \text{for } x \in [a, a+L/3], \\ -D(x-a) + \phi(a) & \text{for } x \in [a+L/3, a+2L/3], \\ 2D(x-b) + \phi(b) & \text{for } x \in [a+2L/3, b], \end{cases}$$

for $\phi \in \mathcal{D}$, where $L = b - a$ and $D = (\phi(b) - \phi(a))/L$. For a nonnegative integer k introduce $\chi_{j,k}$ as the characteristic function of the interval $I_{j,k} = [j/3^k, (j+1)/3^k]$, $j = 0, \dots, 3^{k+1}-1$. We define functions $\{v_k\}$ recursively as follows. Let

$$v_0(x) = \begin{cases} 0 & \text{for } x \le 0, \\ x & \text{for } 0 \le x \le 1, \\ 1 & \text{for } 1 \le x \le 2, \\ 3 - x & \text{for } 2 \le x \le 3, \\ 0 & \text{for } 3 \le x. \end{cases}$$

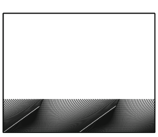
Assume that $v_{j,k}$ is linear on $I_{j,k}$ and let

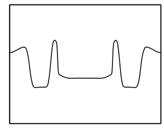
$$v_k = \sum_{j=-3^k}^{3^k-1} v_{j,k}\chi_{j,k}, \tag{3.121}$$

and define the next function v_{k+1} by

$$v_{k+1} = \sum_{j=0}^{3^{k+1}-1} F(v_{j,k})\chi_{j,k} = \sum_{j=0}^{3^{k+2}-1} v_{j,k+1}\chi_{j,k+1}. \tag{3.122}$$

In the left part of Fig. 3.9 we show the effect of the map F, and on the right we show $v_5(x)$ (which is piecewise linear on $3^6 = 729$ segments).





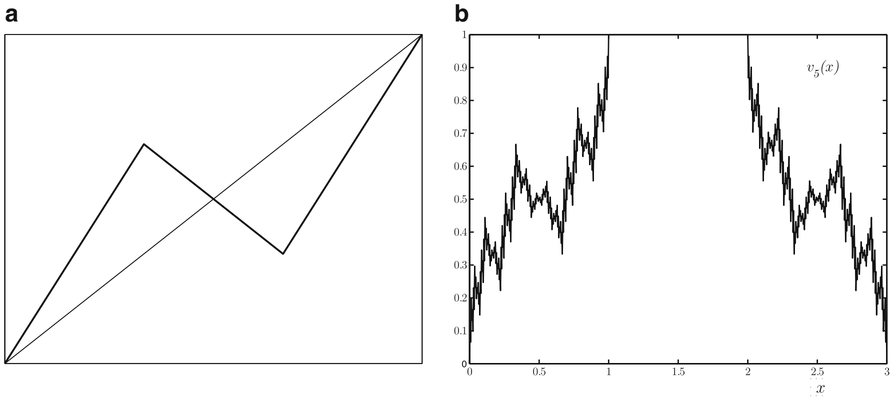


Fig. 3.9 **a** The construction of $F(\phi)$ from ϕ. **b** $v_5(x)$

(a) Show that the sequence $\{v_k\}_{k>1}$ is a Cauchy sequence in the supremum norm, and hence we can define a continuous function v by setting

$$v(x) = \lim_{k\to\infty} v_k(x).$$

(b) Show that v is not of bounded variation, and determine the total variation of v_k.

(c) Show that

$$v(j/3^k) = v_k(j/3^k),$$

for all integers $j = 0, \dots, 3^{k+1}$, $k \in \mathbb{N}$.

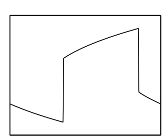
(d) Assume that f is a C^1 function on $[0, 1]$ with $0 \le f'(u) \le 1$. We are interested in solving the conservation law

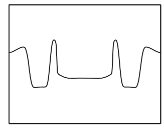
$$u_t + f(u)_x = 0, \quad u_0(x) = v(x).$$

To this end we shall use the upwind scheme defined by (3.10), with $\Delta t = \Delta x = 1/3^k$, and

$$u_j^0 = v(j\Delta x).$$

Show that $u_{\Delta t}(x, t)$ converges to an entropy solution of the conservation law above.





Chapter 4

Multidimensional Scalar Conservation Laws

Just send me the theorems, then I shall find the proofs.[1]
— Chrysippus told Cleanthes, 3rd century BC

Our analysis has so far been confined to scalar conservation laws in one dimension. Clearly, the multidimensional case is considerably more important. Luckily enough, the analysis in one dimension can be carried over to higher dimensions by essentially treating each dimension separately. This technique is called *dimensional splitting*. The final results are very much the natural generalizations one would expect.

The same splitting techniques of dividing complicated differential equations into several simpler parts can in fact be used to handle other problems. These methods are generally called *operator splitting methods* or *fractional steps methods*.

4.1 Dimensional Splitting Methods

We will show in this section how one can analyze scalar multidimensional conservation laws by dimensional splitting, which amounts to solving one space direction at a time. To be more concrete, let us consider the two-dimensional conservation law

$$u_t + f(u)_x + g(u)_y = 0, \qquad u(x,y,0) = u_0(x,y). \tag{4.1}$$

If we let $S_t^{f,x} u_0$ denote the solution of

$$v_t + f(v)_x = 0, \qquad v(x,y,0) = u_0(x,y)$$

(where y is a passive parameter), and similarly let $S_t^{g,y} u_0$ denote the solution of

$$w_t + g(w)_y = 0, \qquad w(x,y,0) = u_0(x,y)$$

(x is a parameter), then the idea of dimensional splitting is to approximate the solution of (4.1) as follows:

$$u(x,y,n\Delta t) \approx \left[S_{\Delta t}^{g,y} \circ S_{\Delta t}^{f,x} \right]^n u_0. \tag{4.2}$$

[1] Lucky guy! Paraphrased from Diogenes Laertius, *Lives of Eminent Philosophers*, c. A.D. 200.

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
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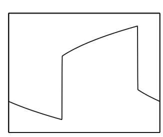
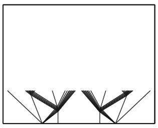




◇ Example 4.1 (A single discontinuity)
We first show how this works on a concrete example. Let

$$f(u) = g(u) = \frac{1}{2}u^2$$

and

$$u_0(x, y) = \begin{cases} u_l & \text{for } x < y, \\ u_r & \text{for } x \geq y, \end{cases}$$

with $u_r > u_l$. The solution in the x-direction for fixed y gives a rarefaction wave, the left and right parts moving with speeds u_l and u_r, respectively. With a quadratic flux, the rarefaction wave is a linear interpolation between the left and right states. Thus

$$u^{1/2} := S^{f,x}_{\Delta t} u_0 = \begin{cases} u_l & \text{for } x < y + u_l \Delta t, \\ (x - y)/\Delta t & \text{for } y + u_l \Delta t < x < y + u_r \Delta t, \\ u_r & \text{for } x > y + u_r \Delta t. \end{cases}$$

The solution in the y-direction for fixed x with initial state $u^{1/2}$ will exhibit a focusing of characteristics. The left state, which now equals u_r, will move with speed given by the derivative of the flux function, in this case u_r, and hence overtake the right state, given by u_l, which moves with smaller speed, namely u_l. The characteristics interact at a time t given by

$$u_r t + x - u_r \Delta t = u_l t + x - u_l \Delta t,$$

or $t = \Delta t$. At that time we are back to the original Riemann problem between states u_l and u_r at the point $x = y$. Thus

$$u^1 := S^{g,y}_{\Delta t} u^{1/2} = u_0.$$

Another application of $S^{f,x}_{\Delta t}$ will of course give

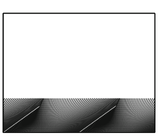
$$u^{3/2} := S^{f,x}_{\Delta t} u^1 = u^{1/2}.$$

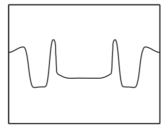
So we have that $u^n = u_0$ for all $n \in \mathbb{N}$. When we introduce coordinates

$$\xi = \frac{1}{\sqrt{2}}(x + y), \qquad \eta = \frac{1}{\sqrt{2}}(x - y),$$

the equation transforms into

$$u_t + \left(\frac{1}{\sqrt{2}} u^2\right)_\xi = 0, \quad u(\xi, \eta, 0) = \begin{cases} u_l & \text{for } \eta \leq 0, \\ u_r & \text{for } \eta > 0. \end{cases}$$





We see that $u(x,y,t) = u_0(x,y)$, and consequently $\lim_{\Delta t\to 0} u^n = u_0$ (where we keep $n\Delta t = t$ fixed). Thus the dimensional splitting procedure produces approximate solutions converging to the right solution in this case. ◇

We will state all results for the general case of arbitrary dimension, while proofs will be carried out in two dimensions only, to keep the notation simple. We first need to define precisely what is meant by a weak entropy solution of the initial value problem

$$u_t + \operatorname{div} f(u) = 0, \quad u|_{t=0} = u_0, \tag{4.3}$$

where $f = (f_1, \dots, f_m)$, and the spatial variables are denoted by $(x_1, \dots, x_m) \in \mathbb{R}^m$. Here we adopt the *Kružkov entropy condition* from Chapt. 2, and say that u is a (weak) Kružkov entropy solution of (4.3) for time $[0,T]$ if u is a bounded function that satisfies[2]

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^m} \Big(|u-k|\, \varphi_t + \operatorname{sign}(u-k) \sum_{j=1}^m \big(f_j(u) - f_j(k) \big)\, \varphi_{x_j} \Big)\, dx_1 \cdots dx_m\, dt \\ + \int_{\mathbb{R}^m} \Big(\varphi|_{t=0}\, |u_0 - k| - (|u-k|\, \varphi)|_{t=T} \Big)\, dx_1 \cdots dx_m \ge 0,\end{aligned} \tag{4.4}$$

for all constants $k \in \mathbb{R}$ and all nonnegative test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0,T])$. It certainly follows as in the one-dimensional case that u is a weak solution, i.e.,

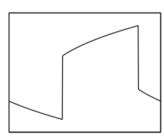
$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^m} \Big(u\varphi_t + f(u)\cdot\nabla\varphi \Big)\, dx_1 \cdots dx_m\, dt \\ + \int_{\mathbb{R}^m} \varphi|_{t=0} u_0\, dx_1 \cdots dx_m = 0,\end{aligned} \tag{4.5}$$

for all test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0,\infty))$.

Our analysis aims at two different goals. We first show that the dimensional splitting indeed produces a sequence of functions that converges to a solution of the multidimensional equation (4.3). Our discussion will here be based on the more or less standard argument using Kolmogorov's compactness theorem. The argument is fairly short. In order to obtain stability in the multidimensional case in the sense of Theorem 2.14, we show that dimensional splitting preserves this stability. Furthermore, we show how one can use front tracking as our solution operator in one dimension in combination with dimensional splitting. Finally, we determine the appropriate convergence rate of this procedure. This analysis strongly uses Kuznetsov's theory from Sect. 3.3, but matters are more complicated and technical than in one dimension.

We shall now show that dimensional splitting produces a sequence that converges to the entropy solution u of (4.3); that is, the limit u should satisfy (4.4).

[2] If we want a solution for all time, we disregard the last term in (4.4) and integrate t over $[0,\infty)$.



As promised, our analysis will be carried out in the two-dimensional case only, i.e., for equation (4.1). Assume that u_0 is a function in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ (consult Definition A.2 for a definition of $BV(\mathbb{R}^2)$; see also (A.11)). Let $t_n = n\Delta t$ and $t_{n+1/2} = \left(n + \frac{1}{2}\right)\Delta t$. Define

$$u^0 = u_0, \qquad u^{n+1/2} = S^{f,x}_{\Delta t} u^n, \qquad u^{n+1} = S^{g,y}_{\Delta t} u^{n+1/2}, \tag{4.6}$$

for $n \in \mathbb{N}_0$. We shall also be needing an approximate solution for $t \neq t_n$. We want the approximation to be an exact solution to a one-dimensional conservation law in each interval $\left[t_j, t_{j+1/2}\right]$, $j = k/2$, and $k \in \mathbb{N}_0$. The way to do this is to make "time go twice as fast" in each such interval; i.e., let $u_{\Delta t}$ be defined by[3]

$$u_{\Delta t}(x,t) = \begin{cases} S^{f,x}_{2(t-t_n)} u^n & \text{for } t_n \le t \le t_{n+1/2}, \\ S^{g,y}_{2(t-t_{n+1/2})} u^{n+1/2} & \text{for } t_{n+1/2} \le t \le t_{n+1}. \end{cases} \tag{4.7}$$

We will use Theorem A.11, that is, we show that the sequence $\{u_{\Delta t}\}$ is compact. Since neither the operator $S^{f,x}$ nor $S^{g,y}$ increases the L^∞ norm, $u_{\Delta t}$ will be uniformly bounded, i.e.,

$$\|u_{\Delta t}\|_{L^\infty(\mathbb{R}^2)} \le \|u_0\|_{L^\infty(\mathbb{R}^2)} \tag{4.8}$$

independent of Δt.

Next we study the total variation. We start by considering

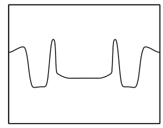
$$\begin{aligned} \int \mathrm{T.V.}_{\cdot y}\left(S^{f,x}_{\Delta t} u^n\right) dx &= \int \lim_{h\to 0} \frac{1}{h} \int \left|u^{n+1/2}(x,y+h) - u^{n+1/2}(x,y)\right| \, dy\, dx \\ &= \lim_{h\to 0} \frac{1}{h} \iint \left|u^{n+1/2}(x,y+h) - u^{n+1/2}(x,y)\right| \, dx\, dy \\ &\le \lim_{h\to 0} \frac{1}{h} \iint \left|u^{n}(x,y+h) - u^{n}(x,y)\right| \, dx\, dy \\ &= \int \lim_{h\to 0} \frac{1}{h} \int \left|u^{n}(x,y+h) - u^{n}(x,y)\right| \, dy\, dx \\ &= \int \mathrm{T.V.}_{\cdot y}\left(u^n\right) dx, \end{aligned} \tag{4.9}$$

where we used Lemma A.1 and the L^1-contractivity; cf. Theorem 2.15 (vi). The interchange of integrals and limits is justified using Lebesgue's dominated convergence theorem.

For the solution constructed from dimensional splitting we have

$$\begin{aligned} \mathrm{T.V.}_{\cdot x,y}\left(u^{n+1/2}\right) &= \int \mathrm{T.V.}_{\cdot x}\left(S^{f,x}_{\Delta t} u^n\right) dy + \int \mathrm{T.V.}_{\cdot y}\left(S^{f,x}_{\Delta t} u^n\right) dx \\ &\le \int \mathrm{T.V.}_{\cdot x}\left(u^n\right) dy + \int \mathrm{T.V.}_{\cdot y}\left(u^n\right) dx \\ &= \mathrm{T.V.}_{\cdot x,y}\left(u^n\right), \end{aligned} \tag{4.10}$$

[3] We will keep the ratio $\lambda = \Delta t/\Delta x$ fixed, and thus we index only with Δt.



using the TVD property of $S^{f,x}$ and (4.9). Similarly,

$$\mathrm{T.V.}_{x,y}\left(u^{n+1}\right) \le \mathrm{T.V.}_{x,y}\left(u^{n+1/2}\right),$$

and thus

$$\mathrm{T.V.}_{x,y}\left(u^{n}\right) \le \mathrm{T.V.}_{x,y}\left(u_0\right)$$

follows by induction. This extends to

$$\mathrm{T.V.}_{x,y}\left(u_{\Delta t}\right) \le \mathrm{T.V.}_{x,y}\left(u_0\right). \tag{4.11}$$

We now want to establish Lipschitz continuity in time of the L^1-norm, i.e.,

$$\|u_{\Delta t}(t) - u_{\Delta t}(s)\|_{L^1(\mathbb{R}^2)} \le C\,|t-s| \tag{4.12}$$

for some constant C. By repeated use of the triangle inequality it suffices to estimate

$$\begin{aligned}
\|u_{\Delta t}(t_{n+1}) - u_{\Delta t}(t_n)\|_{L^1(\mathbb{R}^2)} &\le \left\|u^{n+1} - u^{n+1/2}\right\|_1 + \left\|u^{n+1/2} - u^n\right\|_{L^1(\mathbb{R}^2)} \\
&= \left\|S^{f,x}_{\Delta t} u^n - u^n\right\|_{L^1(\mathbb{R}^2)} \\
&\quad + \left\|S^{g,y}_{\Delta t} u^{n+1/2} - u^{n+1/2}\right\|_{L^1(\mathbb{R}^2)}.
\end{aligned} \tag{4.13}$$

Using Theorem 2.15 (vi), we conclude that the first term in (4.13) is bounded by $\|f\|_{\mathrm{Lip}}\Delta t\ \mathrm{T.V.}_{x,y}\left(u^n\right)$. For the second term. we obtain, using in addition (4.9), the bound $\|g\|_{\mathrm{Lip}}\Delta t\ \mathrm{T.V.}_{x,y}\left(u^n\right)$. This proves

$$\|u_{\Delta t}(t_{n+1}) - u_{\Delta t}(t_n)\|_1 \le \Delta t \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right). \tag{4.14}$$

Using interpolation, we obtain the estimate

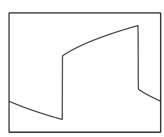
$$\begin{aligned}
\|u_{\Delta t}(t) - u_{\Delta t}(s)\|_1 &\le \|u_{\Delta t}(t) - u_{\Delta t}(t_n)\|_1 \\
&\qquad + \|u_{\Delta t}(t_n) - u_{\Delta t}(t_m)\|_1 + \|u_{\Delta t}(s) - u_{\Delta t}(t_m)\|_1 \\
&\le \left(|t_n - t_m| + 2\Delta t\right) \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right) \\
&\le \left(|t - s| + 4\Delta t\right) \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}\mathrm{T.V.}_{x,y}\left(u_0\right),
\end{aligned} \tag{4.15}$$

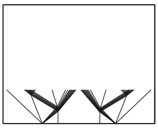
where $t \in [t_n, t_{n+1})$ and $s \in [t_m, t_{m+1})$.

Using Theorem A.11, we conclude the existence of a convergent subsequence, also labeled $\{u_{\Delta t}\}$, and set $u = \lim_{\Delta t \to 0} u_{\Delta t}$. Next we have to prove that the limit u is a weak entropy solution.

Let $\phi = \phi(x, y, t)$ be a nonnegative test function, and define φ by $\varphi(x, y, t) = \phi(x, y, \frac{1}{2}t + t_n)$. By defining $\tau = 2(t - n\Delta t)$, we have that for each y, the function $u_{\Delta t}$ is a weak solution in x on the strip $t \in [t_n, t_{n+1/2}]$ satisfying the inequality

$$\begin{aligned}
&\int_0^{\Delta t}\!\!\int \left(|u_{\Delta t} - k|\,\varphi_\tau + q^f(u_{\Delta t}, k)\varphi_x\right)\,dx\,d\tau \\
&\qquad \ge \int \left|u^{n+1/2} - k\right|\varphi|_{t=\Delta t}\,dx - \int |u^n - k|\,\varphi|_{t=0}\,dx,
\end{aligned} \tag{4.16}$$





for all constants k. Here $q^f(u,k) = \operatorname{sign}(u-k)\,(f(u)-f(k))$. Changing back to the t variable, we find that

$$2\int_{t_n}^{t_{n+1/2}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t + q^f(u_{\Delta t},k)\phi_x\right)dx\,dt \geq \int\left|u^{n+1/2}-k\right|\phi|_{t=t_{n+1/2}}\,dx - \int\left|u^n-k\right|\phi|_{t=t_n}\,dx. \tag{4.17}$$

Similarly,

$$2\int_{t_{n+1/2}}^{t_{n+1}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t + q^g(u_{\Delta t},k)\phi_y\right)dy\,dt \geq \int\left|u^{n+1}-k\right|\phi|_{t=t_{n+1}}\,dy - \int\left|u^{n+1/2}-k\right|\phi|_{t=t_{n+1/2}}\,dy. \tag{4.18}$$

Here q^g is defined similarly to q^f, using g instead of f. Integrating (4.17) over y and (4.18) over x and adding the two results and summing over n, we obtain

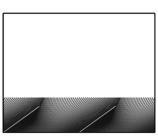
$$\begin{aligned}2\int_0^T\iint\Bigg(\frac{1}{2}\left|u_{\Delta t}-k\right|\phi_t &+ \sum_n \chi_n q^f(u_{\Delta t},k)\phi_x \\ &+ \sum_n \tilde{\chi}_n q^g(u_{\Delta t},k)\phi_y\Bigg)\,dx\,dy\,dt \\ &\geq \iint\left(\left|u_{\Delta t}-k\right|\phi\right)|_{t=T}\,dx\,dy - \iint\left|u_0-k\right|\phi(0)\,dx\,dy,\end{aligned}$$

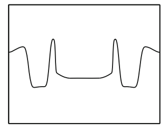
where χ_n and $\tilde{\chi}_n$ denote the characteristic functions of the strips $t_n \leq t \leq t_{n+1/2}$ and $t_{n+1/2} \leq t \leq t_{n+1}$, respectively. As Δt tends to zero, it follows that

$$\sum_n \chi_n \overset{*}{\rightharpoonup} \frac{1}{2}, \quad \sum_n \tilde{\chi}_n \overset{*}{\rightharpoonup} \frac{1}{2}.$$

Specifically, for continuous functions ψ we see that

$$\begin{aligned}\sum_n\int_0^T \chi_n\psi\,dt &= \sum_n\int_{t_n}^{t_{n+1/2}}\psi\,dt \\ &= \sum_n \psi(t_n^*)\frac{\Delta t}{2} \\ &= \frac{1}{2}\sum_n \psi(t_n^*)\Delta t \\ &\to \frac{1}{2}\int_0^T\psi\,dt \text{ as } \Delta t \to 0\end{aligned}$$





(where t_n^* is in $[t_n, t_{n+1/2}]$), by definition of the Riemann integral. The general case follows by approximation.

Letting $\Delta t \to 0$, we thus obtain

$$\int_0^T \iint \left(|u-k|\phi_t + q^f(u,k)\phi_x + q^g(u,k)\phi_y \right) dx\, dy\, dt \\ + \iint |u_0 - k|\, \phi|_{t=0}\, dx\, dy \geq \iint (|u-k|\, \phi)|_{t=T}\, dx\, dy,$$

which proves that $u(x, y, t)$ is a solution to (4.1) satisfying the Kružkov entropy condition.

Next, we want to prove uniqueness of solutions of multidimensional conservation laws. Let u and v be two Kružkov entropy solutions of the conservation law

$$u_t + f(u)_x + g(u)_y = 0 \tag{4.19}$$

with initial data u_0 and v_0, respectively. The argument in Sect. 2.4 leads, with no fundamental changes in the multidimensional case, to the same result (2.65), namely,

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^2)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^2)}, \tag{4.20}$$

thereby proving uniqueness. Using the fact that if every subsequence of a sequence has a further subsequence converging to the same limit, the whole sequence converges to that (unique) limit, we find that the whole sequence $\{u_{\Delta t}\}$ converges, not just a subsequence. We have proved the following result.

Theorem 4.2 *Let f_j be piecewise twice continuously differentiable functions, and furthermore, let u_0 be an integrable and bounded function in $BV(\mathbb{R}^m)$. Define the sequence of functions $\{u^n\}$ by $u^0 = u_0$ and*

$$u^{n+j/m} = S_{\Delta t}^{f_j, x_j} u^{n+(j-1)/m}, \quad j = 1, \dots, m, \quad n \in \mathbb{N}_0.$$

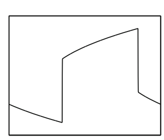
Introduce the function (where $t_r = r\Delta t$ for a rational number r)

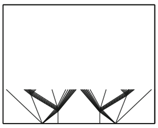
$$u_{\Delta t}(x_1, \dots, x_m, t) = S_{m(t - t_{n+(j-1)/m})}^{f_j, x_j} u^{n+(j-1)/m},$$

for $t \in [t_{n+(j-1)/m}, t_{n+j/m}]$. Fix $T > 0$. Then for every sequence $\{\Delta t\}$ such that $\Delta t \to 0$, for all $t \in [0, T]$ the function $u_{\Delta t}(t)$ converges to the unique weak solution $u(t)$ of (4.3) satisfying the Kružkov entropy condition (4.4). The limit is in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$.

To prove stability of the solution with respect to flux functions, we will show that the one-dimensional stability result (2.80) in Sect. 2.4 remains valid with obvious modifications in several dimensions. Let u and v denote the unique solutions of

$$u_t + f(u)_x + g(u)_y = 0, \quad u|_{t=0} = u_0,$$





and

$$v_t + \tilde{f}(v)_x + \tilde{g}(v)_y = 0, \quad v|_{t=0} = v_0,$$

respectively, that satisfy the Kružkov entropy condition. We want to estimate the L^1-norm of the difference between the two solutions. To this end, we first consider

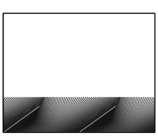
$$\begin{aligned}
\left\|u^{n+1/2} - v^{n+1/2}\right\|_{L^1(\mathbb{R}^2)} &= \iint \left|u^{n+1/2} - v^{n+1/2}\right| \, dx\, dy \\
&\le \int \Big(\int |u^n - v^n| \, dx \\
&\qquad + \Delta t \min\{\mathrm{T.V.}_x\left(u^n\right), \mathrm{T.V.}_x\left(v^n\right)\} \|f - \tilde{f}\|_{\mathrm{Lip}} \Big)\, dy \\
&= \|u^n - v^n\|_{L^1(\mathbb{R}^2)} \\
&\quad + \Delta t \|f - \tilde{f}\|_{\mathrm{Lip}} \int \min\{\mathrm{T.V.}_x\left(u^n\right), \mathrm{T.V.}_x\left(v^n\right)\}\, dy.
\end{aligned}$$

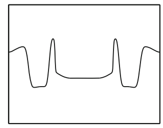
Next we employ the trivial, but useful, inequality

$$a \wedge b + c \wedge d \le (a + c) \wedge (b + d), \quad a, b, c, d \in \mathbb{R}.$$

Thus

$$\begin{aligned}
&\left\|u^{n+1} - v^{n+1}\right\|_{L^1(\mathbb{R}^2)} = \iint \left|u^{n+1} - v^{n+1}\right| \, dx\, dy \\
&\le \int \Big(\int \left|u^{n+1/2} - v^{n+1/2}\right| \, dy \\
&\qquad + \Delta t \min\left\{\mathrm{T.V.}_y\left(u^{n+1/2}\right), \mathrm{T.V.}_y\left(v^{n+1/2}\right)\right\} \|g - \tilde{g}\|_{\mathrm{Lip}} \Big)\, dx \\
&\le \left\|u^{n+1/2} - v^{n+1/2}\right\|_{L^1(\mathbb{R}^2)} \\
&\quad + \Delta t \|g - \tilde{g}\|_{\mathrm{Lip}} \int \min\left\{\mathrm{T.V.}_y\left(u^{n+1/2}\right), \mathrm{T.V.}_y\left(v^{n+1/2}\right)\right\} \, dx \\
&\le \|u^n - v^n\|_{L^1(\mathbb{R}^2)} + \Delta t \max\left\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\right\} \\
&\qquad \times \bigg(\min\left\{ \int \mathrm{T.V.}_x\left(u^n\right)\, dy, \int \mathrm{T.V.}_x\left(v^n\right)\, dy \right\} \\
&\qquad\qquad + \min\left\{ \int \mathrm{T.V.}_y\left(u^n\right)\, dx, \int \mathrm{T.V.}_y\left(v^n\right)\, dx \right\} \bigg) \\
&\le \|u^n - v^n\|_{L^1(\mathbb{R}^2)} \\
&\quad + \Delta t \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \\
&\qquad \times \min \left\{ \begin{matrix} \int \mathrm{T.V.}_x\left(u^n\right)\, dy + \int \mathrm{T.V.}_y\left(u^n\right)\, dx, \\ \int \mathrm{T.V.}_x\left(v^n\right)\, dy + \int \mathrm{T.V.}_y\left(v^n\right)\, dx \end{matrix} \right\} \\
&= \|u^n - v^n\|_{L^1(\mathbb{R}^2)} \\
&\quad + \Delta t \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \min\left\{\mathrm{T.V.}\left(u^n\right), \mathrm{T.V.}\left(v^n\right)\right\},
\end{aligned}$$





which implies

$$\begin{aligned}\|u^n - v^n\|_{L^1(\mathbb{R}^2)} &\le \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} \\ &\quad + n\Delta t \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \min\{\mathrm{T.V.}\,(u_0)\,, \mathrm{T.V.}\,(v_0)\}. \end{aligned} \tag{4.21}$$

Consider next $t \in [t_n, t_{n+1/2})$. Then the continuous interpolants defined by (4.7) satisfy

$$\begin{aligned}\|u_{\Delta t}(t) - v_{\Delta t}(t)\|_{L^1(\mathbb{R}^2)} &= \left\| S^{f,x}_{2(t-t_n)} u^n - S^{\tilde{f},x}_{2(t-t_n)} v^n \right\|_{L^1(\mathbb{R}^2)} \\ &\le \int \Big[\int |u^n - v^n| \, dx \\ &\qquad + 2(t - t_n) \min\{\mathrm{T.V.}_{\cdot x}\,(u^n)\,, \mathrm{T.V.}_{\cdot x}\,(v^n)\} \|f - \tilde{f}\|_{\mathrm{Lip}} \Big] \, dy \\ &= \|u^n - v^n\|_{L^1(\mathbb{R}^2)} \\ &\quad + 2(t - t_n)\|f - \tilde{f}\|_{\mathrm{Lip}} \int \min\{\mathrm{T.V.}_{\cdot x}\,(u^n)\,, \mathrm{T.V.}_{\cdot x}\,(v^n)\} \, dy \\ &\le \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} \\ &\quad + t_n \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \min\{\mathrm{T.V.}\,(u_0)\,, \mathrm{T.V.}\,(v_0)\} \\ &\quad + 2(t - t_n) \min\{\mathrm{T.V.}\,(u_0)\,, \mathrm{T.V.}\,(v_0)\} \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\} \\ &\le \|u_0 - v_0\|_{L^1(\mathbb{R}^2)} \\ &\quad + (t + \Delta t) \min\{\mathrm{T.V.}\,(u_0)\,, \mathrm{T.V.}\,(v_0)\} \max\{\|f - \tilde{f}\|_{\mathrm{Lip}}, \|g - \tilde{g}\|_{\mathrm{Lip}}\}. \end{aligned} \tag{4.22}$$

Observe that the above argument also holds mutatis mutandis in the general case of a scalar conservation law in any dimension. We summarize our results in the following theorem.

Theorem 4.3 *Let u_0 be in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$, and let f_j be piecewise twice continuously differentiable functions for $j = 1, \ldots, m$, and set $f = (f_1, \ldots, f_m)$. Then there exists a unique solution $u = u(x_1, \ldots, x_m, t)$ of the initial value problem*

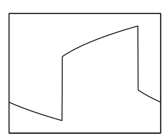
$$u_t + \operatorname{div} f(u) = 0, \quad u|_{t=0} = u_0, \tag{4.23}$$

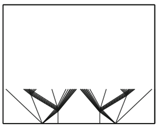
that satisfies the Kružkov entropy condition (4.4). The solution satisfies

$$\begin{aligned} \|u(t)\|_{L^\infty(\mathbb{R}^m)} &\le \|u_0\|_{L^\infty(\mathbb{R}^m)}\,, \\ \mathrm{T.V.}\,(u(t)) &\le \mathrm{T.V.}\,(u_0)\,, \\ \|u(t) - u(s)\|_{L^1(\mathbb{R}^m)} &\le |t - s| \max_j \{\, \|f_j\|_{\mathrm{Lip}}\,\} \mathrm{T.V.}\,(u_0)\,. \end{aligned} \tag{4.24}$$

Furthermore, if v_0 and g share the same properties as u_0 and f, respectively, then the unique weak Kružkov entropy solution of

$$v_t + \operatorname{div} g(v) = 0, \quad v|_{t=0} = v_0, \tag{4.25}$$





satisfies

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^m)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^m)} + t \min\{\text{T.V.}\,(u_0)\,, \text{T.V.}\,(v_0)\} \max_j\{\,\|f_j - g_j\|_{\text{Lip}}\,\}. \tag{4.26}$$

If $u_0 \le v_0$ *and* $f = g$, *then also* $u \le v$ *on all of* $\mathbb{R}^m \times [0, \infty)$.

Proof The proof of the Lipschitz continuity in time follows from (4.15). The monotonicity statement at the end follows using the L^1-contractivity (the special case of (4.26) with $f = g$) as in the one-dimensional case by employing the Crandall–Tartar lemma. □

(See also Exercise 4.1.)

4.2 Dimensional Splitting and Front Tracking

It doesn't matter if the cat is black or white. As long as it catches rats, it's a good cat.
— Deng Xiaoping (1904–1997)

In this section we will study the case in which we use front tracking to solve the one-dimensional conservation laws. More precisely, we replace the flux functions f and g (in the two-dimensional case) by piecewise linear continuous interpolations f_δ and g_δ, with the interpolation points spaced a distance δ apart. The aim is to determine the convergence rate toward the solution of the full two-dimensional conservation law as $\delta \to 0$ and $\Delta t \to 0$.

With the front-tracking approximation, the one-dimensional solutions will be piecewise constant if the initial condition is piecewise constant. In order to prevent the number of discontinuities from growing without bound, we will project the one-dimensional solution $S^{f_\delta,x}u$ onto a fixed grid in the (x, y)-plane before applying the operator $S^{g_\delta,y}$.

To be more concrete, let the grid spacing in the x- and y- directions be given by Δx and Δy, respectively, and let I_{ij} denote the grid cell

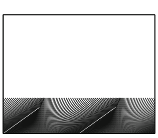
$$I_{ij} = [x_i, x_{i+1}) \times [y_j, y_{j+1}).$$

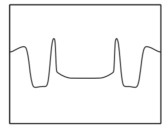
The projection operator π is defined by

$$\pi u(x, y) = \frac{1}{\Delta x \Delta y} \iint\limits_{I_{ij}} u\, dx\, dy \text{ for } (x, y) \in I_{ij}.$$

Let the approximate solution at the discrete times t_l be defined as

$$u^{n+1/2} = \pi \circ S^{f_\delta,x}_{\Delta t} u^n \text{ and } u^{n+1} = \pi \circ S^{g_\delta,y}_{\Delta t} u^{n+1/2},$$





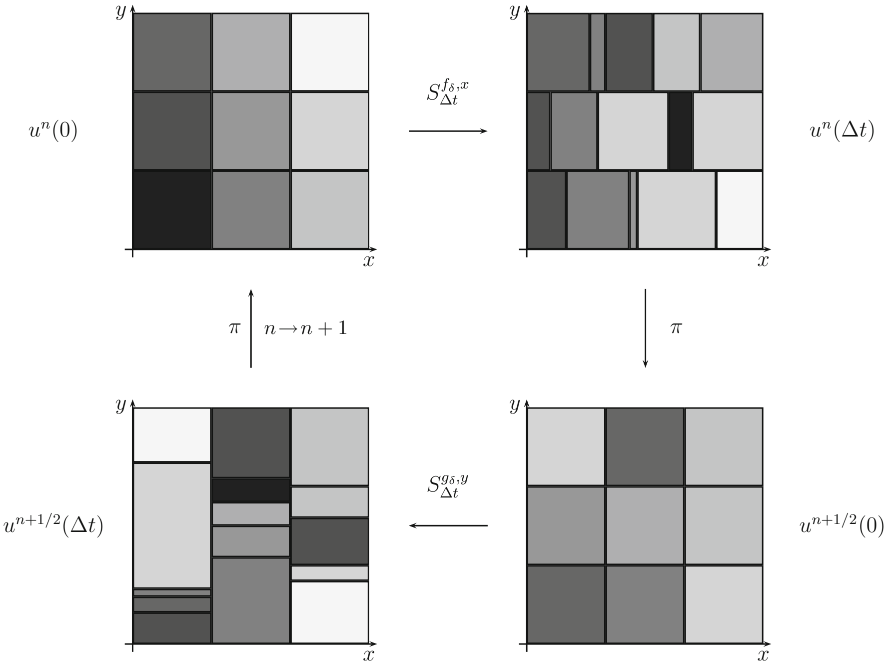


Fig. 4.1 Front tracking and dimensional splitting on a 3×3 grid

for $n = 0, 1, 2, \dots$, with $u^0 = \pi u_0$. We collect the discretization parameters in $\eta = (\delta, \Delta x, \Delta y, \Delta t)$. In analogy to (4.7), we define u_η as

$$u_\eta(t) = \begin{cases} S^{f_\delta,x}_{2(t-t_n)} u^n & \text{for } t_n \le t < t_{n+1/2}, \\ u^{n+1/2} & \text{for } t = t_{n+1/2}, \\ S^{g_\delta,y}_{2(t-t_{n+1/2})} u^{n+1/2} & \text{for } t_{n+1/2} \le t < t_{n+1}, \\ u^{n+1} & \text{for } t = t_{n+1}. \end{cases} \tag{4.27}$$

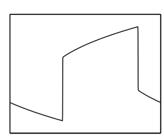
In Fig. 4.1 we illustrate how this works. Starting in the upper left corner, the operator $S^{f_\delta,x}_{\Delta t}$ takes us to the upper right corner; then we apply π and move to the lower right corner. Next, $S^{g_\delta,y}_{\Delta t}$ takes us to the lower left corner, and finally π takes us back to the upper left corner, this time with n incremented by 1.

To prove that u_η converges to the unique solution u as $\eta \to 0$, we essentially mimic the approach we just used to prove Theorem 4.2. First of all we observe that

$$\left\| u_\eta(t) \right\|_{L^\infty(\mathbb{R}^2)} \le \left\| u^0 \right\|_{L^\infty(\mathbb{R}^2)}, \tag{4.28}$$

since $S^{f_\delta,x}$, $S^{g_\delta,y}$, and π all obey a maximum principle. On each rectangle I_{ij} the function u_η is constant for $t = \Delta t$. In a desperate attempt to simplify the notation, we write

$$u^n_{ij} = u_\eta(x, y, n\Delta t) \text{ for } (x, y) \in I_{ij}.$$



Next we go carefully through one full time step in this construction, starting with u^n_{ij}. At each step we define a shorthand notation that we will use in the estimates. When we consider u^n_{ij} as a function of x only, we write

$$u^n_j(0) = u^n_{ij} = u_\eta(\,\cdot\,, j\Delta y, n\Delta t).$$

(The argument "0" on the left-hand side indicates the start of the time variable before we advance time an interval Δt using $S^{f_\delta,x}_{\Delta t}$.) Advancing the solution in time by Δt by applying front tracking in the x-variable produces

$$u^n_j(\Delta t) = \left(S^{f_\delta,x}_{\Delta t} u^n_j\right)(x).$$

(The x-dependence is suppressed in the notation on the left-hand side.) We now apply the projection π, which yields

$$u^{n+1/2}_{ij} = \pi u^n_j(\Delta t).$$

After this sweep in the x-variable, it is time to do the y-direction. Considering $u^{n+1/2}_{ij}$ as a function of y, we write

$$u^{n+1/2}_i(0) = u^{n+1/2}_{ij} = u_\eta\Big(i\Delta x, \cdot, \Big(n + \frac{1}{2}\Big)\Delta t\Big),$$

to which we apply the front-tracking solution operator in the y-direction

$$u^{n+1/2}_i(\Delta t) = \left(S^{g_\delta,y}_{\Delta t} u^{n+1/2}_i\right)(y).$$

(The y-dependence is suppressed in the notation on the left-hand side.) One full time step is completed by a final projection

$$u^{n+1}_{ij} = \pi u^{n+1/2}_i(\Delta t).$$

Using this notation, we first want to prove that the total variation is bounded in the sense that

$$\text{T.V.}\,(u^n) \le \text{T.V.}\,(u_0)\,. \tag{4.29}$$

We will show that

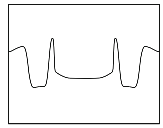
$$\text{T.V.}\,\big(u^{n+1/2}\big) \le \text{T.V.}\,(u^n)\,; \tag{4.30}$$

an analogous argument gives $\text{T.V.}\,\big(u^{n+1}\big) \le \text{T.V.}\,\big(u^{n+1/2}\big)$, from which we conclude that

$$\text{T.V.}\,\big(u^{n+1}\big) \le \text{T.V.}\,(u^n)\,,$$

and (4.29) follows by induction. By definition,

$$\text{T.V.}\,\big(u^{n+1/2}\big) = \sum_{i,j}\left(\left|u^{n+1/2}_{i+1,j} - u^{n+1/2}_{i,j}\right|\Delta y + \left|u^{n+1/2}_{i,j+1} - u^{n+1/2}_{i,j}\right|\Delta x\right), \tag{4.31}$$



while

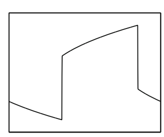
$$\mathrm{T.V.}\,(u^n) = \sum_{i,j} \left(\left| u^n_{i+1,j} - u^n_{i,j} \right| \Delta y + \left| u^n_{i,j+1} - u^n_{i,j} \right| \Delta x \right). \tag{4.32}$$

We first consider

$$\begin{aligned} \sum_i \left| u^{n+1/2}_{i+1,j} - u^{n+1/2}_{i,j} \right| &= \mathrm{T.V.}_{\cdot x} \left(\pi u^n_j(\Delta t) \right) \\ &\le \mathrm{T.V.}_{\cdot x} \left(u^n_j(\Delta t) \right) \le \mathrm{T.V.}_{\cdot x} \left(u^n_j(0) \right) \\ &= \sum_i \left| u^n_{i+1,j} - u^n_{i,j} \right|, \end{aligned} \tag{4.33}$$

where we first used that $\mathrm{T.V.}_{\cdot x}\,(\pi\phi) \le \mathrm{T.V.}_{\cdot x}\,(\phi)$ for step functions ϕ. This follows from the following argument: Let ϕ_c be a continuous function equal to ϕ except close to each jump, where we use a linear interpolation. Then $\mathrm{T.V.}_{\cdot x}\,(\phi) = \mathrm{T.V.}_{\cdot x}\,(\phi_c) \ge \mathrm{T.V.}_{\cdot x}\,(\pi\phi)$, since $\pi\phi$ is just a particular partition of ϕ_c; cf. (A.1). Subsequently we used that $\mathrm{T.V.}\,(v) \le \mathrm{T.V.}\,(v_0)$ for solutions v of one-dimensional conservation laws with initial data v_0. For the second term in the definition of $\mathrm{T.V.}\left(u^{n+1/2}\right)$ we obtain (cf. (4.10))

$$\begin{aligned} \sum_{i,j} \left| u^{n+1/2}_{i,j+1} - u^{n+1/2}_{i,j} \right| \Delta x \Delta y &= \sum_{i,j} \int_{I_{ij}} \left| u^{n+1/2}_{i,j+1} - u^{n+1/2}_{i,j} \right| \, dx\, dy \\ &= \sum_{i,j} \int_{I_{ij}} \left| \pi \left(u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right) \right| \, dx\, dy \\ &\le \sum_{i,j} \int_{I_{ij}} \pi \left(\left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \right) \, dx\, dy \\ &= \sum_{i,j} \int_{I_{ij}} \left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \, dx\, dy \\ &= \sum_{i,j} \Delta y \int_{i\Delta x}^{(i+1)\Delta x} \left| u^n_{j+1}(\Delta t) - u^n_j(\Delta t) \right| \, dx \\ &= \sum_{j} \Delta y \int_{\mathbb{R}} \left| u^n_{j+1}(x, \Delta t) - u^n_j(x, \Delta t) \right| \, dx \\ &\le \sum_{j} \Delta y \int_{\mathbb{R}} \left| u^n_{j+1}(x, 0) - u^n_j(x, 0) \right| \, dx \\ &= \sum_{i,j} \left| u^n_{i,j+1} - u^n_{i,j} \right| \Delta x \Delta y. \end{aligned} \tag{4.34}$$



The first inequality follows from $|\pi\phi| \leq \pi\,|\phi|$; thereafter, we use $\int_{I_{ij}} \pi\phi = \int_{I_{ij}} \phi$, and finally we use the L^1-contractivity, $\|v - w\|_{L^1(\mathbb{R})} \leq \|v_0 - w_0\|_{L^1(\mathbb{R})}$, of solutions of one-dimensional conservation laws. Multiplying (4.33) by Δy, summing over j, dividing (4.34) by Δx, and finally adding the results gives (4.30).

Finally, we want to show the analogue of Lipschitz continuity in time of the spatial L^1-norm as expressed in (4.12). We want to prove the following result:

$$
\begin{aligned}
\left\|u_\eta(t_m) - u_\eta(t_n)\right\|_{L^1(\mathbb{R}^2)} &= \sum_{i,j} \left|u_{ij}^m - u_{ij}^n\right| \Delta x \Delta y \\
&\leq \Big(\max\{\,\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\Delta t + 2(\Delta x + \Delta y)\Big) \\
&\quad \times \mathrm{T.V.}\left(u^0\right)|m - n|\,.
\end{aligned} \tag{4.35}
$$

To prove (4.35), it suffices to show that

$$
\sum_{i,j} \left|u_{ij}^{n+1} - u_{ij}^n\right| \Delta x \Delta y \leq \left(\max\{\,\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\Delta t + 2(\Delta x + \Delta y)\right)\mathrm{T.V.}\left(u^0\right). \tag{4.36}
$$

We start by writing

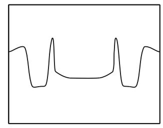
$$
\begin{aligned}
\left|u_{ij}^{n+1} - u_{ij}^n\right| &\leq \left|u_{ij}^{n+1} - u_i^{n+1/2}(\Delta t)\right| + \left|u_{ij}^{n+1/2} - u_j^n(\Delta t)\right| \\
&\qquad + \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right| + \left|u_j^n(\Delta t) - u_j^n(0)\right| \\
&= \left|\pi u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(\Delta t)\right| + \left|\pi u_j^n(\Delta t) - u_j^n(\Delta t)\right| \\
&\qquad + \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right| + \left|u_j^n(\Delta t) - u_j^n(0)\right|.
\end{aligned}
$$

Integrating this inequality over $\mathbb{R}^2$ gives

$$
\begin{aligned}
\sum_{i,j} \left|u_{ij}^{n+1} - u_{ij}^n\right| \Delta x \Delta y \leq &\iint \left|\pi u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(\Delta t)\right| \,dx\,dy \\
&+ \iint \left|\pi u_j^n(\Delta t) - u_j^n(\Delta t)\right| \,dx\,dy \\
&+ \iint \left|u_i^{n+1/2}(\Delta t) - u_i^{n+1/2}(0)\right| \,dx\,dy \\
&+ \iint \left|u_j^n(\Delta t) - u_j^n(0)\right| \,dx\,dy.
\end{aligned} \tag{4.37}
$$

We see that two terms involve the projection operator π. For these terms we prove the estimate

$$
\iint |\pi\psi - \psi| \,dx\,dy \leq (\Delta x + \Delta y)\,\mathrm{T.V.}(\psi)\,. \tag{4.38}
$$



We will prove (4.38) in the one-dimensional case only (See Exercise 4.3). Consider (where $I_i = [x_i, x_{i+1})$)

$$
\begin{aligned}
\int |\pi\psi - \psi|\, dx &= \sum_i \int_{I_i} |\pi\psi(x) - \psi(x)|\, dx \\
&= \sum_i \int_{I_i} \left| \frac{1}{\Delta x} \int_{I_i} \psi(y)\, dy - \psi(x) \right| dx \\
&= \frac{1}{\Delta x} \sum_i \int_{I_i} \left| \int_{I_i} (\psi(y) - \psi(x))\, dy \right| dx \\
&\le \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{I_i} |\psi(y) - \psi(x)|\, dy\, dx \\
&= \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{-x+I_i} |\psi(x+\xi) - \psi(x)|\, d\xi\, dx \\
&\le \frac{1}{\Delta x} \sum_i \int_{I_i} \int_{-\Delta x}^{\Delta x} |\psi(x+\xi) - \psi(x)|\, d\xi\, dx \\
&= \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} \int_{\mathbb{R}} |\psi(x+\xi) - \psi(x)|\, dx\, d\xi \\
&\le \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} |\xi| \mathrm{T.V.}(\psi)\, d\xi \\
&= \Delta x\, \mathrm{T.V.}(\psi)\,.
\end{aligned} \tag{4.39}
$$

For the two remaining terms in (4.37) we obtain, using the Lipschitz continuity in time in the L^1 norm in the x-variable (see Theorem 2.15), that

$$
\begin{aligned}
\iint \left| u_j^n(\Delta t) - u_j^n(0) \right| dx\, dy &\le \Delta t\, \|f_\delta\|_{\mathrm{Lip}} \int \mathrm{T.V.}_{\cdot x}\left(u_j^n(0)\right) dy \\
&\le \Delta t\, \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}(u^n)\,.
\end{aligned} \tag{4.40}
$$

Combining this result with (4.29), (4.38), we conclude that (4.36), and hence also (4.35), holds.

So far we have obtained the following estimates:

(i) Uniform boundedness,

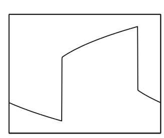
$$
\left\| u_\eta(t) \right\|_{L^\infty(\mathbb{R}^2)} \le \left\| u^0 \right\|_{L^\infty(\mathbb{R}^2)}.
$$

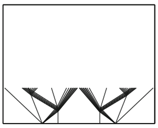
(ii) Uniform bound on the total variation,

$$
\mathrm{T.V.}(u^n) \le \mathrm{T.V.}(u_0)\,.
$$

(iii) Lipschitz continuity in time,

$$
\begin{aligned}
\left\| u_\eta(t_m) - u_\eta(t_n) \right\|_{L^1(\mathbb{R}^2)} \le &\left(\max\{ \|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}} \} + 2\frac{\Delta x + \Delta y}{\Delta t} \right) \\
&\times \mathrm{T.V.}\left(u^0\right) |t_m - t_n|\,.
\end{aligned} \tag{4.41}
$$





From Theorem A.11 we conclude that the sequence $\{u_\eta\}$ has a convergent subsequence as $\eta \to 0$, provided that the ratio $\max\{\Delta x, \Delta y\}/\Delta t$ remains bounded. We let u denote its limit. Furthermore, this sequence converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^2))$ for every positive T.

It remains to prove that the limit is indeed an entropy solution of the full two-dimensional conservation law. We first use that $u_j^n(x,t)$ (suppressing the y-dependence) is a solution of the one-dimensional conservation law in the time interval $[t_n, t_{n+1/2}]$. Hence we know that

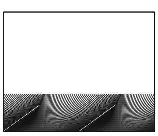
$$
\begin{aligned}
\int_{\mathbb{R}} \int_{t_n}^{t_{n+1/2}} \Big(\frac{1}{2} \left| u_j^n(x,t) - k \right| \phi_t + q^{f_\delta}(u_j^n(x,t), k)\phi_x \Big) \, dt \, dx& \\
- \frac{1}{2} \int_{\mathbb{R}} \left| u_j^n(x, t_{n+1/2}-) - k \right| \phi(x, t_{n+1/2}) \, dx& \\
+ \frac{1}{2} \int_{\mathbb{R}} \left| u_j^n(x, t_n+) - k \right| \phi(x, t_n) \, dx \geq 0&.
\end{aligned}
$$

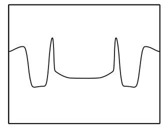
Similarly, we obtain for the y-direction

$$
\begin{aligned}
\int_{\mathbb{R}} \int_{t_{n+1/2}}^{t_{n+1}} \Big(\frac{1}{2} \left| u_i^{n+1/2}(y,t) - k \right| \phi_t + q^{g_\delta}(u_i^{n+1/2}(y,t), k)\phi_y \Big) \, dt \, dy& \\
- \frac{1}{2} \int_{\mathbb{R}} \left| u_i^{n+1/2}(y, t_{n+1}-) - k \right| \phi(y, t_{n+1}) \, dy& \\
+ \frac{1}{2} \int_{\mathbb{R}} \left| u_i^{n+1/2}(y, t_{n+1/2}+) - k \right| \phi(y, t_{n+1/2}) \, dy \geq 0&.
\end{aligned}
$$

Integrating the first inequality over y and the second over x and adding the results as well as adding over n gives, where $T = N\Delta t$,

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} \int_0^T \Big(\frac{1}{2} \left| u_\eta - k \right| \phi_t + \sum_n \chi_n q^{f_\delta}(u_\eta, k)\phi_x + \sum_n \tilde{\chi}_n q^{g_\delta}(u_\eta, k)\phi_y \Big) \, dx \, dy \, dt \\
&\qquad - \frac{1}{2} \Big(\iint_{\mathbb{R}^2} \left| u_\eta(x,y,T) - k \right| \phi(x,y,T) \, dx \, dy \\
&\qquad\qquad - \iint_{\mathbb{R}^2} \left| u_\eta(x,y,0) - k \right| \phi(x,y,0) \, dx \, dy \Big) \\
&\geq -\frac{1}{2} \sum_{n=1}^{2N-1} \iint_{\mathbb{R}^2} \Big(\left| u_\eta(x,y,t_{n/2}+) - k \right| - \left| u_\eta(x,y,t_{n/2}-) - k \right| \Big) \phi(x,y,t_{n/2}) \, dx \, dy \\
&=: -\frac{1}{2} \sum_{n=1}^{2N-1} I_n,
\end{aligned}
$$





and as before, χ_n and $\tilde{\chi}_n$ denote the characteristic functions on $\{(x,y,t) \mid t \in [t_n, t_{n+1/2}]\}$ and $\{(x,y,t) \mid t \in [t_{n+1/2}, t_{n+1}]\}$, respectively. Observe that we have obtained the right-hand side by using a projection at each time step. As $n \to \infty$ and $\Delta t \to 0$ while keeping T fixed, we have that $\sum_n \chi_n \overset{*}{\rightharpoonup} \frac{1}{2}$. To estimate the right-hand side we first observe that

$$u_\eta(x,y,t_{n/2}+) - k = \pi\big(u_\eta(x,y,t_{n/2}-) - k\big),$$

and since the absolute value function is convex, Jensen's inequality implies that

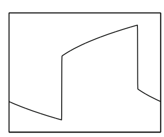
$$\big|u_\eta(x,y,t_{n/2}+) - k\big| - \big|u_\eta(x,y,t_{n/2}-) - k\big| \le 0. \tag{4.42}$$

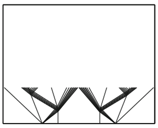
Thus we obtain

$$\begin{aligned}
I_n &= -\iint_{\mathbb{R}^2} \Big(\big|u_\eta(x,y,t_{n/2}+) - k\big| - \big|u_\eta(x,y,t_{n/2}-) - k\big|\Big)\phi(x,y,t_{n/2})\,dx\,dy \\
&= -\sum_{i,j} \iint_{I_{i,j}} \Big(\big|u_\eta(x,y,t_{n/2}+) - k\big| - \big|u_\eta(x,y,t_{n/2}-) - k\big|\Big)\phi(x_i,y_j,t_{n/2})\,dx\,dy \\
&\quad -\sum_{i,j} \iint_{I_{i,j}} \Big(\big|u_\eta(x,y,t_{n/2}+) - k\big| - \big|u_\eta(x,y,t_{n/2}-) - k\big|\Big) \\
&\qquad\qquad \times \big(\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\big)\,dx\,dy \\
&\ge -\sum_{i,j} \iint_{I_{i,j}} \Big(\big|u_\eta(x,y,t_{n/2}+) - k\big| - \big|u_\eta(x,y,t_{n/2}-) - k\big|\Big) \\
&\qquad\qquad \times \big(\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\big)\,dx\,dy \\
&= \tilde{I}_n,
\end{aligned}$$

using (4.42). This implies

$$\begin{aligned}
\big|\tilde{I}_n\big| &\le \sum_{i,j} \iint_{I_{i,j}} \big|u_\eta(x,y,t_{n/2}+) - u_\eta(x,y,t_{n/2}-)\big| \\
&\qquad\qquad \times \big|\phi(x,y,t_{n/2}) - \phi(x_i,y_j,t_{n/2})\big|\,dx\,dy \\
&\le \big(\Delta x + \Delta y\big)\, \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} \\
&\qquad\qquad \times \sum_{i,j} \iint_{I_{i,j}} \big|u_\eta(x,y,t_{n/2}+) - u_\eta(x,y,t_{n/2}-)\big|\,dx\,dy \\
&\le \big(\Delta x + \Delta y\big) \iint_{\mathbb{R}^2} \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} \big|\pi u_\eta(x,y,t_{n/2}-) - u_\eta(x,y,t_{n/2}-)\big|\,dx\,dy \\
&\le \big(\Delta x + \Delta y\big)^2 \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}\, \mathrm{T.V.}\,(u_0)\,,
\end{aligned}$$





since

$$
\begin{aligned}
\left|\phi(x, y)-\phi(x_i, y_j)\right| &\leq \left|(x-x_i, y-y_j)\right| \int_0^1 \left|\nabla\phi(r(x-x_i, y-y_j))\right| \, dr \\
&\leq \left(\Delta x+\Delta y\right) \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}, \quad (x, y) \in I_{i,j},
\end{aligned}
$$

where we have used (4.38). Thus

$$
\sum_{n=1}^{2N} \left|\tilde{I}_n\right| \leq \frac{(\Delta x+\Delta y)^2}{\Delta t} \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{T.V.}\,(u_0)\,. \tag{4.43}
$$

In order to conclude that u is an entropy solution, we need the right-hand side of (4.43) to vanish as $\Delta x, \Delta y, \Delta t \to 0$; that is, we need to assume that

$$
\frac{\Delta x+\Delta y}{\Delta t} \text{ remains bounded}
$$

as $\eta \to 0$. Under this assumption,

$$
\begin{aligned}
\iint_{\mathbb{R}^2} \int_0^T &\left(|u-k|\,\phi_t + q^f(u,k)\phi_x + q^g(u,k)\phi_y\right) \, dt\, dx\, dy \\
&- \iint_{\mathbb{R}^2} |u(x,y,T)-k|\,\phi(x,y,T)\, dx\, dy \\
&+ \iint_{\mathbb{R}^2} |u(x,y,0)-k|\,\phi(x,y,0)\, dx\, dy \geq 0,
\end{aligned}
$$

which shows that u indeed satisfies the Kružkov entropy condition. We summarize the result.

Theorem 4.4 *Let u_0 be an integrable and bounded function in $L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$, and let f_j be piecewise twice continuously differentiable functions for $j = 1, \dots, m$. Construct an approximate solution u_η using front tracking by defining*

$$
u^0 = \pi u_0, \quad u^{n+j/m} = \pi \circ S_{\Delta t}^{f_{j,\delta}, x_j} u^{n+(j-1)/m}, \quad j = 1, \dots, m, \quad n \in \mathbb{N},
$$

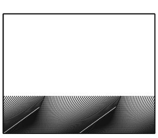
and

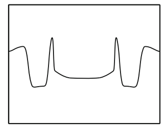
$$
u_\eta(x,t) = \begin{cases} S^{f_{j,\delta}, x_j}_{m(t-t_{n+(j-1)/m})} u^{n+(j-1)/m}, & \text{for } t \in [t_{n+(j-1)/m}, t_{n+j/m}), \\ u^{n+j/m} & \text{for } t = t_{n+j/m}, \end{cases}
$$

where $x = (x_1, \dots, x_m)$.

For every sequence $\{\eta\}$, with $\eta = (\Delta x_1, \dots, \Delta x_m, \Delta t, \delta)$, where $\eta \to 0$ and

$$
\max_j \left\{\Delta x_j\right\} / \Delta t \textit{ remains bounded,}
$$





we have that $\{u_\eta\}$ *converges to the unique solution* $u = u(x,t)$ *of the initial value problem*

$$u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = 0, \quad u(x,0) = u_0(x), \tag{4.44}$$

which satisfies the Kružkov entropy condition.

4.3 Convergence Rates

Now I think I'm wrong on account of those damn partial integrations.
I oscillate between right and wrong.
— Letter from Feynman to Welton (1936)

In this section we show how fast front tracking plus dimensional splitting converges to the exact solution. The analysis is based on Kuznetsov's lemma.

We start by generalizing Kuznetsov's lemma, Theorem 3.14, to the present multidimensional setting. Although the argument carries over, we will present the relevant definitions in arbitrary dimension.

Let the class $\mathcal{K}$ consist of maps $u\colon [0,\infty) \to L^1(\mathbb{R}^m) \cap BV(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ such that:

(i) The limits $u(t\pm)$ exist.
(ii) The function u is right continuous, i.e., $u(t+) = u(t)$.
(iii) $\|u(t)\|_{L^\infty(\mathbb{R}^m)} \le \|u(0)\|_{L^\infty(\mathbb{R}^m)}$.
(iv) $\mathrm{T.V.}\,(u(t)) \le \mathrm{T.V.}\,(u(0))$.

Recall the following definition of moduli of continuity in time (cf. (3.54)):

$$\begin{aligned} \nu_t(u,\sigma) &= \sup_{|\tau|\le\sigma} \|u(t+\tau) - u(t)\|_{L^1(\mathbb{R}^m)}, \quad \sigma > 0, \\ \nu(u,\sigma) &= \sup_{0\le t\le T} \nu_t(u,\sigma). \end{aligned}$$

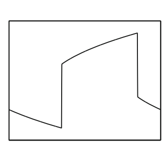
The estimate (3.55) is replaced by

$$\nu(u,\sigma) \le |\sigma|\,\mathrm{T.V.}\,(u_0) \max_j \{\, \|f_j\|_{\mathrm{Lip}} \,\},$$

for a solution u of (4.23).

In several space dimensions, the Kružkov form reads

$$\begin{aligned} \Lambda_T(u,\phi,k) = \iint\limits_{\mathbb{R}^m\times[0,T]} &\Big(|u-k|\,\phi_t + \sum_j q^{f_j}(u,k)\phi_{x_j} \Big)\, dx_1\cdots dx_m \\ &- \int\limits_{\mathbb{R}^m} |u(x,T)-k|\,\phi(x,T)\, dx_1\cdots dx_m\, dt \\ &+ \int\limits_{\mathbb{R}^m} |u_0(x)-k|\,\phi(x,0)\, dx_1\cdots dx_m. \end{aligned} \tag{4.45}$$



In this case, we use the test function

$$\Omega(x, x', s, s') = \omega_{\varepsilon_0}(s - s')\omega_\varepsilon(x_1 - x_1') \cdots \omega_\varepsilon(x_m - x_m'), \qquad (4.46)$$
$$x = (x_1, \ldots, x_m), \quad x' = (x_1', \ldots, x_m').$$

Here ω_ε is the standard mollifier defined by

$$\omega_\varepsilon(x_j) = \frac{1}{\varepsilon}\omega\left(\frac{x_j}{\varepsilon}\right)$$

with

$$0 \le \omega \le 1, \quad \operatorname{supp}\omega \subseteq [-1, 1], \quad \omega(-x_j) = \omega(x_j), \quad \int_{-1}^{1} \omega(z)\, dz = 1.$$

When v is the unique solution of the conservation law (4.25), we introduce

$$\Lambda_{\varepsilon,\varepsilon_0}(u, v) = \int_0^T \int_{\mathbb{R}^m} \Lambda_T\left(u, \Omega(\,\cdot\,, x', \,\cdot\,, s'), v(x', s')\right)\, dx'ds'.$$

Kuznetsov's lemma can be formulated as follows.

Theorem 4.5 *Let u be a function in $\mathcal{K}$, and let v be an entropy solution of* (4.25). *If $0 < \varepsilon_0 < T$ and $\varepsilon > 0$, then*

$$\begin{aligned}
\|u(\,\cdot\,, T-) - v(\,\cdot\,, T)\|_{L^1(\mathbb{R}^m)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^m)} \\
+ \mathrm{T.V.}\,(v_0)\left(2\varepsilon + \varepsilon_0 \max_j\{\, \|f_j\|_{\mathrm{Lip}}\,\}\right) \\
+ \nu(u, \varepsilon_0) - \Lambda_{\varepsilon,\varepsilon_0}(u, v),
\end{aligned} \qquad (4.47)$$

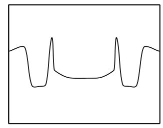
where $u_0 = u(\,\cdot\,, 0)$ and $v_0 = v(\,\cdot\,, 0)$.

The proof of Theorem 3.14 carries over to this setting verbatim.

◇ Example 4.6

Let us first apply this theorem to the case that u is the dimensional splitting approximation, defined with exact solution operators $S^{f,x}_{\Delta t}$ and $S^{g,y}_{\Delta t}$; cf. (4.6). We have established that $\nu(u_{\Delta t}, \varepsilon_0) \le C\varepsilon_0$, where the constant C depends on the total variation of u_0 and the Lipschitz norm of the flux. The inequalities (4.17) and (4.18) imply

$$\begin{aligned}
L_T(u_{\Delta t}, k, \varphi) = \int_0^T \iint_{\mathbb{R}^2} & |u_{\Delta t} - k|\, \varphi_t \\
& + 2\chi_n(t) q^f(u_{\Delta t}, k)\varphi_x + 2\tilde{\chi}_n(t) q^g(u_{\Delta t}, k)\varphi_y \, dx\, dy\, dt \\
& - \iint_{\mathbb{R}^2} |u_{\Delta t} - k|\, \varphi \Big|_{t=T} \, dx\, dy + \iint_{\mathbb{R}^2} |u_{\Delta t} - k|\, \varphi \Big|_{t=0} \, dx\, dy \\
& \ge 0.
\end{aligned}$$



Set

$$L_{\varepsilon_0,\varepsilon} = \iiint L_T(u_{\Delta t}, v(x', y', s), \omega_\varepsilon(\,\cdot - x')\omega(\,\cdot - y')\omega_{\varepsilon_0}(\,\cdot - s)\, dx'\, dy'\, ds \geq 0.$$

In the following we always have that $u_{\Delta t} = u_{\Delta t}(x, y, t)$ and $v = v(x', y', s)$, although we sometimes do not indicate that, or indicate only those variables to which we would like to draw the reader's attention. Then

$$\begin{aligned} -\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) &\leq -\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) + L_{\varepsilon_0,\varepsilon} \\ &= \int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} \left(I^x + I^y\right) dx\, dy\, dt\, dx'\, dy'\, ds, \end{aligned}$$

where

$$\begin{aligned} I^x &= (2\chi_n(t) - 1)\, q^f(u_{\Delta t}, v)\omega_\varepsilon'(x - x')\omega_\varepsilon(y - y')\omega_{\varepsilon_0}(t - s), \\ I^y &= (2\tilde{\chi}_n(t) - 1)\, q^g(u_{\Delta t}, v)\omega_\varepsilon(x - x')\omega_\varepsilon'(y - y')\omega_{\varepsilon_0}(t - s). \end{aligned}$$

We shall estimate $\int I^x$; the estimate for I^y is identical. First observe that

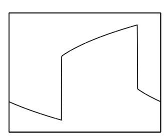
$$2\chi_n(t) - 1 = \begin{cases} 1 & t_n \leq t < t_{n+1/2}, \\ -1 & t_{n+1/2} \leq t < t_{n+1}. \end{cases}$$

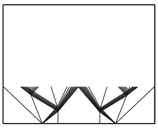
Therefore, if $N\Delta t = T$, then

$$\int_0^T (2\chi_n(t) - 1)\, \psi(t)\, dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \left(\psi(t) - \psi(t + \Delta t/2)\right) dt,$$

for every function ψ. Thus

$$\begin{aligned} &\int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} I^x\, dx\, dy\, dt\, dx'\, dy'\, ds \\ &\quad = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \Big(q^f(u_{\Delta t}(t), v)\omega_{\varepsilon_0}(t - s) \\ &\qquad\quad - q^f(u_{\Delta t}(t + \Delta t/2), v)\omega_{\varepsilon_0}(t + \Delta t/2 - s)\Big) \\ &\qquad\qquad\qquad \times \omega_\varepsilon'(x - x')\omega_\varepsilon(y - y')\, dx\, dy\, dx'\, dy'\, ds\, dt \end{aligned}$$





$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \Big(\omega_{\varepsilon_0}(t-s) - \omega_{\varepsilon_0}(t+\Delta t/2 - s)\Big) \\
&\qquad\qquad \times q^f(u_{\Delta t}(t), v)\omega'_\varepsilon(x-x')\omega_\varepsilon(y-y')\,dx\,dy\,dx'\,dy'\,ds\,dt \\
&\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} \int_0^T \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} \omega_{\varepsilon_0}(t+\Delta t/2 - s) \\
&\qquad\qquad\qquad\qquad \times \Big(q^f(u_{\Delta t}(t+\Delta t), v) - q^f(u_{\Delta t}(t), v)\Big) \\
&\qquad\qquad\qquad\qquad \times \omega'_\varepsilon(x-x')\omega_\varepsilon(y-y')\,dx\,dy\,dx'\,dy'\,ds\,dt \\
&=: A + B.
\end{aligned}
$$

Regarding A,

$$
\begin{aligned}
|A| &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} L \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\;dy \int_0^{\Delta t/2} \int_0^T \left|\omega'_{\varepsilon_0}(t-s+\tau)\right|\,ds\,d\tau dt \\
&\le \frac{CT\Delta t}{\varepsilon_0} \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\;dy.
\end{aligned}
$$

Also

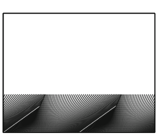
$$
\begin{aligned}
|B| \le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} &\omega_{\varepsilon_0}(t-s+\Delta t/2) \\
&\times L \iint_{\mathbb{R}^2} |u_{\Delta t}(t+\Delta t/2) - u_{\Delta t}(t)|\;dx\,dy\,|\omega'_\varepsilon(x-x')|\;dx'\,ds\,dt \\
\le{}& \nu(u_{\Delta t}, \Delta t/2)\frac{C}{\varepsilon} \\
\le{}& \frac{C\Delta t}{\varepsilon}.
\end{aligned}
$$

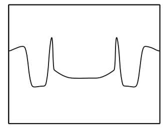
Hence

$$
\begin{aligned}
&\Big|\int_0^T \iint_{\mathbb{R}^2} \int_0^T \iint_{\mathbb{R}^2} I^x\,dx\,dy\,dt\,dx'\,dy'\,ds\Big| \\
&\qquad \le \frac{C\Delta t}{\varepsilon_0} \int_{\mathbb{R}} |u_{\Delta t}(\,\cdot\,, x, t)|_{BV}\;dy + \frac{C\Delta t}{\varepsilon}.
\end{aligned}
$$

We have a similar estimate for the integral of I^y; thus we end up with the estimate

$$
-\Lambda_{\varepsilon_0,\varepsilon}(u_{\Delta t}, v) \le \frac{C\Delta t}{\varepsilon_0}\,|u_0|_{BV(\mathbb{R}^2)} + \frac{C\Delta t}{\varepsilon}.
$$





Since we have $v(0) = u_{\Delta t}(0) = u_0$, Kuznetsov's lemma yields

$$\|u_{\Delta t}(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1(\mathbb{R}^2)} \le C\left(\varepsilon_0 + \varepsilon + \frac{\Delta t}{\varepsilon_0} + \frac{\Delta t}{\varepsilon}\right),$$

which on setting $\varepsilon_0 = \varepsilon = \sqrt{\Delta t}$, yields

$$\|u_{\Delta t}(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1(\mathbb{R}^2)} \le C\sqrt{\Delta t}. \tag{4.48}$$

Since this estimate was obtained using the exact solution operator in each direction, there is no hope of obtaining a better estimate using numerical approximations instead of $S^{f,g}_{\Delta t}$. $\diamond$

Next, we use Kuznetsov's lemma to estimate the rate of convergence for the front tracking approximation. This entails using a first-order (in δ) approximation to the exact solution operators, so from the previous example, the best we can hope for is that the error is bounded by $\mathcal{O}(\delta + \sqrt{\Delta t})$.

We want to estimate

$$\left\|S(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)} \le \|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^m)} + \left\|S_\delta(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)}, \tag{4.49}$$

where $u = S(T)u_0$ and $S_\delta(T)u_0$ denote the exact solutions of the multidimensional conservation law with flux functions f replaced by their piecewise linear and continuous approximations f_δ. The first term can be estimated by

$$\|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^m)} \le T\max_j\left\{\|f_j - f_{j,\delta}\|_{\mathrm{Lip}}\right\}\mathrm{T.V.}\,(u_0)\,, \tag{4.50}$$

while we apply Kuznetsov's lemma, Theorem 4.5, for the second term. For the function u we choose u_η, the approximate solution using front tracking along each dimension and dimensional splitting, while for v we use the exact solution with piecewise linear continuous flux functions f_δ and g_δ, and u_0 as initial data, that is, $v = v_\delta = S_\delta(T)u_0$. Thus we find, using (4.41), that

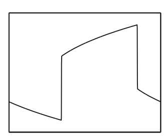
$$\nu(u_\eta, \varepsilon_0) \le \varepsilon_0\left(C + \mathcal{O}\left(\frac{1}{\Delta t}\max_j\left\{\Delta x_j\right\}\right)\right)\mathrm{T.V.}\,(u_0)\,.$$

Kuznetsov's lemma then reads

$$\begin{aligned}\left\|S_\delta(T)u_0 - u_\eta\right\|_{L^1(\mathbb{R}^m)} \le \left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^m)} &+ \Bigg[2\varepsilon + \max_j\left\{\|f_{j,\delta}\|_{\mathrm{Lip}}\right\}\varepsilon_0 \\ &+ \varepsilon_0\Big(C + \mathcal{O}\Big(\frac{\max\left\{\Delta x_j\right\}}{\Delta t}\Big)\Big)\Bigg]\mathrm{T.V.}\,(u_0) \\ &- \Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta),\end{aligned} \tag{4.51}$$

and the name of the game is to estimate $\Lambda_{\varepsilon,\varepsilon_0}$.

To make the estimates more transparent, we start by rewriting $\Lambda_T(u_\eta,\phi,k)$. Since all the complications of several space dimensions are present in two dimensions, we present the argument in two dimensions only, that is, with $m = 2$, and

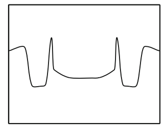


denote the spatial variables by (x, y). All arguments carry over to arbitrary dimensions without any change. By definition we have (in obvious notation, $q^{f_\delta}(u) = \operatorname{sign}(u-k)\,(f_\delta(u) - f_\delta(k))$ and similarly for q^{g_δ})

$$
\begin{aligned}
\Lambda_T(u_\eta, \phi, k) &= \iint \int_0^T \left(\left|u_\eta - k\right| \phi_t + q^{f_\delta}(u_\eta, k)\phi_x + q^{g_\delta}(u_\eta, k)\phi_y \right) dt\, dx\, dy \\
&\quad + \iint \left|u_\eta - k\right| \phi|_{t=0+}\, dx\, dy - \iint \left|u_\eta - k\right| \phi|_{t=T-}\, dx\, dy \\
&= \sum_{n=0}^{N-1} \iint \Big(\int_{t_n}^{t_{n+1/2}} + \int_{t_{n+1/2}}^{t_{n+1}} \Big) \Big(\left|u_\eta - k\right| \phi_t \\
&\qquad\qquad + q^{f_\delta}(u_\eta, k)\phi_x + q^{g_\delta}(u_\eta, k)\phi_y \Big)\, dt\, dx\, dy \\
&\quad + \iint \left|u_\eta - k\right| \phi|_{t=0+}\, dx\, dy - \iint \left|u_\eta - k\right| \phi|_{t=T-}\, dx\, dy \\
&= \sum_{n=0}^{N-1} \iint \int_{t_n}^{t_{n+1/2}} \left(\left|u_\eta - k\right| \phi_t + 2q^{f_\delta}(u_\eta, k)\phi_x \right) dt\, dx\, dy \\
&\quad + \sum_n \iint \int_{t_{n+1/2}}^{t_{n+1}} \left(\left|u_\eta - k\right| \phi_t + 2q^{g_\delta}(u_\eta, k)\phi_y \right) dt\, dx\, dy \\
&\quad + \sum_{n=0}^{N-1} \iint \Big(\int_{t_{n+1/2}}^{t_{n+1}} - \int_{t_n}^{t_{n+1/2}} \Big) q^{f_\delta}(u_\eta, k)\phi_x\, dt\, dx\, dy \\
&\quad + \sum_{n=0}^{N-1} \iint \Big(\int_{t_n}^{t_{n+1/2}} - \int_{t_{n+1/2}}^{t_{n+1}} \Big) q^{g_\delta}(u_\eta, k)\phi_y\, dt\, dx\, dy \\
&\quad + \iint \left|u_\eta - k\right| \phi|_{t=0+}\, dx\, dy - \iint \left|u_\eta - k\right| \phi|_{t=T-}\, dx\, dy.
\end{aligned}
$$

We now use that u_η is an exact solution in the x-direction and the y-direction on each strip $[t_n, t_{n+1/2}]$ and $[t_{n+1/2}, t_{n+1}]$, respectively. Thus we can invoke inequalities (4.17) and (4.18), and we conclude that

$$
\begin{aligned}
\Lambda_T(u_\eta, \phi, k) \geq \sum_{n=0}^{N-1} \iint \Big(&\left|u_\eta - k\right| |_{t=t_{n+1/2}-} \phi(t_{n+1/2}) \\
&- \left|u_\eta - k\right| |_{t=t_n+} \phi(t_n) \Big)\, dx\, dy \\
+ \sum_{n=0}^{N-1} \iint \Big(&\left|u_\eta - k\right| |_{t=t_{n+1}-} \phi(t_{n+1}) \\
&- \left|u_\eta - k\right| |_{t=t_{n+1/2}+} \phi(t_{n+1/2}) \Big)\, dx\, dy
\end{aligned}
$$

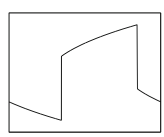


$$
\begin{aligned}
&+\sum_{n=0}^{N-1}\iint\Big(\int_{t_{n+1/2}}^{t_{n+1}}-\int_{t_n}^{t_{n+1/2}}\Big)q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&+\sum_{n=0}^{N-1}\iint\Big(\int_{t_n}^{t_{n+1/2}}-\int_{t_{n+1/2}}^{t_{n+1}}\Big)q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&+\iint\left|u_\eta-k\right|\phi|_{t=0+}\,dx\,dy-\iint\left|u_\eta-k\right|\phi|_{t=T-}\,dx\,dy\\
&=-2\sum_{n=0}^{N-1}\iint\int_{t_n}^{t_{n+1/2}}q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&\quad+\iint\int_0^T q^{f_\delta}(u_\eta,k)\phi_x\,dt\,dx\,dy\\
&\quad-2\sum_{n=0}^{N-1}\iint\int_{t_{n+1/2}}^{t_{n+1}}q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&\quad+\iint\int_0^T q^{g_\delta}(u_\eta,k)\phi_y\,dt\,dx\,dy\\
&\quad+\sum_{n=0}^{N-1}\iint\Big(\left|u_\eta-k\right|\Big|_{t=t_{n+1/2}-}\\
&\qquad\qquad-\left|u_\eta-k\right|\Big|_{t=t_{n+1/2}+}\Big)\phi(t_{n+1/2})\,dx\,dy\\
&\quad+\sum_{n=1}^{N-1}\iint\Big(\left|u_\eta-k\right|\Big|_{t=t_n-}-\left|u_\eta-k\right|\Big|_{t=t_n+}\Big)\phi(t_n)\,dx\,dy\\
&:=-I_1(u_\eta,k)-I_2(u_\eta,k)-I_3(u_\eta,k)-I_4(u_\eta,k).
\end{aligned}
\tag{4.52}
$$

Observe that because we employ the projection operator π between each pair of consecutive times, we solve a conservation law in one dimension; $u^{n+1/2}$ and u^n are in general discontinuous across $t_{n+1/2}$ and t_n, respectively. The terms I_1 and I_2 are due to dimensional splitting, while I_3 and I_4 come from the projections.

Choose now for the constant k the function $v_\delta(x',y',s')$, and for ϕ we use Ω given by (4.46). Integrating over the new variables, we obtain

$$
\begin{aligned}
\Lambda_{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)&=\iint\int_0^T\Lambda_T(u_\eta,\Omega(\,\cdot\,,x',\,\cdot\,,y',\,\cdot\,,s'),v_\delta(x',y',s'))\,ds'\,dx'\,dy'\\
&\geq -I_1^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_2^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_3^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta)-I_4^{\varepsilon,\varepsilon_0}(u_\eta,v_\delta),
\end{aligned}
$$



where $I_j^{\varepsilon,\varepsilon_0}$ are given by

$$
\begin{aligned}
I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \iint \int_0^T \iint \Big(2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1/2}} & q^{f_\delta}(u_\eta, v_\delta)\Omega_x \, ds \\
& - \int_0^T q^{f_\delta}(u_\eta, v_\delta)\Omega_x \, ds \Big) \, dx\, dy\, ds'\, dx'\, dy', \\
I_2^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \iint \int_0^T \iint \Big(2 \sum_{n=0}^{N-1} \int_{t_{n+1/2}}^{t_{n+1}} & q^{g_\delta}(u_\eta, v_\delta)\Omega_y \, ds \\
& - \int_0^T q^{g_\delta}(u_\eta, v_\delta)\Omega_y \, ds \Big) \, dx\, dy\, ds'\, dx'\, dy', \\
I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=1}^{N-1} \iint \int_0^T \iint \Big(& \left|u_\eta - v_\delta\right| |_{s=t_n+} \\
& - \left|u_\eta - v_\delta\right| |_{s=t_n-} \Big) \Omega \, dx\, dy\, ds'\, dx'\, dy', \\
I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=0}^{N-1} \iint \int_0^T \iint \Big(& \left|u_\eta - v_\delta\right| |_{s=t_{n+1/2}+} \\
& - \left|u_\eta - v_\delta\right| |_{s=t_{n+1/2}-} \Big) \Omega \, dx\, dy\, ds'\, dx'\, dy'.
\end{aligned}
$$

We will start by estimating $I_1^{\varepsilon,\varepsilon_0}$ and $I_2^{\varepsilon,\varepsilon_0}$.

Lemma 4.7 *We have the following estimate:*

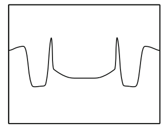
$$
\begin{aligned}
\left|I_1^{\varepsilon,\varepsilon_0}\right| + \left|I_2^{\varepsilon,\varepsilon_0}\right| \leq T \ \max\left\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\right\} \mathrm{T.V.}\,(u_0) & \\
\times \Big(\frac{\Delta t}{\varepsilon_0} + \frac{1}{\varepsilon}\Big(\{\|f\|_{\mathrm{Lip}} + \|g\|_{\mathrm{Lip}}\}\Delta t + \Delta x + \Delta y \Big)\Big). &
\end{aligned}
\tag{4.53}
$$

Proof We will detail the estimate for $\left|I_1^{\varepsilon,\varepsilon_0}\right|$. Writing

$$
\begin{aligned}
q^{f_\delta}(u_\eta(s), v_\delta(s')) &= q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s')) \\
&\quad + \left(q^{f_\delta}(u_\eta(s), v_\delta(s')) - q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s')) \right),
\end{aligned}
$$

we rewrite $I_1^{\varepsilon,\varepsilon_0}$ as

$$
\begin{aligned}
I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=0}^{N-1} \Big[& \left(J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1}) \right) \\
& + \left(J_2(t_n, t_{n+1/2}) - J_2(t_{n+1/2}, t_{n+1}) \right) \Big],
\end{aligned}
\tag{4.54}
$$



with

$$J_1(\tau_1,\tau_2) = \iint \int_0^T \iint \int_{\tau_1}^{\tau_2} q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(x',y',s')) \\ \times \Omega_x(x,x',y,y',s,s')\, ds\, dx\, dy\, ds'\, dx'\, dy',$$

$$J_2(\tau_1,\tau_2) = \iint \int_0^T \iint \int_{\tau_1}^{\tau_2} \Big(q^{f_\delta}(u_\eta(x,y,s), v_\delta(x',y',s')) \\ - q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(x',y',s'))\Big) \\ \times \Omega_x(x,x',y,y',s,s')\, ds\, dx\, dy\, ds'\, dx'\, dy'.$$

Here we have written out all the variables explicitly; however, in the following we will display only the relevant variables. All spatial integrals are over the real line unless specified otherwise. Rewriting

$$\omega_{\varepsilon_0}(s-s') = \omega_{\varepsilon_0}(t_{n+1/2}-s') + \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s},$$

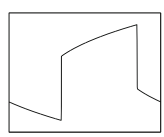
we obtain

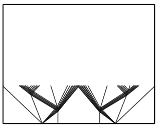
$$J_1(t_n,t_{n+1/2}) = \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon\Big(\int_{t_n}^{t_{n+1/2}} \omega_{\varepsilon_0}(t_{n+1/2}-s')\, ds \\ + \int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds\Big)\, dx\, dy\, ds'\, dx'\, dy' \\ = \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon\Big(\frac{\Delta t}{2}\omega_{\varepsilon_0}(t_{n+1/2}-s') \\ + \int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds\Big)\, dx\, dy\, ds'\, dx'\, dy',$$

where $\Omega^\varepsilon = \omega_\varepsilon(x-x')\omega_\varepsilon(y-y')$ denotes the spatial part of Ω.

If we rewrite $J_1(t_{n+1/2},t_{n+1})$ in the same way, we obtain

$$J_1(t_{n+1/2},t_{n+1}) = \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon\Big(\frac{\Delta t}{2}\omega_{\varepsilon_0}(t_{n+1/2}-s') \\ + \int_{t_{n+1/2}}^{t_{n+1}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s}-s')\, d\bar{s}\, ds\Big)\, dx'\, dy'\, ds'\, dx\, dy,$$





and hence

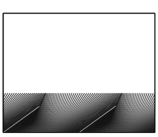
$$
\begin{aligned}
&J_1\big(t_n, t_{n+1/2}\big) - J_1\big(t_{n+1/2}, t_{n+1}\big) \\
&\quad = \iint \int_0^T \iint q^{f_\delta}(u_\eta(t_{n+1/2}), v_\delta(s'))\Omega_x^\varepsilon\Big(\int_{t_n}^{t_{n+1/2}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s} - s')\, d\bar{s}\, ds \\
&\qquad\qquad - \int_{t_{n+1/2}}^{t_{n+1}} \int_{t_{n+1/2}}^{s} \omega'_{\varepsilon_0}(\bar{s} - s')\, d\bar{s}\, ds\Big)\, dx\, dy\, ds'\, dx'\, dy'. \qquad (4.55)
\end{aligned}
$$

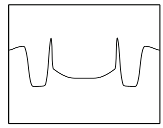
Now using the Lipschitz continuity of q^{f_δ}, we can replace variation in q^{f_δ} by variation in u, and obtain, using $\iint \omega'_{\varepsilon_0}(x-x')\,dx\,dx' = 0$, that

$$
\begin{aligned}
&\left| \iint q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(s'))\omega'_{\varepsilon_0}(x-x')\, dx\, dx' \right| \\
&\quad = \Big| \iint \omega'_{\varepsilon_0}(x-x')\, dx\, dx' \\
&\qquad\qquad \times \big[q^{f_\delta}(u_\eta(x,y,t_{n+1/2}), v_\delta(s')) - q^{f_\delta}(u_\eta(x',y,t_{n+1/2}), v_\delta(s')) \big] \Big| \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \iint \left| \omega'_{\varepsilon_0}(x-x') \right| \\
&\qquad\qquad\qquad \times \left| u_\eta(x,y,t_{n+1/2}) - u_\eta(x',y,t_{n+1/2}) \right|\, dx\, dx' \\
&\quad = \|f_\delta\|_{\mathrm{Lip}} \iint \left| u_\eta(x'+z,y,t_{n+1/2}) - u_\eta(x',y,t_{n+1/2}) \right| \left| \omega'_{\varepsilon_0}(z) \right|\, dx'\, dz \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \int \frac{1}{|z|} \int \left| u_\eta(x'+z,y,t_{n+1/2}) - u_\eta(x',y,t_{n+1/2}) \right|\, dx' \\
&\qquad\qquad\qquad \times \left| z\omega'_{\varepsilon_0}(z) \right|\, dz \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}_{\cdot x}\left(u_\eta(t_{n+1/2})\right) \int \left| z\omega'_{\varepsilon_0}(z) \right|\, dz \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}_{\cdot x}\left(u_\eta(t_{n+1/2})\right),
\end{aligned}
$$

using that $\int \left| z\omega'_{\varepsilon_0}(z) \right|\, dz = 1$. We combine this with (4.55) to get

$$
\begin{aligned}
&\left| J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1}) \right| \\
&\quad \le \|f_\delta\|_{\mathrm{Lip}} \iint \mathrm{T.V.}_{\cdot x}\left(u_\eta(t_{n+1/2})\right) \omega_{\varepsilon_0}(y-y') \\
&\qquad\qquad \times \Big(\int_0^T \int_{t_n}^{t_{n+1/2}} \left| \int_{t_{n+1/2}}^{s} \left| \omega'_{\varepsilon_0}(\bar{s}-s') \right|\, d\bar{s} \right| ds\, ds' \\
&\qquad\qquad\quad + \int_0^T \int_{t_{n+1/2}}^{t_{n+1}} \left| \int_{t_{n+1/2}}^{s} \left| \omega'_{\varepsilon_0}(\bar{s}-s') \right|\, d\bar{s} \right| ds\, ds' \Big)\, dy'\, dy.
\end{aligned}
$$





Inserting the estimate

$$\int_0^T \left|\omega'_{\varepsilon_0}(\bar{s}-s')\right| \, ds' \le \frac{1}{\varepsilon_0} \int |\omega'(z)| \, dz \le 2/\varepsilon_0,$$

we obtain

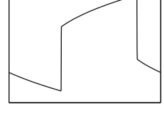
$$\left| J_1(t_n, t_{n+1/2}) - J_1(t_{n+1/2}, t_{n+1}) \right| \le \frac{\|f_\delta\|_{\mathrm{Lip}} (\Delta t)^2}{2\varepsilon_0} \mathrm{T.V.}\left(u_\eta(t_{n+1/2})\right). \tag{4.56}$$

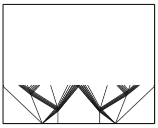
Next we consider the term J_2. We first use the Lipschitz continuity of q^{f_δ}, which yields

$$\begin{aligned}
\left| J_2(t_n, t_{n+1/2}) \right| &\le \|f_\delta\|_{\mathrm{Lip}} \iint \int_0^T \iint \int_{t_n}^{t_{n+1/2}} \left| u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2}) \right| \\
&\qquad \times |\Omega_x| \, ds \, dx' \, dy' \, ds' \, dx \, dy \\
&\le \frac{\|f_\delta\|_{\mathrm{Lip}}}{\varepsilon} \int_{t_n}^{t_{n+1/2}} \iint \left| u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2}) \right| \, ds \, dx \, dy \\
&\le \frac{\|f_\delta\|_{\mathrm{Lip}}}{\varepsilon} \int_{t_n}^{t_{n+1/2}} \iint \left| u_\eta(x,y,s) - u_\eta(x,y,t_{n+1/2}-) \right| \, ds \, dx \, dy \\
&\quad + \frac{\|f_\delta\|_{\mathrm{Lip}} \Delta t}{2\varepsilon} \iint \left| u_\eta(x,y,t_{n+1/2}-) - u_\eta(x,y,t_{n+1/2}) \right| \, dx \, dy \\
&\le \frac{\|f_\delta\|_{\mathrm{Lip}} \Delta t}{\varepsilon} \left(\|f_\delta\|_{\mathrm{Lip}} \Delta t + \Delta x \right) \mathrm{T.V.}\left(u_\eta\left(t_{n+1/2}\right)\right).
\end{aligned}$$

Here we integrated to unity in the variables s' and y', and estimated $\int |\omega'_\varepsilon(x-x')| \, dx'$ by $2/\varepsilon$. Finally, we used the continuity in time of the L^1-norm in the x-direction and estimated the error due to the projection. A similar bound can be obtained for $J_2(t_{n+1/2}, t_{n+1})$, and hence

$$\begin{aligned}
&\left| J_2(t_n, t_{n+1/2}) - J_2(t_{n+1/2}, t_{n+1}) \right| \\
&\qquad \le \left| J_2(t_n, t_{n+1/2}) \right| + \left| J_2(t_{n+1/2}, t_{n+1}) \right| \\
&\qquad \le \frac{\|f\|_{\mathrm{Lip}} \Delta t}{\varepsilon} \left(2\|f\|_{\mathrm{Lip}} \Delta t + \Delta x + \Delta y \right) \mathrm{T.V.}\left(u_\eta(t_n)\right),
\end{aligned} \tag{4.57}$$





where we used that T.V. $\left(u_\eta(t_{n+1/2})\right) \leq$ T.V. $\left(u_\eta(t_n)\right)$. Inserting estimates (4.56) and (4.57) into (4.54) yields

$$\begin{aligned} \left|I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| &\leq \|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}\left(u_\eta(0)\right) \\ &\qquad \times \sum_{n=0}^{N-1}\left(\frac{(\Delta t)^2}{2\varepsilon_0} + \frac{\Delta t}{2\varepsilon}(2\|f_\delta\|_{\mathrm{Lip}}\Delta t + \Delta x + \Delta y)\right) \\ &\leq T\,\|f_\delta\|_{\mathrm{Lip}} \mathrm{T.V.}\left(u_\eta(0)\right) \\ &\qquad \times \left(\frac{\Delta t}{2\varepsilon_0} + \frac{1}{2\varepsilon}(2\|f_\delta\|_{\mathrm{Lip}}\Delta t + \Delta x + \Delta y)\right), \end{aligned}$$

where we again used that T.V. $\left(u_\eta\right)$ is nonincreasing. An analogous argument gives the same estimate for $I_2^{\varepsilon,\varepsilon_0}$. Adding the two inequalities, we conclude that (4.53) holds. □

It remains to estimate $I_3^{\varepsilon,\varepsilon_0}$ and $I_4^{\varepsilon,\varepsilon_0}$. We aim at the following result.

Lemma 4.8 *The following estimate holds*:

$$\left|I_3^{\varepsilon,\varepsilon_0}\right| + \left|I_4^{\varepsilon,\varepsilon_0}\right| \leq \frac{T(\Delta x + \Delta y)^2}{\Delta t\ \varepsilon}\mathrm{T.V.}\left(u_0\right).$$

Proof We discuss the term $I_3^{\varepsilon,\varepsilon_0}$ only. Recall that

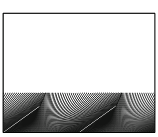
$$\begin{aligned} I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) = \sum_{n=1}^{N-1} \iint \int_0^T \iint \Big(&\left|u_\eta(x,y,t_n) - v_\delta(x',y',s')\right| \\ &- \left|u_\eta(x,y,t_n-) - v_\delta(x',y',s')\right|\Big) \\ &\times \Omega(x,x',y,y',t_n,s')\,dx'\,dy'\,ds'\,dx\,dy. \end{aligned}$$

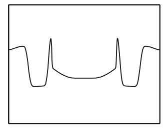
The function $u_\eta(x,y,t_n+)$ is the projection of $u_\eta(x,y,t_n-)$, that is,

$$u_\eta(x,y,t_n+) = \frac{1}{\Delta x \Delta y}\iint_{I_{ij}} u_\eta(\bar{x},\bar{y},t_n-)\,d\bar{x}\,d\bar{y}. \tag{4.58}$$

If we replace $\iint_{\mathbb{R}^2}$ by $\sum_{i,j}\iint_{I_{ij}}$ and use (4.58), we obtain

$$\begin{aligned} &I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) \\ &\quad= \sum_{n=1}^{N-1}\iint\int_0^T \sum_{i,j}\iint_{I_{ij}}\left[\left|\frac{1}{\Delta x\Delta y}\iint_{I_{ij}} u_\eta(\bar{x},\bar{y},t_n-)\,d\bar{x}\,d\bar{y} - v_\delta(x',y',s')\right|\right. \\ &\qquad \left. - \left|u_\eta(x,y,t_n-) - v_\delta(x',y',s')\right|\right]\Omega(x,x',y,y',t_n,s')\,dx\,dy\,ds'\,dx'\,dy' \end{aligned}$$

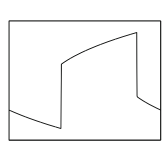


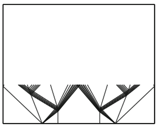


$$
\begin{aligned}
&= \frac{1}{\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(x, x', y, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\left| u_\eta(\bar{x}, \bar{y}, t_n-) - v_\delta(x', y', s') \right| \\
&\qquad - \left| u_\eta(x, y, t_n-) - v_\delta(x', y', s') \right| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy' \\
&= \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(x, x', y, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\left| u_\eta(\bar{x}, \bar{y}, t_n-) - v_\delta(x', y', s') \right| \\
&\qquad - \left| u_\eta(x, y, t_n-) - v_\delta(x', y', s') \right| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy' \\
&\quad + \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\left| u_\eta(x, y, t_n-) - v_\delta(x', y', s') \right| \\
&\qquad - \left| u_\eta(\bar{x}, \bar{y}, t_n-) - v_\delta(x', y', s') \right| \Big) \, dx \, dy \, d\bar{x} \, d\bar{y} \, ds' \, dx' \, dy' \\
&= \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \Big(\Omega(x, x', y, y', t_n, s') - \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \Big) \\
&\qquad \times \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \Big(\left| u_\eta(\bar{x}, \bar{y}, t_n-) - v_\delta(x', y', s') \right| \\
&\qquad - \left| u_\eta(x, y, t_n-) - v_\delta(x', y', s') \right| \Big) \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy'.
\end{aligned}
$$

Estimating $I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)$ using the inverse triangle inequality, we obtain

$$
\begin{aligned}
&\left| I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) \right| \\
&\quad \le \frac{1}{2\Delta x \Delta y} \sum_{n=1}^{N-1} \iint \int_0^T \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \left| u_\eta(\bar{x}, \bar{y}, t_n-) - u_\eta(x, y, t_n-) \right| \\
&\qquad \times \left| \Omega(x, x', y, y', t_n, s') - \Omega(\bar{x}, x', \bar{y}, y', t_n, s') \right| \, d\bar{x} \, d\bar{y} \, dx \, dy \, ds' \, dx' \, dy'.
\end{aligned} \tag{4.59}
$$





The next step is to bound the test functions in (4.59) from above. To this end we first consider, for $x, \bar{x} \in (i\Delta x, (i+1)\Delta x)$,

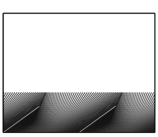
$$\begin{aligned}
\int \left|\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')\right| \, dx' &= \int \left|\omega(z) - \omega(z + (\bar{x}-x)/\varepsilon)\right| \, dz \\
&= \int \left| \int_z^{z+(\bar{x}-x)/\varepsilon} \omega'(\xi)\, d\xi \right| \, dz \\
&\leq \int \int_z^{z+(\bar{x}-x)/\varepsilon} \left|\omega'(\xi)\right| \, d\xi \, dz \\
&\leq \int \int_0^{\Delta x/\varepsilon} \left|\omega'(\alpha+\beta)\right| \, d\alpha \, d\beta = \frac{2\Delta x}{\varepsilon}.
\end{aligned}$$

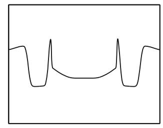
Integrating the time variable to unity, we easily see (really, this is easy!) that

$$\begin{aligned}
&\iint \int_0^T \left|\Omega(x,x',y,y',t_n,s') - \Omega(\bar{x},x',\bar{y},y',t_n,s')\right| \, ds'\, dx' dy' \\
&\quad = \int_0^T \omega_{\varepsilon_0}(s-s')\, ds' \\
&\qquad \times \iint \left|\omega_\varepsilon(x-x')\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{x}-x')\omega_\varepsilon(\bar{y}-y')\right| \, dx' dy' \\
&\quad \leq \iint \left|\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')\right| \omega_\varepsilon(y-y')\, dx' dy' \\
&\qquad + \iint \left|\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{y}-y')\right| \omega_\varepsilon(\bar{x}-x')\, dx' dy' \\
&\quad \leq \int \left|\omega_\varepsilon(x-x') - \omega_\varepsilon(\bar{x}-x')\right| \, dx' + \int \left|\omega_\varepsilon(y-y') - \omega_\varepsilon(\bar{y}-y')\right| \, dy' \\
&\quad \leq (\Delta x + \Delta y)\frac{2}{\varepsilon}.
\end{aligned} \tag{4.60}$$

Furthermore,

$$\begin{aligned}
\left|u_\eta(\bar{x},\bar{y},t_n-) - u_\eta(x,y,t_n-)\right| &= \left|u_\eta(x,\bar{y},t_n-) - u_\eta(x,y,t_n-)\right| \\
&\leq \mathrm{T.V.}_{(j\Delta y,(j+1)\Delta y)}\left(u_\eta(x,\,\cdot\,,t_n-)\right).
\end{aligned} \tag{4.61}$$





Inserting (4.60) and (4.61) into (4.59) yields

$$
\begin{aligned}
&\left|I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \\
&\quad \le \frac{1}{2\Delta x \Delta y} \frac{2(\Delta x + \Delta y)}{\varepsilon} \\
&\qquad \times \sum_{n=1}^{N-1} \sum_{i,j} \iint_{I_{ij}} \iint_{I_{ij}} \mathrm{T.V.}_{(j\Delta y,(j+1)\Delta y)} \left(u_\eta(x,\,\cdot\,,t_n-)\right)\, d\bar{x}\, d\bar{y}\, dx\, dy \\
&\quad \le \frac{\Delta x + \Delta y}{\varepsilon \Delta x \Delta y} \sum_{n=1}^{N-1} \Delta x (\Delta y)^2 \sum_{i,j} \int_{i\Delta x}^{(i+1)\Delta x} \mathrm{T.V.}_{(j\Delta y,(j+1)\Delta y)} \left(u_\eta(x,\,\cdot\,,t_n-)\right)\, dx \\
&\quad \le \frac{(\Delta x + \Delta y)}{\varepsilon} \Delta y \sum_{n=1}^{N-1} \mathrm{T.V.} \left(u_\eta(t_n-)\right) \\
&\quad \le \frac{(\Delta x + \Delta y)}{\varepsilon} \Delta y \frac{T}{\Delta t} \mathrm{T.V.} \left(u_\eta(0)\right), \qquad (4.62)
\end{aligned}
$$

where in the final step we used that $\mathrm{T.V.}\left(u_\eta(t_n-)\right) \le \mathrm{T.V.}\left(u_\eta(0)\right)$.

The same analysis provides the following estimate for $I_4^{\varepsilon,\varepsilon_0}(v_\delta, u_\eta)$:

$$
\left|I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \le \frac{(\Delta x + \Delta y)}{\varepsilon} \Delta x \frac{T}{\Delta t} \mathrm{T.V.} \left(u_\eta(0)\right). \qquad (4.63)
$$

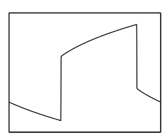
Adding (4.62) and (4.63) proves the lemma. □

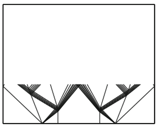
We now return to the proof of the estimate of $\Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)$. Combining Lemma 4.7 and Lemma 4.8, we obtain

$$
\begin{aligned}
-\Lambda_{\varepsilon,\varepsilon_0}(u_\eta, v_\delta) &\le \left|I_1^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_2^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_3^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| + \left|I_4^{\varepsilon,\varepsilon_0}(u_\eta, v_\delta)\right| \\
&\le T\Bigg[\left(\frac{\Delta t}{\varepsilon_0} + \frac{1}{\varepsilon}(\{\|f_\delta\|_{\mathrm{Lip}} + \|g_\delta\|_{\mathrm{Lip}}\}\Delta t + \Delta x + \Delta y)\right) \\
&\qquad \times \max\left\{\|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\right\} + \frac{(\Delta x + \Delta y)^2}{\Delta t\, \varepsilon}\Bigg] \mathrm{T.V.}\,(u_0) \\
&=: T\, \mathrm{T.V.}\,(u_0)\, \Lambda(\varepsilon, \varepsilon_0, \eta). \qquad (4.64)
\end{aligned}
$$

Returning to (4.49), we combine (4.50), (4.51), as well as (4.64), to obtain

$$
\begin{aligned}
&\left\|S(T)u_0 - u_\eta(T)\right\|_{L^1(\mathbb{R}^2)} \\
&\quad \le \|S(T)u_0 - S_\delta(T)u_0\|_{L^1(\mathbb{R}^2)} + \left\|S_\delta(T)u_0 - u_\eta(T)\right\|_{L^1(\mathbb{R}^2)} \\
&\quad \le T \max\{\, \|f - f_\delta\|_{\mathrm{Lip}}, \|g - g_\delta\|_{\mathrm{Lip}}\,\}\mathrm{T.V.}\,(u_0) + \left\|u_0 - u^0\right\|_{L^1(\mathbb{R}^2)} \\
&\qquad + \Big(2\varepsilon + \max\{\, \|f_\delta\|_{\mathrm{Lip}}, \|g_\delta\|_{\mathrm{Lip}}\,\}\varepsilon_0 + \varepsilon_0\Big(C + \mathcal{O}\Big(\frac{\max\{\Delta x, \Delta y\}}{\Delta t}\Big)\Big) \\
&\qquad\qquad + T\, \Lambda(\varepsilon, \varepsilon_0, \eta)\Big)\mathrm{T.V.}\,(u_0). \qquad (4.65)
\end{aligned}
$$





Next we take the minimum over ε and ε_0 on the right-hand side of (4.65). This has the form

$$\min_{\varepsilon,\varepsilon_0}\left(a\,\varepsilon+\frac{b}{\varepsilon}+c\,\varepsilon_0+\frac{d}{\varepsilon_0}\right)=2\sqrt{ab}+2\sqrt{cd}.$$

The minimum is obtained for $\varepsilon=\sqrt{b/a}$ and $\varepsilon_0=\sqrt{d/c}$. We obtain

$$\begin{aligned}
&\left\|S(T)u_0-u_\eta(T)\right\|_{L^1(\mathbb{R}^2)}\\
&\quad\le T\max\left\{\|f-f_\delta\|_{\mathrm{Lip}},\|g-g_\delta\|_{\mathrm{Lip}}\right\}\mathrm{T.V.}\,(u_0)+\left\|u_0-u^0\right\|_{L^1(\mathbb{R}^2)}\\
&\qquad+\mathcal{O}\left(\left((\Delta x+\Delta y)+\Delta t+\frac{(\Delta x+\Delta y)^2}{\Delta t}\right)^{1/2}\right)\mathrm{T.V.}\,(u_0)\,.
\end{aligned}\tag{4.66}$$

We may choose the approximation of the initial data such that $\left\|u_0-u^0\right\|_{L^1(\mathbb{R}^2)}=\mathcal{O}\left(\Delta x+\Delta y\right)\mathrm{T.V.}\,(u_0)$. Furthermore, if the flux functions f and g are piecewise C^2 and Lipschitz continuous, then

$$\|f-f_\delta\|_{\mathrm{Lip}}\le\delta\|f''\|_{L^\infty(\mathbb{R})}.$$

We state the final result in the general case.

Theorem 4.9 *Let u_0 be a function in $L^1(\mathbb{R}^m)\cap L^\infty(\mathbb{R}^m)$ with bounded total variation, and let f_j for $j=1,\ldots,m$ be piecewise C^2 functions that in addition are Lipschitz continuous. Then*

$$\left\|u(T)-u_\eta(T)\right\|_{L^1(\mathbb{R}^m)}\le\mathcal{O}\left(\delta+(\Delta x+\Delta y)^{1/2}\right)$$

as $\eta\to 0$ when

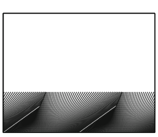
$$\Delta x=K_1\Delta y=K_2\Delta t$$

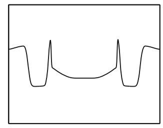
for constants K_1 and K_2.

It is worthwhile to analyze the error terms in the estimate. We are clearly making four approximations with the front-tracking method combined with dimensional splitting. First of all, we are approximating the initial data by step functions. That gives an error of order Δx. Secondly, we are approximating the flux functions by piecewise linear and continuous functions; in this case the error is of order δ. A third source is the intrinsic error in the dimensional splitting, which is of order $(\Delta t)^{1/2}$, and finally, the projection onto the grid gives an error of order $(\Delta x)^{1/2}$.

The advantage of this method over difference methods is the fact that the time step Δt is not bounded by a CFL condition expressed in terms of Δx and Δy. The only relation that must be satisfied is (4.27), which allows for taking large time steps. In practice it is observed that one can choose CFL numbers[4] as high as 10–15 without loss in accuracy. This makes it a very fast method.

[4] In several dimensions the CFL number is defined as $\max_i(\left|f_i'\right|\Delta t/\Delta x_i)$.





4.4 Operator Splitting: Diffusion

The answer, my friend, is blowin' in the wind, the answer is blowin' in the wind.
— Bob Dylan, Blowin' in the Wind (1968)

We show how to use the concept of operator splitting to derive a (weak) solution of the parabolic problem[5] on $\mathbb{R}^m \times [0, T]$,

$$u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = \mu \sum_{j=1}^{m} u_{x_j x_j}, \tag{4.67}$$

by solving

$$u_t + f_j(u)_{x_j} = 0, \quad j = 1, \dots, m, \tag{4.68}$$

and

$$u_t = \mu \Delta u, \tag{4.69}$$

where we employ the notation $\Delta u = \sum_j u_{x_j x_j}$. We augment the equation with initial data $u|_{t=0} = u_0$. Let $S_j(t)u_0$ and $H(t)u_0$ denote the solutions of (4.68) and (4.69), respectively, with initial data u_0. Introducing the heat kernel, we may write

$$\begin{aligned} u(x,t) &= (H(t)u_0)(x,t) \\ &= \int_{\mathbb{R}^m} K(x-y,t)u_0(y)\,dy \\ &= \frac{1}{(4\pi\mu t)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|x-y|^2}{4\mu t}\right) u_0(y)\,dy. \end{aligned}$$

Let Δt be positive and $t_n = n\Delta t$. Define

$$u^0 = u_0, \quad u^{n+1} = \left(H(\Delta t)S_m(\Delta t)\cdots S_1(\Delta t)\right)u^n, \tag{4.70}$$

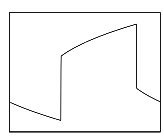
with the idea that u^n approximates $u(x, t_n)$. We will show that u^n converges to the solution of (4.67) as $\Delta t \to 0$.

Lemma 4.10 *The following estimates hold*:

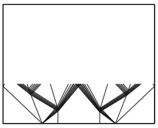
$$\|u^n\|_{L^\infty(\mathbb{R}^m)} \le \left\|u^0\right\|_{L^\infty(\mathbb{R}^m)}, \tag{4.71}$$

$$\mathrm{T.V.}\,(u^n) \le \mathrm{T.V.}\left(u^0\right), \tag{4.72}$$

$$\|u^{n_1} - u^{n_2}\|_{L^1_{\mathrm{loc}}(\mathbb{R}^m)} \le C\left(|n_1 - n_2|\,\Delta t\right)^{1/(m+1)}. \tag{4.73}$$



[5] Although we have used the parabolic regularization to motivate the appropriate entropy condition, we have constructed the solution of the multidimensional conservation law independtly, and hence it is logically consistent to use the solution of the conservation law in combination with operator splitting to derive the solution of the parabolic problem. A different approach, where we start with a solution of the parabolic equation and subsequently show that in the limit of vanishing viscosity the solution converges to the solution of the conservation law, is discussed in Appendix B.



Proof Equation (4.71) is obvious, since both the heat equation and the conservation law obey the maximum principle.

We know that the solution of the conservation law has the TVD property (4.72); see (4.24). Thus it remains to show that this property is shared by the solution of the heat equation. To this end, we have

$$
\begin{aligned}
&\left| H(t)u(x+h) - H(t)u(x) \right| \\
&\qquad = \left| \int_{\mathbb{R}^m} \big(K(x+h-y,t)u(y) - K(x-y,t)u(y) \big)\, dy \right| \\
&\qquad \le \int_{\mathbb{R}^m} \left| K(y,t)u(x+h-y) - K(y,t)u(x-y) \right|\, dy,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \left| H(t)u(x+h) - H(t)u(x) \right| dx \\
&\qquad \le \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| K(y,t)u(x+h-y) - K(y,t)u(x-y) \right|\, dy\, dx \\
&\qquad = \int_{\mathbb{R}^m} K(y,t) \int_{\mathbb{R}^m} \left| u(x+h-y) - u(x-y) \right|\, dx\, dy \\
&\qquad = \int_{\mathbb{R}^m} K(y,t)\, dy \int_{\mathbb{R}^m} \left| u(x+h) - u(x) \right|\, dx \\
&\qquad = \int_{\mathbb{R}^m} \left| u(x+h) - u(x) \right|\, dx.
\end{aligned}
$$

Dividing by $|h|$ and letting $h \to 0$, we conclude that

$$
\mathrm{T.V.}\,(H(t)u) \le \mathrm{T.V.}\,(u)\,,
$$

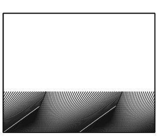
which proves (4.72).

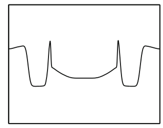
Finally, we consider (4.73). We will first show that the approximate solution obtained by splitting is weakly Lipschitz continuous in time. More precisely, for each ball $\mathcal{B}_r = \{x \mid |x| \le r\}$, we will show that

$$
\left| \int_{\mathcal{B}_r} (u^{n_1} - u^{n_2})\phi \right| \le C_r \left| n_1 - n_2 \right| \Delta t \left(\|\phi\|_\infty + \max_j \|\phi_{x_j}\|_\infty \right), \tag{4.74}
$$

for smooth test functions $\phi = \phi(x)$, where C_r is a constant depending on r. It is enough to study the case $n_2 = n_1 + 1$, and we set $n_1 = n$. Furthermore, we can write

$$
\left| \int (u^{n+1} - u^n)\phi\, dx \right| \le \left| \int (H(\Delta t)\tilde{u}^n - \tilde{u}^n)\phi\, dx \right| + \left| \int (\tilde{u}^n - u^n)\phi\, dx \right|, \tag{4.75}
$$





where $\tilde{u}^n = (S_m(\Delta t) \cdots S_1(\Delta t))\, u^n$. This shows that it suffices to prove this property for the solutions of the conservation law and the heat equation separately. From Theorem 4.3 we know that the solution of the one-dimensional conservation law satisfies the stronger estimate

$$\|S(t)u - u\|_{L^1(\mathbb{R}^m)} \le C\,|t|\,.$$

This implies that (for simplicity with $m = 2$)

$$\begin{aligned}\|S_2(t)S_1(t)u - u\|_{L^1(\mathbb{R}^2)} &\le \|S_2(t)S_1(t)u - S_1(t)u\|_{L^1(\mathbb{R}^2)} + \|S_1(t)u - u\|_{L^1(\mathbb{R}^2)}\\ &\le C\,|t|\,,\end{aligned}$$

and hence we infer that the last term of (4.75) is of order Δt, that is,

$$\|\tilde{u}^n - u^n\|_{L^1(\mathbb{R}^2)} \le C\,\|\phi\|_{L^\infty(\mathbb{R}^2)}\,|\Delta t|\,.$$

The first term can be estimated as follows (for simplicity of notation we assume $m = 1$). Consider

$$\begin{aligned}\left|\int (H(t)u_0 - u_0)\phi\,dx\right| &= \left|\int \int_0^t u_t\,dt\,\phi\,dx\right| = \left|\int \int_0^t u_{xx}\,dt\,\phi\,dx\right|\\ &\le \int \int_0^t |u_x\phi_x|\;dt\,dx\\ &\le \|\phi_x\|_{L^\infty(\mathbb{R})} \int_0^t \int |u_x|\;dx\,dt\\ &\le \|\phi_x\|_{L^\infty(\mathbb{R})} \int_0^t \mathrm{T.V.}\,(u)\;dt \le \|\phi_x\|_{L^\infty(\mathbb{R})}\mathrm{T.V.}\,(u_0)\;t\,.\end{aligned} \tag{4.76}$$

Thus we conclude that (4.74) holds.

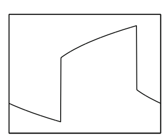
From the TVD property (4.72), we have that

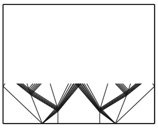
$$\sup_{|\xi|\le\rho} \int |u^n(x+\xi,t) - u^n(x,t)|\;dx \le \rho\,\mathrm{T.V.}\,(u^n)\,. \tag{4.77}$$

Using Kružkov's interpolation lemma (stated and proved right after this proof) we can infer, using (4.74) and (4.77), that

$$\int_{\mathcal{B}_r} |u^{n_1}(x) - u^{n_2}(x)|\;dx \le C_r\left(\varepsilon + \frac{|n_1 - n_2|\,\Delta t}{\varepsilon}\right)$$

for all $\varepsilon \le \rho$. Choosing $\varepsilon = \sqrt{|n_1 - n_2|\,\Delta t}$ proves the result. □





We next state and prove Kružkov's interpolation lemma. It will be convenient to use the multi-index notation. A vector of the form $\alpha = (\alpha_1, \dots, \alpha_m)$, where each component is a nonnegative integer, is called a *multi-index* of order $|\alpha| = \alpha_1 + \dots + \alpha_m$. Given a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.$$

Lemma 4.11 (Kružkov interpolation lemma) *Let $u(x,t)$ be a bounded measurable function defined in the cylinder $\mathcal{B}_{r+\hat{r}} \times [0,T]$, $\hat{r} \ge 0$. For $t \in [0,T]$ and $|\rho| \le \hat{r}$, assume that u possesses a spatial modulus of continuity*

$$\sup_{|\xi| \le |\rho|} \int_{\mathcal{B}_r} |u(x+\xi,t) - u(x,t)| \, dx \le \nu_{r,T,\hat{r}}(|\rho|; u), \tag{4.78}$$

where $\nu_{r,T,\hat{r}}$ does not depend on t. Suppose that for every $\phi \in C_0^\infty(\mathcal{B}_r)$ and $t_1, t_2 \in [0,T]$,

$$\left| \int_{\mathcal{B}_r} \big(u(x,t_2) - u\big(x,t_1\big)\big) \phi(x) \, dx \right| \le \mathrm{Const}_{r,T} \left(\sum_{|\alpha| \le m} \| D^\alpha \phi \|_{L^\infty(\mathcal{B}_r)} \right) |t_2 - t_1|, \tag{4.79}$$

where α denotes a multi-index.

Then for t and $t+\tau \in [0,T]$ and for all $\varepsilon \in (0,\hat{r}]$,

$$\int_{\mathcal{B}_r} |u(x,t+\tau) - u(x,t)| \, dx \le \mathrm{Const}_{r,T} \left(\varepsilon + \nu_{r,T,\hat{r}}(\varepsilon; u) + \frac{|\tau|}{\varepsilon^m} \right). \tag{4.80}$$

Proof Let $\delta \in C_0^\infty$ be a function such that

$$0 \le \delta(x) \le 1, \quad \operatorname{supp} \delta \subseteq \mathcal{B}_1, \quad \int \delta(x) \, dx = 1,$$

and define

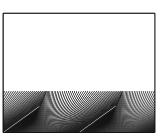
$$\delta_\varepsilon(x) = \frac{1}{\varepsilon^m} \delta\Big(\frac{x}{\varepsilon}\Big).$$

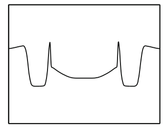
Furthermore, write $f(x) = u(x,t+\tau) - u(x,t)$ (suppressing the time dependence in the notation for f),

$$\sigma(x) = \operatorname{sign}(f(x)) \text{ for } |x| \le r - \varepsilon, \text{ and } 0 \text{ otherwise},$$

and

$$\sigma_\varepsilon(x) = (\sigma * \delta_\varepsilon)(x) = \int \sigma(x-y) \delta_\varepsilon(y) \, dy.$$





By construction, $\sigma_\varepsilon \in C_0^\infty(\mathbb{R}^m)$ and $\operatorname{supp}\sigma_\varepsilon \subseteq \mathcal{B}_r$. Furthermore, $|\sigma_\varepsilon| \le 1$ and

$$\begin{aligned}\left|\frac{\partial}{\partial x_j}\sigma_\varepsilon\right| &\le \frac{1}{\varepsilon^m}\int \left|\frac{\partial}{\partial x_j}\delta(\frac{x-y}{\varepsilon})\right| \sigma(y)\,dy \\ &\le \frac{1}{\varepsilon^{m+1}}\int \left|\delta_{x_j}(\frac{x-y}{\varepsilon})\right| \sigma(y)\,dy \le \frac{C}{\varepsilon}.\end{aligned}$$

This easily generalizes to

$$\|D^\alpha \sigma_\varepsilon\|_{L^\infty(\mathbb{R}^m)} \le \frac{C}{\varepsilon^{|\alpha|}}.$$

Next we have the elementary but important inequality

$$\begin{aligned}\int_{\mathcal{B}_r} |f(x)|\,dx &= \left|\int_{\mathcal{B}_r} |f(x)|\,dx\right| \\ &= \left|\int_{\mathcal{B}_r} (|f(x)| - \sigma_\varepsilon(x) f(x) + \sigma_\varepsilon(x) f(x))\,dx\right| \\ &\le \left|\int_{\mathcal{B}_r} (|f(x)| - \sigma_\varepsilon(x) f(x))\,dx\right| + \left|\int_{\mathcal{B}_r} \sigma_\varepsilon(x) f(x)\,dx\right| \\ &\le \int_{\mathcal{B}_r} \big||f(x)| - \sigma_\varepsilon(x) f(x)\big|\,dx + \left|\int_{\mathcal{B}_r} \sigma_\varepsilon(x) f(x)\,dx\right| \\ &=: I_1 + I_2.\end{aligned}$$

We estimate I_1 and I_2 separately. Starting with I_1, we obtain

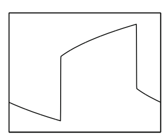
$$\begin{aligned}I_1 &= \int_{\mathcal{B}_r} \big||f(x)| - \sigma_\varepsilon(x) f(x)\big|\,dx \\ &= \int_{\mathcal{B}_r} \left||f(x)|\frac{1}{\varepsilon^m}\int \delta(\frac{x-y}{\varepsilon})\,dy - \frac{1}{\varepsilon^m}\int \delta(\frac{x-y}{\varepsilon})\sigma(y)\,dy\ f(x)\right|\,dx \\ &= \frac{1}{\varepsilon^m}\int\int \delta(\frac{x-y}{\varepsilon})\big||f(x)| - \sigma(y) f(x)\big|\,dy\,dx.\end{aligned}$$

The integrand is integrated over the domain

$$\{(x,y) \mid |x| \le r,\ |x-y| \le \varepsilon\}.$$

We further divide this set into two parts: (i) $|y| \ge r - \varepsilon$, and (ii) $|y| \le r - \varepsilon$; see Fig. 4.2. In case (i) we have

$$\big||f(x)| - \sigma(y) f(x)\big| = |f(x)|,$$



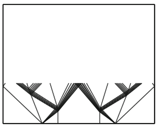
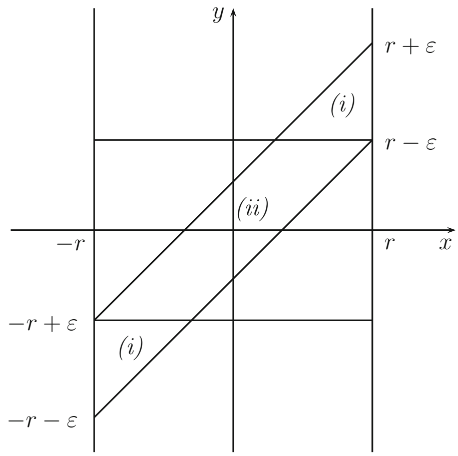


Fig. 4.2 The integration domain



since $\sigma(y) = 0$ whenever $|y| \geq r - \varepsilon$. In case (ii) we have

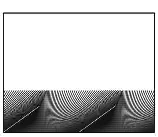
$$\left| |f(x)| - \sigma(y) f(x) \right| = \left| |f(x)| - \operatorname{sign}(f(y))\, f(x) \right| \leq 2 \left| f(x) - f(y) \right|,$$

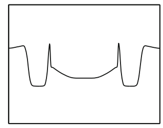
using the elementary inequality

$$\begin{aligned} \left| |a| - \operatorname{sign}(b)\, a \right| &= \left| |a| - |b| + \operatorname{sign}(b)\,(b-a) \right| \\ &\leq \left| |a| - |b| \right| + \left| \operatorname{sign}(b)\,(b-a) \right| \\ &\leq 2\,|a-b|. \end{aligned}$$

Thus

$$\begin{aligned} I_1 &\leq \frac{2}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{\mathcal{B}_{r-\varepsilon}} \delta(\frac{x-y}{\varepsilon})\, |f(x) - f(y)|\; dy\, dx \\ &\quad + \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{|y| \geq r-\varepsilon} \delta(\frac{x-y}{\varepsilon})\, |f(x)|\; dy\, dx \\ &\leq 2 \int\limits_{\mathcal{B}_r} \int\limits_{\mathcal{B}_1} \delta(z)\, |f(x) - f(x - \varepsilon z)|\; dz\; dx \\ &\quad + \|f\|_\infty \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \int\limits_{|y| \geq r-\varepsilon} \delta(\frac{x-y}{\varepsilon})\, dy\, dx \\ &\leq 2 \int\limits_{\mathcal{B}_1} \delta(z) \sup_{|\xi| \leq \varepsilon} \int\limits_{\mathcal{B}_r} |f(x) - f(x+\xi)|\; dx\, dz \\ &\quad + \|f\|_{L^\infty(\mathbb{R}^m)} \int\limits_{\mathcal{B}_{r+\varepsilon} \setminus \mathcal{B}_{r-\varepsilon}} \frac{1}{\varepsilon^m} \int\limits_{\mathcal{B}_r} \delta(\frac{x-y}{\varepsilon})\, dx\, dy \\ &\leq 2\nu(\varepsilon; f) + \|f\|_{L^\infty(\mathbb{R}^m)} \operatorname{vol}(\mathcal{B}_{r+\varepsilon} \setminus \mathcal{B}_{r-\varepsilon}) \\ &\leq 2\nu(\varepsilon; f) + \|f\|_{L^\infty(\mathbb{R}^m)} C_r \varepsilon. \end{aligned}$$





Furthermore,

$$\nu(\varepsilon; f) \leq 2\nu(\varepsilon; u).$$

The second term I_2 is estimated by the assumptions of the lemma, namely,

$$I_2 = \left| \int_{\mathcal{B}_r} \sigma_\varepsilon(x) f(x)\, dx \right| \leq \mathrm{Const}_{r,T} \left(\sum_{|\alpha| \leq m} \|D^\alpha \sigma_\varepsilon\|_{L^\infty(\mathcal{B}_r)} \right) |\tau| \leq C\, \frac{|\tau|}{\varepsilon^m}.$$

Combining the two estimates, we conclude that

$$\int_{\mathcal{B}_r} |u(x, t+\tau) - u(x,t)|\ dx \leq C_r \left(\varepsilon + \nu_{r,T,\hat{r}}(\varepsilon; u) + \frac{|\tau|}{\varepsilon^m} \right). \qquad \square$$

Next we need to extend the function u^n to all times. First, define

$$u^{n+j/(m+1)} = S_j u^{n+(j-1)/(m+1)}, \quad j = 1, \ldots, m.$$

Now let

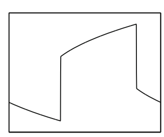
$$u_{\Delta t}(x,t) = \begin{cases} S_j((m+1)(t - t_{n+(j-1)/(m+1)}))u^{n+(j-1)/(m+1)} \\ \qquad\qquad \text{for } t \in [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)}), \\ H((m+1)(t - t_{n+m/(m+1)}))u^{n+m/(m+1)} \\ \qquad\qquad \text{for } t \in [t_{n+m/(m+1)}, t_{n+1}). \end{cases} \tag{4.81}$$

The estimates in Lemma 4.10 carry over to the function $u_{\Delta t}$. Fix $T > 0$. Applying Theorem A.11, we conclude that there exists a sequence of $\Delta t \to 0$ such that for each $t \in [0,T]$, the function $u_{\Delta t}(t)$ converges to a function $u(t)$, and the convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$. It remains to show that u is a weak solution of (4.67), or

$$\int_{\mathbb{R}^m} \int_0^t (u\phi_t + f(u)\cdot\nabla\phi + \nu u \Delta\phi)\ dt\, dx + \int_{\mathbb{R}^m} u_0 \phi|_{t=0}\, dx = \int_{\mathbb{R}^m} (u\phi)|_{t=T}\, dx \tag{4.82}$$

for all smooth and compactly supported test functions ϕ. We have

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_{t_{n+(j-1)/(m+1)}}^{t_{n+j/(m+1)}} \left(\frac{1}{m+1} u_{\Delta t}\, \phi_t + f(u_{\Delta t}) \cdot \nabla\phi \right) dt\, dx \\ &\quad = \frac{1}{m+1} \int_{\mathbb{R}^m} \int_0^{\Delta t} \Bigg(u^{n+(j-1)/(m+1)}(x, \tilde{t})\, \phi_t\left(x, \frac{\tilde{t} - t_{n+(j-1)/(m+1)}}{m+1}\right) \\ &\qquad\qquad + f(u^{n+(j-1)/(m+1)}) \cdot \nabla\phi\left(x, \frac{\tilde{t} - t_{n+(j-1)/(m+1)}}{m+1}\right)\Bigg)\, d\tilde{t}\, dx \\ &\quad = \frac{1}{m+1} \int_{\mathbb{R}^m} (u_{\Delta t}\phi)\, \Big|_{t=t_{n+(j-1)/(m+1)}}^{t=t_{n+j/(m+1)}}\ dx, \end{aligned} \tag{4.83}$$



for $j = 1, \dots, m$, where we have used that $u^{n+(j-1)/(m+1)}$ is a solution of the conservation law on the strip $t \in [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)})$. Similarly, we find for the solution of the heat equation that

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \int\limits_{t_{n+m/(m+1)}}^{t_{n+1}} \left(\frac{1}{m+1} u_{\Delta t}\phi_t + \mu u_{\Delta t}\Delta\phi\right) dt\, dx \\ = \frac{1}{m+1}\int\limits_{\mathbb{R}^m} \left((u_{\Delta t}\phi)\,|_{t=t_{n+m/(m+1)}} - (u_{\Delta t}\phi)\,|_{t=t_{n+1}}\right) dx.\end{aligned} \tag{4.84}$$

Summing (4.83) for $j = 1, \dots, m$, and adding the result to (4.84), we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \int\limits_{0}^{t} \left(\frac{1}{m+1} u_{\Delta t}\phi_t + f_{\Delta t}(u_{\Delta t}) \cdot \nabla\phi + \mu\chi_{m+1} u_{\Delta t}\Delta\phi\right) dt\, dx \\ + \frac{1}{m+1}\int\limits_{\mathbb{R}^m} u_0\phi|_{t=0}\, dx = \frac{1}{m+1}\int\limits_{\mathbb{R}^m} (u_{\Delta t}\phi)|_{t=T}\, dx,\end{aligned} \tag{4.85}$$

where

$$f_{\Delta t} = (\chi_1 f_1, \dots, \chi_m f_m)$$

and

$$\chi_j = \begin{cases} 1 & \text{for } t \in \cup_n [t_{n+(j-1)/(m+1)}, t_{n+j/(m+1)}), \\ 0 & \text{otherwise.} \end{cases}$$

As $\Delta t \to 0$, we have $\chi_j \overset{*}{\rightharpoonup} 1/(m+1)$, which proves (4.82). We summarize the result as follows.

Theorem 4.12 *Let u_0 be a function in $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$, and assume that f_j are piecewise twice continuously differentiable functions for $j = 1, \dots, m$. Define the family of functions $\{u_{\Delta t}\}$ by (4.70) and (4.81). Fix $T > 0$. Then there exists a sequence of $\Delta t \to 0$ such that $\{u_{\Delta t}(t)\}$ converges to a weak solution u of (4.67). The convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$.*

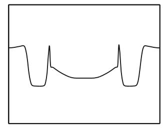
One can prove that a weak solution of (4.67) is indeed a classical solution; see [147]. Hence, by uniqueness of classical solutions, the sequence $\{u_{\Delta t}\}$ converges for *every* sequence $\{\Delta t\}$ tending to zero.

4.5 Operator Splitting: Source

Experience must be our only guide; Reason may mislead us.
— J. Dickinson, the Constitutional Convention (1787)

We will use operator splitting to study the inhomogeneous conservation law

$$u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = g(x,t,u), \quad u|_{t=0} = u_0, \tag{4.86}$$



where the source term g is assumed to be continuous in (x, t) and Lipschitz continuous in u. In this case the Kružkov entropy condition reads as follows. The bounded function u is a weak entropy solution on $[0, T]$ if it satisfies

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} \Big(|u-k|\,\varphi_t + \operatorname{sign}(u-k) \sum_{j=1}^m \big(f_j(u) - f_j(k)\big)\,\varphi_{x_j} \Big)\, dx_1 \cdots dx_m\, dt \\
&\qquad + \int_{\mathbb{R}^m} |u_0 - k|\,\varphi|_{t=0}\, dx_1 \cdots dx_m - \int_{\mathbb{R}^m} (|u-k|\,\varphi)|_{t=T}\, dx_1 \cdots dx_m \\
&\qquad \geq - \int_0^T \int_{\mathbb{R}^m} \operatorname{sign}(u-k)\,\varphi g(x,t,u)\, dx_1 \cdots dx_m\, dt,
\end{aligned}
\tag{4.87}
$$

for all constants $k \in \mathbb{R}$ and all nonnegative test functions $\varphi \in C_0^\infty(\mathbb{R}^m \times [0, T])$.

To simplify the presentation we consider only the case with $m = 1$, and where $g = g(u)$. Thus

$$
u_t + f(u)_x = g(u). \tag{4.88}
$$

The case in which g also depends on (x, t) is treated in Exercise 4.7. Let $S(t)u_0$ and $R(t)u_0$ denote the solutions of

$$
u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{4.89}
$$

and

$$
u_t = g(u), \quad u|_{t=0} = u_0, \tag{4.90}
$$

respectively. Define the sequence $\{u^n\}$ by (we still use $t_n = n\Delta t$)

$$
u^0 = u_0, \quad u^{n+1} = (S(\Delta t)R(\Delta t))\, u^n
$$

for some positive Δt. Furthermore, we need the extension to all times, defined by[6]

$$
u_{\Delta t}(x, t) = \begin{cases} S(2(t - t_n))u^n & \text{for } t \in [t_n, t_{n+1/2}), \\ R\left(2\left(t - t_{n+1/2}\right)\right) u^{n+1/2} & \text{for } t \in [t_{n+1/2}, t_{n+1}), \end{cases} \tag{4.91}
$$

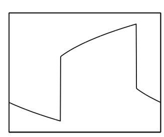
with

$$
u^{n+1/2} = S(\Delta t)u^n, \quad t_{n+1/2} = \left(n + \frac{1}{2}\right)\Delta t.
$$

For this procedure to be welldefined, we must be sure that the ordinary differential equation (4.90) is welldefined. This is the case if g is uniformly Lipschitz continuous in u, i.e.,

$$
|g(u) - g(v)| \leq \|g\|_{\mathrm{Lip}}\, |u - v|\,. \tag{4.92}
$$

[6] Essentially replacing the operator H used in operator splitting with respect to diffusion by R in the case of a source.



For convenience, we set $\gamma = \|g\|_{\text{Lip}}$. This assumption also implies that the solution of (4.90) does not “blow up” in finite time, since

$$|g(u)| \leq |g(0)| + \gamma\, |u| \leq C_g(1 + |u|), \tag{4.93}$$

for some constant C_g. Under this assumption on g we have the following lemma.

Lemma 4.13 *Assume that u_0 is a function in $L^1_{\text{loc}}(\mathbb{R})$, and that u_0 is of bounded variation. Then for $n\Delta t \leq T$, the following estimates hold*:

(i) *There is a constant M_1 independent of n and Δt such that*

$$\|u^n\|_{L^\infty(\mathbb{R})} \leq M_1. \tag{4.94}$$

(ii) *There is a constant M_2 independent of n and Δt such that*

$$\text{T.V.}\,(u^n) \leq M_2. \tag{4.95}$$

(iii) *There is a constant M_3 independent of n and Δt such that for t_1 and t_2, with $0 \leq t_1 \leq t_2 \leq T$, and for each bounded interval $B \subset \mathbb{R}$,*

$$\int_B |u_{\Delta t}(x, t_1) - u_{\Delta t}(x, t_2)|\; dx \leq M_3\, |t_1 - t_2|\,. \tag{4.96}$$

Proof We start by proving (i). The solution operator S_t obeys a maximum principle, so that $\left\|u^{n+1/2}\right\|_\infty \leq \|u^n\|_\infty$. Multiplying (4.90) by sign (u), we find that

$$|u|_t = \text{sign}\,(u)\, g(u) \leq |g(u)| \leq C_g(1 + |u|),$$

where we have used (4.93). By Gronwall’s inequality (see Exercise 1.10), for a solution of (4.90), we have that

$$|u(t)| \leq e^{C_g t}(1 + |u_0|) - 1.$$

This means that

$$\begin{aligned}
\left\|u^{n+1}\right\|_{L^\infty(\mathbb{R})} &\leq e^{C_g \Delta t}\left(1 + \left\|u^{n+1/2}\right\|_{L^\infty(\mathbb{R})}\right) - 1 \\
&\leq e^{C_g \Delta t}\left(1 + \|u^n\|_{L^\infty(\mathbb{R})}\right) - 1,
\end{aligned}$$

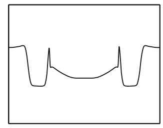
which by induction implies

$$\|u^n\|_{L^\infty(\mathbb{R})} \leq e^{C_g t_n}\left(1 + \|u_0\|_{L^\infty(\mathbb{R})}\right) - 1.$$

Setting

$$M_1 = e^{C_g T}\left(1 + \|u_0\|_{L^\infty(\mathbb{R})}\right) - 1$$

proves (i).



Next, we prove (ii). The proof is similar to that of the last case, since S_t is TVD, $\text{T.V.}\left(u^{n+1/2}\right) \le \text{T.V.}\left(u^n\right)$. As before, let u be a solution of (4.90) and let v be another solution with initial data v_0. Then we have $(u-v)_t = g(u) - g(v)$. Setting $w = u - v$, and multiplying by $\operatorname{sign}(w)$, we find that

$$|w|_t = \operatorname{sign}(w)\,(g(u) - g(v)) \le \gamma\,|w|\,.$$

Then by Gronwall's inequality,

$$|w(t)| \le e^{\gamma t}\,|w(0)|\,.$$

Hence,

$$\left|u^{n+1}(x) - u^{n+1}(y)\right| \le e^{\gamma \Delta t}\left|u^{n+1/2}(x) - u^{n+1/2}(y)\right|.$$

This implies that

$$\text{T.V.}\left(u^{n+1}\right) \le e^{\gamma\Delta t}\,\text{T.V.}\left(u^{n+1/2}\right) \le e^{\gamma\Delta t}\,\text{T.V.}\left(u^{n}\right).$$

Inductively, we then have that

$$\text{T.V.}\left(u^{n}\right) \le e^{\gamma t_n}\,\text{T.V.}\left(u_0\right),$$

and setting $M_2 = e^{\gamma T}$ concludes the proof of (ii).

Regarding (iii), we know that

$$\int_B \left|u^{n+1/2}(x) - u^n(x)\right|\,dx \le C\,\Delta t.$$

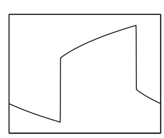
We also have that

$$\begin{aligned}
\int_B \left|u^{n+1}(x) - u^{n+1/2}(x)\right|\,dx &= \int_B \left|\int_0^{\Delta t} g\left(u_{\Delta t}(x, t - t_n)\right)\,dt\right|\,dx \\
&\le \int_B \int_0^{\Delta t} \left|g\left(u_{\Delta t}(x, t - t_n)\right)\right|\,dt\,dx \\
&\le C_g \int_0^{\Delta t}\int_B (1 + M_1)\,dx\,dt \\
&= |B|\,C_g(1 + M_1)\Delta t,
\end{aligned}$$

where $|B|$ denotes the length of B. Setting $M_3 = C + |B|\,C_g(1 + M_1)$ shows that

$$\int_B \left|u^{n+1}(x) - u^n(x)\right| \le M_3\Delta t,$$

which implies (iii). □



Fix $T>0$. Theorem A.11 implies the existence of a sequence $\Delta t\to 0$ such that for each $t\in[0,T]$, the function $u_{\Delta t}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to a bounded function of bounded variation $u(t)$. The convergence is in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^m))$. It remains to show that u solves (4.88) in the sense of (4.87).

Using that $u_{\Delta t}$ is an entropy solution of the conservation law without source term (4.89) in the interval $[t_n,t_{n+1/2}]$, we obtain[7]

$$2\int_{t_n}^{t_{n+1/2}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t+\operatorname{sign}\left(u_{\Delta t}-k\right)\left(f(u_{\Delta t})-f(k)\right)\varphi_x\right)dx\,dt \\ +\int\left(\left|u_{\Delta t}-k\right|\varphi\right)\Big|_{t=t_{n+1/2}}^{t=t_n}dx\geq 0. \tag{4.97}$$

Regarding solutions of (4.90), since $k_t=0$ for every constant k, we find that

$$\left|u-k\right|_t=\operatorname{sign}\left(u-k\right)\left(u-k\right)_t=\operatorname{sign}\left(u-k\right)g(u).$$

Multiplying this by a test function $\phi(t)$ and integrating over $s\in[0,t]$, we find after a partial integration that

$$\int_0^t\left(\left|u-k\right|\phi_s+\operatorname{sign}\left(u-k\right)g(u)\phi\right)ds+u\phi|_{s=0}^{s=t}=0.$$

Since $u_{\Delta t}$ is a solution of the ordinary differential equation (4.90) on the interval $[t_{n+1/2},t_{n+1}]$ (with time running "twice as fast"; see (4.91)), we find that

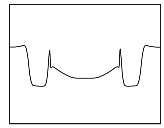
$$2\int_{t_n}^{t_{n+1/2}}\int\left(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t+\operatorname{sign}\left(u_{\Delta t}-k\right)g(u_{\Delta t})\varphi\right)dx\,dt \\ +\int\left(\left|u_{\Delta t}-k\right|\varphi\right)\Big|_{t=t_{n+1}}^{t=t_{n+1/2}}dx=0.$$

Adding this and (4.97), and summing over n, we obtain

$$2\int_0^T\int\Big(\frac{1}{2}\left|u_{\Delta t}-k\right|\varphi_t+\chi_{\Delta t}\operatorname{sign}\left(u_{\Delta t}-k\right)\left(f(u_{\Delta t})-f(k)\right)\varphi_x \\ +\tilde{\chi}_{\Delta t}\operatorname{sign}\left(u_{\Delta t}-k\right)g(u_{\Delta t})\varphi\Big)\,dx\,dt \\ -\int\left(\left|u_{\Delta t}-k\right|\varphi\right)|_{t=0}^{t=T}dx\geq 0,$$

where $\chi_{\Delta t}$ and $\tilde{\chi}_{\Delta t}$ denote the characteristic functions of the sets $\cup_n[t_n,t_{n+1/2})$ and $\cup_n[t_{n+1/2},t_{n+1})$, respectively. We have that $\chi_{\Delta t}\overset{*}{\rightharpoonup}\frac{1}{2}$ and $\tilde{\chi}_{\Delta t}\overset{*}{\rightharpoonup}\frac{1}{2}$, and hence we conclude that (4.87) holds in the limit as $\Delta t\to 0$.

[7] The constants 2 and $\frac{1}{2}$ come from the fact that time is running "twice as fast" in the solution operators S and R in (4.91) (cf. also (4.16)–(4.17)).



Theorem 4.14 *Let $f(u)$ be piecewise twice continuously differentiable, and assume that $g = g(u)$ satisfies the bound (4.92). Let u_0 be a bounded function of bounded variation. Then the initial value problem*

$$u_t + f(u)_x = g(u), \qquad u(x,0) = u_0(x) \tag{4.98}$$

has a weak entropy solution, which can be constructed as the limit of the sequence $\{u_{\Delta t}\}$ defined by (4.91).

4.6 Notes

Dimensional splitting for hyperbolic equations was first introduced by Bagrinovskiĭ and Godunov [7] in 1957. Crandall and Majda made a comprehensive and systematic study of dimensional splitting (or the fractional steps method) in [52]. In [53] they used dimensional splitting to prove convergence of monotone schemes as well as the Lax–Wendroff scheme and the Glimm scheme, i.e., the random choice method. A more general introduction to operator splitting can be found in [91].

There are also methods for multidimensional conservation laws that are intrinsically multidimensional. However, we have here decided to use dimensional splitting as our technique because it is conceptually simple and allows us to take advantage of the one-dimensional analysis.

Another natural approach to the study of multidimensional equations based on the front-tracking concept is first to make the standard front-tracking approximation: Replace the initial data by a piecewise constant function, and replace flux functions by piecewise linear and continuous functions. That gives rise to truly two-dimensional Riemann problems at each grid point $(i\Delta x, j\Delta y)$. However, that approach has turned out to be rather cumbersome even for a single Riemann problem and piecewise linear and continuous flux functions f and g. See Risebro [159].

The one-dimensional front-tracking approach combined with dimensional splitting was first introduced in Holden and Risebro [93]. The theorem on the convergence rate of dimensional splitting was proved independently by Teng [178] and Karlsen [105, 106]. Our presentation here follows Haugse, Lie, and Karlsen [133]. Sect. 4.4, using operator splitting to solve the parabolic regularization, is taken from Karlsen and Risebro [108]. The Kružkov interpolation lemma, Lemma 4.11, is taken from [117]; see also [108].

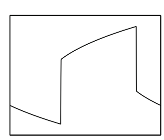
The presentation in Sect. 4.5 can be found in Holden and Risebro [95], where also the case with a stochastic source is treated. The convergence rate in the case of operator splitting applied to a conservation law with a source term is discussed in Langseth, Tveito, and Winther [123].

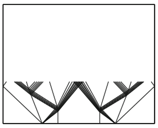
4.7 Exercises

4.1 Consider the initial value problem

$$u_t + f(u)_x + g(u)_y = 0, \quad u|_{t=0} = u_0,$$

where f, g are piecewise twice continuously differentiable functions, and u_0 is a bounded integrable function with finite total variation.





(a) Show that the solution u is Lipschitz continuous in time; that is,

$$\|u(t) - u(s)\|_{L^1(\mathbb{R}^2)} \leq |t - s| \left(\|f\|_{\mathrm{Lip}} \vee \|g\|_{\mathrm{Lip}}\right) \mathrm{T.V.}\,(u_0)\,.$$

(b) Let v_0 be another function with the same properties as u_0. Show that if $u_0 \leq v_0$, then also $u \leq v$ almost everywhere, where v is the solution with initial data v_0.

4.2 Consider the initial value problem

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{4.99}$$

where f is a piecewise twice continuously differentiable function and u_0 is a bounded, integrable function with finite total variation. Write

$$f = f_1 + f_2$$

and let $S_j(t)u_0$ denote the solution of

$$u_t + f_j(u)_x = 0, \quad u|_{t=0} = u_0.$$

Prove that operator splitting converges to the solution of (4.99). Determine the convergence rate.

4.3 Prove (4.38), that is, that

$$\iint |\pi\psi - \psi| \, dx\, dy \leq (\Delta x + \Delta y)\, \mathrm{T.V.}\,(\psi)\,,$$

for all functions ψ of bounded variation.

4.4 Consider the heat equation in $\mathbb{R}^m$,

$$u_t = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}, \qquad u(x,0) = u_0(x). \tag{4.100}$$

Let H_t^i denote the solution operator for the heat equation in the ith direction, i.e., we write the solution of

$$u_t = \frac{\partial^2 u}{\partial x_i^2}, \quad u(x,0) = u_0(x),$$

as $H_t^i u_0$. Define

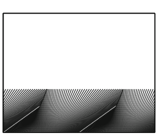
$$u^n(x) = \left[H_{\Delta t}^m \circ \cdots \circ H_{\Delta t}^1\right]^n u_0(x),$$
$$u^{n+j/m}(x) = H_{\Delta t}^j \circ H_{\Delta t}^{j-1} \circ \cdots \circ H_{\Delta t}^1 u^n(x),$$

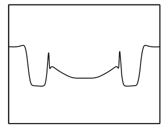
for $j = 1, \ldots, m$, and $n \geq 0$.

For t in the interval $[t_n + ((j-1)/m)\Delta t, t_n + (j/m)\Delta t]$ define

$$u_{\Delta t}(x,t) = H^j_{m(t - t_{n+(j-1)/m})} u^{n+(j-1)/m}(x).$$

If the initial function $u_0(x)$ is bounded and of bounded variation, show that $\{u_{\Delta t}\}$ converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$ to a weak solution of (4.100).





4.5 We consider the viscous conservation law in one space dimension,

$$u_t + f(u)_x = u_{xx}, \quad u(x,0) = u_0(x), \tag{4.101}$$

where f satisfies the "usual" assumptions and u_0 is in $L^1 \cap BV$. Consider the following scheme based on operator splitting:

$$\begin{aligned} U_j^{n+1/2} &= \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \lambda\left(f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right), \\ U_j^{n+1} &= U_j^{n+1/2} + \mu\left(U_{j+1}^{n+1/2} - 2U_j^{n+1/2} + U_{j-1}^{n+1/2}\right), \end{aligned}$$

for $n \geq 0$, where $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/\Delta x^2$. Set

$$U_j^0 = \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_0(x)\, dx.$$

We see that we use the Lax–Friedrichs scheme for the conservation law and an explicit difference scheme for the heat equation. Let

$$u_{\Delta t}(x,t) = U_j^n$$

for $\left(j - \frac{1}{2}\right)\Delta x \leq x < \left(j + \frac{1}{2}\right)\Delta x$ and $n\Delta t < t \leq (n+1)\Delta t$.

(a) Show that this gives a monotone and consistent scheme, provided that a CFL condition holds.
(b) Show that there is a sequence of Δt's such that $u_{\Delta t}$ converges to a weak solution of (4.101) as $\Delta t \to 0$.

(a) Assume that u, f, and g are in $L^1([0,T])$, and that g is nonnegative, while f is strictly positive and nondecreasing. Assume that

$$u(t) \leq f(t) + \int_0^t g(s)u(s)\, ds, \quad t \in [0,T].$$

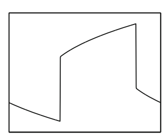
Show that

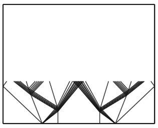
$$u(t) \leq f(t) \exp\left(\int_0^t g(s)\, ds\right), \quad t \in [0,T].$$

4.6 Assume that u and v are entropy solutions of

$$\begin{aligned} u_t + f(u)_x &= g(u), \quad u(x,0) = u_0(x), \\ v_t + f(v)_x &= g(v), \quad v(x,0) = v_0(x), \end{aligned}$$

where u_0 and v_0 are in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and f and g satisfy the assumptions of Theorem 4.14.





(a) Use the entropy formulation (4.87) and mimic the arguments used to prove (2.60) to show that for every nonnegative test function ψ,

$$\begin{aligned}\iint \big(|u(x,t) - v(x,t)|\, \psi_t + q(u,v)\psi_x \big)\, dt\, dx \\ - \int |u(x,T) - v(x,T)|\, \psi(x,T)\, dx \\ + \int |u_0(x) - v_0(x)|\, \psi(x,0)\, dx \\ \geq \iint \operatorname{sign}(u-v)\,(g(u) - g(v))\psi\, dt\, dx.\end{aligned}$$

(b) Define $\psi(x,t)$ by (2.61), and set

$$h(t) = \int |u(x,t) - v(x,t)|\, \psi(x,t)\, dx.$$

Show that

$$h(T) \leq h(0) + \gamma \int_0^T h(t)\, dt,$$

where γ denotes the Lipschitz constant of g. Use the previous exercise to conclude that

$$h(T) \leq h(0)\left(1 + \gamma T e^{\gamma T}\right).$$

(c) Show that

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}\left(1 + \gamma t e^{\gamma t}\right),$$

and hence that entropy solutions of (4.98) are unique. Note that this implies that $\{u_{\Delta t}\}$ defined by (4.91) converges to the entropy solution for *every* sequence $\{\Delta t\}$ such that $\Delta t \to 0$.

4.7 We consider the case that the source depends on (x,t). For $u_0 \in L^1_{\mathrm{loc}} \cap BV$, let u be an entropy solution of

$$u_t + f(u)_x = g(x,t,u), \quad u(x,0) = u_0(x), \tag{4.102}$$

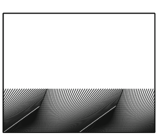
where g is bounded for each fixed u and continuous in t, and satisfies

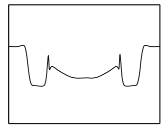
$$\begin{gathered}|g(x,t,u) - g(x,t,v)| \leq \gamma\, |u - v|, \\ \mathrm{T.V.}\,(g(\,\cdot\,,t,u)) \leq b(t),\end{gathered}$$

where the constant γ is independent of x and t, for all u and v and for a bounded function $b(t)$ in $L^1([0,T])$. We let S_t be as before, and let $R(x,t,s)u_0$ denote the solution of

$$u'(t) = g(x,t,u), \quad u(s) = u_0,$$

for $t > s$.



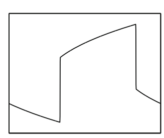


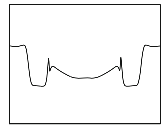
(a) Define an operator splitting approximation $u_{\Delta t}$ using S_t and $R(x,t,s)$.
(b) Show that there is a sequence of Δt's such that $u_{\Delta t}$ converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ to a function of bounded variation u.
(c) Show that u is an entropy solution of (4.102).

4.8 Show that if the initial data u_0 of the heat equation $u_t = \Delta u$ is smooth, that is, $u_0 \in C_0^\infty$, then

$$\|u(t) - u_0\|_{L^1} \le C\,t.$$

Compare this result with (4.76).





Chapter 5

The Riemann Problem for Systems

Diese Untersuchung macht nicht darauf Anspruch, der experimentellen Forschung nützliche Ergebnisse zu liefern; der Verfasser wünscht sie nur als einen Beitrag zur Theorie der nicht linearen partiellen Differentialgleichungen betrachtet zu sehen.[1]
— G. F. B. Riemann [156]

We return to the conservation law (1.2), but now study the case of systems, i.e.,

$$u_t + f(u)_x = 0, \tag{5.1}$$

where $u = u(x,t) = (u_1, \dots, u_n)$ and $f = f(u) = (f_1, \dots, f_n) \in C^2$ are vectors in $\mathbb{R}^n$. (We will not distinguish between row and column vectors, and use whatever is more convenient.) Furthermore, in this chapter we will consider only systems on the line; i.e., the dimension of the underlying physical space is still one. In Chapt. 2 we proved existence, uniqueness, and stability of the Cauchy problem for the scalar conservation law in one space dimension, i.e., well-posedness in the sense of Hadamard. However, this is a more subtle question in the case of systems of hyperbolic conservation laws. We will here first discuss the basic concepts for systems: fundamental properties of shock waves and rarefaction waves. In particular, we will discuss various entropy conditions to select the right solutions of the Rankine–Hugoniot relations.

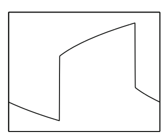
Using these results, we will eventually be able to prove well-posedness of the Cauchy problem for systems of hyperbolic conservation laws with small variation in the initial data.

5.1 Hyperbolicity and Some Examples

Before we start to define the basic properties of systems of hyperbolic conservation laws we discuss some important and interesting examples. The first example is a model for shallow-water waves and will be used throughout this chapter as both a motivation and an example in which all the basic quantities will be explicitly computed.

[1] The present work does not claim to lead to results in experimental research; the author asks only that it be considered as a contribution to the theory of nonlinear partial differential equations.

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_5



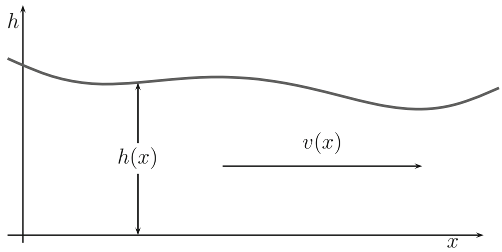


Fig. 5.1 A shallow channel

◇ Example 5.1 (Shallow water)

Water shapes its course according to the nature of the ground over which it flows.
— Sun Tzu, The Art of War (6th–5th century BC)

We will now give a brief derivation of the equations governing shallow-water waves in one space dimension, or, if we want, the long-wave approximation.[2] Consider a one-dimensional channel along the x-axis with a perfect, inviscid fluid with constant density ρ, and assume that the bottom of the channel is horizontal.

In the long-wave or shallow-water approximation we assume that the fluid velocity v is a function only of time and the position along the channel measured along the x-axis. Thus we assume that there is no vertical motion in the fluid. The distance of the surface of the fluid from the bottom is denoted by $h = h(x,t)$. The fluid flow is governed by *conservation of mass* and *conservation of momentum*.

Consider first the conservation of mass of the system. Let $x_1 < x_2$ be two points along the channel. The change of mass of fluid between these points is given by

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{0}^{h(x,t)} \rho\, dy\, dx = -\int_{0}^{h(x_2,t)} \rho v(x_2,t)\, dy + \int_{0}^{h(x_1,t)} \rho v(x_1,t)\, dy.$$

Assuming smoothness of the functions and domains involved, we may rewrite the right-hand side as an integral of the derivative of $\rho v h$. We obtain

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{0}^{h(x,t)} \rho\, dy\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho v(x,t) h(x,t)\right)\, dx,$$

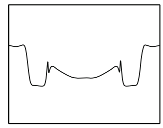
or

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\left(\rho h(x,t)\right) + \frac{\partial}{\partial x}\left(\rho v(x,t) h(x,t)\right)\right] dx = 0.$$

Dividing by $(x_2 - x_1)\rho$ and letting $x_2 - x_1 \to 0$, we obtain the familiar

$$h_t + (vh)_x = 0. \tag{5.2}$$

[2] A word of warning. There are several different equations that are called the shallow-water equations. Also the name Saint-Venant equation is used.



Observe the similarity in the derivations of (5.2) and (1.26). In fact, in the derivation of (1.26) we started by considering individual cars before we made the continuum assumption corresponding to high traffic densities, thereby obtaining (1.26), while in the derivation of (5.2) we simply assumed a priori that the fluid constituted a continuum, and formulated mass conservation directly in the continuum variables.

For the derivation of the equation describing the conservation of momentum we have to assume that the fluid is in hydrostatic balance. For that we introduce the pressure $P = P(x, y, t)$ and consider a small element of the fluid $[x_1, x_2] \times [y, y + \Delta y]$. Hydrostatic balance means that the pressure exactly balances the effect of gravity, or

$$\left(P(\tilde{x}, y + \Delta y, t) - P(\tilde{x}, y, t)\right)(x_2 - x_1) = -(x_2 - x_1)\rho g \Delta y$$

for some $\tilde{x} \in [x_1, x_2]$, where g is the acceleration due to gravity. Dividing by $(x_2 - x_1)\Delta y$ and taking $x_1, x_2 \to x$, $\Delta y \to 0$, we find that

$$\frac{\partial P}{\partial y}(x, y, t) = -\rho g.$$

Integrating and normalizing the pressure to be zero at the fluid surface, we conclude that

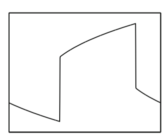
$$P(x, y, t) = \rho g (h(x, t) - y). \tag{5.3}$$

Consider again the fluid between two points $x_1 < x_2$ along the channel. According to Newton's second law, the rate of change of momentum of this part of the fluid is balanced by the net momentum inflow $(\rho v)v = \rho v^2$ across the boundaries $x = x_1$ and $x = x_2$ plus the forces exerted by the pressure at the boundaries. Thus we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho v(x,t)\,dy\,dx = &-\int_0^{h(x_2,t)} P(x_2,y,t)\,dy + \int_0^{h(x_1,t)} P(x_1,y,t)\,dy \\ &-\int_0^{h(x_2,t)} \rho v(x_2,t)^2\,dy + \int_0^{h(x_1,t)} \rho v(x_1,t)^2\,dy.\end{aligned}$$

In analogy with the derivation of the equation for conservation of mass, we may rewrite this, using (5.3), as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho v h\,dx &= -\rho g\left(h(x_2,t)^2 - \frac{1}{2}h(x_2,t)^2\right) \\ &\quad + \rho g\left(h(x_1,t)^2 - \frac{1}{2}h(x_1,t)^2\right) - \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho h v^2\right)\,dx \\ &= -\rho g\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}h^2\right)\,dx - \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho v^2 h\right)\,dx.\end{aligned}$$



Dividing again by $(x_2 - x_1)\rho$ and letting $x_2 - x_1 \to 0$, scaling g to unity, we obtain

$$(vh)_t + \left(v^2 h + \frac{1}{2}h^2\right)_x = 0. \tag{5.4}$$

To summarize, we have the following system of conservation laws:

$$h_t + (vh)_x = 0, \quad (vh)_t + \left(v^2 h + \frac{1}{2}h^2\right)_x = 0, \tag{5.5}$$

where h and v denote the height (depth) and velocity of the fluid, respectively. Introducing the variable q defined by

$$q = vh, \tag{5.6}$$

we may rewrite the shallow-water equations as

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}_x = 0, \tag{5.7}$$

which is the form we will study in detail later on in this chapter. We note in passing that we can write the equation for v as

$$v_t + v v_x + h_x = 0 \tag{5.8}$$

by expanding the second equation in (5.5), and then using the first equation in (5.5).

A different derivation is based on the incompressible Navier–Stokes equations.[3] Consider gravity waves of an incompressible two-dimensional fluid governed by the Navier–Stokes equations

$$\begin{aligned} \bar{v}_{\bar{t}} + (\bar{v}\cdot\nabla)\bar{v} &= \bar{g} - \frac{\bar{p}}{\rho} + \nu\Delta\bar{v}, \\ \nabla\cdot\bar{v} &= 0. \end{aligned} \tag{5.9}$$

Here ρ, $\bar{p}$, $\bar{v} = (\bar{v}_1, \bar{v}_2)$, ν denote the density, pressure, velocity, and viscosity of the fluid, respectively. The first equation describes the momentum conservation, and the second is the incompressibility assumption. We let the y-direction point upward, and thus the gravity $\bar{g}$ is a vector with length equal to g, the acceleration due to gravity, and direction in the negative y-direction. Let L and H denote typical wavelengths of the surface wave and water depth, respectively. The shallow-water assumption (or long-wave assumption) is the following

$$\varepsilon = \frac{H}{L} \ll 1. \tag{5.10}$$

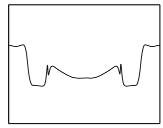
We introduce scaled variables

$$\begin{aligned} x &= L\bar{x}, \quad & y &= H\bar{y}, \quad & t &= T\bar{t}, \\ v &= U\bar{v}_1, \quad & u &= V\bar{v}_2, \quad & p &= \rho g H\bar{p}. \end{aligned} \tag{5.11}$$

The following relations are natural:

$$UT = L, \quad VT = H, \quad U^2 = gH. \tag{5.12}$$

[3] Thanks to Harald Hanche-Olsen.



In addition, we introduce the dimensionless Reynolds number $\mathrm{Re} = UH/\nu$. In the new variables we obtain

$$\begin{aligned} v_t + v v_x + u v_y &= -p_x + \frac{1}{\varepsilon \mathrm{Re}}(\varepsilon^2 v_{xx} + v_{yy}), \\ \varepsilon^2(v u_t + v u v_x + u u_y) &= -1 - p_y + \frac{\varepsilon}{\mathrm{Re}}(\varepsilon^2 u_{xx} + u_{yy}), \\ u_x + v_y &= 0. \end{aligned} \tag{5.13}$$

For typical waves we have $\mathrm{Re} \gg 1$, yet $\varepsilon \mathrm{Re} \gg 1$.[4] Hence a reasonable approximation reads

$$\begin{aligned} v_t + v v_x + u v_y &= -p_x, \\ p_y &= -1, \\ v_x + u_y &= 0. \end{aligned} \tag{5.14}$$

We assume that the bottom is flat and normalize the pressure to vanish at the surface of the fluid, given by $y = h(x,t)$. Hence the pressure equation integrates in the y-direction to yield $p = h(x,t) - y$.

Next we claim that if the horizontal velocity v is independent of y initially, it will remain so, and thus $v_y = 0$. Namely, for a given fluid particle we have that

$$\begin{aligned} \frac{d^2x}{dt^2} = \frac{dv}{dt} &= v_t + v_x \frac{dx}{dt} + v_y \frac{dy}{dt} \\ &= v_t + v v_x + u v_y = -p_x. \end{aligned} \tag{5.15}$$

Since the right-hand side is independent of y, the claim is proved. We can then write

$$v_t + v v_x + h_x = 0. \tag{5.16}$$

A fluid particle at the surface satisfies $y = h(x,t)$, or

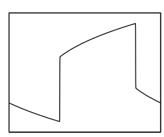
$$u = h_x v + h_t, \text{ whenever } y = h(x,t). \tag{5.17}$$

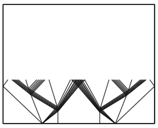
Consider the fluid contained in a domain R between two fixed points x_1 and x_2. By applying Green's theorem on the domain R and on $v_x + u_y = 0$, we obtain

$$\begin{aligned} 0 = \iint_R \big(v_x + u_y\big)\, dx\, dy &= \int_{\partial R} (-u\, dx + v\, dy) \\ &= \int_{x_1}^{x_2} \big((h_x v + h_t)\, dx - v h_x\, dx\big) \\ &\quad + v(x_2,t)h(x_2,t) - v(x_1,t)h(x_1,t) \\ &= \int_{x_1}^{x_2} \big(h_t + (vh)_x\big)\, dx, \end{aligned} \tag{5.18}$$

or $h_t + (vh)_x = 0$, where we used that $v\, dy = v h_x\, dx$ along the curve $y = h(x,t)$.

[4] In tidal waves, say in the North Sea, we have $H \approx 100\,\mathrm{m}$, $T = 6\,\mathrm{h}$, $\nu = 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$, which yields $\varepsilon \approx 2 \cdot 10^{-4}$ and $\mathrm{Re} \approx 3 \cdot 10^9$.





From this we conclude that the shallow-water equations read

$$\begin{aligned} h_t + (vh)_x &= 0, \\ v_t + v v_x + h_x &= 0, \end{aligned} \tag{5.19}$$

in nonconservative form. ◇

◇ Example 5.2 (The wave equation)
Let $\phi = \phi(x,t)$ denote the transverse position away from equilibrium of a one-dimensional string. If we assume that the amplitude of the transversal waves is small, we obtain the wave equation

$$\phi_{tt} = (c^2 \phi_x)_x, \tag{5.20}$$

where c denotes the wave speed. Introducing new variables $u = \phi_x$ and $v = \phi_t$, we find that (5.20) may be written as the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t - \begin{pmatrix} v \\ c^2 u \end{pmatrix}_x = 0. \tag{5.21}$$

If c is constant, we recover the classical linear wave equation $\phi_{tt} = c^2 \phi_{xx}$. See also Example 1.14. ◇

◇ Example 5.3 (The *p*-system)
The p-system is a classical model of an isentropic gas, where one has conservation of mass and momentum, but not of energy. In Lagrangian coordinates it is described by

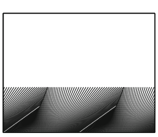
$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0. \tag{5.22}$$

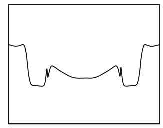
Here v denotes specific volume, that is, the inverse of the density; u is the velocity; and p denotes the pressure. ◇

◇ Example 5.4 (The Euler equations)
The Euler equations are commonly used to model gas dynamics. They can be written in several forms, depending on the physical assumptions used and variables selected to describe them. Let it suffice here to describe the case in which ρ denotes the density, v velocity, p pressure, and E the energy. Conservation of mass and momentum give $\rho_t + (\rho v)_x = 0$ and $(\rho v)_t + (\rho v^2 + p)_x = 0$, respectively. The total energy can be written as $E = \frac{1}{2}\rho v^2 + \rho e$, where e denotes the specific internal energy. Furthermore, we assume that there is a relation between this quantity and the density and pressure, namely $e = e(\rho, p)$. Conservation of energy now reads $E_t + (v(E + p))_x = 0$, yielding finally the system

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0. \tag{5.23}$$

We will return to this system at length in Sect. 5.6. ◇





We will have to make assumptions on the (vector-valued) function f so that many of the properties of the scalar case carry over to the case of systems. In order to have finite speed of propagation, which characterizes hyperbolic equations, we have to assume that the Jacobian of f, denoted by df, has n real eigenvalues

$$df(u)r_j(u) = \lambda_j(u)r_j(u), \quad \lambda_j(u) \in \mathbb{R}, \quad j = 1, \dots, n. \tag{5.24}$$

(We will later normalize the eigenvectors $r_j(u)$.) Furthermore, we order the eigenvalues

$$\lambda_1(u) \le \lambda_2(u) \le \cdots \le \lambda_n(u). \tag{5.25}$$

A system with a full set of eigenvectors with real eigenvalues is called *hyperbolic*, and if all the eigenvalues are distinct, we say that the system is *strictly hyperbolic*.

Let us look at the shallow-water model to see whether that system is hyperbolic.

◇ Example 5.5 (Shallow water (cont'd.))
In the case of the shallow-water equations (5.7) we easily find that

$$\lambda_1(u) = \frac{q}{h} - \sqrt{h} < \frac{q}{h} + \sqrt{h} = \lambda_2(u), \tag{5.26}$$

with corresponding eigenvectors

$$r_j(u) = \begin{pmatrix} 1 \\ \lambda_j(u) \end{pmatrix}, \tag{5.27}$$

and thus the shallow-water equations are strictly hyperbolic away from $h = 0$. ◇

5.2 Rarefaction Waves

Natura non facit saltus.[5]
— Carl Linnaeus, Philosophia Botanica (1751)

Let us consider smooth solutions for the initial value problem

$$u_t + f(u)_x = 0, \tag{5.28}$$

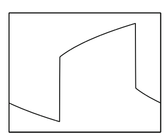
with Riemann initial data

$$u(x, 0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \ge 0. \end{cases} \tag{5.29}$$

First we observe that since both the initial data and the equation are scale-invariant or self-similar, i.e., invariant under the map $x \mapsto kx$ and $t \mapsto kt$, the solution should also have that property. Let us therefore search for solutions of the form

$$u(x, t) = w(x/t) = w(\xi), \quad \xi = x/t. \tag{5.30}$$

[5] Nature does not make jumps.



Inserting this into the differential equation (5.28), we find that

$$-\frac{x}{t^2}\dot{w} + \frac{1}{t}df(w)\dot{w} = 0, \tag{5.31}$$

or

$$df(w)\dot{w} = \xi\dot{w}, \tag{5.32}$$

where $\dot{w}$ denotes the derivative of w with respect to the one variable ξ. Hence we observe that $\dot{w}$ is an eigenvector for the Jacobian $df(w)$ with eigenvalue ξ. From our assumptions on the flux function we know that $df(w)$ has n eigenvectors given by $r_1, \dots, r_n$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. This implies

$$\dot{w}(\xi) = r_j(w(\xi)), \quad \lambda_j(w(\xi)) = \xi, \tag{5.33}$$

for a value of j. Assume in addition that

$$w(\lambda_j(u_l)) = u_l, \quad w(\lambda_j(u_r)) = u_r. \tag{5.34}$$

Thus for a fixed time t, the function $w(x/t)$ will continuously connect the given left state u_l to the given right state u_r. This means that ξ is increasing, and hence $\lambda_j(w(x/t))$ has to be increasing. If this is the case, we have a solution of the form

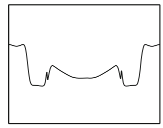
$$u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ w(x/t) & \text{for } t\lambda_j(u_l) \le x \le t\lambda_j(u_r), \\ u_r & \text{for } x \ge t\lambda_j(u_r), \end{cases} \tag{5.35}$$

where $w(\xi)$ satisfies (5.33) and (5.34). We call these solutions *rarefaction waves*, a name that comes from applications to gas dynamics. Furthermore, we observe that the normalization of the eigenvector $r_j(u)$ also is determined from (5.33), namely,

$$\nabla\lambda_j(u) \cdot r_j(u) = 1, \tag{5.36}$$

which follows by taking the derivative with respect to ξ. But this also imposes an extra condition on the eigenvector fields, since we clearly have to have a nonvanishing scalar product between $r_j(u)$ and $\nabla\lambda_j(u)$ to be able to normalize the eigenvector properly. It so happens that in most applications this can be done. However, the Euler equations of gas dynamics have the property that in one of the eigenvector families, the eigenvector and the gradient of the corresponding eigenvalue are orthogonal. We say that the jth family is *genuinely nonlinear* if $\nabla\lambda_j(u) \cdot r_j(u) \neq 0$ and *linearly degenerate* if $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$ for all u under consideration. We will not discuss mixed cases whereby a wave family is linearly degenerate only in certain regions in phase space, e.g., along curves or at isolated points.

Before we discuss these two cases separately, we will make a slight but important change in point of view. Instead of considering given left and right states as in (5.29), we will assume only that u_l is given, and consider those states u_r for which we have a rarefaction wave solution. From (5.33) and (5.35) we see that for each



point u_l in phase space there are n curves emanating from u_l on which u_r can lie allowing a solution of the form (5.35). Each of these curves is given as integral curves of the vector fields of eigenvectors of the Jacobian $df(u)$. Thus our phase space is now the u_r space.

We may sum up the above discussion in the genuinely nonlinear case by the following theorem.

Theorem 5.6 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$ and assume that the equation is genuinely nonlinear in the jth wave family in D. Let the jth eigenvector $r_j(u)$ of $df(u)$ with corresponding eigenvalue $\lambda_j(u)$ be normalized so that $\nabla\lambda_j(u) \cdot r_j(u) = 1$ in D.*

Let $u_l \in D$. Then there exists a curve $R_j(u_l)$ in D, emanating from u_l, such that for each $u_r \in R_j(u_l)$ the initial value problem (5.28), (5.29) has weak solution

$$u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ w(x/t) & \text{for } \lambda_j(u_l)t \le x \le \lambda_j(u_r)t, \\ u_r & \text{for } x \ge \lambda_j(u_r)t, \end{cases} \tag{5.37}$$

where w satisfies $\dot{w}(\xi) = r_j(w(\xi))$, $\lambda_j(w(\xi)) = \xi$, $w(\lambda_j(u_l)) = u_l$, and $w(\lambda_j(u_r)) = u_r$.

Proof The discussion preceding the theorem gives the key computation and the necessary motivation behind the following argument. Assume that we have a strictly hyperbolic, genuinely nonlinear conservation law with appropriately normalized jth eigenvector. Due to the assumptions on f, the system of ordinary differential equations

$$\dot{w}(\xi) = r_j(w(\xi)), \quad w(\lambda_j(u_l)) = u_l \tag{5.38}$$

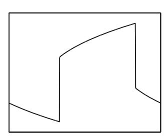
has a solution for all $\xi \in [\lambda_j(u_l), \lambda_j(u_l) + \eta)$ for some $\eta > 0$. For this solution we have

$$\frac{d}{d\xi}\lambda_j(w(\xi)) = \nabla\lambda_j(w(\xi)) \cdot \dot{w}(\xi) = 1, \tag{5.39}$$

proving the second half of (5.33). We denote the orbit of (5.38) by $R_j(u_l)$. If we define $u(x,t)$ by (5.37), a straightforward calculation shows that u indeed satisfies both the equation and the initial data. □

Observe that we can also solve (5.38) for ξ less than $\lambda_j(u_l)$. However, in that case $\lambda_j(u)$ will be decreasing. We remark that the solution u in (5.37) is continuous, but not necessarily differentiable, and hence is not necessarily a regular, but rather a weak, solution.

We will now introduce a different parameterization of the rarefaction curve $R_j(u_l)$, which will be convenient in Section 5.5 when we construct the wave curves for the solution of the Riemann problem. From (5.39) we see that $\lambda_j(u)$ is increasing along $R_j(u_l)$, and hence we may define the positive parameter ϵ by



$\epsilon := \xi - \xi_l = \lambda_j(u) - \lambda_j(u_l) > 0$. We denote the corresponding u by $u_{j,\epsilon}$, that is, $u_{j,\epsilon} = w(\xi) = w(\lambda_j(u)) = w(\epsilon + \lambda_j(u_l))$. Clearly,

$$\left.\frac{du_{j,\epsilon}}{d\epsilon}\right|_{\epsilon=0} = r_j(u_l). \tag{5.40}$$

Assume now that the system is linearly degenerate in the family j, i.e., $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$. Consider the system of ordinary differential equations

$$\frac{du}{d\epsilon} = r_j(u), \quad u|_{\epsilon=0} = u_l, \tag{5.41}$$

with solution $u = u_{j,\epsilon}$ for $\epsilon \in (-\eta, \eta)$ for some $\eta > 0$. We denote this orbit by $C_j(u_l)$, along which $\lambda_j(u_{j,\epsilon})$ is constant, since

$$\frac{d}{d\epsilon}\lambda_j(u_{j,\epsilon}) = \nabla\lambda_j(u_{j,\epsilon}) \cdot r_j(u_{j,\epsilon}) = 0.$$

Furthermore, the Rankine–Hugoniot condition is satisfied on $C_j(u_l)$ with speed $\lambda_j(u_l)$, because

$$\begin{aligned}\frac{d}{d\epsilon}(f(u_{j,\epsilon}) - \lambda_j(u_l)u_{j,\epsilon}) &= df(u_{j,\epsilon})\frac{du_{j,\epsilon}}{d\epsilon} - \lambda_j(u_l)\frac{du_{j,\epsilon}}{d\epsilon} \\ &= (df(u_{j,\epsilon}) - \lambda_j(u_l))r_j(u_{j,\epsilon}) \\ &= (df(u_{j,\epsilon}) - \lambda_j(u_{j,\epsilon}))r_j(u_{j,\epsilon}) = 0,\end{aligned}$$

which implies that $f(u_{j,\epsilon}) - \lambda_j(u_l)u_{j,\epsilon} = f(u_l) - \lambda_j(u_l)u_l$.

Let $u_r \in C_j(u_l)$, i.e., $u_r = u_{j,\epsilon_0}$ for some ϵ_0. It follows that

$$u(x,t) = \begin{cases} u_l & \text{for } x < \lambda_j(u_l)t, \\ u_r & \text{for } x \geq \lambda_j(u_l)t, \end{cases}$$

is a weak solution of the Riemann problem (5.28), (5.29). We call this solution a *contact discontinuity*.

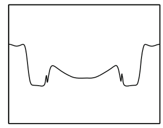
We sum up the above discussion concerning linearly degenerate waves in the following theorem.

Theorem 5.7 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Assume that the equation is linearly degenerate in the jth wave family in D, i.e., $\nabla\lambda_j(u) \cdot r_j(u) \equiv 0$ in D, where $r_j(u)$ is the jth eigenvector of $df(u)$ with corresponding eigenvalue $\lambda_j(u)$.*

Let $u_l \in D$. Then there exists a curve $C_j(u_l)$ in D, passing through u_l, such that for each $u_r \in C_j(u_l)$ the initial value problem (5.28), (5.29) has solution

$$u(x,t) = \begin{cases} u_l & \text{for } x < \lambda_j(u_l)t, \\ u_r & \text{for } x \geq \lambda_j(u_l)t, \end{cases} \tag{5.42}$$

where u_r is determined as follows: Consider the function $\epsilon \mapsto u_\epsilon$ determined by $\frac{du}{d\epsilon} = r_j(u)$, $u|_{\epsilon=0} = u_l$. Then $u_r = u_{\epsilon_0}$ for some ϵ_0.



◇ Example 5.8 (Shallow water (cont'd.))
Let us now consider the actual computation of rarefaction waves in the case of shallow-water waves. Recall that

$$u = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(u) = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix},$$

with eigenvalues $\lambda_j = \frac{q}{h} + (-1)^j \sqrt{h}$, and corresponding eigenvectors $r_j(u) = \binom{1}{\lambda_j(u)}$. With this normalization of r_j, we obtain

$$\nabla \lambda_j(u) \cdot r_j(u) = \frac{3(-1)^j}{2\sqrt{h}}, \tag{5.43}$$

and hence we see that the shallow-water equations are genuinely nonlinear in both wave families. From now on we will renormalize the eigenvectors to satisfy (5.36):

$$r_j(u) = \frac{2}{3}(-1)^j \sqrt{h} \begin{pmatrix} 1 \\ \lambda_j(u) \end{pmatrix}. \tag{5.44}$$

For the 1-family we have that

$$\begin{pmatrix} \dot{h} \\ \dot{q} \end{pmatrix} = -\frac{2}{3}\sqrt{h} \begin{pmatrix} 1 \\ \frac{q}{h} - \sqrt{h} \end{pmatrix}, \tag{5.45}$$

implying that

$$\frac{dq}{dh} = \lambda_1 = \frac{q}{h} - \sqrt{h},$$

which can be integrated to yield

$$q = q(h) = q_l \frac{h}{h_l} - 2h\big(\sqrt{h} - \sqrt{h_l}\,\big). \tag{5.46}$$

Since $\lambda_1(u)$ has to increase along the rarefaction wave, we see from (5.26) (inserting the expression (5.46) for q) that we have to use $h \le h_l$ in (5.46).

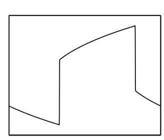
For the second family we again obtain

$$\frac{dq}{dh} = \lambda_2 = \frac{q}{h} + \sqrt{h},$$

yielding

$$q = q(h) = q_l \frac{h}{h_l} + 2h\big(\sqrt{h} - \sqrt{h_l}\,\big). \tag{5.47}$$

In this case we see that we have to use $h \ge h_l$. Observe that (5.46) and (5.47) would follow for *any* normalization of the eigenvector $r_j(u)$. See Fig. 5.2.



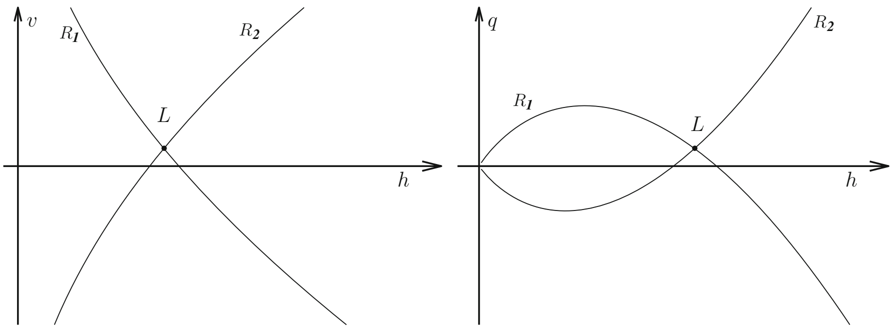


Fig. 5.2 Rarefaction curves in the (h, v)- and (h, q)-planes. We have illustrated the full solution of (5.38) for the shallow-water equations. Only the part given by (5.48) and (5.49) will be actual rarefaction curves

Summing up, we obtain the following rarefaction waves expressed in terms of h:

$$R_1: \quad q = R_1(h; u_l) := q_l \frac{h}{h_l} - 2h\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \in (0, h_l], \tag{5.48}$$

$$R_2: \quad q = R_2(h; u_l) := q_l \frac{h}{h_l} + 2h\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \geq h_l. \tag{5.49}$$

Alternatively, in the (h, v) variables (with $v = q/h$) we have the following:

$$R_1: \quad v = R_1(h; u_l) := v_l - 2\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \in (0, h_l], \tag{5.50}$$

$$R_2: \quad v = R_2(h; u_l) := v_l + 2\big(\sqrt{h} - \sqrt{h_l}\big), \quad h \geq h_l. \tag{5.51}$$

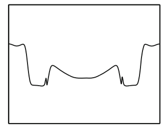
However, if we want to compute the rarefaction curves in terms of the parameter ξ or ϵ, we have to use the proper normalization of the eigenvectors given by (5.44). Consider first the 1-family. We obtain

$$\dot{h} = -\frac{2}{3}\sqrt{h}, \quad \dot{q} = \frac{2}{3}\left(-\frac{q}{\sqrt{h}} + h\right). \tag{5.52}$$

Integrating the first equation directly and inserting the result into the second equation, we obtain

$$\begin{aligned} w_1(\xi) = \begin{pmatrix} h_1 \\ q_1 \end{pmatrix}(\xi) &= R_1(\xi; u_l) \\ &:= \begin{pmatrix} \frac{1}{9}\big(v_l + 2\sqrt{h_l} - \xi\big)^2 \\ \frac{1}{27}\big(v_l + 2\sqrt{h_l} + 2\xi\big)\big(v_l + 2\sqrt{h_l} - \xi\big)^2 \end{pmatrix} \end{aligned} \tag{5.53}$$

for $\xi \in \big[v_l - \sqrt{h_l}, v_l + 2\sqrt{h_l}\big)$.



Similarly, for the second family we obtain

$$
\begin{aligned}
w_2(\xi) &= \begin{pmatrix} h_2 \\ q_2 \end{pmatrix}(\xi) = R_2(\xi; u_l) \\
&:= \begin{pmatrix} \frac{1}{9}(-v_l + 2\sqrt{h_l} + \xi)^2 \\ \frac{1}{27}(v_l - 2\sqrt{h_l} + 2\xi)(-v_l + 2\sqrt{h_l} + \xi)^2 \end{pmatrix}
\end{aligned}
\tag{5.54}
$$

for $\xi \in \big[\lambda_2(u_l), \infty\big)$. Hence the actual solution reads

$$
u(x,t) = \begin{cases} u_l & \text{for } x \le \lambda_j(u_l)t, \\ R_j(x/t; u_l) & \text{for } \lambda_j(u_l)t \le x \le \lambda_j(u_r)t, \\ u_r & \text{for } x \ge \lambda_j(u_r)t. \end{cases} \tag{5.55}
$$

In the (h, v) variables (with $v = q/h$) we obtain

$$
v_1(\xi) = \frac{1}{3}\big(v_l + 2\sqrt{h_l} + 2\xi\big) \tag{5.56}
$$

and

$$
v_2(\xi) = \frac{1}{3}\big(v_l - 2\sqrt{h_l} + 2\xi\big), \tag{5.57}
$$

for the first and the second families, respectively.

In terms of the parameter ϵ we may write (5.53) as

$$
u_{1,\epsilon} = \begin{pmatrix} h_{1,\epsilon} \\ q_{1,\epsilon} \end{pmatrix} = R_{1,\epsilon}(u_l) := \begin{pmatrix} (\sqrt{h_l} - \frac{\epsilon}{3})^2 \\ (v_l + \frac{2\epsilon}{3})(\sqrt{h_l} - \frac{\epsilon}{3})^2 \end{pmatrix} \tag{5.58}
$$

for $\epsilon \in \big[0, 3\sqrt{h_l}\,\big)$, and (5.54) as

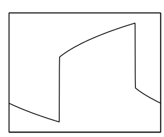
$$
u_{2,\epsilon} = \begin{pmatrix} h_{2,\epsilon} \\ q_{2,\epsilon} \end{pmatrix} = R_{2,\epsilon}(u_l) := \begin{pmatrix} \big(\sqrt{h_l} + \frac{\epsilon}{3}\big)^2 \\ \big(v_l + \frac{2\epsilon}{3}\big)\big(\sqrt{h_l} + \frac{\epsilon}{3}\big)^2 \end{pmatrix} \tag{5.59}
$$

for $\epsilon \in [0, \infty)$. $\diamond$

5.3 The Hugoniot Locus: The Shock Curves

God lives in the details.
— Johannes Kepler (1571–1630)

The discussion in Chapt. 1 concerning weak solutions, and in particular the Rankine–Hugoniot condition (1.27), carries over to the case of systems without restrictions. However, the concept of entropy is considerably more difficult for systems and is still an area of research. Our main concern in this section is the



characterization of solutions of the Rankine–Hugoniot relation. Again, we will take the point of view introduced in the previous section, assuming the left state u_l to be fixed, and consider possible right states u that satisfy the Rankine–Hugoniot condition

$$s(u - u_l) = f(u) - f(u_l), \tag{5.60}$$

for some speed s. We introduce the jump in a quantity ϕ as

$$[\![\phi]\!] = \phi_r - \phi_l,$$

and hence (5.60) takes the familiar form

$$s\,[\![u]\!] = [\![f(u)]\!]\,.$$

The solutions of (5.60), for a given left state u_l, form a set, which we call the *Hugoniot locus* and write $H(u_l)$, i.e.,

$$H(u_l) := \left\{u \mid \exists s \in \mathbb{R} \text{ such that } s\,[\![u]\!] = [\![f(u)]\!]\right\}. \tag{5.61}$$

We start by computing the Hugoniot locus for the shallow-water equations.

◇ Example 5.9 (Shallow water (cont'd.))
The Rankine–Hugoniot condition reads

$$\begin{aligned} s(h - h_l) &= q - q_l, \\ s(q - q_l) &= \left(\frac{q^2}{h} + \frac{h^2}{2}\right) - \left(\frac{q_l^2}{h_l} + \frac{h_l^2}{2}\right), \end{aligned} \tag{5.62}$$

where s as usual denotes the shock speed between the left state $u_l = \binom{h_l}{q_l}$ and right state $u = \binom{h}{q}$:

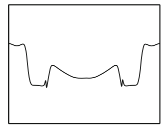
$$\begin{pmatrix} h \\ q \end{pmatrix}(x,t) = \begin{cases} \binom{h_l}{q_l} & \text{for } x < st, \\ \binom{h}{q} & \text{for } x \geq st. \end{cases} \tag{5.63}$$

In the context of the shallow-water equations such solutions are called *bores*. Eliminating s in (5.62), we obtain the equation

$$[\![h]\!]\left(\left[\!\!\left[\frac{q^2}{h}\right]\!\!\right] + \frac{1}{2}\,[\![h^2]\!]\right) = [\![q]\!]^2. \tag{5.64}$$

Introducing the variable v, given by $q = vh$, equation (5.64) becomes

$$[\![h]\!]\left([\![hv^2]\!] + \frac{1}{2}\,[\![h^2]\!]\right) = [\![vh]\!]^2,$$



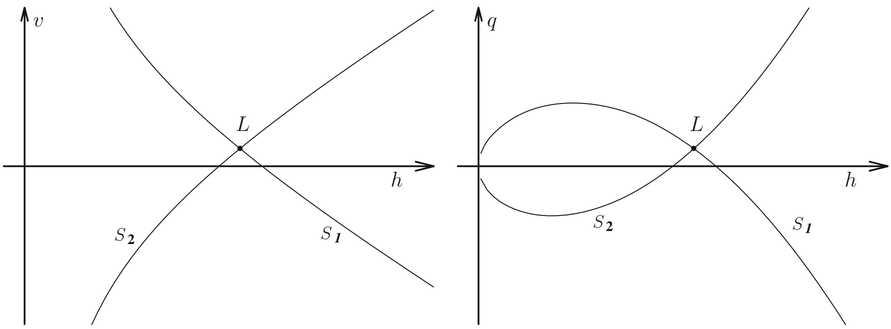


Fig. 5.3 Shock curves in the (h, v)- and (h, q)-planes. Slow (S_1) and fast (S_2) shocks indicated; see Sect. 5.4

with solution

$$v = v_l \pm \frac{1}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}}, \tag{5.65}$$

or alternatively,

$$q = vh = q_l \frac{h}{h_l} \pm \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}}. \tag{5.66}$$

See Fig. 5.3. For later use, we will also obtain formulas for the corresponding shock speeds. We find that

$$\begin{aligned} s = \frac{[\![vh]\!]}{[\![h]\!]} &= \frac{v(h - h_l) + (v - v_l)h_l}{h - h_l} \\ &= v + \frac{[\![v]\!]}{[\![h]\!]}h_l = v \pm \frac{h_l}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}, \end{aligned} \tag{5.67}$$

or

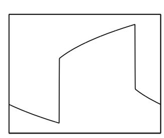
$$s = v + \frac{[\![v]\!]}{[\![h]\!]}h_l = v_l + [\![v]\!] + \frac{[\![v]\!]}{[\![h]\!]}h_l = v_l \pm \frac{h}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}. \tag{5.68}$$

When we want to indicate the wave family, we write

$$\begin{aligned} s_j = s_j(h; v_l) &= v_l + (-1)^j \frac{h}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}} \\ &= v + (-1)^j \frac{h_l}{\sqrt{2}}\sqrt{h^{-1} + h_l^{-1}}. \end{aligned} \tag{5.69}$$

Thus we see that through a given left state u_l there are two curves on which the Rankine–Hugoniot relation holds, namely,

$$H_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \,\middle|\, h > 0 \right\} \tag{5.70}$$



and

$$H_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}} (h - h_l) \sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \,\middle|\, h > 0 \right\}. \tag{5.71}$$

We call the corresponding shocks slow shocks (or 1-shocks) and fast shocks (or 2-shocks), respectively. The Hugoniot locus now reads

$$H(u_l) = \{ u \mid \exists s \in \mathbb{R} \text{ such that } s \, [\![u]\!] = [\![f(u)]\!] \} = H_1(u_l) \cup H_2(u_l). \qquad \diamond$$

We will soon see that the basic features of the Hugoniot locus of the shallow-water equations carry over to the general case of strictly hyperbolic systems at least for small shocks where u is near u_l. The problem to be considered is to solve implicitly the system of n equations

$$\mathcal{H}(s, u; u_l) := s(u - u_l) - (f(u) - f(u_l)) = 0 \tag{5.72}$$

for the $n + 1$ unknowns $u_1, \dots, u_n$ and s for u close to the given u_l. The major problem comes from the fact that we have one equation fewer than the number of unknowns, and that $\mathcal{H}(s, u_l; u_l) = 0$ for all values of s. Hence the implicit function theorem cannot be used without first removing the singularity at $u = u_l$.

Let us first state the relevant version of the implicit function theorem that we will use.

Theorem 5.10 (Implicit function theorem) *Let the function*

$$\Phi = (\Phi_1, \dots, \Phi_p) : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p \tag{5.73}$$

be C^1 in a neighborhood of a point (x_0, y_0), $x_0 \in \mathbb{R}^q$, $y_0 \in \mathbb{R}^p$ with $\Phi(x_0, y_0) = 0$. Assume that the $p \times p$ matrix

$$\frac{\partial \Phi}{\partial y} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial y_1} & \cdots & \frac{\partial \Phi_1}{\partial y_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_p}{\partial y_1} & \cdots & \frac{\partial \Phi_p}{\partial y_p} \end{pmatrix} \tag{5.74}$$

is nonsingular at the point (x_0, y_0).

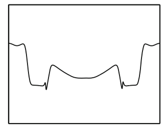
Then there exist a neighborhood N of x_0 and a unique differentiable function $\phi: N \to \mathbb{R}^p$ such that

$$\Phi(x, \phi(x)) = 0, \quad \phi(x_0) = y_0. \tag{5.75}$$

We will rewrite equation (5.72) into an eigenvalue problem that we can study locally around each eigenvalue $\lambda_j(u_l)$. This removes the singularity, and hence we can apply the implicit function theorem.

Theorem 5.11 *Let D be a domain in $\mathbb{R}^n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Let $u_l \in D$.*

Then there exist n smooth curves $H_1(u_l), \dots, H_n(u_l)$ locally through u_l on which the Rankine–Hugoniot relation is satisfied.



Proof Writing

$$\begin{aligned} f(u) - f(u_l) &= \int_0^1 \frac{\partial}{\partial \alpha} f((1-\alpha)u_l + \alpha u)\, d\alpha \\ &= \int_0^1 df((1-\alpha)u_l + \alpha u)(u - u_l)\, d\alpha \\ &= M(u, u_l)(u - u_l), \end{aligned} \tag{5.76}$$

where $M(u, u_l)$ is the averaged Jacobian

$$M(u, u_l) = \int_0^1 df((1-\alpha)u_l + \alpha u)\, d\alpha,$$

we see that (5.72) takes the form

$$\mathcal{H}(s, u, u_l) = (s - M(u, u_l))(u - u_l) = 0. \tag{5.77}$$

Here $u - u_l$ is an eigenvector of the matrix M with eigenvalue s. The matrix $M(u_l, u_l) = df(u_l)$ has n distinct eigenvalues $\lambda_1(u_l), \dots, \lambda_n(u_l)$, and hence we know that there exists an open set N such that the matrix $M(u, u_l)$ has twice-differentiable eigenvectors and distinct eigenvalues, namely,

$$\big(\mu_j(u, u_l) - M(u, u_l)\big)\, v_j(u, u_l) = 0, \tag{5.78}$$

for all $u, u_l \in N$.[6] Let $w_j(u, u_l)$ denote the corresponding left eigenvectors normalized so that

$$w_k(u, u_l) \cdot v_j(u, u_l) = \delta_{jk}. \tag{5.79}$$

In this terminology u and u_l satisfy the Rankine–Hugoniot relation with speed s if and only if there exists a j such that

$$w_k(u, u_l) \cdot (u - u_l) = 0 \text{ for all } k \neq j, \quad s = \mu_j(u, u_l), \tag{5.80}$$

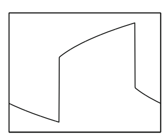
and $w_j(u, u_l) \cdot (u - u_l)$ is nonzero. We define functions $F_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$F_j(u, \epsilon) = \big(w_1(u, u_l) \cdot (u - u_l) - \epsilon \delta_{1j}, \dots, w_n(u, u_l) \cdot (u - u_l) - \epsilon \delta_{nj}\big). \tag{5.81}$$

The Rankine–Hugoniot relation is satisfied if and only if $F_j(u, \epsilon) = 0$ for some ϵ and j. Furthermore, $F_j(u_l, 0) = 0$. A straightforward computation shows that

$$\frac{\partial F_j}{\partial u}(u_l, 0) = \begin{pmatrix} l_1(u_l) \\ \vdots \\ l_n(u_l) \end{pmatrix},$$

[6] The properties of the eigenvalues follow from the implicit function theorem used on the determinant of $\mu I - M(u, u_l)$, and for the eigenvectors by considering the one-dimensional eigenprojections $\int (M(u, u_l) - \mu)^{-1}\, d\mu$ integrated around a small curve enclosing each eigenvalue $\lambda_j(u_l)$.



which is nonsingular. Hence the implicit function theorem implies the existence of a unique solution $u_j(\epsilon)$ of

$$F_j(u_j(\epsilon), \epsilon) = 0 \tag{5.82}$$

for ϵ small. □

Occasionally, in particular in Chapt. 7, we will use the notation

$$H_j(\epsilon)u_l = u_j(\epsilon).$$

We will have the opportunity later to study in detail properties of the parameterization of the Hugoniot locus. Let it suffice here to observe that by differentiating each component of $F_j(u_j(\epsilon), \epsilon) = 0$ at $\epsilon = 0$, we find that

$$l_k(u_l) \cdot u_j'(0) = \delta_{jk} \tag{5.83}$$

for *all* $k = 1, \dots, n$, showing that indeed

$$u_j'(0) = r_j(u_l). \tag{5.84}$$

From the definition of M we see that $M(u, u_l) = M(u_l, u)$, and this symmetry implies that

$$\begin{aligned} \mu_j(u, u_l) &= \mu_j(u_l, u), \quad & \mu_j(u_l, u_l) &= \lambda_j(u_l), \\ v_j(u, u_l) &= v_j(u_l, u), \quad & v_j(u_l, u_l) &= r_j(u_l), \\ w_j(u, u_l) &= w_j(u_l, u), \quad & w_j(u_l, u_l) &= l_j(u_l). \end{aligned} \tag{5.85}$$

Let $\nabla_k h(u_1, u_2)$ denote the gradient of a function $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with respect to the kth variable $u_k \in \mathbb{R}^n$, $k = 1, 2$. Then the symmetries (5.85) imply that

$$\nabla_1 \mu_j(u, u_l) = \nabla_2 \mu_j(u, u_l). \tag{5.86}$$

Hence

$$\nabla \lambda_j(u_l) = \nabla_1 \mu_j(u_l, u_l) + \nabla_2 \mu_j(u_l, u_l) = 2\nabla_1 \mu_j(u_l, u_l). \tag{5.87}$$

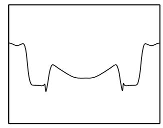
For a vector-valued function $\phi(u) = (\phi_1(u), \dots, \phi_n(u))$ we let $\nabla \phi(u)$ denote the Jacobian matrix,

$$\nabla \phi(u) = \begin{pmatrix} \nabla \phi_1 \\ \vdots \\ \nabla \phi_n \end{pmatrix}. \tag{5.88}$$

Now the symmetries (5.85) imply that

$$\nabla l_k(u_l) = 2\nabla_1 w_k(u_l, u_l) \tag{5.89}$$

in obvious notation.



5.4 The Entropy Condition

. . . and now remains
That we find out the cause of this effect,
Or rather say, the cause of this defect . . .
— W. Shakespeare, Hamlet (1603)

Having derived the Hugoniot loci for a general class of conservation laws in the previous section, we will have to select the parts of these curves that give admissible shocks, i.e., satisfy an entropy condition. This will be considerably more complicated in the case of systems than in the scalar case. To guide our intuition we will return to the example of shallow-water waves.

◇ Example 5.12 (Shallow water (cont'd.))
Let us first study the points on $H_1(u_l)$; a similar analysis will apply to $H_2(u_l)$. We will work with the variables h, v rather than h, q. Consider the Riemann problem where we have a high-water bank at rest to the left of the origin and a lower-water bank to the right of the origin, with a positive velocity; or in other words, the fluid from the lower-water bank moves away from the high-water bank. More precisely, for $h_l > h_r$ we let

$$\begin{pmatrix} h \\ v \end{pmatrix}(x, 0) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq 0, \end{cases}$$

where we have chosen initial data so that the right state is on $H_1(u_l)$, i.e., the Rankine–Hugoniot is already satisfied for a certain speed s. This implies that

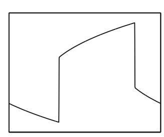
$$\begin{pmatrix} h \\ v \end{pmatrix}(x, t) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < st, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq st, \end{cases}$$

for $h_l > h_r$, where the negative shock speed s given by

$$s = -\frac{h_r \sqrt{h_r^{-1} + h_l^{-1}}}{\sqrt{2}}$$

is a perfectly legitimate weak solution of the initial value problem. However, we see that this is not at all a reasonable solution, since the solution predicts a high-water bank being pushed by a lower one! See Fig. 5.4.

If we change the initial conditions so that the right state is on the other branch of $H_1(u_l)$, i.e., we consider a high-water bank moving into a lower-water bank at rest,



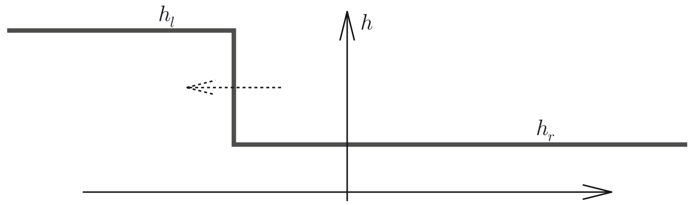


Fig. 5.4 Unphysical solution

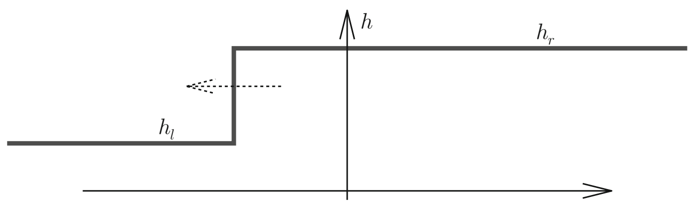


Fig. 5.5 Reasonable solution

or

$$\begin{pmatrix} h \\ v \end{pmatrix}(x,0) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq 0, \end{cases}$$

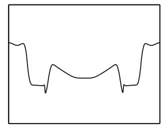
for $h_l < h_r$, we see that the weak solution

$$\begin{pmatrix} h \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < st, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}} \sqrt{h_r^{-1} + h_l^{-1}} \end{pmatrix} & \text{for } x \geq st, \end{cases}$$

for $h_l < h_r$ and with speed $s = -h_r\sqrt{h_r^{-1} + h_l^{-1}}/\sqrt{2}$ is reasonable physically, since the high-water bank now is pushing the lower one. See Fig. 5.5

If you are worried about the fact that the shock is preserved, i.e., that there is no deformation of the shock profile, this is due to the fact that the right state is carefully selected. In general we will have both a shock wave and a rarefaction wave in the solution. This will be clear when we solve the full Riemann problem.

Let us also consider the above examples with energy conservation in mind. In our derivation of the shallow-water equations we used conservation of mass and momentum only. For smooth solutions of these equations, conservation of mechanical energy will follow. Indeed, the kinetic energy of a vertical section of the shallow-water system at a point x is given by $h(x,t)v(x,t)^2/2$ in dimensionless variables, and the potential energy of the same section is given by $h(x,t)^2/2$, and hence the total mechanical energy reads $(h(x,t)v(x,t)^2 + h(x,t)^2)/2$. Consider now a section



of the channel between points $x_1 < x_2$ and assume that we have a smooth (classical) solution of the shallow-water equations. The rate of change of mechanical energy is given by the net energy flow across x_1 and x_2, i.e., $\frac{1}{2}(hv^2 + h^2)v = \frac{1}{2}(hv^3 + h^2 v)$, plus the work done by the pressure. Energy conservation yields

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + \frac{1}{2}h^2 v\right)dx \\
&\quad + \int_0^{h(x_2,t)} P(x_2, y, t)v(x_2, t)\,dy - \int_0^{h(x_1,t)} P(x_1, y, t)v(x_1, t)\,dy \\
&= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + \frac{1}{2}h^2 v\right)dx \\
&\quad + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}h^2 v\right)dx \\
&= \int_{x_1}^{x_2}\frac{\partial}{\partial t}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}hv^3 + h^2 v\right)dx,
\end{aligned}$$

where we have used that $P(x, y, t) = h(x, t) - y$ in dimensionless variables. Hence we conclude that

$$\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)_t + \left(\frac{1}{2}hv^3 + h^2 v\right)_x = 0.$$

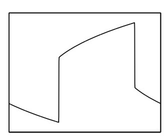
This equation follows easily directly from (5.5) for smooth solutions.

However, for weak solutions, mechanical energy will in general not be conserved. Due to dissipation we expect an energy loss across a bore. Let us compute this change in energy ΔE across the bore in the two examples above, for a time t such that $x_1 < st < x_2$. We obtain

$$\begin{aligned}
\Delta E &= \frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}hv^2 + \frac{1}{2}h^2\right)dx + \left(\frac{1}{2}hv^3 + h^2 v\right)\Big|_{x_1}^{x_2} \\
&= -s\left[\!\left[\frac{1}{2}hv^2 + \frac{1}{2}h^2\right]\!\right] + \left[\!\left[\frac{1}{2}hv^3 + h^2 v\right]\!\right] \\
&= \frac{1}{2}h_r\delta(\left[\!\left[h\right]\!\right]^2\delta^2 h_r + h_r^2 - h_l^2) + (-\left[\!\left[h\right]\!\right]^3\delta^3 h_r - 2\left[\!\left[h\right]\!\right]\delta h_r^2) \\
&= -\frac{1}{4}\left[\!\left[h\right]\!\right]^3\delta,
\end{aligned} \tag{5.90}$$

where we have introduced

$$\delta := \frac{\sqrt{h_r^{-1} + h_l^{-1}}}{\sqrt{2}} = \left(\frac{h_r + h_l}{2h_r h_l}\right)^{1/2}. \tag{5.91}$$



(Recall that $v_l = 0$ and $v_r = [\![v]\!] = -[\![h]\!]\,\delta$ from the Rankine–Hugoniot condition.) Here we have used that we have a smooth solution with energy conservation on each interval $[x_1, st]$ and $[st, x_2]$. In the first case, where we had a low-water bank pushing a high-water bank with $h_r < h_l$, we find indeed that $\Delta E > 0$, while in the other case we obtain the more reasonable $\Delta E < 0$.

From these two simple examples we get a hint that only one branch of $H_1(u_l)$ is physically acceptable. We will now translate this into conditions on existence of viscous profiles and conditions on the eigenvalues of $df(u)$ at $u = u_l$ and $u = u_r$, conditions we will use in cases where our intuition will be more blurred.

In Chapt. 2 we discussed the notion of traveling waves. Recall from (2.7) that a shock between two fixed states u_l and u_r with speed s,

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{5.92}$$

admits a viscous profile if $u(x,t)$ is the limit as $\epsilon \to 0$ of $u^\epsilon(x,t) = U((x - st)/\epsilon) = U(\xi)$ with $\xi = (x - st)/\epsilon$, which satisfies

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon.$$

Integrating this equation, using $\lim_{\epsilon \to 0} U(\xi) = u_l$ if $\xi < 0$, we obtain

$$\dot{U} = A(h,q) := f(U) - f(u_l) - s(U - u_l), \tag{5.93}$$

where the differentiation is with respect to ξ. We will see that it is possible to connect the left state with a viscous profile to a right state only for the branch with $h_r > h_l$ of $H_1(u_l)$, i.e., the physically correct solution.

Computationally it will be simpler to work with viscous profiles in the (h, v) variables rather than (h, q). Using $\dot{q} = \dot{v}h + v\dot{h}$ and (5.93), we find that there is a viscous profile in (h, q) if and only if (h, v) satisfies

$$\begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix} = B(h,v) := \begin{pmatrix} vh - v_l h_l - s(h - h_l) \\ (v - v_l)(v_l - s)\frac{h_l}{h} + \frac{h^2 - h_l^2}{2h} \end{pmatrix}. \tag{5.94}$$

Consider now a slow shock with $s = v_l - h_r\delta$, cf. (5.69). We can write

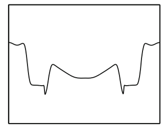
$$B(h,v) = \begin{pmatrix} vh - v_l h_l - s(h - h_l) \\ (v - v_l)\frac{h_l h_r}{h}\delta + \frac{h^2 - h_l^2}{2h} \end{pmatrix}. \tag{5.95}$$

We will analyze the vector field $B(h, v)$ carefully. The Jacobian of B reads

$$dB(h,v) = \begin{pmatrix} v - s & h \\ \frac{h^2 + h_l^2}{2h^2} - (v - v_l)\frac{h_l h_r}{h^2}\delta & \frac{h_l h_r}{h}\delta \end{pmatrix}. \tag{5.96}$$

At the left state u_l we obtain

$$dB(h_l, v_l) = \begin{pmatrix} v_l - s & h_l \\ 1 & h_r\delta \end{pmatrix} = \begin{pmatrix} h_r\delta & h_l \\ 1 & h_r\delta \end{pmatrix}, \tag{5.97}$$



using the value of the shock speed s, equation (5.68). The eigenvalues of $dB(h_l, v_l)$ are $h_r\delta \pm \sqrt{h_l}$, both of which are easily seen to be positive when $h_r > h_l$; thus (h_l, v_l) is a source. Similarly, we obtain

$$dB(h_r, v_r) = \begin{pmatrix} h_l\delta & h_r \\ 1 & h_l\delta \end{pmatrix}, \tag{5.98}$$

with eigenvalues $h_l\delta \pm \sqrt{h_r}$. In this case, one eigenvalue is positive and one negative, and thus (h_r, v_r) is a saddle point. However, we still have to establish an orbit connecting the two states. To this end we construct a region K with (h_l, v_l) and (h_r, v_r) at the boundary of K such that a connecting orbit has to connect the two points within K. The region K will have two curves as boundaries where the first and second components of B vanish, respectively. The first curve, denoted by C_h, is defined by the first component being zero,

$$vh - v_l h_l - s(h - h_l) = 0, \quad h \in [h_l, h_r],$$

which can be simplified to yield

$$v = v_l - (h - h_l)\frac{h_r}{h}\delta, \quad h \in [h_l, h_r]. \tag{5.99}$$

For the second curve, C_v, we have

$$(v - v_l)(v_l - s)\frac{h_l}{h} + \frac{h^2 - h_l^2}{2h} = 0, \quad h \in [h_l, h_r],$$

which can be rewritten as

$$v = v_l - \frac{h^2 - h_l^2}{2h_l h_r \delta}, \quad h \in [h_l, h_r]. \tag{5.100}$$

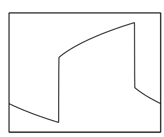
Let us now study the behavior of the second component of B along the curve C_h where the first component vanishes, i.e.,

$$\begin{aligned} &\left[(v - v_l)\frac{h_l h_r}{h}\delta + \frac{h^2 - h_l^2}{2h}\right]\Big|_{C_h} \\ &\qquad = -\frac{h_l}{2h^2}(h_r - h)(h - h_l)(1 + \frac{h + h_r}{h_l}) < 0. \end{aligned} \tag{5.101}$$

Similarly, for the first component of B along C_v, we obtain

$$\begin{aligned} &\left[vh - v_l h_l - s(h - h_l)\right]\Big|_{C_v} \\ &\qquad = \frac{h - h_l}{2h_r h_l \delta}\left(h_r(h_l + h_r) - h(h + h_l)\right) > 0, \end{aligned} \tag{5.102}$$

which is illustrated in Fig. 5.6. The flow of the vector field is leaving the region K along the curves C_h and C_v. Locally, around (h_r, v_r) there has to be an orbit entering



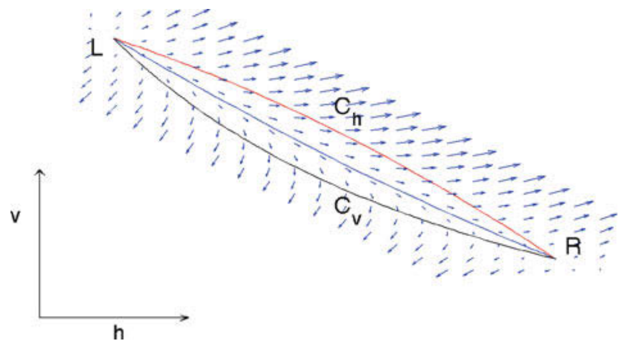


Fig. 5.6 The vector field B, the curves C_v and C_h, as well as the orbit connecting the left and the right states

K as ξ decreases from ∞. This curve cannot escape K and has to connect to a curve coming from (h_l, v_l). Consequently, we have proved existence of a viscous profile.

We saw that the relative values of the shock speed and the eigenvalues of the Jacobian of B, and hence of A, at the left and right states were crucial for this analysis to hold. Let us now translate these assumptions into assumptions on the eigenvalues of dA. The Jacobian of A reads

$$dA(h,q) = \begin{pmatrix} -s & 1 \\ h - \frac{q^2}{h^2} & \frac{2q}{h} - s \end{pmatrix}.$$

Hence the eigenvalues are $-s + \frac{q}{h} \pm \sqrt{h} = -s + \lambda(u)$. At the left state both eigenvalues are positive, and thus u_l is a source, while at u_r one is positive and one negative, and hence u_r is a saddle. We may write this as

$$\lambda_1(u_r) < s < \lambda_1(u_l), \quad s < \lambda_2(u_r). \tag{5.103}$$

We call these the *Lax inequalities*, and say that a shock satisfying these inequalities is a *Lax 1-shock* or a *slow Lax shock*. We have proved that for the shallow-water equations with $h_r > h_l$ there exists a viscous profile, and that the Lax shock conditions are satisfied.

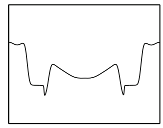
Let us now return to the unphysical shock "solution." In this case we had $u_r \in H_1(u_l)$ with $h_r < h_l$ with the eigenvalues at the left state (h_l, v_l) of different signs. Thus (h_l, v_l) is a saddle. However, for the right state (h_r, v_r) both eigenvalues are positive, and hence that point is a source. Accordingly, there cannot be any orbit connecting the left state with the right state.

A similar analysis can be performed for $H_2(u_l)$, giving that there exists a viscous profile for a shock satisfying the Rankine–Hugoniot relation if and only if the following *Lax entropy conditions* are satisfied:

$$\lambda_2(u_r) < s < \lambda_2(u_l), \quad s > \lambda_1(u_l). \tag{5.104}$$

In that case we have a *fast Lax shock*, or *Lax 2-shock*.

We may sum up the above argument as follows. A shock has a viscous profile if and only if the Lax shock conditions are satisfied. We call such shocks *admissible* and denote the part of the Hugoniot locus where the Lax j conditions are satisfied



by S_j. In the case of shallow-water equations we obtain

$$S_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \;\middle|\; h \geq h_l \right\}, \tag{5.105}$$

$$S_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}}(h - h_l)\sqrt{h^{-1} + h_l^{-1}} \end{pmatrix} \;\middle|\; 0 < h \leq h_l \right\}. \tag{5.106}$$

(These curves are depicted in Sect. 5.3.) We may also want to parameterize the admissible shocks differently. For the slow Lax shocks let

$$h_{1,\epsilon} := h_l - \frac{2}{3}\sqrt{h_l}\,\epsilon, \quad \epsilon < 0. \tag{5.107}$$

This gives

$$q_{1,\epsilon} := q_l \left(1 - \frac{2\epsilon}{3\sqrt{h_l}}\right) + \frac{\epsilon}{9}\sqrt{2h_l \left(6\sqrt{h_l} - 2\epsilon\right)\left(3\sqrt{h_l} - 2\epsilon\right)} \tag{5.108}$$

such that

$$\frac{d}{d\epsilon}\begin{pmatrix} h_{1,\epsilon} \\ q_{1,\epsilon} \end{pmatrix}\Bigg|_{\epsilon=0} = r_1(u_l), \tag{5.109}$$

where $r_1(u_l)$ is given by (5.44). Similarly, for the fast Lax shocks let

$$h_{2,\epsilon} := h_l + \frac{2}{3}\sqrt{h_l}\,\epsilon, \quad \epsilon < 0. \tag{5.110}$$

Then

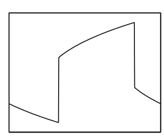
$$q_{2,\epsilon} := q_l \left(1 + \frac{2\epsilon}{3\sqrt{h_l}}\right) + \frac{\epsilon}{9}\sqrt{2h_l \left(6\sqrt{h_l} + 2\epsilon\right)\left(3\sqrt{h_l} + 2\epsilon\right)}, \tag{5.111}$$

such that

$$\frac{d}{d\epsilon}\begin{pmatrix} h_{2,\epsilon} \\ q_{2,\epsilon} \end{pmatrix}\Bigg|_{\epsilon=0} = r_2(u_l), \tag{5.112}$$

where $r_2(u_l)$ is given by (5.44). ◇

In the above example we have seen the equivalence between the existence of a viscous profile and the Lax entropy conditions for the shallow-water equations. This analysis has yet to be carried out for general systems. We will use the above example as a motivation for the following definition, stated for general systems.



Definition 5.13 We say that a shock

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{5.113}$$

is a Lax j-shock if the shock speed s satisfies the Rankine–Hugoniot condition $s [\![u]\!] = [\![f]\!]$ and

$$\lambda_{j-1}(u_l) < s < \lambda_j(u_l), \quad \lambda_j(u_r) < s < \lambda_{j+1}(u_r). \tag{5.114}$$

(Here $\lambda_0 = -\infty$ and $\lambda_{n+1} = \infty$.)

Observe that for strictly hyperbolic systems, for which the eigenvalues are distinct, it suffices to check the inequalities $\lambda_j(u_r) < s < \lambda_j(u_l)$ for small Lax j-shocks if the eigenvalues are continuous in u.

The following result follows from Theorem 5.11.

Theorem 5.14 *Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ in a domain D in $\mathbb{R}^n$. Assume that $\nabla\lambda_j \cdot r_j = 1$. Let $u_l \in D$. A state $u_{j,\epsilon} \in H_j(u_l)$ is a Lax j-shock near u_l if $|\epsilon|$ is sufficiently small and ϵ negative. If ϵ is positive, the shock is not a Lax j-shock.*

Proof Using the ϵ parameterization of the Hugoniot locus, we see that the shock is a Lax j-shock if and only if

$$\lambda_{j-1}(0) < s(\epsilon) < \lambda_j(0), \quad \lambda_j(\epsilon) < s(\epsilon) < \lambda_{j+1}(\epsilon), \tag{5.115}$$

where for simplicity we write $u(\epsilon) = u_{j,\epsilon}$, $s(\epsilon) = s_{j,\epsilon}$, and $\lambda_k(\epsilon) = \lambda_k(u_{j,\epsilon})$. The observation following the definition of Lax shocks shows that it suffices to check the inequalities

$$\lambda_j(\epsilon) < s(\epsilon) < \lambda_j(0). \tag{5.116}$$

Assume first that $u(\epsilon) \in H_j(u_l)$ and that ϵ is negative. We know from the implicit function theorem that $s(\epsilon)$ tends to $\lambda_j(0)$ as ϵ tends to zero. From the fact that also $\lambda_j(\epsilon) \to \lambda_j(0)$ as $\epsilon \to 0$, and

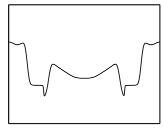
$$\left. \frac{d\lambda_j(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \nabla\lambda_j(0) \cdot r_j(u_l) = 1,$$

it suffices to prove that $0 < s'(0) < 1$. We will in fact prove that $s'(0) = \frac{1}{2}$. Recall from (5.80) that s is an eigenvalue of the matrix $M(u, u_l)$, i.e., $s(\epsilon) = \mu_j(u(\epsilon), u_l)$. Then

$$s'(0) = \nabla_1\mu_j(u_l, u_l) \cdot u'(0) = \frac{1}{2}\nabla\lambda_j(u_l) \cdot r_j(u_l) = \frac{1}{2}, \tag{5.117}$$

using the symmetry (5.87) and the normalization of the right eigenvalue.

If $\epsilon > 0$, we immediately see that $s(\epsilon) > s(0) = \lambda_j(0)$, and in this case we cannot have a Lax j-shock. □



5.5 The Solution of the Riemann Problem

Wie für die Integration der linearen partiellen Differentialgleichungen die fruchtbarsten Methoden nicht durch Entwicklung des allgemeinen Begriffs dieser Aufgabe gefunden worden, sondern vielmehr aus der Behandlung specieller physikalischer Probleme hervorgegangen sind, so scheint auch die Theorie der nichtlinearen partiellen Differentialgleichungen durch eine eingehende, alle Nebenbedingungen berücksichtigende, Behandlung specieller physikalischer Probleme am meisten gefördert zu werden, und in der That hat die Lösung der ganz speciellen Aufgabe, welche den Gegenstand dieser Abhandlung bildet, neue Methoden und Auffassungen erfordert, und zu Ergebnissen geführt, welche wahrscheinlich auch bei allgemeineren Aufgaben eine Rolle spielen werden.[7]
— G. F. B. Riemann [156]

In this section we will combine the properties of the rarefaction waves and shock waves from the previous sections to derive the unique solution of the Riemann problem for small initial data. Our approach will be the following. Assume that the left state u_l is given, and consider the space of all right states u_r. For each right state we want to describe the solution of the corresponding Riemann problem. (We could, of course, reverse the picture and consider the right state as fixed and construct the solution for all possible left states.)

To this end we start by defining *wave curves*. If the jth wave family is genuinely nonlinear, we define

$$W_j(u_l) := R_j(u_l) \cup S_j(u_l), \tag{5.118}$$

and if the jth family is linearly degenerate, we let

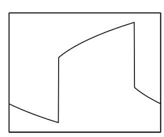
$$W_j(u_l) := C_j(u_l). \tag{5.119}$$

Recall that we have parameterized the shock and rarefaction curves separately with a parameter ϵ such that ϵ positive (negative) corresponds to a rarefaction (shock) wave solution in the case of a genuinely nonlinear wave family. The important fact about the wave curves is that they almost form a local coordinate system around u_l, and this will make it possible to prove existence of solutions of the Riemann problem for u_r close to u_l.

We will commence from the left state u_l and connect it to a nearby intermediate state $u_{m_1} = u_{1,\epsilon_1} \in W_1(u_l)$ using either a rarefaction wave solution ($\epsilon_1 > 0$) or a shock wave solution ($\epsilon_1 < 0$) if the first family is genuinely nonlinear. If the first family is linearly degenerate, we use a contact discontinuity for all ϵ_1. From this state we find another intermediate state $u_{m_2} = u_{2,\epsilon_2} \in W_2(u_{m_1})$. We continue in this way until we have reached an intermediate state $u_{m_{n-1}}$ such that $u_r = u_{n,\epsilon_n} \in W_n(u_{m_{n-1}})$. The problem is to show existence of a unique n-tuple of $(\epsilon_1, \dots, \epsilon_n)$ such that we "hit" u_r starting from u_l using this construction.

As usual, we will start by illustrating the above discussion for the shallow-water equations. This example will contain the fundamental description of the solution, which in principle will carry over to the general case.

[7] The theory of nonlinear equations can, it seems, achieve the most success if our attention is directed to special problems of physical content with thoroughness and with a consideration of all auxiliary conditions. In fact, the solution of the very special problem that is the topic of the current paper requires new methods and concepts and leads to results which probably will also play a role in more general problems.



◇ Example 5.15 (Shallow water (cont'd.))
Fix u_l. For each right state u_r we have to determine one middle state u_m on the first-wave curve through u_l such that u_r is on the second-wave curve with left state u_m, i.e., $u_m \in W_1(u_l)$ and $u_r \in W_2(u_m)$. (In the special case that $u_r \in W_1(u_l) \cup W_2(u_l)$ no middle state u_m is required.) For 2×2 systems of conservation laws it is easier to consider the "backward" second-wave curve $W_2^-(u_r)$ consisting of states u_m that can be connected to u_r on the right with a *fast* wave. The Riemann problem with left state u_l and right state u_r has a unique solution if and only if $W_1(u_l)$ and $W_2^-(u_r)$ have a unique intersection. In that case, clearly the intersection will be the middle state u_m. The curve $W_1(u_l)$ is given by

$$v = v(h) = \begin{cases} v_l - 2\left(\sqrt{h} - \sqrt{h_l}\right) & \text{for } h \in [0, h_l], \\ v_l - \frac{h-h_l}{\sqrt{2}} \sqrt{h^{-1} + h_l^{-1}} & \text{for } h \geq h_l, \end{cases} \tag{5.120}$$

and we easily see that $W_1(u_l)$ is strictly decreasing, unbounded, and starting at $v_l + 2\sqrt{h_l}$. Using (5.49) and (5.106), we find that $W_2^-(u_r)$ reads

$$v = v(h) = \begin{cases} v_r + 2(\sqrt{h} - \sqrt{h_r}) & \text{for } h \in [0, h_r], \\ v_r + \frac{h-h_r}{\sqrt{2}} \sqrt{h^{-1} + h_r^{-1}} & \text{for } h \geq h_r, \end{cases} \tag{5.121}$$

which is strictly increasing, unbounded, with minimum $v_r - 2\sqrt{h_r}$. Thus we conclude that the Riemann problem for shallow water has a unique solution in the region where

$$v_l + 2\sqrt{h_l} \geq v_r - 2\sqrt{h_r}. \tag{5.122}$$

To obtain explicit equations for the middle state u_m we have to make case distinctions, depending on the type of wave curves that intersect, i.e., rarefaction waves or shock curves. This gives rise to four regions, denoted by I, ..., IV. See Fig. 5.7. For completeness we give the equations for the middle state u_m in all cases.

Assume first that $u_r \in$ I. We will determine a unique intermediate state $u_m \in S_1(u_l)$ such that $u_r \in R_2(u_m)$. These requirements give the following equations to be solved for h_m, v_m such that $u_m = (h_m, q_m) = (h_m, h_m v_m)$:

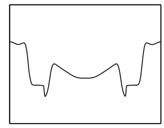
$$v_m = v_l - \frac{1}{\sqrt{2}}(h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}}, \quad v_r = v_m + 2\left(\sqrt{h_r} - \sqrt{h_m}\right).$$

Summing these equations, we obtain the equation

$$\sqrt{2}\, [\![v]\!] = 2\sqrt{2}\left(\sqrt{h_r} - \sqrt{h_m}\right) - (h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}} \quad \text{(I)} \tag{5.123}$$

to determine h_m. Consider next the case with $u_r \in$ III. Here $u_m \in R_1(u_l)$ and $u_r \in S_2(u_m)$, and in this case we obtain

$$\sqrt{2}\, [\![v]\!] = (h_r - h_m)\sqrt{\frac{1}{h_r} + \frac{1}{h_m}} - 2\sqrt{2}\left(\sqrt{h_m} - \sqrt{h_l}\right), \quad \text{(III)} \tag{5.124}$$



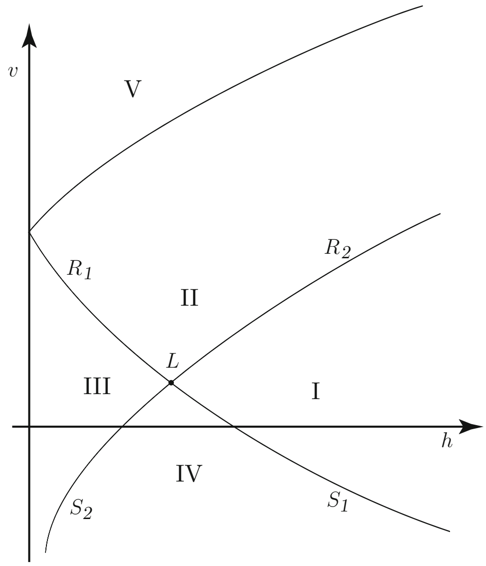


Fig. 5.7 The partition of the (h, v)-plane; see (5.127) and (5.146)

while in the case $u_r \in \text{IV}$, we obtain (here $u_m \in S_1(u_l)$ and $u_r \in S_2(u_m)$)

$$\sqrt{2}\,[\![v]\!] = (h_r - h_m)\sqrt{\frac{1}{h_r} + \frac{1}{h_m}} - (h_m - h_l)\sqrt{\frac{1}{h_m} + \frac{1}{h_l}}. \quad \text{(IV)} \tag{5.125}$$

The case $u_r \in \text{II}$ is special. Here $u_m \in R_1(u_l)$ and $u_r \in R_2(u_m)$. The intermediate state u_m is given by

$$v_m = v_l - 2\left(\sqrt{h_m} - \sqrt{h_l}\right), \quad v_r = v_m + 2\left(\sqrt{h_r} - \sqrt{h_m}\right),$$

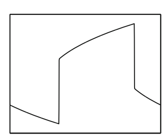
which can easily be solved for h_m to yield

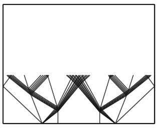
$$\sqrt{h_m} = \frac{2\left(\sqrt{h_r} + \sqrt{h_l}\right) - [\![v]\!]}{4}. \quad \text{(II)} \tag{5.126}$$

This equation is solvable only for right states such that the right-hand side of (5.126) is nonnegative. Observe that this is consistent with what we found above in (5.122). Thus we find that for

$$u_r \in \left\{u \in (0, \infty) \times \mathbb{R} \mid 2(\sqrt{h} + \sqrt{h_l}) \geq [\![v]\!]\right\} \tag{5.127}$$

the Riemann problem has a unique solution consisting of a slow wave followed by a fast wave. Let us summarize the solution of the Riemann problem for the shallow-water equations. First of all, we were not able to solve the problem globally, but only locally around the left state. Secondly, the general solution consists of a composition of elementary waves. More precisely, let $u_r \in \left\{u \in (0, \infty) \times \mathbb{R} \mid 2\left(\sqrt{h} + \sqrt{h_l}\right) \geq [\![v]\!]\right\}$. Let $w_j(x/t; h_m, h_l)$ denote the





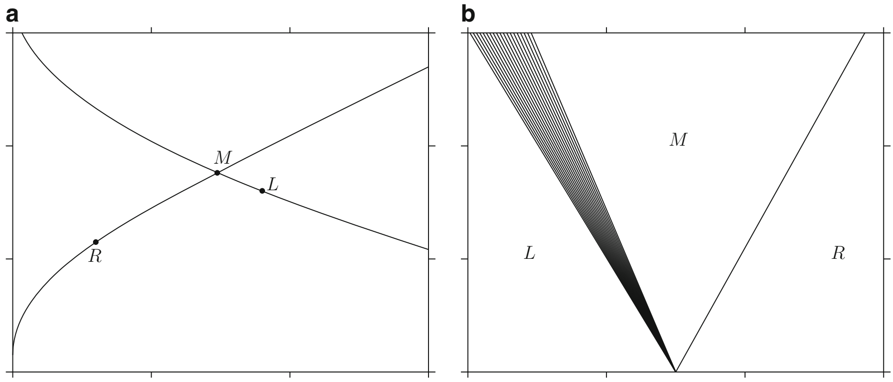


Fig. 5.8 The solution of the Riemann problem in phase space (**a**) and in (x, t)-space (**b**)

solution of the Riemann problem for $u_m \in W_j(u_l)$; here, as in most of our calculations on the shallow-water equations, we use h rather than ϵ as the parameter. We will introduce the notation $\sigma_j^{\pm}$ for the slowest and fastest wave speeds in each family to simplify the description of the full solution. Thus we have that for $j = 1$ $(j = 2)$ and $h_r < h_l$ $(h_r > h_l)$, w_j is a rarefaction-wave solution with slowest speed $\sigma_j^- = \lambda_j(u_l)$ and fastest speed $\sigma_j^+ = \lambda_j(u_r)$. If $j = 1$ $(j = 2)$ and $h_r > h_l$ $(h_r < h_l)$, then w_j is a shock-wave solution with speed $\sigma_j^- = \sigma_j^+ = s_j(h_r, h_l)$. The solution of the Riemann problem reads (see Fig. 5.8)

$$u(x,t) = \begin{cases} u_l & \text{for } x < \sigma_1^- t, \\ w_1(x/t; u_m, u_l) & \text{for } \sigma_1^- t \le x \le \sigma_1^+ t, \\ u_m & \text{for } \sigma_1^+ t < x \le \sigma_2^- t, \\ w_2(x/t; u_r, u_m) & \text{for } \sigma_2^- t \le x \le \sigma_2^+ t, \\ u_r & \text{for } x \ge \sigma_2^+ t. \end{cases} \tag{5.128}$$

We will show later in this chapter how to solve the Riemann problem globally for the shallow-water equations. ◇

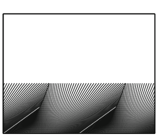
Before we turn to the existence and uniqueness theorem for solutions of the Riemann problem, we will need a certain property of the wave curves that we can explicitly verify for the shallow-water equations.

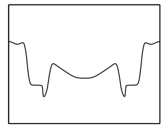
Recall from (5.84) and (5.40) that $\left.\frac{du_\epsilon}{d\epsilon}\right|_{\epsilon=0} = r_j(u_l)$; thus $W_j(u_l)$ is at least differentiable at u_l. In fact, one can prove that $W_j(u_l)$ has a continuous second derivative across u_l.

We introduce the following notation for the *directional derivative* of a quantity $h(u)$ in the direction r (not necessarily normalized) at the point u, which is defined as

$$D_r h(u) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(h(u + \epsilon r) - h(u) \right) = (\nabla h \cdot r)(u). \tag{5.129}$$

(When h is a vector, ∇h denotes the Jacobian.)





Theorem 5.16 *The wave curve $W_j(u_l)$ has a continuous second derivative across u_l. In particular,*

$$u_{j,\epsilon} = u_l + \epsilon r_j(u_l) + \frac{1}{2}\epsilon^2 D_{r_j} r_j(u_l) + \mathcal{O}\left(\epsilon^3\right).$$

Proof In our proof of the admissibility of parts of the Hugoniot loci, Theorem 5.14, we derived most of the ingredients required for the proof of this theorem. The rarefaction curve $R_j(u_l)$ is the integral curve of the right eigenvector $r_j(u)$ passing through u_l, and thus we have (when for simplicity we have suppressed the j-dependence in the notation for u, and write $u(\epsilon) = u_{j,\epsilon}$, etc.)

$$u(0+) = u_l, \quad u'(0+) = r_j(u_l), \quad u''(0+) = \nabla r_j(u_l) r_j(u_l). \tag{5.130}$$

(Here $\nabla r_j(u_l) r_j(u_l)$ denotes the product of the $n \times n$ matrix $\nabla r_j(u_l)$, cf. (5.88), and the (column) vector $r_j(u_l)$.) Recall that the Hugoniot locus is determined by the relation (5.82), i.e.,

$$w_k(u(\epsilon), u_l) \cdot (u(\epsilon) - u_l) = \epsilon \delta_{jk}, \quad k = 1, \dots, n. \tag{5.131}$$

We know already from (5.84) that $u'(0-) = r_j(u_l)$. To find the second derivative of $u(\epsilon)$ at $\epsilon = 0$, we have to compute the second derivative of (5.131). Here we find that[8]

$$2r_j(u_l)\nabla_1 w_k(u_l, u_l) r_j(u_l) + w_k(u_l, u_l) \cdot u''(0-) = 0, \quad k = 1, \dots, n. \tag{5.132}$$

(A careful differentiation of each component may be helpful here; at least we thought so.) In the first term, the matrix $\nabla_1 w_k(u_l, u_l)$ is multiplied from the right by the (column) vector $r_j(u_l)$ and by the (row) vector $r_j(u_l)$ from the left. Using (5.89), i.e., $\nabla_1 w_k(u_l, u_l) = \frac{1}{2}\nabla l_k(u_l)$, we find that

$$r_j(u_l) \cdot \nabla l_k(u_l) r_j(u_l) + l_k(u_l) \cdot u''(0-) = 0. \tag{5.133}$$

The orthogonality of the left and the right eigenvectors, $l_k(u_l) \cdot r_j(u_l) = \delta_{jk}$, shows that

$$r_j(u_l)\nabla l_k(u_l) = -l_k(u_l)\nabla r_j(u_l). \tag{5.134}$$

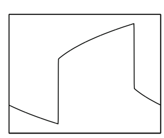
Inserting this into (5.133), we obtain

$$l_k(u_l) \cdot u''(0-) = l_k(u_l)\nabla r_j(u_l) r_j(u_l) \text{ for all } k = 1, \dots, n.$$

From this we conclude that also $u''(0-) = \nabla r_j(u_l) r_j(u_l)$, thereby proving the theorem. □

We will now turn to the proof of the classical Lax theorem about existence of a unique entropy solution of the Riemann problem for small initial data. The assumption of strict hyperbolicity of the system implies the existence of a full set of linearly independent eigenvectors. Furthermore, we have proved that the wave curves are C^2, and hence intersect transversally at the left state. This shows, in

[8] Lo and behold; the second derivative of $w_k(u(\epsilon), u_l)$ is immaterial, since it is multiplied by $u(\epsilon) - u_l$ at $\epsilon = 0$.



a heuristic way, that it is possible to solve the Riemann problem locally. Indeed, we saw that we could write the solution of the corresponding problem for the shallow-water equations as a composition of individual elementary waves that do not interact, in the sense that the fastest wave of one family is slower than the slowest wave of the next family. This will enable us to write the solution in the same form in the general case. In order to do this, we introduce some notation. Let $u_{j,\epsilon_j} = u_{j,\epsilon_j}(x/t; u_r, u_l)$ denote the unique solution of the Riemann problem with left state u_l and right state u_r that consists of a single elementary wave (i.e., shock wave, rarefaction wave, or contact discontinuity) of family j with strength ϵ_j. Furthermore, we need to define notation for speeds corresponding to the fastest and slowest waves of a fixed family. Let

$$\begin{aligned} &\sigma_j^+ = \sigma_j^- = s_{j,\epsilon_j} && \text{if } \epsilon_j < 0, \\ &\left.\begin{aligned} \sigma_j^- &= \lambda_j(u_{j-1,\epsilon_{j-1}}) = \lambda_j(u_{m_{j-1}}), \\ \sigma_j^+ &= \lambda_j(u_{j,\epsilon_j}) = \lambda_j(u_{m_j}) \end{aligned}\right\} && \text{if } \epsilon_j > 0, \end{aligned} \tag{5.135}$$

if the jth wave family is genuinely nonlinear, and

$$\sigma_j^+ = \sigma_j^- = \lambda_j(u_{j,\epsilon_j}) = \lambda_j(u_{m_j}) \tag{5.136}$$

if the jth wave family is linearly degenerate. With these definitions we are ready to write the solution of the Riemann problem as

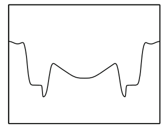
$$u(x,t) = \begin{cases} u_l & \text{for } x < \sigma_1^- t, \\ u_{1,\epsilon_1}(x/t; u_{m_1}, u_l) & \text{for } \sigma_1^- t \le x \le \sigma_1^+ t, \\ u_{m_1} & \text{for } \sigma_1^+ t \le x < \sigma_2^- t, \\ u_{2,\epsilon_2}(x/t; u_{m_2}, u_{m_1}) & \text{for } \sigma_2^- t \le x \le \sigma_2^+ t, \\ u_{m_2} & \text{for } \sigma_2^+ t \le x < \sigma_3^- t, \\ \vdots & \\ u_{n,\epsilon_n}(x/t; u_r, u_{m_{n-1}}) & \text{for } \sigma_n^- t \le x \le \sigma_n^+ t, \\ u_r & \text{for } x \ge \sigma_n^+ t. \end{cases} \tag{5.137}$$

Theorem 5.17 (Lax's theorem) *Assume that $f_j \in C^2(\mathbb{R}^n)$, $j = 1, \dots, n$. Let D be a domain in $\mathbb{R}^n$ and consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ with $u \in D$. Assume that each wave family is either genuinely nonlinear or linearly degenerate.*

Then for $u_l \in D$ there exists a neighborhood $\tilde{D} \subset D$ of u_l such that for all $u_r \in \tilde{D}$ the Riemann problem

$$u(x,0) = \begin{cases} u_l & \textit{for } x < 0, \\ u_r & \textit{for } x \ge 0, \end{cases} \tag{5.138}$$

has a unique solution in $\tilde{D}$ consisting of up to n elementary waves, i.e., rarefaction waves, shock solutions satisfying the Lax entropy condition, or contact discontinuities. The solution is given by (5.137).



Proof Consider the map $W_{j,\epsilon}: u \mapsto u_{j,\epsilon} \in W_j(u)$. We may then write the solution of the Riemann problem using the composition

$$W_{(\epsilon_1,\dots,\epsilon_n)} = W_{n,\epsilon_n} \circ \cdots \circ W_{1,\epsilon_1} \tag{5.139}$$

as

$$W_{(\epsilon_1,\dots,\epsilon_n)} u_l = u_r, \tag{5.140}$$

and we want to prove the existence of a unique vector $(\epsilon_1, \dots, \epsilon_n)$ (near the origin) such that (5.140) is satisfied for $|u_l - u_r|$ small. In our proof we will need the two leading terms, i.e., up to the linear term, in the Taylor expansion for W. For later use we expand to the quadratic term in the next lemma.

Lemma 5.18 *We have*

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_n)}(u_l) = u_l &+ \sum_{i=1}^{n} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &+ \sum_{\substack{i,j=1\\ j<i}}^{n} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right). \end{aligned} \tag{5.141}$$

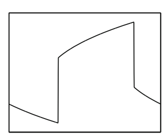
Proof (of Lemma 5.18) We shall show that for $k = 1, \dots, n$,

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l) = u_l &+ \sum_{i=1}^{k} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &+ \sum_{\substack{i,j=1\\ j<i}}^{k} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right) \end{aligned} \tag{5.142}$$

by induction on k. It is clearly true for $k = 1$; cf. Theorem 5.16. Assume (5.142). Now,

$$\begin{aligned} W_{(\epsilon_1,\dots,\epsilon_{k+1},0,\dots,0)}\,(u_l) &= W_{k+1,\epsilon_{k+1}}\left(W_{(\epsilon_1,\dots,\epsilon_k)}(u_l)\right) \\ &= u_l + \sum_{i=1}^{k} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{k} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \epsilon_{k+1} r_{k+1}\left(W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l)\right) \\ &\quad + \frac{1}{2}\epsilon_{k+1}^2 D_{r_{k+1}} r_{k+1}\left(W_{(\epsilon_1,\dots,\epsilon_k,0,\dots,0)}(u_l)\right) + \mathcal{O}\left(|\epsilon|^3\right) \\ &= u_l + \sum_{i=1}^{k+1} \epsilon_i r_i\,(u_l) + \frac{1}{2}\sum_{i=1}^{k+1} \epsilon_i^2 D_{r_i} r_i\,(u_l) \\ &\quad + \sum_{\substack{i,j=1\\ j<i}}^{k+1} \epsilon_i \epsilon_j D_{r_i} r_j\,(u_l) + \mathcal{O}\left(|\epsilon|^3\right) \end{aligned}$$

by Theorem 5.16. □



Let $u_l \in D$ and define the map

$$\mathcal{L}(\epsilon_1, \dots, \epsilon_n, u) = W_{(\epsilon_1,\dots,\epsilon_n)} u_l - u. \tag{5.143}$$

This map $\mathcal{L}$ satisfies

$$\mathcal{L}(0,\dots,0,u_l) = 0, \quad \nabla_\epsilon \mathcal{L}(0,\dots,0,u_l) = (r_1(u_l),\dots,r_n(u_l)),$$

where the matrix $\nabla\mathcal{L}$ has the right eigenvectors r_j evaluated at u_l as columns. This matrix is nonsingular by the strict hyperbolicity assumption.

The implicit function theorem then implies the existence of a neighborhood N around u_l and a unique differentiable function $(\epsilon_1,\dots,\epsilon_n) = (\epsilon_1(u),\dots,\epsilon_n(u))$ such that $\mathcal{L}(\epsilon_1,\dots,\epsilon_n,u) = 0$. If $u_r \in N$, then there exists unique $(\epsilon_1,\dots,\epsilon_n)$ with $W_{(\epsilon_1,\dots,\epsilon_n)} u_l = u_r$, which proves the theorem. □

Observe that we could rephrase the Lax theorem as saying that we may use $(\epsilon_1,\dots,\epsilon_n)$ to measure distances in phase space, and that we indeed have

$$A\,|u_r - u_l| \le \sum_{j=1}^{n} |\epsilon_j| \le B\,|u_r - u_l| \tag{5.144}$$

for constants A and B.

Let us now return to the shallow-water equations and prove the existence of a global solution of the Riemann problem.

◇ Example 5.19 (Shallow water (cont'd.))
We will construct a global solution of the Riemann problem for the shallow-water equations for all left and right states in $D = \{(h,v) \mid h \in [0,\infty), v \in \mathbb{R}\}$. Of course, we will maintain the same solution in the region where we already have constructed a solution, so it remains to construct a solution in the region

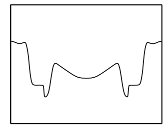
$$u_r \in V := \left\{u_r \in D \mid 2\left(\sqrt{h_r} + \sqrt{h_l}\right) < [\![v]\!]\right\} \cup \{h = 0\}. \tag{5.145}$$

We will work in the (h,v) variables rather than (h,q). Assume first that $u_r = (h_r, v_r)$ in V with h_r positive. We first connect u_l, using a slow rarefaction wave, with a state u_m on the "vacuum line" $h = 0$. This state is given by

$$v_m = v_l + 2\sqrt{h_l}, \tag{5.146}$$

using (5.50). From this state we jump to the unique point v^* on $h = 0$ such that the fast rarefaction starting at $h^* = 0$ and v^* hits u_r. Thus we see from (5.51) that $v^* = v_r - 2\sqrt{h_r}$, which gives the following solution (see Fig. 5.9):

$$u(x,t) = \begin{cases} \binom{h_l}{v_l} & \text{for } x < \lambda_1(u_l)t, \\ R_1(x/t; u_l) & \text{for } \lambda_1(u_l)t < x < (2\sqrt{h_l} + v_l)t, \\ \binom{0}{\tilde{v}(x,t)} & \text{for } (2\sqrt{h_l} + v_l)t < x < v^* t, \\ R_2(x/t; (0, v^*)) & \text{for } v^* t < x < \lambda_2(u_r)t, \\ \binom{h_r}{v_r} & \text{for } x > \lambda_2(u_r)t. \end{cases} \tag{5.147}$$



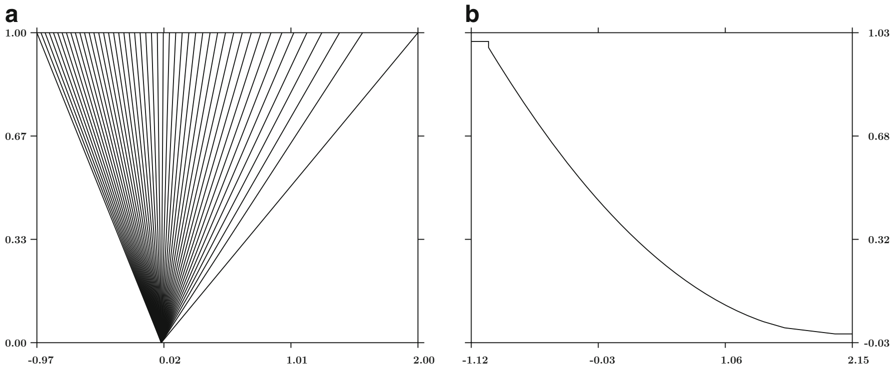


Fig. 5.9 The solution of the dam-breaking problem in (x, t)-space (**a**), and the h-component (**b**)

Physically, it does not make sense to give a value of the speed v of the water when there is no water, i.e., $h = 0$, and mathematically we see that any v will satisfy the equations when $h = 0$. Thus we do not have to associate any value with $\tilde{v}(x, t)$.

If u_r is on the vacuum line $h = 0$, we still connect to a state u_m on $h = 0$ using a slow rarefaction, and subsequently we connect to u_r along the vacuum line. By considering a nearby state $\tilde{u}_r$ with $\tilde{h} > 0$, we see that with this construction we have continuity in the data.

Finally, we have to solve the Riemann problem with the left state on the vacuum line $h = 0$. Now let $u_l = (0, v_l)$, and let $u_r = (h_r, v_r)$ with $h_r > 0$. We now connect u_l to an intermediate state u_m on the vacuum line given by $v_m = v_r - 2\sqrt{h_r}$ and continue with a fast rarefaction to the right state u_r. ◇

We will apply the above theory to one old and two ancient problems:

◇ Example 5.20 (Dam breaking)

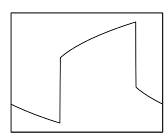
For this problem we consider Riemann initial data of the form (in (h, v) variables)

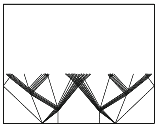
$$u(x,0) = \begin{pmatrix} h(x,0) \\ v(x,0) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < 0, \\ \binom{0}{0} & \text{for } x \geq 0. \end{cases}$$

From the above discussion we know that the solution consists of a slow rarefaction (see Fig. 5.10); thus

$$u(x,t) = \begin{pmatrix} h(x,t) \\ v(x,t) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < -\sqrt{h_l}\,t, \\ \begin{pmatrix} \frac{1}{9}\left(2\sqrt{h_l} - \frac{x}{t}\right)^2 \\ \frac{2}{3}\left(\sqrt{h_l} + \frac{x}{t}\right) \end{pmatrix} & \text{for } -\sqrt{h_l}\,t < x < 2\sqrt{h_l}\,t, \\ \binom{0}{0} & \text{for } x > 2\sqrt{h_l}\,t. \end{cases} \qquad \diamond$$

We shall call the two ancient problems Moses's first and second problems.





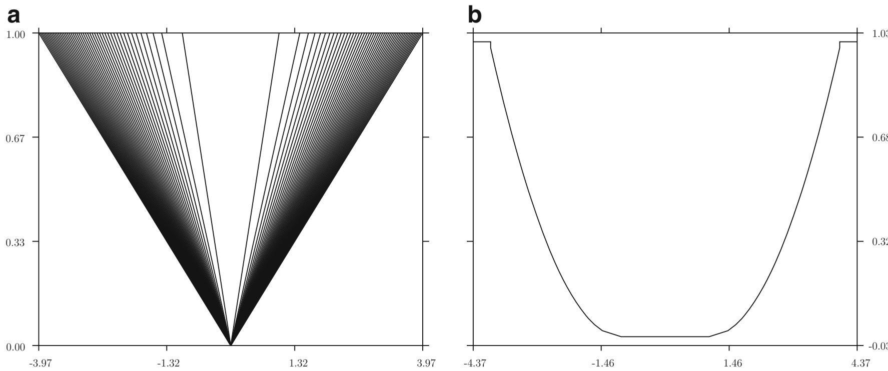


Fig. 5.10 The solution of Moses's first problem in (x, t)-space (**a**), and the h-component (**b**)

◇ Example 5.21 (Moses's first problem)

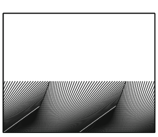
And Moses stretched out his hand over the sea; and the Lord caused the sea to go back by a strong east wind all that night, and made the sea dry land, and the waters were divided. And the children of Israel went into the midst of the sea upon the dry ground: and the waters were a wall unto them on their right hand, and on their left.
— Exodus (14:21–22)

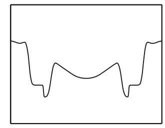
For the first problem we consider initial data of the form (in (h, v) variables)

$$u(x,0) = \begin{cases} \binom{h_0}{-v_0} & \text{for } x < 0, \\ \binom{h_0}{v_0} & \text{for } x \geq 0, \end{cases}$$

for a positive speed v_0. By applying the above analysis, we find that in this case we connect to an intermediate state u_1 on the vacuum line using a slow rarefaction. This state is connected to another state u_2 also on the vacuum line, which subsequently is connected to the right state using a fast rarefaction wave. More precisely, the state u_1 is determined by $v_1 = v(x_1, t_1)$, where $h(x,t) = \frac{1}{9}\big(-v_0 + 2\sqrt{h_0} - \frac{x}{t}\big)^2$ along the slow rarefaction wave (cf. (5.53)) and $h(x_1, t_1) = 0$. We find that $x_1 = \big(2\sqrt{h_0} - v_0\big)t_1$ and thus $v_1 = 2\sqrt{h_0} - v_0$. The second intermediate state u_2 is such that a fast rarefaction wave with left state u_2 hits u_r. This implies that $v_0 = v_2 + 2\sqrt{h_0}$ from (5.51), or $v_2 = v_0 - 2\sqrt{h_0}$. In order for this construction to be feasible, we will have to assume that $v_2 > v_1$ or $v_0 \geq 2\sqrt{h_0}$. If this condition does not hold, we will not get a region without water, and thus the original problem of Moses will not be solved. Combining the above waves in one solution, we obtain

$$h(x,t) = \begin{cases} h_0 & \text{for } x < -\big(v_0 + \sqrt{h_0}\big)t, \\ \frac{1}{9}\big(-v_0 + 2\sqrt{h_0} - \frac{x}{t}\big)^2 & \text{for } -\big(v_0 + \sqrt{h_0}\big)t < x < \big(2\sqrt{h_0} - v_0\big)t, \\ 0 & \text{for } \big(2\sqrt{h_0} - v_0\big)t < x < \big(v_0 - 2\sqrt{h_0}\big)t, \\ \frac{1}{9}\big(v_0 - 2\sqrt{h_0} - \frac{x}{t}\big)^2 & \text{for } \big(v_0 - 2\sqrt{h_0}\big)t < x < \big(v_0 + \sqrt{h_0}\big)t, \\ h_0 & \text{for } x > \big(v_0 + \sqrt{h_0}\big)t, \end{cases}$$





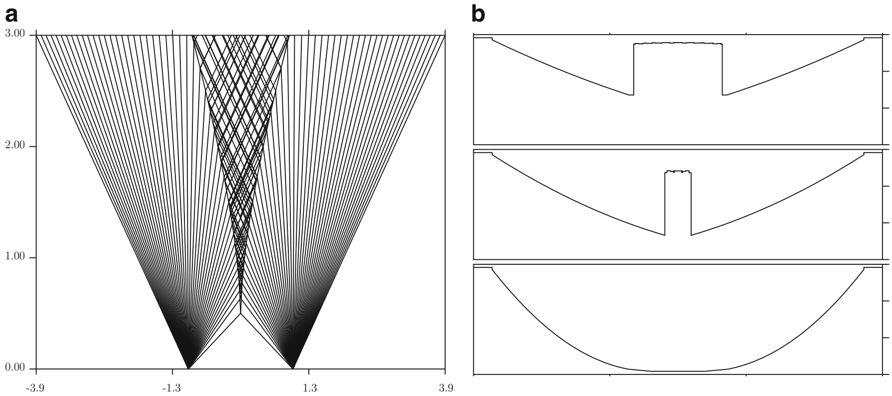


Fig. 5.11 The solution of Moses's second problem in (x,t)-space (**a**), and the h-component (**b**)

$$v(x,t) = \begin{cases} -v_0 & \text{for } x < -\left(v_0 + \sqrt{h_0}\right)t, \\ \frac{1}{3}\left(-v_0 + 2\sqrt{h_0} + 2\frac{x}{t}\right) & \text{for } -\left(v_0 + \sqrt{h_0}\right)t < x < \left(2\sqrt{h_0} - v_0\right)t, \\ 0 & \text{for } \left(2\sqrt{h_0} - v_0\right)t < x < \left(v_0 - 2\sqrt{h_0}\right)t, \\ \frac{1}{3}\left(v_0 - 2\sqrt{h_0} + 2\frac{x}{t}\right) & \text{for } \left(v_0 - 2\sqrt{h_0}\right)t < x < \left(v_0 + \sqrt{h_0}\right)t, \\ v_0 & \text{for } x > \left(v_0 + \sqrt{h_0}\right)t. \end{cases}$$

◇

◇ Example 5.22 (Moses's second problem)

And Moses stretched forth his hand over the sea, and the sea returned to his strength when the morning appeared; and the Egyptians fled against it; and the Lord overthrew the Egyptians in the midst of the sea.
— Exodus (14:27)

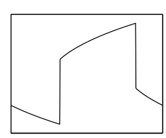
Here we study the multiple Riemann problem given by (in (h,v) variables)

$$u(x,0) = \begin{cases} \binom{h_0}{0} & \text{for } x < 0, \\ \binom{0}{0} & \text{for } 0 < x < L, \\ \binom{h_0}{0} & \text{for } x > L. \end{cases}$$

For small times t, the solution of this problem is found by patching together the solution of two dam-breaking problems. The left problem is solved by a fast rarefaction wave, and the right by a slow rarefaction. At some positive time, these rarefactions will interact, and thereafter explicit computations become harder.

In place of explicit computation we therefore present the numerical solution constructed by front tracking. This method is a generalization of the front-tracking method presented in Chapt. 2, and will be the subject of the next chapter.

In the left part of Fig. 5.11 we see the fronts in (x,t)-space. These fronts are similar to the fronts for the scalar front tracking, and the approximate solution is discontinuous across the lines shown in the figure. Looking at the figure, it is not hard to see why explicit computations become difficult as the two rarefaction waves



interact. The right part of the figure shows the water level as it engulfs the Egyptians. The lower figure shows the water level before the two rarefaction waves interact, and the two upper ones show that two shock waves result from the interaction of the two rarefaction waves. ◇

5.6 The Riemann Problem for the Euler Equations

The Euler equations are often used as a simplification of the Navier–Stokes equations as a model of the flow of a gas. In one space dimension these represent the conservation of mass, momentum, and energy, and read

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0. \tag{5.148}$$

Here ρ denotes the density of the gas, v the velocity, p the pressure, and E the energy. To close this system, i.e., to reduce the number of unknowns to the number of equations, one can add a constitutive "law" relating these. Such laws are often called *equations of state* and are deduced from thermodynamics. For a so-called ideal polytropic gas the equation of state takes the form

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2,$$

where $\gamma > 1$ is a constant spesific to the gas. For air, $\gamma \approx 1.4$. Solving for p, we get

$$p = (\gamma - 1)\, E - \frac{\gamma - 1}{2}\rho v^2 = (\gamma - 1)\, E - \frac{\gamma - 1}{2}\frac{q^2}{\rho}, \tag{5.149}$$

where the momentum q equals ρv. Inserting this in the Euler equations yields

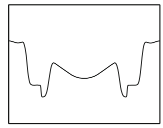
$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \frac{\gamma-3}{2}\rho v^2 + (\gamma - 1)E \\ v\left(\gamma E - \frac{\gamma-1}{2}\rho v^2\right) \end{pmatrix}_x = 0.$$

In the conserved variables ρ, q, and E, this system of conservation laws reads

$$\begin{pmatrix} \rho \\ q \\ E \end{pmatrix}_t + \begin{pmatrix} q \\ \left(\frac{3-\gamma}{2}\right)\frac{q^2}{\rho} + (\gamma - 1)\, E \\ \gamma\frac{Eq}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^3}{\rho^2} \end{pmatrix}_x = 0. \tag{5.150}$$

Set

$$u = \begin{pmatrix} \rho \\ q \\ E \end{pmatrix} \qquad \text{and} \qquad f(u) = \begin{pmatrix} q \\ \left(\frac{3-\gamma}{2}\right)\frac{q^2}{\rho} + (\gamma - 1)\, E \\ \gamma\frac{Eq}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^3}{\rho^2} \end{pmatrix}.$$



Then the Jacobian $df(u)$ reads

$$df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \left(\frac{\gamma-3}{2}\right)\frac{q^2}{\rho^2} & (3-\gamma)\frac{q}{\rho} & \gamma-1 \\ -\gamma\frac{Eq}{\rho^2} + (\gamma-1)\frac{q^3}{\rho^3} & \gamma\frac{E}{\rho} - \frac{3(\gamma-1)}{2}\frac{q^2}{\rho^2} & \gamma\frac{q}{\rho} \end{pmatrix}.$$

Introducing the *enthalpy* as

$$H = \frac{E+p}{\rho} = \gamma\frac{E}{\rho} - \left(\frac{\gamma-1}{2}\right)\frac{q^2}{\rho^2} = \frac{\gamma}{\rho}\left(\frac{p}{\gamma-1}\right) + \frac{1}{2}v^2,$$

the Jacobian can be rewritten as

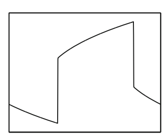
$$df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \left(\frac{\gamma-3}{2}\right)v^2 & (3-\gamma)\,v & \gamma-1 \\ \left(\frac{\gamma-1}{2}\right)v^3 - vH & H - (\gamma-1)\,v^2 & \gamma v \end{pmatrix}.$$

To find its eigenvalues, we compute the determinant

$$\begin{aligned}
\det(\lambda I - df(u)) &= \lambda\left[(\lambda-(3-\gamma)v)\,(\lambda-\gamma v) + (\gamma-1)\left((\gamma-1)\,v^2 - H\right)\right] \\
&\quad + \frac{3-\gamma}{2}v^2\,(\lambda-\gamma v) + (\gamma-1)\left(vH - \frac{\gamma-1}{2}v^3\right) \\
&= \lambda\left[\lambda^2 - 3v\lambda + \gamma(3-\gamma)v^2 + (\gamma-1)^2v^2 + (\gamma-1)H\right] \\
&\quad + \frac{3-\gamma}{2}v^2\lambda - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)Hv \\
&= \lambda\left[\lambda^2 - 3v\lambda + 2v^2 + \frac{1}{2}(\gamma+1)\,v^2 - (\gamma-1)H\right] \\
&\quad - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)vH \\
&= \lambda\Big[(\lambda-v)(\lambda-2v) + \frac{1}{2}(\gamma+1)v^2 - (\gamma-1)H\Big] \\
&\quad - \frac{1}{2}(\gamma+1)v^3 + (\gamma-1)vH \\
&= (\lambda-v)\left[\lambda(\lambda-2v) + \frac{1}{2}(\gamma+1)v^2 - (\gamma-1)H\right] \\
&= (\lambda-v)\left[(\lambda-v)^2 - \left(v^2 - \frac{1}{2}(\gamma+1)v^2 + (\gamma-1)H\right)\right] \\
&= (\lambda-v)\left[(\lambda-v)^2 - \left(\frac{\gamma-1}{2}(2H-v^2)\right)\right].
\end{aligned}$$

This can be simplified further by introducing the *sound speed* c, by

$$c^2 = \frac{\gamma p}{\rho}.$$



We then calculate

$$\begin{aligned}2H - v^2 &= 2\gamma\frac{E}{\rho} - (\gamma-1)v^2 - v^2 = 2\gamma\frac{E}{\rho} - \gamma v^2 = \gamma\left(\frac{2E}{\rho} - v^2\right)\\ &= \frac{\gamma}{\rho}\left(2E - \rho v^2\right) = \frac{\gamma}{\rho}\frac{2p}{\gamma-1}.\end{aligned}$$

Therefore

$$\det(\lambda I - df(u)) = (\lambda - v)\left[(\lambda - v)^2 - c^2\right].$$

Thus the eigenvalues of the Jacobian are

$$\lambda_1(u) = v - c,\ \ \lambda_2(u) = v,\ \ \lambda_3(u) = v + c. \tag{5.151}$$

As for the corresponding eigenvectors, we write these as $r_i = (1, y_i, z_i)$,[9] and we see that $y_i = \lambda_i$, and

$$z_i = \frac{1}{\gamma-1}\left(\lambda_i^2 - \frac{1}{2}(\gamma-3)v^2 + \lambda_i(\gamma-3)v\right).$$

For $i = 1$ we find that

$$\begin{aligned}z_1 &= \frac{1}{\gamma-1}\left(v^2 - \frac{1}{2}(\gamma-3)v^2 + v(\gamma-3)v\right)\\ &\quad + \frac{1}{\gamma-1}\left(c^2 - 2cv - (\gamma-3)cv\right)\\ &= \frac{1}{2}v^2 + \frac{c^2}{\gamma-1} - cv\\ &= \left(\frac{1}{2}v^2 + \frac{\gamma p}{\rho(\gamma-1)}\right) - cv\\ &= H - cv.\end{aligned}$$

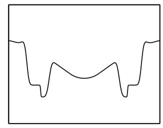
For $i = 3$ we similarly calculate

$$z_3 = H + cv,$$

and for $i = 2$ it is straightforward to see that $z_2 = v^2/2$. Summing up, we have the following eigenvalues and eigenvectors:

$$\begin{aligned}\lambda_1(u) &= v - c, & r_1(u) &= \begin{pmatrix}1\\ v - c\\ H - cv\end{pmatrix},\\ \lambda_2(u) &= v, & r_2(u) &= \begin{pmatrix}1\\ v\\ \frac{1}{2}v^2\end{pmatrix},\\ \lambda_3(u) &= v + c, & r_3(u) &= \begin{pmatrix}1\\ v + c\\ H + cv\end{pmatrix}.\end{aligned} \tag{5.152}$$

[9] Recall that this is in (ρ, q, E) coordinates.



It is important to observe that the second family is linearly degenerate, since

$$\nabla\lambda_2(u) \cdot r_2(u) \equiv 0, \tag{5.153}$$

and hence the solution of the Riemann problem in this family will consist of a contact discontinuity. The first and the third families are both genuinely nonlinear, and we encounter the familiar shock and rarefaction waves.

At this point it is convenient to introduce the concept of an *i-Riemann invariant*. (See Exercise 5.8.) An i-Riemann invariant is a function $R = R(\rho, q, E)$ such that R is constant along the integral curves of r_i. In other words, an i-Riemann invariant satisfies

$$\nabla R(u) \cdot r_i = 0.$$

The usefulness of this is that if we can find for each of the three eigenvectors, two Riemann invariants $R(u)$ and $\tilde{R}(u)$, then we can possibly solve the equations

$$R(\rho, q, E) = R(\rho_l, q_l, E_l), \qquad \tilde{R}(\rho, q, E) = \tilde{R}(\rho_l, q_l, E_l)$$

to obtain a formula for the rarefaction waves. This is equivalent to finding an implicit solution of the ordinary differential equation $\dot{u} = r(u)$ defining the rarefaction curves.

It turns out that we have the following Riemann invariants (see Exercise 5.12):

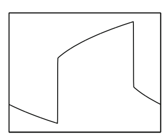
$$\begin{aligned}
&i = 1, \quad \text{Riemann invariants:} \begin{cases} S, \\ v + \frac{2c}{\gamma - 1}, \end{cases} \\
&i = 2, \quad \text{Riemann invariants:} \begin{cases} v, \\ p, \end{cases} \\
&i = 3, \quad \text{Riemann invariants:} \begin{cases} S, \\ v - \frac{2c}{\gamma - 1}, \end{cases}
\end{aligned} \tag{5.154}$$

where we have introduced the *entropy* S by

$$S = -\log\left(\frac{p}{\rho^\gamma}\right). \tag{5.155}$$

Now we can try to obtain solution formulas for the rarefaction curves. For $i = 1$, this curve is given by

$$p = p_l \left(\frac{\rho}{\rho_l}\right)^\gamma, \qquad v = v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right).$$



This curve is parameterized by ρ. We must check which half of the curve to use. This will be the part where $\lambda_1 = v - c$ is increasing. On the curve we have

$$\begin{aligned}
v(\rho) - c(\rho) &= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - \left(\frac{\gamma p(\rho)}{\rho}\right)^{1/2} \\
&= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - \left(\frac{\gamma p_l}{\rho_l}\right)^{1/2}\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2} \\
&= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right) - c_l\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2} \\
&= v_l + \frac{2c_l}{\gamma - 1}\left(1 - \frac{\gamma + 1}{2}\left(\frac{\rho}{\rho_l}\right)^{(\gamma-1)/2}\right).
\end{aligned}$$

Since $\gamma > 1$, we see that $v(\rho) - c(\rho)$ is decreasing in ρ, and for the 1-rarefaction wave we must use $\rho < \rho_l$. Since $p(\rho)$ is increasing in ρ, this also means that we use the part where $p < p_l$. Therefore we can use p as a parameter in the curve for v and write the 1-rarefaction curve as

$$v_1(p) = v_l + \frac{2c_l}{\gamma - 1}\left(1 - \left(\frac{p}{p_l}\right)^{(\gamma-1)/(2\gamma)}\right), \quad p \le p_l.$$

The general theory tells us that (at least for p close to p_l) this curve can be continued smoothly as a 1-shock curve.

To find the rarefaction curve of the third family, we adopt the viewpoint that u_r is fixed, and we wish to find u as a function of u_r (cf. the solution of the Riemann problem for the shallow-water equations). In the same way as for v_1 this leads to the formula

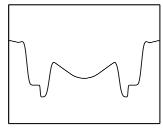
$$v_3(p) = v_r + \frac{2c_r}{\gamma - 1}\left(1 - \left(\frac{p}{p_r}\right)^{(\gamma-1)/(2\gamma)}\right), \quad p \le p_r.$$

To find how the density varies along the rarefaction curves, we can use that the entropy S is constant, leading to

$$\frac{\rho}{\rho_l} = \left(\frac{p}{p_l}\right)^{1/\gamma}.$$

Now we turn to the computation of the Hugoniot loci. We view the left state u_l as fixed, and try to find the right state u; recall the notation $[\![u]\!] = u - u_l$. The Rankine–Hugoniot relations for (5.148) are

$$\begin{aligned}
s\,[\![\rho]\!] &= [\![\rho v]\!], \\
s\,[\![\rho v]\!] &= [\![\rho v^2 + p]\!], \\
s\,[\![E]\!] &= [\![v(E + p)]\!],
\end{aligned} \tag{5.156}$$



where s denotes the speed of the discontinuity. Now we introduce new variables by

$$w = v - s \quad \text{and} \quad m = \rho w.$$

Then the first equation in (5.156) reads

$$s\rho - s\rho_l = \rho w + s\rho - \rho_l w_l - s\rho_l,$$

which implies that $\llbracket m \rrbracket = 0$. Similarly, the second equation reads

$$s\rho w + s^2\rho - s\rho w_l - s^2\rho_l = \rho(w+s)^2 - \rho_l(w_l+s)^2 + \llbracket p \rrbracket,$$

or

$$s\llbracket m \rrbracket + s^2\llbracket \rho \rrbracket = \rho w^2 + 2\rho w + s^2\rho - \rho_l w_l^2 - 2\rho_l w_l - s^2\rho_l + \llbracket p \rrbracket,$$

and subsequently

$$s^2\llbracket \rho \rrbracket = \llbracket \rho w^2 + p \rrbracket + s^2\llbracket \rho \rrbracket.$$

Hence $\llbracket mw + p \rrbracket = 0$. Finally, the third equation in (5.156) reads

$$sE - sE_l = Ew + Es + pw + ps - E_l w_l - E_l s - p_l w_l - p_l s,$$

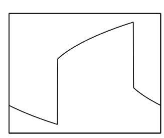
which implies

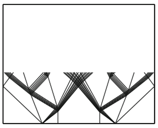
$$\begin{aligned}
0 &= \left(\frac{E}{\rho} - \frac{E_l}{\rho_l}\right) m + pw - p_l w_l + s\llbracket p \rrbracket \\
&= \left(\frac{E}{\rho} - \frac{E_l}{\rho_l} + \frac{p}{\rho} - \frac{p_l}{\rho_l}\right) m - sm\llbracket w \rrbracket \\
&= m\left\llbracket \frac{E+p}{\rho} - sw \right\rrbracket \\
&= m\left\llbracket \frac{c^2}{\gamma-1} + \frac{1}{2}(w+s)^2 - sw \right\rrbracket \\
&= m\left\llbracket \frac{c^2}{\gamma-1} + \frac{1}{2}w^2 \right\rrbracket.
\end{aligned}$$

Hence the Rankine–Hugoniot conditions are equivalent to

$$\begin{aligned}
\llbracket m \rrbracket &= 0, \\
\llbracket mw + p \rrbracket &= 0, \\
m\left\llbracket \frac{c^2}{\gamma-1} + \frac{1}{2}w^2 \right\rrbracket &= 0.
\end{aligned} \tag{5.157}$$

We immediately find one solution by setting $m = 0$, which implies $\llbracket p \rrbracket = 0$ and $\llbracket v \rrbracket = 0$. This is the contact discontinuity. Hence we assume that $m \neq 0$ to find the other Hugoniot loci.





Now we introduce auxiliary parameters

$$\pi = \frac{p}{p_l} \quad \text{and} \quad z = \frac{\rho}{\rho_l}.$$

Using these, we have that

$$\frac{c^2}{c_l^2} = \frac{\pi}{z} \quad \text{and} \quad \frac{w}{w_l} = \frac{1}{z}. \tag{5.158}$$

Then the third equation in (5.157) reads

$$\frac{c_l^2}{\gamma - 1} + \frac{1}{2}w_l^2 = \frac{c_l^2}{\gamma - 1}\frac{\pi}{z} + \frac{1}{2}w_l^2\frac{1}{z^2},$$

which can be rearranged as

$$c_l^2 \frac{2}{\gamma - 1}\left(1 - \frac{\pi}{z}\right) = w_l^2\left(\frac{1}{z^2} - 1\right),$$

so that

$$\left(\frac{w_l}{c_l}\right)^2 = \frac{2}{\gamma - 1}\frac{z(z - \pi)}{1 - z^2}. \tag{5.159}$$

Next recall that $p = \rho c^2/\gamma$. Using this, the second equation in (5.157) reads

$$\frac{\rho c^2}{\gamma} + \rho w^2 = \frac{\rho_l c_l^2}{\gamma} + \rho_l w_l^2,$$

or

$$z\left(\frac{c^2}{\gamma} + w^2\right) = \frac{c_l^2}{\gamma} + w_l^2,$$

which again can be rearranged as

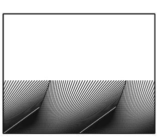
$$z\left(\frac{c_l^2 \pi}{\gamma z} + w_l^2 \frac{1}{z^2}\right) = \frac{c_l^2}{\gamma} + w_l^2.$$

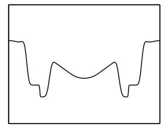
Dividing by c_l^2, we can solve for $(w_l/c_l)^2$:

$$\left(\frac{w_l}{c_l}\right)^2 = \frac{1}{\gamma}\frac{z(\pi - 1)}{z - 1}. \tag{5.160}$$

Equating (5.160) and (5.159) and solving for z yields

$$z = \frac{\beta\pi + 1}{\pi + \beta}, \tag{5.161}$$





where

$$\beta = \frac{\gamma+1}{\gamma-1}. \tag{5.162}$$

Using this expression for z in (5.159), we get

$$\begin{aligned}
\left(\frac{w_l}{c_l}\right)^2 &= \frac{2}{\gamma-1}\frac{\frac{\pi\beta+1}{\pi+\beta}\left(\frac{\pi\beta+1}{\pi+\beta}-\pi\right)}{1-\frac{(\pi\beta+1)^2}{(\pi+\beta)^2}} \\
&= \frac{2}{\gamma-1}\frac{(\pi\beta+1)(1-\pi^2)}{(\pi^2-1)(1-\beta^2)} \\
&= \frac{2}{\gamma-1}\frac{\pi\beta+1}{\beta^2-1}.
\end{aligned}$$

Note that $\gamma > 1$ implies $\beta > 1$, so that this is always well defined. Since $w_l = v_l - s$, we can use this to get an expression for the shock speed,

$$s = v_l \mp c_l\sqrt{\frac{2}{\gamma-1}\frac{\beta\pi+1}{\beta^2-1}}, \tag{5.163}$$

where we use the minus sign for the first family and the plus sign for the third.

Next, using (5.158), we get

$$\frac{v-s}{v_l-s} = \frac{1}{z},$$

which can be used to express v as a function of π:

$$\begin{aligned}
v &= v_l \mp c_l\sqrt{\frac{2}{\gamma-1}\frac{\beta\pi+1}{(\beta^2-1)}} \pm \frac{\pi+\beta}{\pi\beta+1}c_l\sqrt{\frac{2}{\gamma-1}\frac{\pi\beta+1}{(\beta^2-1)}} \\
&= v_l \mp c_l\sqrt{\frac{2}{\gamma-1}\frac{1}{(\beta^2-1)}(\pi\beta+1)}\left(\frac{(\beta-1)(\pi-1)}{\pi\beta+1}\right) \\
&= v_l \mp 2c_l\frac{1}{\sqrt{2\gamma(\gamma-1)}}\frac{\pi-1}{(\pi\beta+1)^{1/2}},
\end{aligned}$$

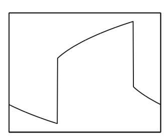
where we take the minus sign for the first family and the plus sign for the third. To see how the density varies along the Hugoniot loci, we use that $\rho = \rho_l z$, or

$$\rho = \rho_l\frac{\pi\beta+1}{\pi+\beta}, \tag{5.164}$$

which holds for both the first and third families.

Next we have to verify the Lax entropy condition, Definition 5.13. Consider the Lax 1-shock condition

$$s < \lambda_1(u_l), \quad \lambda_1(u) < s < \lambda_2(u).$$



Since the shock speed $s = s(u)$ given by (5.163) satisfies

$$s(u_l) = v_l - c_l = \lambda_1(u_l)$$

and is a decreasing function in π, we infer that $s < \lambda_1(u_l)$ holds when $p \geq p_l$, that is, $\pi > 1$. As for the inequality involving the right state, it is advantageous to rewrite the shock speed (5.163) in terms of the right state (see Exercise 5.13); thus

$$s = v \mp c\sqrt{\frac{2}{\gamma - 1}\frac{\beta/\pi + 1}{\beta^2 - 1}}. \tag{5.165}$$

Since $\pi > 1$, we see that

$$\sqrt{\frac{2}{\gamma - 1}\frac{\beta/\pi + 1}{\beta^2 - 1}} < 1,$$

thereby proving $\lambda_1(u) < s < \lambda_2(u)$. This shows that the part of the Hugoniot locus with $p \geq p_l$ satisfies the Lax 1-shock condition. A similar argument applies to the third family.

This means that the whole solution curve for waves of the first family is given by

$$v_1(p) = v_l + 2c_l \begin{cases} \frac{1}{\gamma-1}\left(1 - \left(\frac{p}{p_l}\right)^{(\gamma-1)/(2\gamma)}\right), & p \leq p_l, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1 - \frac{p}{p_l}\right)\left(1 + \beta\frac{p}{p_l}\right)^{-1/2}, & p \geq p_l. \end{cases} \tag{5.166}$$

To find the density along this solution curve, we have the formula

$$\rho_1(p) = \rho_l \begin{cases} \left(\frac{p}{p_l}\right)^{1/\gamma}, & p \leq p_l, \\ \frac{1+\beta\frac{p}{p_l}}{\beta+\frac{p}{p_l}}, & p \geq p_l. \end{cases} \tag{5.167}$$

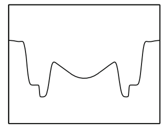
In terms of the parameter $\pi = p/p_l$, the wave curve of the first family reads

$$\begin{aligned} \rho_1(\pi) &= \rho_l \begin{cases} \pi^{1/\gamma}, & \pi \leq 1, \\ \frac{1+\beta\pi}{\beta+\pi}, & \pi \geq 1, \end{cases} \\ v_1(\pi) &= v_l + 2c_l \begin{cases} \frac{1}{\gamma-1}\left(1 - \pi^{(\gamma-1)/(2\gamma)}\right), & \pi \leq 1, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}(1-\pi)(1+\beta\pi)^{-1/2}, & \pi \geq 1. \end{cases} \end{aligned} \tag{5.168}$$

Similar formulas can also be computed for the variables q and E.

Since the second family is linearly degenerate, we can use the whole integral curve of r_2. Using the Riemann invariants, this is given simply as

$$v = v_l, \quad p = p_l, \tag{5.169}$$



and thus only the density ρ varies. The contact discontinuity is often called a slip line.

For the third family, we take the same point of view as for the shallow-water equations; we keep u_r fixed and look for states u such that the Riemann problem

$$u(x,0) = \begin{cases} u & x < 0, \\ u_r & x > 0, \end{cases}$$

is solved by a wave (shock or rarefaction) of the third family. By much the same calculations as for the first family we end up with

$$v_3(p) = v_r - 2c_r \begin{cases} \frac{1}{\gamma-1}\left(1 - \left(\frac{p}{p_r}\right)^{(\gamma-1)/(2\gamma)}\right), & p \le p_r \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1 - \frac{p}{p_r}\right)\left(1 + \beta\frac{p}{p_r}\right)^{-1/2}, & p \ge p_r, \end{cases} \tag{5.170}$$

where the rarefaction part is for $p \le p_r$ and the shock part for $p \ge p_r$. Regarding the density along this curve, it will change according to

$$\rho_3(p) = \rho_r \begin{cases} \left(\frac{p}{p_r}\right)^{1/\gamma}, & p \le p_r, \\ \left(\frac{1+\beta\frac{p}{p_r}}{\beta + \frac{p}{p_r}}\right), & p \ge p_r. \end{cases} \tag{5.171}$$

In terms of the parameter $\pi_r = p/p_r$, the wave curve of the third family reads

$$\begin{aligned} \rho_3(\pi_r) &= \rho_r \begin{cases} \pi^{1/\gamma}, & \pi_r \le 1, \\ \frac{1+\beta\pi}{\beta+\pi}, & \pi_r \ge 1, \end{cases} \\ v_3(\pi_r) &= v_r - 2c_r \begin{cases} \frac{1}{\gamma-1}\left(1-\pi_r\right)^{(\gamma-1)/(2\gamma)}, & \pi_r \le 1, \\ \frac{1}{\sqrt{2\gamma(\gamma-1)}}\left(1-\pi_r\right)\left(1+\beta\pi_r\right)^{-1/2}, & \pi_r \ge 1. \end{cases} \end{aligned} \tag{5.172}$$

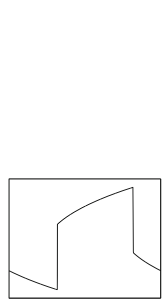
Now for every ρ_l, the curve $v_1(p)$ is a strictly decreasing function of p (or π) for nonnegative density p taking values in the set $(-\infty, v_l + 2c_l/(\gamma-1)]$. Similarly, for every ρ_r, we have that $v_3(p)$ is a strictly increasing function of p (or π_r) taking values in the set $[v_r - 2c_r/(\gamma-1), \infty)$. It follows that these curves will intersect in one point (p_m, v_m) if

$$v_r - \frac{2c_r}{\gamma-1} \le v_l + \frac{2c_l}{\gamma-1},$$

or

$$\frac{1}{2}(\gamma-1)\,[\![v]\!] \le c_l + c_r.$$

In this case we obtain a unique solution of the Riemann problem as the pressure jumps from the value to the left of the slip line to the value on the right-hand side, while the pressure p and velocity v remain unchanged and equal to p_m and v_m, respectively, across the slip line. If this does not hold, then v_1 does not intersect v_3, and we have a solution with vacuum.



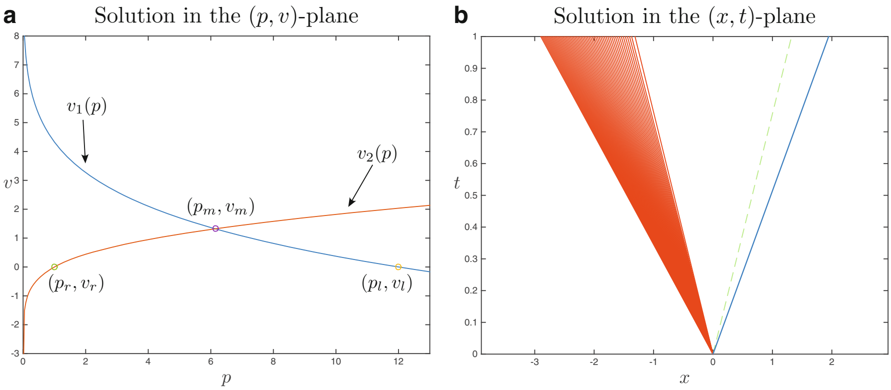


Fig. 5.12 The solution of the Riemann problem (5.173)

◇ Example 5.23 (Sod's shock tube problem)
We consider an initial value problem similar to the dam-breaking problem for shallow water. The initial velocity is everywhere zero, but the pressure to the left is higher than the pressure on the right. Specifically, we set

$$p(x,0) = \begin{cases} 12 & x < 0, \\ 1 & x \geq 0, \end{cases} \qquad v(x,0) = 0, \qquad \rho(x,0) = 2. \tag{5.173}$$

We have used $\gamma = 1.4$.

In Fig. 5.12 we show the solution to this Riemann problem in the (p, v)-plane and in the (x, t)-plane. We see that the solution consists of a leftward-moving rarefaction wave of the first family, followed by a contact discontinuity and a shock wave of the third family. In Fig. 5.13 we show the pressure, velocity, density, and the Mach number as functions of x/t. The Mach number is defined to be $|v|/c$, so that if this is larger than 1, the flow is called supersonic. The solution found here is actually supersonic between the contact discontinuity and the shock wave. ◇

The Euler Equations and Entropy

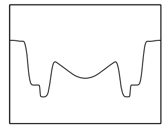
We shall show that the physical entropy is in fact also a mathematical entropy for the Euler equations, in the sense that

$$(\rho S)_t + (v \rho S)_x \leq 0, \tag{5.174}$$

weakly for every weak solution $u = (\rho, q, E)$ that is the limit of solutions to the viscous approximation.

To this end, it is convenient to introduce the *internal specific energy*, defined by

$$e = \frac{1}{\rho}\left(E - \frac{1}{2}\rho v^2\right).$$



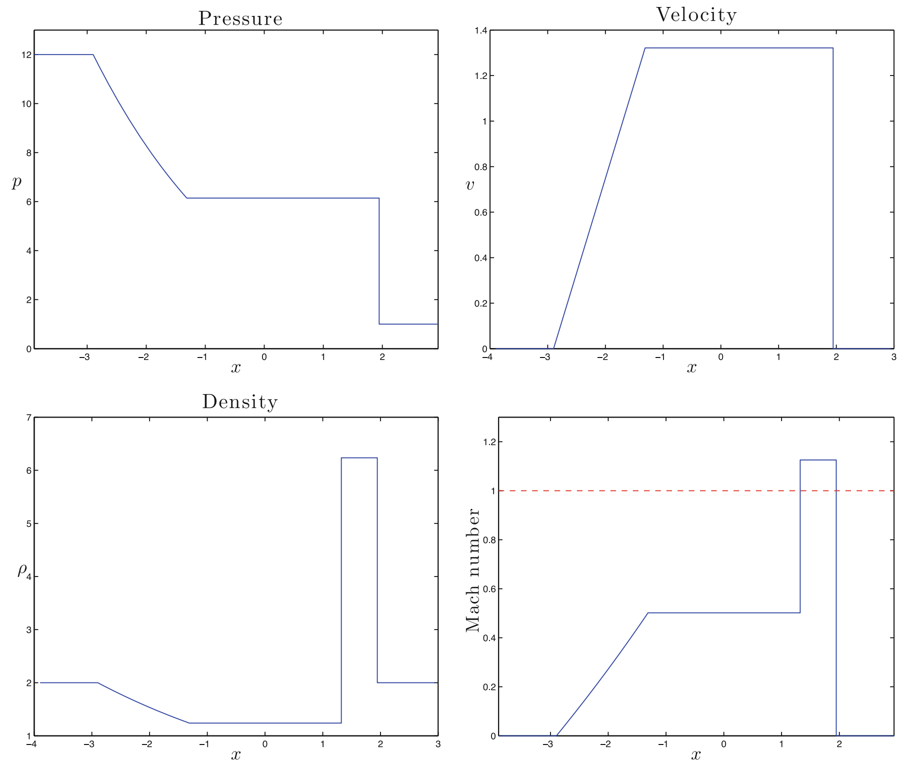


Fig. 5.13 Pressure, velocity, density, and the Mach number for the solution of (5.173)

Then the Euler equations read

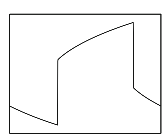
$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + \left(\rho v^2 + p\right)_x &= 0, \\ \left(\rho\left(e + \frac{1}{2}v^2\right)\right)_t + \left(\frac{1}{2}\rho v^2 + \rho e v + p v\right)_x &= 0. \end{aligned} \tag{5.175}$$

For classical solutions, this is equivalent to the nonconservative form (see Exercise 5.12)

$$\begin{aligned} \rho_t + v\rho_x + \rho v_x &= 0, \\ v_t + v v_x + \frac{1}{\rho} p_x &= 0, \\ e_t + v e_x + \frac{p}{\rho} v_x &= 0. \end{aligned} \tag{5.176}$$

We have that

$$\begin{aligned} S &= -\log\left(\frac{p}{\rho^\gamma}\right) \\ &= -\log\left(\frac{(\gamma-1)e}{\rho^{\gamma-1}}\right) \\ &= (\gamma-1)\log(\rho) - \log(e) - \log(\gamma-1). \end{aligned} \tag{5.177}$$



Thus we see that

$$S_\rho = \frac{\gamma - 1}{\rho} > 0 \ \text{ and } \ S_e = -\frac{1}{e} < 0.$$

These inequalities are general, and thermodynamic mumbo jumbo implies that they hold for every equation of state, not only for polytropic gases.

For classical solutions we can compute

$$\begin{aligned} S_t &= S_\rho \rho_t + S_e e_t \\ &= -\frac{\gamma - 1}{\rho}(v\rho_x + \rho v_x) + \frac{1}{e}\left(v e_x + \frac{p}{\rho} v_x\right) \\ &= -\left((\gamma - 1) - \frac{p}{e\rho}\right) v_x - \left((\gamma - 1)\frac{\rho_x}{\rho} - \frac{e_x}{e}\right) v \\ &= -v S_x . \end{aligned}$$

Therefore

$$S_t + v S_x = 0$$

for smooth solutions to the Euler equations. This states that the entropy of a "particle" of the gas remains constant as the particle is transported with velocity v. Furthermore,

$$\begin{aligned} (\rho S)_t &= \rho_t S + \rho S_t \\ &= -(\rho v)_x S - \rho v S_x \\ &= -(v \rho S)_x . \end{aligned}$$

Thus for smooth solutions the *specific entropy* $\eta(u) = \rho S(u)$ is conserved:

$$(\rho S)_t + (\rho v S)_x = 0. \tag{5.178}$$

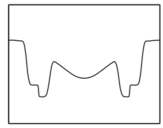
The existence of such an entropy/entropy flux pair is rather exceptional for a system of three hyperbolic conservation laws; see Exercise 5.10. Of course, combining this with (5.175) and viewing the entropy as an independent unknown, we have four equations for three unknowns, so we cannot automatically expect to have a solution. Sometimes one considers models in which the energy is not conserved but the entropy is, so-called *isentropic* flow. In models of isentropic flow the third equation in (5.175) is replaced by the conservation of entropy (5.178).

To show that (5.174) holds for viscous limits, we first show that the map

$$u \mapsto \eta(u) = \rho S(\rho, e(u))$$

is convex. We have that η is convex if its Hessian $d^2\eta$ is a positive definite matrix. For the moment we use the convention that all vectors are column vectors, and for a vector a, a^T denotes its transpose. We first obtain

$$\begin{aligned} \nabla\eta &= S\nabla\rho + \rho\nabla S \\ &= S\nabla\rho + \rho(S_\rho\nabla\rho + S_e\nabla e) \\ &= (S + \rho S_\rho)\nabla\rho + \rho S_e \nabla e . \end{aligned}$$



Trivially we have that $\nabla\rho = (1,0,0)^T$. Furthermore,

$$e(u) = \frac{E}{\rho} - \frac{1}{2}\frac{q^2}{\rho^2},$$

so we have

$$\nabla e = \left(-\frac{E}{\rho^2} + \frac{q^2}{\rho^3}, -\frac{q}{\rho^2}, \frac{1}{\rho}\right)^T = \frac{1}{\rho}\left(-e + \frac{1}{2}v^2, -v, 1\right)^T.$$

Next we compute

$$\begin{aligned}
d^2\eta &= d^2\left(\rho S(\rho,e)\right) \\
&= \nabla\rho\,(\nabla S)^T + \nabla S\,(\nabla\rho)^T + \rho d^2 S \\
&= \nabla\rho\left(S_\rho\nabla\rho + S_e\nabla e\right)^T + \left(S_\rho\nabla\rho + S_e\nabla e\right)(\nabla\rho)^T + \rho d^2 S \\
&= 2S_\rho\nabla\rho(\nabla\rho)^T + S_e\big(\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T\big) + \rho d^2 S.
\end{aligned}$$

To compute the Hessian of S we first compute its gradient:

$$\nabla S(\rho,e) = S_\rho\nabla\rho + S_e\nabla e.$$

Thus[10]

$$\begin{aligned}
d^2 S(\rho,e) &= \nabla(S_\rho\nabla\rho) + \nabla(S_e\nabla e) \\
&= \nabla\rho(\nabla S_\rho)^T + \nabla e(\nabla S_e)^T + S_e d^2 e \\
&= \nabla\rho(S_{\rho\rho}\nabla\rho + S_{\rho e}\nabla e)^T + \nabla e(S_{e\rho}\nabla\rho + S_{ee}\nabla e)^T + S_e d^2 e \\
&= S_{\rho\rho}\nabla\rho(\nabla\rho)^T + S_{\rho e}\big(\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T\big) + S_{ee}\nabla e(\nabla e)^T + S_e d^2 e.
\end{aligned}$$

If we use this in the previous equation, we end up with

$$\begin{aligned}
d^2\eta(u) = {} & \big(\rho S_{\rho\rho} + 2S_\rho\big)\,\nabla\rho\,(\nabla\rho)^T \\
& + \rho S_{\rho e}\left(\nabla\rho\,(\nabla e)^T + \nabla e\,(\nabla\rho)^T\right) + \rho S_{ee}\nabla e\,(\nabla e)^T - S_e C,
\end{aligned}$$

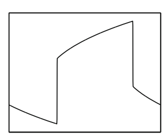
where C is given by

$$C = -\left(\rho d^2 e + \nabla\rho\,(\nabla e)^T + \nabla e\,(\nabla\rho)^T\right).$$

The Hessian of e is given by

$$d^2 e = \begin{pmatrix} 2\frac{E}{\rho^3} - 3\frac{q^2}{\rho^4} & 2\frac{q}{\rho^3} & -\frac{1}{\rho^2} \\ 2\frac{q}{\rho^3} & -\frac{1}{\rho^2} & 0 \\ -\frac{1}{\rho^2} & 0 & 0 \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} 2e - 2v^2 & 2v & -1 \\ 2v & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

[10] In our notation we have $\nabla(f(u)V(u)) = V(u)(\nabla f(u))^T + f(u)\nabla V(u)$, where f is a scalar-valued function and V is (column) vector-valued. The result $\nabla(fV)$ is a matrix.



Next,

$$
\begin{aligned}
\nabla\rho(\nabla e)^T + \nabla e(\nabla\rho)^T &= \frac{1}{\rho}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}-e+\frac{1}{2}v^2, & -v, & 1\end{pmatrix}\\
&\quad + \frac{1}{\rho}\begin{pmatrix}-e+\frac{1}{2}v^2\\-v\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\\
&= \frac{1}{\rho}\begin{pmatrix}-2e+v^2 & -v & 1\\ -v & 0 & 0\\ 1 & 0 & 0\end{pmatrix}.
\end{aligned}
$$

Then

$$
\begin{aligned}
C &= -\frac{1}{\rho}\begin{pmatrix}2e-2v^2 & 2v & -1\\ 2v & -1 & 0\\ -1 & 0 & 0\end{pmatrix} - \frac{1}{\rho}\begin{pmatrix}-2e+v^2 & -v & 1\\ -v & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\\
&= \frac{1}{\rho}\begin{pmatrix}v^2 & -v & 0\\ -v & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}
$$

Now introduce the matrix D by

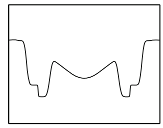
$$
D = \begin{pmatrix}1 & v & \frac{1}{2}v^2+e\\ 0 & \rho & \rho v\\ 0 & 0 & \rho\end{pmatrix}.
$$

We have that D is invertible, and thus $d^2\eta$ is positive definite if and only if $Dd^2\eta D^T$ is positive definite. Then

$$
\begin{aligned}
Dd^2\eta(u)D^T &= \left(\rho S_{\rho\rho} + 2S_\rho\right) D\nabla\rho\,(D\nabla\rho)^T\\
&\quad + \rho S_{\rho e}\left(D\nabla\rho\,(D\nabla e)^T + D\nabla e\,(D\nabla\rho)^T\right)\\
&\quad + \rho S_{ee} D\nabla e\,(D\nabla e)^T - S_e DCD^T.
\end{aligned}
$$

We compute

$$
D\nabla\rho = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad D\nabla e = \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad DCD^T = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},
$$



and using this,

$$
\begin{aligned}
Dd^2\eta(u)D^T &= \left(\rho S_{\rho\rho} + 2S_\rho\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + \rho S_{\rho e}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\\
&\qquad + \rho S_{ee}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} - S_e\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\\
&= \begin{pmatrix}\rho S_{\rho\rho}+2S_\rho & 0 & S_{\rho e}\\ 0 & -S_e & 0\\ S_{\rho e} & 0 & S_{ee}\end{pmatrix}\\
&= \begin{pmatrix}\frac{\gamma-1}{\rho} & 0 & 0\\ 0 & \frac{1}{e} & 0\\ 0 & 0 & \frac{1}{e^2}\end{pmatrix}.
\end{aligned}
$$

Hence $Dd^2\eta(u)D^T$ has three positive eigenvalues and is positive definite. Therefore, also $d^2\eta$ is positive definite, and η is convex. From the general identity

$$\eta(u)_{xx} = (u_x)^T d^2\eta(u)u_x + (\nabla\eta(u))^T u_{xx}, \quad u = u(x) = (u_1, \ldots, u_n), \tag{5.179}$$

we get from the convexity of $d^2\eta$ that

$$\eta(u)_{xx} \geq (\nabla\eta(u))^T u_{xx}. \tag{5.180}$$

Consider now a smooth solution of the regularized Euler equations

$$u^\varepsilon_t + f(u^\varepsilon)_x = \epsilon u^\varepsilon_{xx}. \tag{5.181}$$

We multiply from the left by $(\nabla\eta)^T$, which yields[11]

$$
\begin{aligned}
0 &= (\nabla\eta(u^\varepsilon))^T u^\varepsilon_t + (\nabla\eta(u^\varepsilon))^T df(u^\varepsilon)u^\varepsilon_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\
&= \eta(u^\varepsilon)_t + \left(\nabla(v^\epsilon\eta(u^\varepsilon))\right)^T u^\varepsilon_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\
&= \eta(u^\varepsilon)_t + \left(v^\epsilon\eta(u^\varepsilon)\right)_x - \epsilon(\nabla\eta(u^\varepsilon))^T u^\varepsilon_{xx}\\
&\geq \eta(u^\varepsilon)_t + \left(v^\epsilon\eta(u^\varepsilon)\right)_x - \epsilon\eta(u^\varepsilon)_{xx}.
\end{aligned}
$$

By assuming that $u^\varepsilon \to u$ as $\epsilon \to 0$, we see that

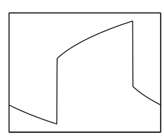
$$\eta_t + (v\eta)_x \leq 0$$

holds in the weak sense (cf. (2.15)). Hence we conclude that (5.174), that is,

$$(\rho S)_t + (v\rho S)_x \leq 0, \tag{5.182}$$

holds weakly.

[11] A word of caution: To show that $(\nabla\eta(u))^T df(u) = (\nabla(v\eta(u)))^T$ is strenuous. It is better done in nonconservative coordinates; see Exercise 5.12.



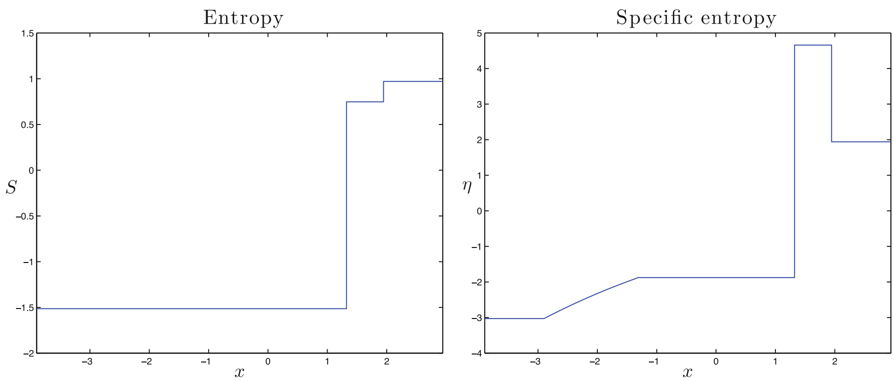


Fig. 5.14 The entropy and specific entropy for the solution of the Riemann problem (5.173)

In Fig. 5.14 we show the entropy and the specific entropy for the solution of Riemann problem (5.173). The entropy decreases as the shock and the contact discontinuity pass, while it is constant across the rarefaction wave.

Analogously to the shallow-water equations, we can also check whether (5.174) holds for the solution of the Riemann problem. We know that this will hold if and only if

$$-s\,[\![\rho S]\!] + [\![\rho v S]\!] \le 0.$$

Using the expression giving the shock speed, (5.163), we calculate

$$\begin{aligned}
-s\,[\![\rho S]\!] + [\![\rho v S]\!] &= S\left(-s\,[\![\rho]\!] + [\![\rho v]\!]\right) + \rho_l\left(-s\,[\![S]\!] + v_l\,[\![S]\!]\right)\\
&= \pm\rho_l c_l \sqrt{\frac{2}{\gamma-1}\frac{\beta\pi+1}{\beta^2-1}}\,[\![S]\!]\,,
\end{aligned}$$

where we use the plus sign for the first family and the minus sign for the second. Hence the entropy will decrease if and only if $[\![S]\!] < 0$ for the first family, and $[\![S]\!] > 0$ for the third family.

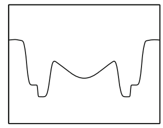
Note in passing that for the contact discontinuity, $s = v$, and thus

$$-s\,[\![\rho S]\!] + [\![\rho v S]\!] = -v\,[\![\rho S]\!] + v\,[\![\rho S]\!] = 0.$$

Therefore, as expected, entropy is conserved across a contact discontinuity.

We consider shocks of the first family, and view $[\![S]\!]$ as a function of $\pi = p/p_l$. Recall that for these shocks, we have $\pi > 1$. Thus

$$\begin{aligned}
[\![S]\!] &= S - S_l\\
&= \log\left(\frac{\rho^\gamma}{\rho_l^\gamma}\right) - \log\left(\frac{p}{p_l}\right)\\
&= \gamma\log(z) - \log(\pi)\\
&= \gamma\log\left(\frac{\beta\pi+1}{\pi+\beta}\right) - \log(\pi)\\
&=: h(\pi).
\end{aligned}$$



To check whether $h(\pi) < 0 = h(1)$, we differentiate, using (5.162):

$$
\begin{aligned}
h'(\pi) &= \gamma \frac{\beta^2 - 1}{(\pi + \beta)(\beta\pi + 1)} - \frac{1}{\pi} \\
&= \frac{1}{\pi(\pi + \beta)(\beta\pi + 1)} \big(\gamma(\beta^2 - 1)\pi - (\pi + \beta)(\beta\pi + 1)\big) \\
&= \frac{1}{\pi(\pi + \beta)(\beta\pi + 1)} \left(\frac{\beta + 1}{\beta - 1}(\beta^2 - 1)\pi - (\pi + \beta)(\beta\pi + 1)\right) \\
&= \frac{\beta}{\pi(\pi + \beta)(\beta\pi + 1)}(2\pi - \pi^2 - 1) \\
&= -\frac{\beta}{\pi(\pi + \beta)(\beta\pi + 1)}(\pi - 1)^2 < 0.
\end{aligned}
$$

Thus S is monotonically decreasing along the Hugoniot locus of the first family. We see also that (5.174) holds only if $p \geq p_l$ for waves of the first family.

For shocks of the third family, an identical computation shows that (5.174) holds only if $p \leq p_l$.

5.7 Notes

The fundamentals of the Riemann problem for systems of conservation laws were presented in the seminal paper by Lax [125], where also the Lax entropy condition was introduced. We refer to Smoller [169] as a general reference for this chapter. Our proof of Theorem 5.11 follows Schatzman [165]. This also simplifies the proof of the classical result that $s'(0) = \frac{1}{2}$ in Theorem 5.14. The parameterization of the Hugoniot locus introduced in Theorem 5.11 makes the proof of the smoothness of the wave curves, Theorem 5.16, quite simple.

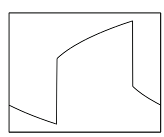
We have used shallow-water equations as our prime example in this chapter. This model can be found in many sources; a good presentation is in Kevorkian [112]. Our treatment of the vacuum for these equations can be found in Liu and Smoller [138].

There is extensive literature on the Euler equations; see, e.g., [51], [169], [167], and [42]. The computations on the Euler equations and entropy are taken from [85].

Our version of the implicit function theorem, Theorem 5.10, was taken from Cheney [40]. See Exercise 5.11 for a proof.

5.8 Exercises

5.1 In this exercise we consider the shallow-water equations in the case of a variable bottom. Make the same assumptions regarding the fluid as in Example 5.1 except that the bottom is given by the function $\bar{y} = \bar{b}(\bar{x}, \bar{t})$. Assume that the characteristic depth of the water is given by H and the characteristic depth



of the bottom is A. Let $\delta = A/H$. Show that the shallow-water equations read

$$\begin{aligned} h_t + (vh)_x &= 0, \\ (vh)_t + (v^2 h + \frac{1}{2} h^2 + \delta h b)_x &= 0. \end{aligned} \tag{5.183}$$

5.2 What assumption on p is necessary for the p-system to be hyperbolic?

5.3 Solve the Riemann problem for the p-system in the case $p(v) = 1/v$. For what left and right states does this Riemann problem have a solution?

5.4 Repeat Exercise 5.3 in the general case where $p = p(v)$ is such that p' is negative and p'' is positive.

5.5 Solve the following Riemann problem for the shallow-water equations:

$$u(x,0) = \begin{pmatrix} h(x,0) \\ v(x,0) \end{pmatrix} = \begin{cases} \binom{h_l}{0} & \text{for } x < 0, \\ \binom{h_r}{0} & \text{for } x \geq 0, \end{cases}$$

with $h_l > h_r > 0$.

5.6 Let $w = (u, v)$ and let $\varphi(w)$ be a smooth scalar function. Consider the system of conservation laws

$$w_t + (\varphi(w) w)_x = 0. \tag{5.184}$$

(a) Find the characteristic speeds λ_1 and λ_2 and the associated eigenvectors r_1 and r_2 for the system (5.184).
(b) Let $\varphi(w) = |w|^2/2$. Then find the solution of the Riemann problem for (5.184).
(c) Now let

$$\varphi(w) = \frac{1}{1 + u + v},$$

and assume that u and v are positive. Find the solution of the Riemann problem of (5.184) in this case.

5.7 Let us consider the Lax–Friedrichs scheme for systems of conservation laws. As in Chapt. 3 we write this as

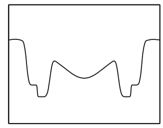
$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\lambda}{2}\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right),$$

where $\lambda = \Delta t/\Delta x$, and we assume that the CFL condition

$$\lambda \leq \max_k |\lambda_k|$$

holds. Let $v_j^n(x,t)$ denote the solution of the Riemann problem with initial data

$$\begin{cases} u_{j-1}^n & \text{for } x < j\Delta x, \\ u_{j+1}^n & \text{for } x \geq j\Delta x. \end{cases}$$



Show that

$$u_j^{n+1} = \frac{1}{2\Delta x} \int\limits_{(j-1)\Delta x}^{(j+1)\Delta x} v_j^n(x, \Delta t)\, dx.$$

5.8 A smooth function $w : \mathbb{R}^n \to \mathbb{R}$ is called a *k-Riemann invariant* if

$$\nabla w(u) \cdot r_k(u) = 0,$$

where r_k is the kth right eigenvector of the Jacobian matrix df, which is assumed to be strictly hyperbolic.

(a) Show that locally there exist precisely $(n-1)$ k-Riemann invariants whose gradients are linearly independent.
(b) Let $R_k(u_l)$ denote the kth rarefaction curve through a point u_l. Then show that all $(n-1)$ k-Riemann invariants are constant on $R_k(u_l)$. This gives an alternative definition of the rarefaction curves.
(c) We say that we have a coordinate system of Riemann invariants if there exist n scalar-valued functions $w_1, \dots, w_n$ such that w_j is a k-Riemann invariant for $j, k = 1, \dots, n$, $j \neq k$, and

$$\nabla w_j(u) \cdot r_k(u) = \gamma_j(u)\delta_{j,k}, \tag{5.185}$$

for some nonzero function g_j. Why cannot we expect to find such a coordinate system if $n > 2$?
(d) Find the Riemann invariants for the shallow-water system, and verify parts **b** and **c** in this case.

5.9 We study the p-system with $p(v) = 1/v$ as in Exercise 5.3.

(a) Find the two Riemann invariants w_1 and w_2 in this case.
(b) Introduce coordinates

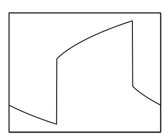
$$\mu = w_1(v, u) \quad \text{and} \quad \tau = w_2(v, u),$$

and find the wave curves in (μ, τ) coordinates.
(c) Find the solution of the Riemann problem in (μ, τ) coordinates.
(d) Show that the wave curves W_1 and W_2 are *stiff* in the sense that if a point (μ, τ) is on a wave curve through (μ_l, τ_l), then the point $(\mu + \Delta\mu, \tau + \Delta\tau)$ is on a wave curve through $(\mu_l + \Delta\mu, \tau_l + \Delta\tau)$. Hence the solution of the Riemann problem can be said to be translation-invariant in (μ, τ) coordinates.
(e) Show that the 2-shock curve through a point (μ_l, τ_l) is the reflection about the line $\mu - \mu_l = \tau - \tau_l$ of the 1-shock curve through (μ_l, τ_l).

5.10 As for scalar equations, we define an entropy/entropy flux pair (η, q) as scalar functions of u such that for smooth solutions,

$$u_t + f(u)_x = 0 \quad \Rightarrow \quad \eta_t + q_x = 0,$$

and η is supposed to be a convex function.



(a) Show that η and q are related by

$$\nabla_u q = \nabla_u \eta \, df. \tag{5.186}$$

(b) Why cannot we expect to find entropy/entropy flux pairs if $n > 2$?
(c) Find an entropy/entropy flux pair for the p-system if $p(v) = 1/v$.
(d) Find an entropy/entropy flux pair for the shallow-water equations.

5.11 This exercise outlines a proof of the implicit function theorem, Theorem 5.10.

(a) Define T to be a mapping $\mathbb{R}^p \to \mathbb{R}^p$ such that for y_1 and y_2,

$$|T(y_1) - T(y_2)| \leq c\,|y_1 - y_2|\,, \quad \text{for some constant } c < 1.$$

Such mappings are called contractions. Show that there exists a unique y such that $T(y) = y$.

(b) Let $u : \mathbb{R}^p \to \mathbb{R}^p$, and assume that u is C^1 in some neighborhood of a point y_0, and that $du(y_0)$ is nonsingular. We are interested in solving the equation

$$u(y) = u(y_0) + v \tag{5.187}$$

for some v where $|v|$ is sufficiently small. Define

$$T(y) = y - du(y_0)^{-1}\left(u(y) - u(y_0) - v\right).$$

Show that T is a contraction in a neighborhood of y_0, and consequently that (5.187) has a unique solution $x = \varphi(v)$ for small v, and that $\varphi(0) = y_0$.

(c) Now let $\Phi(x, y)$ be as in Theorem 5.10. Show that for x close to x_0 we can find $\varphi(x, v)$ such that

$$\Phi(x, \varphi(x, v)) = \Phi(x, y_0) + v$$

for small v.

(d) Choose a suitable $v = v(x)$ to conclude the proof of the theorem.

5.12 Many calculations for the Euler equations become simpler in nonconservative variables. Introduce $w = (\rho, v, e)$, where

$$e = \frac{1}{\rho}\left(E - \frac{1}{2}\rho v^2\right)$$

is the internal specific energy.

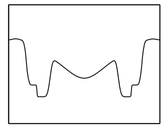
(a) Show that in these variables we have

$$p = (\gamma - 1)e\rho, \quad c^2 = \gamma(\gamma - 1)e. \tag{5.188}$$

(b) Show that w satisfies an equation of the form

$$w_t + A(w)w_x = 0, \tag{5.189}$$

and determine A.



(c) Compute the eigenvalues and eigenvectors for A and determine whether the wave families are linearly degenerate or genuinely nonlinear.
(d) Compute the Riemann invariants in these variables.
(e) Show that

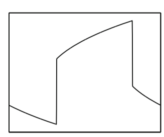
$$\left(\frac{\partial \rho S}{\partial w}\right)^T A(w) = \left(\frac{\partial \rho v S}{\partial w}\right)^T, \tag{5.190}$$

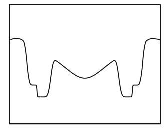
where S denotes the entropy and is given by (5.155) or (5.177). Here

$$\left(\frac{\partial f}{\partial w}\right)^T = (f_\rho, f_v, f_e)$$

for any scalar function f.

5.13 Prove (5.165).





Chapter 6

Existence of Solutions of the Cauchy Problem

Faith is an island in the setting sun. But proof, yes.
Proof is the bottom line for everyone.
— Paul Simon, Proof (1990)

In this chapter we study the generalization of the front-tracking algorithm to systems of conservation laws, and how this generalization generates a convergent sequence of approximate weak solutions. We shall then proceed to show that the limit is a weak solution. Thus we shall study the initial value problem

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{6.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and u_0 is a function in $L^1(\mathbb{R})$.

In doing this, we are in the setting of Lax's theorem (Theorem 5.17); we have a system of strictly hyperbolic conservation laws, where each characteristic field is either genuinely nonlinear or linearly degenerate, and the initial data are close to a constant. This restriction is necessary, since the Riemann problem may fail to have a solution for initial states far apart, which is analogous to the appearance of a "vacuum" in the solution of the shallow-water equations.

The convergence part of the argument follows the traditional method of proving compactness in the context of conservation laws, namely, via Kolmogorov's compactness theorem or Helly's theorem.

Again, the basic ingredient in front tracking is the solution of Riemann problems, or in this case, the approximate solution of Riemann problems. Therefore, we start by defining these approximations.

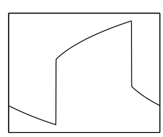
6.1 Front Tracking for Systems

Nisi credideritis, non intelligetis.[1]
— Saint Augustine, De Libero Arbitrio (387/9)

In order for us to define front tracking in the scalar case, the solution of the Riemann problem had to be a piecewise constant function. For systems, this is possible only if all waves are shock waves or contact discontinuities. Consequently, we need to approximate the continuous parts of the solution, the rarefaction waves, by functions that are piecewise continuous in x/t.

[1] Soft on Latin? It says, "If you don't believe it, you won't understand it."

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_6



There are, of course, several ways to make this approximation. We use the following: Let δ be a small parameter. For the rest of this chapter, δ will always denote a parameter that controls the accuracy of the approximation. We start with the system of conservation laws (6.1), and the Riemann problem

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0. \end{cases} \tag{6.2}$$

We have seen (Theorem 5.17) that the solution of this Riemann problem consists of at most $n+1$ constant states, separated by either shock waves, contact discontinuities, or rarefaction waves. We wish to approximate this solution by a piecewise constant function in (x/t).

When the solution has shocks or contact discontinuities, it is already a step function for some range of (x/t), and we set the approximation equal to the exact solution u for such x and t.

Thus, if the jth wave is a shock or a contact discontinuity, we let

$$u^{\delta}_{j,\epsilon_j}(x,t) = u_{j,\epsilon_j}(x,t), \qquad t\sigma_j^+ < x < t\sigma_{j+1}^-,$$

where the right-hand side is given by (5.137).

A rarefaction wave is a smooth transition between two constant states, and we will replace this by a step function whose "steps" are no farther apart than δ and lie on the correct rarefaction curve R_j. The discontinuity between two steps is defined to move with a speed equal to the characteristic speed of the left state.

More precisely, let the solution to (6.2) be given by (5.137). Assume that the jth wave is a rarefaction wave; that is, the solutions u and u_{m_j} lie on the jth rarefaction curve $R_j\left(u_{m_{j-1}}\right)$ through $u_{m_{j-1}}$, or

$$u(x,t) = u_{j,\epsilon_j}\left(x,t\,;\, u_{m_j}, u_{m_{j-1}}\right), \qquad \text{for } t\sigma_j^- \leq x \leq t\sigma_j^+.$$

Let $k = \operatorname{rnd}\left(\epsilon_j/\delta\right)$ for the moment, where $\operatorname{rnd}(z)$ denotes the integer closest[2] to z, and let $\hat{\delta} = \epsilon_j/k$. The step values of the approximation are now defined as

$$u_{j,l} = R_j\big(l\hat{\delta}\,;\, u_{m_{j-1}}\big), \qquad \text{for } l = 0,\dots,k. \tag{6.3}$$

We have that $u_{j,0} = u_{m_{j-1}}$ and $u_{j,k} = u_{m_j}$. We set the speed of the steps equal to the characteristic speed to the left, and hence the piecewise constant approximation we make is the following:

$$u^{\delta}_{j,\epsilon_j}(x,t) := u_{j,0} + \sum_{l=1}^{k} \left(u_{j,l} - u_{j,l-1}\right) H\left(x - \lambda_j\left(u_{j,l-1}\right)t\right), \tag{6.4}$$

where H now denotes the Heaviside function. Equation (6.4) is to hold for $t\sigma_j^+ < x < \sigma_{j+1}^- t$. Loosely speaking, we step along the rarefaction curve with steps of size at most δ. Observe that the discontinuities that occur as a result of the approximation of the rarefaction wave will not satisfy the Rankine–Hugoniot condition, and hence the function will not be a weak solution. However, we will prove that u^{δ} converges to a weak solution as $\delta \to 0$. Fig. 6.1 illustrates this in phase space and in (x,t)-space.

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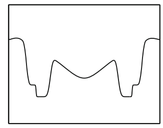
[2] Such that $z - \frac{1}{2} \leq \operatorname{rnd}(z) < z + \frac{1}{2}$.



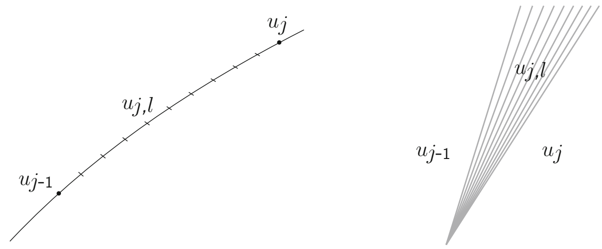


Fig. 6.1 An approximated rarefaction wave in phase space and in (x, t)-space

The approximate solution to the Riemann problem is then found by inserting a superscript δ at the appropriate places in (5.137), resulting in

$$u^\delta(x,t) = \begin{cases} u_l & \text{for } x \le \sigma_1^- t, \\ u^\delta_{1,\epsilon_1}(x/t; u_{m_1}, u_l) & \text{for } \sigma_1^- t \le x \le \sigma_1^+ t, \\ u_{m_1} & \text{for } \sigma_1^+ t \le x \le \sigma_2^- t, \\ u^\delta_{2,\epsilon_2}(x/t; u_{m_2}, u_{m_1}) & \text{for } \sigma_2^- t \le x \le \sigma_2^+ t, \\ u_{m_2} & \text{for } \sigma_2^+ t \le x \le \sigma_3^- t, \\ \vdots & \\ u^\delta_{n,\epsilon_n}(x/t; u_r, u_{m_{n-1}}) & \text{for } \sigma_n^- t \le x \le \sigma_n^+ t, \\ u_r & \text{for } x \ge \sigma_n^+ t. \end{cases} \tag{6.5}$$

It is clear that u^δ converges pointwise to the exact solution given by (5.137). Indeed,

$$\left| u^\delta(x,t) - u(x,t) \right| = \mathcal{O}(\delta).$$

Therefore, we also have that $\left\| u^\delta(t) - u(t) \right\|_{L^1} = \mathcal{O}(\delta)$, since u^δ and u are equal outside a finite interval in x.

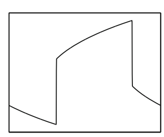
Now we are ready to define the front-tracking procedure to (approximately) solve the initial value problem (6.1).

Our first step is to approximate the initial function u_0 by a piecewise constant function u_0^δ (we let δ denote this approximation parameter as well) such that

$$\lim_{\delta \to 0} \left\| u_0^\delta - u_0 \right\|_{L^1} = 0. \tag{6.6}$$

We then generate approximations, given by (6.5), to the solutions of the Riemann problems defined by the discontinuities of u_0^δ. Already here we see one reason why we must assume T.V. (u_0) to be small: The initial Riemann problems must be solvable. Therefore, we assume our initial data u_0, as well as the approximation u_0^δ, to be in some small neighborhood D of a constant $\bar{u}$. Without loss of generality, $\bar{u}$ can be chosen to be zero.

Since the initial discontinuities interact at some later time, we can solve the Riemann problems defined by the states immediately to the left and right of the collisions. These solutions are then replaced by approximations, and we may continue to propagate the front-tracking construction until the next interaction.



However, as in the scalar case, it is not obvious that this procedure will take us up to any predetermined time. A priori, it is not even clear whether the number of discontinuities will blow up at some finite time, that is, that the collision times will converge to some finite time. This problem is much more severe in the case of a system of conservation laws than in the scalar case, since a collision of two discontinuities generically will result in at least $n-2$ new discontinuities. So for $n > 2$, the number of discontinuities seems to be growing without bound as t increases. As in the scalar case, the key to the solution of these problems lies in the study of interactions of discontinuities. To keep the number of waves finite, we shall eliminate small waves emanating from Riemann problems. However, there is a trade-off: The more waves we eliminate, the easier it is to prove convergence, but the less likely it is that the limit is a solution of the differential equation.

The method we shall use to eliminate discontinuities is taken from [9]. Let $\nu > 0$ be some small number whose precise value will be determined later, cf. (6.36). Henceforth, we shall call all discontinuities in the approximate Riemann solution *fronts*. The *family* of a front separating states u_L and u_R is the unique number j such that

$$u_R \in W_j(u_L),$$

where, as in Chapt. 5, $W_j\,(u)$ denotes the jth wave curve through the point u. These are parameterized as in Theorem 5.16. (Observe that we still have this relation for fronts approximating a rarefaction wave.) The strength of a front is ϵ, where we have

$$u_R = W_{j,\epsilon} u_L.$$

Note that the total strength of a rarefaction wave remains unchanged in the front-tracking approximation.

If a front of strength ϵ_l collides from the left with a front from the right of strength ϵ_r, and $|\epsilon_l \epsilon_r| \le \nu$, then we shall *not* use the approximate Riemann solver given by (6.5), but the following construction.

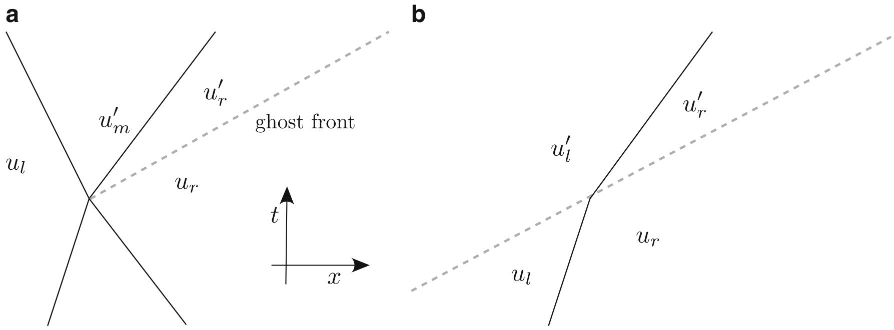
Let $\hat{l}$ denote the family of the front ϵ_l and $\hat{r}$ the family of ϵ_r. Let the state to the left of the collision be u_l and the state to the right be u_r. Observe that since we have a collision, $\hat{l} \ge \hat{r}$. If $\hat{l} > \hat{r}$, define the states u'_m and u'_r by

$$u'_m = W_{\hat{r},\epsilon_r} u_l, \quad u'_r = W_{\hat{l},\epsilon_l} u'_m. \tag{6.7}$$

If $\hat{l} = \hat{r}$, then we define

$$u'_r = W_{\hat{r},\epsilon_l+\epsilon_r} u_l. \tag{6.8}$$

The piecewise constant approximation to the Riemann problem defined by the collision of a left front ϵ_l and right front ϵ_r consists of two fronts if $\hat{l} > \hat{r}$ and of one front if $\hat{l} = \hat{r}$. We define the front-tracking approximation to this problem to be the piecewise constant approximation to the Riemann problem defined by u_l and u'_r, followed by a discontinuity traveling at a fixed speed $\Lambda > \max_u |\lambda_n(u)|$ separating u'_r and u_r. This front we call a *ghost front*. Other fronts we call *physical fronts*. Regarding ghost fronts, we label these ϵ_g, and define the strength of a ghost front ϵ_g to be $\epsilon_g = |u'_r - u_r|$. If N physical fronts, $\gamma_1, \ldots, \gamma_N$, interact at the same point,



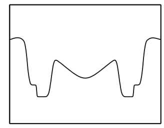


Fig. 6.2 **a** A collision producing a ghost front. **b** Collision between a ghost front and a physical front

then we use an analogous construction if

$$\sum_{\substack{i,j \\ i \le j}} \left|\gamma_i \gamma_j\right| \le \nu,$$

so that the result of this interaction will not be more than N physical fronts of the same families as the incoming fronts, followed by a ghost front. More specifically, we use the following construction. First observe that since the fronts are colliding, their families are nonincreasing from left to right. We sum the strengths of fronts belonging to the same family, i.e., $\tilde{\gamma}_k = \sum_{j,\hat{j}=k} \gamma_j$ for $k = 1, \dots, n$. If the kth family is absent, the corresponding $\tilde{\gamma}_k$ vanishes. Next we construct the new states after the collision, starting from the left. We define $u'_{m_1} = W_{1,\tilde{\gamma}_1} u_l$. Next we let $u'_{m_2} = W_{2,\tilde{\gamma}_2} u'_{m_1}$, and so on until $u'_r = u'_{m_n} = W_{n,\tilde{\gamma}_n} u'_{m_{n-1}}$. The strength of the ghost front will be $\varepsilon_g = |u_r - u'_r|$.

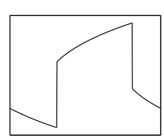
Two ghost fronts will never interact, since they travel at the same speed. In order to complete our description of the front-tracking algorithm, we must define how a collision between a ghost front and a physical front is resolved. If a ghost front separating states u'_l and u_l collides with a physical front ϵ_r separating u_l and u_r, we define

$$u'_r = W_{\hat{r},\epsilon_r} u'_l.$$

Then the solution consists of a physical front of family $\hat{r}$ and strength ϵ_r, followed by a ghost front separating u'_r and u_r, traveling at speed Λ. In particular, note that the strength of a physical front is not changed if it collides with a ghost front. See Fig. 6.2.

If a ghost front interacts with several physical fronts, $\gamma_1, \dots, \gamma_N$ at some point (x_c, t_c), we define $u'_r = W_{1,\tilde{\gamma}_1} \circ \cdots \circ W_{n,\tilde{\gamma}_n} u'_l$,[3] where is $\tilde{\gamma}_k$ is as above. Then we solve the Riemann problem with left state u'_l and right state u'_r by the general procedure. If $\sum_{i \le j} \left|\gamma_i \gamma_j\right| > \nu$, we use the full solution of the Riemann problem to define the fronts. If $\sum_{i \le j} \left|\gamma_i \gamma_j\right| \le \nu$, we should solve the Riemann problem using the middle states $u'_{m_k} = W_{k,\tilde{\gamma}_k} u'_{m_{k-1}}$ for $k = 1, \dots, n$, with $u'_{m_0} = u'_l$ followed by a ghost front separating u'_r and u_r. Note that this solution equals the one we would

[3] Observe that the order of families is reversed.



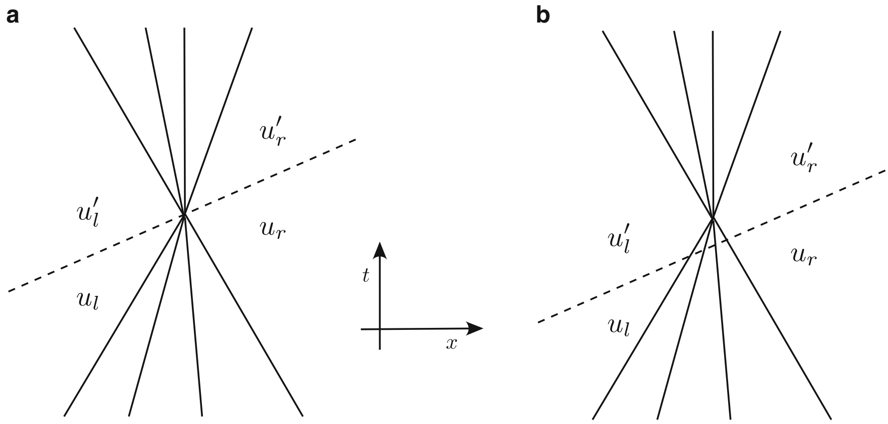


Fig. 6.3 **a** A collision between a ghost front and several physical fronts. **b** How this collision is resolved by considering a sequence of collisions

have obtained if we had let the ghost front first interact with the leftmost of the interacting fronts, γ_N, then let the resulting ghost front interact with γ_{N-1} and so on, until the interaction between a ghost front and the rightmost front γ_1, and after this, resolve the collision between $\gamma_N, \dots, \gamma_1$. Thus, a collision between a ghost front and several physical fronts can be viewed as a succession of collisions, first between the ghost front and each physical front, and then between the physical fronts. For an illustration of this, see Fig. 6.3. This perspective will be useful when we obtain *interaction estimates*, cf. (6.25).

Since ghost fronts have a speed larger than that of other fronts, we define them to be of family $n+1$.

Front tracking in a box (systems)

(i) Given a one-dimensional strictly hyperbolic system of conservation laws,

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0, \tag{6.9}$$

where u_0 has small total variation.

(ii) Approximate the initial data u_0 by a piecewise constant function u_0^δ.

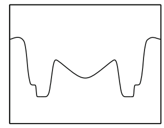
(iii) Approximate the solution of each Riemann problem by a piecewise constant function by sampling points at distance δ apart on the rarefaction curve and using the exact shocks and contact discontinuities.

(iv) Track fronts (discontinuities).

(v) Solve new Riemann problems as in (iii), or if $|\epsilon_l \epsilon_r| \le \nu$ or one of the colliding fronts is a ghost front, use (6.7)–(6.8).

(vi) Continue to solve Riemann problems approximately as in (v). Denote an approximate solution by u^δ.

(vii) The function u^δ is well defined, and as $\delta \to 0$, the approximate solution u^δ will converge to u, the solution of (6.9).[4]



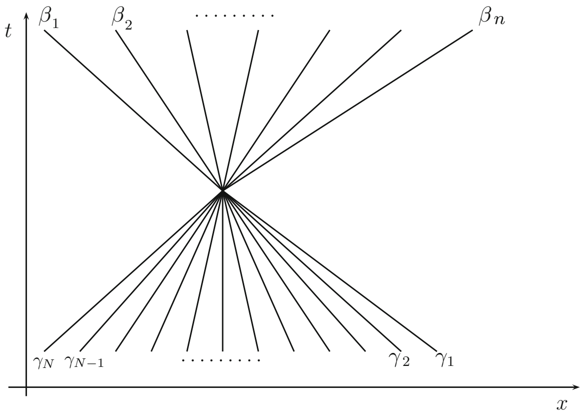


Fig. 6.4 A collision of N physical fronts

We wish to estimate the strengths of the fronts resulting from a collision in terms of the strengths of the colliding fronts. With some abuse of notation we shall refer to both the front itself and its strength by ϵ_i.

Consider therefore once more N physical fronts $\gamma_N, \dots, \gamma_1$ interacting at a single point as in Fig. 6.4. We will have to keep track of the associated family of each front. As before, we denote by $\hat{\imath}$ the family of wave γ_i. Thus if $\gamma_1, \dots, \gamma_4$ all come from the first family, we have $\hat{1} = \dots = \hat{4} = 1$. Since the speed of γ_j is greater than the speed of γ_i for $j > i$, we have $\hat{\jmath} \geq \hat{\imath}$. We label the waves resulting from the collision $\beta_1, \dots, \beta_n$.

Let β denote the vector of waves in solution of the Riemann problem, defined by the collision of $\gamma_1, \dots, \gamma_N$, i.e., $\beta = (\beta_1, \dots, \beta_n)$, and let

$$\alpha = \Big(\sum_{\hat{\imath}=1} \gamma_i \,, \sum_{\hat{\imath}=2} \gamma_i \,, \dots, \sum_{\hat{\imath}=n} \gamma_i \Big).$$

For simplicity, also set $\gamma = (\gamma_1, \dots, \gamma_N)$. Note that β is a function of γ, that is, $\beta = \beta(\gamma)$. For $i < j$ we define

$$\beta_{i,j}(\sigma, \tau) := \frac{\partial^2 \beta}{\partial \gamma_i \partial \gamma_j} \big(\gamma_1, \dots, \gamma_{i-1}, \sigma\gamma_i, 0, \dots, 0, \tau\gamma_j, 0, \dots, 0\big).$$

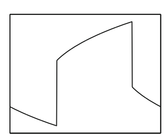
Then

$$\begin{aligned} &\gamma_i \gamma_j \int_0^1 \int_0^1 \beta_{i,j}(\sigma, \tau)\, d\sigma\, d\tau \qquad (6.10) \\ &\quad = \beta\big(\gamma_1, \dots, \gamma_i, 0, \dots, 0, \gamma_j, 0, \dots, 0\big) + \beta\left(\gamma_1, \dots, \gamma_{i-1}, 0, \dots, 0\right) \\ &\qquad - \beta\left(\gamma_1, \dots, \gamma_i, 0, \dots, 0\right) - \beta\big(\gamma_1, \dots, \gamma_{i-1}, 0, \dots, 0, \gamma_j, 0, \dots, 0\big). \end{aligned}$$

Furthermore,

$$\beta(0, \dots, 0, \gamma_k, 0, \dots, 0) = (0, \dots, 0, \gamma_k, 0, \dots, 0), \qquad (6.11)$$

[4] Proved in Sect. 6.2.



where γ_k on the right is at the $\hat{k}$th place, since in this case we have no collision. Summing (6.10) for all $i < j$, we obtain

$$\sum_{i<j}^{N} \gamma_i \gamma_j \int_0^1 \int_0^1 \beta_{i,j}(\sigma, \tau)\, d\sigma\, d\tau$$
$$= \beta(\gamma_1, \dots, \gamma_N) - \sum_{i=1}^{N} \beta(0, \dots, 0, \gamma_i, 0, \dots, 0) = \beta - \alpha. \tag{6.12}$$

By the solution of the general Riemann problem, see Lax's theorem 5.17, we have that $\beta_{i,j}$ is bounded; hence

$$|\beta - \alpha| \le \mathcal{O}(1) \sum_{i,j;i<j}^{N} |\gamma_i \gamma_j|, \tag{6.13}$$

or

$$\beta = \alpha + \mathcal{O}(1) \sum_{\substack{i,j \\ i<j}}^{N} |\gamma_i \gamma_j|. \tag{6.14}$$

Note that if the incoming fronts γ_k are small, then the fronts resulting from the collision will be very small for those families that are not among the incoming fronts.

If we have a collision between a ghost front ϵ_g, separating states u_l' and u_l, and a physical front with strength ϵ of family j separating states u_r and u_r, the result will be a physical front of strength ϵ separating states u_l' and u_r', and a ghost front ϵ_g' separating u_r' and u_r; see the right part of Fig. 6.2. Since $u_r' = W_{j,\epsilon} u_l'$ and $u_r = W_{j,\epsilon} u_l$,

$$\begin{aligned}
u_r - u_r' &= W_{j,\epsilon} u_l - W_{j,\epsilon} u_l' \\
&= u_l - u_l' + \int_0^{\epsilon} \frac{\partial}{\partial \xi} \left(W_{j,\xi} u_l - W_{j,\xi} u_l' \right) d\xi \\
&= u_l - u_l' + \int_0^{\epsilon} \left(\frac{\partial W_{j,\xi}}{\partial \xi}(u_l) - \frac{\partial W_{j,\xi}}{\partial \xi}(u_l') \right) d\xi \\
&= u_l - u_l' + \mathcal{O}(1)\, |\epsilon|\, |u_l - u_l'|.
\end{aligned}$$

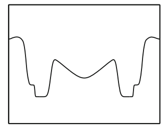
Therefore

$$\left| \epsilon_g' \right| \le \left| \epsilon_g \right| + K\, |\epsilon| \left| \epsilon_g \right|. \tag{6.15}$$

◇ Example 6.1 (Higher-order estimates)

The estimate (6.13) is enough for our purposes, but we can extract some more information from (6.12) by considering higher-order terms. Firstly, note that

$$\beta = \alpha + \sum_{i<j} \gamma_i \gamma_j \beta_{i,j}(0,0) + \mathcal{O}(1) \sum_{i<j} |\gamma_i \gamma_j|\, |\gamma|. \tag{6.16}$$



Therefore, we evaluate $\beta_{i,j}(0,0)$. To do this, observe that

$$u_r = \mathcal{R}_{\beta(\gamma)}\, u_l = \mathcal{R}_{\gamma_N} \circ \mathcal{R}_{\gamma_{N-1}} \circ \cdots \circ \mathcal{R}_{\gamma_1}\, u_l, \tag{6.17}$$

where $\mathcal{R}_\beta$ is defined as in (5.141), and u_l and u_r are the states to the left and right of the collision, respectively. If we define

$$\beta_{\gamma_j} := \frac{\partial \beta}{\partial \gamma_j},$$

(6.11) implies

$$\beta_{\gamma_j}(0,\dots,0) = e_{\hat{j}},$$

where e_k denotes the kth standard basis vector in $\mathbb{R}^n$. Also note that

$$\frac{\partial}{\partial \gamma_i}\mathcal{R}_{\beta(\gamma)} = \nabla_\beta \mathcal{R}_\beta \cdot \beta_{\gamma_i}.$$

Furthermore, from Lemma 5.18 and (5.141), we have that

$$\nabla_\beta \mathcal{R}_\beta = \left(\dots, r_k + \sum_{j=1}^{n} \beta_j\, D_{r_{\min(j,k)}} r_{\max(j,k)}, \dots \right) + \mathcal{O}\left(|\beta|^2\right).$$

Here the first term on the right-hand side is the $n \times n$ matrix with the kth column equal to $r_k + \sum_{j=1}^{n} \beta_j\, D_{r_{\min(j,k)}} r_{\max(j,k)}$. Consequently,[5]

$$\frac{\partial}{\partial \gamma_j}\nabla_\beta \mathcal{R}_{(0,\dots,0)} = \left(D_{r_1} r_{\hat{j}}, D_{r_2} r_{\hat{j}}, \dots, D_{r_{\hat{j}}} r_{\hat{j}}, D_{r_{\hat{j}}} r_{\hat{j}+1}, \dots, D_{r_{\hat{j}}} r_n\right)$$

evaluated at u_l. Differentiating (6.17) with respect to γ_i, we obtain

$$\left(\nabla_\beta \mathcal{R}_\beta \cdot \beta_{\gamma_i}\right)|_{\gamma=\left(0,\dots,0,\gamma_j,0\dots,0\right)}(u_l) = r_{\hat{\imath}}\left(\mathcal{R}_{\gamma_j} u_l\right)$$

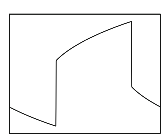
for $j > i$. Differentiating this with respect to γ_j, we obtain

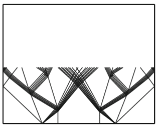
$$\left(\frac{\partial}{\partial \gamma_j}\nabla_\beta \mathcal{R}_\beta\right)|_{\gamma=(0,\dots,0)} e_{\hat{j}} + \nabla_\beta \mathcal{R}_{(0,\dots,0)} \beta_{i,j}(0,0) = D_{r_{\hat{j}}} r_{\hat{\imath}}(u_l).$$

Inserting this in (6.16), we finally obtain

$$\beta = \alpha + \sum_{i<j}^{N} \gamma_i \gamma_j \left(\nabla_\beta \mathcal{R}_\beta\right)^{-1}\left(D_{r_{\hat{j}}} r_{\hat{\imath}} - D_{r_{\hat{\imath}}} r_{\hat{j}}\right) + \mathcal{O}(1) \sum_{i<j} \left|\gamma_i \gamma_j\right| |\gamma|, \tag{6.18}$$

[5] The right-hand side denotes the $n \times n$ matrix whose first $\hat{j}$ columns equal $D_{r_k} r_{\hat{j}}$, $k = 1, \dots, \hat{j}$, and the remaining $(n - \hat{j})$ columns equal $D_{r_{\hat{j}}} r_k$, $k = \hat{j}+1, \dots, n$.





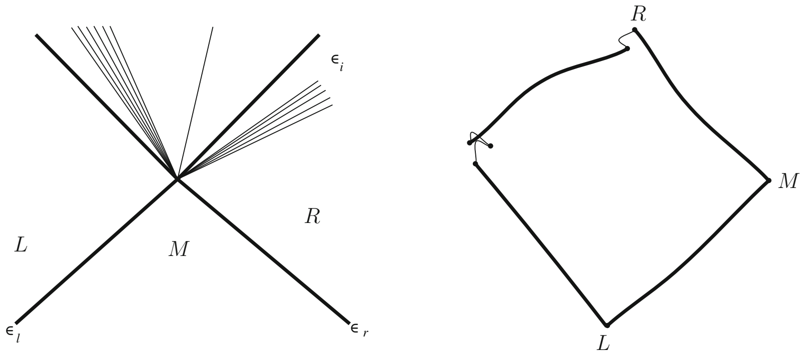


Fig. 6.5 An interaction in (x, t)-space and in phase space

which we call the *interaction estimate*. One can also use (6.12) to obtain estimates of higher order.

In passing, we note that if the integral curves of the eigenvectors form a coordinate system near M, then

$$\left(D_{r_j} r_i - D_{r_i} r_j\right) = 0$$

for all i and j, and we obtain a third-order estimate. The estimate (6.13) will prove to be the key ingredient in our analysis of front tracking.

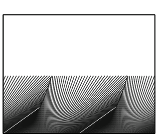
For the reader with knowledge of differential geometry, the estimate (6.18) is no surprise. Assume that only two fronts collide, ϵ_l and ϵ_r, separating states u_L, u_M, and u_R. Let the families of the two fronts be l and r, respectively. The states u_L, u_M, and u_R are almost connected by the integral curves of r_l and r_r, respectively. If we follow the integral curve of r_l a (parameter) distance $-\epsilon_l$ from u_R, and then follow the integral curve of r_r a distance $-\epsilon_r$, we end up with, up to third order in ϵ_l and ϵ_r, half the Lie bracket of $\epsilon_l r_l$ and $\epsilon_r r_r$ away from u_L. This Lie bracket is given by

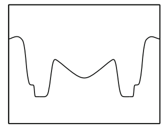
$$[\epsilon_l r_l, \epsilon_r r_r] := \epsilon_l \epsilon_r \left(D_{r_l} r_r - D_{r_r} r_l\right).$$

This means that if we start from u_L and follow r_r a distance ϵ_r, and then r_l a distance ϵ_l, we finish a distance $\mathcal{O}\left([\epsilon_l r_l, \epsilon_r r_r]\right)$ away from u_R. Consequently, up to $\mathcal{O}\left([\epsilon_l r_l, \epsilon_r r_r]\right)$, the solution of the Riemann problem with right state u_R and left state u_L is given by a wave of family r of strength ϵ_r, followed by a wave of family l of strength ϵ_l. While not a formal proof, these remarks illuminate the mechanism behind the calculation leading up to (6.18). See Fig. 6.5. ◇

Before we proceed further, we introduce some notation. Front tracking will produce a piecewise constant function labeled $u^\delta(x, t)$ that has, at least initially, some finite number N of fronts. These fronts have strengths ϵ_i, $i = 1, \dots, N$. We will refer to the ith front by its strength ϵ_i, and label the left and right states u_{l_i} and u_{r_i}, respectively. The position of ϵ_i is denoted by $x_i(t)$, and with a slight abuse of notation we have that

$$x_i(t) = x_i + s_i\,(t - t_i)\,, \tag{6.19}$$





where s_i is the speed of the front, and (x_i, t_i) is the position and time it originated. In this notation, u^δ can be written

$$u^\delta(x,t) = u_{l_1} + \sum_{i=1}^{N} (u_{r_i} - u_{l_i})\, H\,(x - x_i(t))\,. \tag{6.20}$$

The interaction estimate (6.13) shows that the "amount of change" produced by a collision is proportional to the product of the strengths of the colliding fronts. Therefore, in order to obtain some estimate of what will happen as fronts collide, we define the *interaction potential* Q. The idea is that Q should (over)estimate the amount of change in u^δ caused by all future collisions. Hence by (6.13), Q should involve terms of type $|\epsilon_l \epsilon_r|$. We say that two fronts are *approaching* if the front to the left has a larger family than the front to the right, or if both fronts are of the same family and at least one of the fronts is a shock wave. Note that this means that a ghost front is approaching all physical fronts to its right. We collect all pairs of approaching fronts in the *approaching set* $\mathcal{A}$, that is,

$$\mathcal{A} := \left\{ \left(\epsilon_i, \epsilon_j\right) \text{ such that } \epsilon_i \text{ and } \epsilon_j \text{ are approaching} \right\}. \tag{6.21}$$

The set $\mathcal{A}$ will, of course, depend on time. All future collisions will now involve two fronts from $\mathcal{A}$ due to the hyperbolicity of the equation. Observe that two approximate rarefaction waves of the same family never collide unless there is another front between, all colliding at the same point (x,t). Therefore, we define Q as

$$Q := \sum_{\mathcal{A}} \left|\epsilon_i \epsilon_j\right|. \tag{6.22}$$

For scalar equations we saw that the total variation of the solution of the conservation law was not greater than the total variation of the initial data. From the solution of the Riemann problem, we know that this is not true for systems. Nevertheless, we shall see that if the initial total variation is small enough, the total variation of the solution is bounded. To measure the total variation we use another time-dependent functional T defined by

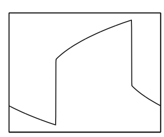
$$T := \sum_{i=1}^{N} |\epsilon_i|\,, \tag{6.23}$$

where N is the number of fronts. Lax's theorem (Theorem 5.17) implies that T is equivalent to the total variation as long as the total variation is small.

Let t_1 denote the first time two fronts collide. At this time we will have another Riemann problem, which can be solved up to the next collision time t_2, etc. In this way we obtain an increasing sequence of collision times t_i, $i \in \mathbb{N}$. To show that front tracking is well defined, we need to show that the sequence $\{t_i\}$ is finite, or if infinite, not convergent. In the scalar case we saw that indeed this sequence is finite.

We will analyze more closely the changes in Q and T when fronts collide. Clearly, they change only at collisions. Let t_c be some fixed collision time.

Assume then that the situation is as in Fig. 6.6: N fronts $\epsilon_1, \dots, \epsilon_N$ are colliding at some point (x_c, t_c), giving N' fronts $\epsilon'_1, \dots, \epsilon'_{N'}$. Observe that if one of the colliding fronts is a ghost front, then it must be the leftmost one, ϵ_N. Furthermore, if ϵ_N is



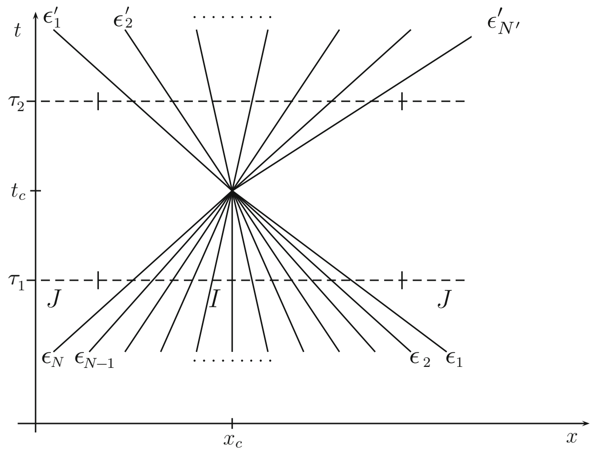


Fig. 6.6 A collision of N fronts

a ghost front, then this collision can be viewed as a sequence of collisions between the ghost front and each physical front $\epsilon_{N-1}, \dots, \epsilon_1$, followed by the interaction of $\epsilon_{N-1}, \dots, \epsilon_1$ as depicted in Fig. 6.3. Thus for interaction estimates, we can assume that if ϵ_N is a ghost front, then there are only two fronts colliding; ϵ_2 (the ghost front) and ϵ_1 (the physical front).

Let I be a small interval containing x_c, and let J be the complement of I. Then we may write $Q = Q(I) + Q(J) + Q(I, J)$, where $Q(I)$ and $Q(J)$ indicate that the summation is restricted to those pairs of fronts that both lie in I and that both lie in J, respectively. Similarly, $Q(I, J)$ means that the summation is over those pairs where one front is in I and the other in J. Let $\tau_1 < t_c < \tau_2$ be two times, chosen such that no other collisions occur in the interval $[\tau_1, \tau_2]$, and such that no fronts other than $\epsilon_1, \dots, \epsilon_N$ are crossing the interval I at time τ_1, and only waves emanating from the collision at t_c, i.e., waves denoted by $\epsilon_1', \dots, \epsilon_n'$, cross I at time τ_2. Let Q_i and T_i denote the values of Q and T at time τ_i. By construction, $Q_2(I) = 0$ and $Q_2(J) = Q_1(J)$, and hence

$$Q_2 - Q_1 = Q_2(I, J) - Q_1(I, J) - Q_1(I). \tag{6.24}$$

We now want to bound the increase in $Q(I, J)$ from time τ_1 to τ_2. More precisely, we want to prove that

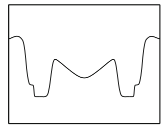
$$Q_2(I, J) \leq Q_1(I, J) + \mathcal{O}(1)\, Q_1(I) T_1(J). \tag{6.25}$$

Let $\left|\epsilon \epsilon_i'\right|$ be a term in $Q_2(I, J)$, i.e., $(\epsilon, \epsilon_i') \in \mathcal{A}$ at time τ_2. This means that $\epsilon_i' < 0$ or $\epsilon < 0$. With a slight abuse of notation we denote the family of ϵ_i' by i. Let $\mathcal{I}_i$ be the set of indices of the colliding fronts in I at time τ_1 with family i, i.e.,

$$\mathcal{I}_i = \{ j \mid \hat{\jmath} = i, \quad j = 1, \dots, N \}.$$

Now the interaction estimate (6.14) reads

$$\epsilon_i' = \sum_{j \in \mathcal{I}_i} \epsilon_j + \mathcal{O}(1) Q_1(I).$$



To prove (6.25) we study different cases. First we consider the three possibilities that can occur if neither ϵ_i' nor ϵ is a ghost front:

(a) The family of ϵ is not i. In this case, $(\epsilon_j, \epsilon) \in \mathcal{A}$ at time τ_1 for all $j \in \mathcal{I}_i$. Therefore

$$|\epsilon_i'\epsilon| \leq \sum_{\substack{\hat{j}=i \\ (\epsilon_j,\epsilon)\in\mathcal{A}}} \left|\epsilon_j \epsilon\right| + \mathcal{O}(1)\, Q_1(I)\, |\epsilon|\,. \tag{6.26}$$

(b) The family of ϵ is i, and $\epsilon < 0$. In this case, since $(\epsilon_i', \epsilon) \in \mathcal{A}$ at time τ_2, also $(\epsilon_j, \epsilon) \in \mathcal{A}$ at time τ_1 for all $j \in \mathcal{I}_i$. Hence (6.26) holds.

(c) The family of ϵ is i, and $\epsilon > 0$. Since $(\epsilon_i', \epsilon) \in \mathcal{A}$, we infer that $\epsilon_i' < 0$. Let $\mathcal{I}_{i,-} = \{k \in \mathcal{I}_i \mid \varepsilon_k < 0\}$. Then

$$|\epsilon_i'| = \sum_{j\in\mathcal{I}_{i,-}} \left|\epsilon_j\right| - \sum_{j\in\mathcal{I}_i\setminus\mathcal{I}_{i,-}} \left|\epsilon_j\right| + \mathcal{O}(1) Q_1(I).$$

Also, for $j \in \mathcal{I}_i$, $(\epsilon_j, \epsilon) \in \mathcal{A}$ if and only if $j \in \mathcal{I}_{i,-}$. Hence (6.26) holds also in this case.

Next we consider the situation when either ϵ_i' or ϵ is a ghost front.

(d) ϵ is a ghost front. In this case ϵ must be to the left of I since $(\epsilon_i', \epsilon) \in \mathcal{A}$. Thus $(\epsilon_j, \epsilon) \in \mathcal{A}$ for all $j \in K$ and (6.26) holds.

(e) ϵ_i' is a ghost front. Then ϵ must be to the right of I for (ϵ_i', ϵ) to be in $\mathcal{A}$. This situation is depicted in Fig. 6.2. In the right case, ϵ_i is a ghost front, and in the left case, $\epsilon_i' = \mathcal{O}(1)\, Q_1(I)$ and there were no ghost fronts in I at τ_1. In the latter case, clearly (6.26) holds. If ϵ_i is a ghost front, then there are only two fronts colliding in I. By (6.15), $\left|\epsilon_i'\right| \leq |\epsilon_i| + \mathcal{O}(1)\, |\epsilon_i|\, T_1(I)$ and $(\epsilon_i, \epsilon) \in \mathcal{A}$. Thus (6.26) holds.

Therefore, for all pairs $(\epsilon_i', \epsilon_k) \in \mathcal{A}$ with ϵ_k in J, (6.26) holds. Summing over i and k gives (6.25).

Inserting (6.25) into (6.24), using the constant K to replace the order symbol, we obtain

$$Q_2 - Q_1 \leq K Q_1(I) T_1 - Q_1(I) = Q_1(I)\,(K T_1 - 1) \leq -\frac{1}{2} Q_1(I) \tag{6.27}$$

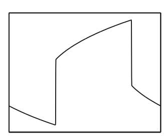
if T_1 is smaller than $1/(2K)$. By the estimate (6.15), (6.27) holds also for collisions involving a ghost front. We summarize the above discussion in the following lemma.

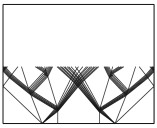
Lemma 6.2 *Assume that $T_1 \leq 1/(2K)$. Then*

$$Q_2 - Q_1 \leq -\frac{1}{2} Q_1(I)$$

for every δ and ν.

We will use this lemma to deduce that the total variation remains bounded if it initially is sufficiently small, or in other words, if the initial data are sufficiently close to a constant state.





Lemma 6.3 *If T is sufficiently small at $t = 0$, then there is some constant c independent of δ such that*

$$G = T + c\,Q$$

is nonincreasing. We call G the Glimm functional. Consequently, T and T.V. $\left(u^\delta\right)$ *are bounded independently of δ and ν.*

Proof Let T_n and Q_n denote the values of T and Q, respectively, before the nth collision of fronts at t_n, with $0 < t_1 < t_2 < \cdots$. Using the interaction estimate (6.13), we first infer that

$$T_{n+1} = \sum_j \left|\epsilon'_j\right| \le T_n + K Q_n(I). \tag{6.28}$$

Let $c \ge 2K$ and assume that $T_1 + cT_1^2 \le 1/(2K)$. Assume furthermore that $T + c\,Q$ is nonincreasing for all t less than t_n, and that $T_n \le 1/(2K)$. Lemma 6.2 and (6.28) imply that

$$\begin{aligned} T_{n+1} + c\,Q_{n+1} &\le T_n + KQ_n(I) + c\,Q_n - \frac{c}{2}Q_n(I) \\ &= T_n + c\,Q_n + \left(K - \frac{c}{2}\right) Q_n(I) \\ &\le T_n + c\,Q_n, \end{aligned}$$

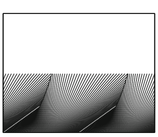
since $K - \frac{c}{2} \le 0$. Consequently,

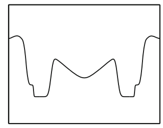
$$T_{n+1} \le T_{n+1} + c\,Q_{n+1} \le \cdots \le T_1 + c\,Q_1 \le T_1 + c\,T_1^2 \le 1/(2K),$$

which by induction proves the result. □

We still have not shown that the front-tracking approximation can be continued up to any desired time. Now, however, this is clear. Since only collisions between physical fronts that have strengths ϵ_l and ϵ_r such that $|\epsilon_l\epsilon_r| > \nu$ will produce new fronts, and Q decreases by at least $|\epsilon_l\epsilon_r|/2$ in such a collision, there can be at most $2Q(0)/\nu$ collisions producing new nonghost fronts. Since fronts of each family will travel in a wedge in the (x, t)-plane, eventually all physical fronts of different families will have interacted. After this time, two rarefaction fronts (fronts approximating rarefaction waves) of the same family will not collide, and the collision of two shock fronts of the same family will produce a single shock front of the same family and a ghost front. Thus in such collisions the number of physical fronts decreases by at least one. Therefore, there can be only a finite number of this type of collision. Since ghost fronts all have the same speed, they will not interact among themselves. Therefore, for fixed δ and ν, there will be only a finite number of interactions for all $t > 0$. Hence the front-tracking approximation is well defined, and we can calculate the approximation $u^\delta(x, t)$ for all $t > 0$ using a finite number of steps. Thus front tracking for systems is also a *hyperfast* method.

Summing up our results so far, we have proved the following result.





Theorem 6.4 *Let $f_j \in C^2(\mathbb{R}^n)$, $j = 1, \dots, n$. Let D be a domain in $\mathbb{R}^n$ and consider the strictly hyperbolic equation $u_t + f(u)_x = 0$ in D. Assume that f is such that each wave family is either genuinely nonlinear or linearly degenerate. Assume also that the function $u_0(x)$ has sufficiently small total variation.*

Then the front-tracking approximation $u^\delta(x,t)$, defined by (6.5), (6.6) and constructed by the front-tracking procedure described above, is well defined. Furthermore, the method is hyperfast, i.e., it requires only a finite number of computations to define $u^\delta(x,t)$ for all t. The total variation of u^δ is uniformly bounded, and there is a constant C such that

$$\mathrm{T.V.}\left(u^\delta(\,\cdot\,,t)\right) \le C,$$

for all $t \ge 0$ and all $\delta > 0$ and all $\nu > 0$.

6.2 Convergence

The Devil is in the details.
— English proverb

At this point we could proceed, as in the scalar case, by showing that front tracking is stable with respect to L^1 perturbations of the initial data. This would then imply that the sequence of approximations $\{u^\delta\}$ has a unique limit as $\delta \to 0$. For systems, however, this analysis is rather complicated. In this section we shall instead prove that the sequence $\{u^\delta\}$ is compact and that every (there is really only one) limit is a weak solution. The reader willing to accept this, or primarily interested in front tracking, may skip ahead to the next chapter.

To show that a subsequence of the sequence $\{u^\delta\}_{\delta>0}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T])$, we use Theorem A.11 from Appendix A. We have already shown that $u^\delta(x,t)$ is bounded, and we have that

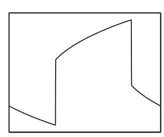
$$\int_{\mathbb{R}} \left|u^\delta(x+\rho,t) - u^\delta(x,t)\right|\,dx \le \rho\,\mathrm{T.V.}\left(u^\delta(\,\cdot\,,t)\right) \le C\rho,$$

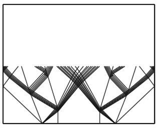
for some C independent of δ. Hence, by Theorem A.11, to conclude that a subsequence of $\{u^\delta\}$ converges, we must show that

$$\int_{-R}^{R} \left|u^\delta(x,t) - u^\delta(x,s)\right|\,dx \le C(t-s),$$

where $t \ge s \ge 0$, for every $R > 0$ and for some C independent of δ. Since u^δ is bounded, we have that Λ (the speed of the ghost fronts) is bounded, and (recall that $\lambda_1 < \cdots < \lambda_n$)

$$\Lambda > \max_{|u|\le\sup\left|u^\delta\right|} \left\{\left|\lambda_n(u)\right|, \left|\lambda_1(u)\right|\right\}.$$





Let t_i and t_{i+1} be two consecutive collision times. For $t \in (t_i, t_{i+1}]$ we write u^δ in the form

$$u^\delta(x,t) = u_1 + \sum_{k=1}^{N_i} \left(u_k^i - u_{k-1}^i\right) H\left(x - x_k^i(t)\right), \tag{6.29}$$

where $x_k^i(t)$ denotes the position of the kth front from the left, and H the Heaviside function. Here $u^\delta(x,t) = u_k^i$ for x between x_k^i and x_{k+1}^i. Assume now that $t \in [t_i, t_{i+1}]$ and $s \in [t_j, t_{j+1}]$, where $j \le i$ and $s \le t$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} \left|u^\delta(x,t) - u^\delta(x,t_i)\right| dx \\
&\qquad = \int_{\mathbb{R}} \left| \int_{t_i}^{t} \frac{d}{d\tau} u^\delta(x,\tau)\, d\tau \right| dx \\
&\qquad \le \int_{\mathbb{R}} \int_{t_i}^{t} \sum_{k=1}^{N_i} \left|u_{k-1}^i - u_k^i\right| \left|x_k^{i\,\prime}(\tau)\right| \left|H'\left(x - x_k^i(\tau)\right)\right| \, d\tau\, dx \\
&\qquad \le \Lambda \int_{t_i}^{t} \sum_{k=1}^{N_i} \left|u_{k-1}^i - u_k^i\right| \int_{\mathbb{R}} \left|H'\left(x - x_k^i(\tau)\right)\right| \, dx\, d\tau \\
&\qquad \le \Lambda\,(t - t_i)\,\mathrm{T.V.}\left(u^\delta(\,\cdot\,,t)\right) \\
&\qquad \le \Lambda C\,(t - t_i),
\end{aligned}$$

since $\left|x_k^{i\,\prime}\right| \le \Lambda$. Similarly, we show that

$$\int_{\mathbb{R}} \left|u^\delta(x,t_i) - u^\delta\left(x,t_{j+1}\right)\right| dx \le \Lambda C\left(t_i - t_{j+1}\right) \quad \text{if } j+1 < i,$$

and

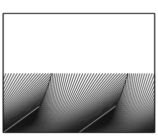
$$\int_{\mathbb{R}} \left|u^\delta\left(x,t_{j+1}\right) - u^\delta(x,s)\right| dx \le \Lambda C\left(t_{j+1} - s\right).$$

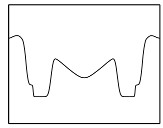
Therefore,

$$\left\|u^\delta(\,\cdot\,,t) - u^\delta(\,\cdot\,,s)\right\|_{L^1} \le C\,|t - s|,$$

for some constant C independent of t and δ. Hence, we can use Theorem A.11 to conclude that there exist a function $u(x,t)$ and a subsequence $\{\delta_j\} \subset \{\delta\}$ such that $u^{\delta_j} \to u(x,t)$ in L^1_{loc} as $j \to \infty$.

As in the scalar case, it is by no means obvious that the limit function $u(x,t)$ is a weak solution of the original initial value problem (6.1). For a single conservation





law, this was not difficult to show, using that the approximations were weak solutions of approximate problems. This is not so in the case of systems, so we must analyze how close the approximations are to being weak solutions.

There are three sources of error in the front-tracking approximation. Firstly, the initial data are approximated by a step function. Secondly, there is the approximation of rarefaction waves by step functions, and finally, ghost fronts are not weak solutions locally.

In the following, the next lemma will be useful.

Lemma 6.5 *Let the sequence* $\{a_i\}_{i=1}^{\infty}$ *be defined by*

$$a_1 = 1, \quad a_m = \sum_{j=1}^{m-1} a_{m-j} a_j, \quad m = 2, 3, \dots. \tag{6.30}$$

Then

$$a_m = 2\frac{(2m-3)!}{m!(m-2)!} = \mathcal{O}(1)\, 4^m m^{-1/2}.$$

Proof We use the notation

$$\binom{1/2}{m} = \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdots(\frac{1}{2}-m+1)}{m!}.$$

Define the function

$$y(x) = \sum_{m=1}^{\infty} a_m x^m.$$

Then, using (6.30),

$$y^2 = \sum_{m=2}^{\infty}\left(\sum_{j=1}^{m-1} a_{m-j}a_j\right)x^m = y - x,$$

and we infer that (recall that $y(0) = 0$)

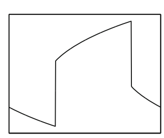
$$y(x) = \frac{1}{2}\left(1-\sqrt{1-4x}\right) = \sum_{m=1}^{\infty}(-1)^{m+1}\binom{1/2}{m}2^{2m-1}x^m,$$

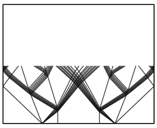
which implies

$$a_m = (-1)^{m+1}\binom{1/2}{m}2^{2m-1}.$$

We may rewrite this as

$$a_m = 2\frac{(2m-3)!}{m!(m-2)!}.$$





To estimate a_m as $m \to \infty$, we apply Stirling's formula [188, p. 253]

$$n! = \sqrt{2\pi} \exp\left(\left(n - \frac{1}{2}\right) \ln(n+1) - (n+1) + \frac{\theta}{12(n+1)}\right),$$

for $0 \leq \theta \leq 1$. We obtain

$$a_m = 2\frac{(2m-3)!}{m!(m-2)!} = \mathcal{O}(1)\, 4^m\, m^{-1/2}. \qquad \square$$

We begin the error analysis by estimating how much we "throw away" by the ghost fronts. To do this, it is useful to introduce the concept of the *generation* of a front. We say that each initial front starting at $t = 0$ belongs to the first generation. Consider two first-generation fronts of families l and r, respectively, that collide. The resulting fronts of families l and r will still belong to the first generation, while all the remaining fronts resulting from this collision will be called second-generation fronts. More generally, if a front of family l and generation m interacts with a front of family r and generation n, the resulting fronts of families l and r are still assigned generations m and n, respectively, while the remaining fronts resulting from this collision are given generation $n + m$. If k fronts, of generations $\breve{1}, \dots, \breve{k}$ and families $\hat{1}, \dots, \hat{k}$ collide, then the resulting fronts of family $\hat{\imath}$ have generation $\breve{\imath}$, while resulting fronts of families *not* in the set $\left\{\hat{1}, \dots, \hat{k}\right\}$ will have generation $\min_{i,j} \{\breve{\imath} + \breve{\jmath}\}$. The motivation behind this concept is that fronts of high generations will have small strength.

For fixed δ and ν, there will be only a finite number of fronts in $u^\delta(x,t)$. We can use Lemma 6.5 to estimate the number of fronts of generation m. If we let G_m denote this number, we have that

$$G_{m+1} \leq (n-2)\frac{T}{\delta} \sum_{j=1}^{m} G_{m+1-j} G_j, \quad m \geq 1, \quad G_1 = N \leq \mathcal{O}(1)\, \frac{T}{\delta}. \tag{6.31}$$

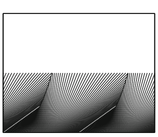
This holds since there will be at most $(n-2)$ waves of new generations at each collision, each of which can consist of at most T/δ rarefaction fronts.

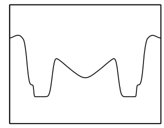
Set $C = (n-2)T/\delta$ and

$$a_m = \frac{G_m}{C^{m-1}}.$$

Then a_m satisfies

$$\begin{aligned} a_{m+1} &= \frac{G_{m+1}}{C^m} \\ &\leq \frac{1}{C^{m-1}} \sum_{j=1} G_{m+1-j} G_j \\ &= \frac{1}{C^{m-1}} \sum_{j=1}^{m} a_{m+1-j} a_j C^{m+1-j-1} K^{j-1} \\ &= \sum_{j=1}^{m} a_{m+1-j} a_j. \end{aligned}$$





We can use Lemma 6.5 and conclude that

$$\begin{aligned} G_m &\le \mathcal{O}(1) \left(\frac{(n-2)T}{\delta} \right)^{m-1} (4N)^m m^{-1/2} \\ &\le \mathcal{O}(1) \frac{4^m (n-2)^{m-1} T^{2m-1}}{\delta^{2m-1} m^{1/2}}. \end{aligned} \tag{6.32}$$

We also need to estimate the total variation of the fronts belonging to a given generation. Let $\mathcal{G}_m$ denote the set of all fronts of generation m, and let $\mathcal{T}_m$ denote the sum of the strengths of fronts of generation m. Thus

$$\mathcal{T}_m(t) = \sum_{\epsilon_j \in \mathcal{G}_m} \left| \epsilon_j \right|.$$

Since there are no fronts of generation more than N (see the discussion of Theorem 6.4),

$$T(t) = \sum_{m=1}^{N} \mathcal{T}_m(t).$$

Lemma 6.6 *We have that*

$$\mathcal{T}_m(t) \le C(4KT(t))^m$$

for some constant C.

Proof Using the interaction estimate, we obtain

$$\begin{aligned} \mathcal{T}_{m+1} &= \sum_{j=1}^{m} \sum_{\epsilon_l \in \mathcal{G}_{m+1-j}} \sum_{\epsilon_r \in \mathcal{G}_j} \mathcal{O}\left(|\epsilon_l| \, |\epsilon_r| \right) \\ &\le K \sum_{j=1}^{m} \sum_{\epsilon_l \in \mathcal{G}_{m+1-j}} \sum_{\epsilon_r \in \mathcal{G}_j} |\epsilon_l| \, |\epsilon_r| \\ &= K \sum_{j=1}^{m} \mathcal{T}_{m+1-j} \mathcal{T}_j . \end{aligned}$$

By introducing $\tilde{T}_m(t) = \mathcal{T}_m(t)/(T(t)^m K^{m-1})$, we see that $\tilde{T}_m(t)$ satisfies

$$\tilde{T}_{m+1}(t) \le \sum_{j=1}^{m} \tilde{T}_{m+1-j}(t) \tilde{T}_j(t),$$

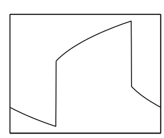
with $\tilde{T}_1(t) \le 1$. Now we can use Lemma 6.5 to conclude that

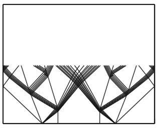
$$\tilde{T}_m \le C 4^m m^{-1/2},$$

and thus

$$\mathcal{T}_m \le C \frac{(4KT)^m}{\sqrt{m}}, \tag{6.33}$$

and the lemma follows. □





Next we must estimate the change in the strength of a ghost front as it collides with other fronts. We denote the strength of the ghost front after colliding with m other fronts by ϵ_m. First we claim that

$$|\epsilon_0| \leq K\nu. \tag{6.34}$$

To see this, assume that a front ϵ_l of family $\hat{l}$ and a front ϵ_r of family $\hat{r}$ collide and produce a ghost front; see Fig. 6.2. If $\hat{l} > \hat{r}$, then (6.7) holds, and if $\hat{l} = \hat{r}$, (6.8) holds. If we solve the Riemann problem exactly, obtaining n waves of strengths $\epsilon_1', \dots, \epsilon_n'$, we have that

$$u_r = W_{n,\epsilon_n'} \circ W_{n-1,\epsilon_{n-1}'} \circ W_{1,\epsilon_1'} u_l,$$

as well as the interaction estimate

$$\epsilon_i' = \delta_{i,\hat{l}}\epsilon_l + \delta_{i,\hat{r}}\epsilon_r + \mathcal{O}(1)\,|\epsilon_l \epsilon_r|\,.$$

With a slight abuse of notation, write

$$W(\epsilon_1, \epsilon_2, \dots, \epsilon_n)u_l := W_{n,\epsilon_n} \circ W_{n-1,\epsilon_{n-1}} \circ W_{1,\epsilon_1} u_l,$$

so that

$$u_r = W(\epsilon_1', \epsilon_2', \dots, \epsilon_n')u_l$$

and

$$u_r' = W(0, \dots, 0, \epsilon_r, 0, \dots, 0, \epsilon_l, 0, \dots, 0)u_l.$$

The function W has bounded derivatives with respect to all its arguments, whence

$$\begin{aligned} |\epsilon_0| = |u_r' - u_r| &\leq K \sum_{i=1}^{n} \left|\epsilon_i' - \delta_{i,\hat{l}}\epsilon_l - \delta_{i,\hat{r}}\epsilon_r\right| \\ &\leq K\,|\epsilon_l \epsilon_r| \leq K\nu, \end{aligned}$$

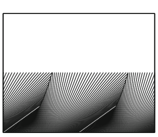
and (6.34) holds. The proof of (6.34) if several fronts interact to produce a ghost front is analogous.

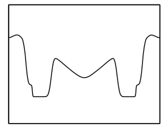
To estimate how the strength of a ghost front evolves as it collides with physical fronts, we use the interaction estimate (6.15),

$$|\epsilon_{m+1}| \leq (1 + K\,|\epsilon_r|)\,|\epsilon_m|\,,$$

after the next collision with a front ϵ_r. Using this repeatedly, after collisions with $\epsilon_{r,1}, \dots, \epsilon_{r,m}$, yields

$$\begin{aligned} |\epsilon_m| &\leq (1 + K\,|\epsilon_{r,1}|) \cdots (1 + K\,|\epsilon_{r,m}|)\,|\epsilon_0| \\ &\leq |\epsilon_0| \exp\Big(K \sum_{k=1}^{m} |\epsilon_{r,k}|\Big). \end{aligned}$$





Assume that the ghost front started at (x_0, t_0), and let $Y(x)$ be the curve coinciding with the trajectory of the ghost front for $t > t_0$ and t_0 otherwise, i.e.,

$$Y(x) = \begin{cases} t_0 & x \le x_0, \\ t_0 + \frac{x - x_0}{\Lambda} & x > x_0. \end{cases}$$

Then we have that

$$\sum_{k=1}^{m} |\epsilon_{r,k}| \le T \Big|_{Y(x)^-} \le G(t_0) \le T(0) + cT(0)^2 \le \frac{1}{2K},$$

since $Y(x)$ is "spacelike." Hence, for all ghost fronts,

$$|\epsilon| \le K\nu e^{1/2}, \tag{6.35}$$

since their initial strength is by definition bounded by $K\nu$.

Now we can finally determine ν. Let $\mathcal{G}$ denote the set of all ghost fronts. We want to choose ν such that the variation of u^δ across the ghost fronts vanishes as δ becomes small. Let T_g denote this variation. We have that

$$\begin{aligned} T_g &= \sum_{g \in \mathcal{G}} |\epsilon_g| \\ &= \sum_{\breve{g}=1}^{k_0-1} |\epsilon_g| + \sum_{\breve{g} \ge k_0} |\epsilon_g| \\ &\le K e^{1/2} \nu \sum_{k=1}^{k_0-1} G_k + \sum_{k \ge k_0} C\,(4KT)^k\,, \end{aligned}$$

where G_k is the total number of fronts of generation k, and T is the total variation over all fronts. Now we assume that $T(0)$ is so small that

$$4KT(t) \le \kappa < 1.$$

By (6.32),

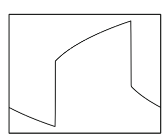
$$G_k \le C(C/\delta)^{2k-1}.$$

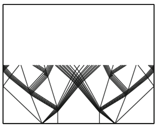
Using this, we have that

$$T_g \le C\nu \sum_{k=1}^{k_0-1} \left(\frac{C}{\delta}\right)^{2k-1} + C\frac{\kappa^{k_0}}{1-\kappa}.$$

Now we first choose k_0 such that

$$C\frac{\kappa^{k_0}}{1-\kappa} \le \frac{\delta}{2},$$





and then choose ν such that

$$C\nu \sum_{k=1}^{k_0-1} \left(\frac{C}{\delta}\right)^{2k-1} \le \frac{\delta}{2}. \tag{6.36}$$

Thus $T_g \le \delta$, and the total strength of the ghost fronts is small.

Now we can estimate how far u^δ is from being a weak solution. Recall that shock fronts are local weak solutions, while we are making errors across fronts approximating rarefaction waves and across ghost fronts.

To bound the error coming from a ghost front, we use

$$|f(u_l) - f(u_r) - \Lambda(u_l - u_r)| \le C\,|u_l - u_r|\,. \tag{6.37}$$

This follows from the Lipschitz continuity of f.

To bound the error coming from a rarefaction front separating u_l and u_r, we note that $u_r = W_{j,\epsilon} u_l$ for some $\epsilon \le \delta$, and we shall need to estimate

$$\begin{aligned}\phi(\epsilon) &= f(u_r) - f(u_l) - \lambda_j(u_l)(u_r - u_l)\\ &= f\left(W_{j,\epsilon}u_l\right) - f(u_l) - \lambda(u_l)\left(W_{j,\epsilon}u_l - u_l\right).\end{aligned}$$

We have that $\phi(0) = 0$ and that

$$\phi'(0) = df(u_l)r_j(u_l) - \lambda_j(u_l)r_j(u_l) = 0.$$

Hence, $\phi(\epsilon) = \mathcal{O}\left(\epsilon^2\right)$, or

$$\left|f(u_r) - f(u_l) - \lambda_j(u_l)(u_r - u_l)\right| \le C\delta^2, \tag{6.38}$$

if u_l and u_r are the left and right states of a rarefaction front.

By construction, if u_l and u_r are the states to the left and right of a shock front traveling with a speed σ, then

$$f(u_r) - f(u_l) - \sigma(u_r - u_l) = 0.$$

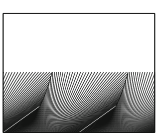
For a fixed time, we have that u^δ is piecewise constant in x, and that the discontinuities of u^δ are located at x_i and move with speed σ_i for $i = 1, \dots, N$. This holds for all times t that are not collision times. Using this, we can write

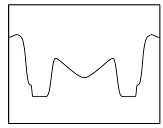
$$\begin{aligned}u^\delta(x,t) &= u_L + \sum_i H\left(x - x_i(t)\right) [\![u]\!]_i\,,\\ f\left(u^\delta\right) &= f\left(u_L\right) + \sum_i H\left(x - x_i(t)\right) [\![f(u)]\!]_i\,,\end{aligned}$$

where H denotes the Heaviside function and $[\![u]\!]_i = u_r - u_l$ if u_r is the state to the right of the discontinuity, and u_l the state to the left. Thus in the distributional sense,

$$\begin{aligned}u^\delta_t(x,t) &= -\sum_i \sigma_i\, [\![u]\!]_i\ \delta_{x_i(t)}(x),\\ f\left(u^\delta(x,t)\right)_x &= \sum_i [\![f(u)]\!]_i\ \delta_{x_i(t)}(x),\end{aligned} \tag{6.39}$$

where $\delta_{x_i(t)}$ denotes the Dirac delta distribution located at $x_i(t)$.





We can use this to estimate how far u^δ is from a being a weak solution. Recall that u is a weak solution of (6.1) if

$$\int_0^\infty \int_{\mathbb{R}} \left(u\varphi_t + f(u)\varphi_x\right) dx\, dt + \int_{\mathbb{R}} u(x,0)\varphi(x,0)\, dx = 0.$$

Since $u = \lim_{\delta\to 0} u^\delta$, we need to show that

$$0 = \lim_{\delta\to 0}\Big(\int_0^\infty \int_{\mathbb{R}} \left(u^\delta\varphi_t + f(u^\delta)\varphi_x\right) dx\, dt + \int_{\mathbb{R}} u^\delta(x,0)\varphi(x,0)\, dx\Big), \tag{6.40}$$

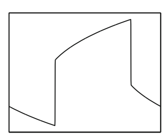
for all test functions φ. We have constructed the initial data $u^\delta(x,0)$ such that the last integral in the limit approaches $\int u_0\varphi(x,0)\,dx$. Regarding the double integral, using the representation of u^δ as a sum of Heaviside functions and (6.39), we have

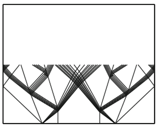
$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \left(u^\delta\varphi_t + f(u^\delta)\varphi_x\right) dx\, dt \\
&\qquad = -\int_0^T \sum_i \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt \\
&\qquad = -\sum_{i\in\mathcal{S}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt \\
&\qquad\quad - \sum_{i\in\mathcal{R}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt \\
&\qquad\quad - \sum_{i\in\mathcal{G}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt,
\end{aligned}$$

where $\mathcal{S}$ denotes the set of shock fronts, $\mathcal{R}$ the set of rarefaction fronts, and $\mathcal{G}$ the set of ghost fronts. Here, T is chosen so that φ is zero for $t > 0$. We have that

$$\begin{aligned}
&\sum_{i\in\mathcal{S}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt = 0, \\
&\Big|\sum_{i\in\mathcal{R}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt\Big| \le C \sum_{i\in\mathcal{R}} \left|[\![u]\!]_i\right|^2 \le C\delta, \\
&\Big|\sum_{i\in\mathcal{G}} \int_0^T \left(\sigma_i [\![u]\!]_i - [\![f(u)]\!]_i\right) \varphi(x_i(t),t)\, dt\Big| \le C \sum_{i\in\mathcal{G}} \left|[\![u]\!]_i\right| \le C\delta.
\end{aligned}$$

Thus the limit is a weak solution.





We can actually extract some more information about the limit u by examining the approximate solutions u^δ. More precisely, we would like to show that isolated jump discontinuities of u satisfy the Lax entropy condition

$$\lambda_m(u_l) \geq \sigma \geq \lambda_m(u_r) \tag{6.41}$$

for some m between 1 and n, where σ is the speed of the discontinuity, and

$$u_l = \lim_{y \to x-} u(y,t) \quad \text{and} \quad u_r = \lim_{y \to x+} u(y,t).$$

To show this, we assume that u has an isolated discontinuity at (x,t), with left and right limits u_l and u_r. We can enclose (x,t) by a trapezoid E_δ with corners defined as follows. Start by finding points

$$x^k_{\delta,l} \to x-, \quad x^k_{\delta,r} \to x+, \quad t^1_\delta \uparrow t, \quad t^2_\delta \downarrow t,$$

for $k = 1,2$ as $\delta \to 0$. We let E_δ denote the trapezoid with corners $(x^1_{\delta,l}, t^1_\delta)$, $(x^1_{\delta,r}, t^1_\delta)$, $(x^2_{\delta,r}, t^2_\delta)$, $(x^2_{\delta,l}, t^2_\delta)$. Recall that convergence in L^1_{loc} implies pointwise convergence almost everywhere, so we choose these points such that

$$\left.\begin{matrix} u^\delta(x^1_{\delta,l}, t^1_\delta) \\ u^\delta(x^2_{\delta,l}, t^2_\delta) \end{matrix}\right\} \to u_l \quad \text{and} \quad \left.\begin{matrix} u^\delta(x^1_{\delta,r}, t^1_\delta) \\ u^\delta(x^2_{\delta,r}, t^2_\delta) \end{matrix}\right\} \to u_r$$

as $\delta \to 0$. We can also choose points such that the diagonals of E_δ have slopes not too different from σ; precisely,

$$\left| \frac{x^1_{\delta,l} - x^2_{\delta,r}}{t^1_\delta - t^2_\delta} - \sigma \right| \leq \varepsilon(\delta) \quad \text{and} \quad \left| \frac{x^1_{\delta,l} - x^2_{\delta,r}}{t^1_\delta - t^2_\delta} - \sigma \right| \leq \varepsilon(\delta), \tag{6.42}$$

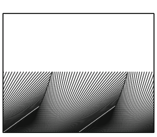
where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Next for $k = 1,2$, set

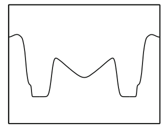
$$M^k_\delta = \frac{\sum |\epsilon_i|}{x^k_{\delta,r} - x^k_{\delta,l}},$$

where the sum is over all rarefaction fronts in the interval $\left[x^k_{\delta,l}, x^k_{\delta,r}\right]$. If M^k_δ is unbounded as $\delta \to 0$, then u contains a centered rarefaction wave at (x,t), i.e., a rarefaction wave starting at (x,t). In this case the discontinuity will not be isolated, and hence M^k_δ remains bounded as $\delta \to 0$. Next observe that

$$\begin{aligned} \frac{\left| u^\delta(x^k_{\delta,l}, t^k_\delta) - u^\delta(x^k_{\delta,r}, t^k_\delta) \right|}{x^k_{\delta,r} - x^k_{\delta,l}} &\leq C \frac{\sum |\text{rarefaction fronts}| + \sum |\text{shock fronts}|}{x^k_{\delta,r} - x^k_{\delta,l}} \\ &= C M^k_\delta + C \frac{\sum |\text{shock fronts}|}{x^k_{\delta,r} - x^k_{\delta,l}}. \end{aligned}$$

Here the sums are over fronts crossing the interval $\left[x^k_{\delta,l}, x^k_{\delta,r}\right]$. Since the fraction on the left is unbounded as $\delta \to 0$, there must be shock fronts crossing the top and





bottom of E_δ for all $\delta > 0$. Furthermore, since the discontinuity is isolated, the total strength of all fronts crossing the left and right sides of E_δ must tend to zero as $\delta \to 0$.

Next we define a *shock line* as a sequence of shock fronts of the same family in u^δ. Assume that a shock line has been defined for $t < t_n$, where t_n is a collision time, and in the interval $[t_{n-1}, t_n)$ consists of the shock front ϵ. In the interval $[t_n, t_{n+1})$, this shock line continues as the front ϵ if ϵ does not collide at t_n. If ϵ collides at t_n, and the approximate solution of the Riemann problem determined by this collision contains an approximate shock front of the same family as ϵ, then the shock line continues as this front. Otherwise, it stops at t_n. Note that we can associate a unique family to each shock line.

From the above reasoning it follows that for all δ there must be shock lines entering E_δ through the bottom that do not exit E_δ through the sides; hence such shock lines must exit E_δ through the top. Assume that the leftmost of these shock lines enters E_δ at $y^1_{\delta,l}$ and leaves E_δ at $y^2_{\delta,l}$. Similarly, the rightmost of the shock lines enters E_δ at $y^1_{\delta,r}$ and leaves E_δ at $y^2_{\delta,r}$. Set

$$v^k_{\delta,l} = u^\delta\left(y^k_{\delta,l}-, t^k_\delta\right) \quad \text{and} \quad v^k_{\delta,r} = u^\delta\left(y^k_{\delta,r}+, t^k_\delta\right).$$

Between $y^k_{\delta,l}$ and $x^k_{\delta,l}$, the function u^δ varies over rarefaction fronts or over shock lines that must enter or leave E_δ through the left or right side. Since the discontinuity is isolated, the total strength of such waves must tend to zero as $\delta \to 0$. Because $u^\delta(x^k_{\delta,l}, t^k_\delta) \to u_l$ as $\delta \to 0$, we have that $v^k_{\delta,l} \to u_l$ as $\delta \to 0$. Similarly, $v^k_{\delta,r} \to u_r$. Since $\varepsilon(\delta) \to 0$, by strict hyperbolicity, the family of all shock lines not crossing the left or right side of E_δ must be the same, say m. The speed of an approximate m-shock front with speed $\tilde\sigma$ and left state $v^k_{\delta,l}$ satisfies

$$\lambda_{m-1}\left(v^k_{\delta,l}\right) < \tilde\sigma + \mathcal{O}(\delta) < \lambda_m\left(v^k_{\delta,l}\right). \tag{6.43}$$

Similarly, an approximate m-shock front with right state $v^k_{\delta,r}$ and speed $\hat\sigma$ satisfies

$$\lambda_m\left(v^k_{\delta,r}\right) < \hat\sigma + \mathcal{O}(\delta) < \lambda_{m+1}\left(v^k_{\delta,r}\right). \tag{6.44}$$

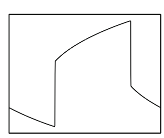
Then (6.41) follows by noting that $\tilde\sigma$ and $\hat\sigma$ both tend to σ as $\delta \to 0$, and then letting $\delta \to 0$ in (6.43) and (6.44).

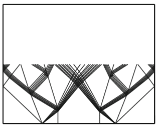
To summarize the results of this chapter we have the following theorem:

Theorem 6.7 *Consider the strictly hyperbolic system of equations*

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x),$$

and assume that $f \in C^2$ is such that each characteristic wave family is either linearly degenerate or genuinely nonlinear. If T.V.(u_0) *is sufficiently small, there exists a global weak solution $u(x,t)$ to this initial value problem. This solution may be constructed by the front-tracking algorithm described in Sect. 6.1. Furthermore, if u has an isolated jump discontinuity at a point (x,t), then the Lax entropy condition* (6.41) *holds for some m between* 1 *and n.*





We have seen that for each $\delta > 0$ there is only a finite number of collisions between the fronts in u^δ for all $t > 0$. Hence there exists a finite time T_δ such that for $t > T_\delta$, the fronts in u^δ will move apart, and not interact. This has some similarity to the solution of the Riemann problem. One can intuitively make the change of variables $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$ without changing the equation, but the initial data is changed to $u_0(x/\varepsilon)$. Sending $\varepsilon \to 0$, or alternatively $t \to \infty$, we see that u solves the Riemann problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x \geq 0, \end{cases} \tag{6.45}$$

where $u_L = \lim_{x\to-\infty} u_0(x)$ and $u_R = \lim_{x\to\infty} u_0(x)$. Thus in some sense, for very large times, u should solve this Riemann problem. Next, we shall show that this (very imprecise statement) is true, but first we need some more information about u^δ.

For $t > T_\delta$, the function u^δ will consist of a finite number of constant states, say u_i^δ, for $i = 0, \dots, M$. If u_{i-1}^δ is connected with u_i^δ by a wave of a different family from the one connecting u_i^δ to u_{i+1}^δ, we call u_i^δ a real state, and we let $\{\bar{u}_i\}_{i=0}^N$ be the set of real states of u^δ. Since the discontinuities of u^δ are moving apart, we must have

$$N \leq n, \tag{6.46}$$

by strict hyperbolicity. Furthermore, to each pair $(\bar{u}_{i-1}, \bar{u}_i)$ we can associate a family k_i such that $1 \leq k_i < k_{i+1} \leq n$, and we define $k_0 = 0$ and $k_{N+1} = n$. We write the solution of the Riemann problem with left and right data $\bar{u}_0$ and $\bar{u}_N$, respectively, as u, and define ϵ_j, $j = 1, \dots, n$, by

$$\bar{u}_N = W_n(\epsilon_n) W_{n-1}(\epsilon_{n-1}) \cdots W_1(\epsilon_1) \bar{u}_0,$$

and define the intermediate states

$$u_0 = \bar{u}_0 \quad \text{and} \quad u_j = W_j(\epsilon_j) u_{j-1} \quad \text{for } j = 1, \dots, n.$$

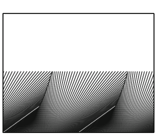
Now we claim that

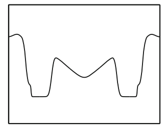
$$\left| u_j - \bar{u}_i \right| \leq \mathcal{O}(\delta), \quad \text{for } k_{i-1} \leq j \leq k_i. \tag{6.47}$$

If $N = 1$, this clearly holds, since in this case u^δ consists of two states for $t > T_\delta$, and by construction of u^δ, the pair $(\bar{u}_0, \bar{u}_1)$ is the solution of the same Riemann problem as u is, but possibly with waves of a high generation ignored.

Now assume that (6.47) holds for some $N > 1$. We shall show that it holds for $N + 1$ as well. Let v be the solution of the Riemann problem with initial data given by

$$v(x,0) = \begin{cases} \bar{u}_0 & \text{for } x < 0, \\ \bar{u}_N & \text{for } x \geq 0, \end{cases}$$





and let w be the solution of the Riemann problem with initial data

$$w(x,0) = \begin{cases} \bar{u}_N & \text{for } x < 0, \\ \bar{u}_{N+1} & \text{for } x \geq 0. \end{cases}$$

We denote the waves in v and w by ϵ_j^v and ϵ_j^w, respectively. Then by the induction hypothesis,

$$\left|\bar{\epsilon}_i - \epsilon_{k_i}^v\right| \leq \mathcal{O}(\delta), \qquad \left|\bar{\epsilon}_{N+1} - \epsilon_{k_{N+1}}^w\right| \leq \mathcal{O}(\delta),$$
$$\sum_{i \notin \{k_1,\ldots,k_N\}} |\epsilon_i^v| \leq \mathcal{O}(\delta), \quad \text{and} \quad \sum_{i \neq k_{N+1}} |\epsilon_i^w| \leq \mathcal{O}(\delta),$$

where $\bar{\epsilon}_i$ denotes the strength of the wave separating $\bar{u}_{i-1}$ and $\bar{u}_i$. Notice now that u can be viewed as the interaction of v and w; hence by the interaction estimate,

$$\sum_i |\epsilon_i - \epsilon_i^v| \leq \mathcal{O}(\delta) \quad \text{for} \quad i \leq k_N, \quad \text{and} \quad \left|\epsilon_{k_{N+1}} - \epsilon_{k_{N+1}}^w\right| \leq \mathcal{O}(\delta).$$

Thus (6.47) holds for $N+1$ real states, and therefore for every $N \leq n$. Now we can conclude that for $u = \lim_{\delta \to 0} u^\delta$ the following result holds.

Theorem 6.8 *Assume that* $u_L = \lim_{x \to -\infty} u_0(x)$ *and* $u_R = \lim_{x \to \infty} u_0(x)$ *exist. Then as* $t \to \infty$, *u will consist of a finite number of states* $\{u_i\}_{i=0}^N$, *where* $N \leq n$. *These states are the intermediate states in the solution of the Riemann problem* (6.45), *and they will be separated by the same waves as the corresponding states in the solution of the Riemann problem.*

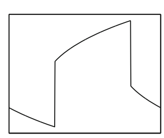
Proof By the calculations preceding the lemma, for $t > T_\delta$ we can define a function $\bar{u}_\delta$ that consists of a number of constant states separated by elementary waves, shocks, rarefactions, or contact discontinuities such that these constant states are the intermediate states in the solution of the Riemann problem defined by $\lim_{x \to -\infty} u^\delta(x,t)$ and $\lim_{x \to \infty} u^\delta(x,t)$, and such that for every bounded interval I,

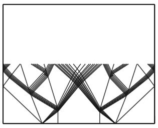
$$\left\|\bar{u}_\delta(\,\cdot\,,t) - u^\delta(\,\cdot\,,t)\right\|_{L^1(I)} \to 0 \quad \text{as } \delta \to 0.$$

Then for $t > T_\delta$,

$$\begin{aligned}\|u(\,\cdot\,,t) - \bar{u}_\delta(\,\cdot\,,t)\|_{L^1(I)} \leq{}& \left\|u(\,\cdot\,,t) - u^\delta(\,\cdot\,,t)\right\|_{L^1(I)} \\ &+ \left\|\bar{u}_\delta(\,\cdot\,,t) - u^\delta(\,\cdot\,,t)\right\|_{L^1(I)}.\end{aligned}$$

Set $t = T_\delta + 1$, and let $\delta \to 0$. Then both terms on the right tend to zero, and $\bar{u}_0 \to u_L$ and $\bar{u}_N \to u_R$. Hence the lemma holds. Note, however, that u does not necessarily equal some $\bar{u}_\delta$ in finite time. □





Remark 6.9 Here is another way to interpret heuristically the asymptotic result for large times. Consider the set

$$\{u^\delta(x,t) \mid x \in \mathbb{R}\}$$

in phase space. There is a certain ordering of that set given by the ordering of x. As $\delta \to 0$, this set will approach some set

$$\{u(x,t) \mid x \in \mathbb{R}\}.$$

Theorem 6.8 states that as $t \to \infty$ this set approaches the set that consists of the states in the solution of the Riemann problem (6.45) with the same order. No statements are made as to how fast this limit is obtained. In particular, if $u_L = u_R = 0$, then $u(x,t) \to 0$ for almost all x as $t \to \infty$.

6.3 Notes

The fundamental result concerning existence of solutions of the general Cauchy problem is due to Glimm [72], where the fundamental approach was given, and where all the basic estimates can be found. Glimm's result for small initial data uses the *random choice method*. The random element is not really essential to the random choice method, as was shown by Liu in [135]. The existence result has been extended for some 2×2 systems, allowing for initial data with large total variation; see [144, 170]. These systems have the rather special property that the solution of the Riemann problem is translation-invariant in phase space.

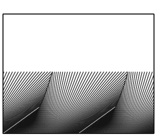
Our proof of the interaction estimate (6.13) is a modified version of Yong's argument [190].

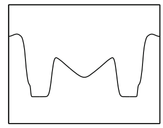
Front tracking for systems was first used by DiPerna in [60]. In this work a front-tracking process was presented for 2×2 systems, and shown to be well defined and to converge to a weak solution. Although DiPerna states that "the method is adaptable for numerical calculation," numerical examples of front tracking were first presented by Swartz and Wendroff in [172], in which front tracking was used as a component in a numerical code for solving problems of gas dynamics.

The front tracking presented here contains elements from the front-tracking methods of Bressan [21] and, in particular, of Risebro [160]. In [160] the generation concept was not used. Instead, one "looked ahead" to see whether a buildup of collision times was about to occur. In [9] Baiti and Jenssen showed that one does not really need to use the generation concept or look ahead in order to decide which fronts to ignore.

The large-time behavior of u was shown to hold for the limit of the Glimm scheme by Liu in [136].

The front-tracking method presented in [160] has been used as a numerical method; see Risebro and Tveito [162, 163] and Langseth [121, 122] for examples of problems in one space dimension. In several space dimensions, front tracking has also been used in conjunction with dimensional splitting with some success for systems; see [92] and [132].





6.4 Exercises

6.1 Assume that $f:\mathbb{R}^n \to \mathbb{R}^n$ is three-times differentiable, with bounded derivatives. We study the solution of the system of ordinary differential equations

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

We write the unique solution as $x(t) = \exp(tf)x_0$.

(a) Show that

$$\exp(\varepsilon f)x_0 = x_0 + \varepsilon f\,(x_0) + \frac{\varepsilon^2}{2} df\,(x_0)\, f\,(x_0) + \mathcal{O}\left(\varepsilon^3\right).$$

(b) If g is another vector field with the same properties as f, show that

$$\begin{aligned}\exp(\varepsilon g)\exp(\varepsilon f)x_0 = x_0 &+ \varepsilon\,(f\,(x_0) + g\,(x_0)) \\ &+ \frac{\varepsilon^2}{2}\Big(df\,(x_0)\, f\,(x_0) + \, dg\,(x_0)\, g\,(x_0)\Big) \\ &+ \varepsilon^2 dg\,(x_0)\, f\,(x_0) + \mathcal{O}\left(\varepsilon^3\right).\end{aligned}$$

(c) The *Lie bracket* of f and g is defined as

$$[f, g](x) = dg(x)f(x) - df(x)g(x).$$

Show that

$$[f, g]\,(x_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\Big(\exp(\varepsilon g)\exp(\varepsilon f)x_0 - \exp(\varepsilon f)\exp(\varepsilon g)x_0\Big).$$

(d) Indicate how this can be used to give an alternative proof of the interaction estimate (6.13).

6.2 We study the p system with $p(u_1)$ as in Exercise 5.3, and we use the results of Exercise 5.9. Define a front-tracking scheme by introducing a grid in the (μ, τ)-plane. We approximate rarefaction waves by choosing intermediate states that are not farther apart than δ in (μ, τ). If ϵ is a front with left state (μ_l, τ_l) and right state (μ_r, τ_r), define

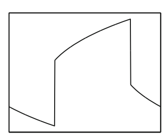
$$T(\epsilon) = \begin{cases} |[\![\mu]\!]| - |[\![\tau]\!]| & \text{if } \epsilon \text{ is a 1-wave,} \\ |[\![\tau]\!]| - |[\![\mu]\!]| & \text{if } \epsilon \text{ is a 2-wave,} \end{cases}$$

and define T additively for a sequence of fronts.

(a) Define a front-tracking algorithm based on this, and show that

$$T_{n+1} \le T_n,$$

where T_n denotes the T value of the front-tracking approximation between collision times t_n and t_{n+1}.



(b) Find a suitable condition on the initial data so that the front-tracking algorithm produces a convergent subsequence.
(c) Show that the limit is a weak solution.

6.3 Assume that the flux function $f(u)$ admits an entropy/entropy flux pair (η, q), that is, η and q are functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$\nabla_u \eta(u) = \nabla_u q(u) df(u).$$

Assume also that for the solution of the Riemann problem

$$u_t + f(u)_x = 0, \quad u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases}$$

we have that

$$\begin{aligned} \eta(u)_t + q(u)_x &= 0 \text{ if the solution is a rarefaction wave} \\ &\qquad \text{or contact discontinuity,} \\ \eta(u)_t + q(u)_x &< 0 \text{ in the distributional sense if the solution is a shock.} \end{aligned}$$

Let now $u = \lim u^\delta$, where u^δ is the front-tracking approximation. Show that

$$\eta(u)_t + q(u)_x \le 0,$$

in the distributional sense.

Chapter 7

Well-Posedness of the Cauchy Problem

Ma per seguir virtute e conoscenza.[1]
— Dante Alighieri (1265–1321), La Divina Commedia

The goal of this chapter is to show that the limit found by front tracking, that is, the weak solution of the initial value problem

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \tag{7.1}$$

is stable in L^1 with respect to perturbations in the initial data. In other words, if $v = v(x,t)$ is another solution found by front tracking, then

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_1 \le C\,\|u_0 - v_0\|_1$$

for some constant C. Furthermore, we shall show that under some mild extra entropy conditions, every weak solution coincides with the solution constructed by front tracking.

◇ Example 7.1 (A special system)
As an example for this chapter we shall consider the special 2×2 system

$$\begin{aligned} u_t + \left(vu^2\right)_x &= 0,\\ v_t + \left(uv^2\right)_x &= 0. \end{aligned} \tag{7.2}$$

For simplicity assume that $u > 0$ and $v > 0$. The Jacobian matrix reads

$$\begin{pmatrix} 2uv & u^2 \\ v^2 & 2uv \end{pmatrix}, \tag{7.3}$$

with eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 &= uv, \qquad & r_1 &= \begin{pmatrix} -u/v \\ 1 \end{pmatrix},\\ \lambda_2 &= 3uv, \qquad & r_2 &= \begin{pmatrix} u/v \\ 1 \end{pmatrix}. \end{aligned} \tag{7.4}$$

[1] Hard to comprehend? It means "[but to] pursue virtue and knowledge."

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_7

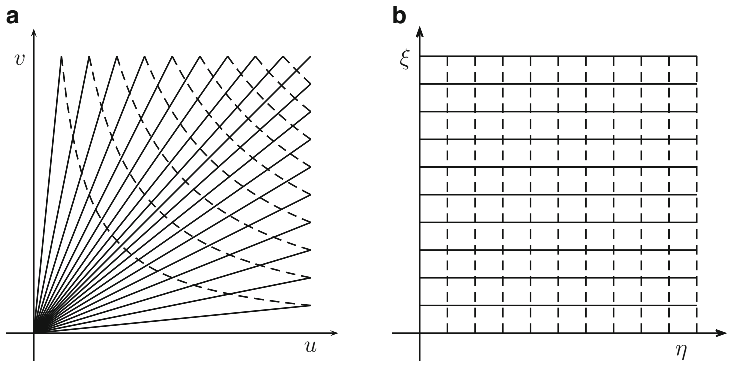


Fig. 7.1 The curves W in (u, v) coordinates (**a**) and (η, ξ) coordinates (**b**)

The system is clearly strictly hyperbolic. Observe that

$$\nabla\lambda_1 \cdot r_1 = 0,$$

and hence the first family is linearly degenerate. The corresponding wave curve $W_1(u_l, v_l) = C_1(u_l, v_l)$ is given by (cf. Theorem 5.7)

$$\frac{du}{dv} = -\frac{u}{v}, \quad u\,(v_l) = u_l,$$

or (see Fig. 7.1)

$$W_1(u_l, v_l) = C_1(u_l, v_l) = \{(u, v) \mid uv = u_l v_l\}.$$

The corresponding eigenvalue λ_1 is constant along each hyperbola.

With the chosen normalization of r_2 we find that

$$\nabla\lambda_2 \cdot r_2 = 6u,$$

and hence the second-wave family is genuinely nonlinear. The rarefaction curves of the second family are solutions of

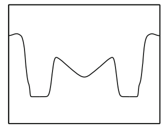
$$\frac{du}{dv} = \frac{u}{v}, \quad u\,(v_l) = u_l,$$

and thus

$$\frac{u}{v} = \frac{u_l}{v_l}.$$

We see that these are straight lines emanating from the origin, and λ_2 increases as u increases. Consequently, R_2 consists of the ray

$$v = u\frac{v_l}{u_l}, \quad u \geq u_l.$$



The rarefaction speed is given by

$$\lambda_2\left(u; u_l, v_l\right) = 3u^2 \frac{v_l}{u_l}.$$

To find the shocks in the second family, we use the Rankine–Hugoniot relation

$$\begin{aligned} s\left(u - u_l\right) &= v u^2 - v_l u_l^2, \\ s\left(v - v_l\right) &= v^2 u - v_l^2 u_l, \end{aligned}$$

which implies

$$\frac{u}{u_l} = \frac{1}{2}\left(\frac{v}{v_l} + \frac{v_l}{v} \pm \left(\frac{v}{v_l} - \frac{v_l}{v}\right)\right) = \begin{cases} v_l/v, \\ v/v_l. \end{cases}$$

(Observe that the solution with $u/u_l = v_l/v$ coincides with the wave curve of the linearly degenerate first family.) The shock part of this curve S_2 consists of the line

$$S_2(u_l, v_l) = \left\{(u, v) \,\middle|\, v = u\frac{v_l}{u_l}, \quad 0 < u \le u_l\right\}.$$

The shock speed is given by

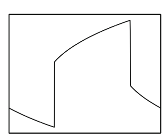
$$s := \mu_2\left(u; u_l, v_l\right) = \left(u^2 + u u_l + u_l^2\right)\frac{v_l}{u_l}.$$

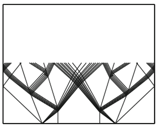
Hence the Hugoniot locus and rarefaction curves coincide for this system. Systems with this property are called *Temple class systems* after Temple [177]. Furthermore, the system is linearly degenerate in the first family and genuinely nonlinear in the second. Summing up, the solution of the Riemann problem for (7.2) is as follows: First the middle state is given by

$$u_m = \sqrt{u_l u_r \frac{v_l}{v_r}}, \quad v_m = \sqrt{v_l v_r \frac{u_l}{u_r}}.$$

If $u_l/v_l \le u_r/v_r$, the second wave is a rarefaction wave, and the solution can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}(x, t) = \begin{cases} \begin{pmatrix} u_l \\ v_l \end{pmatrix} & \text{for } x/t \le u_l v_l, \\ \begin{pmatrix} u_m \\ v_m \end{pmatrix} & \text{for } u_l v_l < x/t \le 3u_m v_m, \\ \sqrt{\frac{x}{3t}\frac{v_m}{u_m}}\begin{pmatrix} u_m/v_m \\ 1 \end{pmatrix} & \text{for } 3u_m v_m < x/t \le 3u_r v_r, \\ \begin{pmatrix} u_r \\ v_r \end{pmatrix} & \text{for } 3u_r v_r < x/t. \end{cases} \tag{7.5}$$





In the shock case, that is, when $u_l/v_l > u_r/v_r$, the solution reads

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \begin{cases} \begin{pmatrix} u_l \\ v_l \end{pmatrix} & \text{for } x/t \le u_l v_l, \\ \begin{pmatrix} u_m \\ v_m \end{pmatrix} & \text{for } u_l v_l < x/t \le \mu_2(u_r; u_m, v_m), \\ \begin{pmatrix} u_r \\ v_r \end{pmatrix} & \text{for } \mu_2(u_r; u_m, v_m) < x/t. \end{cases} \tag{7.6}$$

If we set

$$\eta = uv, \quad \xi = \frac{u}{v},$$

and thus

$$u = \sqrt{\eta\xi}, \quad v = \sqrt{\eta/\xi},$$

the solution of the Riemann problem will be especially simple in (η, ξ) coordinates. See Fig. 7.1. Given left and right states (η_l, ξ_l), (η_r, ξ_r), the middle state is given by (η_l, ξ_r). Consequently, measured in (η, ξ) coordinates, the total variation of the solution of the Riemann problem equals the total variation of the initial data. This means that we do not need the Glimm functional to show that a front-tracking approximation to the solution of (7.2) has bounded total variation. With this in mind it is easy to show (using the methods of the previous chapters) that there exists a weak solution to the initial value problem for (7.2) whenever the total variation of the initial data is bounded.

We may use these variables to parameterize the wave curves as follows:

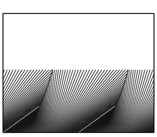
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_l v_l/\eta \\ \eta \end{pmatrix} \text{ (first family)},$$
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_l \eta/v_l \\ \eta \end{pmatrix} \text{ (second family)}.$$

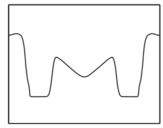
For future use we note that the rarefaction and shock speeds are as follows:

$$\lambda_1(\eta) = \mu_1(\eta) = \eta,$$
$$\lambda_2(\eta) = 3\eta, \quad \text{and} \quad \mu_2(\eta_l, \eta_r) = \left(\eta_l + \sqrt{\eta_l \eta_r} + \eta_r\right). \qquad \diamond$$

As a reminder we now summarize some properties of the front-tracking approximation for a fixed δ.

1. For all positive times t, $u^\delta(x,t)$ has finitely many discontinuities, each having position $x_i(t)$. These discontinuities can be of two types: shock fronts or approximate rarefaction fronts. Furthermore, only finitely many interactions between discontinuities occur for $t \ge 0$.





2. Along each shock front, the left and right states

$$u_{l,r} = u^{\delta}\,(x_i \mp, t) \tag{7.7}$$

are related by

$$u_r = S_{\hat{\imath}}\,(\epsilon_i)\,u_l + e_i\,,$$

where ϵ_i is the strength of the shock and $\hat{\imath}$ is the family of the shock. The "error" e_i is a vector of small magnitude. Furthermore, the speed of the shock, $\dot{x}$, satisfies

$$|\dot{x} - \mu_{\hat{\imath}}\,(u_l, u_r)| \le \mathcal{O}\,(1)\,\delta, \tag{7.8}$$

where $\mu_{\hat{\imath}}\,(u_l, u_r)$ is the $\hat{\imath}$th eigenvalue of the averaged matrix

$$M\,(u_l, u_r) = \int_0^1 df\,\left((1-\alpha)u_l + \alpha u_r\right)\,d\alpha;$$

cf. (5.76)–(5.77).
3. Along each rarefaction front, the values u_l and u_r are related by

$$u_r = R_{\hat{\imath}}\,(\epsilon_i)\,u_l + e_i\,. \tag{7.9}$$

Also,

$$|\dot{x} - \lambda_{\hat{\imath}}\,(u_r)| \le \mathcal{O}\,(1)\,\delta \quad \text{and} \quad |\dot{x} - \lambda_{\hat{\imath}}\,(u_l)| \le \mathcal{O}\,(1)\,\delta, \tag{7.10}$$

where $\lambda_{\hat{\imath}}\,(u)$ is the $\hat{\imath}$th eigenvalue of $df\,(u)$.
4. The total magnitude of all errors is small:

$$\sum_i |e_i| \le \delta. \tag{7.11}$$

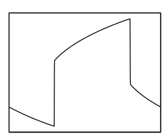
Also, recall that for a suitable constant C_0 the Glimm functional

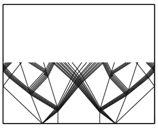
$$G\left(u^{\delta}(\cdot\,,t)\right) = T\left(u^{\delta}(\cdot\,,t)\right) + C_0 Q\left(u^{\delta}(\cdot\,,t)\right)$$

is nonincreasing for each collision of fronts, where T and Q are defined by (6.23) and (6.22), respectively, and that the interaction potential

$$Q\left(u^{\delta}(\cdot\,,t)\right)$$

is strictly decreasing for each collision of fronts.





7.1 Stability

Details are always vulgar.
— Oscar Wilde, The Picture of Dorian Gray (1891)

Now let v^δ be another front-tracking solution with initial condition v_0. To compare u^δ and v^δ in the L^1-norm, i.e., to estimate $\left\|u^\delta - v^\delta\right\|_1$, we introduce the vector $q = q(x,t) = (q_1, \dots, q_n)$ by

$$v^\delta(x,t) = H_n(q_n) H_{n-1}(q_{n-1}) \cdots H_1(q_1) u^\delta(x,t) \tag{7.12}$$

and the intermediate states ω_i,

$$\omega_0 = u^\delta(x,t), \quad \omega_i = H_i(q_i) w_{i-1}, \quad \text{for } 1 \le i \le n, \tag{7.13}$$

with velocities

$$\mu_i = \mu_i(\omega_{i-1}, \omega_i). \tag{7.14}$$

As in Chapt. 5, $H_k(\epsilon)u$ denotes the kth Hugoniot curve through u, parameterized such that

$$\frac{d}{d\epsilon} H_k(\epsilon) u\Big|_{\epsilon=0} = r_k(u).$$

Note that in the definition of q we use *both* parts of this curve, not only the part where $\epsilon < 0$. The vector q represents a "solution" of the Riemann problem with left state u^δ and right state v^δ using only shocks. (For $\epsilon > 0$ these will be weak solutions; that is, they satisfy the Rankine–Hugoniot condition. However, they will not be Lax shocks.)

Later in this section we shall use the fact that genuine nonlinearity implies that $\mu_k(u, H_k(\epsilon)u)$ will be increasing in ϵ, i.e.,

$$\frac{d}{d\epsilon} \mu_k(u, H_k(\epsilon)u) \ge c > 0,$$

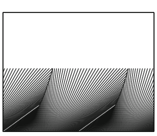
for some constant c depending only on f.

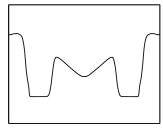
As our model problem showed, the L^1 distance is more difficult to control than the "q-distance." However, it turns out that even the q-distance is not quite enough, and we need to introduce a weighted form. We let $\mathcal{D}\left(u^\delta\right)$ and $\mathcal{D}\left(v^\delta\right)$ denote the sets of all discontinuities in u and v, respectively, and define the functional $\Phi\left(u^\delta, v^\delta\right)$ as

$$\Phi(u^\delta, v^\delta) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} |q_k(x)| \, W_k(x) \, dx. \tag{7.15}$$

Here the weights W_k are defined as

$$W_k = 1 + \kappa_1 A_k + \kappa_2 \left(Q\left(u^\delta\right) + Q\left(v^\delta\right)\right), \tag{7.16}$$





where $Q\left(u^\delta\right)$ and $Q\left(v^\delta\right)$ are the interaction potentials of u^δ and v^δ, respectively; cf. (6.22). The quantity A_k is the *total strength of all waves in* u^δ *or* v^δ *that approach the* k*-wave* $q_k(x)$. More precisely, if the kth field is linearly degenerate, then

$$A_k(x) = \sum_{\substack{i,\, x_i<x \\ \hat{\imath}>k}} |\epsilon_i| + \sum_{\substack{i,\, x>x_i \\ \hat{\imath}<k}} |\epsilon_i| . \tag{7.17}$$

The summation is over all discontinuities $x_i \in \mathcal{D}\left(u^\delta\right) \cup \mathcal{D}(v^\delta)$. If the kth field is genuinely nonlinear, we must also account for waves of the same family approaching each other, and define

$$\begin{aligned} A_k(x) = & \sum_{\substack{i,\, x_i<x \\ \hat{\imath}>k}} |\epsilon_i| + \sum_{\substack{i,\, x>x_i \\ \hat{\imath}<k}} |\epsilon_i| \\ & + \begin{cases} \displaystyle\sum_{\substack{i\in\mathcal{D}(u^\delta) \\ \hat{\imath}=k,\, x_i<x}} |\epsilon_i| + \sum_{\substack{i\in\mathcal{D}(v^\delta) \\ \hat{\imath}=k,\, x<x_i}} |\epsilon_i| & \text{if } q_k(x) < 0, \\ \displaystyle\sum_{\substack{i\in\mathcal{D}(v^\delta) \\ \hat{\imath}=k,\, x_i<x}} |\epsilon_i| + \sum_{\substack{i\in\mathcal{D}(u^\delta) \\ \hat{\imath}=k,\, x<x_i}} |\epsilon_i| & \text{if } q_k(x) > 0. \end{cases} \end{aligned} \tag{7.18}$$

In plain words, a q_k shock is approached by k-waves in u^δ from the left, and k-waves in v^δ from the right. Similarly, a q_k rarefaction wave is approached by k-waves in v^δ from the left and k-waves in u^δ from the right.

Once the values of the constants κ_1 and κ_2 are determined, we will assume that the total variations of u^δ and v^δ are so small that

$$1 \le W_k(x) \le 2. \tag{7.19}$$

In this case we see that Φ is equivalent to the L^1 norm; i.e., there exists a finite constant C_1 such that

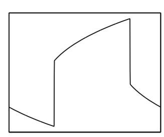
$$\frac{1}{C_1}\left\|u^\delta - v^\delta\right\|_1 \le \Phi\left(u^\delta, v^\delta\right) \le C_1\left\|u^\delta - v^\delta\right\|_1. \tag{7.20}$$

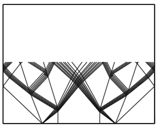
We can also define, with obvious modifications, $\Phi(u^{\delta_1}(t), v^{\delta_2}(t))$ with two different parameters δ_1 and δ_2. Our first goal will be to show that

$$\Phi\left(u^{\delta_1}(t), v^{\delta_2}(t)\right) - \Phi\left(u^{\delta_1}(s), v^{\delta_2}(s)\right) \le C_2(t-s)\big(\delta_1 \vee \delta_2\big), \tag{7.21}$$

for all $0 \le t \le s$. Once this inequality is in place, we can show that the sequence of front-tracking approximations is a Cauchy sequence in L^1 for

$$\begin{aligned} \left\|u^{\delta_1}(t) - u^{\delta_2}(t)\right\|_1 &\le C_1\Phi\left(u^{\delta_1}(t), u^{\delta_2}(t)\right) \\ &\le C_1\Phi\left(u^{\delta_1}(0), u^{\delta_2}(0)\right) + C_1C_2t\big(\delta_1 \vee \delta_2\big) \\ &\le C_1^2\left\|u^{\delta_1}(0) - u^{\delta_2}(0)\right\|_1 + C_1C_2t\big(\delta_1 \vee \delta_2\big). \end{aligned}$$





Letting δ_1 and δ_2 tend to zero, we have the convergence of the whole sequence, and not only a subsequence.

The first step in order to prove (7.21) is to choose κ_2 so large that the weights W_k do not increase when fronts in u^{δ_1} or v^{δ_2} collide. This is possible, since the total variations of both u^{δ_1} and v^{δ_2} are uniformly small; hence the terms $\kappa_1 A_k$ are uniformly bounded, and by the interaction estimate, Q decreases for all collisions. This ensures the inequalities (7.19).

Then we must examine how Φ changes between collisions. Observe that $\Phi(t)$ is piecewise linear and continuous in t. Let

$$\mathcal{D} = \mathcal{D}\left(u^{\delta_1}\right) \cup \mathcal{D}\left(v^{\delta_2}\right).$$

We differentiate Φ and find that

$$\begin{aligned}
\frac{d}{dt}\Phi\left(u^{\delta_1}, v^{\delta_2}\right) &= \sum_{i\in\mathcal{D}}\sum_{k=1}^{n}\left\{\left|q_k\left(x_i-\right)\right| W_k\left(x_i-\right) - \left|q_k\left(x_i+\right)\right| W_k\left(x_i+\right)\right\}\dot{x}_i \\
&= \sum_{i\in\mathcal{D}}\sum_{k=1}^{n}\left\{\left|q_k^{i,+}\right| W_k^{i,+}\left(\mu_k^{i,+}-\dot{x}_i\right) - \left|q_k^{i,-}\right| W_k^{i,-}\left(\mu_k^{i,-}-\dot{x}_i\right)\right\}, \\
&=: \sum_{i\in\mathcal{D}}\sum_{k=1}^{n} E_{i,k},
\end{aligned} \tag{7.22}$$

where

$$\begin{aligned}
\mu_k^{i,\pm} &= \mu_k\left(x_i\pm\right), \quad \mu_k(x) = \mu_k\left(\omega_{k-1}(x), \omega_k(x)\right), \\
q_k^{i,\pm} &= q_k\left(x_i\pm\right), \quad \text{and} \quad W_k^{i,\pm} = W_k\left(x_i\pm\right).
\end{aligned}$$

The second equality in (7.22) is obtained by adding terms

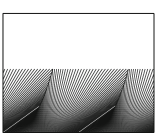
$$\left|q_k^{i,-}\right| W_k^{i,-}\mu_k^{i,-} - \left|q_k^{(i-1),+}\right| W_k^{(i-1),+}\mu_k^{(i-1),+} = 0,$$

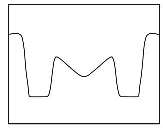
and observing that there is only a finite number of terms in the sum in (7.22).

◇ Example 7.2 (Example 7.1 (cont'd.))
Let us check how this works for our special system. The two front-tracking approximations are denoted by u and v, and for simplicity we omit the superscript δ. These are made by approximating a rarefaction wave between $\eta_l = n\delta$ and $\eta_r = m\delta$, $m > n$, by a series of discontinuities with speed $3j\delta$, $j = n, \dots, m-1$. In other words, we use the characteristic speed to the left of the discontinuity. The functions u and v are well defined by standard techniques.

Since we managed this far without the interaction potential, we define the weights also without these (they are needed only to bound the weights, anyway). Hence for the example we use

$$W_k(x) = 1 + \kappa A_k(x). \tag{7.23}$$





Now we shall estimate

$$\frac{d}{dt}\Phi(u,v) = \sum_{i\in\mathcal{D}}(E_{i,1}+E_{i,2}). \tag{7.24}$$

To this end we consider a fixed discontinuity at x (to simplify the notation we do not use a subscript on this discontinuity) in one of the functions, say v. This discontinuity gives a contribution to the right-hand side of (7.24), denoted by $E_1 + E_2$, where

$$E_j = W_j^+ \left|q_j^+\right| \left(\mu_j^+ - \dot{x}\right) - W_j^- \left|q_j^-\right| \left(\mu_j^- - \dot{x}\right), \quad j = 1, 2.$$

For this 2×2 system we have

$$A_1(x) = \sum_{x_i < x,\, \hat{\imath}=2} |\epsilon_i|,$$

$$A_2(x) = \sum_{x_i > x,\, \hat{\imath}=1} |\epsilon_i| + \begin{cases} \displaystyle\sum_{\substack{\hat{\imath}=2,\, x_i<x \\ x_i\in\mathcal{D}(u)}} |\epsilon_i| + \sum_{\substack{\hat{\imath}=2,\, x_i>x \\ x_i\in\mathcal{D}(v)}} |\epsilon_i| & \text{if } q_2 < 0, \\ \displaystyle\sum_{\substack{\hat{\imath}=2,\, x_i<x \\ x_i\in\mathcal{D}(v)}} |\epsilon_i| + \sum_{\substack{\hat{\imath}=2,\, x_i>x \\ x_i\in\mathcal{D}(u)}} |\epsilon_i| & \text{if } q_2 > 0. \end{cases}$$

To estimate $E_1 + E_2$ we study several cases.

Case 1 Assume first that the jump at x is a *contact discontinuity*, that is, of the first family, in which case

$$A_1^+ = A_1^-,$$

and consequently,

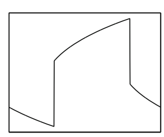
$$W_1^+ = W_1^-. \tag{7.25}$$

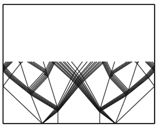
Furthermore,

$$q_1^+ = q_1^- + \epsilon \quad \text{and} \quad \mu_1^+ = \mu_1^- = \dot{x} - q_2^-.$$

Then

$$\begin{aligned} E_1 &= W_1^+ \left|q_1^+\right| \left(\mu_1^+ - \dot{x}\right) - W_1^- \left|q_1^-\right| \left(\mu_1^- - \dot{x}\right) \\ &= W_1^- \left\{\left|q_1^- + \epsilon\right| - \left|q_1^-\right|\right\} \left(-q_2^-\right) \\ &\le W_1^- \left|q_2^-\right| |\epsilon|. \end{aligned} \tag{7.26}$$





For the weights of the second family we find that

$$A_2^+ = A_2^- - |\epsilon|, \quad W_2^+ = W_2^- - \kappa\,|\epsilon|, \quad q_2^+ = q_2^-, \quad \mu_2^- = \mu_2^+.$$

To estimate $\mu_2^- - \dot{x}$ we exploit that μ_2^- is a discontinuity of the second family, while $\dot{x}$ is a contact discontinuity of the first family. Thus we can estimate from below their difference by the smallest difference in speeds between waves in the first- and second-wave families. We find that $\mu_2^- - \dot{x} \geq c = \min_{u,v}\{\eta\} > 0$. Hence

$$\begin{aligned} E_2 &= W_2^+\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - W_2^-\left|q_2^-\right|\left(\mu_2^- - \dot{x}\right) \\ &= \left|q_2^-\right|\left(\mu_2^- - \dot{x}\right)\left(-\kappa\,|\epsilon|\right) \\ &\leq -\kappa c\left|q_2^-\right||\epsilon|. \end{aligned} \tag{7.27}$$

Then

$$E_1 + E_2 = \left|q_2^-\right||\epsilon|\left(W_1^- - \kappa c\right) \leq 0 \tag{7.28}$$

if $\kappa c \geq \sup_x W_1(x)$. (Throughout this argument we will choose larger and larger κ.) This inequality (7.28) is the desired estimate when x is a contact discontinuity.

Case 2 The case that x is a *genuinely nonlinear wave*, that is, belongs to the second family, is more complicated. There are two distinct cases, that of an (approximate) rarefaction wave and that of a shock wave. First we treat the term E_1, which is common to the two cases. Here

$$\begin{aligned} &A_1^+ = A_1^- + |\epsilon|, \quad W_1^+ = W_1^- + \kappa\,|\epsilon|, \quad q_1^+ = q_1^-, \\ &\mu_1^+ = \mu_1^-, \quad \text{and} \quad \mu_1^- - \dot{x} < -c. \end{aligned}$$

Consequently,

$$\begin{aligned} E_1 &= W_1^+\left|q_1^+\right|\left(\mu_1^+ - \dot{x}\right) - W_1^-\left|q_1^-\right|\left(\mu_1^- - \dot{x}\right) \\ &= \kappa\,|\epsilon|\left|q_1^-\right|\left(\mu_1^- - \dot{x}\right) \\ &\leq -\kappa c\left|q_1^-\right||\epsilon| \leq 0. \end{aligned} \tag{7.29}$$

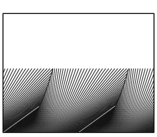
We split the estimate for E_2 into several cases.

Case 2a (rarefaction wave) First we consider the case that x is an approximate rarefaction wave. By the construction of v we have

$$\epsilon = \delta > 0 \quad \text{and} \quad q_2^+ = q_2^- + \epsilon.$$

The speeds appearing in E_2 are given by

$$\begin{aligned} \mu_2^+ &= 2\eta_u + q_2^- + \epsilon + \sqrt{\eta_u\left(\eta_u + q_2^- + \epsilon\right)}, \\ \mu_2^- &= 2\eta_u + q_2^- + \sqrt{\eta_u\left(\eta_u + q_2^-\right)}, \\ \dot{x} &= 3\left(\eta_u + q_2^-\right). \end{aligned}$$



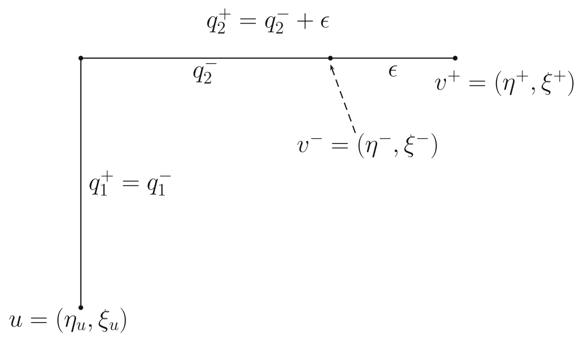


Fig. 7.2 $q_2^- > 0$

We define the auxiliary speed

$$\tilde{\mu} = \mu_2\left(v^-, v^+\right) = 2\eta_u + 2q_2^- + \epsilon + \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)}.$$

It is easily seen that

$$0 \leq \epsilon \leq \tilde{\mu} - \dot{x} \leq 2\epsilon.$$

We have several subcases. First we assume that $q_2^- > 0$, in which case $q_2^+ > 0$ as well; see Fig. 7.2.

In this case $A_2^+ = A_2^- + |\epsilon|$. Hence

$$\begin{aligned} E_2 &= W_2^+ \left|q_2^+\right| \left(\mu_2^+ - \dot{x}\right) - \left(W_2^+ - \kappa\,|\epsilon|\right) |q_2^-| \left(\mu_2^- - \dot{x}\right) \\ &= W_2^+ \left\{\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right\} \\ &\quad + W_2^+ \left(q_2^+ - q_2^-\right)(\tilde{\mu} - \dot{x}) + \kappa\,|\epsilon|\,|q_2^-|\,(\mu_2^- - \dot{x}). \end{aligned}$$

We need to estimate the term $\left\{\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right\}$. This estimate is contained in Lemma 7.4 in the general case, and it is verified directly for this model right after the proof of Lemma 7.4. We obtain

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \tilde{\mu}\right) - q_2^-\left(\mu_2^- - \tilde{\mu}\right)\right| \leq \mathcal{O}\left(1\right)|\epsilon|\,|q_2^-|\left(|q_2^-| + |\epsilon|\right),$$

and thus

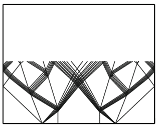
$$\begin{aligned} E_2 &\leq \mathcal{O}\left(1\right)|\epsilon|\,|q_2^-|\left(|q_2^-| + |\epsilon|\right) + W_2^+\,|\epsilon|\,|\tilde{\mu} - \dot{x}| + \kappa\,|\epsilon|\,|q_2^-|\,(\mu_2^- - \dot{x}) \\ &\leq \mathcal{O}\left(1\right)|\epsilon|\,|q_2^-|\left(|q_2^-| + |\epsilon|\right) + 2W_2^+\,|\epsilon|^2 + \kappa\,|\epsilon|\,|q_2^-|\,(\mu_2^- - \dot{x}). \end{aligned}$$

We estimate $\mu_2^- - \dot{x} \leq -q_2^- \leq 0$, and hence

$$E_2 \leq |\epsilon|\,|q_2^-|^2\left(\mathcal{O}\left(1\right) - \kappa\right) + \mathcal{O}\left(1\right)|\epsilon|^2\,|q_2^-| + \mathcal{O}\left(1\right)|\epsilon|^2 \leq M\,|\epsilon|\,\delta,$$

for some constant M if we choose κ big enough. We have used that W_2^+ is bounded. Therefore,

$$E_1 + E_2 \leq M\,|\epsilon|\,\delta.$$



Now for the case $q_2^- < 0$. Here we have two further subcases, $q_2^+ < 0$ and $q_2^+ > 0$. First we assume that $q_2^+ < 0$, and thus both q_2^- and q_2^+ are negative. Note that

$$\left|q_2^+\right| = |q_2^-| - |\epsilon|\,, 0 \le -q_2^- \le \mu_2^- - \dot{x} \le -2q_2^-, \quad \text{and} \quad A_2^+ = A_2^- - |\epsilon|\,.$$

Thus

$$\begin{aligned}
E_2 &= \left(W_2^- - \kappa\,|\epsilon|\right)\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - W_2^-\,|q_2^-|\left(\mu_2^- - \dot{x}\right) \\
&= W_2^-\left\{\left(q_2^+ - \epsilon\right)\left(\mu_2^- - \tilde{\mu}\right) - q_2^+\left(\mu_2^+ - \tilde{\mu}\right)\right\} \\
&\quad - W_2^-\,|\epsilon|\,(\tilde{\mu} - \dot{x}) - \kappa\,|\epsilon|\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) \\
&\le \mathcal{O}\,(1)\,|\epsilon|\left|q_2^+\right|\left(\left|q_2^+\right| + |\epsilon|\right) + \mathcal{O}\,(1)\,|\epsilon|^2 - \kappa\,|\epsilon|\left|q_2^+\right|^2 \\
&\le |\epsilon|\,|q_2^-|^2\,(\mathcal{O}\,(1) - \kappa) + \mathcal{O}\,(1)\,|\epsilon|^2 \\
&\le M\,|\epsilon|\,\delta,
\end{aligned}$$

where we have used Lemma 7.4 (with $\varepsilon = \epsilon$, $\varepsilon' = q_2^+$) and chosen κ sufficiently large. Thus we conclude that $E_1 + E_2 \le M\,|\epsilon|\,\delta$ in this case as well.

Now for the last case in which $\epsilon > 0$, namely $q_2^- < 0 < q_2^+$. Since $q_2^+ = q_2^- + \epsilon$, we have

$$\left|q_2^+\right| \le \delta, \qquad |q_2^-| \le \delta.$$

Furthermore, $A_2^+ = A_2^-$, and thus $W_2^+ = W_2^-$. We see that

$$0 \le -q_2^- \le \mu_2^- - \dot{x} \le -2q_2^-, \qquad \mu_2^+ - \dot{x} \le 2\epsilon - q_2^-,$$

and hence

$$\begin{aligned}
E_2 &= W_2^+\left\{q_2^+\left(\mu_2^+ - \dot{x}\right) + |q_2^-|\left(\mu_2^- - \dot{x}\right)\right\} \\
&\le W_2^+\left\{q_2^+\left(2\,|\epsilon| + q_2^-\right) + |q_2^-|\;2\;|q_2^-|\right\} \\
&\le M\,|\epsilon|\,\delta,
\end{aligned}$$

for some constant M.

Case 2b (shock wave) When x is a shock front, we have $\epsilon < 0$. In this case,

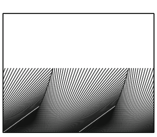
$$\dot{x} = \tilde{\mu} = \mu_2\left(v^-, v^+\right) = 2\eta_u + 2q_2^- + \epsilon + \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)}.$$

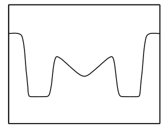
We first consider the case $q_2^- < 0$. Then

$$q_2^+ = q_2^- + \epsilon < 0, \quad \left|q_2^+\right| = |q_2^-| + |\epsilon|\,, \quad \text{and} \quad A_2^+ = A_2^- - |\epsilon|\,,$$

and we obtain

$$\begin{aligned}
E_2 &= \left(W_2^- - \kappa\,|\epsilon|\right)\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - W_2^-\,|q_2^-|\left(\mu_2^- - \dot{x}\right) \\
&= -W_2^-\left((q_2^- + \epsilon)(\mu_2^+ - \dot{x}) - q_2^-(\mu_2^- - \dot{x})\right) \\
&\quad - \kappa\,|\epsilon|\,(|q_2^-| + |\epsilon|)(\mu_2^+ - \dot{x}) \\
&\le \mathcal{O}\,(1)\,|\epsilon|\,|q_2^-|\,(|q_2^-| + |\epsilon|) - \kappa\,|\epsilon|\,(|q_2^-| + |\epsilon|)\,|q_2^-| \\
&\le |\epsilon|\,|q_2^-|\,(|q_2^-| + |\epsilon|)\,(\mathcal{O}\,(1) - \kappa) \le 0.
\end{aligned}$$





Lemma 7.4 (with $\varepsilon' = \epsilon$, $\varepsilon = q_2^-$) implies

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \dot{x}\right) - q_2^-\left(\mu_2^- - \dot{x}\right)\right| \le \mathcal{O}(1)\,|\epsilon|\,|q_2^-|\left(|q_2^-| + |\epsilon|\right).$$

Furthermore,

$$\begin{aligned}
\mu_2^+ - \dot{x} &= -q_2^- + \sqrt{\eta_u\left(\eta_u + q_2^- + \epsilon\right)} \\
&\qquad - \sqrt{\left(\eta_u + q_2^-\right)\left(\eta_u + q_2^- + \epsilon\right)} \\
&= -q_2^-\left(1 + \frac{\sqrt{\eta_u + q_2^+}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) \\
&\ge -q_2^- = |q_2^-|.
\end{aligned}$$

If $q_2^- > 0$, then there are two further cases to be considered, depending on the sign of q_2^+. We first consider the case $q_2^+ < 0$, and thus $q_2^+ < 0 < q_2^-$. Now $A_2^+ = A_2^-$. Furthermore,

$$\begin{aligned}
&\mu_2^- - \dot{x} \ge -2q_2^- \ge 0, \\
&\mu_2^+ - \dot{x} = -q_2^-\left(1 + \frac{\sqrt{\eta_u + q_2^+}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) < -|q_2^-|.
\end{aligned}$$

Thus

$$\mu_2^+ < \dot{x} < \mu_2^-,$$

and we easily obtain

$$E_2 = W_2^-\left\{\left|q_2^+\right|\left(\mu_2^+ - \dot{x}\right) - |q_2^-|\left(\mu_2^- - \dot{x}\right)\right\} < 0.$$

This leaves the final case $q_2^\pm > 0$. In this case we have that $A_2^+ = A_2^- + |\epsilon|$. We still have

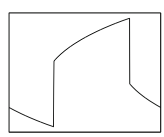
$$\mu_2^- - \dot{x} = -q_2^+\left(1 + \frac{\sqrt{\eta_u + q_2^-}}{\sqrt{\eta_u} + \sqrt{\eta_u + q_2^+}}\right) \le -q_2^+ < 0,$$

and thus

$$|\dot{x} - \mu_2^-| \ge q_2^+.$$

Furthermore, by Lemma 7.4, we have that

$$\left|\left(q_2^- + \epsilon\right)\left(\mu_2^+ - \dot{x}\right) - q_2^-\left(\mu_2^- - \dot{x}\right)\right| \le \mathcal{O}(1)\left|q_2^+\right||\epsilon|\left(\left|q_2^+\right| + |\epsilon|\right).$$



Then we calculate

$$\begin{aligned}
E_2 &= W_2^+ \left|q_2^+\right| \left(\mu_2^+ - \dot{x}\right) - \left(W_2^+ - \kappa\,|\epsilon|\right) |q_2^-| \left(\mu_2^- - \dot{x}\right) \\
&= W_2^+ \left((q_2^- + \epsilon)(\mu_2^+ - \dot{x}) - q_2^- (\mu_2^- - \dot{x})\right) + \kappa\,|\epsilon|\,|q_2^-|\,(\mu_2^- - \dot{x}) \\
&\le W_2^+ \left|q_2^+ \left(\mu_2^+ - \dot{x}\right) - q_2^- \left(\mu_2^- - \dot{x}\right)\right| - \kappa\,|\epsilon|\,|\mu_2^- - \dot{x}|\,|q_2^-| \\
&\le \mathcal{O}(1)\,|\epsilon|\,|q_2^-| \left(|q_2^-| + |\epsilon|\right) - \kappa\,|\epsilon|\,|q_2^-| \left|q_2^+\right| \\
&\le \mathcal{O}(1)\,|\epsilon|^2 + |\epsilon|\,|q_2^-|^2 \left(\mathcal{O}(1) - \kappa\right) \\
&\le M\,|\epsilon|\,\delta
\end{aligned}$$

if κ is sufficiently large. This is the last case.

Now we have shown that in all cases,

$$E_1 + E_2 \le M\,|\epsilon|\,\delta.$$

Summing over all discontinuities in u and v we conclude that

$$\frac{d}{dt}\Phi(u, v) \le C'\delta,$$

for some finite constant C' independent of δ.

We shall now show that

$$\sum_{k=1}^{n} E_{i,k} \le \mathcal{O}(1)\,|\epsilon_i| \left(\delta_1 \vee \delta_2\right) + \mathcal{O}(1)\,|e_i|\,, \tag{7.30}$$

and this estimate is easily seen to imply (7.21). To prove (7.30) we shall need some preliminary results:

Lemma 7.3 *Assume that the vectors $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, $\epsilon' = \left(\epsilon_1', \dots, \epsilon_n'\right)$, and $\epsilon'' = \left(\epsilon_1'', \dots, \epsilon_n''\right)$ satisfy*

$$H(\epsilon)\,u = H\left(\epsilon''\right) H\left(\epsilon'\right) u$$

for some vector u, where

$$H(\epsilon) = H_n(\epsilon_n)\,H_{n-1}(\epsilon_{n-1}) \cdots H_1(\epsilon_1)\,.$$

Then

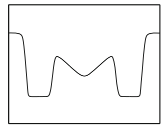
$$\sum_{k=1}^{n} |\epsilon_k - \epsilon_k' - \epsilon_k''| = \mathcal{O}(1) \left(\sum_j \left|\epsilon_j' \epsilon_j''\right| \left(\left|\epsilon_j'\right| + \left|\epsilon_j''\right|\right) + \sum_{\substack{k,l \\ k \ne l}} \left|\epsilon_j' \epsilon_l''\right| \right). \tag{7.31}$$

If the scalar ϵ and the vector $\epsilon' = \left(\epsilon_1', \dots, \epsilon_n'\right)$ satisfy

$$R_l(\epsilon)\,u = H\left(\epsilon'\right) u,$$

where R_l denotes the lth rarefaction curve, then

$$|\epsilon - \epsilon_l'| + \sum_{k \ne l} |\epsilon_k'| = \mathcal{O}(1)\,|\epsilon| \left(|\epsilon_l'| \left(|\epsilon| + |\epsilon_l'|\right) + \sum_{k \ne l} |\epsilon_k'| \right). \tag{7.32}$$



Proof The proof of this lemma is a straightforward modification of the proof of the interaction estimate (6.18). □

Lemma 7.4 *Let $\bar{\omega} \in \Omega$ be sufficiently small, and let ε and ε' be real numbers. Define*

$$\begin{aligned} \omega &= H_k(\varepsilon)\bar{\omega}, & \mu &= \mu_k\left(\bar{\omega}, \omega\right), \\ \omega' &= H_k\left(\varepsilon'\right)\omega, & \mu' &= \mu_k\left(\omega, \omega'\right), \\ \omega'' &= H_k\left(\varepsilon + \varepsilon'\right)\bar{\omega}, & \mu'' &= \mu_k\left(\bar{\omega}, \omega''\right). \end{aligned}$$

Then one has

$$|(\varepsilon + \varepsilon')(\mu'' - \mu') - \varepsilon(\mu - \mu')| \leq \mathcal{O}(1)\,|\varepsilon\varepsilon'|\left(|\varepsilon| + |\varepsilon'|\right). \tag{7.33}$$

Proof The proof of this is again in the spirit of the proof of the interaction estimate, equation (6.13). Let the function Ψ be defined as

$$\Psi\left(\varepsilon, \varepsilon'\right) = \left(\varepsilon + \varepsilon'\right)\mu'' - \varepsilon\mu - \varepsilon'\mu'.$$

Then Ψ is at least twice differentiable, and satisfies

$$\Psi\left(\varepsilon, 0\right) = \Psi\left(0, \varepsilon'\right) = 0, \qquad \frac{\partial^2 \Psi}{\partial\varepsilon\partial\varepsilon'}(0, 0) = 0.$$

Consequently,

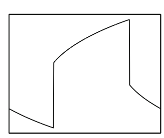
$$\Psi\left(\varepsilon, \varepsilon'\right) = \int_0^{\varepsilon}\int_0^{\varepsilon'} \frac{\partial^2 \Psi}{\partial\varepsilon\partial\varepsilon'}(r, s)\,ds\,dr = \mathcal{O}(1)\int_0^{|\varepsilon|}\int_0^{|\varepsilon'|} \left(|r| + |s|\right)\,dr\,ds.$$

From this the lemma follows. □

◇ Example 7.5 (Lemma 7.4 for Example 7.1)
If $k = 2$, let $\bar{\omega}$, ω', and ω'' denote the η-coordinate, since only this will influence the speeds. Then a straightforward calculation yields

$$\begin{aligned} &\left|(\varepsilon + \varepsilon')(\mu'' - \mu') - \varepsilon(\mu - \mu')\right| \\ &\quad = |\varepsilon|\,|\varepsilon'|\left(|\varepsilon| + |\varepsilon'|\right) \\ &\qquad \times \frac{\sqrt{\bar{\omega}} + \sqrt{\omega'} + \sqrt{\omega''}}{\bar{\omega}\left(\sqrt{\omega'} + \sqrt{\omega''}\right) + \omega'\left(\sqrt{\bar{\omega}} + \sqrt{\omega''}\right) + \omega''\left(\sqrt{\bar{\omega}} + \sqrt{\omega'}\right) + 2\sqrt{\bar{\omega}\omega'\omega''}} \\ &\quad \leq \frac{|\varepsilon|\,|\varepsilon'|\left(|\varepsilon| + |\varepsilon'|\right)}{\min\left\{\bar{\omega}, \omega', \omega''\right\}}, \end{aligned}$$

verifying the lemma in this case. ◇



If the kth characteristic field is genuinely nonlinear, then the characteristic speed $\lambda_k\left(H_k(\epsilon)\omega\right)$ is increasing in ϵ, and we can even choose the parameterization such that

$$\lambda_k\left(H_k(\epsilon)\omega\right) - \lambda_k(\omega) = \epsilon,$$

for all sufficiently small ϵ and ω. This also implies that $\mu_k(\omega, H_k(\epsilon)\omega)$ is strictly increasing in ϵ. However, the Hugoniot locus through the point ω does not in general coincide with the Hugoniot locus through the point $H_k(q)\omega$. Therefore, it is not so straightforward comparing speeds defined on different Hugoniot loci. When proving (7.30) we shall need to do this, and we repeatedly use the following lemma:

Lemma 7.6 *For some state ω define*

$$\Psi(q) = \mu_k\left(H_k(q)\omega, H_k(\epsilon)H_k(q)\omega\right) - \mu_k\left(\omega, H_k(\epsilon+q)\omega\right).$$

Then Ψ is at least twice differentiable for all $k = 1, \ldots, n$. Furthermore, if the kth characteristic field is genuinely nonlinear, then for sufficiently small $|q|$ and $|\epsilon|$,

$$\Psi'(q) \geq c > 0, \tag{7.34}$$

where c depends only on f for all sufficiently small $|\omega|$.

Proof Let the vector ϵ' be defined by $\mathcal{H}(\epsilon')\omega = H_k(\epsilon)H_k(q)\omega$. Then by Lemma 7.3,

$$\left|\epsilon'_k - (q+\epsilon)\right| + \sum_{i\neq k}\left|\epsilon'_i\right| \leq \mathcal{O}(1)\left|q\epsilon\right|\left(|\epsilon| + |q|\right).$$

Consequently,

$$H_k(\epsilon+q)\omega = H_k(\epsilon)H_k(q)\omega + \mathcal{O}(1)\left|q\epsilon\right|\left(|\epsilon| + |q|\right).$$

Using this we find that

$$\left|\frac{H_k(\epsilon)H_k(q)\omega - H_k(\epsilon)\omega}{q}\right| = \left|\frac{H_k(\epsilon+q)\omega - H_k(\epsilon)\omega}{q}\right| + \mathcal{O}(1)\left|\epsilon\right|\left(|\epsilon| + |q|\right).$$

Therefore,

$$\frac{d}{dq}\left\{H_k(\epsilon)H_k(q)\omega\right\}\Big|_{q=0} = \frac{d}{d\epsilon}\left\{H_k(\epsilon)\omega\right\} + \mathcal{O}(1)\left|\epsilon\right|^2. \tag{7.35}$$

Hence, we compute

$$\begin{aligned}
\Psi'(0) &= \nabla_1\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot r_k(\omega)\\
&\qquad - \nabla_2\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot\left(\frac{d}{d\epsilon}\left\{H_k(\epsilon)\omega\right\} - \frac{d}{dq}\left\{H_k(\epsilon)H_k(q)\omega\right\}\Big|_{q=0}\right)\\
&= \nabla_1\mu_k\left(\omega, H_k(\epsilon)\omega\right)\cdot r_k(\omega) + \mathcal{O}(1)\left|\epsilon\right|^2\\
&\geq c' > 0,
\end{aligned}$$

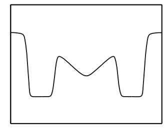
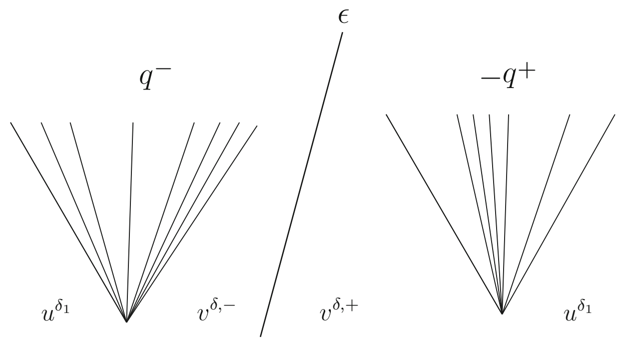


Fig. 7.3 The setting in the proof of (7.30)



for sufficiently small $|\epsilon|$. The value of the constant c' (and its existence) depends on the genuine nonlinearity of the system and hence on f. Since Ψ' is continuous for small $|q|$, the lemma follows. □

We shall prove (7.30) in the case that the front at x_i is a front in v^{δ_2}; the case in which it is a front in u^{δ_1} is completely analogous. We therefore fix i, and study the relation between q_k^- and q_k^+. Since the front is going to be fixed from now on, we drop the subscript i. For simplicity we write $\delta = \delta_2$. Assume the the family of the front x is l and the front has strength ϵ. The situation is as in Fig. 7.3.

A key observation is that we can regard the waves q_k^+ as the result of an interaction between the waves q_k^- and ϵ; similarly, the waves $-q_k^-$ are the result of an interaction between ϵ and $-q_k^+$.

Regarding the weights, from (7.16) and (7.18) we find that

$$W_k^+ - W_k^- = \begin{cases} \kappa_1 |\epsilon| & \text{if } k < l, \\ -\kappa_1 |\epsilon| & \text{if } k > l, \end{cases} \tag{7.36}$$

while for $k = l$ we obtain

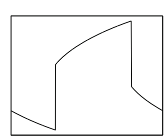
$$W_l^+ - W_l^- = \begin{cases} \kappa_1 |\epsilon| & \text{if } \min\left\{q_l^-, q_l^+\right\} > 0, \\ -\kappa_1 |\epsilon| & \text{if } \max\left\{q_l^-, q_l^+\right\} < 0, \\ \mathcal{O}(1) & \text{if } q_l^- q_l^+ < 0. \end{cases} \tag{7.37}$$

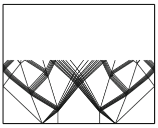
The proof of (7.30) is a study of cases. We split the estimate into two subgroups, depending on whether the front at x is an approximate rarefaction wave or a shock. Within each subgroup we discuss three subcases depending on the signs of $q_l^{\pm}$. In all cases we discuss the terms E_k $(k \neq l)$ and E_l separately. For $k \neq l$ we write E_k (recall that we dropped the subscript i) as

$$\begin{aligned} E_k = {} & \left(\left|q_k^+\right| - \left|q_k^-\right|\right) W_k^+ \left(\mu_k^+ - \dot{x}\right) \\ & + \left|q_k^-\right| \left(W_k^+ - W_k^-\right) \left(\mu_k^+ - \dot{x}\right) + \left|q_k^-\right| W_k^- \left(\mu_k^+ - \mu_k^-\right). \end{aligned} \tag{7.38}$$

By the strict hyperbolicity of the system, we have that

$$\begin{aligned} \mu_k^+ - \dot{x} \le -c < 0, & \quad \text{for } k < l, \\ \mu_k^+ - \dot{x} \ge c > 0, & \quad \text{for } k > l, \end{aligned}$$





where c is some fixed constant depending on the system. Thus we always have that

$$\left(W_k^+ - W_k^-\right)\left(\mu_k^+ - \dot{x}\right) \leq -c\kappa_1 \left|\epsilon\right|, \quad k \neq l. \tag{7.39}$$

We begin with the case that the front at x is an approximate rarefaction wave ($\epsilon > 0$). In this case,

$$R_l(\epsilon)v^{\delta,-} + e = H\left(q^+\right)u^{\delta_1} = H\left(q^+\right)H\left(-q^-\right)v^{\delta,-} = H\left(\tilde{q}\right)v^{\delta,-}$$

for some vector $\tilde{q}$. Hence

$$H\left(-q^-\right)v^{\delta,-} = H\left(-q^+\right)H\left(\tilde{q}\right)v^{\delta,-}, \tag{7.40}$$

$$R_l(\epsilon)v^{\delta,-} + e = H\left(\tilde{q}\right)v^{\delta,-}. \tag{7.41}$$

From (7.31) and (7.40) we obtain

$$\sum_k \left|q_k^+ - q_k^- - \tilde{q}_k\right| \mathcal{O}(1)\Big(\sum_k \left|q_k^+\tilde{q}_k\right|\left(\left|q_k^+\right| + \left|\tilde{q}_k\right|\right) + \sum_{\substack{k,j\\k\neq j}} \left|q_k^+\tilde{q}_j\right|\Big), \tag{7.42}$$

and from (7.32) and (7.41) we obtain

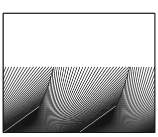
$$\left|\tilde{q}_l - \epsilon\right| + \sum_{k\neq l}\left|\tilde{q}_k\right| = \mathcal{O}(1)\left|\epsilon\right|\Big(\left|\tilde{q}_l\right|\left(\left|\tilde{q}_l\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|\tilde{q}_k\right|\Big) + \mathcal{O}(1)\left|e\right|.$$

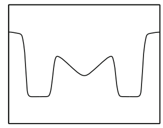
This implies that

$$\begin{aligned}\left|\tilde{q}_l - \epsilon\right| &\leq \mathcal{O}(1)\left|\epsilon\right| + \mathcal{O}(1)\left|e\right|,\\ \sum_{k\neq l}\left|\tilde{q}_k\right| &\leq \mathcal{O}(1)\left|\epsilon\right| + \mathcal{O}(1)\left|e\right|.\end{aligned} \tag{7.43}$$

Furthermore, since ϵ is an approximate rarefaction, $0 \leq \epsilon \leq \delta$. Therefore, we can replace $\tilde{q}_l$ with ϵ and $\tilde{q}_k$ ($k \neq l$) with zero on the right-hand side of (7.42), making an error of $\mathcal{O}(1)\,\delta$. Indeed,

$$\begin{aligned}&\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k\neq l}\left|q_k^+ - q_k^-\right|\\ &\quad\leq \sum_k \left|q_k^+ - q_k^- - \epsilon\right| + \left|\tilde{q}_l - \epsilon\right| + \sum_{k\neq l}\left|\tilde{q}_k\right|\\ &\quad\leq \mathcal{O}(1)\Bigg(\sum_k \left|q_k^+\tilde{q}_k\right|\left(\left|q_k^+\right| + \left|\tilde{q}_k\right|\right) + \sum_{\substack{k,j\\k\neq j}}\left|q_k^+\tilde{q}_j\right|\Bigg)\\ &\qquad + \mathcal{O}(1)\left|\epsilon\right|\Bigg(\left|\tilde{q}_l\right|\left(\left|\tilde{q}_l\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|\tilde{q}_k\right|\Bigg) + \mathcal{O}(1)\left|e\right|.\end{aligned}$$





Using (7.43) and the fact that $\epsilon \le \delta$, we conclude that

$$\begin{aligned} &\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k\neq l}\left|q_k^+ - q_k^-\right| \\ &\quad = \mathcal{O}(1)\,|\epsilon|\left(\delta + \left|q_l^+\right|\left(\left|q_l^+\right| + |\epsilon|\right) + \sum_{k\neq l}\left|q_k^+\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.44}$$

Similarly,

$$\begin{aligned} &\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k\neq l}\left|q_k^+ - q_k^-\right| \\ &\quad = \mathcal{O}(1)\,|\epsilon|\left(\delta + \left|q_l^-\right|\left(\left|q_l^-\right| + |\epsilon|\right) + \sum_{k\neq l}\left|q_k^-\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.45}$$

Since in this case $0 \le \epsilon \le \delta$, and the total variation is small, we can assume that the right-hand sides of (7.44)–(7.45) are smaller than $\epsilon + \mathcal{O}(1)\,|e|$. Also, the error e is small; cf. (7.11). Then

$$0 < q_l^+ - q_l^- < 2\epsilon + \mathcal{O}(1)\,|e| \le 2\delta + \mathcal{O}(1)\,|e|\,. \tag{7.46}$$

We can also use the estimates (7.44) and (7.45) to make a simplifying assumption throughout the rest of our calculations. Since the total variation of $u - v$ is uniformly bounded, we can assume that the right-hand sides of (7.44) and (7.45) are bounded by

$$\frac{1}{2}\,|\epsilon| + \mathcal{O}(1)\,|e|\,.$$

In particular, we then find that

$$\epsilon - \frac{1}{2}\,|\epsilon| - \mathcal{O}(1)\,|e| \le q_\ell^+ - q_\ell^- \le \epsilon + \frac{1}{2}\,|\epsilon| + \mathcal{O}(1)\,|e|\,.$$

Hence if $\epsilon > 0$, from the left inequality we find that

$$q_\ell^+ > q_\ell^-$$

or

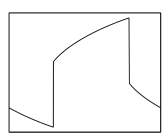
$$|\epsilon| \le \mathcal{O}(1)\,|e|\,,$$

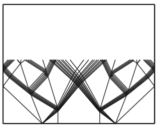
and if $\epsilon < 0$, from the right inequality above,

$$q_\ell^+ < q_\ell^-$$

or

$$|\epsilon| \le \mathcal{O}(1)\,|e|\,.$$





If $\epsilon > 0$ and $q_\ell^- \geq q_\ell^+$ or $\epsilon < 0$ and $q_\ell^+ \geq q_\ell^-$, then $|\epsilon| \leq \mathcal{O}(1)\,|e|$. In this case we find for $k \neq l$, or $k = l$ and $q_\ell^- q_\ell^+ > 0$, that

$$\begin{aligned} E_k &= \left\{|q_k^-|\left(W_k^- - W_k^+\right) + W_k^+\left(|q_k^-| - |q_k^+|\right)\right\}\dot{x} \\ &\leq \left\{|q_k^-|\,\kappa_1\,|\epsilon| + \left|W_k^+\right|\left(|\epsilon|/2 + \mathcal{O}(1)\,|e|\right)\right\}|\dot{x}| \\ &\leq \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.47}$$

If $k = l$ and $q_\ell^- q_\ell^+ < 0$, then for $\epsilon > 0$ we have that $q_\ell^+ - q_\ell^- \geq \mathcal{O}(1)\,|e|$, so if $q_\ell^+ < q_\ell^-$, we must have that

$$\left|q_\ell^+\right| \leq \mathcal{O}(1)\,|e| \quad \text{and} \quad q_\ell^- \leq \mathcal{O}(1)\,|e|\,.$$

Similarly, if $\epsilon < 0$ and $q_\ell^+ > q_\ell^-$, we obtain

$$q_\ell^+ < \mathcal{O}(1)\,|e| \quad \text{and} \quad |q_\ell^-| \leq \mathcal{O}(1)\,|e|\,.$$

Then we find that

$$E_l = \left\{|q_\ell^-|\,W_l^- - \left|q_\ell^+\right| W_l^+\right\}\dot{x} \leq \mathcal{O}(1)\,|e|\,. \tag{7.48}$$

These observations imply that if $|\epsilon| = \mathcal{O}(1)\,|e|$, we have that

$$\sum_k E_k = \mathcal{O}(1)\,|e|\,,$$

which is what we want to show. Thus in the following we can assume that either

$$\epsilon > 0 \quad \text{and} \quad q_\ell^+ > q_\ell^-,$$

or

$$\epsilon < 0 \quad \text{and} \quad q_\ell^+ < q_\ell^-. \tag{7.49}$$

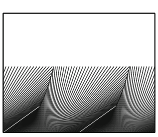
Now follows a discussion of several different cases, depending on whether the front is an approximate rarefaction wave or a shock wave, and on the signs of q_ℓ^- and q_ℓ^+.

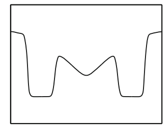
Case R1 $0 < q_l^- < q_l^+$, $\epsilon > 0$.
For $k \neq l$ we recall (7.38) that

$$\begin{aligned} E_k &= \left(\left|q_k^+\right| - |q_k^-|\right) W_k^+\left(\mu_k^+ - \dot{x}\right) \\ &\quad + |q_k^-|\left(W_k^+ - W_k^-\right)\left(\mu_k^+ - \dot{x}\right) + |q_k^-|\,W_k^-\left(\mu_k^+ - \mu_k^-\right). \end{aligned} \tag{7.50}$$

The second term in (7.50) is less than or equal to (cf. (7.39))

$$-c\kappa_1\,|q_k^-|\,|\epsilon|\,.$$





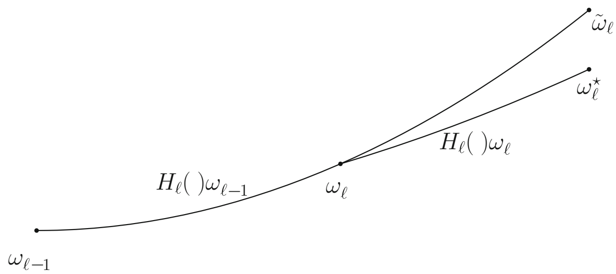


Fig. 7.4 The situation for $0 < q_l^- < q_l^+, \epsilon > 0$, and $k = l$

Furthermore, by (7.45),

$$\left|q_k^+\right| - |q_k^-| \leq \mathcal{O}(1)\,|\epsilon| \left(\delta + |q_l^-|\left(|q_l^-| + |\epsilon|\right) + \sum_{k \neq l} |q_k^-|\right) + \mathcal{O}(1)\,|e|\,.$$

By the continuity of μ_k,

$$\left|\mu_k^+ - \mu_k^-\right| = \mathcal{O}(1)\left(|\epsilon| + |e|\right).$$

Hence from (7.38), we find that

$$\begin{aligned}
E_k &\leq \mathcal{O}(1)\,|\epsilon| \left(\delta + |q_l^-|\left(|q_l^-| + |\epsilon|\right) + \sum_{\tilde{k} \neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e| - c\kappa_1\,|q_k^-|\,|\epsilon| \\
&\leq \mathcal{O}(1)\,|\epsilon| \left(\delta + \sum_{\tilde{k} \neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e| \\
&\quad - c\kappa_1\,|\epsilon|\,|q_k^-| + \mathcal{O}(1)\,|\epsilon|\,|q_l^-|\left(|q_l^-| + |\epsilon|\right).
\end{aligned} \tag{7.51}$$

For $k = l$ the situation is more complicated. We define states and speeds

$$\begin{aligned}
\tilde{\omega}_\ell &= H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-, & \tilde{\mu}_\ell &= \mu_l\left(\omega_{l-1}^-, \tilde{\omega}_\ell\right), \\
\omega_\ell^\star &= H_l(\epsilon)\,\omega_l^-, & \mu_\ell^\star &= \mu_l\left(\omega_l^-, \omega_\ell^\star\right);
\end{aligned} \tag{7.52}$$

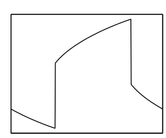
see Fig. 7.4.

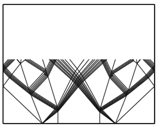
Recall that

$$\mu_l^\pm = \mu_l\left(\omega_{l-1}^\pm, \omega_l^\pm\right).$$

Now by Lemma 7.4, with $\omega = \omega_{l-1}^-$, $\varepsilon = q_\ell^-$, and $\varepsilon'' = q_\ell^- + \epsilon$,

$$\left|\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^-\left(\mu_l^- - \mu_\ell^\star\right)\right| = \mathcal{O}(1)\,|q_\ell^-|\,|\epsilon|\left(|q_\ell^-| + |\epsilon|\right). \tag{7.53}$$





We also find that (cf. (7.10) and the fact that $\mu_l(u,u) = \lambda_l(u)$)

$$
\begin{aligned}
|\mu^\star_\ell - \dot{x}| &\leq \left|\mu_l\left(\omega_l^-, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,-}, v^{\delta,-}\right)\right| + \mathcal{O}(1)\,\delta \\
&= \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l\left(\omega_n^-, \omega_n^-\right)\right| + \mathcal{O}(1)\,\delta \\
&\leq \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l(\omega_l^-, \omega_l^-)\right| \\
&\quad + \left|\mu_l(\omega_l^-, \omega_l^-) - \mu_l(\omega_l^-, \omega_{l+1}^-)\right| \\
&\quad + \left|\mu_l(\omega_l^-, \omega_{l+1}^-) - \mu_l(\omega_{l+1}^-, \omega_{l+2}^-)\right| + \cdots \\
&\quad + |\mu_l(\omega_{n-1}^-, \omega_n^-) - \mu_l(\omega_n^-, \omega_n^-)| + \mathcal{O}(1)\,\delta \\
&\leq \mathcal{O}(1)\left(|\epsilon| + \left|\omega_l^- - \omega_{l+1}^-\right| + \cdots + |\omega_{n-1}^-, \omega_n^-|\right) + \mathcal{O}(1)\,\delta \\
&\leq \mathcal{O}(1)\left(|\delta| + |q_l^-| + \sum_{k>l} |q_k^-|\right).
\end{aligned}
\tag{7.54}
$$

Furthermore,

$$
\begin{aligned}
\left|\mu_l^+ - \tilde{\mu}_\ell\right| &= \left|\mu_l\left(\omega_{l-1}^+, H_l\left(q_\ell^+\right)\omega_{l-1}^+\right) - \mu_l\left(\omega_{l-1}^-, H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right)\right| \\
&\leq \left|\mu_l\left(\omega_{l-1}^+, H_l\left(q_\ell^+\right)\omega_{l-1}^+\right) - \mu_l\left(H_l\left(q_\ell^+\right)\omega_{l-1}^+, \omega_{l-1}^-\right)\right| \\
&\quad + \left|\mu_l\left(H_l\left(q_\ell^+\right)\omega_{l-1}^+, \omega_{l-1}^-\right) - \mu_l\left(\omega_{l-1}^+, H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right)\right| \\
&\leq \mathcal{O}(1)\left(\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|H_l\left(q_\ell^+\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^-\right|\right) \\
&\leq \mathcal{O}(1)\left(\,\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|H_l\left(q_\ell^+\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^+\right|\right. \\
&\quad \left. + \left|H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^+ - H_l\left(q_l^- + \epsilon\right)\omega_{l-1}^+\right|\right) \\
&\leq \mathcal{O}(1)\left(\left|\omega_{l-1}^+ - \omega_{l-1}^-\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\
&\leq \mathcal{O}(1)\left(\left|q_{l-2}^+ - q_{l-2}^-\right| + \cdots + \left|q_1^+ - q_1^-\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\
&= \mathcal{O}(1)\,\epsilon\Big(\delta + |q_\ell^-|\left(|q_\ell^-| + |\epsilon|\right) + \sum_{k\neq l} |q_k^-|\Big) + \mathcal{O}(1)\,|e|\,.
\end{aligned}
\tag{7.55}
$$

Since the lth field is genuinely nonlinear, then by Lemma 7.6,

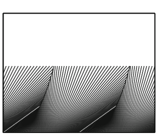
$$
\mu^\star_\ell - \tilde{\mu}_\ell \geq c\,|q_\ell^-|
\tag{7.56}
$$

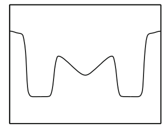
for some constant $c > 0$ depending only on the system. Recall that in this case,

$$
W_\ell^+ = W_\ell^- + \kappa_1\,|\epsilon|\,.
$$

Moreover, ϵ, q_ℓ^+, and q_ℓ^- are positive. Using the above inequalities, we compute

$$
\begin{aligned}
E_l &= W_\ell^+ q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- q_\ell^-\left(\mu_\ell^- - \dot{x}\right) \\
&= \left(W_\ell^- + \kappa_1\,|\epsilon|\right) q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- q_\ell^-\left(\mu_\ell^- - \dot{x}\right) \\
&= \kappa_1 \epsilon q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \epsilon\left\{\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - \left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right\} \\
&\quad + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \epsilon\left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \epsilon\left\{\left(q_\ell^- + \epsilon\right)\left(\mu_\ell^+ - \dot{x} - \left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right) + \left(q_\ell^+ - q_\ell^- - \epsilon\right)\left(\mu_\ell^+ - \dot{x}\right)\right\} \\
&\quad + W_\ell^-\left\{q_\ell^+\left(\mu_\ell^+ - \dot{x}\right) - q_\ell^-\left(\mu_\ell^- - \dot{x}\right)\right\}
\end{aligned}
$$





$$
\begin{aligned}
&\le \kappa_1 \epsilon \left(q_\ell^- + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + \kappa_1 \epsilon \left(q_\ell^- + \epsilon\right) \left(\left|\mu_\ell^+ - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right) \\
&\quad + \kappa_1 \epsilon \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^+ - \dot{x}\right| + W_\ell^- \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\kappa_1 q_\ell^- \epsilon \left(q_\ell^- + \epsilon\right) \\
&\quad + \kappa_1 \epsilon \left(q_\ell^- + \epsilon\right) \Bigg(\mathcal{O}\left(1\right) \epsilon \Bigg(\delta + q_\ell^- \left(q_\ell^- + \epsilon\right) + \sum_{k \neq l} \left|q_\ell^-\right| \Bigg) \\
&\qquad\qquad + \delta + \mathcal{O}\left(1\right) \sum_{k > l} \left|q_\ell^-\right| + \mathcal{O}\left(1\right) \left|e\right| \Bigg) \\
&\quad + \mathcal{O}\left(1\right) \kappa_1 \epsilon^2 \Bigg(\delta + q_\ell^- \left(q_\ell^- + \epsilon\right) + \sum_{k \neq l} \left|q_k^-\right| \Bigg) + \mathcal{O}\left(1\right) \left|e\right| \\
&\quad + W_\ell^- \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\kappa_1 q_\ell^- \epsilon \left(q_\ell^- + \epsilon\right) + \mathcal{O}\left(1\right) \kappa_1 \epsilon \Bigg(\delta + q_\ell^- \left(q_\ell^- + \epsilon\right) + \sum_{k \neq l} \left|q_k^-\right| \Bigg) + \mathcal{O}\left(1\right) \left|e\right| \\
&\quad + W_\ell^- \Big\{ \left|\left(q_\ell^- + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^- \left(\mu_\ell^- - \mu_\ell^\star\right)\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^+ - \dot{x}\right| \\
&\qquad\quad \epsilon \left|\mu_\ell^\star - \dot{x}\right| + \left(q_\ell^- + \epsilon\right) \left|\mu_\ell^+ - \tilde{\mu}_\ell\right| \Big\} \\
&\le -c\kappa_1 \left|q_\ell^-\right| \left|\epsilon\right| \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \mathcal{O}\left(1\right) \left|\epsilon\right| \Bigg(\delta + \left|q_\ell^-\right| \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{k \neq l} \left|q_k^-\right| \Bigg) \\
&\quad + \mathcal{O}\left(1\right) \left|e\right| \\
&\le \mathcal{O}\left(1\right) \left|\epsilon\right| \Bigg(\delta + \sum_{k \neq l} \left|q_k^-\right| \Bigg) + \mathcal{O}\left(1\right) \left|e\right| + \left|\epsilon\right| \, \left|q_\ell^-\right| \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) \left(\mathcal{O}\left(1\right) - c\kappa_1\right).
\end{aligned}
\tag{7.57}
$$

Adding (7.57) and (7.51), we obtain

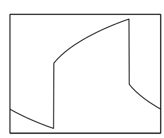
$$
\begin{aligned}
\sum_k E_k = E_l + \sum_{k \neq l} E_k \\
&\le \mathcal{O}\left(1\right) \epsilon \delta + \mathcal{O}\left(1\right) \left|e\right| + \epsilon \sum_{k \neq l} \left|q_k^-\right| \left(\mathcal{O}\left(1\right) - c\kappa_1\right) \\
&\quad + \epsilon \left|q_\ell^-\right| \left(\left|q_\ell^-\right| + \epsilon\right) \left(\mathcal{O}\left(1\right) - c\kappa_1\right) \\
&\le \mathcal{O}\left(1\right) \epsilon \delta + \mathcal{O}\left(1\right) \left|e\right|,
\end{aligned}
\tag{7.58}
$$

which holds for sufficiently large κ_1. This implies (7.30) in Case R1.

Case R2 $q_l^- < q_l^+ < 0, \epsilon > 0$.

Writing E_k as in (7.38), and using (7.44) (instead of (7.45) as in the previous case), we find for $k \neq l$ that

$$
E_k \le \mathcal{O}\left(1\right) \Bigg(\delta + \left|q_l^+\right| \left(\left|q_l^+\right| + \left|\epsilon\right|\right) + \sum_{\tilde{k} \neq l} \left|q_{\tilde{k}}^+\right| \Bigg) + \mathcal{O}\left(1\right) \left|e\right| - c\kappa_1 \left|q_k^+\right| \left|\epsilon\right|.
\tag{7.59}
$$



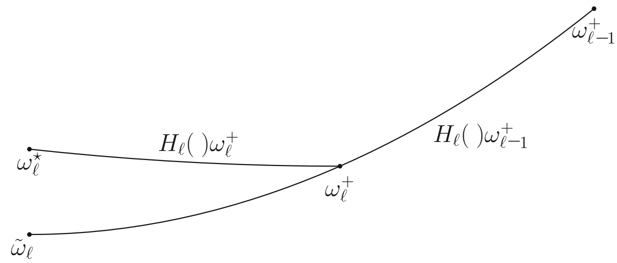


Fig. 7.5 The situation for $q_l^- < q_l^+ < 0$, $\epsilon > 0$, and $k = l$

For $k = l$ the situation is similar to the previous case. We define auxiliary states and speeds

$$\begin{aligned} \tilde{\omega}_\ell &= H_l\left(q_\ell^+ - \epsilon\right)\omega_{l-1}^+, & \tilde{\mu}_\ell &= \mu_l\left(\omega_{l-1}^+, \tilde{\omega}_\ell\right), \\ \omega_\ell^\star &= H_l\left(-\epsilon\right)\omega_l^+, & \mu_\ell^\star &= \mu_l\left(\omega_l^+, \omega_\ell^\star\right); \end{aligned} \tag{7.60}$$

see Fig. 7.5.

Recall that

$$\omega_\ell^+ = H_l\left(q_\ell^+\right)\omega_{l-1}^+ \quad \text{and} \quad \mu_\ell^+ = \mu_l\left(\omega_{l-1}^+, \omega_\ell^+\right).$$

In this case we use (7.33) with $\bar{\omega} = \omega_\ell^+$, $\varepsilon = q_\ell^+$, and $\varepsilon' = -\epsilon$. This gives

$$\left|\left(q_\ell^+ - \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - q_\ell^+\left(\mu_\ell^+ - \mu_\ell^\star\right)\right| = \mathcal{O}\left(1\right)\left|q_\ell^+\right|\,|\epsilon|\left(\left|q_\ell^+\right| + |\epsilon|\right). \tag{7.61}$$

As in (7.54), we find that

$$\begin{aligned} \left|\mu_\ell^\star - \dot{x}\right| &\le \left|\mu_l\left(\omega_\ell^+, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| + \mathcal{O}\left(1\right)\delta \\ &= \left|\mu_l\left(\omega_l^+, H_l(-\epsilon)\omega_l^+\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| + \mathcal{O}\left(1\right)\delta \\ &\le \mathcal{O}\left(1\right)\delta + \mathcal{O}\left(1\right)\left(\left|\omega_\ell^+ - \omega_n^+\right| + \epsilon\right) \\ &\le \mathcal{O}\left(1\right)\delta + \mathcal{O}\left(1\right)\sum_{k\ne l}\left|q_k^+\right|. \end{aligned} \tag{7.62}$$

We also obtain the analogue of (7.55), namely,

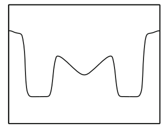
$$\begin{aligned} \left|\mu_l^- - \tilde{\mu}_\ell\right| &= \left|\mu_l\left(\omega_{l-1}^-, H_l\left(q_\ell^-\right)\omega_{l-1}^-\right) - \mu_l\left(\omega_{l-1}^+, H_l\left(q_l^+ - \epsilon\right)\omega_{l-1}^+\right)\right| \\ &= \mathcal{O}\left(1\right)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\ &= \mathcal{O}\left(1\right)\left(\delta + \left|q_\ell^+\right|\left(\left|q_\ell^+\right| + |\epsilon|\right) + \sum_{k\ne l}\left|q_k^+\right|\right) + \mathcal{O}\left(1\right)|e|. \end{aligned} \tag{7.63}$$

By genuine nonlinearity, using Lemma 7.6, we find that

$$\tilde{\mu}_\ell - \mu_\ell^\star > c\left|q_\ell^+\right|, \tag{7.64}$$

for some constant c. Now

$$W_\ell^- = W_\ell^+ + \kappa_1|\epsilon|.$$



Using the above estimates (7.61)–(7.64), we compute

$$
\begin{aligned}
E_l &= W_\ell^+ \left|q_\ell^+\right| \left(\mu_\ell^+ - \dot{x}\right) - \left(W_\ell^+ + \kappa_1 \epsilon\right) \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) \\
&= -\kappa_1 \epsilon \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\leq -\kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) + \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\left|\mu_\ell^- - \tilde{\mu}_\ell\right| + \left|\dot{x} - \mu_\ell^\star\right|\right) \\
&\quad + \kappa_1 \epsilon \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\leq -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \\
&\quad + \mathcal{O}\left(1\right) \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \neq l} \left|q_k^+\right| + |e|\right) \\
&\quad - W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\leq -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \\
&\quad + \mathcal{O}\left(1\right) \kappa_1 \epsilon \left(\left|q_\ell^+\right| + \epsilon\right) \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \neq l} \left|q_k^+\right| + |e|\right) \\
&\quad + W_\ell^+ \Big\{ \left| \left|q_\ell^+\right| \left(\mu_\ell^+ - \mu_\ell^\star\right) - \left(\left|q_\ell^+\right| + \epsilon\right) \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \right| \\
&\qquad\quad + \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| + \epsilon \left|\mu_\ell^\star - \dot{x}\right| + \left(\left|q_\ell^+\right| + \epsilon\right) \left|\mu_\ell^- - \tilde{\mu}_\ell\right| \Big\} \\
&\leq -c\epsilon\kappa_1 \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \mathcal{O}\left(1\right) \epsilon \left(\delta + \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) + \sum_{k \neq l} \left|q_k^+\right|\right) \\
&\qquad + \mathcal{O}\left(1\right) |e| \\
&\leq \mathcal{O}\left(1\right) \epsilon \left(\delta + \sum_{k \neq l} \left|q_k^+\right|\right) + \mathcal{O}\left(1\right) |e| + \epsilon \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \left(\mathcal{O}\left(1\right) - c\kappa_1\right).
\end{aligned}
\tag{7.65}
$$

Finally,

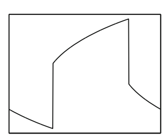
$$
\begin{aligned}
\sum_k E_k &= E_l + \sum_{k \neq l} E_k \\
&\leq \mathcal{O}\left(1\right) \epsilon\, \delta + \mathcal{O}\left(1\right) |e| + \epsilon \sum_{k \neq l} \left|q_k^+\right| \left(\mathcal{O}\left(1\right) - c\kappa_1\right) \\
&\quad + \epsilon \left|q_\ell^+\right| \left(\left|q_\ell^+\right| + \epsilon\right) \left(\mathcal{O}\left(1\right) - c\kappa_1\right) \\
&\leq \mathcal{O}\left(1\right) \epsilon\, \delta + \mathcal{O}\left(1\right) |e|
\end{aligned}
\tag{7.66}
$$

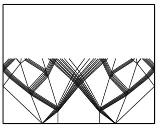
by choosing κ_1 larger if necessary. Hence (7.30) holds in this case as well.

Case R3 $q_l^- < 0 < q_l^+, \epsilon > 0$.

Since the front at x is a rarefaction front, both estimates (7.51) and (7.59) hold. Moreover, we have that

$$
q_\ell^+ - q_\ell^- = \left|q_\ell^+\right| + \left|q_\ell^-\right| < 2\epsilon \leq 2\delta.
$$





Then from $AD + BC \leq (A + B)(D + C)$ for positive A, B, C, and D, we obtain

$$\begin{aligned}
E_l &= W_\ell^+ \left|q_\ell^+\right| \left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) \\
&\leq \mathcal{O}(1) \left(\left|q_\ell^+\right| + \left|q_\ell^-\right|\right) \left(\left|\mu_\ell^+ - \dot{x}\right| + \left|\mu_\ell^- - \dot{x}\right|\right) \\
&\leq \mathcal{O}(1)\, \epsilon \left(\left|\mu_\ell^+ - \dot{x}\right| + \left|\mu_\ell^- - \dot{x}\right|\right) \\
&= \mathcal{O}(1)\, \epsilon \Big(\left|\mu_l\left(\omega_{l-1}^+, \omega_\ell^+\right) - \mu_l\left(v^{\delta,+}, v^{\delta,+}\right)\right| \\
&\qquad\qquad + \left|\mu_l\left(\omega_{l-1}^-, \omega_\ell^-\right) - \mu_l\left(v^{\delta,-}, v^{\delta,-}\right)\right|\Big) \\
&= \mathcal{O}(1)\, \epsilon \left(\delta + \left|q_\ell^+\right| + \left|q_\ell^-\right| + \sum_{k>l} \left|q_k^+\right| + \sum_{k<l} \left|q_k^-\right|\right) \\
&= \mathcal{O}(1)\, \epsilon \left(\delta + \sum_{k>l} \left|q_k^+\right| + \sum_{k<l} \left|q_k^-\right|\right).
\end{aligned} \tag{7.67}$$

Using (7.51) for $k < l$ and (7.59) for $k > l$, and choosing κ_1 sufficiently large, we obtain (7.30).

Now we shall study the cases in which the front at x is a shock front. Also, here we prove (7.30) in three cases depending on q_ℓ^- and q_ℓ^+. If the front at x is a shock front, then by the construction of the front-tracking approximation, we have

$$H_l(\epsilon) v^{\delta,-} = v^{\delta,+} + e,$$

or

$$H_l(\epsilon) H\left(q^-\right) u^{\delta_1} = H\left(q^+\right) u^{\delta_1} + e,$$

where $q^\pm = \left(q_1^\pm, \dots, q_n^\pm\right)$, and e is the error of the front at x. Then we can use (7.31) and continuity of the mapping H to find that

$$\begin{aligned}
&\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\
&\quad = \mathcal{O}(1) \left|\epsilon\right| \left(\left|q_l^-\right| \left(\left|q_l^-\right| + \left|\epsilon\right|\right) + \sum_{k \neq l} \left|q_k^-\right|\right) + \mathcal{O}(1) \left|e\right|.
\end{aligned} \tag{7.68}$$

We also have that

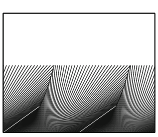
$$u^{\delta_1} = H\left(-q^+\right) v^{\delta,+} = H\left(-q^+\right) \left(H_l(\epsilon) v^{\delta,-} + e\right),$$

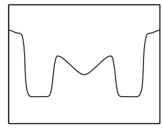
or

$$H\left(-q^-\right) v^{\delta,-} = H\left(-q^+\right) H_l(\epsilon) v^{\delta,-} + \mathcal{O}(1) \left|e\right|,$$

by the continuity of H. From this we obtain

$$\begin{aligned}
&\left|q_l^+ - q_l^- - \epsilon\right| + \sum_{k \neq l} \left|q_k^+ - q_k^-\right| \\
&\quad = \mathcal{O}(1) \left|\epsilon\right| \left(\left|q_l^+\right| \left(\left|q_l^+\right| + \left|\epsilon\right|\right) + \sum_{k \neq l} \left|q_k^+\right|\right) + \mathcal{O}(1) \left|e\right|.
\end{aligned} \tag{7.69}$$





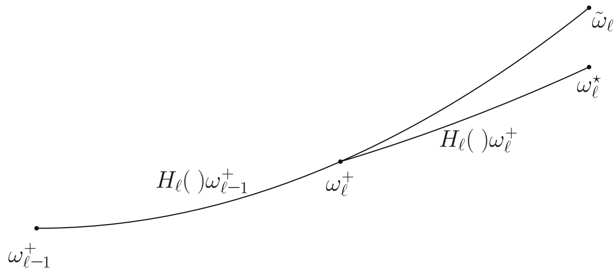


Fig. 7.6 The situation for $0 < q_\ell^+ < q_\ell^-, \epsilon < 0$, and $k = l$

Case S1 $0 < q_\ell^+ < q_\ell^-, \epsilon < 0$.

If $k \neq l$, then we can write E_k as (7.38) and use the arguments leading to (7.51) and the estimate (7.69) to obtain

$$E_k \leq \mathcal{O}(1)\,|\epsilon| \left(\left|q_l^+\right| \left(\left|q_l^+\right| + |\epsilon|\right) + \sum_{\tilde{k}\neq l} \left|q_{\tilde{k}}^+\right| \right) + \mathcal{O}(1)\,|e| - c\kappa_1 \left|q_k^+\right| |\epsilon| \,. \tag{7.70}$$

For $k = l$ we define the auxiliary states and speeds as in (7.60); see Fig. 7.6.

Then the estimate (7.61) holds. Also, using (7.69) we find that

$$\begin{aligned} \left|\mu_l^- - \tilde{\mu}_\ell\right| &= \mathcal{O}(1)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right) \\ &= \mathcal{O}(1)\,|\epsilon| \left(\left|q_\ell^+\right| \left(\left|q_\ell^+\right| + |\epsilon|\right) + \sum_{k\neq l} \left|q_k^+\right| \right) + \mathcal{O}(1)\,|e| \,. \end{aligned} \tag{7.71}$$

Moreover,

$$\begin{aligned} \left|\mu_\ell^\star - \dot{x}\right| &= \left|\mu_l\left(\omega_\ell^+, \omega_\ell^\star\right) - \mu_l\left(v^{\delta,-}, v^{\delta,+}\right)\right| \\ &\leq \left|\mu_l\left(\omega_\ell^+, H_l(-\epsilon)\omega_\ell^+\right) - \mu_l\left(v^{\delta,+}, H_l\left(-\epsilon\right) v^{\delta,+}\right)\right| + \mathcal{O}(1)\,|e| \\ &= \mathcal{O}(1)\left(\left|\omega_\ell^+ - \omega_n^+\right|\right) + \mathcal{O}(1)\,|e| \\ &= \mathcal{O}(1)\left(\sum_{k>l} \left|q_k^+\right| + |e| \right). \end{aligned} \tag{7.72}$$

By Lemma 7.6, we have

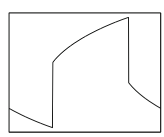
$$\mu_\ell^\star - \tilde{\mu}_\ell > c q_\ell^+ . \tag{7.73}$$

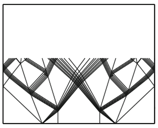
In this case

$$W_\ell^+ = W_\ell^- + \kappa_1\,|\epsilon| \,, \tag{7.74}$$

and

$$\epsilon < 0 < q_\ell^+ < q_\ell^- .$$





We estimate

$$
\begin{aligned}
E_l &= W_\ell^+ q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - \left(W_\ell^+ - \kappa_1 \left|\epsilon\right|\right) q_\ell^- \left(\mu_\ell^- - \dot{x}\right) \\
&= \kappa_1 \left|\epsilon\right| q_\ell^- \left(\mu_\ell^- - \dot{x}\right) + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \left|\epsilon\right| \left\{\left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mu_\ell^- - \mu_\ell^\star\right) + q_\ell^- \left(\mu_\ell^- - \dot{x}\right) - \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mu_\ell^- - \mu_\ell^\star\right)\right\} \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&= \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left\{\left(q_\ell^+ + \left|\epsilon\right|\right)\left(\left(\mu_\ell^- - \dot{x}\right) - \left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right)\right. \\
&\qquad\qquad \left. + \left(q_\ell^- - q_\ell^+ - \left|\epsilon\right|\right)\left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\left|\mu_\ell^- - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right) \\
&\quad + \kappa_1 \left|\epsilon\right| \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| \\
&\quad + W_\ell^+ \left\{q_\ell^+ \left(\mu_\ell^+ - \dot{x}\right) - q_\ell^- \left(\mu_\ell^- - \dot{x}\right)\right\} \\
&\le -c\kappa_1 \left(q_\ell^+ + \left|\epsilon\right|\right) \left|\epsilon\right| q_\ell^+ \\
&\quad + \mathcal{O}\left(1\right) \kappa_1 \left|\epsilon\right|^2 \left(q_\ell^+ \left(q_\ell^+ + \left|\epsilon\right|\right) + \sum_{k\neq l} \left|q_k^+\right|\right) + \mathcal{O}\left(1\right) \left|e\right| \\
&\quad + \mathcal{O}\left(1\right) \kappa_1 \left|\epsilon\right| \left(\sum_{k>l} \left|q_k^+\right|\right) + \mathcal{O}\left(1\right) \left|e\right| \\
&\quad + W_\ell^+ \Big\{\left|q_\ell^+ \left(\mu_\ell^+ - \mu_\ell^\star\right) - \left(q_\ell^+ - \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right| \\
&\quad + \left|q_\ell^+ - q_\ell^- - \epsilon\right| \left|\mu_\ell^- - \dot{x}\right| + \left|\epsilon\right| \left|\mu_\ell^\star - \dot{x}\right| + \left(q_\ell^+ + \left|\epsilon\right|\right) \left|\mu_\ell^- - \tilde{\mu}_\ell\right| \Big\} \\
&\le -c\kappa_1 \left(q_\ell^+ + \left|\epsilon\right|\right) \left|\epsilon\right| q_\ell^+ + \mathcal{O}\left(1\right) \left|\epsilon\right| \left(q_\ell^+ \left(q_\ell^+ + \left|\epsilon\right|\right) + \sum_{k\neq l} \left|q_k^+\right|\right) \\
&\quad + \mathcal{O}\left(1\right) \left|e\right| \\
&\le \mathcal{O}\left(1\right) \sum_{k\neq l} \left|q_k^+\right| + \left|\epsilon\right| \, \left|q_\ell^+\right| \left(q_\ell^+ + \left|\epsilon\right|\right)\left(\mathcal{O}\left(1\right) - c\kappa_1\right) + \mathcal{O}\left(1\right) \left|e\right| .
\end{aligned}
\tag{7.75}
$$

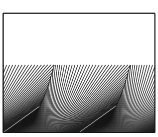
As before, setting κ_1 sufficiently large, (7.75) and (7.70) imply

$$
\sum_k E_k = E_l + \sum_{k\neq l} E_k \le \mathcal{O}\left(1\right) \left|e\right| ,
\tag{7.76}
$$

which is (7.30).

Case S2 $q_\ell^+ < q_\ell^- < 0, \epsilon < 0$.
In this case we proceed as in Case S1, but using (7.68) instead of (7.69). For $k \neq l$ this gives the estimate

$$
E_k \le \mathcal{O}\left(1\right) \left|\epsilon\right| \left(\left|q_\ell^-\right| \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{\tilde{k}\neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}\left(1\right) \left|e\right| - c\kappa_1 \left|q_k^-\right| \left|\epsilon\right| .
\tag{7.77}
$$



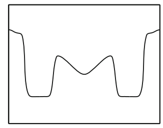
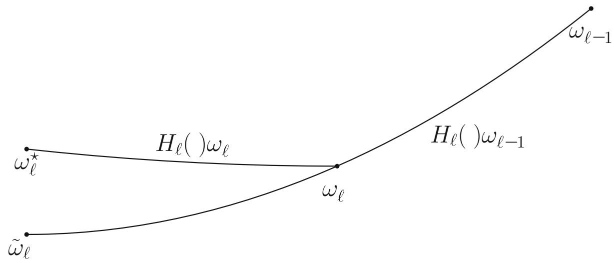


Fig. 7.7 The situation for $q_\ell^+ < q_\ell^- < 0$, $\epsilon < 0$, and $k = l$



We now define the intermediate states $\tilde{\omega}_\ell$, $\omega_\ell^\star$ and the speeds $\tilde{\mu}_\ell$ and $\mu_\ell^\star$ as in (7.52); see Fig. 7.7.

Then the estimate (7.53) holds. As in Case R1, we compute

$$\begin{aligned}\left|\mu_\ell^+ - \tilde{\mu}_\ell\right| &= \mathcal{O}(1)\left(\left|\omega_{l-1}^- - \omega_{l-1}^+\right| + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\right)\\ &= \mathcal{O}(1)\left|\epsilon\right|\left(\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|q_k^-\right|\right) + \mathcal{O}(1)\left|e\right| \qquad (7.78)\end{aligned}$$

and

$$\begin{aligned}\left|\mu_\ell^\star - \dot{x}\right| &\leq \left|\mu_l\left(\omega_l^-, H_l(\epsilon)\omega_l^-\right) - \mu_l\left(v^{\delta,-}, H_l(\epsilon)v^{\delta,-}\right)\right| + \mathcal{O}(1)\left|e\right|\\ &\leq \mathcal{O}(1)\,\delta + \mathcal{O}(1)\left|\omega_l^- - \omega_0^-\right| + \mathcal{O}(1)\left|e\right|\\ &\leq \mathcal{O}(1)\left|e\right| + \mathcal{O}(1)\sum_{k<l}\left|q_k^-\right|. \qquad (7.79)\end{aligned}$$

In this case, genuine nonlinearity and Lemma 7.6 imply that

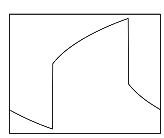
$$\tilde{\mu}_\ell - \mu_\ell^\star > c\left|q_\ell^-\right|, \qquad (7.80)$$

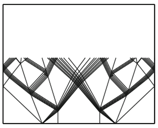
with $c > 0$. Moreover, now

$$W_\ell^+ = W_\ell^- - \kappa_1\left|\epsilon\right|.$$

Now we can use the (by now) familiar technique of estimating E_l:

$$\begin{aligned}E_l &= \left(W_\ell^- - \kappa_1\left|\epsilon\right|\right)\left|q_\ell^+\right|\left(\mu_\ell^+ - \dot{x}\right) - W_\ell^-\left|q_\ell^-\right|\left(\mu_\ell^- - \dot{x}\right)\\ &\leq -\kappa_1\left|\epsilon\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\\ &\quad + \kappa_1\left|\epsilon\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\left(\left|\mu_\ell^+ - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right)\\ &\quad + \kappa_1\left|\epsilon\right|\left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot{x}\right|\\ &\quad + W_\ell^-\left\{\left|q_\ell^+\right|\left(\mu_\ell^+ - \dot{x}\right) - \left|q_\ell^-\right|\left(\mu_\ell^- - \dot{x}\right)\right\}\\ &\leq -c\kappa_1\left|q_\ell^-\right|\left|\epsilon\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\\ &\quad + \mathcal{O}(1)\,\kappa_1\left|\epsilon\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\left(\left|q_\ell^-\right|\left(\left|q_\ell^-\right| + \left|\epsilon\right|\right) + \sum_{k\neq l}\left|q_k^-\right|\right)\\ &\quad + W_\ell^-\Big\{\left|q_\ell^-\left(\mu_\ell^- - \mu_\ell^\star\right) - \left(q_\ell^- + \epsilon\right)\left(\tilde{\mu}_\ell - \mu_\ell^\star\right)\right|\\ &\qquad\qquad + \left|q_\ell^+ - q_\ell^- - \epsilon\right|\left|\mu_\ell^+ - \dot{x}\right|\\ &\qquad\qquad + \left|\epsilon\right|\left|\mu_\ell^\star - \dot{x}\right| + \left(\left|q_\ell^-\right| + \left|\epsilon\right|\right)\left|\mu_\ell^+ - \tilde{\mu}_\ell\right|\Big\} + \mathcal{O}(1)\left|e\right|\end{aligned}$$





$$
\begin{aligned}
&\leq -c\kappa_1 \left|q_\ell^-\right| |\epsilon| \left(\left|q_\ell^-\right| + |\epsilon|\right) + \mathcal{O}(1) \left(\left|q_\ell^-\right| \left(\left|q_\ell^-\right| + |\epsilon|\right) + \sum_{k\neq l} \left|q_k^-\right|\right) \\
&\quad + \mathcal{O}(1) |e| \\
&\leq \mathcal{O}(1) \sum_{k\neq l} \left|q_k^-\right| + |\epsilon| \left|q_\ell^-\right| \left(\left|q_\ell^-\right| + |\epsilon|\right) \left(\mathcal{O}(1) - c\kappa_1\right) + \mathcal{O}(1) |e| .
\end{aligned} \tag{7.81}
$$

Combining (7.81) and (7.77), we obtain

$$
\sum_k E_k = E_l + \sum_{k\neq l} E_k \leq \mathcal{O}(1) |e| , \tag{7.82}
$$

which is (7.30).

Case S3 $q_\ell^+ < 0 < q_\ell^-$, $\epsilon < 0$.

For $k \neq l$, the estimate (7.77) remains valid.

Next we consider the case $k = l$. The $\mathcal{O}(1)$ that multiplies $|\epsilon|$ in (7.69) (or (7.69)) is proportional to the total variation of the initial data. Hence we can assume that this is arbitrarily small by choosing T.V. (u_0) sufficiently small. Since all terms $q_j^\pm$ are bounded, we can and will assume that

$$
\left|q_l^+ - q_l^- - \epsilon\right| \leq \frac{1}{2} |\epsilon| + \mathcal{O}(1) |e| . \tag{7.83}
$$

Without loss of generality we may assume that $\left|q_l^+\right| \geq \left|q_l^-\right|$. This implies that

$$
\left|q_l^+ - q_l^- - \epsilon\right| \geq \left|q_l^- - q_l^+\right| - |\epsilon| = q_l^- - q_l^+ + \epsilon \geq 2q_l^- + \epsilon. \tag{7.84}
$$

Thus

$$
2q_l^- + \epsilon \leq \frac{1}{2} |\epsilon| + \mathcal{O}(1) |e| , \tag{7.85}
$$

or

$$
q_l^- + \epsilon \leq -\frac{1}{4} |\epsilon| + \mathcal{O}(1) |e| , \tag{7.86}
$$

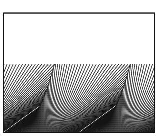
which can be rewritten as

$$
\left|q_l^- + \epsilon - \mathcal{O}(1) |e|\right| \geq \frac{1}{4} |\epsilon| . \tag{7.87}
$$

From this we conclude that

$$
\left|q_l^- + \epsilon\right| \geq \frac{1}{4} |\epsilon| - \mathcal{O}(1) |e| . \tag{7.88}
$$

We define the auxiliary states $\tilde{\omega}_\ell$, $\omega_\ell^\star$ and the speeds $\tilde{\mu}_\ell$ and $\mu_\ell^\star$ as in (7.52); see Fig. 7.8. Then estimates (7.78) and (7.79) hold.



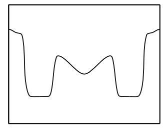
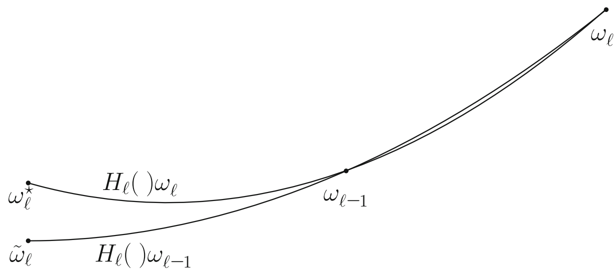


Fig. 7.8 The situation for $q_\ell^+ < 0 < q_\ell^-, \epsilon < 0$, and $k = l$



By Lemma 7.6 we have that

$$\tilde{\mu}_\ell - \mu_\ell^\star \le 0, \tag{7.89}$$
$$\mu_\ell^- - \mu_\ell^\star \ge c\,|q_\ell^- + \epsilon|\,, \tag{7.90}$$

for a positive constant c. Recalling that $W_\ell^- \ge 1$, and using (7.89), (7.90), and the estimates (7.78) and (7.79) (which remain valid in this case), we compute

$$\begin{aligned} E_l &= W_\ell^+ \left|q_\ell^+\right| \left(\mu_\ell^+ - \dot{x}\right) - W_\ell^- \left|q_\ell^-\right| \left(\mu_\ell^- - \dot{x}\right) \\ &\le W_\ell^+ \left|q_\ell^+\right| \left(\tilde{\mu}_\ell - \mu_\ell^\star\right) - W_\ell^- \left|q_\ell^-\right| \left(\mu_\ell^- - \mu_\ell^\star\right) \\ &\quad + W_\ell^+ \left|q_\ell^+\right| \left(\left|\mu_\ell^+ - \tilde{\mu}_\ell\right| + \left|\mu_\ell^\star - \dot{x}\right|\right) + W_\ell^- \left|q_\ell^-\right| \left|\mu_\ell^\star - \dot{x}\right| \\ &\le -\left|q_\ell^-\right| c \left|q_\ell^- + \epsilon\right| + \mathcal{O}(1)\,|\epsilon| \left(q_\ell^- \left(q_\ell^- + |\epsilon|\right) + \sum_{\tilde{k}\neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e| \\ &\le \frac{-c}{4} \left|q_\ell^-\right| |\epsilon| + \mathcal{O}(1)\,|\epsilon| \left(q_\ell^- \left(q_\ell^- + |\epsilon|\right) + \sum_{\tilde{k}\neq l} \left|q_{\tilde{k}}^-\right|\right) + \mathcal{O}(1)\,|e|\,. \end{aligned} \tag{7.91}$$

Now (7.77) and (7.91) are used to balance the terms containing the factor $\sum_{k\neq l} \left|q_k^-\right|$. The remaining term,

$$\left|q_\ell^-\right| |\epsilon| \left(-\frac{1}{4}c + \mathcal{O}(1)\left(q_\ell^- + |\epsilon|\right)\right),$$

can be made negative by choosing T.V. (u_0) (and hence $\mathcal{O}(1)$) sufficiently small.

Hence also in this case (7.30) holds.

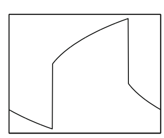
Finally, if q_ℓ^- or q_ℓ^+ is zero, (7.30) can easily be shown to be a limit of one of the previous cases.

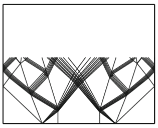
Summing up, we have proved the following theorem:

Theorem 7.7 *Let u^{δ_1} and v^{δ_2} be front-tracking approximations, with accuracies defined by δ_1, δ_2,*

$$G\left(u^{\delta_1}(t)\right) < M, \quad \text{and} \quad G\left(v^{\delta_2}(t)\right) < M, \qquad \text{for } t \ge 0. \tag{7.92}$$

For sufficiently small M there exist constants κ_1, κ_2, and C_2 such that the functional Φ defined by (7.15) and (7.16) satisfies (7.21). Furthermore, there exists





a constant C (independent of δ_1 and δ_2) such that

$$\left\|u^{\delta_1}(t) - v^{\delta_2}(t)\right\|_1 \le C \left\|u^{\delta_1}(0) - v^{\delta_2}(0)\right\|_1 + Ct\left(\delta_1 \vee \delta_2\right). \tag{7.93}$$

To state the next theorem we need the following definition. Let the domain $\mathcal{D}$ be defined as the L^1 closure of the set

$$\mathcal{D}_0 = \left\{u \in L^1(\mathbb{R};\mathbb{R}^n) \mid u \text{ is piecewise constant and } G(u) < M\right\}; \tag{7.94}$$

that is, $\mathcal{D} = \overline{\mathcal{D}}_0$. Since the total variation is small, we will assume that all possible values of u are in a (small) neighborhood $\Omega \subset \mathbb{R}^n$.

Theorem 7.8 *Let $f_j \in C^2(\mathbb{R}^n)$, $j = 1,\dots,n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$. Assume that each wave family is either genuinely nonlinear or linearly degenerate. For all initial data u_0 in $\mathcal{D}$, defined by (7.94), every sequence of front-tracking approximations u^δ converges to a unique limit u as $\delta \to 0$. Furthermore, let u and v denote solutions*

$$u_t + f(u)_x = 0,$$

with initial data u_0 and v_0, respectively, obtained as a limit of a front-tracking approximation. Then

$$\|u(t) - v(t)\|_1 \le C\|u_0 - v_0\|_1. \tag{7.95}$$

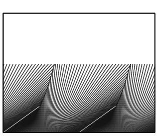
Proof First we use (7.93) to conclude that every front-tracking approximation u^δ has a unique limit u as $\delta \to 0$. Then we take the limit $\delta \to 0$ in (7.93) to conclude that (7.95) holds. □

Note that this also gives an error estimate for front tracking for systems. If we denote the limit of the sequence $\left\{u^\delta\right\}$ by u and $v^{\delta_2} = u^\delta$, then by letting $\delta_2 \to 0$ in (7.93)

$$\left\|u^\delta(\cdot,t) - u(\cdot,t)\right\|_1 \le C\left(\left\|u_0^\delta - u_0\right\|_1 + \delta t\right) = \mathcal{O}(1)\,\delta$$

for some finite constant C. Hence front tracking for systems is a first-order method.

7.2 Uniqueness



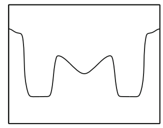
Let S_t denote the map that maps initial data u_0 into the solution u of

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0$$

at time t, that is, $u = S_t u_0$. In Chapt. 6 we showed the existence of the semigroup S_t, and in the previous section its stability for initial data in the class $\mathcal{D}$ as limits of approximate solutions obtained by front tracking. Thus we know that it satisfies

$$S_0 u = u, \qquad S_t S_s u = S_{t+s} u,$$
$$\|S_t u - S_s v\|_1 \le L\left(|t-s| + \|u - v\|_1\right)$$

for all $t, s \ge 0$ and u, v in $\mathcal{D}$.



In this section we prove uniqueness of solutions that have initial data in $\mathcal{D}$.

We want to demonstrate that every other solution u coincides with this semigroup. To do this we will basically need three assumptions. The first is that u is a weak solution, the second is that it satisfies Lax's entropy conditions across discontinuities, and the third is that it has locally bounded variation on a certain family of curves. Concretely, we define an *entropy solution* of

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0,$$

to be a bounded measurable function $u = u(x,t)$ of bounded total variation satisfying the following conditions:

A The function $u = u(x,t)$ is a weak solution of the Cauchy problem (7.1) taking values in $\mathcal{D}$, i.e.,

$$\int_0^T \int_{\mathbb{R}} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{\mathbb{R}} \varphi(x,0)u_0(x)\,dx = 0 \tag{7.96}$$

for all test functions φ whose support is contained in the strip $[0,T\rangle$.

B Assume that u has a jump discontinuity at some point (x,t), i.e., there exist states $u_{l,r} \in \Omega$ and speed σ such that if we let

$$U(y,s) = \begin{cases} u_l & \text{for } y < x + \sigma(s-t), \\ u_r & \text{for } y \geq x + \sigma(s-t), \end{cases} \tag{7.97}$$

then

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \int_{t-\rho}^{t+\rho} \int_{x-\rho}^{x+\rho} |u(y,s) - U(y,s)|\;dy\,ds = 0. \tag{7.98}$$

Furthermore, there exists k such that

$$\lambda_k\,(u_l) \geq \sigma \geq \lambda_k\,(u_r)\,. \tag{7.99}$$

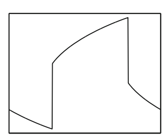
C There exists a $\theta > 0$ such that for all Lipschitz functions γ with Lipschitz constant not exceeding θ, the total variation of $u(x,\gamma(x))$ is locally bounded.

Remark 7.9 One can prove, see Exercise 7.1, that the front-tracking solution constructed in the previous chapter is an entropy solution of the conservation law.

There is a direct argument showing that any weak solution, whether it is a limit of a front-tracking approximation or not, satisfies a Lipshitz continuity in time of the spatial L^1-norm, as long as the solution has a uniform bound on the total variation. We present that argument here.

Theorem 7.10 *Let $u_0 \in \mathcal{D}$, and let u denote any weak solution of* (7.1) *such that* $\mathrm{T.V.}\,(u(t)) \leq C$. *Then*

$$\|u(\,\cdot\,,t) - u(\,\cdot\,,s)\|_1 \leq C\,\|f\|_{\mathrm{Lip}}\,|t-s|\,, \quad s,t \geq 0. \tag{7.100}$$



Proof Let $0 < s < t < T$, and let α_h be a smooth approximation to the characteristic function of the interval $[s, t]$, so that

$$\lim_{h\to 0} \alpha_h = \chi_{[s,t]}.$$

Furthermore, define

$$\varphi_h(y, \tau) = \alpha_h(\tau)\phi(y),$$

where ϕ is any smooth function with compact support. If we insert this into the weak formulation

$$\int_0^T \int_{\mathbb{R}} (u\varphi_{h,t} + f(u)\varphi_{h,x})\, dx\, dt + \int_{\mathbb{R}} \varphi_h(x, 0)u(x, 0)\, dx = 0, \tag{7.101}$$

and let $h \to 0$, we obtain

$$\int \phi(y) \left(u(y, t) - u(y, s)\right)\, dy + \int_s^t \int \phi_y\, f(u)\, dy\, ds = 0.$$

From this we obtain

$$\begin{aligned}
\|u(\,\cdot\,, t) - u(\,\cdot\,, s)\|_1 &= \sup_{|\phi|\le 1} \int \phi(y) \left(u(y, t) - u(y, s)\right)\, dy \\
&= -\sup_{|\phi|\le 1} \int_s^t \int \phi(y)_y\, f(u)\, dy\, ds \\
&\le \int_s^t \mathrm{T.V.}\,(f(u))\, ds \\
&\le C\, \|f\|_{\mathrm{Lip}}(t - s),
\end{aligned}$$

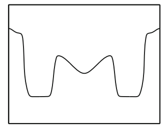
which proves the claim. Here we first used Exercise A.1, Theorem A.4, subsequently the definition (A.1) for $\mathrm{T.V.}\,(f)$, and finally the Lipschitz continuity of f and the bound on the total variation on u. □

Remark 7.11 This argument provides an alternative to the proof of the Lipschitz continuity in Theorem 2.15 in the scalar case.

Before we can compare an arbitrary entropy solution to the semigroup solution, we need some preliminary results. Firstly, Theorem 7.10 says that every function $u(\,\cdot\,, t)$ taking values in $\mathcal{D}$ and satisfying **A** is L^1 Lipschitz continuous:

$$\|u(\,\cdot\,, t) - u(\,\cdot\,, s)\|_1 \le L(t - s),$$

for $t \ge s$.



Furthermore, by the structure theorem for functions of bounded variation [193, Theorem 5.9.6], u is continuous almost everywhere. For the sake of definiteness, we shall assume that all functions in $\mathcal{D}$ are right continuous. Also, there exists a set $\mathcal{N}$ of zero Lebesgue measure in the interval $[0, T]$ such that for $t \in [0, T] \backslash \mathcal{N}$, the function $u(\,\cdot\,, t)$ either is continuous at x or has a jump discontinuity there. Intuitively, the set $\mathcal{N}$ can be thought of as the set of times when collisions of discontinuities occur.

Lemma 7.12 *If* (7.96)–(7.98) *hold, then*

$$u_l = \lim_{y \to x-} u(y, t), \quad u_r = \lim_{y \to x+} u(y, t),$$
$$\textit{and} \quad \sigma\,(u_l - u_r) = f\,(u_l) - f\,(u_r)\,.$$

Proof Let P_λ denote the parallelogram

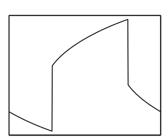
$$P_\lambda = \big\{(y, s) \mid |t - s| \le \lambda,\ |y - x - \sigma(s - t)| \le \lambda\big\}.$$

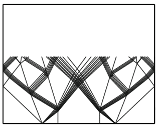
Integrating the conservation law over P_λ, we obtain

$$\begin{aligned}
&\Bigg(\int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y, t + \lambda)\, dy - \int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y, t - \lambda)\, dy\Bigg) \\
&\quad + \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x + \lambda + \sigma(s - t), s)\, ds \\
&\quad - \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x - \lambda + \sigma(s - t), s)\, ds = 0.
\end{aligned}$$

If we furthermore integrate this with respect to λ from $\lambda = 0$ to $\lambda = \rho$, and divide by ρ^2, we obtain

$$\begin{aligned}
&\frac{1}{\rho^2}\Bigg(\int_0^\rho \int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y, t + \lambda)\, dy\, d\lambda - \int_0^\rho \int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y, t - \lambda)\, dy\, d\lambda\Bigg) \\
&\quad + \frac{1}{\rho^2}\Bigg(\int_0^\rho \int_{t-\lambda}^{t+\lambda} (f(u) - \sigma u)(x + \lambda + \sigma(s - t), s)\, ds\, d\lambda \\
&\quad - \int_0^\rho \int_{t-\sigma}^{t+\sigma} (f(u) - \sigma u)(x - \lambda + \sigma(s - t), s)\, ds\, d\lambda\Bigg) = 0.
\end{aligned}$$





Now let $\rho \to 0$. Then

$$\frac{1}{\rho^2}\int_0^{\rho}\int_{x-\lambda+\lambda\sigma}^{x+\lambda+\lambda\sigma} u(y,t+\lambda)\,dy\,d\lambda \to \frac{1}{2}(u_l+u_r),$$

$$\frac{1}{\rho^2}\int_0^{\rho}\int_{x-\lambda-\lambda\sigma}^{x+\lambda-\lambda\sigma} u(y,t-\lambda)\,dy\,d\lambda \to \frac{1}{2}(u_l+u_r),$$

$$\frac{1}{\rho^2}\int_0^{\rho}\int_{t-\lambda}^{t+\lambda} (f(u)-\sigma u)(x+\lambda+\sigma(s-t),s)\,ds\,d\lambda \to f\,(u_r)-\sigma u_r,$$

$$\frac{1}{\rho^2}\int_0^{\rho}\int_{t-\lambda}^{t+\lambda} (f(u)-\sigma u)(x-\lambda+\sigma(s-t),s)\,ds\,d\lambda \to f\,(u_l)-\sigma u_l.$$

Hence

$$\frac{1}{2}(u_l+u_r)-\frac{1}{2}(u_l+u_r)+(f\,(u_r)-\sigma u_r)-(f\,(u_l)-\sigma u_l)=0.$$

This concludes the proof of the lemma. □

The next lemma states that if u satisfies **C**, then the discontinuities cannot cluster too tightly together.

Lemma 7.13 *Assume that* $u\colon[0,T]\to\mathcal{D}$ *satisfies* **C**. *Let* $t\in[0,T]$ *and* $\varepsilon>0$. *Then the set*

$$B_{t,\varepsilon}=\Big\{x\in\mathbb{R}\mid \limsup_{s\to t+,\,y\to x}|u(x,t)-u(y,s)|>\varepsilon\Big\} \tag{7.102}$$

has no limit points.

Proof Assume that $B_{t,\varepsilon}$ has a limit point, denoted by x_0. Then there is a monotone sequence $\{x_i\}_{i=1}^{\infty}$ in $B_{t,\varepsilon}$ converging to x_0. Without loss of generality we assume that the sequence is decreasing. Since $u(x,t)$ is right continuous, we can find a point z_i in (x_i,x_{i-1}) such that

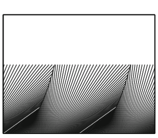
$$|u\,(z_i,t)-u\,(x_i,t)|\le\frac{\varepsilon}{2}.$$

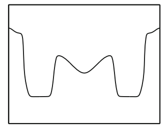
Now choose $s_i>t$ and $y_i\in(z_{i+1},z_i)$ such that

$$|u\,(y_i,s_i)-u\,(x_i,t)|\ge\varepsilon,\quad |s_i-t|\le\theta\max\{|y_i-z_i|\;,\;|y_i-z_{i+1}|\}\,.$$

We define a curve $\gamma(x)$ for $x\in[x_0,x_1]$ passing through all the points (z_i,t) and (y_i,s_i) by

$$\gamma(x)=\begin{cases} t & \text{for } x=x_0 \text{ or } x\ge z_1,\\ s_i-(x-y_i)\,\frac{s_i-t}{z_i-y_i} & \text{for } x\in[y_i,z_i],\\ t+(x-z_{i+1})\,\frac{s_i-t}{y_i-z_{i+1}} & \text{for } x\in[z_{i+1},y_i]. \end{cases} \tag{7.103}$$





Then γ is Lipschitz continuous with Lipschitz constant θ, and we have that

$$|u(y_i, s_i) - u(z_i, t)| \geq \frac{\varepsilon}{2}$$

for all $i \in \mathbb{N}$. This means that the total variation of $u(x, \gamma(x))$ is infinite, violating **C**, concluding the proof of the lemma. $\square$

In the following, we let $\sigma^\star$ be a number strictly larger than the absolute value of every characteristic speed, and we also demand that $\sigma^\star \geq 1/\theta$, where θ is the constant in **C**. The next lemma says that if u satisfies **C**, then discontinuities cannot propagate faster than $\sigma^\star$. Precisely, we have the following result.

Lemma 7.14 *Assume that* $u\colon [0, T] \to \mathcal{D}$ *satisfies* **C**. *Then for* $(x, t) \in (0, T) \times \mathbb{R}$,

$$\lim_{\substack{s \to t^+,\, y \to x\pm \\ |x-y| > \sigma^\star(s-t)}} u(y, s) = u(x\pm, t). \tag{7.104}$$

Proof We assume that the lemma does not hold. Then, for some (x_0, t) there exist decreasing sequences $s_j \to t$ and $y_j \to x_0$ such that

$$\left|y_j - x_0\right| \geq \sigma^\star \left(s_j - t\right), \quad \left|u\left(y_j, s_j\right) - u(x_0, t)\right| \geq \varepsilon$$

for some $\varepsilon > 0$ and $j \in \mathbb{N}$. Now let

$$z_0 = y_1 + \frac{s_1 - t}{\theta},$$

where as before θ is defined by **C**. Now we define a subsequence of $\left\{(y_j, s_j)\right\}$ as follows. Set $j_1 = 1$ and for $i \geq 1$ define

$$\begin{cases} z_i = y_{j_i} - \frac{s_{j_i} - t}{\theta}, \\ j_{i+1} = \min\left\{k \;\middle|\; s_k \leq t - \theta\,(y_k - z_i)\right\}. \end{cases}$$

Then

$$y_{j_i} \in (z_{i+1}, z_i) \quad \text{and} \quad \left|s_{j_i} - t\right| \leq \theta \max\left\{\left|y_{j_i} - z_i\right|, \left|y_{j_i} - z_{i+1}\right|\right\}$$

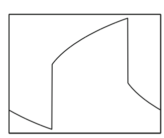
for all i. Let γ be the curve defined in (7.103). Since we have that $z_i \to x_0$, we have that

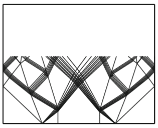
$$|u(z_i, t) - u(x_0, t)| \leq \frac{\varepsilon}{2}$$

for sufficiently large i. Consequently,

$$\left|u(z_i, t) - u\left(y_{j_i}, s_{j_i}\right)\right| \geq \frac{\varepsilon}{2},$$

and the total variation of $u(x, \gamma(x))$ is infinite, contradicting **C**. $\square$





The next lemma concerns properties of the semigroup S_t. We assume that u is a continuous function $u\colon [0,T] \to \mathcal{D}$, and wish to estimate $S_T u(0) - u(T)$. Let h be a small number such that $Nh = T$. Then we can calculate

$$\begin{aligned}\|S_T u(0) - u(T)\|_1 &\le \sum_{i=1}^{N} \left\| S_{T-(i-1)h} u((i-1)h) - S_{T-ih} u(ih) \right\|_1 \\ &\le L \sum_{i=1}^{N} \left\| \frac{1}{h}\big(u(ih) - S_h u((i-1)h)\big) \right\|_1 h.\end{aligned}$$

Letting h tend to zero, we obtain the following lemma:

Lemma 7.15 *Assume that $u\colon [0,T] \to \mathcal{D}$ is Lipschitz continuous in the L^1-norm. Then for every interval $[a,b]$, we have*

$$\begin{aligned}&\|S_T u(0) - u(T)\|_{L^1([a+\sigma^\star T, b-\sigma^\star T]\,;\,\mathbb{R}^n)} \\ &\quad \le \mathcal{O}(1) \int_0^T \left\{ \liminf_{h\to 0+} \frac{1}{h} \|S_h u(t) - u(t+h)\|_{L^1([a+\sigma^\star(t+h), b-\sigma^\star(t+h)]\,;\,\mathbb{R}^n)} \right\} dt.\end{aligned}$$

Proof For ease of notation we set

$$\|\cdot\| = \|\cdot\|_{L^1([a+\sigma^\star(t+h), b-\sigma^\star(t+h)]\,;\,\mathbb{R}^n)}.$$

Observe that by finite speed of propagation, we can define $u(x,0)$ to be zero outside of $[a,b]$, and the Lipschitz continuity of the semigroup will look identical written in the norm $\|\cdot\|$ to how it looked before. Let

$$\phi(t) = \liminf_{h\to 0+} \frac{1}{h} \|u(t+h) - S_h u(t)\|.$$

Note that ϕ is measurable, and for all $h > 0$, the function

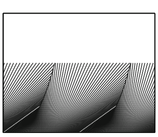
$$\phi_h(t) = \frac{1}{h} \|u(t+h) - S_h u(t)\|$$

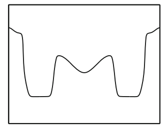
is continuous. Hence we have that

$$\phi(t) = \lim_{\varepsilon\to 0+} \inf_{h\in\mathbb{Q}\cap[0,\varepsilon]} \phi_h(t),$$

and therefore ϕ is Borel measurable. Define functions

$$\begin{aligned}\Psi(t) &= \|S_{T-t} u(t) - S_T u(0)\|, \\ \psi(t) &= \Psi(t) - L \int_0^t \phi(s)\, ds. \end{aligned} \tag{7.105}$$





The function ψ is a Lipschitz function, and hence

$$\psi(T) = \int_0^T \psi'(s)\,ds. \tag{7.106}$$

Furthermore, Rademacher's theorem[2] implies that there exists a null set $\mathcal{N}_1 \subseteq [0,T]$ such that Ψ and ψ are differentiable outside $\mathcal{N}_1$. Furthermore, using that Lebesgue measurable functions are approximately continuous almost everywhere (see [64, p. 47]), we conclude that there exists another null set $\mathcal{N}_2$ such that ϕ is continuous outside $\mathcal{N}_2$. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$. Outside $\mathcal{N}$ we have

$$\psi'(t) = \lim_{h\to 0} \frac{1}{h}\big(\Psi(t+h) - \Psi(t)\big) - L\psi(t). \tag{7.107}$$

Using properties of the semigroup we infer

$$\begin{aligned}
\Psi(t+h) - \Psi(t) &= \|S_{T-t-h}u(t+h) - S_T u(0)\| - \|S_{T-t}u(t) - S_T u(0)\| \\
&\le \|S_{T-t-h}u(t+h) - S_{T-t}u(t)\| \\
&= \|S_{T-t-h}u(t+h) - S_{T-t-h}S_h u(t)\| \\
&\le L\,\|u(t+h) - S_h u(t)\|,
\end{aligned}$$

which implies

$$\lim_{h\to 0} \frac{1}{h}\big(\Psi(t+h) - \Psi(t)\big) \le L \liminf_{h\to 0} \frac{1}{h}\,\|u(t+h) - S_h u(t)\| = L\phi(t).$$

Thus $\psi' \le 0$ almost everywhere, and we conclude that

$$\psi(T) \le 0, \tag{7.108}$$

which proves the lemma. □

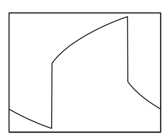
The next two lemmas are technical results valid for functions satisfying (7.97) and (7.98).

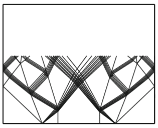
Lemma 7.16 *Assume that $u\colon [0,T] \to \mathcal{D}$ is Lipschitz continuous, and that for some (x,t) equations (7.97) and (7.98) hold. Then for all positive α we have*

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_0^\alpha |u(x+\lambda h+\rho y, t+h) - u_r|\,dy = 0, \tag{7.109}$$

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_{-\alpha}^0 |u(x+\lambda h+\rho y, t+h) - u_l|\,dy = 0. \tag{7.110}$$

[2] Rademacher's theorem states that a Lipschitz function is differentiable almost everywhere; see [64, p. 81].





Proof We assume that the limit in (7.109) is not zero. Then there exist sequences $\rho_i \to 0$ and $|h_i| < \rho_i$ and a $\delta > 0$ such that

$$\int_0^\alpha |u\left(x + \lambda h_i + \rho_i y, t + h_i\right) - u_r|\; dy > \delta \tag{7.111}$$

for all i. Without loss of generality we assume that $h_i > 0$, and let

$$v(z, h) = u(x + \lambda h + z, t + h).$$

Then the map $h \mapsto v(\cdot, h)$ is Lipschitz continuous with respect to the L^1 norm, since

$$\begin{aligned}\|v(\cdot, h) - v(\cdot, \eta)\|_1 &= \int |u(z, t + h) - u(\lambda(\eta - h) + z, t + \eta)|\; dz \\ &\le \int |u(z, t + h) - u(z, t + \eta)|\; dz \\ &\quad + \int |u(z, t + \eta) - u(\lambda(\eta - h) + z, t + \eta)|\; dz \\ &\le M\,|h - \eta| + \lambda\,|\eta - h|\,\mathrm{T.V.}\,(u(t + \eta)) \\ &\le \widetilde{M}\,|\eta - h|\,.\end{aligned}$$

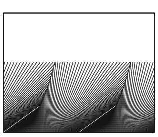
From (7.111) we obtain

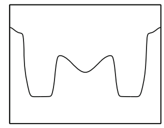
$$\begin{aligned}&\int_0^{\alpha\rho_i} |u(x + \lambda h + z, t + h) - u_r|\; dz \\ &\quad\ge \int_0^{\alpha\rho_i} |u\left(x + \lambda h_i + z, t + h_i\right) - u_r|\; dz \\ &\qquad - \int_0^{\alpha\rho_i} |u\left(x + \lambda h_i + z, t + h_i\right) - u(x + \lambda h + z, t + h)|\; dz \\ &\quad\ge \delta\rho_i - \widetilde{M}\,|h_i - h|\,.\end{aligned}$$

We can (safely) assume that $\delta/\widetilde{M} < 1$ (if this is not so, then (7.111) will hold for smaller δ as well). We integrate the last inequality with respect to h, for h in $[-\rho_i, \rho_i]$. Since $\left[h_i - \rho_i\delta/\widetilde{M}, h_i\right] \subset [-\rho_i, \rho_i]$, we obtain

$$\begin{aligned}\int_{-\rho_i}^{\rho_i}\int_0^{\alpha\rho_i} |u(x + \lambda h + z, t + h) - u_r|\; dz\, dh &\ge \int_{h_i - \rho_i\delta/\widetilde{M}}^{h_i} \left(\delta\rho_i - \widetilde{M}(h_i - h)\right) dh \\ &= (\delta^2\rho_i^2)/(2\widetilde{M}).\end{aligned}$$

Comparing this with (7.97) and (7.98) yields a contradiction. The limit (7.111) is proved similarly. □





For the next lemma, recall that a (signed) Radon measure is a (signed) regular Borel measure[3] that is finite on compact sets.

Lemma 7.17 *Assume that w is in $L^1((a,b);\mathbb{R}^n)$ such that for some Radon measure μ, we have that*

$$\left|\int_{x_1}^{x_2} w(x)\,dx\right| \le \mu([x_1,x_2]) \quad \text{for all } a < x_1 < x_2 < b. \tag{7.112}$$

Then

$$\int_a^b |w(x)|\,dx \le \mu((a,b)). \tag{7.113}$$

Proof First observe that the assumptions of the lemma also hold if the closed interval on the right-hand side of (7.112) is replaced by an open interval. We have that

$$\begin{aligned}\left|\int_{x_1}^{x_2} w(x)\,dx\right| &= \lim_{\varepsilon\to 0}\left|\int_{x_1+\varepsilon}^{x_2-\varepsilon} w(x)\,dx\right| \\ &\le \lim_{\varepsilon\to 0}\mu([x_1+\varepsilon,x_2-\varepsilon]) = \mu((x_1,x_2)).\end{aligned}$$

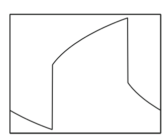
Secondly, since w is in L^1, it can be approximated by piecewise constant functions. Let v be a piecewise constant function with discontinuities located at $a = x_0 < x_1 < \cdots < x_N = b$, and

$$\int_a^b |w(x)-v(x)|\,dx \le \varepsilon.$$

Then we have

$$\begin{aligned}\int_a^b |w(x)|\,dx &\le \int_a^b |w(x)-v(x)| + \int_a^b |v(x)|\,dx \\ &\le \varepsilon + \sum_i \int_{x_{i-1}}^{x_i} |v(x)|\,dx \\ &= \varepsilon + \sum_i \left|\int_{x_{i-1}}^{x_i} v(x)\,dx\right|\end{aligned}$$

[3] A Borel measure μ is regular if it is outer regular on all Borel sets (i.e., $\mu(B) = \inf\{\mu(A) \mid A \supseteq B, A \text{ open}\}$ for all Borel sets B) and inner regular on all open sets (i.e., $\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}$ for all open sets U).



$$
\begin{aligned}
&\leq \varepsilon + \sum_i \left| \int\limits_{x_{i-1}}^{x_i} (v(x) - w(x))\, dx \right| + \sum_i \left| \int\limits_{x_{i-1}}^{x_i} w(x)\, dx \right| \\
&\leq \varepsilon + \int\limits_a^b |v(x) - w(x)|\, dx + \sum_i \mu\left((x_{i-1}, x_i)\right) \\
&\leq 2\varepsilon + \mu\left((a, b)\right).
\end{aligned}
$$

Since ε can be made arbitrarily small, this proves the lemma. □

Next we need two results that state how well the semigroup is approximated firstly by the solution of a Riemann problem with states that are close to the initial state for the semigroup, and secondly by the solution of the linearized equation. To define this precisely, let ω_0 be a function in $\mathcal{D}$, fix a point x on the real line (which will remain fixed throughout the next lemma and its proof), and let $\omega(y, t)$ be the solution of the Riemann problem

$$
\omega_t + f(\omega)_y = 0, \qquad \omega(y, 0) = \begin{cases} \omega_0(x-) & \text{for } y < 0, \\ \omega_0(x+) & \text{for } y \geq 0. \end{cases}
$$

(If ω_0 is continuous at x, then $\omega(y, t) = \omega_0(x)$ is constant.) Define $\tilde{A} = df(\omega_0(x+))$, and let $\tilde{u}$ be the solution of the linearized equation

$$
\tilde{u}_t + \tilde{A}\tilde{u}_y = 0, \qquad \tilde{u}(y, 0) = \omega_0(y). \tag{7.114}
$$

Furthermore, define $\hat{u}(y, t)$ by

$$
\hat{u}(y, t) = \begin{cases} \omega(y - x, t) & \text{for } |y - x| \leq \sigma^\star t, \\ \omega_0(y) & \text{otherwise.} \end{cases} \tag{7.115}
$$

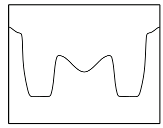
Then we can state the following lemma.

Lemma 7.18 *Let $\omega_0 \in \mathcal{D}$. Then we have*

$$
\frac{1}{h} \int\limits_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} \left| (S_h \omega_0)(y) - \hat{u}(y, h) \right| \, dy = \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right), \tag{7.116}
$$

$$
\frac{1}{h} \int\limits_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} \left| (S_h \omega_0)(y) - \tilde{u}(y, h) \right| \, dy = \mathcal{O}(1) \left(\mathrm{T.V.}_{(x-\rho,x+\rho)}\left(\omega_0\right)\right)^2, \tag{7.117}
$$

for all x and all positive h and ρ such that $x - \rho + h\sigma^\star < x + \rho - h\sigma^\star$.



Proof We first prove (7.117). In the proof of this we shall need the following general result:

Let $\bar{v}$ be the solution of $\bar{v}_t + f(\bar{v})_y = 0$ with Riemann initial data

$$\bar{v}(y,0) = \begin{cases} u_l & \text{for } y < 0, \\ u_r & \text{for } y \geq 0, \end{cases}$$

for some states $u_{l,r} \in \Omega$. We have that this Riemann problem is solved by waves separating constant states $u_l = v_0, v_1, \ldots, v_n = u_r$. Let u^c be a constant in Ω and set $A^c = df(u^c)$. Assume that u_l and u_r satisfy

$$A^c\,(u_l - u_r) = \lambda_k^c\,(u_l - u_r)\,;$$

i.e., λ_k^c is the kth eigenvalue and $u_l - u_r$ is the kth eigenvector of A^c. Let $\tilde{v}$ be defined by

$$\tilde{v}(y,t) = \begin{cases} u_l & \text{for } y < \lambda_k^c t, \\ u_r & \text{for } y \geq \lambda_k^c t \end{cases}$$

($\tilde{v}$ solves $u_t + A^c u_y = 0$ with a single jump at $y = 0$ from u_l to u_r as initial data). We wish to estimate

$$I = \frac{1}{t}\int_{-\sigma^\star t}^{\sigma^\star t} |\bar{v}(y,t) - \tilde{v}(y,t)|\,dy.$$

Note that since $\bar{v}$ and $\tilde{v}$ are equal outside the range of integration, the limits in the integral can be replaced by $\mp\infty$.

Due to the hyperbolicity of the system, the vectors $\{r_j(u)\}_{j=1}^n$ form a basis in $\mathbb{R}^n$, and hence we can find unique numbers $\bar{\varepsilon}_j^{l,r}$ such that

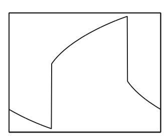
$$u_r - u_l = \sum_{j=1}^{n} \bar{\varepsilon}_j^l r_j(u_l) = \sum_{j=n}^{1} \bar{\varepsilon}_j^r r_j(u_r). \tag{7.118}$$

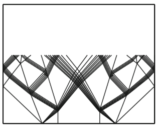
From $u_r - u_l = \varepsilon^c r_k(u^c)$ for some ε^c it follows that

$$\begin{aligned} \bar{\varepsilon}_i^l &= l_i(u_l)\cdot \sum_{j=1}^{n} \bar{\varepsilon}_j^l r_j(u_l) \\ &= l_i(u_l)\cdot(u_l - u_r) \\ &= (l_i(u_l) - l_i(u^c))\cdot(u_l - u_r) + l_i(u^c)\cdot(u_l - u_r) \\ &= (l_i(u_l) - l_i(u^c))\cdot(u_l - u_r) + \varepsilon^c l_i(u^c)\cdot r_k(u^c) \\ &= (l_i(u_l) - l_i(u^c))\cdot(u_l - u_r), \quad i \neq k. \end{aligned}$$

Thus we conclude (using an identical argument for the right state) that

$$\begin{aligned} \left|\bar{\varepsilon}_i^l\right| &\leq C\ |u_l - u_r|\ |u_l - u^c|\,, \quad i \neq k, \\ \left|\bar{\varepsilon}_i^r\right| &\leq C\ |u_l - u_r|\ |u_r - u^c|\,, \quad i \neq k. \end{aligned} \tag{7.119}$$





Let ε_i denote the strength of the ith wave in $\bar{v}$. Then, by construction of the solution of the Riemann problem, for $i < k$ we have that

$$\left|\varepsilon_i - \bar{\varepsilon}_i^l\right| \le C \left(|v_{i-1} - u_l|^2 + |v_i - u_l|^2\right) \le C\, |u_l - u_r|^2,$$

while for $i > k$ we find that

$$|\varepsilon_i - \bar{\varepsilon}_i^r| \le C\, |u_l - u_r|^2,$$

for some constant C. Assume that the k-wave in $\bar{v}$ moves with speed in the interval $[\underline{\lambda}_k, \bar{\lambda}^k]$; i.e., if the k-wave is a shock, then $\underline{\lambda}_k = \bar{\lambda}^k = \mu_k\,(v_{k-1}, v_k)$, and if the wave is a rarefaction wave, then $\underline{\lambda}_k = \lambda_k(v_{k-1})$ and $\bar{\lambda}_k = \lambda_k(v_k)$. Set $\underline{s} = \min(\underline{\lambda}_k, \tilde{\lambda}_k)$ and $\bar{s} = \max(\bar{\lambda}_k, \tilde{\lambda}_k)$. Then we can write I as

$$\begin{aligned} I &= \frac{1}{t}\Bigg(\int_{-\infty}^{\underline{s}} |u_l - \bar{v}(y,t)|\; dy \\ &\qquad + \int_{\underline{s}}^{\bar{s}} |\hat{v}(y,t) - \bar{v}(y,t)|\; dy + \int_{\bar{s}}^{\infty} |u_r - \bar{v}(y,t)|\; dy\Bigg) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Next we note that the first integral above can be estimated as

$$I_1 \le C \sum_{i=1}^{k-1} |v_i - u_l| \le C \sum_{i=1}^{k-1} |\varepsilon_i| \le C \left(\sum_{i=1}^{k-1} \left|\bar{\varepsilon}_i^l\right| + |u_r - u_l|^2\right),$$

and similarly,

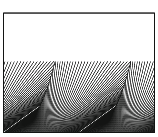
$$I_3 \le C \left(\sum_{i=k+1}^{n} |\bar{\varepsilon}_i^r| + |u_l - u_r|^2\right).$$

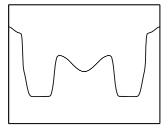
Using (7.119), we obtain

$$\begin{aligned} I_1 + I_3 &\le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c| + |u_l - u_r|\right) \\ &\le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c|\right), \end{aligned} \tag{7.120}$$

for some constant C. It remains to estimate I_2. We first assume that the k-wave in $\bar{v}$ is a shock wave and that $\lambda_k^c > \mu_k(v_{k-1}, v_k)$. Then

$$\begin{aligned} I_3 &= \left(\lambda_k^c - \mu_k\,(v_{k-1}, v_k)\right) |u_l - v_k| \\ &\le C\, |u_l - v_k| \left(|u^c - v_{k-1}| + |u^c - v_k|\right) \\ &\le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c| + |v_k - u_r| + |v_{k-1} - u_l|\right), \\ &\le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c| + C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c|\right)\right) \\ &\le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c|\right) \end{aligned} \tag{7.121}$$





by the above estimates for $|v_k - u_r|$ and $|v_{k-1} - u_l|$. If $\lambda_k^c \le \mu_k(v_{k-1}, v_k)$ or the k-wave is a rarefaction wave, we similarly establish (7.121). Combining this with (7.120), we find that

$$I \le C\, |u_l - u_r| \left(|u_l - u^c| + |u_r - u^c|\right). \tag{7.122}$$

Having established this preliminary estimate, we turn to the proof of (7.117). Let $\bar{\omega}_0$ be a piecewise constant approximation to ω_0 such that

$$\begin{gathered} \bar{\omega}_0(x\pm) = \omega_0(x\pm), \quad \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \le \epsilon, \\ \mathrm{T.V.}_{(x-\rho,x+\rho)}(\bar{\omega}_0) \le \mathrm{T.V.}_{(x-\rho,x+\rho)}(\omega_0). \end{gathered} \tag{7.123}$$

Furthermore, let v be the solution of the linear hyperbolic problem

$$v_t + \tilde{A} v_y = 0, \quad v(y,0) = \bar{\omega}_0(y),$$

where again $\tilde{A} = df(\omega_0(x+))$. Let the eigenvalues and the right and left eigenvectors of $\tilde{A}$ be denoted by $\tilde{\lambda}_k$, $\tilde{r}_k$, and $\tilde{l}_k$, respectively, for $k = 1, \dots, n$, normalized so that

$$\left|\tilde{l}_k\right| = 1, \quad \tilde{l}_k \cdot \tilde{r}_j = \begin{cases} 0 & \text{for } j \ne k, \\ 1 & \text{for } j = k. \end{cases} \tag{7.124}$$

Then it is not too difficult to verify (see Sect. 1.1) that $v(y,t)$ is given by

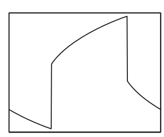
$$v(y,t) = \sum_k \left(\tilde{l}_k \cdot \bar{\omega}_0(y - \tilde{\lambda}_k t)\right) \tilde{r}_k. \tag{7.125}$$

We can also construct v by front tracking. Since the eigenvalues are constant and the initial data piecewise constant, front tracking will give the exact solution. Hence v will be piecewise constant with a finite number of jumps occurring at $x_i(t)$, where we have that

$$\begin{gathered} \frac{d}{dt} x_i(t) = \tilde{\lambda}_k, \\ \left(\tilde{A} - \tilde{\lambda}_k I\right)\left(v(x_i(t)+, t) - v(x_i(t)-, t)\right) = 0, \end{gathered}$$

for all t where we do not have a collision of fronts, that is, for all but a finite number of t's. Now we apply the estimate (7.122) to each individual front x_i. Then we obtain, introducing $v_i^{\pm} = v(x_i(t)\pm, t)$,

$$\begin{aligned} &\int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} |(S_\varepsilon v(\,\cdot\,,\tau))(y) - v(y, \tau+\varepsilon)|\, dy \\ &\quad \le \varepsilon\, \mathcal{O}(1) \sum_i \left|v_i^+ - v_i^-\right| \left(\left|v_i^+ - \omega_0(x+)\right| + |v_i^- - \omega_0(x+)|\right) \\ &\quad \le \varepsilon\, \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x+\rho)}(\bar{\omega}_0) \sum_i \left|v_i^+ - v_i^-\right| \\ &\quad \le \varepsilon\, \mathcal{O}(1) \left(\mathrm{T.V.}_{(x-\rho,x+\rho)}(\omega_0)\right)^2. \end{aligned} \tag{7.126}$$



Recall that $\tilde{A} = df(\omega_0(x+))$ and that $\tilde{u}$ was defined by (7.114), that is,

$$\tilde{u}_t + \tilde{A}\tilde{u}_y = 0, \qquad \tilde{u}(y,0) = \omega_0(y). \tag{7.127}$$

In analogy to formula (7.125) we have that $\tilde{u}$ satisfies

$$\tilde{u}(y,t) = \sum_k \big(\tilde{l}_k \cdot \omega_0(y - \tilde{\lambda}_k t)\big)\tilde{r}_k. \tag{7.128}$$

Regarding the difference between $\tilde{u}$ and v, we find that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |v(y,h) - \tilde{u}(y,h)|\, dy \tag{7.129}$$

$$\begin{aligned}
&= \int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} \left|\sum_k \Big(\tilde{l}_k \cdot (\bar{\omega}_0 - \omega_0)\,(y - \tilde{\lambda}_k h)\Big)\,\tilde{r}_k\right| dy \\
&\le \mathcal{O}(1) \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \\
&\le \mathcal{O}(1)\,\epsilon.
\end{aligned} \tag{7.130}$$

By the Lipschitz continuity of the semigroup we have that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |S_h\bar{\omega}_0(y) - S_h\omega_0(y)|\, dy \le L \int_{x-\rho}^{x+\rho} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \le L\epsilon. \tag{7.131}$$

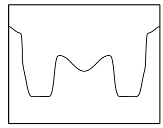
Furthermore, by Lemma 7.15 with $a = x - \rho$, $b = x + \rho$, $T = h$, and $t = 0$, and using (7.126), we obtain

$$\begin{aligned}
&\frac{1}{h} \int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |(S_h\bar{\omega}_0)\,(y) - v(y,h)|\, dy \\
&\le \frac{\mathcal{O}(1)}{h} \int_0^h \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} |(S_\varepsilon v\,(\,\cdot\,,\tau))\,(y) - v(y,\tau+\varepsilon)|\, dy\, d\tau \\
&\le \mathcal{O}(1)\,\big(\mathrm{T.V.}_{(x-\rho,x+\rho)}\,(\omega_0)\big)^2.
\end{aligned} \tag{7.132}$$

Consequently, using (7.132), (7.131), and (7.130), we find that

$$\begin{aligned}
&\frac{1}{h} \int_{x-\rho+h\sigma^\star}^{x+\rho-h\sigma^\star} |(S_h\omega_0)\,(y) - \tilde{u}(y,h)|\, dy \\
&\qquad \le \mathcal{O}(1)\,\big(\mathrm{T.V.}_{(x-\rho,x+\rho)}\,(\omega_0)\big)^2 + \frac{L\epsilon}{h} + \mathcal{O}(1)\,\frac{\epsilon}{h}.
\end{aligned}$$

Since ϵ is arbitrary, this proves (7.117).



Now we turn to the proof of (7.116). First we define z to be the function

$$z(y,t) = \begin{cases} u_l & \text{for } y < \lambda t, \\ u_r & \text{for } y \geq \lambda t, \end{cases}$$

where $|\lambda| \leq \sigma^\star$. Recall that $\bar{v}(y,t)$ denotes the solution of $\bar{v}_t + f(\bar{v})_y = 0$ with Riemann initial data

$$\bar{v}(y,0) = \begin{cases} u_l & \text{for } y < 0, \\ u_r & \text{for } y \geq 0. \end{cases}$$

Then trivially we have that

$$\int_{-\sigma^\star t}^{\sigma^\star t} |z(y,t) - \bar{v}(y,t)|\, dy \leq t\, \mathcal{O}(1)\, |u_l - u_r|\,. \tag{7.133}$$

Let $\bar{\omega}_0$ be as (7.123) but replacing the TV bound by

$$\mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\bar{\omega}_0\right) \leq \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right).$$

Recall that $\hat{u}(y,t)$ was defined in (7.115) by

$$\hat{u}(y,t) = \begin{cases} \omega(y-x,t) & \text{for } |y-x| \leq \sigma^\star t, \\ \omega_0(y) & \text{otherwise.} \end{cases}$$

Let J_h be the set

$$J_h = \left\{ y \mid h\sigma^\star < |y-x| < \rho - h\sigma^\star \right\},$$

and let $\hat{v}$ be the function defined by

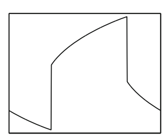
$$\hat{v}(y,t) = \begin{cases} \hat{u}(y,t) & \text{for } |x-y| \leq \sigma^\star t, \\ \bar{\omega}_0(y) & \text{otherwise.} \end{cases}$$

Then we have that

$$\int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} |\hat{v}(y,h) - \hat{u}(y,h)|\, dy \leq \int_{J_h} |\bar{\omega}_0(y) - \omega_0(y)|\, dy \leq \epsilon. \tag{7.134}$$

Note that the bound (7.131) remains valid. We need a replacement for (7.126). In this case we wish to estimate

$$I = \int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} \left|\left(S_\varepsilon v(\,\cdot\,,\tau)\right)(y) - \bar{v}(y,\tau+\varepsilon)\right|\, dy.$$



For $|x-y| > \sigma^\star t$, the function $\bar{v}(y,t)$ is discontinuous across lines located at x_i. In addition, it may be discontinuous across the lines $|x-y| = \sigma^\star t$. Inside the region $|x-y| \le \sigma^\star t$, v is an exact entropy solution, coinciding with the semigroup solution. Using this and (7.133), we find that

$$\begin{aligned}
I &= \left(\int_{x-\rho+\sigma^\star\varepsilon}^{x-\sigma^\star\tau} + \int_{x+\sigma^\star\tau}^{x+\rho-\sigma^\star\varepsilon} \right) \left| \left(S_{\tau+\varepsilon}\bar{\omega}_0\right)(y) - \bar{\omega}_0(y) \right| \, dy \\
&\quad + \int_{x-\sigma^\star\tau}^{x+\sigma^\star\tau} \left| \left(S_\varepsilon \hat{u}(\,\cdot\,,\tau)\right)(y) - \hat{u}(y,\tau+\varepsilon) \right| \, dy \\
&\le \varepsilon\, \mathcal{O}(1) \left(\sum_{|x_i - x| < \sigma^\star\tau} \left| \bar{\omega}_0(x_i+) - \bar{\omega}_0(x_i-) \right| \right) \\
&\quad + L\left(\int_{x-2\sigma^\star\tau}^{x} \left| \bar{\omega}_0(y) - u_l \right| \, dy + \int_{x}^{x+2\sigma^\star\tau} \left| \bar{\omega}_0(y) - u_r \right| \, dy \right) \\
&\le \varepsilon\, \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right). \qquad (7.135)
\end{aligned}$$

Now using Lemma 7.15, we find that

$$\begin{aligned}
&\frac{1}{h} \int_{x-\rho+\sigma^\star h}^{x+\rho-\sigma^\star h} \left| \left(S_h\bar{\omega}_0\right)(y) - \bar{v}(y,t) \right| \, dy \\
&\quad \le \frac{\mathcal{O}(1)}{h} \int_0^h \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{x-\rho+\sigma^\star\varepsilon}^{x+\rho-\sigma^\star\varepsilon} \left| \left(S_\varepsilon \bar{v}(\,\cdot\,,\tau)\right)(y) - \bar{v}(y,\tau+\varepsilon) \right| \, dy \, d\tau \\
&\quad \le \mathcal{O}(1)\, \mathrm{T.V.}_{(x-\rho,x)\cup(x,x+\rho)}\left(\omega_0\right). \qquad (7.136)
\end{aligned}$$

As before, since ϵ is arbitrary, (7.131), (7.134), and (7.136) imply (7.116). $\square$

Remark 7.19 Note that if ω_0 is continuous at x, then Lemma 7.18 and (7.117) say that the linearized equation gives a good local approximation of the action of the semigroup. If ω_0 has a discontinuity at x, then

$$\mathrm{T.V.}_{\langle x-\rho,x+\rho\rangle}\left(\omega_0\right)$$

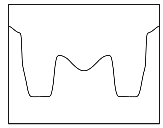
does not become small as ρ tends to zero; hence we must resort to (7.116) in this case. Since the total variation of every function in $\mathcal{D}$ is small, (7.117) is a much stronger estimate than (7.116).

Now that the preliminary technicalities are out of the way, we can set about proving that an entropy solution coincides with the semigroup.

Let u be an entropy solution. To prove that $u(\,\cdot\,,t) = S_t u_0$, it suffices to show, applying Lemma 7.15, that

$$\liminf_{h\to 0} \frac{1}{h} \left\| S_h u(\,\cdot\,,t) - u(\,\cdot\,,t+h) \right\|_{L^1([a,b])} = 0, \qquad (7.137)$$

for all $a < b$, and for all $t \in [0,T] \setminus \mathcal{N}$.



Assume therefore that $t \notin \mathcal{N}$. Then by the structure theorem, see [193, Theorem 5.9.6], there exists a null set $\mathcal{N} \subset [0, T]$ such that outside that set, u either is continuous or has a jump discontinuity (as a function of x). Therefore, we split the argument into two cases, one in which u has a jump discontinuity, and one in which u is continuous or has a small jump in the sense that it is not in the set $B_{t,\varepsilon}$.

Consider first a point (x, t) where u has jump discontinuity.[4] By condition **B** there exist $u_{l,r} \in \Omega$ and σ such that the limit (7.98) holds when U is defined by (7.97). Using a change of variables, we find that

$$\begin{aligned}
&\lim_{h\to 0+} \frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |u(y,t+h) - U(y,t+h)|\, dy \\
&\qquad = \lim_{h\to 0+} \sigma^\star \Bigg[\int_{-1-\lambda/\sigma^\star}^{0} |u\,(x+\lambda h+\sigma^\star h y, t+h) - u_l|\, dy \\
&\qquad\qquad + \int_{0}^{1-\lambda/\sigma^\star} |u\,(x+\lambda h+\sigma^\star h y, t+h) - u_r|\, dy \Bigg] = 0,
\end{aligned}$$

by Lemma 7.16. Hence for small positive h, we have that

$$\frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |u(y,t+h) - U(y,t+h)|\, dy \le \tilde{\varepsilon}, \tag{7.138}$$

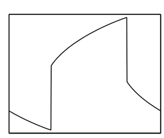
for some small $\tilde{\varepsilon}$ to be determined later. By Lemma 7.14 we have $U(y, s) = \hat{u}(y, s-t)$, where $\hat{u}$ is defined by (7.115) with $\omega_0(y) = u(y, t)$, and U is defined by (7.97), in some forward neighborhood of (x, t). Then using (7.138) and (7.116), we obtain

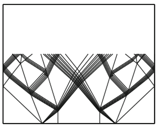
$$\begin{aligned}
&\frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y,t+h)|\, dy \\
&\qquad \le \tilde{\varepsilon} + \frac{1}{h} \int_{x-\sigma^\star h}^{x+\sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - U(y,t+h)|\, dy \\
&\qquad \le \tilde{\varepsilon} + \mathcal{O}\,(1)\, \mathrm{T.V.}_{(x-2\sigma^\star h,x)\cup(x,x+2\sigma^\star h)}\,(u(\,\cdot\,,t)) \\
&\qquad \le 2\tilde{\varepsilon},
\end{aligned} \tag{7.139}$$

for all h sufficiently small, since we compute the total variation on a shrinking interval excluding the jump in u at x.

Now we consider points (x, t) where u either is continuous or has a small jump discontinuity. Hence we can choose an interval $\langle c, d \rangle$ centered at x such that $B_{t,\varepsilon} \cap$

[4] The following argument is valid for every jump discontinuity, but will be applied only to jumps in $B_{t,\varepsilon}$.





$(c, d) = \emptyset$. Recall that $B_{t,\varepsilon}$, defined in (7.102), is the set of points where $u(\cdot\,, t)$ has a jump larger than ε. Let the family of trapezoids Γ_h be defined by

$$\Gamma_h = \left\{(y, s) \mid s \in [t, t+h],\ y \in (c + \sigma^\star(s-t), d - \sigma^\star(s-t))\right\}.$$

Now we claim that for h sufficiently small, we have that for all $(y, s) \in \Gamma_h$,

$$|u(y, s) - u(x, t)| \le 2\varepsilon + \mathrm{T.V.}_{(c,d)}\left(u(\cdot\,, t)\right). \tag{7.140}$$

To prove this, we argue as follows: By Lemma 7.14, discontinuities in u cannot propagate faster than $\sigma^\star$; hence $u(\cdot\,, t)$ is continuous in the lower corners of Γ_h, and the estimate surely holds for (y, s) located there. We must prove (7.140) for (y, s) in a region $[c+h', d-h'] \times [t, t+h]$, where h' is given and we can be free to choose h small. Now also $[c+h', d-h'] \cap B_{\varepsilon,t} = \emptyset$; hence for each $y \in [c+h', d-h']$ we can find ξ_y, h_y such that the estimate (7.140) is valid for

$$(y, s) \in (y - \xi_y, y + \xi_y) \times \left[t, t + h_y\right].$$

Now we can cover the compact interval $[c+h', d-h']$ with a finite number of intervals of the form $(y_i - \xi_{y_i}, y_i + \xi_{y_i})$, and choose

$$h = \min_i h_{y_i}.$$

Then we obtain (7.140) for (y, s) in $[c+h', d-h'] \times [t, t+h]$.

Now we must compare u and $\tilde{u}$ near (x, t). The eigenvectors of $\tilde{A} = df(u(x, t))$ are normalized according to (7.124). Observe that trivially

$$u = \sum_k \left(\tilde{l}_k \cdot u\right) \tilde{r}_k.$$

Then

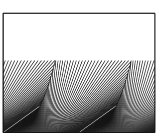
$$\begin{aligned}&\int_{c+\sigma^\star h}^{d-\sigma^\star h} |u(y, t+h) - \tilde{u}(y, t+h)|\, dy \\ &\qquad \le \sum_k \int_{c+\sigma^\star h}^{d-\sigma^\star h} \left|\tilde{l}_k \cdot \left(u(y - \tilde{\lambda}_k h, t) - u(y, t+h)\right)\right| dy.\end{aligned} \tag{7.141}$$

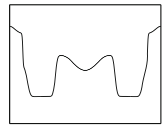
To aid us here we use Lemma 7.17. Let $x_1 < x_2$ be in the interval $(c+\sigma^\star h, d-\sigma^\star h)$. Then we shall estimate

$$E_k = \int_{x_1}^{x_2} \tilde{l}_k \cdot \left(u(y, t+h) - u(y - \tilde{\lambda}_k h, t)\right) dy.$$

If we integrate the conservation law over the region

$$\left\{(y, s) \mid y \in [x_1 - (s - (t+h))\tilde{\lambda}_k, x_2 + (s - (t+h))\tilde{\lambda}_k],\ s \in [t, t+h]\right\},$$





we find that

$$\int_{x_1}^{x_2} u(y,t+h)\,dy - \int_{x_1-\tilde{\lambda}_k h}^{x_2+\tilde{\lambda}_k h} u(y,t)\,dy + \int_t^{t+h} (f(u)-\tilde{\lambda}_k u)(x_2+(s-(t+h))\tilde{\lambda}_k, s)\,ds$$
$$- \int_t^{t+h} (f(u)-\tilde{\lambda}_k u)(x_1-(s-(t+h))\tilde{\lambda}_k, s)\,ds = 0.$$

Taking the inner product with $\tilde{l}_k$, we obtain

$$\begin{aligned} E_k &= \int_t^{t+h} \tilde{l}_k \cdot (f(u)-\tilde{\lambda}_k u)(x_2+(s-(t+h))\tilde{\lambda}_k, s)\,ds \\ &\quad - \int_t^{t+h} \tilde{l}_k \cdot (f(u)-\tilde{\lambda}_k u)(x_1-(s-(t+h))\tilde{\lambda}_k, s)\,ds \\ &= \int_t^{t+h} \tilde{l}_k \cdot \big(f(u_2)-f(u_1)-\tilde{\lambda}_k(u_2-u_1)\big)\,ds, \end{aligned} \tag{7.142}$$

where we have defined

$$u_1 = u\big(x_1-(s-(t+h))\tilde{\lambda}_k, s\big), \quad u_2 = u\big(x_2+(s-(t+h))\tilde{\lambda}_k, s\big).$$

Let $A^\star$ denote the matrix

$$A^\star = \int_0^1 df\left(su_2+(1-s)u_1\right)ds - \tilde{A}.$$

Then

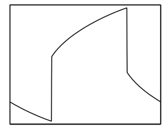
$$\begin{aligned} \tilde{l}_k \cdot \big(f(u_2)-f(u_1)-\tilde{\lambda}_k(u_2-u_1)\big) &= \tilde{l}_k \cdot \big(A^\star(u_2-u_1) \\ &\qquad + \tilde{A}(u_2-u_1) - \tilde{\lambda}_k\,(u_2-u_1)\big) \\ &= \tilde{l}_k \cdot A^\star(u_2-u_1). \end{aligned} \tag{7.143}$$

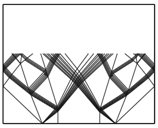
Since

$$\|A^\star\| \le \mathcal{O}(1)\left(|u_1-u(x,t)|+|u_2-u(x,t)|\right),$$

(7.142) and (7.143) yield

$$\begin{aligned} |E_k| &\le \mathcal{O}(1) \int_t^{t+h} \left(|u_1-u(x,t)|+|u_2-u(x,t)|\right)\,|u_2-u_1|\;ds \\ &\le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s)-u(x,t)| \\ &\qquad \times \int_t^{t+h} \mathrm{T.V.}_{(x_1-(s-(t+h))\tilde{\lambda}_k, x_2+(s-(t+h))\tilde{\lambda}_k)}\left(u(\,\cdot\,,s)\right)\;ds. \end{aligned}$$





Therefore,

$$
\begin{aligned}
&\left| \int_{x_1}^{x_2} \left(u(y,t+h) - \tilde{u}(y,t+h)\right)\,dy \right| \\
&\qquad \le \sum_k |E_k| \\
&\qquad \le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s) - u(x,t)| \\
&\qquad\quad \times \int_t^{t+h} \sum_k \mathrm{T.V.}_{(x_1-(s-(t+h))\tilde{\lambda}_k,\, x_2+(s-(t+h))\tilde{\lambda}_k)}\left(u(\,\cdot\,,s)\right)\,ds. \qquad (7.144)
\end{aligned}
$$

Returning to (7.141) and using Lemma 7.17, we find that

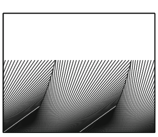
$$
\begin{aligned}
&\int_{c+\sigma^\star h}^{d-\sigma^\star h} \left|u(y,t+h) - \tilde{u}(y,t+h)\right|\,dy \\
&\qquad \le \mathcal{O}(1) \sup_{(y,s)\in\Gamma_h} |u(y,s) - u(x,t)| \qquad (7.145) \\
&\qquad\quad \times \int_t^{t+h} \mathrm{T.V.}_{[c+\sigma^\star(s-t),\, d-\sigma^\star(s-t)]}\left(u(\,\cdot\,,s)\right)\,ds.
\end{aligned}
$$

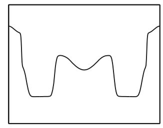
Now we use (7.117), (7.145), and (7.140) to obtain

$$
\begin{aligned}
&\int_{c+\sigma^\star h}^{d-\sigma^\star h} \left|\left(S_h u(\,\cdot\,,t)\right)(y) - u(y,t+h)\right|\,dy \\
&\quad \le \int_{c+\sigma^\star h}^{d-\sigma^\star h} \left(\left|\left(S_h u(\,\cdot\,,t)\right)(y) - \tilde{u}(y,t+h)\right| + \left|\tilde{u}(y,t+h) - u(y,t+h)\right| \right) dy \\
&\quad \le \mathcal{O}(1)\, h \left(\mathrm{T.V.}_{(c,d)}\left(u(\,\cdot\,,t)\right)\right)^2 \\
&\qquad + \mathcal{O}(1) \left(2\varepsilon + \mathrm{T.V.}_{[c,d]}\left(u(\,\cdot\,,t)\right)\right) \\
&\qquad \times \int_t^{t+h} \mathrm{T.V.}_{[c+\sigma^\star(s-t),\, d-\sigma^\star(s-t)]}\left(u(\,\cdot\,,s)\right)\,ds. \qquad (7.146)
\end{aligned}
$$

By Lemma 7.13, the set $B_{t,\varepsilon}$ contains only a finite number of points; $x_1 < x_2 < \cdots < x_N$, where $u(\,\cdot\,,t)$ has a discontinuity larger than ε. We can cover the set $[a,b] \setminus \cup_i \{x_i\}$ by a finite number of open intervals (c_j, d_j), $j = 1, \dots, M$, such that:

(**a**) $x_i \notin \cup_j (c_j, d_j) = \emptyset$ for $i = 1, \dots, N$.
(**b**) $\mathrm{T.V.}_{(c_j,d_j)}\left(u\,(\,\cdot\,,t)\right) \le 2\varepsilon$ for $j = 1, \dots, M$.
(**c**) Every $x \in [a,b]$ is contained in at most two distinct intervals (c_i, d_i).





We have established that for sufficiently small h,

$$\frac{1}{h} \int_{x_i - \sigma^\star h}^{x_i + \sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y, t+h)| \leq \frac{\varepsilon}{N},$$

by (7.139) choosing $\tilde{\varepsilon} = \varepsilon/(2N)$. Also,

$$\begin{aligned}
&\int_{c_j + \sigma^\star h}^{d_j - \sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y, t+h)|\ dy \\
&\qquad \leq \mathcal{O}(1)\,\varepsilon \int_t^{t+h} \mathrm{T.V.}_{(c_j + \sigma^\star (s-t), d_j - \sigma^\star (s-t))}\,(u(\,\cdot\,,s))\ ds \\
&\qquad\quad + \mathcal{O}(1)\,h\varepsilon \mathrm{T.V.}_{(c_j, d_j)}\,(u(\,\cdot\,,t))
\end{aligned}$$

for all i, j, and $\varepsilon > 0$. Combining this, we find that

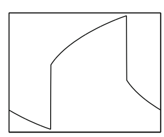
$$\begin{aligned}
&\frac{1}{h} \int_a^b |(S_h u(\,\cdot\,,t))\,(y) - u(y, t+h)|\ dy \\
&\qquad \leq \sum_i \frac{1}{h} \int_{x_i - \sigma^\star h}^{x_i + \sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y, t+h)|\ dy \\
&\qquad\quad + \sum_j \int_{c_j + \sigma^\star h}^{d_j - \sigma^\star h} |(S_h u(\,\cdot\,,t))\,(y) - u(y, t+h)|\ dy \\
&\qquad \leq \varepsilon + \mathcal{O}(1)\,\frac{\varepsilon}{h} \int_t^{t+h} \mathrm{T.V.}\,(u(\,\cdot\,,s))\ ds + \varepsilon \mathrm{T.V.}\,(u(\,\cdot\,,t)) \\
&\qquad \leq \mathcal{O}(1)\ \varepsilon.
\end{aligned}$$

Since ε can be arbitrarily small, (7.137) holds, and we have proved the following theorem:

Theorem 7.20 *Let $f_j \in C^2(\mathbb{R}^n)$, $j = 1, \ldots, n$. Consider the strictly hyperbolic equation $u_t + f(u)_x = 0$. Assume that each wave family is either genuinely nonlinear or linearly degenerate. For every $u_0 \in \mathcal{D}$, defined by (7.94), the initial value problem*

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x),$$

has a unique weak entropy solution satisfying conditions **A**–**C**, *see Sect. 7.2. Furthermore, this solution can be found by the front-tracking construction.*



7.3 Notes

The material in Sect. 7.1 is taken almost entirely from the fundamental result of Bressan, Liu, and Yang [33]; there is really only an $\mathcal{O}(|e|)$ difference.

Stability of front-tracking approximations to systems of conservation laws was first proved by Bressan and Colombo in [28], in which they used a pseudopolygon technique to "differentiate" the front-tracking approximation with respect to the initial location of the fronts. This approach was later used to prove stability for many special systems; see [47], [8], [3], [4].

The same results as those in Sect. 7.1 of this chapter have also been obtained by Bressan, Crasta, and Piccoli, using a variant of the pseudopolygon approach [29]. This leads to *many* technicalities, and [29] is heavy reading indeed!

The material in Sect. 7.2 is taken from the works of Bressan [23–26] and coworkers, notably Lewicka [32], Goatin [30], and LeFloch [31].

There are few earlier results on uniqueness of solutions to systems of conservation laws; most notable are those by Bressan [20], where uniqueness and stability are obtained for Temple class systems where every characteristic field is linearly degenerate, and in [22] for more general Temple class systems.

Continuity in L^1 with respect to the initial data was also proved by Hu and LeFloch [100] using a variant of Holmgren's technique. See also [77].

Stability for some non-strictly hyperbolic systems of conservation laws (these are really only "quasisystems") has been proved by Winther and Tveito [185] and Klingenberg and Risebro [114].

We end this chapter with a suitable quotation:

This is really easy:

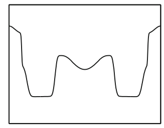
$$|\textit{what you have}| \leq |\textit{what you want}| + |\textit{what you have} - \textit{what you want}|$$

— Rinaldo Colombo, private communication

7.4 Exercises

7.1 Show that the solution of the Cauchy problem obtained by the front-tracking construction of Chapt. 6 is an entropy solution in the sense of conditions **A**–**C** in Sect. 7.2.

7.2 The proof of Theorem 7.8 was carried out in detail only in the genuinely nonlinear case. Do the necessary estimates in the case of a linearly degenerate wave family.



Chapter 8

Conservation Laws with Discontinuous Flux Functions

Of course it is happening inside your head, Harry,
but why on earth should that mean it is not real?
— Albus Dumbledore, in Harry Potter and the Deathly Hallows

The aim of this chapter is to give a brief introduction to scalar conservation laws with a space-dependent flux function, where the spatial dependence of the flux can have discontinuities. We shall restrict ourselves to one spatial dimension, both for reasons of simplicity and because the theory is more complete in one dimension.

In one spatial dimension, a conservation law with a space-dependent flux can be written

$$u_t + f(x,u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{8.1}$$

Since the interpretation of f is the flux of u at the point x, there are many applications where the flux depends on the location. We give some simple examples that are modeled by such conservation laws.

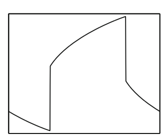
◇ Example 8.1
Traffic flow is a simple model in whcih conservation laws with space-dependent coefficients arise naturally. We refer to Example 1.6, and repeat the necessary notation here.

Let ρ denote the density of cars on a long "one way" road. We normalize units, so that $\rho = 1$ if the cars are packed bumper to bumper. Assume that the speed of the cars is a decreasing function of the density $v = v(\rho)$. The speed of the cars on an empty road ($\rho = 0$) is governed by the road conditions and the speed limits, so that $v(0) = \gamma$, where γ is a function of the position on the road. Furthermore, it is reasonable to assume that $v(1) = 0$. For simplicity we can then define v as $v(\rho) = \gamma(1-\rho)$. If the road conditions, and thereby γ, vary with the position x, then we end up with the conservation law

$$\rho_t + (\gamma(x)\rho(1-\rho))_x = 0, \tag{8.2}$$

which is an example of a conservation law with an x-dependent flux function. On the scale of continuum traffic, where the natural lengths are *many* times that of a single car, the road conditions often vary discontinuously. ◇

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2_8



◇ Example 8.2
When extracting oil from an oil reservoir, one often injects water in order to maintain the pressure, and thereby to force out more oil. Assuming that we have two phases, oil and water, present, the mass conservation of oil and water reads,

$$\phi s_t + u_x = 0 \quad \text{and} \quad \phi(1-s)_t - v_x = 0,$$

where (the unknown) s denotes the saturation, i.e., the fraction of the available pore space occupied by oil, and u and v are the Darcy velocities of oil and water respectively. The factor ϕ denotes the fraction of the void space in the material, commonly called the *porosity*. One often assumes that Darcy's law holds,

$$u = -k\lambda_{\text{oil}} P'_{\text{oil}} - g\rho_{\text{oil}} \quad \text{and} \quad v = -k\lambda_{\text{water}} P'_{\text{water}} - g\rho_{\text{water}},$$

where k denotes the absolute permeability of the medium, g the gravitational constant, λ_{phase} the mobility, P_{phase} the pressure, and ρ_{phase} the density. Here the subscript "phase" denotes either water or oil. If we assume that the two pressures are the same, and that the total velocity $q = u + v$ is constant (incompressibilty), we can add the two conservation equations to obtain

$$\phi s_t + \left(\frac{\lambda_{\text{oil}}(s)}{\lambda_{\text{oil}}(s) + \lambda_{\text{water}}(s)} \left(q - k(x) g \lambda_{\text{water}}(s)\Delta\rho\right) \right)_x = 0, \tag{8.3}$$

where $\Delta\rho = \rho_{\text{water}} - \rho_{\text{oil}}$. The absolute permeability of the rock is often modeled as a piecewise constant function of x, and therefore this is another example of a conservation law in which the flux function varies discontinuously. ◇

◇ Example 8.3
Since oil is much more viscous than water, water injection can lead to the formation of thin "fingers" of water. In order to prevent this, one sometimes injects a mixture of polymer and water instead of water only. This polymer is passively transported with the water. In a "one-dimensional" homogeneous oil reservoir, conservation of water and polymer is expressed through the system of conservation laws

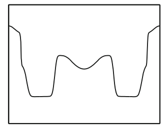
$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc)_t + (cf(s,c))_x &= 0, \end{aligned} \tag{8.4}$$

where c denotes the concentration of the polymer in the water, and the flux function $f(s,c)$ is a known function of the type in (8.3), where λ_{water} is now a function of both s and c. Introducing new coordinates (y,τ) by

$$\frac{\partial y}{\partial x} = s, \quad \frac{\partial y}{\partial t} = -f(s,c), \quad \frac{\partial \tau}{\partial x} = 0, \quad \text{and} \quad \frac{\partial \tau}{\partial t} = 1,$$

the system (8.4) reads

$$\begin{aligned} \left(\frac{1}{s}\right)_\tau - \left(\frac{f(s,c)}{s}\right)_y &= 0, \\ c_\tau &= 0. \end{aligned} \tag{8.5}$$



This change of independent variables is valid only for differentiable (classical) solutions, whereas we know that we cannot expect solutions of conservation laws to be even continuous. Therefore, we must interpret solutions in the *weak* sense. Nevertheless, by [187, Thm. 2] there is a one-to-one correspondence between weak solutions of (8.4) and weak solutions of (8.5). Hence if the initial polymer concentration is discontinuous, (8.5) is another example of a conservation law with a flux function depending discontinuously on the spatial location. ◇

We can always view an x-dependent flux as a flux function depending on a parameter γ that in turn depends on x. In this way we write (8.1) as a system

$$u_t + f(\gamma, u)_x = 0, \quad \gamma_t = 0. \tag{8.6}$$

This is a hyperbolic system with a Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial \gamma} \\ 0 & 0 \end{pmatrix},$$

which has the eigenvalues

$$\lambda_1 = \frac{\partial f}{\partial u}, \quad \lambda_2 = 0.$$

So if $\frac{\partial f}{\partial u} = 0$ for some values of γ and u, the system is not strictly hyperbolic. This is the cause of many difficulties when one is working with conservation laws with x-dependent fluxes. In [176], Temple exhibited a simple example of a sequence of approximate solutions without any uniform bound on the variation. This means that when studying conservation laws of the type (8.6), one must use more powerful (and complicated) tools. The "z-transform" used in this chapter is perhaps the simplest (and least powerful) example of such a tool. Recently, compensated compactness and variants of the "div-curl" lemma have taken the place of the "z-transform" in proving convergence of approximations; see [107] for a recent example.

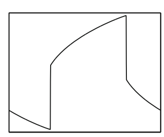
We emphasize that this chapter is meant to be an introduction to this topic and does not contain the most general results.

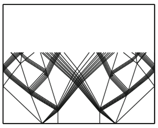
8.1 The Riemann Problem

In this section we shall study the Riemann problem, that is, the initial value problem in which the initial values consist of two constants separated by a jump discontinuity. More precisely, this is the problem

$$\begin{cases} u_t + f(\gamma_l, u)_x = 0, & u(x,0) = u_l, \quad \text{for } x < 0, \\ u_t + f(\gamma_r, u)_x = 0, & u(x,0) = u_r, \quad \text{for } x > 0, \end{cases} \tag{8.7}$$

where γ_l, γ_r, u_l, and u_r are constants. Riemann problems for conservation laws have the simplest solutions that are not constant. Furthermore, by studying the





solution of Riemann problems, we gain insight into the local behavior of typical solutions. It turns out that solutions of Riemann problems can be used as a building block in many numerical methods, in particular front tracking.

By a solution of (8.7) we mean a weak solution in the usual sense, i.e., $u \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0,\infty))$ is called a weak solution if for every test function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$,

$$\int_0^\infty \Big(\int_{-\infty}^0 \big(u\varphi_t + f(\gamma_l, u)\,\varphi_x\big)\,dx + \int_0^\infty \big(u\varphi_t + f(\gamma_r, u)\,\varphi_x\big)\,dx \Big)\,dt \\ + \int_{\mathbb{R}} u(x,0)\varphi(x,0)\,dx = 0. \tag{8.8}$$

Now we shall first show that under reasonable assumptions on f, weak solutions exist, and that if we require that weak solutions satisfy an additional entropy condition, then there exists only one weak solution.

Existence of a Solution

To show the existence of a solution, we start by observing that for x negative, $u(x,t)$ must be the solution of a scalar conservation law

$$v_t + f(\gamma_l, v)_x = 0, \tag{8.9}$$

with $v(x,0)$ given by

$$v(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_l' & \text{for } x = 0, \end{cases}$$

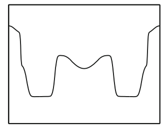
where u_l' is a value to be determined. Similarly, for x positive, u must be the solution of the scalar initial value problem

$$w_t + f(\gamma_r, w)_x = 0, \qquad w(x,0) = \begin{cases} u_r' & \text{for } x = 0, \\ u_r & \text{for } x > 0, \end{cases} \tag{8.10}$$

where u_r' is to be determined. Setting

$$u(x,t) = \begin{cases} v(x,t) & \text{for } x < 0, \\ w(x,t) & \text{for } x > 0, \end{cases} \tag{8.11}$$

provided that $v(0-,t)$ and $w(0+,t)$ satisfy some extra condition, we find that this will give a weak solution, since both v and w are weak solutions. Therefore, to find a weak solution, we must find solutions of scalar Riemann problems v and w such that this construction is possible.



Now recall, or reread Sect. 2.2, that the solution to the scalar Riemann problem

$$v_t + g(v)_x = 0, \quad v(x,0) = \begin{cases} v_l & x < 0, \\ v_r & x \geq 0, \end{cases}$$

is found by constructing the lower convex (if $v_l < v_r$) or upper concave (if $v_l > v_r$) envelope of g between v_l and v_r; cf. Sect. 2.2. To make the notation less cumbersome we introduce

$$\bar{g}\,(v; v_l, v_r) = \begin{cases} g_{\frown}(v; v_l, v_r) & \text{if } v_r < v_l, \\ g_{\smile}(v; v_l, v_r) & \text{if } v_l < v_r. \end{cases} \tag{8.12}$$

In this notation the entropy solution v is given by

$$v(x,t) = \bar{g}'^{-1}\left(\frac{x}{t}; v_l, v_r\right), \qquad t > 0. \tag{8.13}$$

Now we turn to the Riemann problem (8.7). The left and right parts of u are v, given by (8.9), and w, given by (8.10). If we are to form u by gluing together v and w, then v must equal u_l' for $x > 0$, and w must equal u_r' for $x < 0$. In other words, v must contain only waves of nonpositive speed, and w only waves of nonnegative speed. To utilize these observations, we introduce the notation

$$f_l(u) = f\,(\gamma_l, u) \quad \text{and} \quad f_r(u) = f\,(\gamma_r, u)$$

and define $\bar{f}_l(u; u_l, \tilde{u})$ and $\bar{f}_r(u; \tilde{u}, u_r)$ analogously to (8.12).

Since v contains only waves of nonpositive speed, we must choose u_l' from the set

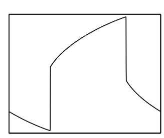
$$H_l\,(u_l) = \left\{\tilde{u} \mid \bar{f}_l'(u; u_l, \tilde{u}) \leq 0 \quad \text{for all } u \text{ between } u_l \text{ and } \tilde{u}\right\}. \tag{8.14}$$

Similarly, since w must contain waves of nonnegative speed, we must choose u_r' from the set

$$H_r\,(u_r) = \left\{\tilde{u} \mid \bar{f}_r'(u; \tilde{u}, u_r) \geq 0 \quad \text{for all } u \text{ between } u_r \text{ and } \tilde{u}\right\}. \tag{8.15}$$

There is another characterization of the admissible sets H_l and H_r that will be useful. Let h_l be defined by

$$h_l\,(u; u_l) = \begin{cases} \inf\left\{h(u) \;\middle|\; \begin{matrix} h(u) \geq f_l(u), \quad h'(u) \leq 0, \\ \text{and} \quad h(u_l) = f_l(u_l) \end{matrix}\right\} & \text{if } u \leq u_l, \\ \sup\left\{h(u) \;\middle|\; \begin{matrix} h(u) \leq f_l(u), \quad h'(u) \leq 0, \\ \text{and} \quad h(u_l) = f_l(u_l) \end{matrix}\right\} & \text{if } u \geq u_l, \end{cases} \tag{8.16}$$



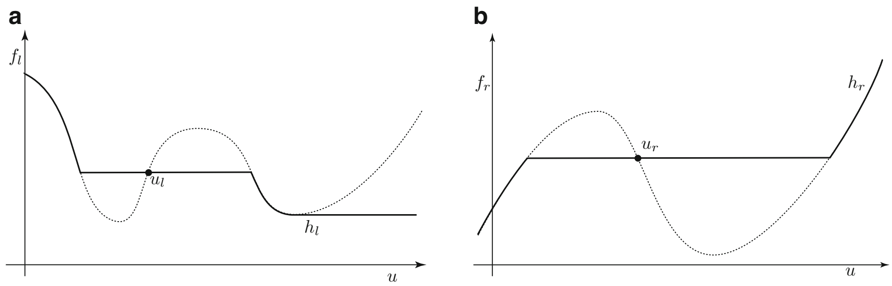


Fig. 8.1 **a** h_l (*solid line*) and f_l (*dotted line*). **b** h_r (*solid line*) and f_r (*dotted line*)

and define h_r by

$$h_r\,(u;u_r) = \begin{cases} \sup\left\{ h(u) \;\middle|\; \begin{array}{c} h(u) \le f_r(u), \quad h'(u) \ge 0, \\ \text{and} \quad h(u_r) = f_r(u_r) \end{array} \right\} & \text{if } u \le u_r, \\ \inf\left\{ h(u) \;\middle|\; \begin{array}{c} h(u) \ge f_r(u), \quad h'(u) \le 0, \\ \text{and} \quad h(u_l) = f_l(u_l) \end{array} \right\} & \text{if } u \ge u_l. \end{cases} \tag{8.17}$$

In these definitions, the function h appearing in the infima and suprema is assumed to be continuous. In Fig. 8.1 we show an example of h_l and h_r. Using h_l and h_r we can use the following alternative definition of the admissible sets H_l and H_r, namely

$$H_l\,(u_l) = \{u \mid h_l\,(u;u_l) = f_l(u)\}\,, \tag{8.18}$$

$$H_r\,(u_r) = \{u \mid h_r\,(u;u_r) = f_r(u)\}\,. \tag{8.19}$$

Since the jump in u at $x = 0$ is a discontinuity with zero speed, the Rankine–Hugoniot condition says that for every weak solution we must have

$$f\left(\gamma_l, u_l'\right) = f\left(\gamma_r, u_r'\right) =: f^{\times}. \tag{8.20}$$

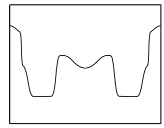
We now have $u_l' \in H_l(u_l)$ and $u_r' \in H_r(u_r)$, using (8.18) and (8.19). This can be restated as

$$h_l\left(u_l', u_l\right) = h_r\left(u_r', u_r\right). \tag{8.21}$$

Since the mapping $u \mapsto h_l(u;u_l)$ is nonincreasing and $u \mapsto h_r(u;u_r)$ is nondecreasing, the above equality, (8.21), will hold for at most one h value. Therefore, if the graphs of h_l and h_r intersect, the flux value at $x = 0$ is determined by the flux value at this intersection point. We label this flux value $f^{\times}$.

From these observations it also follows that if the graph of h_l does *not* intersect the graph of h_r, we cannot hope to find a weak solution to the Riemann problem (8.7). For instance, if

$$f_l(u) = e^{-u^2} \quad \text{and} \quad f_r(u) = 2 + e^{-u^2},$$



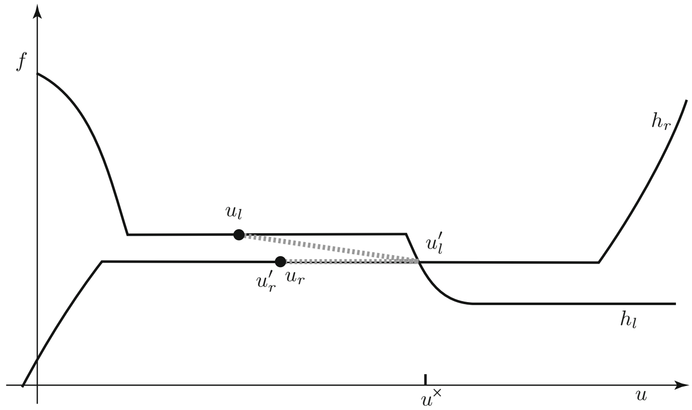


Fig. 8.2 An example showing how to solve a Riemann problem of the type (8.7)

we cannot find any weak solution. Another important example for which we cannot find any solution to the Riemann problem is

$$f_l'(u) \geq 0 \quad \text{and} \quad f_r'(u) \leq 0.$$

In this case $h_l(u; u_l) = f_l(u_l)$ and $h_r(u; u_r) = f_r(u_r)$, so unless these happen to be equal, we cannot find any solution.

Furthermore, even if the flux value at the intersection is uniquely determined, the actual values u_l' and u_r' need not be. This is so since for $u \notin H_l(u_l)$ we have $h_l'(u; u_l) = 0$, and similarly, if $u \notin H_r(u_r)$, then $h_r'(u; u_r) = 0$. In other words, h_l and h_r may both be constant on the interval where their graphs cross. In order to resolve this nonuniqueness problem, we propose that u_l' and u_r' be chosen such that the variation of the solution u is minimal, subject to the above restrictions.

To be more concrete, we choose u_l' to be the unique value such that

$$|u_l - u_l'| \text{ is minimized, provided } u_l' \in H_l(u_l) \text{ and } h_l(u_l'; u_l) = f^\times. \tag{8.22}$$

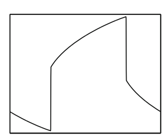
Similarly, we choose u_r' to be the unique value such that

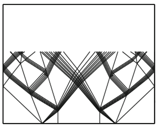
$$|u_r - u_r'| \text{ is minimized, provided } u_r' \in H_r(u_r) \text{ and } h_l(u_r'; u_r) = f^\times. \tag{8.23}$$

These criteria for choosing u_l' and u_r' are called the *minimal jump entropy condition*.

It is perhaps instructive to examine this condition in a little more detail. If the graphs of h_l and h_r intersect in a single point $u^\times$, then $u^\times \in H_l(u_l)$ or $u^\times \in H_r(u_r)$. If $u^\times \in H_l(u_l)$, then $u_l' = u^\times$, and if $u^\times \in H_r(u_r)$, then $u_r' = u^\times$. Assuming for definiteness that $u_l < u^\times$ and $u^\times \notin H_l(u_l)$, then there will be a smallest point $\tilde{u}$ in the interval $[u_l, u^\times]$ such that the interval $(\tilde{u}, u^\times]$ is not contained in $H_l(u_l)$, and $\tilde{u} \in H_l(u_l)$. It is clear that according to (8.22) we must choose $u_l' = \tilde{u}$.

In Fig. 8.2 we show how the Riemann problem from Fig. 8.1 is solved in this way. Here $u^\times \in H_l(u_l)$ so $u_l' = u^\times$. Also the point minimizing $|u_r' - u_r|$ is clearly u_r, so that $u_r' = u_r$. Finally the Riemann problem is solved by a shock of negative





speed from u_l to u_l', and then by a discontinuity at $x = 0$ from u_l' to u_r. There is some more important information to be extracted from the minimal jump entropy condition. Since the Riemann problem with $u_l = u_l'$ and $u_r = u_r'$ is solved by a single stationary discontinuity, in the interval spanned by u_l' and u_r', we must have

$$h_l\left(u; u_l'\right) = f^\times, \text{ or } h_r\left(u; u_r'\right) = f^\times. \tag{8.24}$$

If $u_l' < u_r'$, since $h_l(\,\cdot\,; u_l')$ is the largest nonincreasing continuous function less than or equal to f_l such that $h_l(u_l'; u_l') = f_l(u_l')$, then

$$h_l\left(u; u_l'\right) = f^\times \quad \Rightarrow \quad f_l(u) > f^\times \text{ for } u \in (u_l', u_r')$$

and

$$h_r\left(u; u_r'\right) = f^\times \quad \Rightarrow \quad f_r(u) > f^\times \text{ for } u \in (u_l', u_r'),$$

since $h_r(\,\cdot\,; u_r')$ is the largest continuous nondecreasing function smaller than or equal to f_r. Similarly, if $u_r' < u_l'$, then

$$h_l\left(u; u_l'\right) = f^\times \quad \Rightarrow \quad f_l(u) < f^\times \text{ for } u \in (u_r', u_l')$$

and

$$h_r\left(u; u_r'\right) = f^\times \quad \Rightarrow \quad f_r(u) < f^\times \text{ for } u \in (u_r', u_l').$$

Summing up, we have

$$u_l' \le u_r' \Rightarrow \begin{cases} f_l(u) \ge f_l(u_l') & \text{for all } u \in [u_l', u_r'] \text{ or} \\ f_r(u) \ge f_r(u_r') & \text{for all } u \in [u_l', u_r'], \end{cases} \tag{8.25}$$

$$u_r' \le u_l' \Rightarrow \begin{cases} f_l(u) \le f_l(u_l') & \text{for all } u \in [u_r', u_l'] \text{ or} \\ f_r(u) \le f_r(u_r') & \text{for all } u \in [u_r', u_l']. \end{cases} \tag{8.26}$$

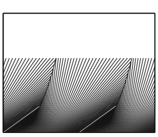
Furthermore, the implications (8.25) and (8.26) actually imply that u_l' and u_r' are chosen according to the minimal jump entropy condition.

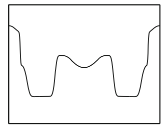
Lemma 8.4 *If the values u_l' and u_r' are chosen according to the minimal jump entropy condition* (8.22), (8.23), *then for every constant c,*

$$F_r\left(u_r', c\right) - F_l\left(u_l', c\right) \le \left|f_r(c) - f_l(c)\right|, \tag{8.27}$$

where F_l and F_r are the Kružkov entropy fluxes. Thus

$$F_l(u, c) = \operatorname{sign}(u - c)\left(f_l(u) - f_l(c)\right),$$
$$F_r(u, c) = \operatorname{sign}(u - c)\left(f_r(u) - f_r(c)\right).$$





Proof If $\operatorname{sign}\left(u_l'-c\right)=\operatorname{sign}\left(u_r'-c\right)$, then the left-hand side of (8.27) equals

$$\operatorname{sign}\left(u_l'-c\right)\left(f_r\left(u_r'\right)-f_r(c)-f_l\left(u_l'\right)+f_l(c)\right)=\operatorname{sign}\left(u_l'-c\right)\left(f_l(c)-f_r(c)\right),$$

and the inequality clearly holds.

If $u_l'\le c\le u_r'$, then (8.27) reads

$$2f^\times-f_l(c)-f_r(c)\le\left|f_r(c)-f_l(c)\right|,$$

or

$$\begin{aligned}2f^\times-\max\left\{f_l(c),f_r(c)\right\}-\min\left\{f_l(c),f_r(c)\right\}\\ \le\max\left\{f_l(c),f_r(c)\right\}-\min\left\{f_l(c),f_r(c)\right\}.\end{aligned}$$

In other words, (8.27) is the same as

$$f^\times\le\max\left\{f_l(c),f_r(c)\right\},$$

and it is immediate that (8.25) implies this.

If $u_r'\le c\le u_l'$, then (8.27) reads

$$f^\times\ge\min\left\{f_l(c),f_r(c)\right\},$$

which is implied by (8.26). □

From the proof of Lemma 8.4 it is also transparent that the condition (8.27) does *not* imply the minimal jump entropy condition (8.25) and (8.26). However, define the pair of "constants" c_l and c_r (these numbers depend on u_l' and u_r') by requiring

$$c_l\left(u_l',u_r'\right)=\begin{cases}\min\arg_{[u_l',u_r']}f_l(u) & \text{if } u_l'\le u_r',\\ \max\arg_{[u_r',u_l']}f_l(u) & \text{if } u_l'\ge u_r',\end{cases}\tag{8.28}$$

$$c_r\left(u_l',u_r'\right)=\begin{cases}\min\arg_{[u_l',u_r']}f_r(u) & \text{if } u_l'\le u_r',\\ \max\arg_{[u_r',u_l']}f_r(u) & \text{if } u_l'\ge u_r'.\end{cases}\tag{8.29}$$

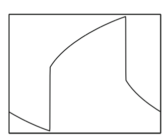
Using the arguments of the proof of Lemma 8.4, it readily follows that the minimal jump entropy condition is equivalent to

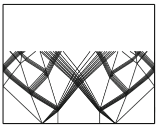
$$F_r\left(u_r',c_r\right)-F_l\left(u_l',c_l\right)\le\left|f_r\left(c_r\right)-f_l\left(c_l\right)\right|.\tag{8.30}$$

Furthermore, for every c between u_l' and u_r', the inequality

$$F_r\left(u_r',c\right)-F_l\left(u_l',c\right)\le F_r\left(u_r',c_r\right)-F_l\left(u_l',c_l\right),$$

holds.





Remark 8.5 In a special case (8.27) actually implies that the values u_l' and u_r' are chosen according to the minimal jump entropy condition. Assume that there is a value $\hat{u}$ such that both $f_l(u)$ and $f_r(u)$ have a global maximum (minimum) at $\hat{u}$, and that $f_{l,r}$ is increasing (decreasing) for $u < \hat{u}$ and decreasing (increasing) for $u > \hat{u}$. To see this, we recall that (8.27) holds trivially if c is not between u_l' and u_r',while if c is between these values, (8.27) reads

$$\begin{cases} f^\times \leq \max\{f_l(c), f_r(c)\}, & \text{if } u_l' < u_r', \\ f^\times \geq \max\{f_l(x), f_r(c)\}, & \text{if } u_l' > u_r'. \end{cases} \tag{8.31}$$

By assuming that $f_l(u_l') = f_r(u_r')$, that the above holds, and that the flux functions $f_{l,r}$ have a single common maximum, the reader can check that (8.31) implies (8.25) and (8.26). Actually, this implication holds for more general flux functions as well; cf. the notorious "crossing condition" in [110].

Although it seemingly has nothing to do with the solution of the Riemann problem, at this point it is convenient to state and prove the following lemma, which will play an important role in proving well-posedness in Sect. 8.3.

Lemma 8.6 *Assume that the pairs* (u_l', u_r') *and* (v_l', v_r') *are both chosen according to the minimal jump entropy condition. Then*

$$Q = F_r\left(u_r', v_r'\right) - F_l\left(u_l', v_l'\right) \leq 0. \tag{8.32}$$

Proof Since $f_l(v_l') = f_r(v_r')$ and $f_l(u_l') = f_r(u_r')$, if

$$\operatorname{sign}\left(u_l' - v_l'\right) = \operatorname{sign}\left(u_r' - v_r'\right),$$

then $Q = 0$. Assume therefore that

$$\operatorname{sign}\left(u_l' - v_l'\right) = -1 \quad \text{and} \quad \operatorname{sign}\left(u_r' - v_r'\right) = 1.$$

In this case,

$$\begin{aligned} Q &= \left[f_r\left(u_r'\right) - f_r\left(v_r'\right)\right] + \left[f_l\left(u_l'\right) - f_l\left(v_l'\right)\right] \\ &= 2\left(f_r\left(u_r'\right) - f_r\left(v_r'\right)\right) && (8.33) \\ &= 2\left(f_l\left(u_l'\right) - f_l\left(v_l'\right)\right), && (8.34) \end{aligned}$$

since $f_l(v_l') = f_r(v_r')$ and $f_l(u_l') = f_r(u_r')$. Moreover

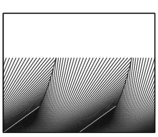
$$u_l' \leq v_l' \quad \text{and} \quad v_r' \leq u_r'.$$

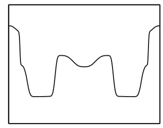
Then either u_l' and u_r' are both in the interval $[v_r', v_l']$ (case **a**), or v_l' and v_r' are in the interval $[u_l', u_r']$ (case **b**), or $v_r' \leq u_l' \leq v_l' \leq u_r'$ (case **c**), or $u_l' \leq v_r' \leq u_r' \leq v_l'$ (case **d**).

If case **a** holds, then (8.26) for v_l' and v_r' gives that either

$$f_l\left(u_l'\right) \leq f_l\left(v_l'\right) \quad \text{or} \quad f_r\left(u_r'\right) \leq f_r\left(v_r'\right).$$

It is easy to see that this coupled with either (8.33) or (8.34) will give $Q \leq 0$.





If case **b** holds, then (8.26) for u gives that either

$$f_l\left(v_l'\right) \geq f_l\left(u_l'\right) \quad \text{or} \quad f_r\left(v_r'\right) \geq f_r\left(u_r'\right).$$

So again $Q \leq 0$.

Recall that case **c** is defined to hold if

$$v_r' \leq u_l' \leq v_l' \quad \text{and} \quad u_l' \leq v_l' \leq u_r'.$$

Using the first inequality and (8.26) for v, we find that

$$f_l\left(u_l'\right) \leq f_l\left(v_l'\right) \quad \text{or} \quad f_r\left(u_r'\right) \leq f_r\left(v_r'\right),$$

both of which give the desired conclusion.

Finally, in case **d**, we have

$$u_l' \leq v_r' \leq u_r' \quad \text{and} \quad v_r' \leq u_r' \leq v_l'.$$

Using the first inequality with (8.25) gives

$$f_l\left(v_l'\right) \geq f_l\left(u_l'\right) \quad \text{or} \quad f_r\left(v_r'\right) \geq f_r\left(u_r'\right),$$

thereby completing the proof. □

◇ Example 8.7

Now we pause to consider two examples. First consider the Riemann problem for the equation

$$u_t + \left(\frac{1}{2}u^2 + \gamma\right)_x = 0, \tag{8.35}$$

where

$$u_0(0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0. \end{cases}$$

If $u_l \leq 0$, then

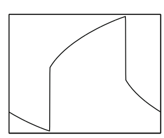
$$H_l\left(u_l\right) = (-\infty, 0],$$

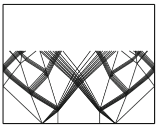
and if $u_l \geq 0$, then

$$H_l\left(u_l\right) = \left(-\infty, -u_l\right] \cup \left\{u_l\right\}.$$

Similarly, if $u_r \leq 0$, then

$$H_r\left(u_r\right) = \left\{-u_r\right\} \cup \left[-u_r, \infty\right),$$





and if $u_r \geq 0$, then

$$H_r(u_r) = [0, \infty).$$

Now it is easy to construct the solution for any initial data and any γ. Assume that $\gamma_l = -1$, $\gamma_r = 1$, $u_l = 1$, and $u_r = 1$. Then

$$h_l(u; -1) = \begin{cases} \frac{1}{2}u^2 - 1 & \text{if } u \leq -1, \\ -\frac{1}{2} & \text{if } u \geq -1, \end{cases} \quad \text{and} \quad h_r(u; 1) = \begin{cases} 1 & \text{if } u \leq 0, \\ \frac{1}{2}u^2 + 1 & \text{if } u \geq 0. \end{cases}$$

The graphs of h_l and h_r intersect in a single point where the flux equals 1 and $u < 0$. Thus we obtain u_l' as the solution of

$$h_l\left(u_l'; -1\right) = 1, \quad u_l' < 0,$$

and thus $u_l' = -2$. Following the general construction, we see that $u_r' = 0$. The complete solution thus consists of the solution of a scalar Riemann problem for the equation

$$v_t + \left(\frac{1}{2}v^2\right)_x = 0, \quad v(x, 0) = \begin{cases} 1 & \text{for } x \leq 0, \\ -2 & \text{for } x \geq 0, \end{cases}$$

glued together with the solution of the scalar Riemann problem

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0, \quad w(x, 0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

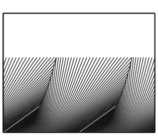
From the general solution procedure for scalar Riemann problems, i.e., taking envelopes, we see that

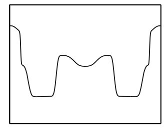
$$v(x, t) = \begin{cases} 1 & \text{for } x \leq -t/2, \\ -2 & \text{for } x > -t/2, \end{cases} \quad \text{and} \quad w(x, t) = \begin{cases} 0 & \text{for } x \leq 0, \\ x/t & \text{for } 0 < x \leq t, \\ 1 & \text{for } t < x. \end{cases}$$

Finally, we set

$$u(x, t) = \begin{cases} v(x, t) & \text{for } x < 0, \\ w(x, t) & \text{for } x > 0. \end{cases}$$

This solution is depicted in Fig. 8.3. To the left we see the solution path in the (u, f)-plane, and to the right $u(x, t)$. Perhaps the most important lesson to be learned from this example is that the variation of the solution u is *not* bounded by the variation of the initial data $u(x, 0)$. Even though this is so, it is natural to ask whether the variation of u is bounded by the variation of u_0 plus the variation of γ. From the construction of the solution of the Riemann problem, the total variation of $f(\gamma, u)$ is bounded by the total variation of $f(\gamma, u_0)$. Nevertheless, an explicit example shows that it may happen that the total variation of u_0 is finite, yet for a finite $T > 0$, we have T.V. $(u(\,\cdot\,, T)) = \infty$; see [1]. We shall return to these observations in a later section. ◇





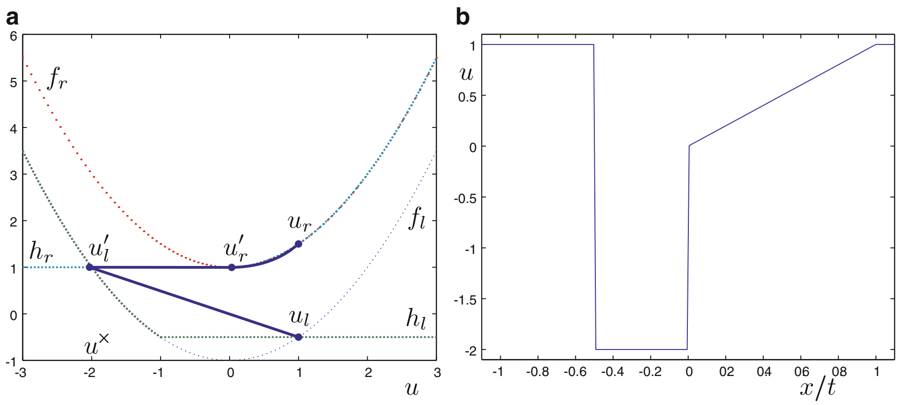


Fig. 8.3 An example of the solution of a Riemann problem. **a** The solution path in (u, f) space. **b** $u(x,t)$

◇ Example 8.8

As a second example we study the traffic flow model

$$u_t + (\gamma(x)4u(1-u))_x = 0, \tag{8.36}$$

where

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x \geq 0. \end{cases}$$

For simplicity, we assume that γ_l and γ_r are positive. Now

$$H_l(u_l) = \begin{cases} \{u_l\} \cup [1-u_l, \infty) & \text{if } u_l \leq 1/2, \\ [1/2, \infty) & \text{if } u_l \geq 1/2, \end{cases}$$

and

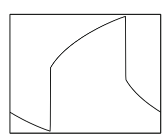
$$H_r(u_r) = \begin{cases} (-\infty, 1/2] & \text{if } u_r \leq 1/2, \\ (-\infty, 1-u_r] \cup \{u_r\} & \text{if } u_r \geq 1/2. \end{cases}$$

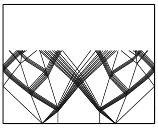
We shall now detail the complete solution of the Riemann problem in this case. This is instructive, since (8.36) exhibits many of the features of Riemann solutions for general flux functions.

We describe the solution by listing what happens in various cases, depending on γ_l, γ_r, u_l, and u_r. Note first that $f(\gamma, u)$ has a maximum at $u = 1/2$ for all γ and that $f(\gamma, 1/2) = \gamma$. We start by assuming that

$$u_l \leq \frac{1}{2}. \tag{8.37}$$

In this case the structure of the solution will depend on whether $\gamma_l < \gamma_r$. We start by examining the case $\gamma_l < \gamma_r$ and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$. The situation is depicted in Fig. 8.4. Here we show the h_l and h_r functions as dotted lines, and





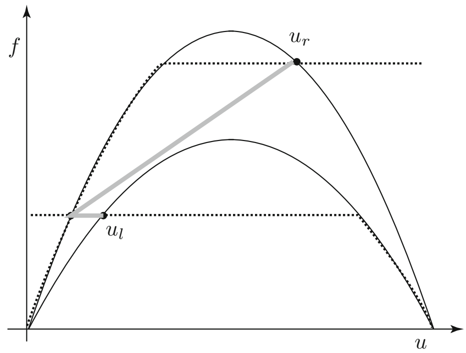


Fig. 8.4 The solution of the Riemann problem if $u_l < 1/2$, $\gamma_l < \gamma_r$, and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$

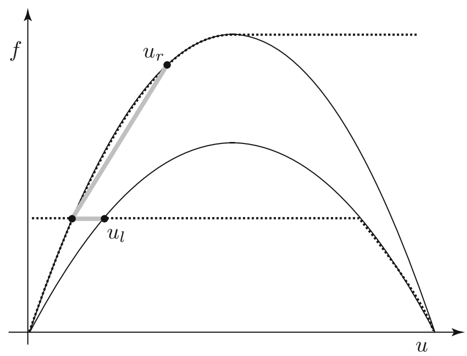


Fig. 8.5 The solution of the Riemann problem if $u_l < 1/2$, $\gamma_l < \gamma_r$, and $f(\gamma_l, u_l) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$

the solution path as a gray line. In this case $u_l' = u_l$, and u_r' is the solution of

$$f\left(\gamma_r, u_r'\right) = f\left(\gamma_l, u_l\right), \quad u_r' < \frac{1}{2}.$$

In our case, this means that

$$u_r' = \frac{1}{2}\left(1 - \sqrt{1 - \frac{\gamma_l}{\gamma_r} 4u_l\left(1 - u_l\right)}\right).$$

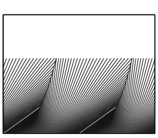
The solution consists of a stationary discontinuity separating (u_l', γ_l) and (u_r', γ_r), which we shall call a γ-wave, followed by a shock in u moving to the right. This we call a u-wave. For clarity we also show the solution if $u_r \leq 1/2$ in Fig. 8.5.

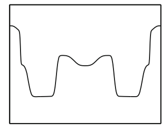
Next, we turn to the case that $\gamma_l < \gamma_r$ and $f(\gamma_l, u_l) \geq f(\gamma_r, u_r)$, depicted in Fig. 8.6. The solution consists of a u-wave with negative speed followed by a γ-wave separating u_l' and u_r. In other words, we have $u_r' = u_r$, and u_l' is the solution of

$$f\left(\gamma_l, u_l'\right) = f\left(\gamma_r, u_r\right), \quad u_l' \geq \frac{1}{2}.$$

In the next case we assume that $u_l \geq 1/2$. In this case, if $u_r \leq 1/2$, or $f(\gamma_r, u_r) > f(\gamma_l, 1/2)$, then $u_l' = 1/2$, and u_r' solves

$$f\left(\gamma_r, u_r'\right) = f\left(\gamma_l, u_l'\right) = \gamma_l, \quad u_r' < \frac{1}{2}.$$





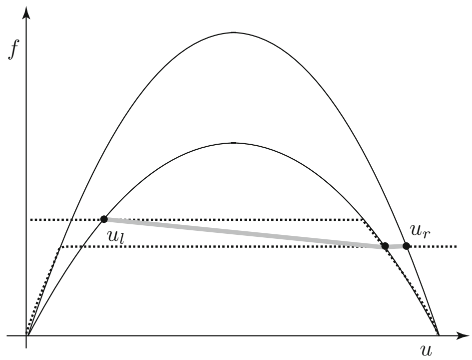


Fig. 8.6 The solution of the Riemann problem if $u_l < 1/2$, $\gamma_l < \gamma_r$, $f(\gamma_l, u_l) \geq f(\gamma_r, u_r)$, and $u_r \geq 1/2$

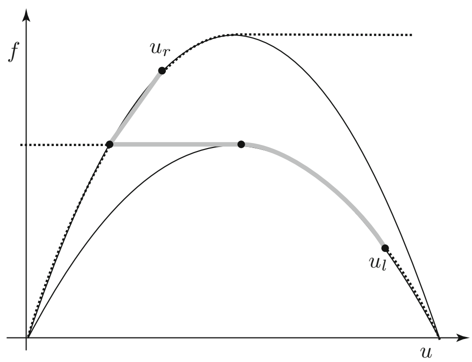


Fig. 8.7 The solution of the Riemann problem if $u_l \geq 1/2$, $\gamma_l < \gamma_r$, and $f(\gamma_l, 1/2) < f(\gamma_r, u_r)$ or $u_r \leq 1/2$

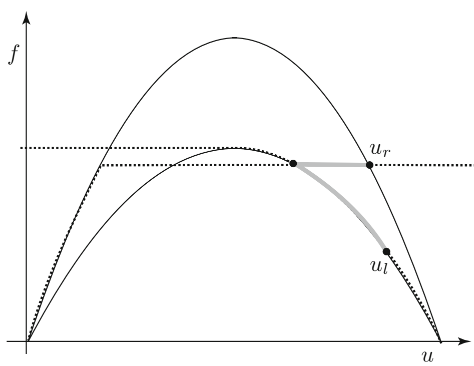
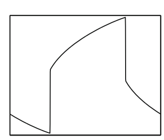


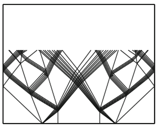
Fig. 8.8 The solution of the Riemann problem if $u_l \geq 1/2$, $\gamma_l < \gamma_r$, $f(1/2, u_l) \geq f(\gamma_r, u_r)$, and $u_r > 1/2$

This is illustrated in Fig. 8.7. Now the solution consists of a u-wave moving to the left, this u-wave is a rarefaction wave, followed by a γ-wave. The last wave is a u-wave moving to the right; this is a shock wave.

Next, if $u_l \geq 1/2$, $u_r \geq 1/2$, and $f(\gamma_r, u_r) \leq f(\gamma_l, 1/2)$, the solution is shown in Fig. 8.8. In this case u consists of a leftward moving u-wave followed by a γ-wave. This exhausts the case $\gamma_l < \gamma_r$.

The case $\gamma_l > \gamma_r$ is analogous, and we show the four different possibilities in Fig. 8.9.





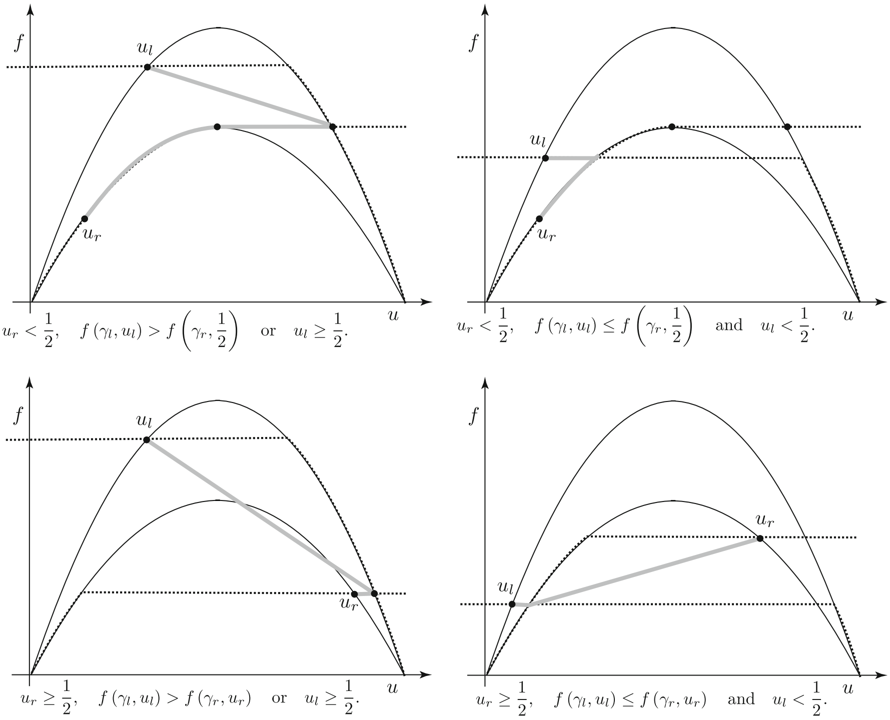


Fig. 8.9 The different possibilities for a solution of the Riemann problem if $u_r \geq 1/2$. The solution path is the *gray line*

In order to determine a particular solution, follow the gray path from u_l to u_r. If the path follows the graph of f_l or f_r, the wave is a rarefaction wave, and, if not, it is a shock wave. The horizontal segments joining f_l and f_r are γ-waves. In these figures, the dotted lines indicate the functions h_l and h_r.

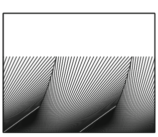
From the above diagrams, we observe that if u_l and u_r are in the interval [0, 1], then also the solution $u(x,t)$ will take values in [0, 1]. In many applications involving similar conservation laws, u is interpreted as a density; hence it is natural to require that u be between 0 and 1.

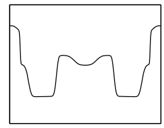
There is another and much more compact way to depict the solution of the general Riemann problem for this conservation law. Let $z = z(\gamma, u)$ be defined as

$$\begin{aligned} z(\gamma, u) &= \operatorname{sign}\left(\frac{1}{2} - u\right)\left[f(\gamma, u) - f\left(\gamma, \frac{1}{2}\right)\right] \qquad (8.38) \\ &= \gamma \operatorname{sign}\left(u - \frac{1}{2}\right)(2u-1)^2 \\ &= \int_{1/2}^{u} \left|\frac{\partial f}{\partial u}(\gamma, \xi)\right| d\xi. \end{aligned}$$

This mapping takes the rectangle $[\gamma_1, \gamma_2] \times [0, 1]$ into the region

$$\{(z, \gamma) \mid \gamma_1 \leq \gamma \leq \gamma_2 \text{ and } -\gamma \leq z \leq \gamma\}.$$





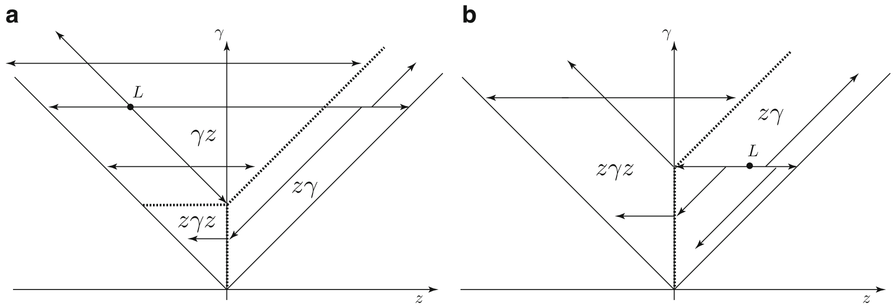


Fig. 8.10 The solution of the Riemann problem. **a** $z_l \leq 0$. **b** $z_l \geq 0$

Furthermore, $u \mapsto z(\gamma, u)$ is injective, and strictly increasing. In (z, γ) coordinates, γ-waves are straight lines of slope -1 if $u \leq 1/2$ and slope 1 if $u \geq 1/2$, and u-waves are horizontal lines. In Fig. 8.10 we show how the solution looks in the various cases in the (z, γ)-plane. To read the diagram, start at the point $L = (z(u_l, \gamma_l), \gamma_l)$ and follow the arrows to the right location. The dotted lines mark the boundaries where the solution type is constant. Since we are working with (z, γ) coordinates, we call u-waves z-waves, and the solution types are $z\gamma$, $z\gamma z$, and γz. If a solution type is, e.g., γz, this means that the solution consists of a z-wave (u-wave) followed by a γ-wave. This finishes the second example. ◇

Actually, our two examples are more similar than it might seem at a first glance. The inverse of the mapping (8.38) is

$$u = \frac{1}{2}\left(1 + \operatorname{sign}(z)\sqrt{\frac{|z|}{\gamma}}\right),$$

and

$$f(\gamma, u) = |z| + \gamma.$$

Inserting this into equation (8.36), we find that

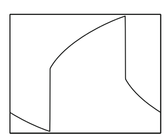
$$\left(\frac{1}{2}\left(1 + \operatorname{sign}(z)\sqrt{\frac{|z|}{\gamma}}\right)\right)_t + (|z| + \gamma)_x = 0.$$

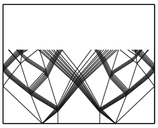
Since γ is independent of t, we can rearrange this as

$$\left(\operatorname{sign}(z)\sqrt{|z|}\right)_t + 2\sqrt{\gamma}\,(|z| + \gamma)_x = 0.$$

If we now introduce $w = \operatorname{sign}(z)\sqrt{|z|}$ and a new time coordinate τ such that $\partial/\partial\tau = \sqrt{2\gamma}\,\partial/\partial t$, then

$$w_\tau + \left(\frac{1}{2}w^2 + \gamma\right)_x = 0.$$





Now we return to the discussion of the Riemann problem for the general conservation law; cf. (8.7). We have seen that we cannot always find a weak solution to this problem, but if the graphs of the functions $H_l(\,\cdot\,; u_l)$ and $H_r(\,\cdot\,; u_r)$ intersect, we can choose a unique pair (u_l', u_r'), which in turn gives us a unique solution of the Riemann problem. We call this solution, satisfying the minimal jump entropy condition, an *entropy solution* of the Riemann problem.

It seems complicated to give a general criterion for f_l and f_r to guarantee the intersection of h_l and h_r, but for two important classes of flux functions we always have an intersection.

Lemma 8.9 *Consider the Riemann problem*

$$u_t + f(\gamma, u)_x = 0, \quad t > 0,$$
$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0. \end{cases} \tag{8.39}$$

(i) *Let $f = f(\gamma, u)$ be a continuously differentiable function on the set*

$$(\gamma, u) \in [\gamma_1, \gamma_2] \times [u_1, u_2] = \Omega.$$

Assume that

$$\frac{\partial f}{\partial \gamma}(\gamma, u_1) = \frac{\partial f}{\partial \gamma}(\gamma, u_2) = 0,$$

so that $f(\gamma, u_1) = C_1$ and $f(\gamma, u_2) = C_2$ for some constants C_1 and C_2. Then the Riemann problem (8.39) has a unique entropy solution for all (γ_l, u_l) and (γ_r, u_r) in Ω. Furthermore, $u(x,t) \in \Omega$ for all x and t.

(ii) *Let $f = f(\gamma, u)$ be a locally Lipschitz continuous function for $\gamma \in [\gamma_1, \gamma_2]$ and $u \in \mathbb{R}$. Assume that*

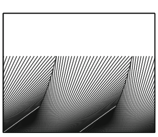
$$\lim_{u \to \pm\infty} f(\gamma, u) = \infty \quad \text{or} \quad \lim_{u \to \pm\infty} f(\gamma, u) = -\infty,$$

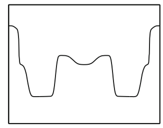
for all $\gamma \in [\gamma_1, \gamma_2]$. Then the Riemann problem (8.39) has a unique entropy solution for all (γ_l, u_l) and (γ_r, u_r) in $[\gamma_1, \gamma_2] \times \mathbb{R}$.

Our first example is of the second type mentioned in the lemma, and the second example is of the first type. This lemma is proved simply by constructing the functions h_l and h_r in the two cases.

Vanishing Viscosity and Smoothing

We would like to motivate the minimal jump entropy condition. In our construction of the solution of the Riemann problem, it emerged naturally as a candidate for finding a unique solution. In this section we shall give two partial motivations for





this entropy condition. Both of these motivations are based on the study of equations that formally have (8.7) as a limit, but whose solutions, or the equations themselves, possess more regularity than the conservation law with a discontinuous coefficient. When doing this, we hope that the minimal jump condition will be a consequence of requiring that the solutions to the perturbed equations tend to the solution of the Riemann problem as the size of the perturbations tends to zero.

It is common to motivate entropy conditions for conservation laws by requiring that the solution of Riemann problems be limits of traveling wave solutions to the singularly perturbed equation

$$v_t + f(v)_x = \varepsilon v_{xx},$$

as $\varepsilon \downarrow 0$. For a scalar equation in which the flux function does not depend on x, the "lower convex envelope" criterion is indeed a consequence of such an approach. We also say that the weak solution found by taking envelopes satisfies the *vanishing viscosity* entropy condition; see Sects. 2.1 and 2.2.

Let now u^ε be a traveling wave solution of the initial value problem

$$u^\varepsilon_t + f(\gamma, u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \quad \gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0 \end{cases} \tag{8.40}$$

(with $\gamma_l \neq \gamma_r$). We hope that

$$\lim_{x\to-\infty} u^\varepsilon(x,t) = u'_l, \text{ and } \lim_{x\to\infty} u^\varepsilon(x,t) = u'_r \tag{8.41}$$

for some values u'_l, u'_r. Since γ depends on x, we cannot expect to find a traveling wave solution, i.e., a solution that depends on $(x - st)/\varepsilon$, unless it is stationary, that is, $s = 0$. Thus we consider a function that depends on space only, $u^\varepsilon(x,t) = u(x/\varepsilon)$. Introduce $\xi = x/\varepsilon$, to obtain

$$\dot{f}(\gamma, u) = \ddot{u},$$

where $\dot{f} = df/d\xi$. The equation can be integrated once, and if we assume that the limits (8.41) are reached in a suitable manner, we get

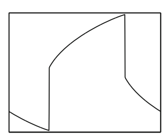
$$\dot{u} = f(\gamma, u) - f\left(\gamma_l, u'_l\right) = f(\gamma, u) - f\left(\gamma_r, u'_r\right),$$

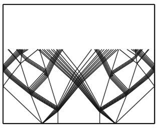
which also gives us the Rankine–Hugoniot condition

$$f\left(\gamma_l, u'_l\right) = f\left(\gamma_r, u'_r\right) =: f^\times. \tag{8.42}$$

Summing up, we say that the discontinuity separating (γ_l, u'_l) and (γ_r, u'_r) admits a viscous profile, or that this discontinuity satisfies the viscous profile entropy conditions, if the ordinary differential equation

$$\frac{du}{d\xi} = \begin{cases} f\left(\gamma_l, u\right) - f^\times & \text{if } \xi < 0, \\ f\left(\gamma_r, u\right) - f^\times & \text{if } \xi > 0, \end{cases} \tag{8.43}$$





has a (at least one) solution $u(\xi)$ such that either

$$\lim_{\xi\to-\infty} u(\xi) = u_l' \quad \text{and} \quad u(\bar{\xi}) = u_r'$$

or

$$u(\bar{\xi}) = u_r' \quad \text{and} \quad \lim_{\xi\to\infty} u(\xi) = u_r',$$

where $\bar{\xi}$ can be finite or infinite. This means that one of two alternatives must hold: *Either* the ordinary differential equation

$$\dot{v} = f(\gamma_l, v) - f^\times, \quad \xi < 0, \quad v(0) = u_r',$$

has a solution such that

$$\lim_{\xi\to-\infty} v(\xi) = u_l',$$

in which case we say that v is a left viscous profile, *or* the equation

$$\dot{w} = f(\gamma_r, u) - f^\times, \quad \xi > 0, \quad w(0) = u_l',$$

has a solution such that

$$\lim_{\xi\to\infty} w(\xi) = u_r',$$

in which case we call w a right viscous profile.
Hence the discontinuity satisfies the viscous profile entropy condition if there exists a left or right viscous profile.

If $u_l' < u_r'$, we will have a left viscous profile if and only if

$$f(\gamma_l, u) > f\left(\gamma_l, u_l'\right), \quad \text{for all } u \in (u_l', u_r').$$

Similarly, we will have a right viscous profile if and only if

$$f(\gamma_r, u) > f\left(\gamma_r, u_r'\right), \quad \text{for all } u \in (u_l', u_r').$$

Also, if $u_l' > u_r'$, we will have a left viscous profile if and only if

$$f(\gamma_l, u) < f\left(\gamma_l, u_l'\right), \quad \text{for all } u \in (u_l', u_r').$$

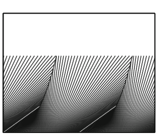
Similarly, we will have a right viscous profile if and only if

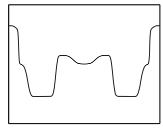
$$f(\gamma_r, u) < f\left(\gamma_r, u_r'\right), \quad \text{for all } u \in (u_l', u_r').$$

Summing up, the viscous profile entropy condition is equivalent to

$$u_l' \le u_r' \Rightarrow \quad \begin{cases} f(\gamma_l, u) > f^\times & \text{for all } u \in (u_l', u_r') \text{ or} \\ f(\gamma_r, u) > f^\times & \text{for all } u \in (u_l', u_r'), \end{cases} \tag{8.44}$$

$$u_r' \le u_l' \Rightarrow \quad \begin{cases} f(\gamma_l, u) < f^\times & \text{for all } u \in (u_r', u_l') \text{ or} \\ f(\gamma_r, u) < f^\times & \text{for all } u \in (u_r', u_l'). \end{cases} \tag{8.45}$$





This condition implies the minimal jump entropy condition, and thus provides a motivation.

If the coefficient γ is a continuous function of x, then the classical theory of scalar conservation laws applies, and the initial value problem has a unique weak solution. If we let γ^ε denote a smooth approximation to

$$\gamma(x) = \begin{cases} \gamma_l & \text{for } x < 0, \\ \gamma_r & \text{for } x > 0, \end{cases}$$

such that $\gamma^\varepsilon \to \gamma$ as $\varepsilon \to 0$, and let u^ε denote the weak solution to

$$u_t^\varepsilon + f\,(\gamma^\varepsilon, u^\varepsilon)_x = 0, \quad u^\varepsilon(x,0) = \begin{cases} u_l' & \text{for } x < 0, \\ u_r' & \text{for } x > 0, \end{cases} \tag{8.46}$$

it is natural to ask whether u^ε tends to the minimal jump entropy solution as $\varepsilon \to 0$.

◇ Example 8.10

We shall consider this in an example. Define

$$\begin{aligned} f_l(u) &= 4 - (u+1)^2, \\ f_r(u) &= 4 - (u-1)^2, \\ f(\gamma, u) &= (1-\gamma) f_l(u) + \gamma f_r(u), \end{aligned}$$

and consider the Riemann problem

$$u_t + f(\gamma, u)_x = 0, \quad u(x,0) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases} \quad \gamma(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

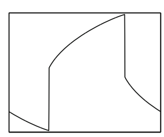
In this case we find that

$$h_l(u;-1) = \begin{cases} 4 & \text{if } u < -1, \\ 4-(u+1)^2 & \text{if } u \ge -1, \end{cases}$$

$$h_r(u;1) = \begin{cases} 4-(u+1)^2 & \text{if } u \le 1, \\ 0 & \text{if } u > 1. \end{cases}$$

Furthermore, the discontinuity separating the u and γ values $(-1,0)$ and $(1,1)$ satisfies the minimal jump entropy condition, and hence $u(x,0)$ is a weak solution satisfying the minimal jump entropy condition. Now set

$$\gamma^\varepsilon(x) = \begin{cases} 0 & \text{for } x \le -\varepsilon, \\ \frac{x+\varepsilon}{2\varepsilon} & \text{for } -\varepsilon < x < \varepsilon, \\ 1 & \text{for } \varepsilon \le x, \end{cases}$$



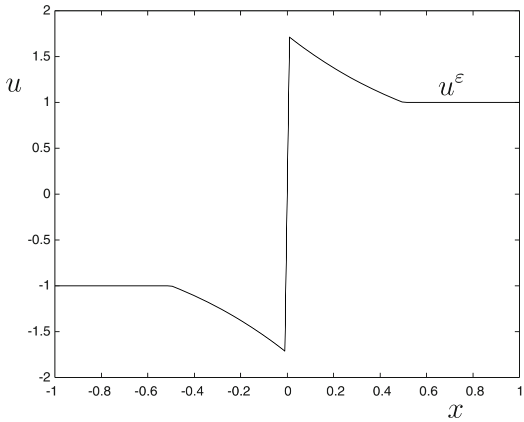


Fig. 8.11 The stationary solution of (8.46), $\varepsilon = 1/2$, and the discontinuity at $x = 0$

and let u^{ε} denote the stationary solution to (8.46) with $u_l' = -1$ and $u_r' = 1$. We have that u^{ε} satisfies

$$f\,(\gamma^{\varepsilon}, u^{\varepsilon})_x = 0,$$

and thus

$$f\,(\gamma^{\varepsilon}, u^{\varepsilon}) = f(0, -1) = 0.$$

Solving this for u^{ε}, we find that

$$u^{\varepsilon}(x) = 1 - 2\gamma^{\varepsilon}(x) \pm \sqrt{(1 - 2\gamma^{\varepsilon}(x))^2 + 3}.$$

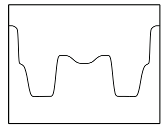
Since $u^{\varepsilon} = -1$ for $x \leq -\varepsilon$ and $u^{\varepsilon} = 1$ for $x \geq \varepsilon$, we can choose the negative sign for x close to $-\varepsilon$ and the positive sign for x close to ε. Furthermore, since for every (fixed) γ, $f(\gamma, u)$ is concave in u, we can jump from the negative to the positive solution if this will give a shock with zero speed (recall that u^{ε} is stationary). But since $f(\gamma^{\varepsilon}, u^{\varepsilon})$ is constant, we can jump at any value of x! For instance, we can choose to jump at $x = 0$, giving

$$u^{\varepsilon} = \begin{cases} -1 & \text{for } x \leq -\varepsilon, \\ 1 - \frac{2x}{\varepsilon} - \sqrt{\left(1 - \frac{2x}{\varepsilon}\right)^2 + 3} & \text{for } -\varepsilon < x < 0, \\ 1 - \frac{2x}{\varepsilon} + \sqrt{\left(1 - \frac{2x}{\varepsilon}\right)^2 + 3} & \text{for } 0 < x < \varepsilon, \\ 1 & \text{for } \varepsilon \leq x. \end{cases}$$

We show a plot of this solution in Fig. 8.11, and we note that although $u^{\varepsilon} \to u$, the variation of the approximate solution is larger than that of u. $\diamond$

This example readily generalizes to the following case. Assume that the map

$$u \mapsto f(\gamma, u)$$



has a single global maximum for all γ, and

$$\lim_{u\to-\infty} f(\gamma,u) = -\infty \quad \text{and} \quad \lim_{u\to\infty} f(\gamma,u) = -\infty.$$

Let $u^{\pm}(\gamma, y)$ denote the two solutions of

$$y = f\left(\gamma, u^{\pm}\right),$$

such that $u^- \le u^+$. As before, let u^ε denote the stationary solution of (8.46), where

$$\gamma^\varepsilon(x) = \gamma_l + \frac{x+\varepsilon}{2\varepsilon}\left(\gamma_r - \gamma_l\right), \quad -\varepsilon < x < \varepsilon.$$

Then it is possible to find a weak solution u^ε if and only if

$$u^-\left(\gamma_l, f\left(\gamma_l, u_l'\right)\right) = u_l' \quad \text{or} \quad u^+\left(\gamma_r, f\left(\gamma_r, u_r'\right)\right) = u_r'. \tag{8.47}$$

Recall that we are always assuming that u_l' and u_r' satisfy the Rankine–Hugoniot condition, i.e., $f(\gamma_l, u_l') = f(\gamma_r, u_r') = f^\times$. If both of the conditions in (8.47) hold, then this solution is given by

$$u^\varepsilon(x) = \begin{cases} u_l' & \text{for } x < -\varepsilon, \\ u^-\left(\gamma^\varepsilon(x), f^\times\right) & \text{for } -\varepsilon \le x \le x_J, \\ u^+\left(\gamma^\varepsilon(x), f^\times\right) & \text{for } x_J < x \le \varepsilon, \\ u_r' & \text{for } \varepsilon < x, \end{cases} \tag{8.48}$$

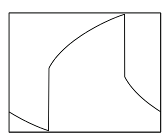
for every $x_J \in [-\varepsilon, \varepsilon]$. Since we are jumping from u^- to u^+, this jump is allowed since $u^- \le u^+$ and $f(\gamma, u) > f^\times$ in the interval (u^-, u^+). If only one of the conditions in (8.47) holds, then we stay on u^+ or u^- throughout the interval $[-\varepsilon, \varepsilon]$. If

$$u_l' = u^+\left(\gamma_l, f\left(\gamma_l, u_l'\right)\right) \quad \text{and} \quad u_r' = u^-\left(\gamma_r, f\left(\gamma_r, u_r'\right)\right),$$

we must at some point jump *from* u^+ *to* u^-, and this will give an entropy-violating weak solution. Looking at the shapes of the graphs of $f(\gamma_l, u)$ and $f(\gamma_r, u)$, we see that (8.47) is equivalent to the minimal jump entropy condition in this case. Hence, if (u_l', u_r') satisfies the minimal jump entropy condition, there exist entropy solutions u^ε of (8.46) such that u^ε tends to the minimal jump entropy condition when $\varepsilon \to 0$ (if the flux f has the properties assumed above).

Remark 8.11 The minimal jump entropy condition is not always reasonable. Entropy conditions are based on extra information, such as physics or common sense. To illustrate this, consider the equation

$$\begin{gathered} u_t + \left(\frac{1}{2}(u+\gamma)^2\right)_x = 0, \\ \gamma(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases} \quad u(x,0) = 0. \end{gathered} \tag{8.49}$$



In this case,

$$h_l(u;0) = \begin{cases} \frac{1}{2}(u-1)^2 & \text{if } u \le 1, \\ 0 & \text{if } u > 1, \end{cases} \quad h_r(u;0) = \begin{cases} 0 & \text{if } u \le -1, \\ \frac{1}{2}(u+1)^2 & \text{if } u > -1. \end{cases}$$

We see that there is a unique crossing value $f^{\times} = 1/2$, and the minimal jump entropy condition gives the solution $u(x,t) = 0$.

One can also try to find a solution of (8.49) by making the substitution $w = u + \gamma$, which turns (8.49) into

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0, \quad w(x,0) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

The entropy solution to this, found by taking the lower convex envelope, reads

$$w(x,t) = \begin{cases} -1 & \text{for } x < -t, \\ x/t & \text{for } -t \le x \le t, \\ 1 & \text{for } x > t. \end{cases}$$

Since $u = w - \gamma$, we obtain the alternative solution

$$\tilde{u}(x,t) = \begin{cases} 0 & \text{for } |x| > t, \\ \frac{x}{t} - \operatorname{sign}(x) & \text{otherwise.} \end{cases} \tag{8.50}$$

So which of these solutions shall we choose? We have already seen that the minimal jump solution, $u = 0$, is the limit of the viscous approximations u^{ε} satisfying

$$u^{\varepsilon}_t + \left(\frac{1}{2}(u^{\varepsilon} + \gamma)^2\right)_x = \varepsilon u^{\varepsilon}_{xx}. \tag{8.51}$$

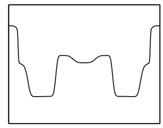
We know that w is the limit of the viscous approximation w^{ε} satisfying

$$w^{\varepsilon}_t + \left(\frac{1}{2}w^{\varepsilon 2}\right)_x = \varepsilon w^{\varepsilon}_{xx}.$$

This means that $\tilde{u}$ is the limit of $\tilde{u}^{\varepsilon}$, where $\tilde{u}^{\varepsilon}$ and γ^{ε} satisfy the viscous approximation for the system (8.6), i.e.,

$$\begin{aligned} \tilde{u}^{\varepsilon}_t + \left(\frac{1}{2}(\tilde{u}^{\varepsilon} + \gamma^{\varepsilon})^2\right)_x &= \varepsilon \tilde{u}^{\varepsilon}_{xx}, \\ \gamma^{\varepsilon}_t &= \varepsilon \gamma^{\varepsilon}_{xx}. \end{aligned} \tag{8.52}$$

Therefore, it is reasonable to choose $u = 0$ if (8.49) is an approximation of (8.51) and $\tilde{u}$ if (8.49) is an approximation of (8.52).



8.2 The Cauchy Problem

In this section we shall demonstrate the existence of an entropy solution to the conservation law where the flux function depends on a discontinuous coefficient. To be concrete, this is the initial value problem

$$\begin{cases} u_t + f(\gamma, u)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{8.53}$$

where $\gamma = \gamma(x)$ is a function of bounded variation. Fix an arbitrary $T > 0$, and set $\Pi_T = \mathbb{R} \times [0,T)$. By a solution of (8.53) we mean a weak solution, that is, a function u in $L^1_{\mathrm{loc}}(\Pi_T) \cap C([0,T); L^1_{\mathrm{loc}}(\mathbb{R}))$ such that

$$\iint\limits_{\mathbb{R}\times(0,\infty)} \big(u\varphi_t + f(\gamma,u)\varphi_x\big)\,dt\,dx + \int\limits_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx = 0, \tag{8.54}$$

for all test functions $\varphi \in C^1_0(\Pi_T)$. In order to demonstrate existence we shall assume that f and γ have additional properties; for instance, we must be assured that the Riemann problem has a solution for all relevant initial data.

To show that a solution exists, we shall construct it as a limit of a sequence of approximations. This can be done using difference approximations, front-tracking approximations, or the limits of parabolic regularizations, but we shall use front tracking.

Front Tracking for the Model Equation

In this section we will restrict ourselves to the model equation with $f(\gamma,u) = 4\gamma u(1-u)$, i.e.,

$$u_t + (4\gamma u(1-u))_x = 0, \quad u(x,0) = u_0(x). \tag{8.55}$$

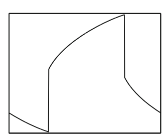
We assume that $\gamma\colon \mathbb{R} \to \mathbb{R}$ is a function of bounded variation that is continuously differentiable on a finite set of intervals. In particular, we assume that there exists a finite number of intervals

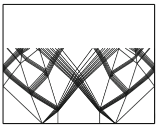
$$I_m = (\xi_m, \xi_{m+1}) \quad \text{for } m = 0, \dots, M,$$

where $\xi_0 = -\infty$, $\xi_{M+1} = \infty$, such that

$$\gamma' \big|_{I_m} \quad \text{is continuous and bounded for } m = 0, \dots, M. \tag{8.56}$$

For the moment, we also assume that the initial function u_0 is of bounded variation and such that $u_0(x) \in [0,1]$ for all x. Now we shall design a front-tracking scheme to approximate solutions of (8.55).





In order to prove convergence of the front-tracking approximations in the scalar case, we used that the variation of $\{u^\delta\}_{\delta>0}$ was uniformly bounded. As Example 8.13 will show, such a bound does not exist if γ is not constant.

In order to circumvent this obstacle, we shall work with the variable z defined by (8.38). The reason that this is a good idea is outlined in the remark below.

Remark 8.12 Assume that u^ε and v^ε are smooth solutions of the regularized equations

$$u_t^\varepsilon + f(\gamma, u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad v_t^\varepsilon + f(\gamma, v^\varepsilon)_x = \varepsilon v_{xx}^\varepsilon,$$

with smooth initial data u_0^ε and v_0^ε, respectively. Let η be a smooth convex function. We subtract these equations and multiply the result by $\eta'(u^\varepsilon - v^\varepsilon)$ to obtain

$$\begin{aligned}
\eta(u^\varepsilon - v^\varepsilon)_t &= -\eta'(u^\varepsilon - v^\varepsilon)\left[f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)\right]_x \\
&\quad + \varepsilon\eta(u^\varepsilon - v^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon - v^\varepsilon)(u^\varepsilon - v^\varepsilon)_x^2 \\
&\leq -\left[\eta'(u^\varepsilon - v^\varepsilon)(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon))\right]_x \\
&\quad + \varepsilon\eta(u^\varepsilon - v^\varepsilon)_{xx} + \eta'(u^\varepsilon - v^\varepsilon)_x(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon)).
\end{aligned}$$

Now we let $\eta = \eta_\kappa$ be a continuously differentiable approximation to $|\cdot|$, explicitly

$$\eta_\kappa(u) = \int_0^u \max\left(-1, \min\left(\frac{v}{\kappa}, 1\right)\right) dv.$$

Assuming that $u^\varepsilon - v^\varepsilon$ has compact support in x, we can integrate the above inequality over $x \in \mathbb{R}$, and get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \eta_\kappa(u^\varepsilon - v^\varepsilon)\, dx &\leq \int_{\mathbb{R}} \eta_\kappa''(u^\varepsilon - v^\varepsilon)(f(\gamma, u^\varepsilon) - f(\gamma, v^\varepsilon))(u^\varepsilon - v^\varepsilon)_x\, dx \\
&\leq L \int_{|u^\varepsilon - v^\varepsilon| < \kappa} |(u^\varepsilon - v^\varepsilon)_x|\, dx,
\end{aligned}$$

where $L = \sup|f_u|$, since

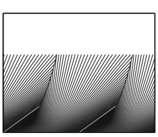
$$\eta_\kappa''(u) = \begin{cases} \frac{1}{\kappa} & \text{for } |u| \leq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

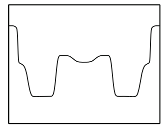
By Lemma B.5,

$$\lim_{\kappa \to 0} \int_{|u^\varepsilon - v^\varepsilon| < \kappa} |(u^\varepsilon - v^\varepsilon)_x|\, dx = 0.$$

Thus we can send κ to zero, and obtain for any two solutions of the regularized equation

$$\|u^\varepsilon(\cdot, t) - v^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0^\varepsilon - v_0^\varepsilon\|_{L^1(\mathbb{R})}. \tag{8.57}$$





Now we can set $v^\varepsilon(\cdot\,,t) = u^\varepsilon(\cdot\,,t+\tau)$ in (8.57), then divide by τ and let $\tau \to 0$, to deduce that

$$\|u_t^\varepsilon(\cdot\,,t)\|_{L^1(\mathbb{R})} \le \|u_t^\varepsilon(\cdot\,,0+)\|_{L^1(\mathbb{R})} = \left| f\left(\gamma, u_0^\varepsilon\right)\right|_{BV}. \tag{8.58}$$

Without loss of generality we can construct u_0^ε so that $\left|f(\gamma,u_0^\varepsilon)\right|_{BV} \le |f(\gamma,u_0)|_{BV}$. This means that the total variation of $f(\gamma,u^\varepsilon)$ is bounded independently of ε, i.e.,

$$|f\left(\gamma, u^\varepsilon(\cdot\,,t)\right)|_{BV} \le |f(\gamma,u_0)|_{BV}. \tag{8.59}$$

If $f_u(\gamma,u) \ge c > 0$ for all γ and u, then this would imply that also u^ε had uniformly bounded variation.[1] For the flux function in our example, $f_u(\gamma,1/2) = 0$, so we cannot deduce that u^ε is of bounded variation. This is precisely where the z-mapping comes to the rescue. We write (8.38) as

$$z(\gamma,u) = \operatorname{sign}\left(u - \frac{1}{2}\right)\left(f\left(\gamma,u\right) - f\left(\gamma,\frac{1}{2}\right)\right).$$

Now

$$\begin{aligned}|z\left(\gamma,u^\varepsilon\right)|_{BV} &\le |f\left(\gamma,u^\varepsilon\right)|_{BV} + \left\|f_\gamma\right\|_{L^\infty}|\gamma|_{BV}\\ &\le |f\left(\gamma,u_0\right)|_{BV} + \left\|f_\gamma\right\|_{L^\infty}|\gamma|_{BV}.\end{aligned}$$

Thus $z^\varepsilon = z(\gamma,u^\varepsilon)$ has uniformly bounded variation, and the mapping $u \mapsto z(\gamma,u)$ is continuous and invertible. The next step in this strategy is to attempt to show that $\{z^\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(\mathbb{R}\times[0,\infty))$, and thus (for a subsequence) $z^\varepsilon \to \bar{z}$ as $\varepsilon \to 0$. Then we define

$$u = z^{-1}\left(\gamma,\bar{z}\right) = \lim_{\varepsilon\to 0} z^{-1}\left(\gamma,z^\varepsilon\right) = \lim_{\varepsilon\to 0} u^\varepsilon.$$

The final step will then be to show that the limit u is a weak solution. See, e.g., [114] for an example where this strategy has been carried out.

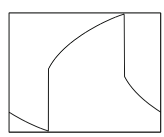
This remark is meant to indicate how the z-mapping could be used to show existence via viscous regularizations, and to motivate the use of the z-mapping also for front-tracking approximations.

As in the case without a coefficient, we start with a discussion of an approximate solution to the Riemann problem, or rather with the exact solution of the Riemann problem for an approximate equation. In the simple scalar case, we saw that the exact solution of the Riemann problem was piecewise constant in x/t if the flux function was piecewise linear. We shall now define an approximate flux function g^δ such that $g^\delta(\gamma,u) \approx 4\gamma u(1-u)$ and the solution of the Riemann problem with flux g^δ is piecewise constant.

From Sect. 8.1 we saw that the solution of the Riemann problem consisted of a sequence of straight lines in the (z,γ)-plane, where

$$z(\gamma,u) = \operatorname{sign}\left(u - \frac{1}{2}\right)\gamma\left(1-2u\right)^2. \tag{8.60}$$

[1] This assumption excludes resonances, i.e., coinciding eigenvalues.



There were z-waves, over which γ is constant, and γ-waves, over which γ was not constant. Now fix a (small) positive number δ, and set

$$\gamma_i = i\delta, \quad i > 0, \quad i \in \mathbb{N}, \tag{8.61}$$

and for integers j such that $-i \le j \le i$, $z_{i,j} = j\delta$, and

$$u_{i,j} = z^{-1}\left(\gamma_i, z_{i,j}\right) = \frac{1}{2}\left(1 + \operatorname{sign}\left(z_{i,j}\right)\sqrt{\frac{\left|z_{i,j}\right|}{\gamma_i}}\right). \tag{8.62}$$

Note that the set $\left\{(z_{i,j}, \gamma_i)\right\}$ defines a grid in the (z, γ)-plane. We define g^δ to be the linear interpolation to f on this grid, i.e.,

$$g^\delta\left(\gamma_i, u\right) = f_{i,j} + \left(u - u_{i,j}\right)\frac{f_{i,j+1} - f_{i,j}}{u_{i,j+1} - u_{i,j}}, \quad \text{for } u \in [u_{i,j}, u_{i,j+1}], \tag{8.63}$$

where $f_{i,j} = f(\gamma_i, u_{i,j}) = 4\gamma_i u_{i,j}(1 - u_{i,j})$. For each fixed i, $g^\delta(\gamma_i, u)$ will be a concave function with a maximum for $u = 1/2$. Therefore the solution of the Riemann problem

$$\begin{gathered} u_t + g^\delta\left(\gamma(x), u\right)_x = 0, \\ u(x,0) = \begin{cases} u_{i,j} & \text{for } x < 0, \\ u_{m,n} & \text{for } x > 0, \end{cases} \quad \gamma(x) = \begin{cases} \gamma_i & \text{for } x < 0, \\ \gamma_m & \text{for } x > 0, \end{cases} \end{gathered} \tag{8.64}$$

can be found from the diagrams in Fig. 8.10. Furthermore, since g^δ is piecewise linear in u, this solution will be piecewise constant in x/t. Also, by our choice of interpolation points in constructing g^δ, *all* the intermediate values of $u(x,t)$ will be grid points, i.e.,

$$z(\gamma(x), u(x,t)) = \left(z_{i',j'}, \gamma_{i'}\right), \quad \text{where } i' = i \text{ or } i' = m.$$

We label the grid points in the (u, γ)-plane, or when there is no danger of misunderstanding, in the (z, γ)-plane $\mathcal{U}^\delta$. Hence, the solution of the Riemann problem takes pointwise values in $\mathcal{U}^\delta$ if the "initial" states $(u(x,0), \gamma(x))$ take values in $\mathcal{U}^\delta$.

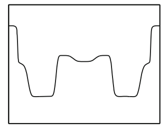
Once we have the solution of the approximate Riemann problem (8.64), we can use this to design a front-tracking scheme. To this end, let $\left\{u_0^\delta\right\}_{\delta>0}$ and $\left\{\gamma^\delta\right\}_{\delta>0}$ be two sequences of piecewise constant functions such that

$$\left(u_0^\delta(x), \gamma^\delta(x)\right) \in \mathcal{U}^\delta \quad \text{for all but a finite number of } x\text{-values.}$$

Furthermore, we demand that

$$\lim_{\delta\to 0}\left\|u_0^\delta - u_0\right\|_1 = 0, \tag{8.65}$$

$$\lim_{\delta\to 0}\left\|\gamma^\delta - \gamma\right\|_1 = 0. \tag{8.66}$$



We label the discontinuity points of γ^δ by $y_1 < \cdots < y_N$. Of course, these depend on δ, but we suppress this dependency in our notation. At each point of discontinuity of either u_0^δ or γ^δ, we have a Riemann problem whose solution will give a sequence of z-waves and γ-waves. We define the front-tracking approximation as in the scalar case, by following discontinuities, called fronts, and solve the Riemann problems (using the approximate flux g^δ) defined by their collisions. We call the resulting piecewise constant function u^δ. As in the scalar case, in order to show that we can define $u^\delta(\,\cdot\,, t)$ for every $t > 0$, we must study the interaction of fronts.

The front-tracking solution u^δ has two types of fronts, z-fronts and γ-fronts, where z-fronts are those fronts whose left and right γ-values are equal. Regarding the collision of two or more z-fronts, we have seen that such a collision always results in *one* z-front. Hence, the number of fronts in u^δ decreases when z-fronts collide.

Moreover, γ-fronts have zero speed (recall that these are the discontinuities of γ^δ), and therefore two γ-fronts will never collide. It remains to study collisions between z-fronts and γ-fronts. This turns out to be complicated, and simple examples show that we can have such collisions that result in three outgoing fronts. Furthermore, even if such collisions always result in two outgoing fronts, it is in general not possible to bound the total variation of u^δ independently of δ, as the next example shows.

◇ Example 8.13
Assume for the moment that

$$u_0(x) = \frac{1}{2}, \quad \gamma(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 + x & \text{for } 0 < x \leq 2, \\ 2 & \text{for } 2 < x. \end{cases} \tag{8.67}$$

In this case $z(\gamma(x), u_0(x)) = 0$, and we can set

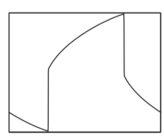
$$\gamma^\delta(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 + i\delta & \text{for } i\delta < x \leq (i+1)\delta,\ i = 0, \dots, 2/(\delta - 1), \\ 2 & \text{for } 2 < x. \end{cases}$$

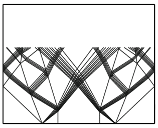
The z-component of the solution of each of the Riemann problems defined by $(u_0^\delta, \gamma^\delta)$ at $x = i\delta$ reads

$$(z, \gamma) = \begin{cases} (0, 1 + (i-1)\delta) & \text{for } x < i\delta, \\ (-\delta, 1 + i\delta) & \text{for } i\delta \leq x < ts_i + i\delta, \\ (0, 1 + i\delta) & \text{for } i\delta + ts_i \leq x, \end{cases}$$

where

$$s_i = \sqrt{\delta(1 + i\delta)}.$$





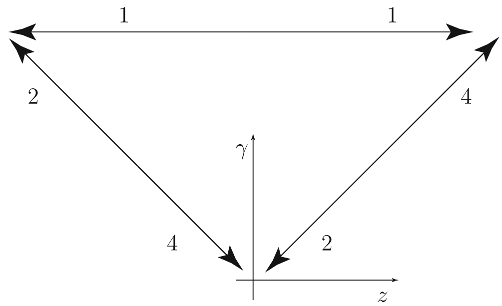


Fig. 8.12 The weights in the Temple functional, (8.68)

This follows from the diagram in Fig. 8.10. Hence, before any interaction of fronts, the total variation of u^δ reads

$$\left|u^\delta\right|_{BV} = \sum_{i=1}^{1/\delta} \sqrt{\frac{\delta}{1+i\delta}} \geq \sum_{i=1}^{1/\delta} \sqrt{\frac{\delta}{2}} = \frac{1}{\delta}\sqrt{\frac{\delta}{2}} = \frac{1}{\sqrt{2\delta}} \to \infty \quad \text{as } \delta \to 0.$$

Despite this, since $\gamma(x)$ is Lipschitz continuous, the total variation of the exact solution to this problem is uniformly bounded for $t < T$ for every finite time T; see, e.g., Kružkov [118] or Karlsen and Risebro [109]. As an indication of things to come, we observe in passing that

$$\left|z^\delta\right|_{BV} = \sum_{i=1}^{1/\delta} |\delta| = 1,$$

where $z^\delta = z(\gamma^\delta, u^\delta)$. So the total variation of the transformed variable z is uniformly bounded for this example, at least until the first interaction. ◇

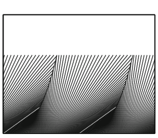
For reasons outlined in the above example and in Remark 8.12, we shall work with the z variable instead of u. In the above example, it was trivial to show that the variation of z was bounded independently of δ, but this becomes more cumbersome in general, so to help us we use the Temple functional.[2] For a single front, which we label $\mathfrak{f}$, this is defined as

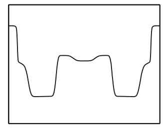
$$T(\mathfrak{f}) = \begin{cases} |\Delta z| & \text{if } \mathfrak{f} \text{ is a } z\text{-front,} \\ 4\,|\Delta z| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l < z_r, \\ 2\,|\Delta z| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l > z_r, \end{cases} \tag{8.68}$$

where z_l is the z value to the left of the front, z_r the value to the right, and $\Delta z = z_r - z_l$. Fig. 8.12 will perhaps be useful later. The figure shows the weights given to $|\Delta z|$ in the various cases. Recall also that if $\mathfrak{f}$ is a γ-front, then

$$|\Delta z| = |\Delta\gamma|\,,$$

[2] This, or rather a similar functional, was first used in the paper of Temple [176].





and thus an alternative definition of T is

$$T(\mathfrak{f}) = \begin{cases} |\Delta z| & \text{if } \mathfrak{f} \text{ is a } z\text{-front}, \\ 4\,|\Delta\gamma| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l < z_r, \\ 2\,|\Delta\gamma| & \text{if } \mathfrak{f} \text{ is a } \gamma\text{-front and } z_l > z_r. \end{cases}$$

For a sequence of fronts, we define T additively, and with a slight abuse of notation we write

$$T\left(u^\delta\right) = \sum_{\mathfrak{f}\in u^\delta} T\left(\mathfrak{f}\right).$$

With this definition of T we have the obvious inequalities

$$\left|z^\delta\right|_{BV} \le T\left(u^\delta\right) \le 4\left(\left|z^\delta\right|_{BV} + \left|\gamma^\delta\right|_{BV}\right). \tag{8.69}$$

We also have for every front $\mathfrak{f} \in u^\delta$ that

$$T\left(\mathfrak{f}\right) \ge \delta.$$

With a further abuse of notation we shall write $T(t) = T(u^\delta(\,\cdot\,,t))$.

Lemma 8.14 *If* $0 < s < t$*, then*

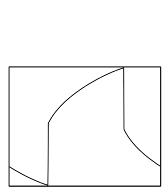
$$T(t) \le T(s). \tag{8.70}$$

Hence $\left|z^\delta(\,\cdot\,,t)\right|_{BV} \le T(0+)$.

Proof The value of T will change only when fronts collide. From the analysis of collisions of z-fronts, we have established that T does not increase at such collisions. To prove the lemma, it therefore remains to study collisions between z-fronts and γ-fronts. We say that a γ-front is nonpositive if it connects points in the half-plane $z \le 0$, and similarly, we say that it is nonnegative if it connects points in the half-plane $z \ge 0$.

We shall study the collision between z-fronts and a γ-fronts, and we thus have three points in the (z,γ)-plane, (z_l,γ_l), (z_m,γ_m), and (z_r,γ_r), which lie to the left of, in between, and to the right of the colliding fronts respectively. If we have more than one z-front colliding with the γ-front, we can reduce to the two-front collision type as follows. If we have several z-fronts colliding with the γ-front from the same side, then we can resolve the collision between the z-fronts first, and then the collision between the (single) resulting z-front and the γ-front.

Therefore, we consider the case that we have two z-fronts colliding with one γ-front. One z-front collides from the left, the other from the right. We label the states to the left of the left z-front $L = (z_l,\gamma_l)$, the one to the left of the γ-front $M_- = (z_-,\gamma_l)$, the state to the left of the right z-front $M_+ = (z_+,\gamma_r)$, and finally, the state to the right of this z-front $R = (z_r,\gamma_r)$. Of course we may have $z_l = z_-$ or $z_+ = z_r$, in which case we have only two colliding fronts. In order to study how T



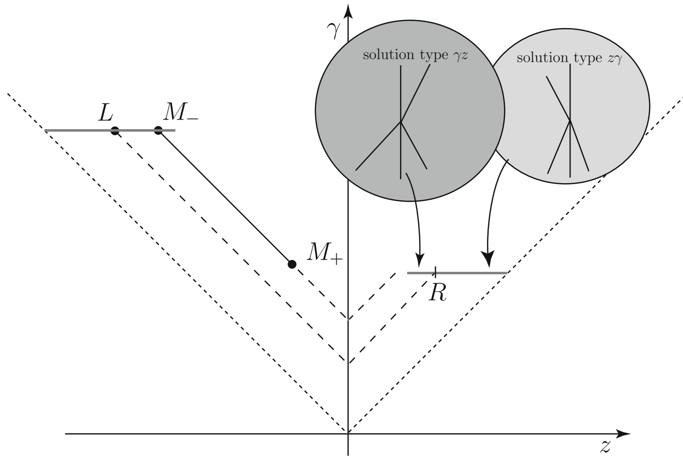


Fig. 8.13 The possible locations of L and R if the γ-front is nonpositive and $\gamma_l > \gamma_r$

changes by this collision, we study a number of cases. These are distinguished by whether the γ-front lies in the left (it is *nonpositive*) or the right (it is *nonnegative*) half-spaces and by whether $\gamma_l < \gamma_r$.

Case 1: The γ-front is nonpositive and $\gamma_l > \gamma_r$. Consult Fig. 8.13 in what follows. Now we regard the z-front, and hence M_- and M_+, as fixed. Since the γ-front is negative, $z_+ \leq 0$, and since $\gamma_l > \gamma_r$, $z_- \leq -\delta$. The z-front between z_l and z_- moves with positive speed, and it is the solution of the Riemann problem defined by these two states with a flux function $f^\delta(\gamma_l, \cdot)$. Hence z_l cannot be larger than "one breakpoint to the right" of z_-. If it were, then the solution would contain more than one front. Furthermore, $u_l = z^{-1}(\gamma_l, z_l) \geq 0$, which is the same as $z_l \geq -\gamma_l$. Thus

$$z_l \in [-\gamma_l, z_- + \delta].$$

This interval is indicated by the upper left horizontal gray line in Fig. 8.13. Reasoning in the same way, we see that the right z-front must have negative speed and thus that

$$z_r \in \{z_+\} \cup [-z_+ + \delta, \gamma_r].$$

This interval is indicated by the lower right horizontal gray line in Fig. 8.13. We have two alternatives. First if $-z_l + \gamma_l \geq z_r + \gamma_r$, then the solution of the Riemann problem defined by (z_l, γ_l) and (z_r, γ_r) is of type γz, and if $-z_l + \gamma_l < z_r + \gamma_r$, then this Riemann problem has a solution of type $z\gamma$. This is indicated in Fig. 8.13, where the dashed line passing through L is the line where $|z| + \gamma = -z_l + \gamma_l$.

If $z_l = z_-$, i.e., we have a collision between a γ-front and a z-front from the right, then the solution type is always $z\gamma$. In other words, the wave is transmitted. Consulting Fig. 8.12, we see that if $z_l \leq z_-$, then T is unchanged by the collision. If $z_l = z_- + \delta$ (which is the maximum value for z_l), and the solution type is $z\gamma$, then T decreases by 2δ. Otherwise, T is unchanged. In the special case that $z_r = z_- = 0$ and $z_l = z_- + \delta$, the z-front is reflected. Thus we see that a reflection results in a decrease of T by 2δ. The reader is urged to check these statements.

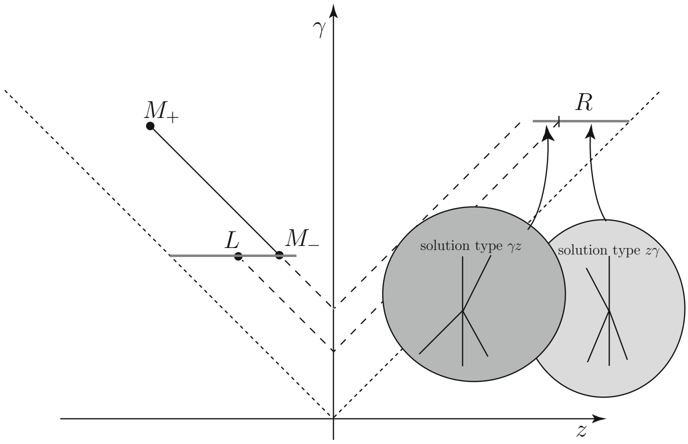


Fig. 8.14 The possible locations of L and R if the γ-front is nonpositive and $\gamma_l < \gamma_r$

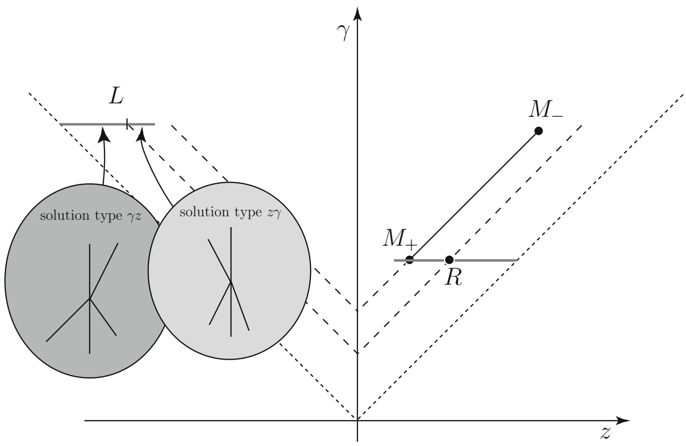


Fig. 8.15 The possible locations of L and R if the γ-front is nonnegative and $\gamma_l > \gamma_r$

Case 2: The γ-front is nonpositive and $\gamma_l < \gamma_r$. Consult Fig. 8.14 in what follows. Since the fronts are colliding, the speed of the left z-front is positive and that of the right z-front is negative. Hence $z_l \in [-\gamma_l, z_- + \delta]$ and $z_r \in \{z_+\} \cup [-z_+ + \delta, \gamma_r]$. These intervals are indicated in Fig. 8.14 by the lower left and upper right horizontal lines. If $z_r + \gamma_r < -z_l + \gamma_l$, then the solution type is $z\gamma$, and if $z_r + \gamma_r \geq -z_l + \gamma_l$, the solution type is γz. In both of these cases T is unchanged. If $z_r = z_+$, then the solution type is γz, and if $z_l = z_-$, then the solution type is $z\gamma$. Thus there are no reflected fronts in this case.

Case 3: The γ-front is nonnegative and $\gamma_l > \gamma_r$. Consult Fig. 8.15 in what follows. This case is similar to Case 2 above. By considering the speeds of the colliding fronts, we find that

$$z_l \in [-z_- - \delta, -\gamma_l] \cup \{z_-\} \quad \text{and} \quad z_r \in [z_+ - \delta, \gamma_r].$$

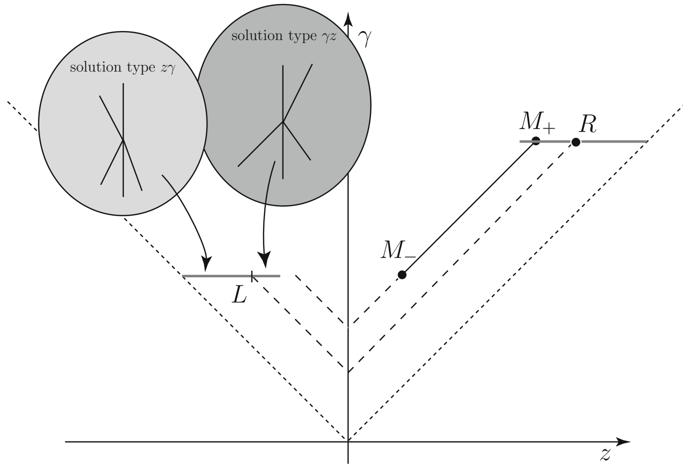


Fig. 8.16 The possible locations of L and R if the γ-front is nonnegative and $\gamma_l < \gamma_r$

If $|z_l| + \gamma_l < z_r + \gamma_r$, then the solution is of type γz, and if $|z_l| + \gamma_l \geq z_r + \gamma_r$, the solution is of type $z\gamma$. Note that if $z_r = z_+$, then the solution type is γz, while if $z_l = z_-$, the solution type is $z\gamma$. So also in this case a front cannot be reflected. Furthermore, T is unchanged.

Case 4: The γ-front is nonnegative and $\gamma_l < \gamma_r$. Consult Fig. 8.16 in what follows. This case is similar to Case 1 above. We find that

$$z_l \in [-z_- - \delta, -\gamma_l] \cup \{z_-\} \quad \text{and} \quad z_r \in [z_+ - \delta, \gamma_r].$$

If $|z_l| + \gamma_l > z_r + \gamma_r$, then the solution type is $z\gamma$, while if $|z_l| + \gamma_l \leq z_r + \gamma_r$, the type is γz. If $z_r = z_+ - \delta$ and the solution type is γz, then T decreases by 2δ; otherwise, it is unchanged. If $z_+ = z_r$, then the solution type is $z\gamma$, while if $z_l = z_-$ and $z_r = z_+ - \delta$, we have a reflection, and in this case T decreases by 2δ.

This finishes the proof of Lemma 8.14. □

Remark 8.15 Recall that we have used the term "reflection" for a collision between a z-front and a γ-front if the z-front collides from the left and the solution of the Riemann problem is of type $z\gamma$, or if the z-front collides from the right and the solution type is γz. From the proof of the above lemma, it is clear that whenever we have a reflection, T decreases by 2δ. Hence, if $T(0+)$ is finite, we can have only a finite number of reflections in u^δ.

One immediate consequence of Lemma 8.14 and (8.69) is the following result.

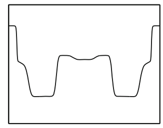
Corollary 8.16 *If*

$$\left|\gamma^\delta\right|_{BV} \leq |\gamma|_{BV} \quad \textit{and} \quad \left|z(u_0^\delta, \gamma^\delta)\right|_{BV} \leq |z(u_0, \gamma)|_{BV}, \tag{8.71}$$

then for $t \geq 0$,

$$\left|z^\delta(\,\cdot\,, t)\right|_{BV} \leq |z(u_0, \gamma)|_{BV} + 4\,|\gamma|_{BV},$$

and thus $\left|z^\delta(\,\cdot\,, t)\right|_{BV}$ *is bounded independently of* δ *and* t.



Note that this corollary in itself does not imply that the front-tracking construction u^δ can be defined up to an arbitrary time t. In order to show this, we have to do some more work. For a z-front $\mathfrak{f}_z$ let $\mathcal{A}(\mathfrak{f}_z)$ be the set of γ-fronts $\mathfrak{f}_\gamma$ that approach $\mathfrak{f}_z$, i.e.,

$$\mathfrak{f}_\gamma \in \mathcal{A}\left(\mathfrak{f}_z\right) \quad \text{if} \quad \begin{cases} x(\mathfrak{f}_z) < x(\mathfrak{f}_\gamma) & \text{and } s(\mathfrak{f}_z) \geq 0 \text{ or} \\ x(\mathfrak{f}_z) > x(\mathfrak{f}_\gamma) & \text{and } s(\mathfrak{f}_z) \leq 0, \end{cases}$$

where $x(\mathfrak{f})$ denotes the position of $\mathfrak{f}$, and $s(\mathfrak{f})$ its speed. For every z-front $\mathfrak{f}_z$ define

$$J\left(\mathfrak{f}_z\right) = \sum_{\mathfrak{f}_\gamma \in \mathcal{A}(\mathfrak{f}_z)} |\Delta\gamma|\,, \tag{8.72}$$

where $\Delta\gamma$ denotes the difference in γ over the front.

Lemma 8.17 *Assume that* (8.71) *holds. Then for each fixed* δ*, the functional*

$$F(t) = \delta \sum_{\mathfrak{f}_z} J\left(\mathfrak{f}_z\right) + T(t)\,|\gamma|_{BV} \tag{8.73}$$

is nonincreasing, and it decreases by at least δ^2 *when a* z*-front collides with a* γ*-front.*

Proof Let $N_{\mathfrak{f}}(t)$ denote the number of fronts in u^δ at time t. For each front we have $|\Delta z| \geq \delta$, and thus

$$N_{\mathfrak{f}} \leq \frac{\left|z^\delta\right|_{BV}}{\delta}.$$

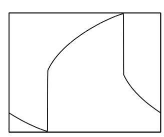
Recall that T is bounded and $J\left(\mathfrak{f}_z\right) \leq |\gamma|_{BV}$. Hence F is bounded by

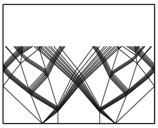
$$\begin{aligned} F(t) &\leq \delta\,|\gamma|_{BV}\, N_{\mathfrak{f}} + 2T(0+)\,|\gamma|_{BV} \\ &\leq 4\,|\gamma|_{BV}\left(|z(u_0,\gamma)|_{BV} + 4\,|\gamma|_{BV}\right). \end{aligned} \tag{8.74}$$

Thus F is bounded independently of δ and t. We must show that F is decreasing by at least δ^2 for collisions between z-fronts and γ-fronts, and nonincreasing when z-fronts collide.

First consider a collision between one (or two) z-front(s) and a γ-front. From the proof of Lemma 8.14 we saw that either (**a**) a z-front "passes through" the γ-front in the collision, or (**b**) we have a reflection, and T decreases by 2δ. If (**a**) holds, then the sum in (8.73) will "lose" at least one term (two terms if one z-front is lost in the collision) of size $|\Delta\gamma|$, and the second term in (8.73), does not increase. Thus F decreases by at least $\delta\,|\Delta\gamma| \geq \delta^2$. If (**b**) holds, then T decreases by 2δ, and the sum increases by at most $|\gamma|_{BV}$. Hence F decreases by a least $\delta\,|\gamma|_{BV} \geq \delta^2$.

Next we consider a collision between two (or more) z-fronts. Recall that this collision will result in one z-front. If more than two fronts collide, we can consider this as several collisions between two fronts occurring at the same point. Therefore,





we consider a collision between two z-fronts, $\mathfrak{f}_l$ and $\mathfrak{f}_r$, separating values z_l, z_m and z_r. We label the resulting front $\mathfrak{f}$. If z_m is between z_l, and z_r, then T does not change by the collision. However, the speed of $\mathfrak{f}$ is between the speeds of $\mathfrak{f}_l$ and $\mathfrak{f}_r$. If the speed of $\mathfrak{f}$ is different from 0, then $\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_l)$ or $\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_r)$. Hence the sum in (8.73) loses one term, and F decreases by at least δ^2. If the speed of $\mathfrak{f}$ is 0, then the speed of $\mathfrak{f}_l$ is positive, and the speed of $\mathfrak{f}_r$ negative, whence

$$\mathcal{A}(\mathfrak{f}) = \mathcal{A}(\mathfrak{f}_l) \cup \mathcal{A}(\mathfrak{f}_r),$$

and thus F is constant.

If z_m is not between z_l and z_r, then either $z_r = z_m - \delta$ or $z_l = z_m + \delta$. This is so because g^δ is convex. In this case T decreases by δ, and the first term in equation (8.73) increases by at most $\delta \left|\gamma^\delta\right|_{BV}$. This concludes the proof of the lemma. ☐

Note that an immediate consequence of equation (8.74) and Lemma 8.17 is that for a fixed δ, the number of collisions of z-fronts and γ-fronts is bounded by

$$4\left|\gamma\right|_{BV} \frac{\left|z(u_0,\gamma)\right|_{BV} + 4\left|\gamma\right|_{BV}}{\delta^2}.$$

Also, the *smallest* absolute value of the speed of any z-front having speed different from zero is bounded below by

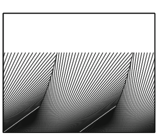
$$\sqrt{\min\left(\gamma^\delta\right)\delta}.$$

Hence, after some *finite* time T_1, collisions between z-fronts and γ-fronts cannot occur. This means that there must be a time $T_2 \geq T_1$ such that all z-fronts in the interval (y_1, y_N) (recall that γ^δ has discontinuities at $y_1, \dots, y_N$) have zero speed, that all z-fronts to the left of y_1 have nonpositive speed and that the z-fronts to the right of y_N have nonnegative speed for all $t > T_2$. Outside the interval $[y_1, y_N]$, u^δ is the front-tracking approximation to a scalar conservation law with a constant coefficient, and there can be only a finite number of collisions between fronts in u^δ there. Therefore, there exists a finite time $T_3 \geq T_2$ such that there will be no further collisions between fronts in u^δ for $t > T_3$. Thus, the front-tracking method is hyperfast.

◇ Example 8.18

Now we pause for a moment in order to exhibit an example of how front tracking looks in practice. We wish to find the front-tracking approximation to the initial value problem

$$\begin{gathered} u_t + \left[4\gamma(x)u(1-u)\right]_x = 0, \quad t > 0, \\ \gamma(x) = \begin{cases} e^{|x|} & \text{for } -1 \leq x \leq 1, \\ \sin(\pi x^2) + 2 & \text{for } 1 < |x| < 2, \\ 1 & \text{otherwise}, \end{cases} \\ u(x,0) = \begin{cases} \frac{1}{2}(1 + e^{-|x|}) & \text{for } -1 \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \end{gathered} \tag{8.75}$$



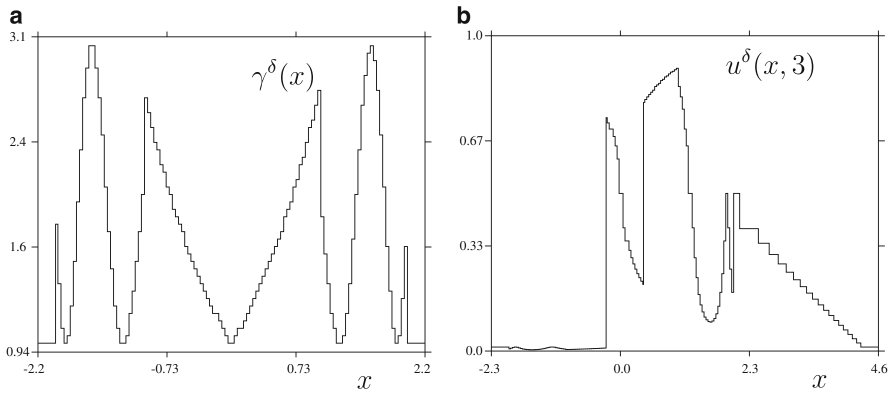


Fig. 8.17 **a** $\gamma^\delta(x)$. **b** $u^\delta(x,3)$ for $\delta = 0.05$

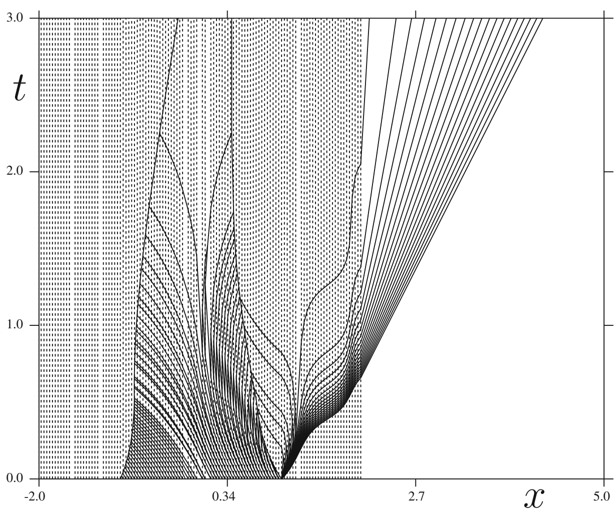


Fig. 8.18 The fronts in the (x,t)-plane for Example 8.18

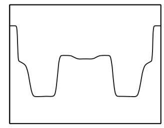
In Fig. 8.17, we show the approximation γ^δ for $\delta = 0.05$, and $u^\delta(\cdot\,,3)$. In Fig. 8.18, we show the fronts in u^δ in the (x,t)-plane. Here z-fronts are marked with solid lines, and γ-fronts with dashed lines. We see that the number of fronts decreases rapidly, and there do not seem to be many collisions after $t = 3$. ◇

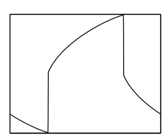
Returning now to the more general case, we claim that the sequence $\{z^\delta\}_{\delta>0}$ satisfies the following bounds:

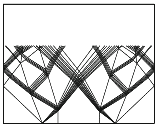
$$\left\|z^\delta\right\|_{L^\infty(\mathbb{R})} \le \left\|\gamma^\delta\right\|_{L^\infty(\mathbb{R})} \le C, \tag{8.76}$$

$$\left\|z^\delta(\cdot\,,t)\right\|_{L^1_{\mathrm{loc}}} \le C, \quad t < T, \tag{8.77}$$

$$\|z(\cdot\,,t) - z(\cdot\,,s)\|_{L^1(\mathbb{R})} \le C(t-s), \tag{8.78}$$







where the constant C does not depend on t or on δ. The first bound (8.76) follows by the definition of z, (8.60), and the fact that u^δ takes values in the interval $[0,1]$. Regarding (8.77), we have that u^δ is a weak solution of

$$u_t^\delta + g^\delta\left(\gamma^\delta, u^\delta\right)_x = 0, \quad u^\delta(x,0) = u_0^\delta(x). \tag{8.79}$$

Thus we can repeat the argument used in the proof of Theorem 7.10, to obtain

$$\begin{aligned}\left\|u^\delta(\,\cdot\,,t) - u^\delta(\,\cdot\,,s)\right\|_{L^1(\mathbb{R})} &\le \max_{\tau\in[s,t]} \left|g^\delta\left(\gamma^\delta, u^\delta(\,\cdot\,,\tau)\right)\right|_{BV}(t-s)\\ &\le \max_{\tau\in[s,t]} \left|z^\delta(\,\cdot\,,\tau)\right|_{BV}(t-s)\\ &\le C(t-s),\end{aligned} \tag{8.80}$$

for some constant not depending on t, s, or δ. Setting $s=0$, we obtain

$$\left\|u^\delta(\,\cdot\,,t)\right\|_{L^1(\mathbb{R})} \le \left\|u_0^\delta\right\|_{L^1(\mathbb{R})} + Ct, \tag{8.81}$$

and thus $u^\delta(\,\cdot\,,t)$ is in $L^1(\mathbb{R})$ for all finite t. Now

$$\begin{aligned}\left|z\left(u^\delta,\gamma^\delta\right)\right| &= \left|z\left(0,\gamma^\delta\right) + z_u\left(\xi,\gamma^\delta\right)u^\delta\right|\\ &\le \left|\gamma^\delta\right| + C\left|u^\delta\right|,\end{aligned}$$

for some positive constant C, where ξ is in the interval $[0,u^\delta]$. Since γ^δ is in L^1_{loc}, equation (8.77) follows. Actually, in our case, since $u^\delta(x,t)\in[0,1]$, we have that

$$\left\|u^\delta(\,\cdot\,,t)\right\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} u^\delta(x,t)\,dx = \int_{\mathbb{R}} u_0^\delta(x)\,dx = \left\|u_0^\delta\right\|_1,$$

which is stronger than (8.81).

To prove (8.78) we use the equality

$$\begin{aligned}z^\delta(x,t) - z^\delta(x,s) &= z\left(u^\delta(x,t),\gamma^\delta\right) - z\left(u^\delta(x,s),\gamma^\delta\right)\\ &= z_u\left(\xi,\gamma^\delta\right)\left(u^\delta(x,t) - u^\delta(x,s)\right).\end{aligned}$$

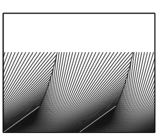
Since z_u is bounded, by (8.80) the bound (8.78) holds.

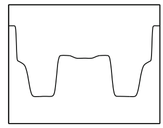
Hence, by standard techniques as in the case with constant coefficients, it follows that there exists a subsequence of $\{\delta\}$ (which we also label $\{\delta\}$) and a function $z\in L^1_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))\cap L^\infty((0,\infty);BV(\mathbb{R}))$ such that

$$\lim_{\delta\to0} z^\delta = z \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T]). \tag{8.82}$$

Since $z^\delta = z(u^\delta,\gamma^\delta)$, it also follows that there is a function $u\in L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T])$ such that $u^\delta\to u$, and $u=z^{-1}(z,\gamma)$. Furthermore, for this subsequence also $g^\delta(\gamma^\delta,u^\delta)\to f(\gamma,u)$. Thus

$$\begin{aligned}&\lim_{\delta\to0}\iint\left(u^\delta\varphi_t + g^\delta(\gamma^\delta,u^\delta)\varphi_x\right)dx\,dt\\ &\qquad = \iint\left(u\varphi_t + f(\gamma,u)\varphi_x\right)dx\,dt,\end{aligned}$$





and by construction,

$$\lim_{\delta \to 0} \int_{\mathbb{R}} u^\delta(x,0)\varphi(x,0)\,dx = \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx.$$

Since u^δ is a weak solution to (8.79), it follows from this that u is a weak solution to (8.53).

Furthermore, it is transparent that although we performed the analysis for $f(\gamma, u) = 4\gamma u(1-u)$, our results could be (slightly) extended to include flux functions that are similar to f. To be precise, assume that:

A.1 There is an interval $[a,b]$ such that $f(\gamma,a) = f(\gamma,b) = C$ for all γ.
A.2 There is a point $u^\star(\gamma) \in (a,b)$ such that $f_u(\gamma,u) > 0$ for $a < u < u^\star(\gamma)$ and $f_u(\gamma,u) < 0$ for $u^\star(\gamma) < u < b$.
A.3 The map $\gamma \mapsto f(\gamma,u)$ is strictly monotone for all $u \in (a,b)$.
A.4 The flux function f belongs to $C^2(\mathbb{R} \times [a,b])$.

If f satisfies these assumptions, we can define the mapping z as

$$z(\gamma,u) = \operatorname{sign}(u - u^\star(\gamma))\,(f(\gamma,u^\star(\gamma)) - f(\gamma,u)), \tag{8.83}$$

and use this to show that the front-tracking approximation is well defined. This analysis is only a slight modification of the analysis in the case $f(\gamma,u) = 4\gamma u(1-u)$. Hence, mutatis mutandis, we have proved the following theorem.

Theorem 8.19 *Let f be a function satisfying A.1–A.4, and assume that $u_0(x)$ is a function in L^1_{loc} taking values in the interval $[a,b]$, and that γ is a function in $BV(\mathbb{R}) \cup L^1_{\mathrm{loc}}(\mathbb{R})$. Then there exists a weak solution to the initial value problem*

$$u_t + f(\gamma,u)_x = 0, \quad x \in \mathbb{R} \quad t > 0, \qquad u(x,0) = u_0(x).$$

Furthermore, this solution is the limit of a sequence of front-tracking approximations.

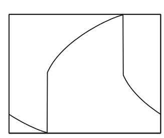
An Entropy Inequality

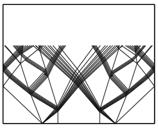
Now we shall show that the limit of every front-tracking approximation to the general conservation law (8.53) satisfies a Kružkov-type entropy condition. Thus we let u^δ be a weak solution to the approximate problem

$$\begin{cases} u^\delta_t + g^\delta\left(\gamma^\delta, u^\delta\right)_x = 0, & x \in \mathbb{R} \quad t > 0, \\ u^\delta(x,0) = u^\delta_0(x), & x \in \mathbb{R}, \end{cases} \tag{8.84}$$

where $g^\delta(\gamma,\cdot)$ is a piecewise linear continuous approximation of $f(\gamma,u)$ such that $g^\delta \to f$ as $\delta \to 0$. Here γ^δ is a piecewise constant approximation to γ, such that $\gamma^\delta \to \gamma$ in L^1 as $\delta \to 0$. We *assume* that u^δ can be constructed by front tracking, and that for each fixed $T > 0$,

$$u^\delta \to u \text{ in } L^1(\mathbb{R} \times [0,T]) \text{ as } \delta \to 0. \tag{8.85}$$





Furthermore, we let

$$z(\gamma,u)=\int_0^u |f_u(\gamma,v)|\,dv, \tag{8.86}$$

and set $z^\delta = z(\gamma^\delta,u^\delta)$. We shall also assume that for each t the family $\{z^\delta(\,\cdot\,,t)\}$ is a sequence of uniformly bounded variation in x and satisfies the three basic estimates (8.76), (8.77), and (8.78), so that we have convergence of z^δ along a subsequence.

Using that u^δ is a weak solution to (8.84), it is not hard to show that u is a weak solution to (8.53) if $u_0^\delta \to u$ as $\delta \to 0$. We would like to show that the limit u satisfies a generalization of the Kružkov entropy condition. Recall that if γ is continuous, then an entropy solution to (8.53) in the strip $\Pi_T = \mathbb{R}\times[0,T]$ satisfies

$$\begin{aligned}&\iint\limits_{\Pi_T}\big(|u-c|\,\varphi_t + F(\gamma,u,c)\varphi_x\big)\,dx\,dt \\ &\qquad - \iint\limits_{\Pi_T} \operatorname{sign}(u-c)\,\partial_x f(\gamma,c)\varphi\,dx\,dt + \int_{\mathbb{R}} |u_0(x)-c|\,\varphi(x,0)\,dx \geq 0,\end{aligned} \tag{8.87}$$

for all constants c and all nonnegative test functions φ such that $\varphi(\,\cdot\,,T)=0$. Here F is the Kružkov entropy flux defined by

$$F(\gamma,u,c)=\operatorname{sign}(u-c)\,(f(\gamma,u)-f(\gamma,c))\,. \tag{8.88}$$

We would like to show that the front-tracking limit u satisfies (8.87) if γ is continuous, and if γ has discontinuities, find a suitable generalization that is satisfied by the front-tracking limit. The condition (8.87) does not make sense for discontinuous γ's, since the second integral is undefined.

We shall assume that γ is piecewise continuous on a *finite* number of intervals, i.e., that γ has a finite number of discontinuities. We call this set of discontinuities $\mathcal{D}_\gamma = \{\xi_0,\dots,\xi_N\}$, and we assume that $\gamma(x)$ is continuously differentiable for $x \notin \mathcal{D}_\gamma$. Thus γ and γ' have left and right limits at each discontinuity point $\xi_i \in \mathcal{D}_\gamma$.

Next, we shall require that the approximation $\gamma^\delta(x)$ also have discontinuity points for all $x \in \mathcal{D}_\gamma$ for all relevant δ. In addition to these discontinuities, for a fixed δ, γ^δ has discontinuities at $\{y_{i,j}\}$. These are ordered so that

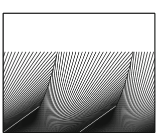
$$\xi_i = y_{i,0} < y_{i,1} < \cdots < y_{i,N_i} < y_{i,N_i+1} = \xi_{i+1},$$

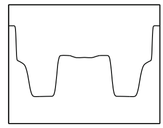
for $i=0,\dots,N$. Let $\gamma_{i,j+1/2}$ denote the value of γ^δ in the interval $(y_{i,j},y_{i,j+1})$, and set

$$\Delta x_{i,j} = \frac{1}{2}\left(y_{i,j+1}-y_{i,j-1}\right), \quad j=1,\dots,N_i.$$

Of course, these quantities all depend on δ, but for simplicity we omit this in our notation. We also assume that

$$\lim_{\delta\to 0}\frac{g^\delta\left(\gamma_{i,j+1/2},c\right)-g^\delta\left(\gamma_{i,j-1/2},c\right)}{\Delta x_{i,j}}\chi_{I_{i,j}}(x) = \frac{\partial f\,(\gamma(x),c)}{\partial x}, \tag{8.89}$$





where $\chi_{I_{i,j}}$ denotes the characteristic function of the interval

$$I_{i,j} = \left(\frac{y_{i,j-1} + y_{i,j}}{2}, \frac{y_{i,j} + y_{i,j+1}}{2} \right).$$

This is not unreasonable, since γ is continuously differentiable in (ξ_i, ξ_{i+1}). In what follows, we let $u_i^{\mp}$ and $u_{i,j}^{\mp}$ denote the left and right limits of u^δ at the points ξ_i and $y_{i,j}$, respectively. Since $u^\delta(\,\cdot\,, t)$ is piecewise constant, these limits exist.

In each interval $(y_{i,j}, y_{i,j+1})$ the function u^δ is an entropy solution of the conservation law

$$u_t^\delta + g^\delta(\gamma_{i,j+1/2}, u^\delta)_x = 0,$$

and hence

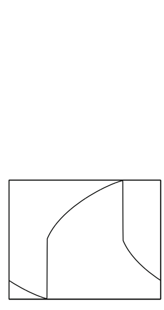
$$\begin{aligned}
&-\int_0^T \int_{y_{i,j}}^{y_{i,j+1}} \left(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma_{i,j+1/2}, u^\delta, c\right) \varphi_x \right) dx\, dt \\
&\quad + \int_0^T \left(F^\delta \left(\gamma_{i,j+1/2}, u_{i,j+1}^-, c\right) \varphi\left(y_{i,j+1}, t\right) - F^\delta\left(\gamma_{i,j+1/2}, u_{i,j}^+, c\right) \varphi\left(y_{i,j}, t\right) \right) dt \\
&\qquad\qquad - \int_{y_{i,j}}^{y_{i,j+1}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\, dx \le 0,
\end{aligned} \tag{8.90}$$

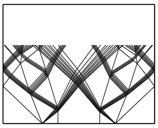
where

$$F^\delta(\gamma, u, c) = \operatorname{sign}(u - c)\left(g^\delta(\gamma, u) - g^\delta(\gamma, c)\right).$$

Summing this for $j = 0, \ldots, N_i$, we find that

$$\begin{aligned}
&-\int_0^T \int_{\xi_i}^{\xi_{i+1}} \left(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta, u^\delta, c\right) \varphi_x \right) dx\, dt - \int_{\xi_i}^{\xi_{i+1}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\, dx \\
&\qquad + \int_0^T \left(F^\delta\left(\gamma_{i,N_i+1/2}, u_{i+1}^-, c\right) \varphi(\xi_i, t) - F^\delta\left(\gamma_{i,1/2}, u_i^+, c\right) \varphi(\xi_{i+1}, t) \right) dt \\
&\qquad - \int_0^T \sum_{j=1}^{N_i} \left[F^\delta\left(\gamma_{i,j+1/2}, u_{i,j}^+, c\right) - F^\delta\left(\gamma_{i,j-1/2}, u_{i,j}^-, c\right) \right] \varphi(y_{i,j}, t)\, dt \\
&\quad \le 0.
\end{aligned} \tag{8.91}$$





Regarding the terms in the integrand in the last term in (8.91), we can write

$$
\begin{aligned}
&F^\delta\left(\gamma_{i,j+1/2}, u_{i,j}^+, c\right) - F^\delta\left(\gamma_{i,j-1/2}, u_{i,j}^-, c\right) \\
&= \begin{cases}
-\operatorname{sign}\left(u_{i,j}^+ - c\right)\left[f\left(\gamma_{i,j+1/2}, c\right) - f\left(\gamma_{i,j-1/2}, c\right)\right] \\
\qquad + \left\{\operatorname{sign}\left(u_{i,j}^+ - c\right) - \operatorname{sign}\left(u_{i,j}^- - c\right)\right\}\left(f_{i,j}^\times - f\left(\gamma_{i,j-1/2}\right)\right) \\
\text{or} \\
-\operatorname{sign}\left(u_{i,j}^- - c\right)\left[f\left(\gamma_{i,j+1/2}, c\right) - f\left(\gamma_{i,j-1/2}, c\right)\right] \\
\qquad + \left\{\operatorname{sign}\left(u_{i,j}^+ - c\right) - \operatorname{sign}\left(u_{i,j}^- - c\right)\right\}\left(f_{i,j}^\times - f\left(\gamma_{i,j+1/2}\right)\right),
\end{cases}
\end{aligned}
$$

where $f_{i,j}^\times = f(\gamma_{i,j+1/2}, u_{i,j}^+) = f(\gamma_{i,j-1/2}, u_{i,j}^-)$. If $\operatorname{sign}(u_{i,j}^+ - c) = \operatorname{sign}(u_{i,j}^- - c)$, the last terms in the above expressions are zero, while if $u_{i,j}^- \le c \le u_{i,j}^+$, then since these values are chosen according to the minimal jump entropy condition (8.25), we have that

$$
\operatorname{sign}\left(u_{i,j}^+ - c\right) - \operatorname{sign}\left(u_{i,j}^- - c\right) = 2 \quad \text{and} \quad \begin{cases} f(\gamma_{i,j-1/2}, c) \ge f_{i,j}^\times & \text{or} \\ f(\gamma_{i,j+1/2}, c) \ge f_{i,j}^\times, \end{cases}
$$

and thus in this case one of the last terms must be nonpositive. If $u_{i,j}^+ < c < u_{i,j}^-$, we use (8.26) to conclude that

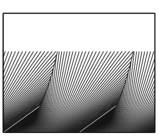
$$
\operatorname{sign}\left(u_{i,j}^+ - c\right) - \operatorname{sign}\left(u_{i,j}^- - c\right) = -2 \quad \text{and} \quad \begin{cases} f(\gamma_{i,j-1/2}, c) \le f_{i,j}^\times & \text{or} \\ f(\gamma_{i,j+1/2}, c) \le f_{i,j}^\times, \end{cases}
$$

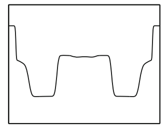
and again we find that one of the last terms is nonpositive. If the first of these last terms is nonpositive for c between $u_{i,j}^-$ and $u_{i,j}^+$, we define $u_{i,j} = u^\delta(y_{i,j}, t) = u_{i,j}^+$. Otherwise, we define $u_{i,j} = u^\delta(y_{i,j}, t) = u_{i,j}^-$. Using these observations, we find that

$$
\begin{aligned}
&-\int_0^T \int_{\xi_i}^{\xi_{i+1}} \left(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta, u^\delta, c\right)\varphi_x\right) dx\,dt - \int_{\xi_i}^{\xi_{i+1}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\,dx \\
&\qquad + \int_0^T \left(F^\delta\left(\gamma_{i,N_i+1/2}, u_{i+1}^-, c\right)\varphi\left(\xi_i, t\right) - F^\delta\left(\gamma_{i,1/2}, u_i^+, c\right)\varphi\left(\xi_{i+1}, t\right)\right)dt \\
&\qquad + \int_0^T \sum_{j=1}^{N_i} \operatorname{sign}\left(u_{i,j} - c\right)\left[f\left(\gamma_{i,j+1/2}, c\right) - f\left(\gamma_{i,j-1/2}, c\right)\right]\varphi\left(y_{i,j}, t\right)\,dt \\
&\le 0.
\end{aligned} \tag{8.92}
$$

Now $u_{i,j} = u^\delta(y_{i,j}^-, \cdot)$ or $u_{i,j} = u^\delta(y_{i,j}^+, \cdot)$; hence if we define $\bar{u}^\delta(x,t) = u_{i,j}(t)\chi_{I_{i,j}}(x)$, and set $\bar{z}^\delta = z_{i,j}^\delta(t)\chi_{I_{i,j}}$, we have that

$$
\bar{z}^\delta\left(y_{i,j}, t\right) = z^\delta\left(y_{i,j}, t\right).
$$





Now we claim that the sequence $\{\bar{z}^\delta\}$ is compact in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,T])$. Trivially we have that

$$\left\|\bar{z}^\delta\right\|_{L^\infty(\mathbb{R})} \le \left\|z^\delta\right\|_{L^\infty(\mathbb{R})} < C \tag{8.93}$$

and

$$\left|\bar{z}^\delta(\,\cdot\,,t)\right|_{BV} \le \left|z^\delta(\,\cdot\,,t)\right|_{BV} \le C. \tag{8.94}$$

Furthermore,

$$\begin{aligned}
\left\|\bar{z}^\delta(\,\cdot\,,t)-z^\delta(\,\cdot\,,t)\right\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \left|\bar{z}^\delta(x,t)-z^\delta(x,t)\right|\,dx \\
&= \sum_{i,j}\int_{y_{i,j-1/2}}^{y_{i,j+1/2}} \left|z^\delta\left(y_{i,j},t\right)-z^\delta(y,t)\right|\,dy \\
&\le \sum_{i,j}\int_{y_{i,j-1/2}}^{y_{i,j+1/2}}\int_{y}^{y_{i,j}} \left|z^\delta_x(x,t)\right|\,dx\,dy \\
&\le \max_{i,j}\left|\Delta x_{i,j}\right|\left|z^\delta(\,\cdot\,,t)\right|_{BV}.
\end{aligned}$$

Setting $\Delta x = \max_{i,j}\Delta x_{i,j}$, we therefore find that

$$\begin{aligned}
\left\|\bar{z}^\delta(\,\cdot\,,t)-\bar{z}^\delta(\,\cdot\,,s)\right\|_{L^1(\mathbb{R})} &\le \left\|z^\delta(\,\cdot\,,t)-z^\delta(\,\cdot\,,s)\right\|_{L^1(\mathbb{R})} + 2\Delta x\left|z^\delta(\,\cdot\,,t)\right|_{BV} \\
&\le C((t-s)+\Delta x).
\end{aligned} \tag{8.95}$$

By the bounds (8.93), (8.94), and (8.95), the sequence $\{\bar{z}^\delta\}$ converges along a subsequence (also labeled δ), and

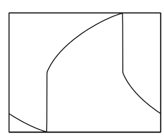
$$\lim_{\delta\to 0}\bar{z}^\delta = \lim_{\delta\to 0} z^\delta = z.$$

Therefore, also $\lim_{\delta\to 0}\bar{u}^\delta = u$. Now define

$$\Delta_x g^\delta(x,c) = \frac{1}{\Delta x_{i,j}}\left(f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right),\quad \text{for } x\in I_{i,j}.$$

Using this notation, the inequality (8.92) reads

$$\begin{aligned}
&-\int_0^T\int_{\xi_i}^{\xi_{i+1}} \left(\left|u^\delta-c\right|\varphi_t + F^\delta\left(\gamma^\delta,u^\delta,c\right)\varphi_x\right)dx\,dt - \int_{\xi_i}^{\xi_{i+1}}\left|u^\delta(x,0)-c\right|\varphi(x,0)\,dx \\
&\qquad -\int_0^T \left(F^\delta\left(\gamma_i^+,u_i^+,c\right)\varphi\left(\xi_i,t\right) - F^\delta\left(\gamma_{i+1}^-,u_{i+1}^-,c\right)\varphi\left(\xi_{i+1},t\right)\right)dt \\
&\qquad +\int_0^T\int_{\xi_i}^{\xi_{i+1}} \operatorname{sign}\left(\bar{u}^\delta-c\right)\Delta_x g^\delta(y,c)\sum_{j=1}^{N_i}\varphi\left(y_{i,j},t\right)\chi_{I,j}(y)\,dy\,dt \\
&\le 0.
\end{aligned} \tag{8.96}$$



Now we can add this for $i = 0, \dots, M$ to obtain

$$
\begin{aligned}
&- \iint\limits_{\Pi_T} \left(\left| u^\delta - c \right| \varphi_t + F^\delta \left(\gamma^\delta, u^\delta, c \right) \varphi_x \right) dx\, dt - \int\limits_{\mathbb{R}} \left| u^\delta(x,0) - c \right| \varphi(x,0)\, dx \\
&\qquad - \int_0^T \sum_{i=1}^{M} \left[F^\delta \left(\gamma_i^+, u_i^+, c \right) - F^\delta \left(\gamma_i^-, u_i^-, c \right) \right] \varphi \left(\xi_i, t \right)\, dt \\
&\qquad + \int_0^T \sum_{i=0}^{M} \int_{\xi_i}^{\xi_{i+1}} \operatorname{sign} \left(\bar{u}^\delta - c \right) \Delta_x g^\delta(y,c) \sum_{j=1}^{N_i} \varphi \left(y_{i,j}, t \right) \chi_{I_{i,j}}(y)\, dy dt \\
&\qquad \le 0.
\end{aligned}
\tag{8.97}
$$

At this point it is convenient to state the following general lemma.

Lemma 8.20 *Let $\Omega \in \mathbb{R}$ be a bounded open set, $g \in L^1(\Omega)$, and suppose that $g_n(x) \to g(x)$ almost everywhere. Then there exists a set $\Theta \subseteq \mathbb{R}$, which is a most countable, such that for every $c \in \mathbb{R} \setminus \Theta$,*

$$
\operatorname{sign}(g_n(x) - c) \to \operatorname{sign}(g(x) - c) \quad \text{a.e. in } \Omega.
$$

Furthermore, let $c \in \Theta$ and define

$$
\mathcal{E}_c = \{ x \in \Omega \mid g(x) = c \}.
$$

Then it is possible to define sequences $\{\underline{c}_m\}_{m=1}^\infty \subset \mathbb{R} \setminus \Theta$ and $\{\bar{c}_m\}_{m=1}^\infty \subset \mathbb{R} \setminus \Theta$ such that

$$
\underline{c}_m \uparrow c \quad \text{and} \quad \operatorname{sign}\left(g(x) - \underline{c}_m\right) \to \operatorname{sign}(g(x) - c) \quad \text{a.e. in } \Omega \setminus \mathcal{E}_c, \tag{8.98}
$$

$$
\bar{c}_m \downarrow c \quad \text{and} \quad \operatorname{sign}(g(x) - \bar{c}_m) \to \operatorname{sign}(g(x) - c) \quad \text{a.e. in } \Omega \setminus \mathcal{E}_c, \tag{8.99}
$$

as $m \to \infty$.

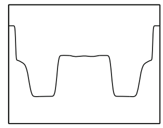
Proof Fix $c \in \mathbb{R}$ and a point $x \in \Omega$ such that $g_n(x) \to g(x)$ and $g(x) \neq c$. For sufficiently large n, $\operatorname{sign}(g_n(x) - c) = \operatorname{sign}(g(x) - c)$, i.e., $\operatorname{sign}(g_n(x) - c)$ is constant in n, and therefore converges to the correct limit. Thus for each $c \in \mathbb{R}$, $\operatorname{sign}(g_n(x) - c) \to \operatorname{sign}(g(x) - c)$ almost everywhere in $\Omega \setminus \mathcal{E}_c$. It remains to show that all but countably many of the sets $\mathcal{E}_c$ have zero measure. To this end, define

$$
C_k = \left\{ c \in \mathbb{R} \mid \operatorname{meas}(\mathcal{E}_c) \ge \frac{1}{k} \right\}.
$$

Since Ω is bounded, C_k contains only a finite number of points. Therefore, the set

$$
\{ c \in \mathbb{R} \mid \operatorname{meas}(\mathcal{E}_c) > 0 \} = \bigcup_{k>0} C_k
$$

is at most countable.



To prove (8.98), fix $c \in \Theta$. Since Θ is at most countable, we can find a sequence $\underline{c}_n \uparrow c$ such that $c_n \notin \Theta$. For $x \in \Omega \setminus \mathcal{E}_c$, we have that $g(x) \neq c$, and thus $\operatorname{sign}\left(g(x) - \underline{c}_n\right) = \operatorname{sign}\left(g(x) - c\right)$ for n sufficiently large. Thus (8.98) holds. The existence of $\{\bar{c}_n\}$ and (8.99) is proved in the same way. $\square$

Now clearly

$$\Delta_x g^\delta(y,c) \sum_{j=1}^{N_i} \varphi\left(y_{i,j}, t\right) \chi_{I_{i,j}}(y) \to \partial_x f(\gamma(y), c)\varphi(y,t) \text{ as } \delta \to 0$$

in each interval (ξ_i, ξ_{i+1}). Furthermore, by Lemma 8.20,

$$\operatorname{sign}\left(\bar{u}^\delta - c\right) \to \operatorname{sign}\left(u - c\right),$$

for almost all (x,t) and all but at most a countable set of c's.

Regarding the middle term of (8.97), by Lemma 8.4 each summand is bounded by

$$\left| g^\delta\left(\gamma_i^+, c\right) - g^\delta\left(\gamma_i^-, c\right) \right|,$$

since (u_i^-, u_i^+) satisfies the minimal jump entropy condition. Therefore, by sending δ to 0 in (8.97), we find that

$$\begin{aligned}
&- \iint\limits_{\Pi_T} \left(|u - c|\, \varphi_t + F(\gamma, u, c)\varphi_x \right) dx\, dt + \underbrace{\iint\limits_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}\left(u - c\right) \partial_x f(\gamma, c)\varphi \, dx\, dt}_{I(c)} \\
&\quad - \int_0^T \sum_{x \in \mathcal{D}_\gamma} \left| f(\gamma(x^+), c) - f(\gamma(x^-), c) \right| \varphi(x,t)\, dt - \int_{\mathbb{R}} |u_0 - c|\, \varphi(x,0)\, dx \\
&\quad \leq 0
\end{aligned} \tag{8.100}$$

for all but a countable set of c's and all nonnegative test functions φ. This can be rewritten as

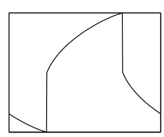
$$I(c) \leq G(c),$$

where G is a *continuous* function of c. Let Θ denote the set where the convergence of $\operatorname{sign}\left(\bar{u}^\delta - c\right) \to \operatorname{sign}\left(u - c\right)$ does not hold. Fix some $c \in \Theta$ and define the two sequences $\{\underline{c}_n\}$ and $\{\bar{c}_n\}$ as in Lemma 8.20. Set

$$\mathcal{E}_c = \left\{(x,t) \;\middle|\; u(x,t) = c\right\}.$$

Since (8.100) holds for $\underline{c}_n$ and $\bar{c}_n$, we can write $I(c)$ as

$$\begin{aligned}
&\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}\left(u - \underline{c}_n\right) \partial_x f(\gamma, u)\varphi \, dx\, dt \\
&\qquad + \iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \operatorname{sign}\left(u - \underline{c}_n\right) \partial_x f(\gamma, u)\varphi \, dx\, dt \leq G(c),
\end{aligned} \tag{8.101}$$



where $\hat{\Pi}_T = \Pi_T \setminus \mathcal{D}_\gamma$. Since $\underline{c}_n < c$, the last integral can be rewritten as

$$\iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dxdt.$$

Since f is continuous, by sending n to ∞, we find that

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt + \iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c). \tag{8.102}$$

Similarly, using the sequence $\{\bar{c}_n\}$, we arrive at

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt - \iint\limits_{\mathcal{E}_c \setminus \mathcal{D}_\gamma} \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c). \tag{8.103}$$

Adding (8.102) and (8.103) and dividing by 2, we find that

$$\iint\limits_{\hat{\Pi}_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c).$$

Since $\operatorname{sign}(0) = 0$, $\operatorname{sign}(u-c) = 0$ on $\mathcal{E}_c$, and therefore, we can conclude that

$$\iint\limits_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}(u-c)\, \partial_x f(\gamma, u)\varphi \, dx\, dt \le G(c) \tag{8.104}$$

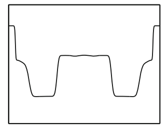
for *all* constants c. We have proved the following theorem.

Theorem 8.21 *Assume that the flux function satisfies A.1–A.4, and let u^δ be a weak solution of (8.84), constructed by front tracking, such that u^δ converges to u in $L^1(\Pi_T)$. Then the entropy condition (8.100) holds for all constants c.*

8.3 Uniqueness of Entropy Solutions

Now we shall use the Kružkov entropy formulation, (8.100), to show that there exists at most one entropy solution. For convenience, we restate this condition,

$$\begin{aligned}\iint\limits_{\Pi_T} \left(|u-c|\,\varphi_t + F(\gamma, u, c)\varphi_x\right) dt\, dx - \iint\limits_{\Pi_T \setminus \mathcal{D}_\gamma} \operatorname{sign}(u-c)\, \partial_x f(\gamma, c)\varphi \, dt\, dx \\ + \int_0^T \sum_i \left| f\left(\gamma_i^+, c\right) - f\left(\gamma_i^-, c\right)\right| \varphi\left(\xi_i, t\right)\, dt + \int_{\mathbb{R}} |u_0 - c|\,\varphi(x, 0)\, dx \ge 0,\end{aligned} \tag{8.105}$$



for all nonnegative test functions $\varphi \in C_0^1(\mathbb{R} \times [0,T))$ and all real constants c, and where we write $\gamma_i^\pm = \gamma(\xi_i\pm)$.

In addition to satisfying this entropy inequality, we demand[3] that an entropy solution be a weak solution, i.e., that it satisfy (8.54) and be slightly more regular in the sense described below.

If $w \in L^\infty(\Pi_T)$, by the left and right traces of $w(\,\cdot\,,t)$ at a point x_0 we understand functions $t \mapsto w(x_0\pm,t) \in L^\infty([0,T])$ that satisfy a.e. $t \in [0,T)$,

$$\begin{aligned} \operatorname{ess\,lim}_{x\downarrow x_0} |w(x,t) - w(x_0+,t)| = 0,\\ \operatorname{ess\,lim}_{x\uparrow x_0} |w(x,t) - w(x_0-,t)| = 0. \end{aligned} \tag{8.106}$$

When comparing two entropy solutions, we shall need that they have traces at the points ξ_i, i.e., if u is an entropy solution, then we assume that the following traces exist:

$$u_i^\pm(t) = u\,(x_i\pm,t)\,, \tag{8.107}$$

in the sense of (8.106) for almost all t and for $i = 1,\dots,N$.

An entropy solution of (8.53) is a function in $L^1_{\mathrm{loc}}(\Pi_T)\cap C([0,T);L^1_{\mathrm{loc}}(\mathbb{R}))$ such that (8.54), (8.105), and the regularity assumption (8.107) all hold.

We have already shown that an entropy solution exists for our model problem, since the existence of traces follows by noting that $z(\,\cdot\,,t) \in BV(\mathbb{R})$, which implies that z has traces. Since $u = z^{-1}(\gamma,z)$, the same applies to u.

Let now $w = w(x)$ be any function on $\mathbb{R}$, and fix a point y. We use the following notation:

$$\operatorname{L-lim}_{x\downarrow y} w(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_y^{y+\varepsilon} w(x)\,dx,$$

$$\operatorname{L-lim}_{x\uparrow y} w(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{y-\varepsilon}^{y} w(x)\,dx.$$

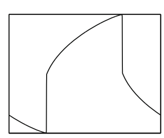
Lemma 8.22 *Let $w \in L^\infty(\Pi_T)$, and fix a point $x_0 \in \mathbb{R}$. If the left and right traces $t \mapsto w(x_0\pm,t)$ exist in the sense of (8.106), then for a.e. $t \in [0,t)$ we have that*

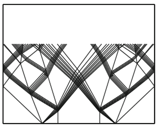
$$\operatorname{L-lim}_{x\downarrow x_0} w(x,t) = w(x_0+,t), \quad \operatorname{L-lim}_{x\uparrow x_0} w(x,t) = w(x_0-,t).$$

Proof We prove the first limit as follows:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{x_0}^{x_0+\varepsilon} |w(x,t) - w(x_0+,t)|\,dx \\ &\qquad \le \frac{1}{\varepsilon}\int_{x_0}^{x_0+\varepsilon} \operatorname{ess\,sup}_{y\in(x_0,x_0+\varepsilon)} |w(y,t) - w(x_0+,t)|\,dx \\ &\qquad = \operatorname{ess\,sup}_{y\in(x_0,x_0+\varepsilon)} |w(y,t) - w(x_0+,t)| \quad \to 0 \quad \text{as } \varepsilon \downarrow 0. \qquad \square \end{aligned}$$

[3] This does not follow easily from the entropy condition, which is in contrast to the case in which the flux function is space-independent.





As a consequence of this lemma, the following limits exist for every entropy solution u:

$$\begin{aligned} \text{L-lim}_{x\downarrow\xi_i} f\,(\gamma(x), u(x,t)) &= f\,(\gamma\,(\xi_i+)\,, u(\xi_i+,t))\,,\\ \text{L-lim}_{x\uparrow\xi_i} f\,(\gamma(x), u(x,t)) &= f\,(\gamma\,(\xi_i-)\,, u(\xi_i-,t))\,, \end{aligned} \tag{8.108}$$

and therefore, if v is another entropy solution,

$$\begin{aligned} \text{L-lim}_{x\downarrow\xi_i} F\,(\gamma(x), u(x,t), v(x,t)) &= F\,(\gamma\,(\xi_i+)\,, u(\xi_i+,t), v(\xi_i+,t))\,,\\ \text{L-lim}_{x\uparrow\xi_i} F\,(\gamma(x), u(x,t), v(x,t)) &= F\,(\gamma\,(\xi_i-)\,, u(\xi_i-,t), v(\xi_i-,t))\,, \end{aligned} \tag{8.109}$$

where F is the Kružkov entropy flux (8.88). Before we continue, let us define the following compactly supported Lipschitz continuous function:

$$\theta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}(\varepsilon + x) & \text{if } x \in (-\varepsilon, 0],\\ \frac{1}{\varepsilon}(\varepsilon - x) & \text{if } x \in [0, \varepsilon),\\ 0 & \text{otherwise.} \end{cases} \tag{8.110}$$

Lemma 8.23 *Let u be an entropy solution. Then for a.e. $t \in [0,t)$ and for all constants c,*

$$\begin{gathered} f\left(\gamma_i^+, u_i^+(t)\right) = f\left(\gamma_i^-, u_i^-(t)\right),\\ F\left(\gamma_i^+, u_i^+, c\right) - F\left(\gamma_i^-, u_i^-\right) \le \left| f\left(\gamma_i^+, c\right) - f\left(\gamma_i^-, c\right)\right|, \end{gathered}$$

where F is the Kružkov entropy flux (8.88).

Proof Since $u \in L^\infty(\Pi_T)$, a density argument shows that

$$\varphi(x,t) = \theta_\varepsilon\,(x - \xi_i)\,\psi(t),$$

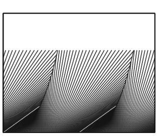
where $\psi \in C_0^1((0,T))$ is an admissible test function that can be used in the weak formulation (8.54). If $\varepsilon < \min_i \{\xi_{i+1} - \xi_i\}$, we get

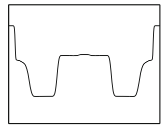
$$\begin{aligned} &\iint\limits_{\Pi_T} u\theta_\varepsilon\,(x - \xi_i)\,\psi'(t)\,dx\,dt\\ &\qquad = \int_0^T \Big(\frac{1}{\varepsilon}\int_{\xi_i}^{\xi_i+\varepsilon} f(\gamma(x), u)\,dx - \frac{1}{\varepsilon}\int_{\xi_i-\varepsilon}^{\xi_i} f(\gamma(x), u)\,dx\Big)\psi(t)\,dt. \end{aligned}$$

By sending $\varepsilon \downarrow 0$ and using Lemma 8.23, we obtain

$$\int_0^T \left(f\left(\gamma_i^+, u_i^+\right) - f\left(\gamma_i^-, u_i^-\right)\right)\psi(t)\,dt = 0.$$

Since this holds for every test function ψ, the integrand must be zero.





To prove the inequality in the lemma, we choose the same test function, but restrict ψ to be nonnegative. By the entropy condition, (8.105), we get

$$
\begin{aligned}
&\iint\limits_{\Pi_T} |u-c|\theta_\varepsilon\,(x-\xi_i)\,\psi'(t)\,dx\,dt \\
&\qquad - \int_0^T \Big(\frac{1}{\varepsilon}\int_{\xi_i}^{\xi_i+\varepsilon} F(\gamma(x),u,c)\,dx - \frac{1}{\varepsilon}\int_{\xi_i-\varepsilon}^{\xi_i} F(\gamma(x),u,c)\,dx\Big)\psi(t)\,dt \\
&\qquad - \iint\limits_{\Pi_T} \operatorname{sign}(u-c)\,\partial_x f(\gamma(x),c)\theta_\varepsilon\,(x-\xi_i)\,\psi(t)\,dx\,dt \\
&\qquad + \int_0^T \left|f\left(\gamma_i^+,c\right) - f\left(\gamma_i^-,c\right)\right|\psi(t)\,dt \ge 0.
\end{aligned}
$$

Again, by sending $\varepsilon \downarrow 0$,

$$
\int_0^T \left|f\left(\gamma_i^+,c\right) - f\left(\gamma_i^-,c\right)\right|\psi(t)\,dt \ge \int_0^T \left(F\left(\gamma_i^+,u_i^+,c\right) - F\left(\gamma_i^-,u_i^-,c\right)\right)\psi(t)\,dt,
$$

which implies the inequality. □

This has the following immediate corollary.

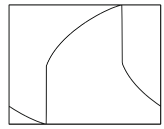
Corollary 8.24 *Assume that the flux function f satisfies A.1–A.4. If u is an entropy solution, then the pairs (u_i^-,u_i^+) satisfy the minimal jump entropy condition (8.25)–(8.26) for $i=1,\dots,N$.*

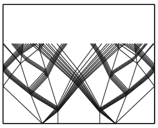
For any test function φ that has support away from $\mathcal{D}_\gamma$, we can double variables in the sense of Kružkov.

Lemma 8.25 *For every two entropy solutions u and v and nonnegative test function $\varphi \in C_0^1(\Pi_T \setminus \mathcal{D}_\gamma)$, we have that*

$$
\begin{aligned}
&-\iint\limits_{\Pi_T} \left(|u-v|\,\varphi_t + F(\gamma,u,v)\varphi_x\right)dt\,dx \\
&\qquad \le C\iint\limits_{\Pi_T} |u-v|\,\varphi\,dt\,dx + \int_{\mathbb{R}} |u_0-v_0|\,\varphi(x,0)\,dx,
\end{aligned}
\tag{8.111}
$$

where the constant C is zero if γ is piecewise constant.





Proof The proof is a classical doubling of variables argument. It uses exactly the same arguments as in Sect. 2.4, but adapted to our situation.

Let ϕ be a nonnegative test function in $C_0^1(\Pi_T \times \Pi_T)$. We use the notation $u = u(y,s)$, $v = v(x,t)$. Then using $c = u(y,s)$ as the constant in the entropy inequality for v and then integrating over (y,s), we get

$$\begin{aligned} - \iiiint\limits_{\Pi_T\times\Pi_T} \big(|u-v|\,\phi_t + F(\gamma(x),u,v)\phi_x \big)\, dt\, dx\, ds\, dy \\ + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\, f\,(\gamma(x),u)_x\, \phi\, dt\, dx\, dy\, ds \\ \le \iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |v_0-u|\,\phi(x,0,y,s)\, dx\, ds\, dy. \end{aligned} \tag{8.112}$$

Similarly, starting with the entropy inequality for u and integrating over (x,t), we arrive at

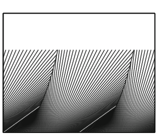
$$\begin{aligned} - \iiiint\limits_{\Pi_T\times\Pi_T} \big(|u-v|\,\phi_s + F(\gamma(y),u,v)\phi_y \big)\, ds\, dy\, dt\, dx \\ + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\, f\,(\gamma(y),v)_y\, \phi\, ds\, dy\, dt\, dx \\ \le \iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |u_0-v|\,\phi(x,t,y,0)\, dy\, dt\, dx. \end{aligned} \tag{8.113}$$

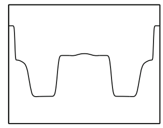
Since γ is differentiable outside $\mathcal{D}_\gamma$, for $(x,t) \in \Pi_T \setminus \mathcal{D}_\gamma$ we have

$$\begin{aligned} F\,(\gamma(x),v,u)\,\phi_x - &\operatorname{sign}(v-u)\, f\,(\gamma(x),u)_x\,\phi \\ &= \operatorname{sign}(v-u)\,\big(f\,(\gamma(x),v) - f\,(\gamma(y),u)\big)\,\phi_x \\ &\quad - \operatorname{sign}(v-u)\,\big(\big(f\,(\gamma(x),u) - f\,(\gamma(y),u)\big)\,\phi\big)_x\,. \end{aligned}$$

Using this, we find that

$$\begin{aligned} &- \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \big(F\,(\gamma(x),v,u)\,\phi_x - \operatorname{sign}(v-u)\, f\,(\gamma(x),u)_x\,\phi\big)\, dt\, dx\, ds\, dy \\ &= - \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\,\big(f\,(\gamma(x),v) - f\,(\gamma(y),u)\big)\,\phi_x\, dt\, dx\, ds\, dy \\ &\quad + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(v-u)\,\big(\big(f\,(\gamma(x),u) - f\,(\gamma(y),u)\big)\,\phi\big)_x\; dt\, dx\, ds\, dy. \end{aligned}$$





We also have a similar equality for u,

$$
\begin{aligned}
&- \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \left(F\left(\gamma(y), v, u\right)\phi_y - \operatorname{sign}(u-v)\, f\left(\gamma(y), v\right)_y \phi\right) ds\, dy\, dt\, dx \\
&\quad = - \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\left(f\left(\gamma(y), u\right) - f\left(\gamma(x), v\right)\right)\phi_y \, ds\, dy\, dt\, dx \\
&\qquad + \iiiint\limits_{(\Pi_T\setminus\mathcal{D}_\gamma)\times(\Pi_T\setminus\mathcal{D}_\gamma)} \operatorname{sign}(u-v)\left(\left(f\left(\gamma(y), v\right) - f\left(\gamma(x), v\right)\right)\phi\right)_y \, ds\, dy\, dt\, dx.
\end{aligned}
$$

Now we introduce the notation

$$\partial_{t+s} = \partial_t + \partial_s, \quad \partial_{x+y} = \partial_x + \partial_y.$$

We use the above result and add (8.113) and (8.112) to obtain

$$
\begin{aligned}
&- \iiiint\limits_{\Pi_T\times\Pi_T} \Big(|v-u|\, \partial_{t+s}\phi \\
&\qquad\qquad + \operatorname{sign}(v-u)\left(f\left(\gamma(x), v\right) - f\left(\gamma(y), u\right)\right)\partial_{x+y}\phi\Big)\, dt\, dx\, ds\, dy \\
&+ \iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\Big[\left(\left(f\left(\gamma(x), u\right) - f\left(\gamma(y), u\right)\right)\phi\right)_x \\
&\qquad\qquad + \left(\left(f\left(\gamma(y), v\right) - f\left(\gamma(x), v\right)\right)\phi\right)_y\Big]\, dt\, dx\, ds\, dy \\
&\leq \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |v_0 - u|\, \phi(x, 0, y, s)\, dx\, ds\, dy \\
&\quad + \iint\limits_{\Pi_T}\int\limits_{\mathbb{R}} |u_0 - v|\, \phi(x, t, y, 0)\, dy\, dt\, dx.
\end{aligned}
\tag{8.114}
$$

Now we shall choose a suitable test function. First let $\omega \in C_0^\infty(\mathbb{R})$ be a function such that $\omega(-a) = \omega(a)$, $\omega'(a) \leq 0$ for $a > 0$, $|\omega'(a)| \leq 2$, $\omega(a) = 0$ for $|a| \geq 1$, and $\int \omega(a)\, da = 1$. For positive ε, set

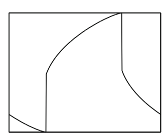
$$\omega_\varepsilon(a) = \frac{1}{\varepsilon}\omega\left(\frac{a}{\varepsilon}\right).$$

Let $\varphi(x, t)$ be a test function such that

$$\varphi(x, t) = 0 \quad \text{for } |x - \xi_i| \leq \varepsilon_0,\ i = 1, \dots, N,$$

for some positive ε_0. Then we define

$$\phi(x, t, y, s) = \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right), \tag{8.115}$$



for $\varepsilon < \varepsilon_0$. We can easily check that $\phi \in C_0^1((\Pi_T \setminus \mathcal{D}_\gamma)\times(\Pi_T \setminus \mathcal{D}_\gamma))$. Furthermore, we have the useful identities

$$\begin{aligned}
\partial_{t+s}\phi(x,t,y,s) &= \partial_{t+s}\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right),\\
\partial_{x+y}\phi(x,t,y,s) &= \partial_{x+y}\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right).
\end{aligned}$$

If we use these identities in (8.114), we find that

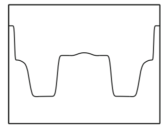
$$\begin{aligned}
-&\iiiint\limits_{\Pi_T\times\Pi_T}\left(I_{\text{time}}(x,t,y,s)+I_{\text{conv}}(x,t,y,s)\right)\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy\\
&\le \iiiint\limits_{\Pi_T\times\Pi_T}\left(I^1_{\text{flux}}(x,t,y,s)+I^2_{\text{flux}}(x,t,y,s)+I^3_{\text{flux}}(x,t,y,s)\right)dt\,dx\,ds\,dy\\
&\quad+\underbrace{\iint\limits_{\Pi_T}\int\limits_{\mathbb{R}}|v_0-u|\phi(x,0,y,s)\,dx\,ds\,dy+\iint\limits_{\Pi_T}\int\limits_{\mathbb{R}}|u_0-v|\phi(x,t,y,0)\,dy\,dt\,dx}_{J_{\text{init}}},
\end{aligned}\tag{8.116}$$

where

$$\begin{aligned}
I_{\text{time}}(x,t,y,s) &= |v-u|\,\partial_{t+s}\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right),\\
I_{\text{conv}}(x,t,y,s) &= \operatorname{sign}(v-u)\left[f(\gamma(x),v)-f(\gamma(y),u)\right]\\
&\qquad\times\partial_{x+y}\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right),\\
I^1_{\text{flux}}(x,t,y,s) &= -\operatorname{sign}(v-u)\,\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\\
&\qquad\times\left[\gamma'(x)f_\gamma(\gamma(x),u)-\gamma'(y)f_\gamma(\gamma(y),v)\right],\\
I^2_{\text{flux}}(x,t,y,s) &= -\operatorname{sign}(v-u)\,\omega_\varepsilon\left(\frac{x-y}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)\\
&\qquad\times\Big[\partial_x\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\left(f(\gamma(x),u)-f(\gamma(y),u)\right)\\
&\qquad\quad+\partial_y\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\left(f(\gamma(x),v)-f(\gamma(y),v)\right)\Big],\\
I^3_{\text{flux}}(x,t,y,s) &= \left[F(\gamma(x),v,u)-F(\gamma(y),v,u)\right]\\
&\qquad\times\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\omega_\varepsilon\left(\frac{t-s}{2}\right)\partial_x\omega_\varepsilon\left(\frac{x-y}{2}\right).
\end{aligned}$$

Introduce new variables

$$\tilde{x}=\frac{x+y}{2},\quad z=\frac{x-y}{2},\quad \tilde{t}=\frac{t+s}{2},\quad \tau=\frac{t-s}{2},$$



which map $\Pi_T \times \Pi_T$ into

$$\Omega_T = \left\{ \left(\tilde{x}, \tilde{t}, z, \tau\right) \in \mathbb{R}^4 \mid 0 \le \tilde{t} \pm \tau \le T \right\},$$

and $(\Pi_T \setminus \mathcal{D}_\gamma) \times (\Pi_T \setminus \mathcal{D}_\gamma)$ into

$$\Omega_{T,\gamma} = \left\{ \left(\tilde{x}, \tilde{t}, z, \tau\right) \in \Omega_T \mid \tilde{x} \pm z \neq \xi_i,\ i = 1, \dots, N \right\}.$$

We start by estimating the terms in J_{init}:

$$\begin{aligned}
&\iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |v_0(x) - u(y,s)|\, \varphi\left(\frac{x+y}{2}, \frac{s}{2}\right) \omega_\varepsilon\left(\frac{x-y}{2}\right) \omega_\varepsilon\left(\frac{-s}{2}\right) dx\, ds\, dy \\
&\quad = \int\limits_0^\varepsilon \int\limits_{\mathbb{R}} \int\limits_{-\varepsilon}^\varepsilon \left| v_0(\tilde{x}+z) - u(\tilde{x}-z, \tilde{t}-\tau) \right| \varphi(\tilde{x},\tau) \omega_\varepsilon(z) \omega_\varepsilon(\tau)\, dz\, d\tilde{x}\, d\tau \\
&\quad \to \frac{1}{2} \int\limits_{\mathbb{R}} |v_0(x) - u(x,0)|\, \varphi(x,0)\, dx
\end{aligned}$$

as $\varepsilon \to 0$. Since $t \mapsto u(x,t)$ is L^1 continuous, we can replace $u(x,0)$ by $u_0(x)$. Similarly, we find that

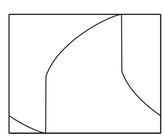
$$\iint\limits_{\Pi_T} \int\limits_{\mathbb{R}} |u_0 - v|\, \phi(x,t,y,0)\, dy\, dt\, dx \to \frac{1}{2} \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x,0)\, dx$$

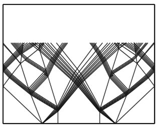
as $\varepsilon \to 0$, and thus

$$\lim_{\varepsilon \to 0} J_{\mathrm{init}} = \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x,0)\, dx. \tag{8.117}$$

In the transformed variables, we have

$$\begin{aligned}
I_{\mathrm{time}}(\tilde{x},\tilde{t},z,\tau) &= \left| v(\tilde{x}+z, \tilde{t}+\tau) - u(\tilde{x}-z, \tilde{t}-\tau) \right| \partial_{\tilde{t}} \varphi\left(\tilde{x},\tilde{t}\right), \\
I_{\mathrm{conv}}(\tilde{x},\tilde{t},z,\tau) &= \operatorname{sign}\left(v(\tilde{x}+z, \tilde{t}+\tau) - u(\tilde{x}-z, \tilde{t}-\tau) \right) \partial_{\tilde{x}} \varphi\left(\tilde{x},\tilde{t}\right) \\
&\qquad \times \Big[f\left(\gamma(\tilde{x}+z), v(\tilde{x}+z, \tilde{t}+\tau)\right) \\
&\qquad\qquad - f\left(\gamma(\tilde{x}-z), u(\tilde{x}-z, \tilde{t}-\tau)\right) \Big], \\
I^1_{\mathrm{flux}}(\tilde{x},\tilde{t},z,\tau) &= \operatorname{sign}\left(v(\tilde{x}+z, \tilde{t}+\tau) - u(\tilde{x}-z, \tilde{t}-\tau) \right) \omega_\varepsilon(z)\, \omega_\varepsilon(\tau) \\
&\qquad \times \Big[\gamma'(\tilde{x}+z) f_\gamma(\gamma(\tilde{x}+z), u(\tilde{x}-z, \tilde{t}-\tau)) \\
&\qquad\qquad - \gamma'(\tilde{x}-z) f_\gamma(\gamma(\tilde{x}-z), v(\tilde{x}+z, \tilde{t}+\tau)) \Big] \varphi\left(\tilde{x},\tilde{t}\right),
\end{aligned}$$





$$
\begin{aligned}
&I^2_{\text{flux}}(\tilde{x},\tilde{t},z,\tau) \\
&\qquad = \operatorname{sign}\left(v(\tilde{x}+z,\tilde{t}+\tau)-u(\tilde{x}-z,\tilde{t}-\tau)\right)\omega_\varepsilon(z)\,\omega_\varepsilon(\tau)\,\partial_{\tilde{x}}\varphi\left(\tilde{x},\tilde{t}\right) \\
&\qquad\quad \times\Big[\left(f(\gamma(\tilde{x}+z),u(\tilde{x}-z,\tilde{t}-\tau))-f(\gamma(\tilde{x}-z),u(\tilde{x}-z,\tilde{t}-\tau))\right) \\
&\qquad\qquad +\left(f(\gamma(\tilde{x}+z),v(\tilde{x}+z,\tilde{t}+\tau))-f(\gamma(\tilde{x}-z),v(\tilde{x}+z,\tilde{t}+\tau))\right)\Big], \\
&I^3_{\text{flux}}(\tilde{x},\tilde{t},z,\tau) = \Big[F\left(\gamma(\tilde{x}+z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right) \\
&\qquad\qquad\qquad -F\left(\gamma(\tilde{x}-z),v(\tilde{x}+z,\tilde{t}+\tau),u(\tilde{x}-z,\tilde{t}-\tau)\right)\Big] \\
&\qquad\qquad \times\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(t)\,\partial_z\omega_\varepsilon(z)\,.
\end{aligned}
$$

It is straightforward to deduce the limits

$$
\lim_{\varepsilon\to 0}\iiiint_{\Omega} I_{\text{time}}(\tilde{x},\tilde{t},z,\tau)\omega_\varepsilon(z)\,\omega_\varepsilon(t)\;d\tau\,dz\,d\tilde{t}\,d\tilde{x} = \iint_{\Pi_T}|u-v|\,\varphi_t\,dt\,dx, \tag{8.118}
$$

$$
\lim_{\varepsilon\to 0}\iiiint_{\Omega} I_{\text{conv}}(\tilde{x},\tilde{t},z,\tau)\omega_\varepsilon(z)\,\omega_\varepsilon(t)\;d\tau\,dz\,d\tilde{t}\,d\tilde{x} = \iint_{\Pi_T}F\left(\gamma(x),u,v\right)\varphi_x\,dt\,dx. \tag{8.119}
$$

Since γ is C^1 outside $\mathcal{D}_\gamma$, we deduce that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\iiiint_{\Omega_\gamma} I^1_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau dz &= \iint_{\Pi_T\setminus\mathcal{D}_\gamma}\gamma'(x)F_\gamma\left(\gamma(x),u,v\right)\,dt\,dx \\
&\le C\iint_{\Pi_T}|u-v|\;dt\,dx,
\end{aligned} \tag{8.120}
$$

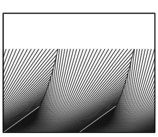
where

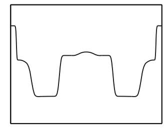
$$
C = \|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\left\|f_{u\gamma}\right\|_{L^\infty}.
$$

In particular, we observe that C can be chosen as zero if γ is a piecewise constant function.

Next we consider I^2_{flux}. Since φ vanishes near $\mathcal{D}_\gamma$, I^2_{flux} also vanishes near $\mathcal{D}_\gamma$. Hence γ is uniformly C^1 where $I^2_{\text{flux}}\neq 0$. Therefore,

$$
\begin{aligned}
&\left|I^2_{\text{flux}}(\tilde{x},\tilde{t},z,\tau)\right| \\
&\quad\le \omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right| \\
&\qquad\times\Big(\left|f\left(\gamma(\tilde{x}+z),u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right)-f\left(\gamma(\tilde{x}-z),u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right)\right| \\
&\qquad\quad+\left|f\left(\gamma(\tilde{x}+z),v\left(\tilde{x}+z,\tilde{t}+\tau\right)\right)-f\left(\gamma(\tilde{x}-z),v\left(\tilde{x}+z,\tilde{t}+\tau\right)\right)\right|\Big) \\
&\quad\le \omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right|2\left\|f_\gamma\right\|_{L^\infty}\left|\gamma(\tilde{x}+z)-\gamma(\tilde{x}-z)\right| \\
&\quad\le 4\left\|f_\gamma\right\|_{L^\infty(\mathbb{R})}\|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\,\omega_\varepsilon(z)\omega_\varepsilon(\tau)\left|\partial_{\tilde{x}}\varphi(\tilde{x},\tilde{t})\right||z|\,.
\end{aligned}
$$





From this we conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigl|\iiiint_{\Omega} I_{\text{flux}}^{2}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz\Bigr| \\ \le \lim_{\varepsilon\to 0} C\int_{-\varepsilon}^{\varepsilon}|z|\,\omega_\varepsilon(z)\,dz = 0.\end{aligned} \tag{8.121}$$

Finally, we turn to I_{flux}^{3}:

$$\begin{aligned}\left|I_{\text{flux}}^{3}(\tilde{x},\tilde{t},z,\tau)\right| &\le \varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\,|\partial_z\omega_\varepsilon(z)| \\ &\quad\times\Bigl|F\left(\gamma(\tilde{x}+z), v(\tilde{x}+z,\tilde{t}+\tau), u(\tilde{x}-z,\tilde{t}-\tau)\right) \\ &\qquad - F\left(\gamma(\tilde{x}-z), v(\tilde{x}+z,\tilde{t}+\tau), u(\tilde{x}-z,\tilde{t}-\tau)\right)\Bigr| \\ &\le \varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\,|\partial_z\omega_\varepsilon(z)|\,2\,\|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\,|z| \\ &\quad\times\left\|f_{\gamma u}\right\|_{L^\infty(\mathbb{R})}\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right| \\ &\le \left\|f_{\gamma u}\right\|_{L^\infty(\mathbb{R})}\|\gamma'\|_{L^\infty(\mathbb{R}\setminus\mathcal{D}_\gamma)}\,\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\frac{8}{2\varepsilon}\chi_{\{z\,|\,|z|\le\varepsilon\}} \\ &\qquad\times\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|.\end{aligned}$$

Now set

$$h_\varepsilon(\tilde{x},\tilde{t}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right)-u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|\varphi\left(\tilde{x},\tilde{t}\right)\omega_\varepsilon(\tau)\,d\tau\,dz.$$

By Lebesgue's differentiation theorem,

$$\lim_{\varepsilon\to 0} h_\varepsilon(x,t) = |v(x,t)-u(x,t)| \quad \text{a.e. } (x,t).$$

Therefore,

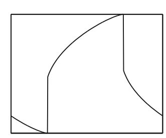
$$\lim_{\varepsilon\to 0}\Bigl|\iiiint_{\Omega} I_{\text{flux}}^{3}\left(\tilde{x},\tilde{t},z,\tau\right)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz\Bigr| \le \lim_{\varepsilon\to 0} C\iint_{\Pi_T}|u-v|\,\varphi\,dt\,dx, \tag{8.122}$$

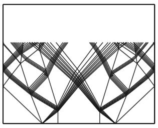
where the constant C is zero if γ is piecewise constant.

Combining (8.118), (8.119), (8.120), (8.121), and (8.122) we get (8.111). □

Equipped with Lemma 8.25, we can continue to prove uniqueness of entropy solutions. Define

$$\psi_\varepsilon(x) = \begin{cases} \frac{2}{\varepsilon}\left(\varepsilon+x\right) & \text{if } x\in[-\varepsilon,-\varepsilon/2], \\ 1 & \text{if } -\varepsilon/2 < x < \varepsilon/2, \\ \frac{2}{\varepsilon}(\varepsilon-x) & \text{if } x\in[\varepsilon/2,\varepsilon], \\ 0 & \text{otherwise,}\end{cases}$$





and set $\Psi_\varepsilon(x) = 1 - \sum_i^N \psi_\varepsilon(x - \xi_i)$. Observe that $\Psi_\varepsilon \to 1$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon \to 0$, and we consider only ε such that $\varepsilon < \min_i \{\xi_{i+1} - \xi_i\}$. Let φ be a nonnegative test function in $C_0^1(\Pi_T)$. Then $\phi = \varphi \Psi_\varepsilon$ is an admissible test function, as a density argument will show. Furthermore, ϕ has support away from $\mathcal{D}_\gamma$. With this test function, (8.111) takes the form

$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \Psi_\varepsilon \varphi_t + F(\gamma, u, v) \Psi_\varepsilon \varphi_x \big)\, dt\, dx - \iint\limits_{\Pi_T} F(\gamma, u, v) \Psi_\varepsilon' \varphi \, dt\, dx \\
\le C \iint\limits_{\Pi_T} |u - v|\, \Psi_\varepsilon \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \Psi_\varepsilon \varphi(x, 0)\, dx.
\end{aligned}
$$

Set

$$
I_\varepsilon = \iint\limits_{\Pi_T} F(\gamma, u, v) \Psi_\varepsilon' \varphi \, dt\, dx,
$$

and let $\varepsilon \downarrow 0$. This yields

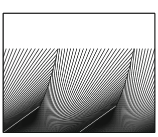
$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \varphi_t + F(\gamma, u, v) \varphi_x \big)\, dt\, dx \\
\le C \iint\limits_{\Pi_T} |u - v|\, \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x, 0)\, dx + \lim_{\varepsilon \downarrow 0} I_\varepsilon.
\end{aligned}
$$

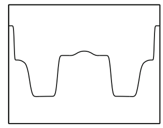
Now we use that (u_i^-, u_i^+) and (v_i^-, v^+) both satisfy the minimal jump entropy condition, and thus Lemma 8.6 applies at each discontinuity in γ. With this in mind, we calculate

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} I_\varepsilon &= \sum_i^N \lim_{\varepsilon \downarrow 0} \int_0^T \Bigg(\frac{2}{\varepsilon} \int_{\xi_i + \varepsilon/2}^{\xi_i + \varepsilon} F(\gamma(x), u, v) \varphi \, dx \\
&\qquad\qquad - \frac{2}{\varepsilon} \int_{\xi_i - \varepsilon}^{\xi_i - \varepsilon/2} F\left(\gamma(x), u, v\right) \varphi \, dx \Bigg)\, dt \\
&= \lim_{\varepsilon \downarrow 0} \sum_i^N \int_0^T \left(F\left(\gamma_i^+, u_i^+, v_i^+\right) - F\left(\gamma_i^-, u_i^-, v_i^-\right) \right) \varphi\left(\xi_i, t\right)\, dt \\
&\le 0.
\end{aligned}
$$

Hence for every nonnegative test function, we have

$$
\begin{aligned}
-\iint\limits_{\Pi_T} \big(|u - v|\, \varphi_t + F(\gamma, u, v) \varphi_x \big)\, dt\, dx \\
\le C \iint\limits_{\Pi_T} |u - v|\, \varphi \, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - v_0|\, \varphi(x, 0)\, dx. \qquad (8.123)
\end{aligned}
$$





This equation is very similar to (2.60), the difference being that F replaces q and that F depends explicitly on x. What follows is therefore analogous to the arguments used after (2.60).

Now let $\alpha_r(x)$ be a smooth function taking values in $[0, 1]$ such that

$$\alpha_r(x) = \begin{cases} 1 & \text{if } |x| \le r, \\ 0 & \text{if } |x| \ge r+1, \end{cases}$$

and $\max |\alpha_r'(x)| \le 2$. Then fix s_0 and s so that $0 < s_0 < s < T$. For all positive κ and τ such that $s_0 + \tau < s + \kappa < T$, let $\beta_{\kappa,\tau}(t)$ be a Lipschitz function that is linear on $[s_0, s_0 + \kappa]$ and on $[s, s + \tau]$ and satisfies

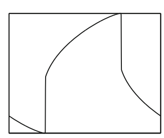
$$\beta_{\kappa,\tau}(t) = \begin{cases} 0 & \text{if } t < s_0 \text{ or } t > s + \kappa, \\ 1 & \text{if } s \in [s_0 + \tau, s]. \end{cases}$$

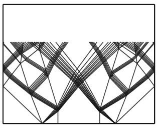
By density arguments, $\varphi = \alpha_r \beta_{\kappa,\tau}$ is an admissible test function, and using this in (8.123) gives

$$\begin{aligned} &\frac{1}{\kappa} \int_s^{s+\kappa} \int_{\mathbb{R}} |u - v|\, \alpha_r \, dx\, dt - \frac{1}{\tau} \int_{s_0}^{s_0+\tau} |u - v|\, \alpha_r \, dx\, dt \\ &\quad \le C \int_{s_0}^{s+\kappa} \int_{\mathbb{R}} |u - v|\, \alpha_r \, dx\, dt + 2 \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F(\gamma, u, v)|\, \beta_{\kappa,\tau} \, dx\, dt. \end{aligned}$$

Next, we let $s_0 \downarrow 0$ and use the triangle inequality to get

$$\begin{aligned} \frac{1}{\kappa} \int_s^{s+\kappa} \int_{\mathbb{R}} |u - v|\, \alpha_r \, dx\, dt \le &\int_{\mathbb{R}} |u_0 - v_0|\, \alpha_r \, dx \\ &+ \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |v(x,t) - v_0(x)|\, \alpha_r(x) \, dx\, dt \\ &+ \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |u(x,t) - u_0(x)|\, \alpha_r(x) \, dx\, dt \\ &+ C \int_{s_0}^{s+\kappa} \int_{\mathbb{R}} |u - v|\, \alpha_r \, dx\, dt \\ &+ 2 \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F(\gamma, u, v)|\, \beta_{\kappa,\tau} \, dx\, dt. \end{aligned}$$





We shall now send $\tau \downarrow 0$ and prove later that for every entropy solution u,

$$\lim_{\tau\downarrow 0} \frac{1}{\tau} \int_0^{\tau} \int_{\mathbb{R}} |u(x,t) - u_0(x)|\, \alpha_r(x)\, dx dt = 0. \tag{8.124}$$

Furthermore, by finite speed of propagation, if $u_0(x) = v_0(x)$ for $|x|$ large, then also $u(x,t) = v(x,t)$ for $|x|$ large. Hence $F(\gamma(x), u(x,t), v(x,t)) = 0$ for $|x|$ large. Thus

$$\lim_{r\to\infty} \int_{s_0}^{s+\kappa} \int_{r<|x|<r+1} |F\,(\gamma, u, v)|\, \beta_{\kappa,\tau}\, dx\, dt = 0.$$

Set

$$\mathcal{E}(t) = \int_{\mathbb{R}} |u(x,t) - v(x,t)|\; dx.$$

By sending $\tau \downarrow 0$ and then $r \uparrow \infty$, we obtain

$$\frac{1}{\kappa} \int_s^{s+\kappa} \mathcal{E}(t)\, dt \le \mathcal{E}(0) + C \int_0^{s+\kappa} \mathcal{E}(t)\, dt. \tag{8.125}$$

Let s be a Lebesgue point for the L^1 function $\mathcal{E}$. Sending $\kappa \downarrow 0$ yields

$$\mathcal{E}(s) \le \mathcal{E}(0) + C \int_0^s \mathcal{E}(t)\, dt.$$

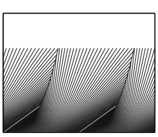
Since the set of Lebesgue points has full measure, we can use Gronwall's inequality to conclude that

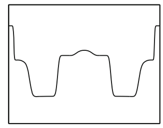
$$\mathcal{E}(t) \le e^{Ct} \mathcal{E}(0),$$

for almost every $t > 0$.

It remains to prove (8.124). To this end, define

$$\beta_\tau(t) = d \begin{cases} \frac{1}{\tau}(\tau - t) & \text{if } 0 \le t \le \tau, \\ 0 & \text{otherwise.} \end{cases}$$





We then use the test function $\omega_\varepsilon(x-y)\beta_\tau(t)\alpha_r(x)$ and the constant $c = u_0(y)$ in the entropy formulation (8.105). The result of this is

$$
\begin{aligned}
&\iint\limits_{\Pi_\tau} |u(x,t) - u_0(y)|\, \omega_\varepsilon(x-y)\alpha_r(x)\beta_\tau'(t)\,dt\,dx \\
&\quad + \iint\limits_{\Pi_\tau} F\left(\gamma(x), u, u_0(y)\right)\left(\omega_\varepsilon(x-y)\alpha_r(x)\right)_x \beta_\tau(t)\,dt\,dx \\
&\quad - \iint\limits_{\Pi_\tau\setminus\mathcal{D}_\gamma} \operatorname{sign}\left(u - u_0(y)\right)\partial_x f\left(\gamma(x), u_0(y)\right)\omega_\varepsilon(x-y)\alpha_r(x)\beta_\tau(t)\,dt\,dx \\
&\quad + \int\limits_0^\tau \sum_i \left| f\left(\gamma_i^+, u_0(y)\right) - f\left(\gamma_i^+, u_0(y)\right)\right| \omega_\varepsilon\left(\xi_i - y\right)\alpha_r\left(\xi_i\right)\beta_\tau(t)\,dt \\
&\quad + \int\limits_{\mathbb{R}} |u_0(x) - u_0(y)|\,\omega_\varepsilon(x-y)\alpha_r(x)\,dx \ge 0.
\end{aligned}
$$

Since $u \in L^1_{\mathrm{loc}}$, on sending $\tau \downarrow 0$, all terms in the above expression containing β_τ will vanish. Recalling that $\beta_\tau'(t) = -1/\tau$ for $t \in (0,\tau)$, after an application of the triangle inequality and an integration over $y \in \mathbb{R}$, we find that

$$
\begin{aligned}
&\lim_{\tau\downarrow 0} \frac{1}{\tau}\int\limits_0^\tau \int\limits_{\mathbb{R}} |u(x,t) - u_0(x)|\,\alpha_r(x)\,dx\,dt \\
&\qquad\qquad \le 2\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |u_0(x) - u_0(y)|\,\omega_\varepsilon(x-y)\alpha_r(x)\,dx\,dy.
\end{aligned}
$$

Since $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$, we can send $\varepsilon \downarrow 0$ to prove (8.124).

We have now proved that the initial value problem (8.53) is well posed in L^1.

Theorem 8.26 *Assume that the flux function f satisfies the assumptions A.1–A.4, and that the initial value u_0 is in $L^1(\mathbb{R})$ and $f(\gamma, u_0) \in BV(\mathbb{R})$. Then there exists a weak entropy solution, in the sense of* (8.54) *and* (8.105)*, to the initial value problem* (8.53).

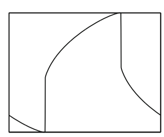
If v is another entropy solution with initial data v_0, then

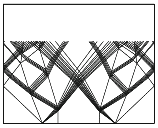
$$
\|v(\cdot\,,t) - u(\cdot\,,t)\|_{L^1(\mathbb{R})} \le e^{Ct}\,\|v_0 - u_0\|_{L^1(\mathbb{R})}\,,
$$

where the constant C depends on $\gamma'(x)$ for $x \notin \mathcal{D}_\gamma$ and is zero if γ is piecewise constant.

8.4 Notes

The presentation here is based on [161]. Over that last twenty years, conservation laws with spatially discontinuous flux functions have been studied in several papers; a very incomplete list includes [2, 36, 59, 71, 110, 111, 166, 181, 182] and other references therein.





The solution of the Riemann problem presented in this chapter is based on [70]. Regarding the admissibility criteria for solutions of the Riemann problem, as already hinted at in the text, there exist many criteria for selecting unique solutions; see, e.g., [2, 59]. It turns out that all these recipes can be used to prove an estimate similar to (8.123), and thus give a unique solution to the Cauchy problem. How this is done is explained in [5]. Example 8.8 is taken from [143].

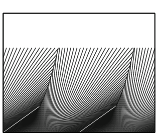
The convergence of the front-tracking algorithm is taken from [113]. In [114] the convergence of front tracking was shown for the polymer model (8.5). Existence proofs based on finite volume methods were first presented in [181], see also [182], and later extended to several dimensions in [107]. For a general overview we refer to [35].

8.5 Exercises

8.1 Solve the Riemann problem for the linear conservation law with discontinuous coefficients,

$$u_t + (a(x)u)_x = 0, \qquad a(x) = \begin{cases} a_l, & x < 0, \\ a_r, & x \geq 0. \end{cases}$$

8.2 Carry out the coordinate change transforming (8.4) into (8.5).



Appendix A
Total Variation, Compactness, Etc.

I hate T.V. I hate it as much as peanuts. But I can't stop eating peanuts.
— Orson Welles, The New York Herald Tribune (1956)

A key concept in the theory of conservation laws is the notion of *total variation*, T.V. (u), of a function u of one variable. We define

$$\text{T.V.}\,(u) := \sup \sum_i |u\,(x_i) - u\,(x_{i-1})|\,. \tag{A.1}$$

We will also use the notation $|u|_{BV} := \text{T.V.}\,(u)$. The supremum in (A.1) is taken over all finite partitions $\{x_i\}$ such that $x_{i-1} < x_i$. The set of all functions with finite total variation on I we denote by $BV(I)$. Clearly, functions in $BV(I)$ are bounded. We shall omit explicit mention of the interval I if (we think that) this is not important, or if it is clear which interval we are referring to.

For any finite partition $\{x_i\}$ we can write

$$\begin{aligned}\sum_i |u\,(x_{i+1}) - u\,(x_i)| &= \sum_i \max\left(u\,(x_{i+1}) - u\,(x_i)\,, 0\right)\\ &\quad - \sum_i \min\left(u\,(x_{i+1}) - u\,(x_i)\,, 0\right)\\ &=: p + n.\end{aligned}$$

Then the total variation of u can be written

$$\text{T.V.}\,(u) = P + N := \sup p + \sup n. \tag{A.2}$$

We call P the positive, and N the negative, variation of u. If for the moment we consider the finite interval $I = [a, x]$ and partitions with $a = x_1 < \cdots < x_n = x$, we have that

$$p_a^x - n_a^x = u(x) - u(a),$$

where we write p_a^x and n_a^x to indicate which interval we are considering. Hence

$$p_a^x \le N_a^x + u(x) - u(a).$$

H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*,
Applied Mathematical Sciences, DOI 10.1007/978-3-662-47507-2

Taking the supremum on the left-hand side, we obtain

$$P_a^x - N_a^x \leq u(x) - u(a).$$

Similarly, we have that $N_a^x - P_a^x \leq u(a) - u(x)$, and consequently

$$u(x) = P_a^x - N_a^x + u(a). \tag{A.3}$$

In other words, every function $u(x)$ in BV can be written as a difference between two increasing functions,[1]

$$u(x) = u_+(x) - u_-(x), \tag{A.4}$$

where $u_+(x) = u(a) + P_a^x$ and $u_-(x) = N_a^x$. Let ξ_j denote the points where u is discontinuous. Then we have that

$$\sum_j \left| u(\xi_j+) - u(\xi_j-) \right| \leq \mathrm{T.V.}\,(u) < \infty,$$

and hence we see that there can be at most a countable set of points where $u(\xi+) \neq u(\xi-)$.

Observe that functions with finite total variation are bounded, since

$$|u(x)| \leq |u(a)| + |u(a) - u(x)| \leq |u(a)| + \mathrm{T.V.}\,(u)\,.$$

Equation (A.3) has the very useful consequence that if a function u in BV is also differentiable, then

$$\int |u'(x)|\; dx = \mathrm{T.V.}\,(u)\,. \tag{A.5}$$

This equation holds, since

$$\int |u'(x)|\; dx = \int \Big(\frac{d}{dx} P_a^x + \frac{d}{dx} N_a^x \Big)\, dx = P + N = \mathrm{T.V.}\,(u)\,.$$

We can also relate the total variation with the shifted L^1-norm. Define

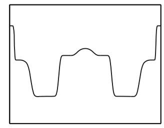
$$\lambda(u, \varepsilon) = \int |u(x+\varepsilon) - u(x)|\; dx. \tag{A.6}$$

If $\lambda(u,\varepsilon)$ is a (nonnegative) continuous function in ε with $\lambda(u,0) = 0$, we say that it is a *modulus of continuity* for u. More generally, we will use the name modulus of continuity for every continuous function $\lambda(u,\varepsilon)$ vanishing at $\varepsilon = 0$[2] such that $\lambda(u,\varepsilon) \geq \|u(\,\cdot\, + \varepsilon) - u\|_p$, where $\|\cdot\|_p$ is the L^p-norm. We will need a convenient characterization of total variation (in one variable), which is described in the following lemma.

---

[1] This decomposition is often called the Jordan decomposition of u.

[2] This is *not* an exponent, but a footnote! Clearly, $\lambda(u,\varepsilon)$ is a modulus of continuity if and only if $\lambda(u,\varepsilon) = o\,(1)$ as $\varepsilon \to 0$.



Lemma A.1 *Let u be a function in $L^1(\mathbb{R})$. If $\lambda(u,\varepsilon)/|\varepsilon|$ is bounded as a function of ε, then u is in BV and*

$$\mathrm{T.V.}\,(u) = \lim_{\varepsilon\to 0} \frac{\lambda(u,\varepsilon)}{|\varepsilon|}. \tag{A.7}$$

Conversely, if u is in BV, then $\lambda(u,\varepsilon)/|\varepsilon|$ is bounded, and thus (A.7) holds. In particular, we shall frequently use

$$\lambda(u,\varepsilon) \le |\varepsilon|\,\mathrm{T.V.}\,(u) \tag{A.8}$$

if u is in BV.

Proof Assume first that u is a smooth function. Let $\{x_i\}$ be a partition of the interval in question. Then

$$|u\,(x_i) - u\,(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} u'(x)\,dx \right| \le \lim_{\varepsilon\to 0} \int_{x_{i-1}}^{x_i} \left| \frac{u(x+\varepsilon)-u(x)}{\varepsilon} \right| dx.$$

Summing this over i, we get

$$\mathrm{T.V.}\,(u) \le \liminf_{\varepsilon\to 0} \frac{\lambda(u,\varepsilon)}{|\varepsilon|}$$

for differentiable functions $u(x)$. Let u be an arbitrary bounded function in L^1, and u_k a sequence of smooth functions such that $u_k(x) \to u(x)$ for almost all x, and $\|u_k - u\|_1 \to 0$. The triangle inequality shows that

$$|\lambda\,(u_k,\varepsilon) - \lambda\,(u,\varepsilon)| \le 2\|u_k - u\|_{L^1} \to 0.$$

Let $\{x_i\}$ be a partition of the interval. We can now choose u_k such that $u_k\,(x_i) = u\,(x_i)$ for all i. Then

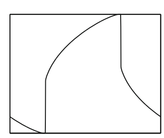
$$\sum |u\,(x_i) - u\,(x_{i-1})| \le \liminf_{\varepsilon\to 0} \frac{\lambda(u_k,\varepsilon)}{|\varepsilon|}.$$

Therefore,

$$\mathrm{T.V.}\,(u) \le \liminf_{\varepsilon\to 0} \frac{\lambda(u,\varepsilon)}{|\varepsilon|}.$$

Furthermore, we have

$$\begin{aligned}
\int |u(x+\varepsilon) - u(x)|\,dx &= \sum_j \int_{(j-1)\varepsilon}^{j\varepsilon} |u(x+\varepsilon) - u(x)|\,dx \\
&= \sum_j \int_0^{\varepsilon} |u(x+j\varepsilon) - u(x+(j-1)\varepsilon)|\,dx \\
&= \int_0^{\varepsilon} \sum_j |u(x+j\varepsilon) - u(x+(j-1)\varepsilon)|\,dx \\
&\le \int_0^{\varepsilon} \mathrm{T.V.}\,(u) \\
&= |\varepsilon|\,\mathrm{T.V.}\,(u)\,.
\end{aligned}$$



Thus we have proved the inequalities

$$\frac{\lambda(u,\varepsilon)}{|\varepsilon|} \le \mathrm{T.V.}\,(u) \le \liminf_{\varepsilon\to 0} \frac{\lambda(u,\varepsilon)}{|\varepsilon|} \le \limsup_{\varepsilon\to 0} \frac{\lambda(u,\varepsilon)}{|\varepsilon|} \le \mathrm{T.V.}\,(u)\,, \tag{A.9}$$

which imply the lemma. □

Observe that we trivially have

$$\tilde{\lambda}(u,\varepsilon) := \sup_{|\sigma|\le|\varepsilon|} \lambda(u,\sigma) \le |\varepsilon|\,\mathrm{T.V.}\,(u)\,. \tag{A.10}$$

For functions in L^p care has to be taken as to which points are used in the supremum, since these functions in general are not defined pointwise. The right choice here is to consider only points x_i that are points of *approximate continuity*[3] of u. Lemma A.1 remains valid.

We measure the variation in the case of a function u of two variables $u = u(x,y)$ as follows:

$$\mathrm{T.V.}_{x,y}\,(u) = \int \mathrm{T.V.}_{x}\,(u)\,(y)\,dy + \int \mathrm{T.V.}_{y}\,(u)\,(x)\,dx. \tag{A.11}$$

The extension to functions of n variables is obvious. We include a useful characterization of total variation.

Definition A.2 Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. We define the set of all functions with finite total variation with respect to Ω as follows:

$$BV(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) \mid \sup_{\phi\in C^1_0(\Omega;\mathbb{R}^n),\|\phi\|_\infty\le 1} \int_\Omega u(x)\,\mathrm{div}\,\phi(x)\,dx < \infty\}.$$

For $u \in BV(\Omega)$ we write

$$\|Du\| = \sup_{\phi\in C^1_0(\Omega;\mathbb{R}^n),\|\phi\|_\infty\le 1} \int_\Omega u(x)\,\mathrm{div}\,\phi(x)\,dx,$$

and for $u \in BV(\Omega) \cap L^1(\Omega)$ we define

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + \|Du\|\,.$$

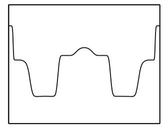
Remark A.3 If u is integrable with weak derivatives that are integrable functions, we clearly have

$$\|Du\| = \int |\nabla u(x)|\,dx.$$

In one space dimension there is a simple relation between $\|Du\|$ and $\mathrm{T.V.}\,(u)$, as the next theorem shows.

---

[3] A function u is said to be approximately continuous at x if there exists a measurable set A such that $\lim_{r\to 0} |[x-r,x+r]\cap A| \,/\, |[x-r,x+r]| = 1$ (here $|B|$ denotes the measure of the set B), and u is continuous at x relative to A. (Every Lebesgue point is a point of approximate continuity.) The supremum (A.1) is then called the essential variation of the function. However, in the theory of conservation laws it is customary to use the name total variation in this case, too, and we will follow this custom here.



Theorem A.4 *Let u be a function in $L^1(I)$, where I is an interval. Then*

$$\mathrm{T.V.}\,(u) = \|Du\|\,. \tag{A.12}$$

Proof Assume that u has finite total variation on I. Let ω be a nonnegative function bounded by unity with support in $[-1, 1]$ and unit integral. Define

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega\Big(\frac{x}{\varepsilon}\Big)$$

and

$$u^\varepsilon = \omega_\varepsilon * u. \tag{A.13}$$

Consider points $x_1 < x_2 < \cdots < x_n$ in I. Then

$$\begin{aligned}\sum_i |u^\varepsilon(x_i) - u^\varepsilon(x_{i-1})| &\le \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(x) \sum_i |u(x_i - x) - u(x_{i-1} - x)|\; dx\\ &\le \mathrm{T.V.}\,(u)\,.\end{aligned} \tag{A.14}$$

Using (A.5) and (A.14), we obtain

$$\begin{aligned}\int |(u^\varepsilon)'(x)|\; dx &= \mathrm{T.V.}\,(u^\varepsilon)\\ &= \sup \sum_i |u^\varepsilon(x_i) - u^\varepsilon(x_{i-1})|\\ &\le \mathrm{T.V.}\,(u)\,.\end{aligned}$$

Let $\phi \in C_0^1$ with $|\phi| \le 1$. Then

$$\begin{aligned}\int u^\varepsilon(x)\phi'(x)\,dx &= -\int (u^\varepsilon)'(x)\phi(x)\,dx\\ &\le \int |(u^\varepsilon)'(x)|\; dx\\ &\le \mathrm{T.V.}\,(u)\,,\end{aligned}$$

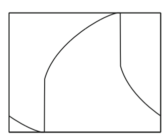
which proves the first part of the theorem.

Now let u be such that

$$\|Du\| := \sup_{\substack{\phi\in C_0^1\\ |\phi|\le 1}} \int u(x)\phi_x(x)\,dx < \infty.$$

First we infer that

$$\begin{aligned}-\int (u^\varepsilon)'(x)\phi(x)\,dx &= \int u^\varepsilon(x)\phi'(x)\,dx\\ &= -\int (\omega_\varepsilon * u)(x)\phi'(x)\,dx\\ &= -\int u(x)(\omega_\varepsilon * \phi)'(x)\,dx\\ &\le \|Du\|\,.\end{aligned}$$



Using that (see Exercise A.1)

$$\|f\|_{L^1(I)} = \sup_{\substack{\phi \in C_0^1(I),\\ |\phi|\le 1}} \int f(x)\phi(x)\,dx,$$

we conclude that

$$\int |(u^\varepsilon)'(x)|\,dx \le \|Du\|\,. \tag{A.15}$$

Next we show that $u \in L^\infty$. Choose a sequence $u_j \in BV \cap C^\infty$ such that (see, e.g., [64, p. 172])

$$u_j \to u \text{ a.e.}, \quad \left\|u_j - u\right\|_{L^1} \to 0, \quad j \to \infty, \tag{A.16}$$

and

$$\int \left|u_j'(x)\right|\,dx \to \|Du\|\,, \quad j \to \infty. \tag{A.17}$$

For all y, z we have

$$u_j(z) = u_j(y) + \int_y^z u_j'(x)\,dx.$$

Averaging over some bounded interval $J \subseteq I$, we obtain

$$\left|u_j\right| \le \frac{1}{|J|}\int_J \left|u_j(y)\right|\,dy + \int_I \left|u_j'(x)\right|\,dx, \tag{A.18}$$

which shows that the u_j are uniformly bounded, and hence $u \in L^\infty$. Thus

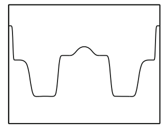
$$u^\varepsilon(x) \to u(x)$$

as $\varepsilon \to 0$ at each point of approximate continuity of u. Using points of approximate continuity $x_1 < x_2 < \cdots < x_n$, we conclude that

$$\begin{aligned}\sum_i |u(x_i) - u(x_{i-1})| &= \lim_{\varepsilon\to 0}\sum_i |u^\varepsilon(x_i) - u^\varepsilon(x_{i-1})| \\ &\le \limsup_{\varepsilon\to 0}\int |(u^\varepsilon)'(x)|\,dx \\ &\le \|Du\|\,. \end{aligned} \tag{A.19}$$

□

The next result shows that the generalization (A.11) of the total variation to higher dimension yields a (semi)norm that is equivalent to the one coming from bounded variation.



Theorem A.5 *Let $u \in L^1(K)$ with $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Then*

$$\|Du\| \leq \mathrm{T.V.}\,(u) \leq n\,\|Du\|\,.$$

Proof Assume first that $u \in BV(K) \cap L^1(K)$, and define the mollifier of u,

$$u^\varepsilon = \omega_\varepsilon * u.$$

Then $u^\varepsilon \to u$ in $L^1(K)$ and $\limsup_\varepsilon \|Du^\varepsilon\| \leq \|Du\| < \infty$; see [193, Thm. 5.3.1]. Let u_k denote the function all of whose variables but the kth remain fixed, namely

$$\begin{aligned} u_k(x', x) &= u(x_1, \ldots, x_{k-1}, x, x_k, \ldots, x_n), \\ x' &= (x_1, \ldots, x_{k-1}, x_k, \ldots, x_n) \in K'. \end{aligned}$$

Then also $u_k^\varepsilon \to u_k$ in $L^1([a_k, b_k])$, which implies, by the lower semicontinuity of the bounded variation [193, Thm. 5.2.1] and Theorem A.4,

$$\mathrm{T.V.}_{[a_k,b_k]}\,(u_k) \leq \liminf_\varepsilon \mathrm{T.V.}_{[a_k,b_k]}\left(u_k^\varepsilon\right).$$

Fatou's lemma and Theorem A.4 then imply

$$\begin{aligned} \int_{K'} \mathrm{T.V.}_{[a_k,b_k]}\,(u_k)\; dx' &\leq \liminf_\varepsilon \int_{K'} \mathrm{T.V.}_{[a_k,b_k]}\left(u_k^\varepsilon\right) dx' \\ &= \liminf_\varepsilon \int_K |D_k u^\varepsilon|\; dx \\ &\leq \limsup_\varepsilon \int_K |D_k u^\varepsilon|\; dx \\ &\leq \|Du\| < \infty. \end{aligned}$$

This implies that

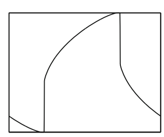
$$\mathrm{T.V.}\,(u) \leq n\,\|Du\|\,.$$

Assume now that $\int_{K'} \mathrm{T.V.}_{[a_k,b_k]}\,(u_k)\; dx' < \infty$ for all $k = 1, \ldots, n$. From Theorem A.4 we have for $\phi \in C_0^1(\Omega; \mathbb{R}^n)$, $\|\phi\|_\infty \leq 1$, that

$$\int_{K'} u \frac{\partial \phi}{\partial x_k}\, dx' \leq \int_{K'} \mathrm{T.V.}_{[a_k,b_k]}\,(u_k)\; dx',$$

from which it follows that $\|Du\| \leq \mathrm{T.V.}\,(u)$. $\square$

Total variation is used to obtain compactness. The appropriate compactness statement is Kolmogorov's compactness theorem. We say that a subset M of a complete metric space X is *compact* if every infinite sequence of points of M contains a (strongly) convergent sequence. A set is *relatively compact* if its closure is compact. A subset of a metric space is called *totally bounded* if it is contained in a finite union of balls of radius ε for every $\varepsilon > 0$ (we call this finite union an ε-net). Our starting theorem is the following result.



Theorem A.6 *A subset M of a complete metric space X is relatively compact if and only if it is totally bounded.*

Proof Consider first the case in which M is relatively compact. Assume that there exists an ε_0 for which there is no finite ε_0-net. For every element $u_1 \in M$ there exists an element $u_2 \in M$ such that $\|u_1 - u_2\| \geq \varepsilon_0$. Since the set $\{u_1, u_2\}$ is not an ε_0-net, there has to be a $u_3 \in M$ such that $\|u_1 - u_3\| \geq \varepsilon_0$ and $\|u_2 - u_3\| \geq \varepsilon_0$. Continuing inductively, construct a sequence $\{u_j\}$ such that

$$\left\|u_j - u_k\right\| \geq \varepsilon_0, \quad j \neq k,$$

which clearly cannot have a convergent subsequence, which yields a contradiction. Hence we conclude that there has to exist an ε-net for every ε.

Assume now that we can find a finite ε-net for M for every $\varepsilon > 0$, and let M_1 be an arbitrary infinite subset of M. Construct an ε-net for M_1 with $\varepsilon = \frac{1}{2}$, say $\{u_1^{(1)}, \dots, u_{N_1}^{(1)}\}$. Now let $M_1^{(j)}$ be the set of those $u \in M_1$ such that $\|u - u_j^{(1)}\| \leq \frac{1}{4}$. At least one of $M_1^{(1)}, \dots, M_1^{(N_1)}$ has to be infinite, since M_1 is infinite. Denote such a set by M_2 and the corresponding element by u_2. On this set we construct an ε-net with $\varepsilon = \frac{1}{4}$. Continuing inductively, we construct a nested sequence of subsets $M_{k+1} \subset M_k$ for $k \in \mathbb{N}$ such that M_k has an ε-net with $\varepsilon = 1/2^k$, say $\{u_1^{(k)}, \dots, u_{N_k}^{(k)}\}$. For arbitrary elements u, v of M_k we have $\|u - v\| \leq \|u - u_k\| + \|u_k - v\| \leq 1/2^{k-1}$. The sequence $\{u_k\}$ with $u_k \in M_k$ is convergent, since

$$\|u_{k+m} - u_k\| \leq \frac{1}{2^{k-1}},$$

proving that M_1 contains a convergent sequence. □

A result that simplifies our argument is the following.

Lemma A.7 *Let M be a subset of a metric space X. Assume that for each $\varepsilon > 0$, there is a totally bounded set A such that* $\operatorname{dist}(f, A) < \varepsilon$ *for each $f \in M$. Then M is totally bounded.*

Proof Let A be such that $\operatorname{dist}(f, A) < \varepsilon$ for each $f \in M$. Since A is totally bounded, there exist points $x_1, \dots, x_n$ in X such that $A \subseteq \cup_{j=1}^n \mathcal{B}_\varepsilon(x_j)$, where

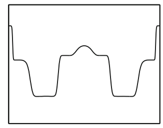
$$\mathcal{B}_\varepsilon(y) = \{z \in X \mid \|z - y\| \leq \varepsilon\}.$$

For every $f \in M$ there exists by assumption some $a \in A$ such that $\|a - f\| < \varepsilon$. Furthermore, $\left\|a - x_j\right\| < \varepsilon$ for some j. Thus $\left\|f - x_j\right\| < 2\varepsilon$, which proves

$$M \subseteq \bigcup_{j=1}^n \mathcal{B}_{2\varepsilon}(x_j).$$

Hence M is totally bounded. □

We can state and prove Kolmogorov's compactness theorem.



Theorem A.8 (Kolmogorov's compactness theorem) *Let M be a subset of $L^p(\Omega)$, $p \in [1,\infty)$, for some open set $\Omega \subseteq \mathbb{R}^n$. Then M is relatively compact if and only if the following three conditions are fulfilled*:

(i) *M is bounded in $L^p(\Omega)$, i.e.,*

$$\sup_{u\in M} \|u\|_{L^p} < \infty.$$

(ii) *We have*

$$\|u(\,\cdot\,+\varepsilon) - u\|_{L^p} \le \lambda(|\varepsilon|)$$

for a modulus of continuity λ that is independent of $u \in M$ (we let u equal zero outside Ω).

(iii)

$$\lim_{\alpha\to\infty} \int_{\{x\in\Omega \mid |x|\ge\alpha\}} |u(x)|^p \, dx = 0 \textit{ uniformly for } u \in M.$$

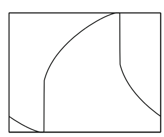
Remark A.9 In the case that Ω is bounded, condition (i) is clearly superfluous.

Proof We start by proving that conditions (i)–(iii) are sufficient to show that M is relatively compact. Let φ be a nonnegative and continuous function such that $\varphi \le 1$, $\varphi(x) = 1$ on $|x| \le 1$, and $\varphi(x) = 0$ whenever $|x| \ge 2$. Write $\varphi_r(x) = \varphi(x/r)$. From condition (iii) we see that $\|\varphi_r u - u\|_{L^p} \to 0$ as $r \to \infty$. Using Lemma A.7, we see that it suffices to show that $M_r = \{\varphi_r u \mid u \in M\}$ is totally bounded. Furthermore, we see that M_r satisfies (i) and (ii). In other words, we need to prove only that (i) and (ii) together with the existence of some R such that $u = 0$ whenever $u \in M$ and $|x| \ge R$ imply that M is totally bounded. Let ω_ε be a mollifier, that is,

$$\omega \in C_0^\infty, \quad 0 \le \omega \le 1, \quad \int \omega \, dx = 1, \quad \omega_\varepsilon(x) = \frac{1}{\varepsilon^n}\omega\Big(\frac{x}{\varepsilon}\Big).$$

Then

$$\begin{aligned}
\|u * \omega_\varepsilon - u\|_{L^p}^p &= \int |u*\omega_\varepsilon(x) - u(x)|^p \, dx \\
&= \int \Big| \int_{\mathcal{B}_\varepsilon} \big(u(x-y) - u(x)\big)\omega_\varepsilon(y)\, dy \Big|^p dx \\
&\le \int \int_{\mathcal{B}_\varepsilon} |u(x-y) - u(x)|^p \, dy \, \|\omega_\varepsilon\|_{L^q}^p \, dx \\
&= \varepsilon^{np/q-p} \|\omega\|_{L^q}^p \int_{\mathcal{B}_\varepsilon} \int |u(x-y) - u(x)|^p \, dx \, dy \\
&\le \varepsilon^{np/q-p} \|\omega\|_{L^q}^p \int_{\mathcal{B}_\varepsilon} \max_{|z|\le\varepsilon} \lambda(|z|)\, dy \\
&= \varepsilon^{n+np/q-p} \|\omega\|_{L^q}^p \, |\mathcal{B}_1| \max_{|z|\le\varepsilon} \lambda(|z|),
\end{aligned}$$



where $1/p + 1/q = 1$ and

$$\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(0) = \{z \in \mathbb{R}^n \mid \|z\| \le \varepsilon\}.$$

Thus

$$\|u * \omega_\varepsilon - u\|_{L^p} \le \varepsilon^{n-1} \|\omega\|_{L^q} \; |\mathcal{B}_1|^{1/p} \max_{|z|\le\varepsilon} \lambda(|z|), \tag{A.20}$$

which together with (ii) proves uniform convergence as $\varepsilon \to 0$ for $u \in M$. Using Lemma A.7, we see that it suffices to show that $N_\varepsilon = \{u * \omega_\varepsilon \mid u \in M\}$ is totally bounded for every $\varepsilon > 0$.

Hölder's inequality yields

$$|u * \omega_\varepsilon(x)| \le \|u\|_{L^p} \|\omega_\varepsilon\|_{L^q},$$

so by (i), functions in N_ε are uniformly bounded. Another application of Hölder's inequality implies

$$\begin{aligned} |u * \omega_\varepsilon(x) - u * \omega_\varepsilon(y)| &= \left| \int (u(x-z) - u(y-z))\omega_\varepsilon(z)\, dz \right| \\ &\le \|u(\cdot + x - y) - u\|_{L^p} \; \|\omega_\varepsilon\|_{L^q}, \end{aligned}$$

which together with (ii) proves that N_ε is equicontinuous. The Arzelà–Ascoli theorem implies that N_ε is relatively compact, and hence totally bounded in $C(\mathcal{B}_{R+r})$. Since the natural embedding of $C(\mathcal{B}_{R+r})$ into $L^p(\mathbb{R}^n)$ is bounded, it follows that N_ε is totally bounded in $L^p(\mathbb{R}^n)$ as well. Thus we have proved that conditions (i)–(iii) imply that M is relatively compact.

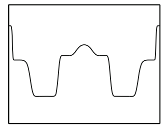
To prove the converse, we assume that M is relatively compact. Condition (i) is clear. Now let $\varepsilon > 0$. Since M is relatively compact, we can find functions $u_1, \dots, u_m$ in $L^p(\mathbb{R}^n)$ such that

$$M \subseteq \bigcup_{j=1}^m \mathcal{B}_\varepsilon(u_j).$$

Furthermore, since $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we may as well assume that $u_j \in C_0(\mathbb{R}^n)$. Clearly, $\left\|u_j(\cdot + y) - u_j\right\|_{L^p} \to 0$ as $y \to 0$, and so there is some $\delta > 0$ such that $\left\|u_j(\cdot + y) - u_j\right\|_{L^p} \le \varepsilon$ whenever $|y| < \delta$. If $u \in M$ and $|y| < \delta$, then pick some j such that $\left\|u - u_j\right\|_{L^p} < \varepsilon$, and obtain

$$\begin{aligned} \|u(\cdot + z) - u\|_{L^p} &\le \left\|u(\cdot + z) - u_j(\cdot + z)\right\|_{L^p} \\ &\quad + \left\|u_j(\cdot + z) - u_j\right\|_{L^p} + \left\|u_j - u\right\|_{L^p} \\ &= 2\left\|u_j - u\right\|_{L^p} + \left\|u_j(\cdot + z) - u_j\right\|_{L^p} \\ &\le 3\varepsilon, \end{aligned}$$

proving (ii).



When r is large enough, $\chi_{\mathcal{B}_r} u_j = u_j$ for all j, and then, with the same choice of j as above, we obtain

$$\|\chi_{\mathcal{B}_r} u - u\|_{L^p} \leq \left\|\chi_{\mathcal{B}_r}(u - u_j)\right\|_{L^p} + \left\|u - u_j\right\|_{L^p} \leq 2\left\|u - u_j\right\|_{L^p} \leq 2\varepsilon,$$

which proves (iii). □

Helly's theorem is a simple corollary of Kolmogorov's compactness theorem.

Corollary A.10 (Helly's theorem) *Let $\{h^\delta\}$ be a sequence of functions defined on an interval $[a, b]$, and assume that this sequence satisfies*

$$\mathrm{T.V.}\left(h^\delta\right) < M \qquad \textit{and} \qquad \left\|h^\delta\right\|_\infty < M,$$

where M is some constant independent of δ. Then there exists a subsequence h^{δ_n} that converges almost everywhere to some function h of bounded variation.

Proof It suffices to apply (A.8) (for $p = 1$) together with the boundedness of the total variation to show that condition (ii) in Kolmogorov's compactness theorem is satisfied. □

We remark that one can prove that the convergence in Helly's theorem is at every point, not only almost everywhere; see Exercise A.2.

The application of Kolmogorov's theorem in the context of conservation laws relies on the following result.

Theorem A.11 *Let $u_\eta \colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a family of functions such that for each positive T,*

$$\left|u_\eta(x, t)\right| \leq C_T, \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

for a constant C_T independent of η. Assume in addition that for all compact $B \subset \mathbb{R}^n$ and for $t \in [0, T]$,

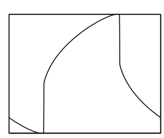
$$\sup_{|\xi| \leq |\rho|} \int_B \left|u_\eta(x + \xi, t) - u_\eta(x, t)\right| \, dx \leq \nu_{B,T}(|\rho|),$$

for a modulus of continuity $\nu_{B,T}$. Furthermore, assume that for s and t in $[0, T]$,

$$\int_B \left|u_\eta(x, t) - u_\eta(x, s)\right| \, dx \leq \omega_{B,T}(|t - s|) \textit{ as } \eta \to 0,$$

for some modulus of continuity $\omega_{B,T}$. Then there exists a sequence $\eta_j \to 0$ such that for each $t \in [0, T]$ the sequence $\{u_{\eta_j}(t)\}$ converges to a function $u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The convergence is in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^n))$.

Remark A.12 If the spatial total variation of u_η is uniformly bounded, then u_η has a spatial modulus of continuity.



Proof Kolmogorov's theorem implies that for each fixed $t \in [0,T]$ and for every sequence $\eta_j \to 0$ there exists a subsequence (still denoted by η_j) $\eta_j \to 0$ such that $\{u_{\eta_j}(t)\}$ converges to a function $u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Consider now a dense countable subset E of the interval $[0,T]$. By possibly taking a further subsequence (which we still denote by $\{u_{\eta_j}\}$), we find that

$$\int_B \left|u_{\eta_j}(x,t) - u(x,t)\right| \, dx \to 0 \text{ as } \eta_j \to 0, \text{ for } t \in E.$$

Now let $\varepsilon > 0$ be given. Then there exists a positive δ such that $\omega_{B,T}(\tilde{\delta}) \le \varepsilon$ for all $\tilde{\delta} \le \delta$. Fix $t \in [0,T]$. We can find a $t_k \in E$ with $|t_k - t| \le \delta$. Thus

$$\int_B \left|u_{\tilde{\eta}}(x,t) - u_{\tilde{\eta}}(x,t_k)\right| \, dx \le \omega_{B,T}(|t - t_k|) \le \varepsilon \text{ for } \tilde{\eta} \le \eta$$

and

$$\int_B \left|u_{\eta_{j_1}}(x,t_k) - u_{\eta_{j_2}}(x,t_k)\right| \, dx \le \varepsilon \text{ for } \eta_{j_1}, \eta_{j_2} \le \eta \text{ and } t_k \in E.$$

The triangle inequality yields

$$\begin{aligned}
&\int_B \left|u_{\eta_{j_1}}(x,t) - u_{\eta_{j_2}}(x,t)\right| \, dx \\
&\qquad \le \int_B \left|u_{\eta_{j_1}}(x,t) - u_{\eta_{j_1}}(x,t_k)\right| \, dx + \int_B \left|u_{\eta_{j_1}}(x,t_k) - u_{\eta_{j_2}}(x,t_k)\right| \, dx \\
&\qquad\quad + \int_B \left|u_{\eta_{j_2}}(x,t_k) - u_{\eta_{j_2}}(x,t)\right| \, dx \\
&\qquad \le 3\varepsilon,
\end{aligned}$$

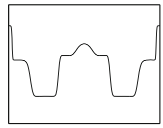
proving that for each $t \in [0,T]$ we have that $u_\eta(t) \to u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The bounded convergence theorem then shows that

$$\sup_{t\in[0,T]} \int_B \left|u_\eta(x,t) - u(x,t)\right| \, dx \to 0 \text{ as } \eta \to 0,$$

thereby proving the theorem. □

A.1 Notes

Extensive discussion about total variation can be found, e.g., in [64], [193], and [6]. The proof of Theorem A.6 is taken from Sobolev [171, pp. 28 ff]. An alternative proof can be found in Yosida [191, p. 13]. The proof of Theorem A.4 is from [64,



Thm. 1, p. 217]. The proof of Theorem A.5 follows [64, Thm. 2, p. 220] and [193, Thm. 5.3.5].

Kolmogorov's compactness theorem, Theorem A.8, was first proved by Kolmogorov in 1931 [115] in the case that Ω is bounded, $p > 1$, and the translation $u(x + \varepsilon)$ of $u(x)$ is replaced by the spherical mean of u over a ball of radius ε in condition (ii). It was extended to the unbounded case by Tamarkin [174] in 1932 and finally extended to the case with $p = 1$ by Tulajkov [184] in 1933. M. Riesz [158] proved the theorem with translations. See also [67]. For a survey, see [82].

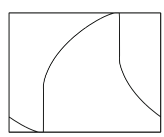
For other proofs of Kolmogorov's theorem, see, e.g., [171, pp. 28 ff], [39, pp. 69 f], [191, pp. 275 f], and [189, pp. 201 f].

A.2 Exercises

A.1 Show that for every $f \in L^1(I)$ we have

$$\|f\|_{L^1(I)} = \sup_{\substack{\phi \in C_0^1(I) \\ |\phi| \le 1}} \int f(x)\phi(x)\,dx.$$

A.2 Show that in Helly's theorem, Corollary A.10, one can find a subsequence h^{δ_n} that converges for all x to some function h of bounded variation.



Appendix B
The Method of Vanishing Viscosity

Details are the only things that interest.
— Oscar Wilde, Lord Arthur Savile's Crime (1891)

In this appendix we will give an alternative proof of existence of solutions of scalar multidimensional conservation laws based on the viscous regularization

$$u_t^\mu + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f_j\left(u^\mu\right) = \mu \Delta u^\mu, \quad u^\mu|_{t=0} = u_0, \tag{B.1}$$

where as usual Δu denotes the Laplacian $\sum_j u_{x_j x_j}$. Our starting point will be the following theorem:

Theorem B.1 *Let $u_0 \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$ with bounded derivatives and $f_j \in C^1(\mathbb{R})$ with bounded derivative. Then the Cauchy problem (B.1) has a classical solution, denoted by u^μ, that satisfies*[1]

$$u^\mu \in C^2(\mathbb{R}^m \times (0, \infty)) \cap C(\mathbb{R}^m \times [0, \infty)). \tag{B.2}$$

Furthermore, the solution satisfies the maximum principle

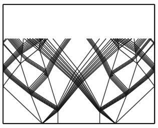
$$\|u^\mu(t)\|_{L^\infty(\mathbb{R}^m)} \leq \|u_0\|_{L^\infty(\mathbb{R}^m)}. \tag{B.3}$$

Let v^μ be another solution with initial data v_0 satisfying the same properties as u_0. Assume in addition that both u_0 and v_0 have finite total variation and are integrable. Then

$$\|u(\,\cdot\,, t) - v(\,\cdot\,, t)\|_{L^1(\mathbb{R}^m)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^m)}, \tag{B.4}$$

for all $t \geq 0$.

[1] The existence and regularity result (B.2) is valid for systems of equations in one spatial dimension as well.



Proof We present the proof in the one-dimensional case only, that is, with $m = 1$. Let K denote the heat kernel, that is,

$$K(x,t) = \frac{1}{\sqrt{4\mu\pi t}} \exp\Big(-\frac{x^2}{4\mu t}\Big). \tag{B.5}$$

Define functions u^n recursively as follows: Let $u^{-1} = 0$, and define u^n to be the solution of

$$u^n_t + f(u^{n-1})_x = \mu u^n_{xx}, \quad u^n|_{t=0} = u_0, \quad n = 0, 1, 2, \dots. \tag{B.6}$$

Then $u^n(t) \in C^\infty(\mathbb{R})$ for t positive. Applying Duhamel's principle, we obtain

$$\begin{aligned} u^n(x,t) &= \int K(x-y,t)u_0(y)\,dy \\ &\qquad - \iint_0^t K(x-y,t-s) f(u^{n-1}(y,s))_y \,ds\,dy \\ &= u^0(x,t) - \iint_0^t \frac{\partial}{\partial x} K(x-y,t-s) f(u^{n-1}(y,s))\,ds\,dy. \end{aligned} \tag{B.7}$$

Define $v^n = u^n - u^{n-1}$. Then

$$v^{n+1}(x,t) = -\iint_0^t \frac{\partial}{\partial x} K(x-y,t-s)\big(f(u^n(y,s)) - f(u^{n-1}(y,s))\big)\,ds\,dy.$$

Using Lipschitz continuity, we obtain

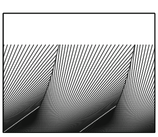
$$\begin{aligned} \left\|v^{n+1}(t)\right\|_{L^\infty(\mathbb{R})} &\le \|f\|_{\mathrm{Lip}} \int_0^t \|v^n(s)\|_{L^\infty(\mathbb{R})} \int \left|\frac{\partial}{\partial x} K(x,t-s)\right| dx\,ds \\ &\le \frac{\|f\|_{\mathrm{Lip}}}{\sqrt{\pi\mu}} \int_0^t (t-s)^{-1/2} \|v^n(s)\|_{L^\infty(\mathbb{R})}\,ds. \end{aligned}$$

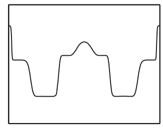
Assume that $|u_0| \le M$ for some constant M. Then we claim that

$$\|v^n(t)\|_{L^\infty(\mathbb{R})} \le M\|f\|^n_{\mathrm{Lip}} \frac{t^{n/2}}{\mu^{n/2}\Gamma(\frac{n+2}{2})}, \tag{B.8}$$

where we have introduced the gamma function defined by

$$\Gamma(p) = \int_0^\infty e^{-s}s^p\,ds.$$





We shall use the following properties of the gamma function. Let the beta function $B(p,q)$ be defined as

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}\,ds.$$

Then

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

After a change of variables, the last equality implies that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Equation (B.8) is clearly correct for $n = 0$. Assume it to be correct for n. Then

$$\begin{aligned}
\left|v^{n+1}(x,t)\right| &\le M\|f\|_{\mathrm{Lip}}^{n+1} \frac{1}{\sqrt{\pi}\mu^{(n+1)/2}\Gamma(\frac{n+2}{2})} \int_0^t (t-s)^{-1/2} s^{n/2}\,ds \\
&= M\|f\|_{\mathrm{Lip}}^{n+1} \frac{t^{(n+1)/2}}{\sqrt{\pi}\mu^{(n+1)/2}\Gamma(\frac{n+2}{2})} \int_0^1 (1-s)^{-1/2} s^{n/2}\,ds \\
&= M\|f\|_{\mathrm{Lip}}^{n+1} \frac{t^{(n+1)/2}}{\mu^{(n+1)/2}\Gamma(\frac{n+3}{2})}.
\end{aligned} \tag{B.9}$$

Hence we conclude that $\sum_n v^n$ converges uniformly on every bounded strip $t \in [0,T]$, and that

$$u = \lim_{n\to\infty} u^n = \lim_{n\to\infty} \sum_{j=0}^n v^j$$

exists. The convergence is uniform on the strip $t \in [0,T]$. It remains to show that u is a classical solution of the differential equation. We immediately infer that

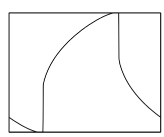
$$u(x,t) = u^0(x,t) - \iint_0^t \frac{\partial}{\partial x} K(x-y, t-s) f(u(y,s))\,ds\,dy. \tag{B.10}$$

It remains to show that (B.10) implies that u satisfies the differential equation

$$u_t + f(u)_x = \mu u_{xx}, \quad u|_{t=0} = u_0. \tag{B.11}$$

Next we want to show that u is differentiable. Define

$$M_n(t) = \sup_{x\in\mathbb{R}} \max_{0\le s\le t} |u_x^n(x,s)|.$$



Clearly,

$$\left|u_x^n(x,t)\right| \le \|f\|_{\mathrm{Lip}} \frac{1}{\sqrt{\pi\mu}} \int_0^t (t-s)^{-1/2} M_{n-1}(s)\,ds + M_0(t).$$

Choose B such that $M^0 \le B/2$. Then

$$M_n(t) \le B\exp(Ct/\mu) \tag{B.12}$$

if C is chosen such that

$$\|f\|_{\mathrm{Lip}} \frac{1}{\sqrt{\pi\mu}} \int_0^\infty s^{-1/2} e^{-Cs/\mu}\,ds \le \frac{1}{2}.$$

Inequality (B.12) follows by induction: It clearly holds for $n=0$. Assume that it holds for n. Then

$$\begin{aligned}
\left|u_x^{n+1}(s,x)\right| &\le \|f\|_{\mathrm{Lip}} \frac{1}{\sqrt{\pi\mu}} \int_0^t (t-s)^{-1/2} M_n(s)\,ds + B/2 \\
&\le Be^{Ct/\mu}\left(\|f\|_{\mathrm{Lip}} \frac{1}{\sqrt{\pi\mu}} \int_0^t s^{-1/2} e^{-Cs/\mu}\,ds + \frac{1}{2}\right) \\
&\le Be^{Ct/\mu}.
\end{aligned}$$

Define

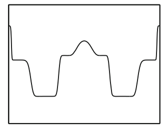
$$N_n(t) = \sup_{x\in\mathbb{R}} \max_{0\le s\le t} \left|u_{xx}^n(x,s)\right|.$$

Choose $\tilde{B} \ge \max\{2N^0, B^2+1\}$ and $\tilde{C} \ge C$ such that

$$2\tilde{B}\left(\|f'\|_{L^\infty} + \|f''\|_{L^\infty}\right) \frac{1}{\sqrt{\pi\mu}} \int_0^\infty s^{-1/2} e^{-2\tilde{C}s/\mu} \le \frac{1}{2}.$$

Then we show inductively that

$$N_n(t) \le \tilde{B} e^{2\tilde{C}t/\mu}.$$



The estimate is valid for $n = 0$. Assume that it holds for n. Then

$$\begin{aligned}
\left|u_{xx}^{n+1}(x,t)\right| &\le \left|u_{xx}^{0}(x,t)\right| \\
&\quad + \int_0^t \left(\|f''\|_{L^\infty} M_n(s)^2 + \|f'\|_{L^\infty} N_n(s)\right) \int \left|\frac{\partial}{\partial x} K(y,t-s)\right| dy\, ds \\
&\le N_0 \\
&\quad + \left(\|f'\|_{L^\infty} + \|f''\|_{L^\infty}\right) \frac{1}{\sqrt{\pi\mu}} \int_0^t \left(M_n(s)^2 + N_n(s)\right)(t-s)^{-1/2}\, ds \\
&\le N_0 \\
&\quad + \left(\|f'\|_{L^\infty} + \|f''\|_{L^\infty}\right) \frac{1}{\sqrt{\pi\mu}} \int_0^t \left(B^2 e^{2Cs/\mu} + e^{\tilde{C}s/\mu}\right)(t-s)^{-1/2}\, ds \\
&\le \tilde{B} e^{2\tilde{C}t/\mu}\left(1 + 2\tilde{B}\left(\|f'\|_{L^\infty} + \|f''\|_{L^\infty}\right) \frac{1}{\sqrt{\pi\mu}} \int_0^t e^{-2\tilde{C}s/\mu}\, ds\right) \\
&\le \tilde{B} e^{2\tilde{C}t/\mu}.
\end{aligned}$$

We have now established that $u^n \to u$ uniformly and that u_x^n and u_{xx}^n both are uniformly bounded (in (x,t) and n). Lemma B.2 (proved after this theorem) implies that indeed u is differentiable and that u_x equals the uniform limit of u_x^n. Performing an integration by parts in (B.10), we find that the limit u satisfies

$$u(x,t) = u^0(x,t) - \iint_0^t K(x-y,t-s)\, f(u(y,s))_y\, ds\, dy.$$

Applying Lemma B.3, we conclude that u satisfies

$$u_t + f(u)_x = \mu u_{xx}, \quad u|_{t=0} = u_0,$$

with the required regularity.[2]

The proof of (B.3) is nothing but the maximum principle. Consider the auxiliary function

$$U(x,t) = u(x,t) - \eta(t + (\eta x)^2/2).$$

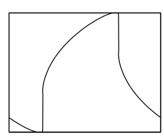
Since $U \to -\infty$ as $|x| \to \infty$, U obtains a maximum on $\mathbb{R} \times [0,T]$, say at the point (x_0,t_0). We know that

$$U(x_0,t_0) = u(x_0,t_0) - \eta(t_0 + (\eta x_0)^2/2) \ge u_0(0).$$

Hence

$$\eta^3 x_0^2 \le 2u(x_0,t_0) - 2u_0(0) - 2\eta t_0 \le \mathcal{O}(1) \tag{B.13}$$

[2] The argument up to this equality is valid for one-dimensional systems as well.



independently of η, since u is bounded on $\mathbb{R} \times [0, T]$ by construction. Assume that $0 < t_0 \le T$. At the maximum point we have

$$u_x(x_0, t_0) = \eta^3 x_0, \quad u_t(x_0, t_0) \ge \eta, \quad \text{and} \quad u_{xx}(x_0, t_0) \le \eta^3,$$

which implies that

$$\begin{aligned} u_t(x_0, t_0) + f'(u(x_0, t_0))u_x(x_0, t_0) - \mu u_{xx}(x_0, t_0) &\ge \eta - \mathcal{O}(1)\, \eta^{3/2} - \mu\eta^3 \\ &> 0 \end{aligned}$$

if η is sufficiently small. We have used that $f'(u)$ is bounded and (B.13). This contradicts the assumption that the maximum was attained for t positive. Thus

$$\begin{aligned} u(x, t) - \eta(t + (\eta x)^2/2) &\le \sup_x U(x, 0) \\ &= \sup_x \left(u_0(x) - \eta^3 x^2/2\right) \\ &\le \sup_x u_0(x), \end{aligned}$$

which implies that $u \le \sup u_0$. By considering η negative, we find that $u \ge \inf u_0$, from which we conclude that $\|u\|_{L^\infty} \le \|u_0\|_{L^\infty}$.

Lemma B.6 implies that every solution u satisfies the property needed for our uniqueness estimate, namely that if u_0 is in L^1, then $u(\,\cdot\,, t)$ is in L^1. This is so, since we have that

$$\|u(\,\cdot\,, t)\|_{L^1} - \|u_0\|_{L^1} \le \|u(\,\cdot\,, t) - u_0\|_{L^1} \le Ct.$$

Furthermore, since u is of bounded variation (which is the case if u_0 is of bounded variation), u_x is in L^1, and thus $\lim_{|x|\to\infty} u_x(x, t) = 0$. Hence, if u_0 is in $L^1 \cap BV$, then we have that

$$\frac{d}{dt}\int u(x, t)\, dx = -\int \left(f(u)_x + \mu u_{xx}\right)\, dx = 0.$$

Hence

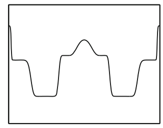
$$\int u(x, t)\, dx = \int u_0(x)\, dx. \tag{B.14}$$

By the Crandall–Tartar lemma, Lemma 2.13, to prove (B.4) it suffices to show that if $u_0(x) \le v_0(x)$, then $u(x, t) \le v(x, t)$. To this end, we first add a constant term to the viscous equation. More precisely, let u^δ denote the solution of (for simplicity of notation we let $\mu = 1$ in this part of the argument)

$$u_t^\delta + f(u^\delta)_x = u_{xx}^\delta - \delta, \quad u^\delta|_{t=0} = u_0.$$

In integral form we may write (cf. (B.10))

$$\begin{aligned} u^\delta(x, t) = &\int K(x - y, t)u_0(y)\, dy \\ &- \iint_0^t \frac{\partial}{\partial x} K(x - y, t - s)\, f(u^\delta(y, s))\, ds\, dy - \delta t. \end{aligned}$$



Furthermore,

$$
\begin{aligned}
&\left|u^\delta(x,t)-u(x,t)\right| \\
&\qquad \le \iint\limits_0^t \left|\frac{\partial}{\partial x}K(x-y,t-s)\right| \left|f(u^\delta(y,s))-f(u(y,s))\right| \,ds\,dy + |\delta|\,t \\
&\qquad \le \|f\|_{\mathrm{Lip}} \iint\limits_0^t \left|\frac{\partial}{\partial x}K(x-y,t-s)\right| \left|u^\delta(y,s)-u(y,s)\right| \,ds\,dy + |\delta|\,t \\
&\qquad \le \|f\|_{\mathrm{Lip}} \int_0^t \left\|u^\delta(s)-u(s)\right\|_{L^\infty} \frac{ds}{\sqrt{\pi(t-s)}} + |\delta|\,t \\
&\qquad \le \int_0^t \left\|u^\delta(s)-u(s)\right\|_{L^\infty} d\mu(s) + |\delta|\,t
\end{aligned}
$$

with the new integrable measure $d\mu(s) = \|f\|_{\mathrm{Lip}}/\sqrt{\pi(t-s)}$. Gronwall's inequality yields that

$$
\left\|u^\delta(t)-u(t)\right\|_{L^\infty} \le t\,|\delta| \exp\Big(\int_0^t d\mu(s)\Big) = t\,|\delta| \exp\left(2\frac{\sqrt{t}\,\|f\|_{\mathrm{Lip}}}{\sqrt{\pi}}\right),
$$

which implies that $u^\delta \to u$ in L^∞ as $\delta \to 0$. Thus it suffices to prove the monotonicity property for u^δ and v^δ, where

$$
v^\delta_t + f(v^\delta)_x = v^\delta_{xx} + \delta, \quad v^\delta|_{t=0} = v_0. \tag{B.15}
$$

Let $u_0 \le v_0$. We want to prove that $u^\delta \le v^\delta$. Assume to the contrary that $u^\delta(x,t) > v^\delta(x,t)$ for some (x,t), and define

$$
\hat{t} = \inf\{t \mid u^\delta(x,t) > v^\delta(x,t) \text{ for some } x\}.
$$

Pick $\hat{x}$ such that $u^\delta(\hat{x},\hat{t}) = v^\delta(\hat{x},\hat{t})$. At this point we have

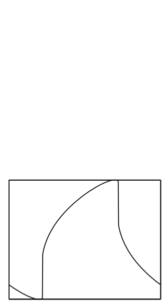
$$
u^\delta_x(\hat{x},\hat{t}) = v^\delta_x(\hat{x},\hat{t}), \quad u^\delta_{xx}(\hat{x},\hat{t}) \le v^\delta_{xx}(\hat{x},\hat{t}), \quad \text{and} \quad u^\delta_t(\hat{x},\hat{t}) \ge v^\delta_t(\hat{x},\hat{t}).
$$

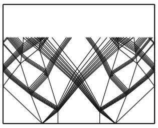
However, this implies the contradiction

$$
-\delta = u^\delta_t + f'(u^\delta)u^\delta_x - u^\delta_{xx} \ge v^\delta_t + f'(v^\delta)v^\delta_x - v^\delta_{xx} \ge \delta \text{ at the point } (\hat{x},\hat{t})
$$

whenever δ is positive.

Hence $u(x,t) \le v(x,t)$ and the solution operator is monotone, and (B.4) holds. □





In the above proof we needed the following two results.

Lemma B.2 *Let $\phi_n \in C^2(I)$ on the interval I, and assume that $\phi_n \to \phi$ uniformly. If $\|\phi_n'\|_{L^\infty}$ and $\|\phi_n''\|_{L^\infty}$ are bounded, then ϕ is differentiable, and*

$$\phi_n' \to \phi'$$

uniformly as $n \to \infty$.

Proof The family $\{\phi_n'\}$ is clearly equicontinuous and bounded. The Arzelà–Ascoli theorem implies that a subsequence $\{\phi_{n_k}'\}$ converges uniformly to some function ψ. Then

$$\phi_{n_k} = \int^x \phi_{n_k}'\,dx \to \int^x \psi\,dx,$$

from which we conclude that $\phi' = \psi$. We will show that the sequence $\{\phi_n'\}$ itself converges to ψ. Assume otherwise. Then we have a subsequence $\{\phi_{n_j}'\}$ that does not converge to ψ. The Arzelà–Ascoli theorem implies the existence of a further subsequence $\{\phi_{n_{j'}}'\}$ that converges to some element $\tilde{\psi}$, which is different from ψ. But then we have

$$\int^x \psi\,dx = \lim_{k\to\infty} \phi_{n_k} = \lim_{j'\to\infty} \phi_{n_{j'}} = \int^x \tilde{\psi}\,dx,$$

which shows that $\psi = \tilde{\psi}$, which is a contradiction. □

Lemma B.3 *Let $F(x,t)$ be a continuous function such that*

$$|F(x,t) - F(y,t)| \leq M\,|x-y|$$

uniformly in x, y, t. Define

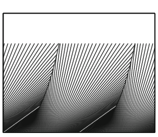
$$u(x,t) = \int K(x-y,t)u_0(y)\,dy + \iint_0^t K(x-y,t-s)F(y,s)\,ds\,dy.$$

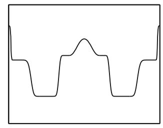
Then u is in $C^2(\mathbb{R}^m \times (0,\infty)) \cap C(\mathbb{R}^m \times [0,\infty))$ and satisfies

$$u_t = u_{xx} + F(x,t), \quad u|_{t=0} = u_0.$$

Proof To simplify the presentation we assume that $u_0 = 0$. First we observe that

$$u(x,t) = \int_0^t F(x,s)\,ds + \iint_0^t K(x-y,t-s)\big(F(y,s) - F(x,s)\big)\,ds\,dy.$$





The natural candidate for the time derivative of u is

$$u_t(x,t) = F(x,t) + \iint_0^t \frac{\partial}{\partial t} K(x-y,t-s)\big(F(y,s) - F(x,s)\big)\, ds\, dy. \quad \text{(B.16)}$$

To show that this is well defined we first observe that

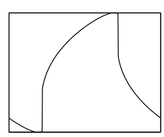
$$\left|\frac{\partial}{\partial t} K(x-y,t-s)\right| \le \frac{\mathcal{O}(1)}{t-s} K(x-y,2(t-s)).$$

Thus

$$\begin{aligned}
&\iint_0^t \left|\frac{\partial}{\partial t} K(x-y,t-s)\right| \, |F(y,s) - F(x,s)| \, ds\, dy \\
&\qquad \le M\,\mathcal{O}(1) \int_0^t \int \frac{1}{t-s} K(x-y,2(t-s))\, |y-x| \, dy\, ds \\
&\qquad \le M\,\mathcal{O}(1) \int_0^t \frac{1}{\sqrt{t-s}}\, ds \le \mathcal{O}(1)\,.
\end{aligned}$$

Consider now

$$\begin{aligned}
&\left|\frac{1}{\Delta t}\big(u(x,t+\Delta t) - u(x,t)\big) - u_t(x,t)\right| \\
&\quad \le \left|\frac{1}{\Delta t} \int_t^{t+\Delta t} F(x,s)\, ds - F(x,t)\right| \\
&\qquad + \int \frac{1}{\Delta t} \int_t^{t+\Delta t} K(x-y,t+\Delta t-s)\, |F(y,s) - F(x,s)| \, ds\, dy \\
&\qquad + \iint_0^t \left|\frac{1}{\Delta t}\big(K(x-y,t+\Delta t-s) - K(x-y,t-s)\big)\right. \\
&\qquad\qquad \left. - \frac{\partial}{\partial t} K(x-y,t-s)\right| |F(y,s) - F(x,s)| \, dy\, ds \\
&\quad \le \left|\frac{1}{\Delta t} \int_t^{t+\Delta t} F(x,s)\, ds - F(x,t)\right| \\
&\qquad + M \frac{1}{\Delta t} \int_t^{t+\Delta t} \int K(y,t+\Delta t-s)\, |y| \, dy\, ds \\
&\qquad + M \iint_0^t \left|\frac{\partial}{\partial t} K(y,t+\theta\Delta t-s) - \frac{\partial}{\partial t} K(y,t-s)\right| \, |y| \, ds\, dy,
\end{aligned}$$



for some $\theta \in [0,1]$. We easily see that the first two terms vanish in the limit as $\Delta t \to 0$. The last term can be estimated as follows (where $\delta > 0$):

$$\begin{aligned}
&\iint_0^t \left|\frac{\partial}{\partial t}K(y,t+\theta\Delta t-s) - \frac{\partial}{\partial t}K(y,t-s)\right| |y|\, dy\, ds \\
&\quad\le \int_0^{t-\delta}\int \left|\frac{\partial}{\partial t}K(y,t+\theta\Delta t-s) - \frac{\partial}{\partial t}K(y,t-s)\right| |y|\, dy\, ds \\
&\qquad + \int_{t-\delta}^{t}\int \left(\left|\frac{\partial}{\partial t}K(y,t+\theta\Delta t-s)\right| + \left|\frac{\partial}{\partial t}K(y,t-s)\right|\right) |y|\, dy\, ds \\
&\quad\le \int_0^{t-\delta}\int \left|\frac{\partial}{\partial t}K(y,t+\theta\Delta t-s) - \frac{\partial}{\partial t}K(y,t-s)\right| |y|\, dy\, ds \\
&\qquad + \mathcal{O}(1)\int_{t-\delta}^{t}\int \left(\frac{1}{t+\theta\Delta t-s}K(y,2(t+\theta\Delta t-s)) \right.\\
&\qquad\qquad\qquad \left. + \frac{1}{t-s}K(y,2(t-s))\right) |y|\, dy\, ds.
\end{aligned}$$

Choosing δ sufficiently small in the second integral, we can make that term less then a prescribed ϵ. For this fixed δ we choose Δt sufficiently small to make that integral less than ϵ. We conclude that indeed (B.16) holds. Using estimates

$$\left|\frac{\partial}{\partial x}K(x,t)\right| \le \frac{\mathcal{O}(1)}{\sqrt{t}}K(x,2t),$$
$$\left|\frac{\partial^2}{\partial x^2}K(x,t)\right| \le \frac{\mathcal{O}(1)}{t}K(x,2t),$$

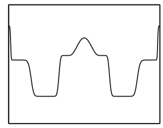
we conclude that the spatial derivatives are given by

$$\begin{aligned}
u_x(x,t) &= \iint_0^t \frac{\partial}{\partial x}K(x-y,t-s)F(y,s)\, ds\, dy, \\
u_{xx}(x,t) &= \iint_0^t \frac{\partial^2}{\partial x^2}K(x-y,t-s)F(y,s)\, ds\, dy,
\end{aligned} \tag{B.17}$$

from which we conclude that

$$\begin{aligned}
&u_t(x,t) - u_{xx}(x,t) \qquad\qquad (\mathrm{B.18})\\
&\quad = F(x,t) + \iint_0^t \left(\frac{\partial}{\partial t}K(x-y,t-s) - \frac{\partial^2}{\partial x^2}K(x-y,t-s)\right)F(y,s)\, ds\, dy \\
&\quad = F(x,t).
\end{aligned}$$

□



Remark B.4 The lemma is obvious if F is sufficiently differentiable; see, e.g., [142, Theorem 3, p. 144].

Next, we continue by showing directly that as $\mu \to 0$, the sequence $\{u^\mu\}$ converges to the unique entropy solution of the conservation law (B.30). We remark that this convergence was already established in Chapt. 3 when we considered error estimates.

In order to establish our estimates we shall need the following technical result.

Lemma B.5 *Let* $v\colon \mathbb{R}^m \to \mathbb{R}$ *such that* $v \in C^1(\mathbb{R}^m)$ *and* $|\nabla v| \in L^1(\mathbb{R}^m)$. *Then*

$$\int\limits_{|v|\le\eta} |\nabla v|\, dx \to 0 \text{ as } \eta \to 0.$$

Proof By the inverse function theorem, the set

$$\Big\{x \;\Big|\; v(x) = 0,\ \nabla v(x) \neq 0\Big\}$$

is a smooth $(m-1)$-dimensional manifold of $\mathbb{R}^m$. Thus

$$\int\limits_{|v|\le\eta} |\nabla v|\, dx = \int\limits_{0<|v|\le\eta} |\nabla v|\, dx.$$

The integrand (the norm of the gradient times the characteristic function of the region where $|v|$ is nonzero and less than η) tends pointwise to zero as $\eta \to 0$. The lemma follows using Lebesgue's dominated convergence theorem. □

The key estimates are contained in the next lemma.

Lemma B.6 *Assume that* $u_0 \in C^2(\mathbb{R}^m)$ *with bounded derivatives and finite total variation. Let* u^μ *denote the solution of equation* (B.1). *Then the following estimates hold*:

$$\mathrm{T.V.}\,(u^\mu(t)) \le \mathrm{T.V.}\,(u^\mu(0))\,, \tag{B.19}$$

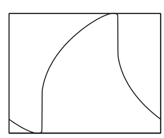
$$\|u^\mu(t) - u^\mu(s)\|_{L^1(\mathbb{R}^m)} \le C\,|t-s|\,. \tag{B.20}$$

Proof We set $w^0 = \partial u^\varepsilon/\partial t$ and $w^i = \partial u^\varepsilon/\partial x_i$ for $i = 1, \dots, m$. Then we find that

$$\frac{\partial w^i}{\partial t} + \sum_{j=1}^m \Big(f_j'(u^\mu) w^i\Big)_{x_j} = \mu \Delta w^i \tag{B.21}$$

for $i = 0, 1, \dots, m$. Define the following continuous approximation to the sign function:

$$\operatorname{sign}_\eta(x) = \begin{cases} 1 & \text{for } x \ge \eta, \\ x/\eta & \text{for } |x| < \eta, \\ -1 & \text{for } x \le -\eta. \end{cases}$$



Multiply (B.21) by $\operatorname{sign}_\eta(w^i)$ and integrate over $\mathbb{R}^m \times [0,T]$ for some T positive. This yields

$$\int_{\mathbb{R}^m}\int_0^T \frac{\partial w^i}{\partial t}\operatorname{sign}_\eta(w^i)\,dt\,dx + \sum_{j=1}^m \int_{\mathbb{R}^m}\int_0^T \left(f_j'(u^\mu)w^i\right)_{x_j}\operatorname{sign}_\eta(w^i)\,dt\,dx$$
$$= \int_{\mathbb{R}^m}\int_0^T \mu\Delta w^i \operatorname{sign}_\eta(w^i)\,dt\,dx. \tag{B.22}$$

The first term in (B.22) can be written

$$\int_{\mathbb{R}^m}\int_0^T \frac{\partial w^i}{\partial t}\operatorname{sign}_\eta(w^i)\,dt\,dx = \int_{\mathbb{R}^m}\int_0^T \left(w^i \operatorname{sign}_\eta(w^i)\right)_t\,dt\,dx$$
$$- \int_{\mathbb{R}^m}\int_0^T w^i \operatorname{sign}_\eta'(w^i)w^i_t\,dt\,dx.$$

Here we have that

$$\left|\int_{\mathbb{R}^m}\int_0^T w^i \operatorname{sign}_\eta'(w^i)w^i_t\,dt\,dx\right| = \frac{1}{\eta}\left|\int_{|w^i|\le\eta}\int_{t\le T} w^i\, w^i_t\,dt\,dx\right|$$
$$\le \int_{|w^i|\le\eta}\int_{t\le T} \left|w^i_t\right|\,dt\,dx \to 0,$$

using Lemma B.5, which implies

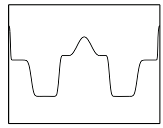
$$\int_{\mathbb{R}^m}\int_0^T \frac{\partial w^i}{\partial t}\operatorname{sign}_\eta(w^i)\,dt\,dx \to \int_{\mathbb{R}^m}\int_0^T \frac{\partial}{\partial t}\left|w^i\right|\,dt\,dx$$
$$= \left\|w^i(T)\right\|_{L^1} - \left\|w^i(0)\right\|_{L^1}, \tag{B.23}$$

as $\eta \to 0$. The second term in (B.22) reads

$$I := \sum_{j=1}^m \int_{\mathbb{R}^m}\int_0^T \left(f_j'(u^\mu)w^i\right)_{x_j}\operatorname{sign}_\eta(w^i)\,dt\,dx$$
$$= -\sum_{j=1}^m \int_{\mathbb{R}^m}\int_0^T f_j'(u^\mu)w^i \operatorname{sign}_\eta'(w^i)\frac{\partial w^i}{\partial x_j}\,dt\,dx$$
$$= -\frac{1}{\eta}\int_{\substack{|w^i|\le\eta\\ t\le T}} w^i\, f'(u^\mu)\cdot\nabla w^i\,dt\,dx,$$

where $f' = \left(f_1', \dots, f_m'\right)$. This can be estimated as follows:

$$|I| \le \sup_u |f'(u)| \int_{|w^i|\le\eta}\int_{t\le T} \left|\nabla w^i\right|\,dt\,dx \to 0, \tag{B.24}$$



as $\eta \to 0$. Here the supremum is over $|u| \le \|u^\mu(0)\|_\infty$. Finally,

$$\mu \int_{\mathbb{R}^m} \int_0^T \Delta w^i \operatorname{sign}_\eta(w^i)\, dt\, dx = -\mu \int_{\mathbb{R}^m} \int_0^T \left|\nabla w^i\right|^2 \operatorname{sign}'_\eta(w^i)\, dt\, dx \le 0. \tag{B.25}$$

Using (B.23), (B.24), and (B.25) in (B.22), we obtain, when $\eta \to 0$,

$$\left\|w^i(T)\right\|_{L^1} - \left\|w^i(0)\right\|_{L^1} \le 0. \tag{B.26}$$

For $i = 0$ this implies

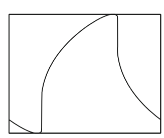
$$\begin{aligned}
\|u^\mu(t) - u^\mu(s)\|_{L^1} &= \int_{\mathbb{R}^m} \left| \int_s^t \frac{\partial u^\mu}{\partial t}\, dt \right| dx \\
&\le \int_{\mathbb{R}^m} \int_s^t \left|w^0(\tilde t)\right| d\tilde t\, dx \\
&= \int_s^t \left\|w^0(\tilde t)\right\|_{L^1} d\tilde t \\
&\le |t-s| \left\|w^0(0)\right\|_{L^1}.
\end{aligned}$$

For $i \ge 1$ we use (B.26) to prove (B.19). Recalling the results from Appendix A, we define

$$\lambda_i(u,\mu) = \int_{\mathbb{R}^m} |u(x+\mu e_i) - u(x)|\, dx \quad \text{and} \quad \lambda(u,\mu) = \sum_{i=1}^m \lambda_i(u,\mu).$$

Then the inequalities (A.10) hold. We have that

$$\begin{aligned}
\lambda_i(u^\varepsilon(\cdot,t),\mu) &= \int_{\mathbb{R}^m} |u^\varepsilon(x+\mu e_i,t) - u^\varepsilon(x,t)|\, dx \\
&= \int_{\mathbb{R}^m} \left| \int_0^\mu w^i(x+\alpha e_i,t)\, d\alpha \right| dx \\
&\le \int_{\mathbb{R}^m} \int_0^\mu \left|w^i(x+\alpha e_i,t)\right| d\alpha\, dx \\
&\le \int_0^\mu \left\|w^i(\cdot,t)\right\|_{L^1} d\alpha \\
&= |\mu| \left\|w^i(\cdot,t)\right\|_{L^1} \\
&\le |\mu| \left\|w^i(\cdot,0)\right\|_{L^1} \\
&= |\mu| \int_{\mathbb{R}^{m-1}} \mathrm{T.V.}_{x_i}(u_0)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_m.
\end{aligned}$$



Thus we find that

$$\mathrm{T.V.}\,(u^{\varepsilon}(\,\cdot\,,t)) = \liminf_{\mu\to 0} \frac{\lambda\,(u^{\varepsilon}(\,\cdot\,,t),\mu)}{|\mu|} \le \mathrm{T.V.}\,(u_0)\,,$$

which proves (B.19). □

From the estimates in Lemma B.6 we may conclude, using Helly's theorem, Corollary A.10, and Theorem A.11, that there exists a (sub)sequence of $\{u^{\mu}\}$ that converges uniformly in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^m))$ to a function that we denote by u. It remains to show that u is an entropy solution of the conservation law.

Let k be in $\mathbb{R}$. Then

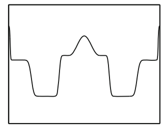
$$(u^{\mu}-k)_t + \nabla\cdot(f(u^{\mu}) - f(k)) = \mu\Delta(u^{\mu}-k). \tag{B.27}$$

Multiply (B.27) by $\mathrm{sign}_{\eta}(u^{\mu}-k)$ times a nonnegative test function ϕ and integrate over $[0,T]\times\mathbb{R}^m$. We find, when we write $U = u^{\mu}-k$, that

$$\begin{aligned}
0 &= \iint \Big(U_t\,\mathrm{sign}_{\eta}(U)\phi \\
&\qquad + \nabla\cdot(f(u^{\mu}) - f(k))\,\mathrm{sign}_{\eta}(U)\phi - \mu\,\mathrm{sign}_{\eta}(U)\Delta U\phi \Big)\,dx\,dt \\
&= \iint \Big(\big(U\,\mathrm{sign}_{\eta}(U)\big)_t\,\phi \\
&\qquad - (f(u^{\mu}) - f(k))\cdot\Big(\mathrm{sign}_{\eta}(U)\nabla\phi + \phi\,\mathrm{sign}'_{\eta}(U)\nabla U\Big)\Big)\,dx\,dt \\
&\quad + \mu\iint \nabla U\cdot\nabla\left(\mathrm{sign}_{\eta}(U)\phi\right)\,dx\,dt - \iint U\,\mathrm{sign}'_{\eta}(U)U_t\,\phi\,dx\,dt \\
&= -\iint \left(U\,\mathrm{sign}_{\eta}(U)\phi_t + \mathrm{sign}_{\eta}(U)\,(f(u^{\mu}) - f(k))\cdot\nabla\phi\right)\,dx\,dt \\
&\quad - \int \left((U\phi)\,\big|_{t=0} - (U\phi)\,\big|_{t=T}\,\right)dx \\
&\quad - \iint \phi\,\mathrm{sign}'_{\eta}(U)\,(f(u^{\mu}) - f(k))\cdot\nabla U\,dx\,dt \\
&\quad - \iint U\phi\,\mathrm{sign}'_{\eta}(U)U_t\,dx\,dt \\
&\quad + \mu\iint \mathrm{sign}_{\eta}(U)\nabla U\cdot\nabla\phi\,dx\,dt + \mu\iint |\nabla U|^2\,\mathrm{sign}'_{\eta}(U)\phi\,dx\,dt.
\end{aligned}$$

The third and the fourth integrals tend to zero as $\eta\to 0$ (since f is Lipschitz and $x\,\mathrm{sign}\,(x)'_{\eta}$ tends weakly to zero), and the last term is nonpositive. Hence

$$\begin{aligned}
&\iint \left(|u^{\mu}-k|\,\phi_t + \mathrm{sign}\,(u^{\mu}-k)\,(f(u^{\mu}) - f(k))\cdot\nabla\phi\right)dx\,dt \\
&\qquad - \int (u^{\mu}(0)-k)\phi|_{t=0}^{t=T}\,dx \ge \mu\iint \mathrm{sign}\,(U)\,\nabla U\cdot\nabla\phi\,dx\,dt.
\end{aligned}\tag{B.28}$$



Taking $\mu \to 0$, we see that the right-hand side tends to zero, and we conclude that

$$\iint \Big(|u-k|\phi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\cdot\nabla\phi \Big)\,dx\,dt \\ + \int (u_0-k)\phi|_{t=0}\,dx - \int (u(T)-k)\phi|_{t=T}\,dx \geq 0, \tag{B.29}$$

which is the Kružkov entropy condition. We have proved the following result.

Theorem B.7 *Let $u_0 \in C^2(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ with bounded derivatives and finite total variation, and let $f_j \in C^1(\mathbb{R})$ with bounded derivative. Let u^μ be the unique solution of* (B.1)*. Then there exists a convergent subsequence of $\{u^\mu\}$ that converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^m))$ to a function u that satisfies the Kružkov entropy condition* (B.29)*, and hence is the unique solution of*

$$u_t + \sum_{j=1}^m \frac{\partial}{\partial x_j} f_j(u) = 0, \quad u|_{t=0} = u_0. \tag{B.30}$$

B.1 Notes

Our proof of Theorem B.1 is taken in part from [99], where a similar result is proved for an equation of the form

$$u_t + \sum_{j=1}^m \psi_j(x,t,u)u_{x_j} = \mu\Delta u.$$

We are grateful to H. Hanche-Olsen for discussions on the proof of this theorem. We have also used [43]. Other proofs can be found; see, e.g., [141]. The conditions of Theorem B.1 can be weakened considerably. Alternative proofs of Theorem B.1 can be obtained using the dimensional splitting construction in Sect. 4.4. Lemma B.6 is familiar; see, e.g., [141]. Our presentation of Lemma B.6 and Theorem B.7 follows in part Bardos et al. [12].

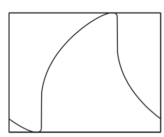
Bianchini and Bressan [16–18] have published results concerning the vanishing viscosity method for general systems. More precisely, consider the solution u^ε of the system

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}, \quad u^\varepsilon|_{t=0} = u_0.$$

They prove that u^ε converges to u, the solution of

$$u_t + A(u)u_x = 0, \quad u|_{t=0} = u_0,$$

as $\varepsilon \to 0$. Their assumptions are the following: the matrices $A(u)$ are smooth and strictly hyperbolic in a neighborhood of a compact set K, the initial data u_0 has sufficiently small total variation, and $\lim_{x\to-\infty} u_0(x) \in K$. The proof uses an ingenious decomposition of the function u^ε_x in terms of gradients of viscous traveling waves selected by a center manifold technique.



B.2 Exercises

B.1 Consider the *system* of parabolic equations

$$u_t + f(u)_x = \mu\, u_{xx}, \quad u|_{t=0} = u_0, \tag{B.31}$$

with $u_0(x) \in \mathbb{R}^n$.

(a) Show that there exists a solution u^μ of (B.31) that satisfies the regularity condition (B.2).

(b) Fix temporarily $\mu = 1$, and consider the equation

$$u_t + f(u)_x = u_{xx} - \delta, \quad u|_{t=0} = u_0, \tag{B.32}$$

with solution u^δ. Show that $\left\| u^\delta(t) - u(t) \right\|_1 \to 0$, where u solves (B.31) (with $\mu = 1$).

(c) Assume that the flux function f satisfies

$$f(u_1, \dots, u_{j-1}, u^*, u_{j+1}, \dots, u_n) = \text{const}, \quad u_i \in \mathbb{R},\ i \neq j,$$

for some j and some $u^* \in \mathbb{R}$. Assume that the jth component $u_{0,j}$ of u_0 satisfies $u_{0,j} \leq u^*$. Show that the jth component u_j of the solution u of (B.31) satisfies

$$u_j(x,t) \leq u^*.$$

(d) Assume that there are constants $u_* < u^*$ and j such that

$$\begin{aligned} f(u_1, \dots, u_{j-1}, u_*, u_{j+1}, \dots, u_n) &= \text{const}, \quad u_i \in \mathbb{R},\ i \neq j, \\ f(u_1, \dots, u_{j-1}, u^*, u_{j+1}, \dots, u_n) &= \text{const}, \quad u_i \in \mathbb{R},\ i \neq j, \end{aligned}$$

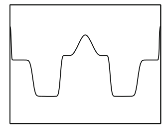
and that $u_* \leq u_{0,j} \leq u^*$. Show that

$$u_* \leq u_j(x,t) \leq u^*,$$

and hence that the region

$$\{u \in \mathbb{R}^n \mid u_* \leq u_j \leq u^*\}$$

is invariant for the solution of (B.31). Systems with this property appear, e.g., in multiphase flow in porous media and chemical chromatography. For more on invariant regions, see [96] and [88].



Appendix C
Answers and Hints

Health Warning

We accept no liability for these hints and answers. Use them at your own risk.

The Authors

The only way to get rid of a temptation is to yield to it.
Resist it, and your soul grows sick with longing
for the things it had forbidden itself.
— Oscar Wilde, The Picture of Dorian Gray (*1891*)

Chapter 1, Sect. 1.3

1.1 The characteristics are given by

$$x = 2\arctan\left(x_0 e^{\xi}\right), \quad t = \xi + t_0, \quad z = \xi + z_0,$$

$$x = 2\arctan\left(\frac{x_0 e^{\xi} - 1}{x_0 e^{\xi} + 1}\right), \quad t = 2\arctan\left(t_0 e^{\xi}\right), \quad z = z_0,$$

$$x = 2\int_0^{\xi} \sin\left(z_0 e^{\sigma}\right) d\sigma + x_0, \quad t = \xi + t_0, \quad z = z_0 e^{\xi},$$

$$x = \cos(z_0)\xi + x_0, \quad t = \sin(z_0)\xi + t_0, \quad z = z_0.$$

1.2 **a.**

$$u(x, y) = y + \sqrt{y^2 - x^2}.$$

b.

$$u(x, y) = \frac{1}{x}\cosh(y).$$

c.

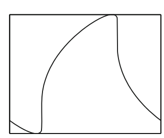
$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)\exp\left(\arctan(y/x)\right).$$

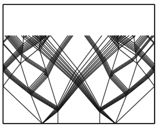
d.

$$u(x, y) = (x + 1)(y - 1).$$

e.

$$u(x, y) = \exp\left(y + (1 - x^2)/2\right) - 1.$$





f.
$$u(x, y) = (y - x^2)^2 \exp\left(x^2/2\right) - 1.$$

g.
$$x = (u - y^2) \exp\left(uy - 2y^3/3\right),$$

which determines u *implicitly* in terms of (x, y).

1.3 **a.** The theory of characteristics yields

$$t = s, \quad t = as + y, \quad z = z_0 + \int_0^y f(a\tau + y, \tau)\, d\tau,$$

which gives the solution.

b. Observe that

$$\frac{d}{ds} u(\xi(s; x_0), s) = f(\xi(s; x_0), s).$$

Integrate this from $s = 0$ to $s = t$.

c. Observe that

$$\frac{d}{ds} u(\zeta(s; x_0), t - s) = f(\zeta(s; x_0), t - s).$$

Integrate this from $s = 0$ to $s = t$.

1.4 You may use that u equals a constant is clearly a classical solution in the two domains $D_\pm = \{(x, t) \mid \pm(x - at) > 0\}$ that satisfies the Rankine–Hugoniot condition across $x = at$, since $f(u) = au$.

1.5 This is identical to the scalar case; work with each component f_i and u_i, etc.

1.6 Set $\mathbf{x} = (x, y)$ and $\mathbf{f} = (f, g)$. The conservation law reads $u_t + \nabla \cdot \mathbf{f} = 0$. Let $(\mathbf{x}, t)$ denote a point on the surface of discontinuity, and let $\mathcal{B}_r = \left\{(\mathbf{z}, \tau) \mid |\mathbf{x} - \mathbf{z}|^2 + (t - \tau)^2 \le r^2\right\}$. Denote by Γ_r the intersection of the surface of discontinuity with $\mathcal{B}_r$. Parameterize the surface as $t = t(x, y)$ with normal $N = (1, -t_x, -t_y)$. The velocity equals $\sigma = (t_x^2 + t_y^2)^{-1/2}$. Choose a test function $\varphi \in C_0^\infty(\mathcal{B}_r)$. Then an application of the divergence theorem yields

$$\int_{\Gamma_r} \varphi(x, y)\, (\llbracket u \rrbracket, \llbracket f \rrbracket, \llbracket g \rrbracket) \cdot N(x, y)\, dx\, dy = 0,$$

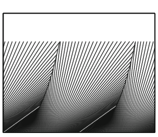
where $\llbracket u \rrbracket$ as usual denotes the jump in u across Γ_r, etc. Since φ is arbitrary, we obtain

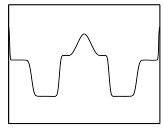
$$(\llbracket u \rrbracket, \llbracket f \rrbracket, \llbracket g \rrbracket) \cdot N = 0,$$

or

$$\sigma \llbracket u \rrbracket = n \cdot \llbracket \mathbf{f} \rrbracket,$$

where $n = (t_x, t_y)\sigma$ is the unit normal in the direction of the propagating discontinuity.





1.7 Observe that we need only consider the extreme characteristics originating at $x = \mp 1$, since before these meet, the solution will be linear between these. These characteristics have speed ± 1, and hence the solution will be continuous until $t = 1$, and is given by $u(x,t) = u_0(x/(t-1))$. The solution of the linearized equation is given by $v(x,t) = u_0(\alpha e^{\alpha t} x)$. From this we find that

$$v_n\left(x,(m+1)/n\right) = v_n\left(\alpha_{m,n} e^{\alpha_{m,n}/n}, m/n\right),$$

and thus $\alpha_{m+1,n} = \alpha_{m,n} e^{\alpha_{m,n}/n}$. Set $1/n = \Delta t$. Assuming that the limit holds, we have $\alpha(m+1,n) = \bar{\alpha}(t+\Delta t)$, and thus

$$\frac{\ln(\bar{\alpha}(t+\Delta t)) - \ln(\bar{\alpha}(t))}{\Delta t} = \bar{\alpha}(t).$$

Letting Δt go to zero, we find that $\bar{\alpha}'(t) = \bar{\alpha}^2(t)$, which gives the conclusion, since $\bar{\alpha}(0) = 1$. For $t \geq 1$, $\alpha_{m,n}$ diverges to $+\infty$, which incidentally gives us the correct solution.

1.8 The solutions in parts **a** and **b** are, respectively,

$$u(x,t) = \begin{cases} -1 & \text{for } x \leq t, \\ x/t & \text{for } |x| < t, \\ 1 & \text{for } x > t, \end{cases} \quad \text{and} \quad u(x,t) = u_0(x).$$

In the first case we directly verify that $u(x,t)$ also solves (1.75). In the second case the Rankine–Hugoniot condition, which in this case reads $s = 2\left[\!\left[u^3\right]\!\right]/(3\left[\!\left[u^2\right]\!\right])$, is violated. Set $v = u^2$. Then $v_t + \frac{2}{3}(v^{3/2})_x = 0$, and $v(x,0) = u(x,0)$. Hence the correct solution is a shock with speed $2/3$, which is different from the square of the solution in part **b**.

1.9 The jumps satisfy the Rankine–Hugoniot condition. That's all.

1.10 **a.** We multiply the inequality by $e^{-\gamma t}$ and find that

$$\frac{d}{dt}\left(e^{-\gamma t} u(t)\right) \leq 0.$$

Thus

$$u(t)e^{-\gamma t} \leq u(0).$$

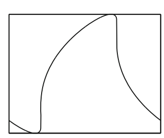
b. Multiply the inequality by e^{-Ct} to find that

$$\frac{d}{dt}\left(u(t)e^{-Ct}\right) \leq Ce^{-Ct}.$$

We integrate from 0 to t:

$$u(t)e^{-Ct} - u(0) \leq \int_0^t Ce^{-Cs}\,ds = 1 - e^{-Ct}.$$

After some rearranging, this is what we want.



c. Multiplying by $\exp(-\int_0^t c(s)ds)$ we find that

$$u(t) \le u(0)e^{\int_0^t c(s)ds} + \int_0^t d(s)e^{\int_s^t c(\tau)d\tau}\, ds,$$

which implies the claim.

d. Set $U(t) = \int_0^t u(t)dt$. Then

$$U'(t) \le C_1 U(t) + C_2,$$

and an application of part **c** gives that

$$\int_0^t u(s)\, ds \le -\frac{C_2}{C_1}\big(1 - e^{C_1 t}\big).$$

Inserting this in the original inequality yields the claim.

e. Introduce $w = u/f$. Then

$$w(t) \le 1 + \int_0^t \frac{f(s)}{f(t)} g(s)w(s)\, ds \le 1 + \int_0^t g(s)w(s)\, ds.$$

Let U be the right-hand side of the above inequality, that is,

$$U = 1 + \int_0^t g(s)w(s)\, ds.$$

Clearly, $U(0) = 1$ and $U'(t) = g(t)w(t) \le g(t)U(t)$. Applying part **c**, we find that

$$u(t) = w(t)f(t) \le f(t)U(t) \le f(t)\exp\Big(\int_0^t g(s)\, ds\Big).$$

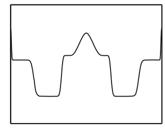
(If f is differentiable, we see that $U(t) = f(t) + \int_0^t g(s)u(s)\, ds$ satisfies $U'(t) \le f'(t) + g(t)U(t)$, and hence we could have used part **c** directly.)

1.11 Taylor's formula implies that

$$\eta'(u_j)D_-u_j = D_-\eta(u_j) + \frac{\Delta x}{2}\eta''(u_{j-1/2})\left(D_-u_j\right)^2.$$

We have that

$$a_j D_-\eta(u_j) = D_-(a_j\eta(u_j)) - \eta(u_{j-1})D_-a_j,$$



which gives

$$\frac{d}{dt}\eta(u_j) + D_-(a_j\eta(u_j)) \le \eta(u_{j-1})D_-a_j.$$

Summing over j gives the desired result. Regarding the Lipschitz continuity we have

$$\|aD_-u - a\ D_-v\| \le \frac{\|a\|_{L^\infty}}{\Delta x}2\,\|u-v\|\,.$$

1.12 The scheme implies the update

$$u_j^{n+1} = u_j^n - a_j^n\lambda\big(u_j^n - u_{j-1}^n\big)$$

with $\lambda = \Delta t/\Delta x$. Using that η is convex, we get (assuming the CFL condition $0 \le a_j^n\lambda \le 1$)

$$\begin{aligned}\eta\big(u_j^{n+1}\big) &= \eta\Big(\big(1-a_j^n\lambda\big)u_j^n + a_j^n\lambda u_{j-1}^n\Big)\\ &\le \big(1-a_j^n\lambda\big)\eta(u_j^n) + a_j^n\lambda\eta\big(u_{j-1}^n\big)\\ &= \eta\big(u_j^n\big) - a_j^n\Delta t D_-\eta\big(u_j^n\big).\end{aligned}$$

Summing over j, using the integration by parts formula

$$\sum_j c_j D_- b_j = -\sum_j b_j D_+ c_j, \quad \text{if} \quad c_{\pm\infty} = 0 \quad \text{or} \quad b_{\pm\infty} = 0,$$

one obtains

$$\begin{aligned}\sum_j \eta(u_j^{n+1}) &\le \sum_j \eta(u_j^n) - \Delta t\sum_j a_j^n D_-\eta(u_j^n)\\ &= \sum_j \eta(u_j^n) + \Delta t\sum_j \eta(u_j^n)D_+a_j^n,\end{aligned}$$

which implies

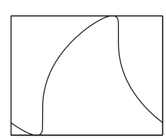
$$\Delta x\sum_j \eta(u_j^{n+1}) \le \Delta x\sum_j \eta(u_j^n) + \Delta t\Delta x\sum_j \eta(u_j^n)D_+a_j^n.$$

Define

$$F(t^{n+1}) = \Delta x\sum_j \eta(u_j^{n+1}).$$

Then

$$F(t^{n+1}) \le F(t^n) + \Delta t\Delta x\sum_j \eta(u_j^n)D_+a_j^n.$$



Also

$$|D_+ a_j^n| = |a_x(\xi)| \le C,$$

where ξ is a suitable intermediate value and C is a bound on a_x. Then

$$F(t^{n+1}) \le F(t^n) + C\Delta t F(t^n).$$

By recursion one obtains

$$F(t^{n+1}) \le F(t^0) + \sum_{\nu=0}^{n} C\Delta t F(t^\nu).$$

Using a discrete version of Gronwall's lemma (see below), identifying $\omega_n = F(t^n)$, $a_n = C\Delta t$, and $b_n = F(t^0)$, which is nondecreasing, we conclude that

$$F(t^n) \le \exp\left(\sum_{\nu=0}^{n-1} C\Delta t\right) F(t^0) = \exp(Ct^n) F(t^0),$$

i.e.,

$$\Delta x \sum_j \eta(u_j^n) \le e^{Ct^n} \Delta x \sum_j \eta(u_j^0).$$

Now we use that

$$u_j^0 = \frac{1}{\Delta x} \int_{I_{j-1/2}} u_0 \, dx.$$

Applying Jensen's inequality on the interval $[x_{j-1}, x_j]$, using that η is convex, we get

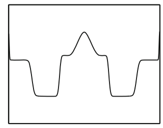
$$\eta(u_j^0) = \eta\left(\frac{1}{\Delta x} \int_{I_{j-1/2}} u_0 \, dx\right) \le \frac{1}{\Delta x} \int_{I_{j-1/2}} \eta(u_0) \, dx.$$

Therefore,

$$\begin{aligned}
\Delta x \sum_j \eta(u_j^0) &\le \Delta x \sum_j \frac{1}{\Delta x} \int_{I_{j-1/2}} \eta(u_0) \, dx \\
&= \sum_j \int_{I_{j-1/2}} \eta(u_0) \, dx = \|\eta(u_0)\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Discrete Gronwall: Let $(\omega_n)_{n\ge 0}$, $(a_n)_{n\ge 0}$, $(b_n)_{n\ge 0}$ be sequences of nonnegative numbers and $\omega_0 \le b_0$. If

$$\omega_n \le \sum_{\nu=0}^{n-1} a_\nu \omega_\nu + b_n, \quad n \in \mathbb{N},$$



and $(b_n)_{n\geq 0}$ is nondecreasing, then

$$\omega_n \leq \exp\Big(\sum_{\nu=0}^{n-1} a_\nu\Big) b_n, \quad n \in \mathbb{N}.$$

1.13 **a.** We find that

$$u_t = \alpha_{tt} + c\alpha_{tx}, \quad u_x = \alpha_{xt} + (c\alpha_x)_x$$

and

$$v_t = \alpha_{tt} - c\alpha_{tx}, \quad v_x = \alpha_{xt} - (c\alpha_x)_x.$$

Therefore,

$$u_t - cu_x = \alpha_{tt} + c\alpha_{tx} - c\alpha_{xt} - c(c\alpha_x)_x = \alpha_{tt} - c(c\alpha_x)_x = 0$$

and

$$v_t + cv_x = \alpha_{tt} - c\alpha_{tx} + c\alpha_{xt} - c(c\alpha_x)_x = \alpha_{tt} - c(c\alpha_x)_x = 0.$$

This implies the set of differential equations

$$\begin{aligned} u_t - cu_x &= 0, \\ v_t + cv_x &= 0, \\ \alpha_t &= \frac{1}{2}(u+v). \end{aligned}$$

We deduce the initial conditions u_0 and v_0 from α_0 and β_0:

$$\begin{aligned} u &= \alpha_t + c\alpha_x \quad \Rightarrow \quad u_0(x) = \beta_0(x) + c\alpha_0'(x), \\ v &= \alpha_t - c\alpha_x \quad \Rightarrow \quad v_0(x) = \beta_0(x) - c\alpha_0'(x). \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,0) &= u_0(x) = \beta_0(x) + c\alpha_0'(x), \\ v(x,0) &= v_0(x) = \beta_0(x) - c\alpha_0'(x), \\ \alpha(x,0) &= \alpha_0(x). \end{aligned}$$

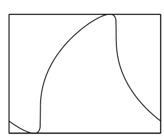
b. We define the backward characteristic equation

$$\frac{d}{d\tau}\zeta_u(\tau;x) = c(\zeta_u(\tau;x)), \quad \zeta_u(0;x) = x.$$

This equation is well defined, since c is Lipschitz. Then $u(x,t) = u_0(\zeta_u(t;x))$. We also define

$$\frac{d}{d\tau}\zeta_v(\tau;x) = -c(\zeta_v(\tau;x)), \quad \zeta_v(0;x) = x.$$

Then $v(x,t) = v_0(\zeta_v(t;x))$.



c. *Alternative 1:* Use upwind (with the appropriate upwind direction) and Euler methods for α:

$$\begin{aligned}
U_j^{n+1} &= U_j^n + \frac{c_j^n \Delta t}{\Delta x}\left(U_{j+1}^n - U_j^n\right),\\
V_j^{n+1} &= V_j^n - \frac{c_j^n \Delta t}{\Delta x}\left(V_j^n - V_{j-1}^n\right),\\
\alpha_j^{n+1} &= \alpha_j^n + \Delta t\frac{U_j^n + V_j^n}{2}.
\end{aligned}$$

Alternative 2: Use the Lax–Friedrichs scheme for (u, v) and Euler method for α:

$$\begin{aligned}
U_j^{n+1} &= \frac{U_{j-1}^n + U_{j+1}^n}{2} + \frac{c_j^n \Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right),\\
V_j^{n+1} &= \frac{V_{j-1}^n + V_{j+1}^n}{2} - \frac{c_j^n \Delta t}{2\Delta x}\left(V_{j+1}^n - V_{j-1}^n\right),\\
\alpha_j^{n+1} &= \alpha_j^n + \Delta t\frac{U_j^n + V_j^n}{2}.
\end{aligned}$$

The CFL condition reads

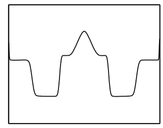
$$\Delta t^n \leq \frac{\Delta x}{\max_j \left|c_j^n\right|}.$$

d. We locate our variables at the cell centers and evaluate c_j^n at the cell centers. See Fig. C.1

1.14 Here is a MATLAB code that will work:

```
function u=charsolve(a,u0,x,T,dt)
% Solves the equation
%           u_t + a(x,t)u_x = 0
%              u(x,0)=u0(x)
% by approximate integration
% of the backward characteristics.

% ------------------------
% Solve the backward characteristic
% equation
%       y'(t)=-a(y(t),T-t),  y(0)=x,
% until t=T by the Euler method.
N=ceil(T/dt); dt=T/N; y=x; t=0;
for i=1:N,
    y=y-dt*a(y,T-t);
    t=t+dt;
end
% ----------------------
u=u0(y);
```



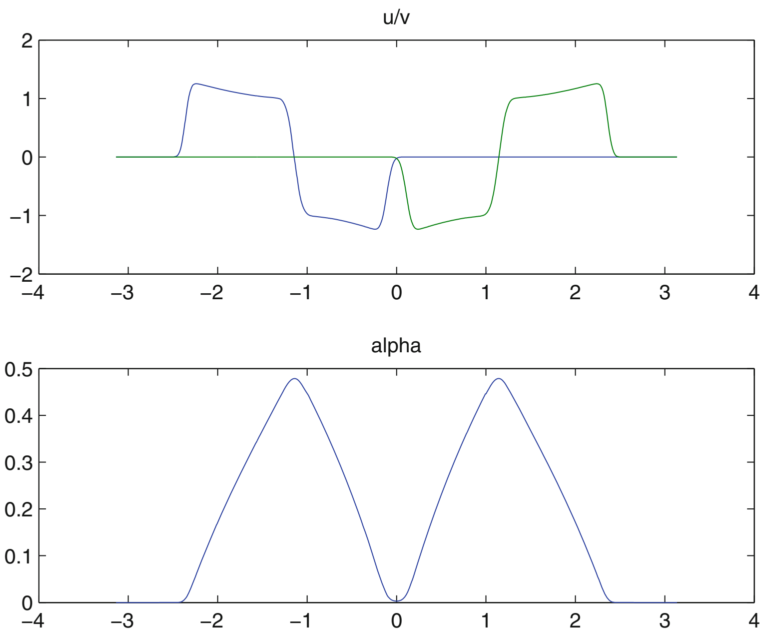


Fig. C.1 Solution at time $t = 1.0$ on an $N = 800$ grid using upwinding

For the initial data in the exercise, it would be invoked as

```
>> a=@(x,t) max(0,min(x,1));
>> u0=@(x) sin(x);
>> x=linspace(-2,2,200);
>> u=charsolve(a,u0,x,2,0.05);
```

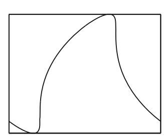
Chapter 2, Sect. 2.6

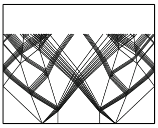
2.1 We find that

$$f'(u) = \frac{2u(1-u)}{(u^2 + (1-u)^2)^2},$$

and that the graph of f is "S-shaped" in the interval $[0, 1]$ with a single inflection point at $u = \frac{1}{2}$. Hence the solution of the Riemann problem will be a rarefaction wave followed by a shock. The left limit of the shock, u_1, will solve the equation $f'(u_1) = (1-f(u_1))/(1-u_1)$, which gives $u_1 = 1-\sqrt{2}/2$, and the speed of the shock will be $\sigma = (1+\sqrt{2})/2$. For $u < u_1$ we must find the inverse of f', and after some manipulation this is found to be

$$(f')^{-1}(\xi) = \frac{1}{2}\left(1 - \sqrt{\frac{1}{\xi}\left(\sqrt{4\xi+1} - 1\right) - 1}\right).$$





Hence the solution will be given by

$$u(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ (f')^{-1}(x/t) & \text{for } 0 \le x \le t(1+\sqrt{2})/2, \\ 1 & \text{for } x > t(1+\sqrt{2})/2. \end{cases}$$

2.2 See Examples 1.6 and 2.3. All velocity models give a concave flux function $f(\rho) = \rho v(\rho)$. For Riemann initial data

$$u(x,0) = \begin{cases} u_l & \text{for } x \le 0, \\ u_r & \text{for } x \ge 0, \end{cases}$$

the solution reads

$$u(x,t) = \begin{cases} u_l & \text{for } x \le st, \\ u_r & \text{for } x \ge st, \end{cases} \qquad s = [\![f]\!]\,/\,[\![\rho]\!]\,,$$

whenever $u_l < u_r$, and

$$u(x,t) = \begin{cases} u_l & \text{for } x \le f'(u_l)t, \\ (f')^{-1}(x/t) & \text{for } f'(u_l)t \le x \le f'(u_r)t, \\ u_r & \text{for } x \ge f'(u_r)t, \end{cases}$$

for $u_l > u_r$. In the case of the California model, the flux function becomes linear, $f(\rho) = v_0(1 - \rho/\rho_{\max})$, and thus the solution reads

$$u(x,t) = \begin{cases} u_l & \text{for } x \le -v_0 t/\rho_{\max}, \\ u_r & \text{for } x \ge -v_0 t/\rho_{\max}, \end{cases}$$

for all initial data.

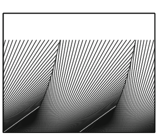
2.3 To show that the function in part **a** is a weak solution, we check that the Rankine–Hugoniot condition holds. In part **b** we find that $u^\varepsilon(x,t) = u_0^\varepsilon(x\varepsilon/(t+\varepsilon))$. Then

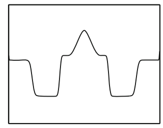
$$\bar{u}(x,t) = \begin{cases} -1 & \text{for } x < -t, \\ x/t & \text{for } |x| \le t, \\ 1 & \text{for } x > t. \end{cases}$$

The solution found in part **a** does not satisfy the entropy condition, whereas $\bar{u}$ does.

2.4 Introduce coordinates (τ, y) by $\tau = t$, $y = x - at$. Then the resulting problem reads

$$u_\tau^\varepsilon = \varepsilon u_{yy}^\varepsilon.$$





The solution to this is found by convolution with the heat kernel and reads

$$u^{\varepsilon}(y,\tau) = u_l + \frac{u_r - u_l}{\sqrt{4\varepsilon\pi\tau}} \int_0^\infty \exp\bigl(-(y-z)^2/(4\varepsilon\pi\tau)\bigr)\, dz.$$

The result follows from this formula.

2.5 Add (2.83) and (2.60); then choose ψ as (2.61).

2.6 **a.** The characteristics are given by

$$\frac{\partial t}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \xi} = c(x) f'(z), \quad \frac{\partial z}{\partial \xi} = 0,$$

or

$$t = \xi + t_0, \quad \frac{\partial x}{\partial \xi} = c(x) f'(z_0), \quad z = z_0.$$

b. The Rankine–Hugoniot condition reads

$$s\, [\![u]\!] = c\, [\![f]\!]\,.$$

c. The characteristics are given by

$$t = \xi + t_0, \quad x = \tan\bigl(z\xi + \arctan(x_0)\bigr), \quad z = z_0.$$

Using all values of z_0 between -1 and 1 for characteristics starting at the origin, and writing u in terms of (x,t), we obtain

$$u(x,t) = \begin{cases} -1 & \text{for } x \le -\tan t, \\ \frac{\arctan x}{t} & \text{for } |x| < \tan t, \\ 1 & \text{for } x \ge \tan t. \end{cases}$$

d. One possibility is to approximate f by a continuous, piecewise linear flux function, and keep the function c. The characteristics will no longer be straight lines, and one will have to solve the ordinary differential equations that come from the jump condition. Another possibility is to approximate c by piecewise constant or piecewise linear functions.

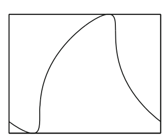
e. The entropy condition reads

$$|u-k|_t + c(\operatorname{sign}(u-k)\,(f(u)-f(k)))_x \le 0$$

weakly for all $k \in \mathbb{R}$.

2.7 **a.** The characteristics are given by

$$\frac{\partial t}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \xi} = c(x) f'(z), \quad \frac{\partial z}{\partial \xi} = -c'(x) f(z). \tag{C.1}$$



b. The entropy condition reads

$$|u-k|_t + (q(u,k)c(x))_x + \operatorname{sign}(u-k)\, f(k)c'(x) \le 0, \tag{C.2}$$

in the distributional sense.

c. Set $\eta(u,v) = |u-v|$ and $q(u,v) = \operatorname{sign}(u-v)\,(f(u)-f(v))$. Starting from the entropy condition (C.2), we get

$$\iint \Big[\eta(u,k)\varphi_t + q(u,k)c(x)\varphi_x - \operatorname{sign}(u-k)\, f(k)c'(x)\varphi\Big]\,dx\,dt \ge 0,$$
$$\iint \Big[\eta(v,k)\varphi_s + q(v,k)c(y)\varphi_y - \operatorname{sign}(v-k)\, f(k)c'(y)\varphi\Big]\,dx\,dt \ge 0.$$

We set $k=v$ in the first equation, $k=u$ in the second, and then add and integrate, obtaining

$$\begin{aligned}
\iint\!\!\iint \Big[&\eta(u,v)(\varphi_t+\varphi_s) + q(u,v)(c(x)\varphi_x + c(y)\varphi_y) \\
&\qquad - \operatorname{sign}(u-v)\,(f(u)c'(y) - f(v)c'(x))\varphi\Big]\,dx\,dt\,dy\,ds \\
+ \iint\!\!\iint &\operatorname{sign}(u-v)\Big[(f(u)-f(v))(c(y)-c(x))\varphi_y \\
&\qquad - c'(x)f(v)\varphi + c'(y)f(u)\varphi\Big]\,dx\,dt\,dy\,ds \ge 0.
\end{aligned}$$

Now, $\varphi_y = \psi_y\omega + \psi\omega_y$. Therefore, the first term in the last integrand above can be split into

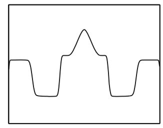
$$\begin{aligned}
&\operatorname{sign}(u-v)\,[(f(u)-f(v))(c(y)-c(x))]\,\omega_y\psi \\
&\qquad + \operatorname{sign}(u-v)\,[(f(u)-f(v))(c(y)-c(x))]\,\omega\psi_y.
\end{aligned}$$

The integral of the last term will vanish as $\varepsilon_1 \to 0$, since c is continuous. What remains is the integral of

$$\begin{aligned}
\psi\,\operatorname{sign}(u-v)\Big[&(f(u)-f(v))(c(y)-c(x))\omega_y \\
&\qquad - c'(x)f(v)\omega + c'(y)f(u)\omega\Big].
\end{aligned}$$

We have that

$$\begin{aligned}
(c(y)-c(x))\omega_y + c'(y)\omega &= \frac{\partial}{\partial y}\Big((c(y)-c(x))\omega\Big), \\
-(c(y)-c(x))\omega_y - c'(x)\omega &= -\Big((c(x)-c(y))\omega_x + c'(x)\omega\Big) \\
&= -\frac{\partial}{\partial x}\Big((c(x)-c(y))\omega\Big).
\end{aligned}$$



Thus the troublesome integrand can be written

$$\psi \operatorname{sign}(u-v)\left(f(u)\frac{\partial}{\partial y}\Big((c(y)-c(x))\omega\Big) \right.$$
$$\left. - f(v)\frac{\partial}{\partial x}\Big((c(x)-c(y))\omega\Big)\right).$$

We add and subtract to find that this equals

$$\psi \operatorname{sign}(u-v)(f(u)-f(v))\left(\frac{\partial}{\partial y}\Big((c(y)-c(x))\omega\Big)\right)$$
$$+ \psi \operatorname{sign}(u-v)\ f(v)\left[c'(y)-c'(x)\right]\omega.$$

Upon integration, the last term will vanish in the limit, since c' is continuous. Thus, after a partial integration we are left with

$$\iint\!\!\iint \frac{\partial}{\partial y}\Big(\psi q(u,v)\Big)(c(x)-c(y))\omega\,dx\,dt\,dy\,ds$$
$$\leq \|c'\|_{L^\infty}\varepsilon_1 \iint\!\!\iint \left|\frac{\partial}{\partial y}\Big(\psi q(u,v)\Big)\right| \omega\,dx\,dt\,dy\,ds$$
$$\leq \text{const}\,\varepsilon_1\left(\text{T.V.}(v)+\text{T.V.}(\psi)\right).$$

By sending ε_0 and ε_1 to zero, we find that

$$\iint |u-v|\,\psi_t + \operatorname{sign}(u-v)\,(f(u)-f(v))c(x)\psi_x\,dx\,dt \geq 0.$$

With this we can continue as in the proof of Proposition 2.10.

2.8 Mimic the proof of the Rankine–Hugoniot condition by applying the computation (1.21).

2.9 The function q satisfies $q' = f'\eta'$. Thus $q = u^3/3$. The entropy condition reads

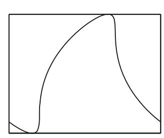
$$\int_{\mathbb{R}}\int_0^T \left(\frac{1}{2}u^2\phi_t + \frac{1}{3}u^3\phi_x\right)dt\,dx \geq -\frac{1}{2}\int_{\mathbb{R}} \left(u_0^2\phi|_{t=0} - (u^2\phi)|_{t=T}\right)dx.$$

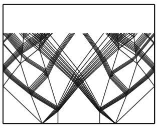
Choose functions ϕ that approximate the identity function appropriately. Then

$$\int_{\mathbb{R}} u^2\,dx \leq \int_{\mathbb{R}} u_0^2.$$

Solutions of conservation laws are not contractive in the L^2-norm in general, as the following counterexample shows. Let

$$u_0 = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad v_0 = \begin{cases} \frac{1}{2} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$





We find that

$$\|u(t) - v(t)\|_2^2 = \frac{1}{4} + \frac{5}{24}t$$

for $t < 2$.

2.10 We have that u is a Kružkov entropy solution. Thus (cf. (2.23))

$$\begin{aligned}\int_0^T \int \left(\eta\phi_t + q\phi_x\right) dx\, dt + \int \eta(u_0)\phi(x,0)\, dx \\ - \int \left(\eta(u)\phi\right)|_{t=T}\, dx \ge 0.\end{aligned} \tag{C.3}$$

In particular, u is a weak solution, and thus

$$\begin{aligned}\int_0^T \int \left((u-k)\phi_t + (f(u) - f(k))\phi_x\right) dx\, dt \\ + \int (u_0 - k)\phi(x,0)\, dx - \int \left((u-k)\phi\right)|_{t=T}\, dx = 0.\end{aligned} \tag{C.4}$$

Adding and subtracting (C.4) to (C.3) we find

$$\begin{aligned}\int_0^T \int \left((u-k)^{\pm}\phi_t + (u-k)^{\pm}(f(u) - f(k))\phi_x\right) dx\, dt \\ + \int (u_0 - k)^{\pm}\phi(x,0)\, dx - \int \left((u-k)^{\pm}\phi\right)|_{t=T}\, dx \ge 0.\end{aligned} \tag{C.5}$$

By following the Kružkov doubling of variables method we obtain the analogue of (2.59) with $\eta = |u - v|$ and $q = q(u,v)$ replaced by

$$\eta^{\pm} = (u-v)^{\pm}, \qquad q^{\pm}(u,v) = (u-v)^{\pm}(f(u) - f(v)).$$

The rest of the argument follows in a similar way.

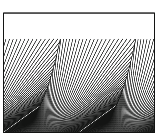
2.11 **a.** The Rankine–Hugoniot relation is the same as before,

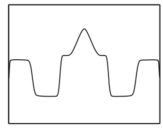
$$s\, [\![u]\!] = [\![f]\!]\,.$$

b.

$$\begin{aligned}\int_{\mathbb{R}} \int_0^T \left(|u-k|\, \phi_t + q(u,k)\phi_x\right) dt\, dx \\ + \int_{\mathbb{R}} \left((|u-k|\, \phi)|_{t=0} - (|u-k|\, \phi)|_{t=T}\right) dx \\ \ge - \int_{\mathbb{R}} \int_0^T \operatorname{sign}(u-k)\, g(u)\phi\, dt\, dx\end{aligned}$$

for all $k \in \mathbb{R}$ and all nonnegative test functions $\phi \in C_0^{\infty}(\mathbb{R} \times [0,T])$. (Recall that $q(u,k) = \operatorname{sign}(u-k)\,(f(u) - f(k))$.)





2.12 First we note that the Rankine–Hugoniot condition implies that v is locally bounded and uniformly continuous. Assume now that $v - \varphi$ has a local maximum at (x_0, t_0), where $t_0 > 0$. Since p is piecewise differentiable, we can define the following limits:

$$\bar{p}_l = \lim_{x \to x_0-} p(x, t_0) \geq \varphi_x(x_0, t_0) \geq \bar{p}_r = \lim_{x \to x_0+} p(x, t_0).$$

The inequalities hold, since $v - \varphi$ has a maximum at (x_0, t_0) and where p is differentiable,

$$v_x = p + x p_x - t H(p)_x = p + \frac{x}{t}\dot{p} + t p_t = p + \frac{x}{t}\dot{p} - \frac{x}{t}\dot{p} = p.$$

Thus $\hat{\varphi}_x = \varphi_x(x_0, t_0)$ is between $\bar{p}_l$ and $\bar{p}_r$. We also take the upper convex envelope. Thus

$$H_l + \sigma(\hat{\varphi}_x - \bar{p}_l) \geq H(\hat{\varphi}_x),$$
$$H_r + \sigma(\hat{\varphi}_x - \bar{p}_r) \geq H(\hat{\varphi}_x),$$

where $H_{l,r} = H(\bar{p}_{l,r})$ and $\sigma = (H_l - H_r)/(\bar{p}_l - \bar{p}_r)$ if $p_l \neq p_r$ and $\sigma = H'(p_{l,r})$ otherwise. We add the two equations to find that

$$\sigma \hat{\varphi}_x \geq H(\hat{\varphi}_x) + \frac{\sigma}{2}(\bar{p}_l + \bar{p}_r) - \frac{1}{2}(H_l + H_r). \tag{C.6}$$

Now we find (x, t) close to (x_0, t_0) such that

$$\sigma = \frac{x_0 - x}{t_0 - t}.$$

Since $v - \varphi$ has a local maximum at (x_0, t_0), we have that

$$\frac{v(x_0, t_0) - v(x, t)}{t_0 - t} \geq \frac{\varphi(x_0, t_0) - \varphi(x, t)}{t_0 - t}.$$

If p is assumed to be left continuous, we can now use this to show that

$$\sigma \bar{p}_l - H_l \geq \hat{\varphi}_t + \sigma \hat{\varphi}_x.$$

Choosing (x, t) slightly to the right of the line $x = \sigma t$, we can also show that

$$\sigma \bar{p}_r - H_r \geq \hat{\varphi}_t + \sigma \hat{\varphi}_x,$$

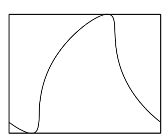
and therefore

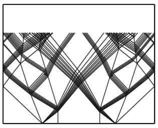
$$\frac{\sigma}{2}(\bar{p}_l + \bar{p}_r) - \frac{1}{2}(H_l + H_r) \geq \hat{\varphi}_t + \sigma \hat{\varphi}_x.$$

Using (C.6), we conclude that

$$\hat{\varphi}_t + H(\hat{\varphi}_x) \leq 0,$$

and v is a subsolution. To show that v is also a supsolution, proceed along similar lines.





2.13 Assuming f to be twice continuously differentiable, we find that

$$\begin{aligned}
\|f - f_\delta\|_{\mathrm{Lip}} &= \sup \frac{|(f(p) - f_\delta(p)) - (f(q) - f_\delta(q))|}{|p-q|} \\
&\le \sup |f'(p) - f_\delta'(p)| \\
&= \sup_{\substack{j,p \\ j\delta \le p \le (j+1)\delta}} \left| f'(p) - \frac{f((j+1)\delta) - f(j\delta)}{\delta} \right| \\
&= \sup_{\substack{p,q \\ |p-q|\le\delta}} |f'(p) - f'(q)| \le \delta \|f''\|_{L^\infty}.
\end{aligned}$$

Assume next that f is twice continuously differentiable on closed intervals I_1, I_2 with $I_1 \cap I_2 = \{\tilde{u}\}$, where $\tilde{u}$ is such that f is not twice differentiable at $\tilde{u}$. Then we get for $u \in I_1, v \in I_2$ that

$$\begin{aligned}
&\left| \frac{(f - f_\delta)(u) - (f - f_\delta)(v)}{u - v} \right| \\
&\quad \le \frac{|(f - f_\delta)(u) - (f - f_\delta)(\tilde{u})|}{|u-v|} + \frac{|(f - f_\delta)(\tilde{u}) - (f - f_\delta)(v)|}{|u-v|} \\
&\quad \le \frac{|(f - f_\delta)(u) - (f - f_\delta)(\tilde{u})|}{|u-\tilde{u}|} + \frac{|(f - f_\delta)(\tilde{u}) - (f - f_\delta)(v)|}{|\tilde{u}-v|} \\
&\quad \le \|f - f_\delta\|_{\mathrm{Lip}(I_1)} + \|f - f_\delta\|_{\mathrm{Lip}(I_2)} \\
&\quad \le \delta \, \|f''\|_{L^\infty(I_1)} + \delta \, \|f''\|_{L^\infty(I_2)} \\
&\quad \le 2\delta \, \|f''\|_{L^\infty(I_1 \cup I_2)} \,.
\end{aligned}$$

Thus in the general case, we get

$$\|f - f_\delta\|_{\mathrm{Lip}(-M,M)} \le C_1 \delta \, \|f''\|_{L^\infty(-M,M)} \,,$$

where C_1 is one plus the number of points where the second derivative does not exist.
Finally, we observe that

$$\begin{aligned}
\|f_{n_1} - f_{n_2}\|_{\mathrm{Lip}(-M,M)} &\le \|f_{n_1} - f\|_{\mathrm{Lip}(-M,M)} + \|f - f_{n_2}\|_{\mathrm{Lip}(-M,M)} \\
&\le 2C_1 \delta_{n_1} \, \|f''\|_{L^\infty(-M,M)}
\end{aligned}$$

when $n_1 \le n_2$.

2.14 **a.** Observe first that

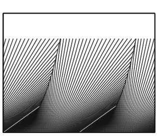
$$f_\smile(u) = \sup_{g \in A} g(u),$$

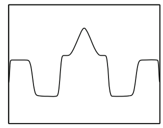
where A is the set of all affine function below f, that is,

$$A = \{g(u) = su + q \mid s, q \in \mathbb{R},\ sv + q \le f(v),\ v \in [a,b]\}.$$

Consider the subset of A with a fixed slope:

$$A_s = \{g \in A \mid g' = s\}.$$





Then

$$f_{\smile}(u) = \sup_{s\in\mathbb{R}} \sup_{g\in A_s} g(u),$$

and there exists $g_s \in A_s$ such that $\sup_{g\in A_s} g(u) = g_s(u)$. Since f is continuous, there exists $\bar{u}$ such that

$$g_s(\bar{u}) = f(\bar{u}),$$

which implies

$$q = g_s(\bar{u}) - s\bar{u} = f(\bar{u}) - s\bar{u}.$$

Since $g_s \le f$, we infer that q must be minimal. Thus

$$q = \min_{v\in[a,b]} \big(f(v) - sv\big) = -\max_{v\in[a,b]} \big(sv - f(v)\big) = -f^*(s).$$

Finally,

$$f_{\smile}(u) = \sup_{s\in\mathbb{R}} g_s(u) = \sup_{s\in\mathbb{R}} \big(su - f^*(s)\big) = f^{**}(u).$$

b. The fact that $u(\xi) = (f_{\smile}')^{-1}(\xi)$ implies that $\xi = f_{\smile}'(u)$. We have that

$$f_{\smile}(u) = f^{**}(u) = \max_{v\in[a,b]} \big(uv - f^*(v)\big).$$

The maximum is attained for $s = f_{\smile}'(u)$, and hence $s = \xi$, and we obtain

$$f_{\smile}(u) = u\xi - f^*(\xi).$$

By differentiation with respect to ξ we obtain

$$f_{\smile}'(u)u'(\xi) = u'(\xi)\xi + u(\xi) - \frac{d}{d\xi} f^*(\xi),$$

which implies, since we have $\xi = f_{\smile}'(u)$, that

$$u(\xi) = \frac{d}{d\xi} f^*(\xi).$$

See [148].

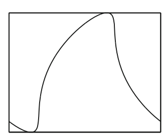
2.15 We first find the characteristics (parameterized using t)

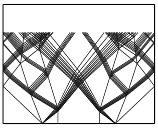
$$x = x(\eta, t) = \begin{cases} \eta + (1 - e^{-t}) & \text{for } \eta \le -\frac{1}{2}, \\ \eta(2e^{-t} - 1) & \text{for } -\frac{1}{2} < \eta < 0, \\ \eta & \text{for } \eta \ge 0, \end{cases}$$

$$u = u(\eta, t) = u_0(\eta)e^{-t}.$$

Characteristics with $\eta \in \big[-\frac{1}{2}, 0\big]$ collide at $t = \ln 2$. At that time a shock forms. The solution reads

$$u(x,t) = \begin{cases} e^{-t} & \text{for } x < \min\big(\frac{1}{2} - e^{-t}, \frac{1}{4} - \frac{1}{2}e^{-t}\big), \\ \frac{2x}{e^t - 2} & \text{for } \frac{1}{2} - e^{-t} \le x \le 0, \\ 0 & \text{for } x \ge \max\big(0, \frac{1}{4} - \frac{1}{2}e^{-t}\big). \end{cases}$$





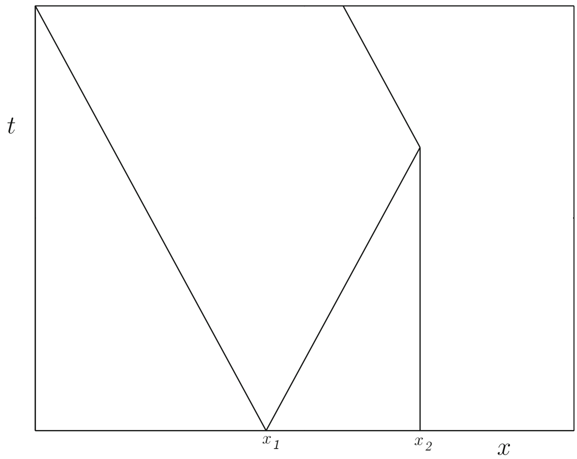


Fig. C.2 The fronts for Exercise 2.19

2.16 The solution reads

$$u(x,t) = \begin{cases} 2 & \text{for } x < \frac{1}{2}(e^2-1)t, \\ 0 & \text{for } x \ge \frac{1}{2}(e^2-1)t. \end{cases}$$

2.17 **a.**

$$u(x,t) = \begin{cases} 1 & \text{for } x < t+2, \\ 0 & \text{for } x \ge t+2. \end{cases}$$

b.

$$u(x,t) = \begin{cases} 0 & \text{for } x \le 2, \\ \left(\frac{x-2}{3t}\right)^{1/2} & \text{for } 2 < x < 3t+2, \\ 1 & \text{for } x \ge 3t+2. \end{cases}$$

2.18 The solution reads

$$u(x,t) = \begin{cases} x/t & \text{for } 0 < x < t, \\ 1 & \text{for } t \le x \le 1+t/2, \\ 0 & \text{otherwise,} \end{cases}$$

when $t \le 2$, and

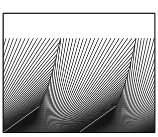
$$u(x,t) = \begin{cases} x/t & \text{for } 0 < x < \sqrt{2t}, \\ 0 & \text{otherwise,} \end{cases}$$

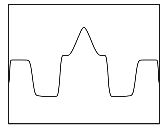
when $t > 2$.

2.19 In Fig. C.2 you can see how the fronts are supposed to move, but you will have to work out the states yourself.

2.20 See, e.g., [58, p. 255].

2.21 Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = \infty$ denote the collision times. On each time interval $[t_i, t_{i+1}]$ the solution satisfies the Kružkov entropy condition in the sense of (2.23); cf. (2.30). By adding all these inequalities, all boundary terms will cancel except the term coming from the initial data, which results in (2.23) with $T = \infty$.





Chapter 3, Sect. 3.7

3.1 The weak entropy solution reads

$$u(x,t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{t} & \text{for } 0 < x < t, \\ 1 & \text{for } x \geq t. \end{cases}$$

3.2 We do the MacCormack method only; the Lax–Wendroff scheme is similar. It simplifies the computation to use repeatedly that

$$\begin{aligned} &\phi\big(u + a\varepsilon + b\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)\big) \\ &\qquad = \phi(u) + \phi'(u)a\varepsilon + \frac{\varepsilon^2}{2}\big(\phi''(u)a^2 + 2b\phi'(u)\big) + \mathcal{O}\left(\varepsilon^3\right) \end{aligned}$$

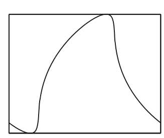
as $\varepsilon \to 0$. Consider an exact classical (smooth) solution u of $u_t + f(u)_x = 0$, and compute (where S_M is the operator defined by the MacCormack scheme)

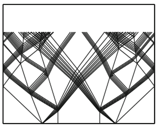
$$\begin{aligned} L_{\Delta t} &= \frac{1}{\Delta t}\big(S(\Delta t)u - S_M(\Delta t)\big) \\ &= \frac{1}{\Delta t}\Big\{u(x, t + \Delta t) - u(x,t) \\ &\qquad + \frac{\lambda}{2}\big[f\big(u(x,t) - \lambda(f(u(x + \Delta x, t)) - f(u(x,t)))\big) \\ &\qquad\qquad - f\big(u(x - \Delta x, t) - \lambda(f(u(x,t)) - f(u(x - \Delta x, t)))\big) \\ &\qquad\qquad + f(u(x,t)) - f(u(x - \Delta x, t))\big]\Big\} \\ &= \frac{1}{\Delta t}\Big\{(u_t + f(u)_x)\Delta t \\ &\qquad + \frac{\lambda^2}{2}\big[u_{tt} - 2f'(u)f''(u)u_x^2 - f'(u)^2u_{xx}\big]\Delta x^2 + \mathcal{O}\left(\Delta x^3\right)\Big\} \\ &= \mathcal{O}\left(\Delta x^2\right), \end{aligned}$$

where we have used that a smooth solution of $u_t + f(u)_x = 0$ satisfies $u_{tt} - 2f'(u)f''(u)u_x^2 - f'(u)^2u_{xx} = 0$ as well.

3.3 **a.** If $u = u_j^n$, $v = u_{j+1}^n$ and $w = u_{j-1}^n$, we have that

$$\begin{aligned} u_j^{n+1} &= g(u, v, w) \\ &= u - \lambda\Big(\int_0^u f'(s) \vee 0\, ds + \int_0^v f'(s) \wedge 0\, ds \\ &\qquad\qquad - \int_0^w f'(s) \vee 0\, ds - \int_0^u f'(s) \wedge 0\, ds\Big) \\ &= u - \lambda\Big(\int_0^u \left|f'(s)\right|\, ds + \int_0^v f'(s) \wedge 0\, ds - \int_0^w f'(s) \vee 0\, ds\Big). \end{aligned}$$





Computing partial derivatives, we find that

$$\begin{aligned}
\frac{\partial g}{\partial u} &= 1 - \lambda\,|f'(u)| \geq 0 \quad \text{if } \lambda\,|f'| \leq 1,\\
\frac{\partial g}{\partial v} &= -\lambda f'(v) \wedge 0 \geq 0,\\
\frac{\partial g}{\partial w} &= \lambda f'(w) \vee 0 \geq 0.
\end{aligned}$$

Consistency is easy to show.

b. If $f' \geq 0$, the scheme coincides with the upwind scheme; hence it is of first order.

c. For any number a we have

$$\left.\begin{aligned} |a| &= a \vee 0 - a \wedge 0,\\ a &= a \vee 0 + a \wedge 0, \end{aligned}\right\} \Rightarrow \begin{cases} a \vee 0 = \dfrac{1}{2}(a + |a|),\\[2ex] a \wedge 0 = \dfrac{1}{2}(a - |a|). \end{cases}$$

Using this the form of the scheme easily follows.

d. We have that

$$\int |u|\; du = \operatorname{sign}(u)\,\frac{u^2}{2}.$$

From this it follows that

$$\begin{aligned}
f^{\mathrm{EO}}(u,v) &= \frac{1}{2}\left(\frac{u^2}{2} + \frac{v^2}{2} - \operatorname{sign}(v)\,\frac{v^2}{2} + \operatorname{sign}(u)\,\frac{u^2}{2}\right)\\
&= \frac{1}{2}\left(\frac{u^2}{2}(1 + \operatorname{sign}(u)) + \frac{v^2}{2}(1 - \operatorname{sign}(v))\right),
\end{aligned}$$

which is what we want to show. If f is convex with a unique minimum at $\bar{u}$, then

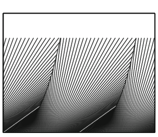
$$f^{\mathrm{EO}}(u,v) = f(u \vee \bar{u}) + f(v \wedge \bar{u}) - f(\bar{u}).$$

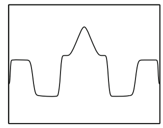
3.4 The scheme is not monotone, since

$$\frac{\partial u_j^{n+1}}{\partial u_{j\pm 1}^n} = \mp \frac{\Delta t}{2\Delta x} f'(u_{j\pm 1}^n).$$

3.5 Assume that waves coming from $x_{j-1/2}$ and $x_{j+1/2}$ at time t_n interact before t_{n+1}, say at $\tilde{t}_n$. Integrating the conservation law over the rectangle $[x_{j-1/2}, x_{j+1/2}] \times [t_n, \tilde{t}_n]$ yields

$$\tilde{u}_j^n = u_j^n - \frac{\tilde{t}_n - t_n}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right),$$





where the Godunov numerical flux $F^n_{j+1/2}$ is given by (3.8), and $\tilde{u}^n_j$ is the average of the solution at time $\tilde{t}_n$. If we now integrate the conservation law over the rectangle $[x_{j-1/2}, x_{j+1/2}] \times [\tilde{t}_n, t_{n+1}]$, we obtain

$$u^{n+1}_j = \tilde{u}^n_j - \frac{t_{n+1} - \tilde{t}_n}{\Delta x}\left(F^n_{j+1/2} - F^n_{j-1/2}\right);$$

the Godunov numerical flux is the same. Adding the two expressions, we get

$$u^{n+1}_j = u^n_j - \lambda\left(F^n_{j+1/2} - F^n_{j-1/2}\right),$$

which is the Godunov method.

3.6 We calculate

$$\begin{aligned}
L_{\Delta t} &= \frac{1}{\Delta t}\Big[u(t+\Delta t) - \frac{1}{2}(u(x+\Delta x) + u(x-\Delta x)) \\
&\qquad + \frac{\Delta t}{2\Delta x}(f(x+\Delta x) - f(x-\Delta x))\Big] \\
&= \frac{1}{\Delta t}\left[u + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} - u - \frac{\Delta x^2}{2}u_{xx} + \Delta t f_x\right] + \mathcal{O}(\Delta t^2) \\
&= \frac{\Delta t}{2}\left(\left(f'(u)^2 u_x\right)_x - \frac{1}{\lambda^2}u_{xx}\right) + \mathcal{O}(\Delta t^2) \\
&= \frac{\Delta t}{2\lambda^2}\left[\left(\left(\lambda f'(u)\right)^2 - 1\right)u_x\right]_x + \mathcal{O}(\Delta t^2).
\end{aligned}$$

3.7 Consider the function

$$G(a,b) = a + b - \lambda(f(a) - f(b)).$$

We have ϵ positive,

$$\begin{aligned}
G(a+\epsilon, b) - G(a,b) &= \epsilon - \lambda(f(a+\epsilon) - f(a)) \\
&\geq \epsilon - \lambda\,|f(a+\epsilon) - f(a)| \\
&\geq \epsilon(1 - \lambda L) > 0
\end{aligned}$$

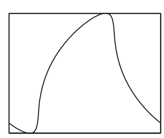
for $\lambda L < 1$, and where L is the Lipschitz constant of f. Similar calculations for the b variable.

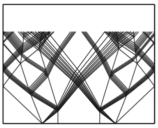
3.8 For the case $f(u) = u$, we find that Heun's method gives

$$u^{n+1}_j = u^n_j - \frac{\lambda}{2}\left(u^n_{j+1} - u^n_{j-1}\right) - \frac{\lambda^2}{4}\left(u^n_{j+1} - 2u^n_j + u^n_{j-2}\right).$$

Using the ansatz $u^n_j = \mu_n e^{ix_j}$. we obtain

$$\begin{aligned}
\mu_{n+1} &= \mu_n\left(1 - \frac{\lambda}{2}\left(e^{i\Delta x} - e^{-i\Delta x}\right) - \left(\frac{\lambda}{2}\left(e^{i\Delta x} - e^{-i\Delta x}\right)\right)^2\right) \\
&= \mu_n\left(1 - i\lambda\sin(\Delta x) + \lambda^2\sin^2(\Delta x)\right).
\end{aligned}$$





Hence

$$|\mu_{n+1}| = |\mu_n| \sqrt{(1+\lambda^2 \sin^2(\Delta x))^2 + \lambda^2 \sin^2(\Delta x)} > |\mu_n|,$$

for all $\lambda > 0$ and $\Delta x > 0$, and the method is unconditionally unstable.

3.9 **a.** We find that

$$u_j^{n+1} = u_j^n / \big(1 + \lambda(u_j^n - u_{j-1}^n)\big),$$

provided that the denominator is nonzero. Thus

$$\frac{\partial u_j^{n+1}}{\partial u_j^n} = \lambda u_j^n / \big(1 + \lambda(u_j^n - u_{j-1}^n)\big)^2,$$
$$\frac{\partial u_j^{n+1}}{\partial u_{j-1}^n} = \big(1 - \lambda u_j^n\big) / \big(1 + \lambda(u_j^n - u_{j-1}^n)\big)^2.$$

Assume $\lambda < 1$. Considering u_j^{n+1} as a function of $u_j^n, u_{j-1}^n \in [0,1]$, we see that u_j^{n+1} takes on its largest value, namely one, when $u_j^n = u_{j-1}^n = 1$. Thus $u_j^{n+1} \in [0,1]$ for all n and j. The same computation shows that the scheme is monotone.

b. A constant is mapped into the same constant by this scheme, which therefore is consistent. A Taylor expansion around a smooth solution shows that the truncation error is of first order.

3.10 **a.** With the obvious notation we have that

$$v_j^{n+1} = v_j^n - \frac{\lambda}{\Delta x}\left(f_j^n - f_{j-1}^n + f_{j-2}^n - f_{j-1}^n\right),$$

and by a Taylor expansion about U_{j-1}^n,

$$f_j^n = f_{j-1}^n + (u_j^n - u_{j-1}^n) f'(u_{j-1}^n) + \frac{1}{2}(u_j^n - u_{j-1}^n)^2 f''(\eta_{j-1/2}),$$
$$f_{j-2}^n = f_{j-1}^n + (u_{j-2}^n - u_{j-1}^n) f'(u_{j-1}^n) + \frac{1}{2}(u_{j-2}^n - u_{j-1}^n)^2 f''(\eta_{j-3/2}).$$

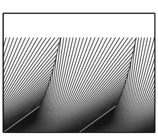
Using this, we get the desired result.

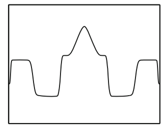
b. Assuming that $v_j^n \geq v_{j-1}^n \geq 0$, we find that

$$v_j^{n+1} \leq v_j^n - c\Delta t \left(v_j^n\right)^2 = g(v_j^n).$$

The function g has a maximum at $1/(2c\Delta t)$. Hence v_j^n is in an interval where g is increasing. Thus

$$v_j^{n+1} \leq g\left(\frac{1}{(2+n)c\Delta t}\right) = \frac{1}{(n+2)c\Delta t}\,\frac{n+1}{n+2} < \frac{1}{(n+3)c\Delta t}.$$





The case $v^n_{j-1} > v^n_j$ is similar, and the case $0 \geq v^n_j \vee v^n_{j-1}$ is trivial. Thus we have completed the induction. Hence for all n, $\hat{v}^n$ will be in an interval where g is increasing, and

$$v^{n+1}_j \leq g\left(\hat{v}^n\right).$$

Taking the maximum over j and 0 on the left completes the claim.

c. Assuming that the claim holds for $n = 0$, we wish to show that it holds for every n by induction. Since $\hat{v}^n$ is in an interval where g is increasing, we find that

$$\begin{aligned}\hat{v}^{n+1}_j &\leq \frac{\hat{v}_0}{1 + cn\Delta t\hat{v}_0}\left(1 - \frac{\hat{v}_0}{1 + cn\Delta t\hat{v}_0}\right)\\ &= \frac{\hat{v}_0}{1 + cn\Delta t\hat{v}_0}\,\frac{1 + c\Delta t\hat{v}_0(n-1)}{1 + c\Delta t\hat{v}_0 n},\end{aligned}$$

so if

$$\frac{1 + \hat{v}_0 c\Delta t(n-1)}{(1 + \hat{v}_0 c\Delta t n)^2} \leq \frac{1}{1 + \hat{v}_0 c\Delta t(n+1)},$$

we are ok. Set $k = \hat{v}_0 c\Delta t$. Since

$$(1 + kn)^2 - k^2 < (1 + kn)^2,$$

the claim follows.

d. Since $v^n_j \leq \hat{v}^n$, the claim follows by noting that

$$u^n_i - u^n_j = \sum_{k=j+1}^{i} v^n_j.$$

e. Since $\left\{u^n_j\right\}$ converges to the entropy solution u for almost every x (and y) and t, we find that the claim holds.

3.11 **a.** We find that $u(\cdot + p, t)$ is another entropy solution with the same initial condition; hence $u(\cdot + p, \cdot) = u$, and u is periodic.

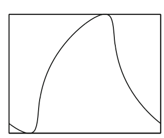
b. Taking the infimum over y and the supremum over x, we find that this holds.

c. Set $u_\varepsilon = u * \omega_\varepsilon$. Then u_ε is differentiable, and satisfies

$$\partial_t u_\varepsilon + \partial_x(f(u) * \omega_\varepsilon) = 0.$$

Thus

$$\frac{d}{dt}\int_0^p u_\varepsilon(x,t)\,dx = (f(u) * \omega_\varepsilon)(0,t) - (f(u) * \omega_\varepsilon)(p,t) = 0,$$



since also $f * \omega_\varepsilon$ is periodic with period p. Therefore,

$$\int_0^p u_\varepsilon(x,t)\,dx = \int_0^p u_{0,\varepsilon}(x)\,dx.$$

We know that u_ε converges to u in $L^1([0,p])$; hence

$$\int_0^p u(x,t)\,dx = \int_0^p u_0(x)\,dx.$$

Now, since $u(x,t) \to \bar{u}$ as t becomes large,

$$p\bar{u} = \int_0^p u_0(x)\,dx.$$

3.12 The limit of $\chi(\lambda, u_n(x))$ will be independent of x, we have that

$$\chi(\lambda, u_n(x)) = \begin{cases} 1 & 0 \le \lambda \le 1, \text{ and } x \in [2k/2n, (2k+1)/2n), \\ -1 & -1 \le \lambda < 0, \text{ and } x \in [(2k+1)/2n, (2k+2)/2n). \end{cases}$$

Therefore

$$\chi(\lambda, u_n(x)) \rightharpoonup f(\lambda) = \begin{cases} \frac{1}{2} & 0 \le \lambda \le 1, \\ -\frac{1}{2} & -1 \le \lambda < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\frac{d}{d\lambda} f(\lambda) = \delta_0(\lambda) - \frac{1}{2}\left(\delta_{-1}(\lambda) + \delta_1(\lambda)\right).$$

Thus $\nu = (\delta_{-1} + \delta_1)/2$.

3.13 **a.** Observe first that v_k is constant and equal to one on the interval $[1,2]$. From the definition of F we see that

$$|F(\phi) - \phi| \le \frac{1}{3}\,|\phi(b) - \phi(a)|.$$

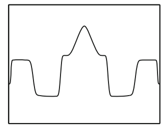
Thus

$$|v_{k+1} - v_k| = \sum_j \left|F(v_{j,k}) - v_{j,k}\right| \chi_{j,k} \le \left(\frac{2}{3}\right)^k,$$

and hence the limit

$$v(x) = \lim_{k\to\infty} v_k(x)$$

exists and is continuous.



b. Observe that $\text{T.V.}\left(v_{j,k}\right) = \left(\frac{2}{3}\right)^k$ (on $[0,1]$ and $[2,3]$), and that

$$\text{T.V.}\left(F(\phi)\right) = \frac{5}{3}\left|\phi(b) - \phi(a)\right| = \frac{5}{3}\,\text{T.V.}\left(\phi\right).$$

Thus

$$\begin{aligned}\text{T.V.}\left(v_{k+1}\right) &= \sum_j \text{T.V.}\left(F(v_{j,k})\right) = \frac{5}{3}\sum_j \text{T.V.}\left(v_{j,k}\right)\\ &= \frac{5}{3}\left(\frac{2}{3}\right)^k \cdot 2 \cdot 3^k = \frac{10}{3}\,2^k.\end{aligned} \tag{C.7}$$

c. We see that

$$v(j/3^k) = v_k(j/3^k)$$

by construction.

d. Define the upwind scheme by

$$u_j^{n+1} = u_j^n + \lambda(f_j^n - f_{j-1}^n), \quad u_j^0 = v(j/3^k) = v_k(j/3^k). \tag{C.8}$$

From the assumptions on the flux function we know that the scheme is TVD with a CFL number at most one. Thus

$$\text{T.V.}\left(u^n\right) \le \text{T.V.}\left(u^0\right) \le \text{T.V.}\left(v_k\right) = \frac{10}{3}\,2^k.$$

We apply Theorem 3.32, and consider

$$(\Delta x)^\beta \sum_n \sum_j \left|u_{j+1}^n - u_j^n\right| \Delta t \le T\,3^{-k\beta}\,\frac{10}{3}\,2^k.$$

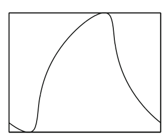
For this to be less than a constant $C(T)$, we need $2/3^\beta \le 1$, or $\beta \ge \ln 2/\ln 3 \approx 0.63$. For Theorem 3.32 to apply we note that (3.110) is satisfied with right-hand side zero.

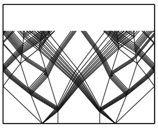
Chapter 4, Sect. 4.7

4.1 **a.** Set

$$\alpha_\varepsilon(t) = \int_0^t \left(\omega_\varepsilon(s - t_1) - \omega_\varepsilon(s - t_2)\right) ds.$$

Then α_ε will tend to the characteristic function of the interval $[t_1, t_2]$ as $\varepsilon \to 0$. Furthermore, set $\psi(x, y, t) = \alpha_\varepsilon(t)\varphi(x, y)$ for some test function





φ with $|\varphi(x,y)| \le 1$. Since u is a weak solution, we find, using ψ as a test function and taking $\varepsilon \to 0$, that

$$\iint \varphi(x,y)(u(x,y,t_1) - u(x,y,t_2))\,dx\,dy + \int_{t_1}^{t_2} \iint \left(f(u)\varphi_x + g(u)\varphi_y\right) dx\,dy\,dt = 0.$$

Then we have that

$$\begin{aligned}
&\|u(\cdot,\cdot,t_1) - u(\cdot,\cdot,t_2)\|_{L^1(\mathbb{R}^2)} \\
&\quad = \sup_{|\varphi|\le 1} \iint \varphi(x,y)(u(x,y,t_1) - u(x,y,t_2))\,dx\,dy \\
&\quad \le \int_{t_1}^{t_2} \sup_{|\varphi|\le 1} \iint \left(f(u)\varphi_x + g(u)\varphi_y\right) dx\,dy\,dt \\
&\quad \le \int_{t_1}^{t_2} \left(\int \mathrm{T.V.}_{\cdot x}\left(f(u(\cdot,y,t))\right)\,dy + \int \mathrm{T.V.}_{\cdot y}\left(g(u(x,\cdot,t))\right)\,dx \right) dt \\
&\quad \le |t_1 - t_2| \left(\|f\|_{\mathrm{Lip}} \vee \|g\|_{\mathrm{Lip}}\right) \mathrm{T.V.}\,(u_0)\,.
\end{aligned}$$

See also Theorem 7.10.

b. Let $u_{\Delta t}$ and $v_{\Delta t}$ denote the dimensional splitting approximations to u and v, respectively. It is easy to show (using monotonicity for the one-dimensional solution operators) that if $u_0 \le v_0$ a.e., then $u_{\Delta t} \le v_{\Delta t}$ a.e. Hence

$$\iint \left[u_{\Delta t}(x,y,t) - v_{\Delta t}(x,y,t)\right] \vee 0\,dx\,dy = 0,$$

and since both $u_{\Delta t}$ and $v_{\Delta t}$ converge strongly in L^1 to u and v, respectively, it follows that

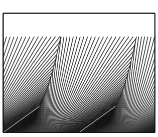
$$\iint \left[u(x,y,t) - v(x,y,t)\right] \vee 0\,dx\,dy = 0,$$

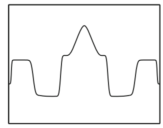
and thus $u \le v$ a.e.

4.2 Let $u = S_t^j u_0$ denote the solution of $u_t + f_j(u)_x = 0$ with initial condition $u|_{t=0} = u_0$. Define $\{u^n\}$ by

$$u^0 = u_0, \quad u^{n+1/2} = S^1_{\Delta t} u^n, \quad u^{n+1} = S^2_{\Delta t} u^{n+1/2}.$$

Interpolate by defining $u_{\Delta t} = S^1_{2(t-t_n)} u^n$ if $t_n \le t \le t_{n+1/2}$ and $u_{\Delta t} = S^1_{2(t-t_{n+1/2})} u^{n+1/2}$ whenever $t_{n+1/2} \le t \le t_{n+1}$. (Here $t_n = n\Delta t$.) By mimicking the multidimensional case, one concludes that (i) $\|u_{\Delta t}\|_{L^\infty} \le C$; (ii)





T.V. $(u_{\Delta t}(t)) \le$ T.V. (u_0); and (iii) $\|u_{\Delta t}(t) - u_{\Delta t}(s)\|_{L^1} \le C\,|t-s|$. Theorem A.11 shows that $u_{\Delta t}$ has a limit u as $\Delta t \to 0$. Write the Kružkov entropy condition for $u_{\Delta t}$ for each time interval $[t_n, t_{n+1/2}]$ (for f_1) and $[t_n, t_{n+1/2}]$ (for f_2), add the results, and let $Dt \to 0$. As in the multidimensional case, the limit is the Kružkov entropy condition for u and the original initial value problem (4.99). The analysis in Sect. 4.3 applies concerning convergence rates.

4.3 Consider first a smooth function ψ. Let

$$I = I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2.$$

For $x_j, y_j \in I_j$ we have

$$\begin{aligned}\psi(x_1, x_2) - \psi(y_1, y_2) &= \psi(x_1, x_2) - \psi(x_1, y_2) + \psi(x_1, y_2) - \psi(y_1, y_2)\\ &= \int_{y_1}^{x_1} \frac{\partial \psi}{\partial x}(\xi, y_2)\,d\xi + \int_{y_2}^{x_2} \frac{\partial \psi}{\partial y}(x_1, \xi)\,d\xi.\end{aligned}$$

By integrating over I we obtain

$$\begin{aligned}&\left|\psi(x_1, x_2) - \frac{1}{|I|}\iint_I \psi(y_1, y_2)\,dy_1\,dy_2\right|\\ &\qquad\le \frac{1}{|I_2|}\iint_I \left|\frac{\partial \psi}{\partial x}(\xi, y_2)\right| d\xi\,dy_2 + \int_{a_2}^{b_2} \left|\frac{\partial \psi}{\partial y}(x_1, \xi)\right| d\xi.\end{aligned}$$

Thus

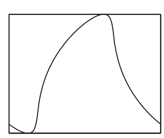
$$\begin{aligned}&\iint_I \left|\psi(x_1, x_2) - \frac{1}{|I|}\iint_I \psi(y_1, y_2)\,dy_1\,dy_2\right| dx_1\,dx_2\\ &\qquad\le |I_1| \iint_I \left|\frac{\partial \psi}{\partial x}(x, y)\right| dx\,dy + |I_2| \iint_I \left|\frac{\partial \psi}{\partial y}(x, y)\right| dx\,dy\\ &\qquad\le \max\{|I_1|, |I_2|\} \iint_I |\nabla \psi(x, y)|\,dx\,dy.\end{aligned}$$

We can approximate every function ψ of bounded variation with smooth functions ψ_k such that (see [64, Thm. 2, p. 172])

$$\|\psi - \psi_k\|_{L^1(I)} \to 0, \quad \|\nabla \psi_k\|_{L^1(I)} \to \|\nabla \psi\|_{L^1(I)}.$$

This implies that

$$\iint_{I_{ij}} |\psi_k - \pi\psi_k|\,dx\,dy \le \max\{\Delta x, \Delta y\} \iint_{I_{ij}} |\nabla \psi_k|\,dx\,dy.$$



By taking limits we have

$$\iint\limits_{I_{ij}} |\psi - \pi\psi| \, dx\, dy \le \max\{\Delta x, \Delta y\} \iint\limits_{I_{ij}} |\nabla\psi| \, dx\, dy,$$

and subsequently

$$\begin{aligned}\iint |\psi - \pi\psi| \, dx\, dy &= \sum_{i,j} \iint\limits_{I_{ij}} |\psi - \pi\psi| \, dx\, dy \\ &\le \max\{\Delta x, \Delta y\} \sum_{i,j} \iint\limits_{I_{ij}} |\nabla\psi| \, dx\, dy \\ &= \max\{\Delta x, \Delta y\} \iint |\nabla\psi| \, dx\, dy.\end{aligned}$$

4.4 For simplicity, we assume that $m = 2$. Using the heat kernel, we can write

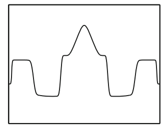
$$\begin{aligned}u^{n+1/2}(x,y) &= \frac{1}{\sqrt{4\pi\,\Delta t}} \int \exp\Big(-\frac{(x-z)^2}{4\Delta t}\Big) u^n(z,y)\, dz \\ u^{n+1}(x,y) &= \frac{1}{\sqrt{4\pi\,\Delta t}} \int \exp\Big(-\frac{(y-w)^2}{4\Delta t}\Big) u^{n+1/2}(x,w)\, dw \\ &= \frac{1}{4\pi\,\Delta t} \iint \exp\Big(-\frac{(x-z)^2 + (y-w)^2}{4\Delta t}\Big) u^n(z,w)\, dz\, dw.\end{aligned}$$

From this we see that $u^{n+1}(x,y)$ is the exact solution of the heat equation with initial data $u^n(x,y)$ after a time Δt. If we let $u(x,y,t)$ denote the exact solution of the original heat equation, we therefore see that

$$u(x,y,t_n) = u_{\Delta t}(x,y,t_n), \quad n = 0,1,2,\dots.$$

We have that u and $u_{\Delta t}$ are L^1 continuous in t; hence it follows that $u_{\Delta t} \to u$ in L^1 as $\Delta t \to 0$.
If we want a rate of this convergence, we first assume that $u(\,\cdot\,,\,\cdot\,,t_n)$ is uniformly continuous. For $t \in (t_n, t_{n+1/2})$ we have that

$$\begin{aligned}&u_{\Delta t}(x,y,t) - u(x,y,t) \\ &\quad= \frac{1}{4\pi(t-t_n)} \iint \exp\Big(-\frac{(x-z)^2 + (y-w)^2}{4(t-t_n)}\Big) \\ &\qquad\qquad \times \big[u(z,y,t_n) - u(z,w,t_n)\big]\, dw\, dz \\ &\quad= \frac{1}{\sqrt{4\pi(t-t_n)}} \int \exp\Big(-\frac{(x-z)^2}{4(t-t_n)}\Big) \\ &\qquad \times \int \frac{1}{\sqrt{4\pi(t-t_n)}} \exp\Big(-\frac{(y-w)^2}{4(t-t_n)}\Big) \\ &\qquad\qquad \times \big[u(z,y,t_n) - u(z,w,t_n)\big]\, dw\, dz \\ &\quad= \frac{1}{\sqrt{4\pi(t-t_n)}} \int \exp\Big(-\frac{(x-z)^2}{4(t-t_n)}\Big) \big[\eta(z,y,t) - \eta(z,y,t_n)\big]\, dz,\end{aligned}$$



where $\eta(z, w, t)$ denotes the solution of

$$\eta_t = \eta_{ww}, \quad \eta(z, w, t_n) = u^n(z, w).$$

If $u^n(z, w)$ is uniformly continuous, then

$$|\eta(z, y, t) - \eta(z, y, t_n)| \leq C\sqrt{\Delta t},$$

and hence

$$|u_{\Delta t}(x, y, t) - u(x, y, t)| \leq C\sqrt{\Delta t},$$

and an identical estimate is available if $t \in (t_{n+1/2}, t_{n+1})$. Hence if $u_0(x, y)$ is uniformly continuous, then

$$\|u_{\Delta t}(\cdot, \cdot, t) - u(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq C\sqrt{\Delta t}.$$

If $u_0(x, y)$ is not assumed to be continuous, but merely of bounded variation, we must use Kružkov's interpolation lemma to conclude that

$$\|\eta(z, \cdot, t) - \eta(z, \cdot, t_n)\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \leq C\sqrt{\Delta t}.$$

Using this, we find that

$$\|u_{\Delta t}(\cdot, \cdot, t) - u(\cdot, \cdot, t)\|_{L^1_{\mathrm{loc}}(\mathbb{R}^2)} \leq C\sqrt{\Delta t}.$$

4.5 **a.** The scheme will be monotone if $U_j^{n+1/2}\,(U^n)$ and $U_j^n\,\left(U^{n+1/2}\right)$ are monotone in all arguments. This is the case if

$$\lambda \|f'\|_1 \leq 1 \quad \text{and} \quad \mu \leq \frac{1}{2}.$$

b. We see that if $\lambda\,|f'| \leq 1$, then

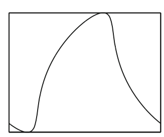
$$\min_j U_j^n \leq U_j^{n+1/2} \leq \max_j U_j^n,$$

and if $\mu \leq \frac{1}{2}$,

$$\min_j U_j^{n+1/2} \leq U_j^{n+1} \leq \max_j U_j^{n+1/2}.$$

Therefore, the sequence $\{u_{\Delta t}\}$ is uniformly bounded. Now let V_j^n be another solution with initial data V_j^0. Set $W_j^n = U_j^n - V_j^n$. Then

$$W_j^{n+1/2} = \frac{1}{2}\left(1 - f'\left(\eta_{j+1}\right)\right) W_{j+1}^n + \frac{1}{2}\left(1 + f'\left(\eta_{j-1}\right)\right) W_{j-1}^n,$$



where η_j is between U_j^n and V_j^n. By the CFL condition, the coefficients of $W_{j\pm 1}^n$ are positive; hence

$$\left|W_j^{n+1/2}\right| \le \frac{1}{2}\left(1 - f'\left(\eta_{j+1}\right)\right)\left|W_{j+1}^n\right| + \frac{1}{2}\left(1 + f'\left(\eta_{j-1}\right)\right)\left|W_{j-1}^n\right|.$$

Summing over j, we find that

$$\sum_j \left|W_j^{n+1/2}\right| \le \sum_j \left|W_j^n\right|.$$

Similarly, we find that

$$\left|W_j^{n+1}\right| \le (1-2\mu)\left|W_j^{n+1/2}\right| + \mu\left|W_{j+1}^{n+1/2}\right| + \mu\left|W_{j-1}^{n+1/2}\right|,$$

so that

$$\sum_j \left|W_j^{n+1}\right| \le \sum_j \left|W_j^{n+1/2}\right| \le \sum_j \left|W_j^n\right|.$$

Setting $V_j^n = U_{j-1}^n$, we see that T.V. $(u_{\Delta t})$ is uniformly bounded, and setting $V_j^0 = 0$, we see that $\|u_{\Delta t}\|_1$ is also uniformly bounded. To apply Theorem A.11 we need to use Kružkov's interpolation lemma, Lemma 4.11, to find a temporal modulus of continuity. Now we find that

$$\begin{aligned}
&\left|\int \phi(x)\big(u_{\Delta t}(x, n\Delta t) - u_{\Delta t}(x, m\Delta t)\big)\,dx\right| \\
&\qquad = \left|\sum_{k=n+1}^{m}\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x)\,dx\left(U_j^{k+1} - U_j^k\right)\right|.
\end{aligned}$$

If we set $\bar{\phi}_j = \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x)\,dx$ and $D_j^k = U_j^k - U_{j-1}^k$, then the sum over j can be written

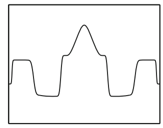
$$\begin{aligned}
&\sum_j \bar{\phi}_j\,\mu\left(D_{j+1}^{k+1/2} - D_j^{k+1/2}\right) + \sum_j \bar{\phi}_j\,\frac{1}{2}\left(D_{j+1}^k - D_j^k\right) \\
&\qquad + \sum_j \bar{\phi}_j\,\lambda\left(f_{j+1}^n - f_{j-1}^n\right).
\end{aligned} \tag{C.9}$$

We do a partial summation in the first sum to find that it equals

$$-\mu \sum_j D_j^{k+1/2}\left(\bar{\phi}_j - \bar{\phi}_{j-1}\right).$$

Now

$$\left|\bar{\phi}_j - \bar{\phi}_{j-1}\right| = \left|\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{x-\Delta x}^{x} \phi'(y)\,dy\,dx\right| \le \|\phi'\|_{L^1}\,\Delta x^2.$$



Since $\mu = \Delta t/\Delta x^2$, the first sum in (C.9) is bounded by

$$\Delta t \|\phi'\|_{L^1} \mathrm{T.V.}\,(u_{\Delta t})\,.$$

Similarly, the second term is bounded by

$$\Delta t \frac{\|\phi'\|_{L^1}}{2} \frac{\Delta x}{\lambda} \mathrm{T.V.}\,(u_{\Delta t})\,.$$

Finally, since $\lambda = \Delta t/\Delta x$, the last term in (C.9) is bounded by

$$\Delta t \|\phi\|_{L^1} \mathrm{T.V.}\,(u_{\Delta t})\,.$$

Therefore,

$$\left| \int \phi(x) \left(u_{\Delta t}(x, n\Delta t) - u_{\Delta t}(x, m\Delta t)\right)\, dx \right| \\ \leq (n-m)\Delta t \text{ const } \left(\|\phi\|_{L^1} + \|\phi'\|_{L^1}\right).$$

Then by Lemma 4.11 and the bound on the total variation,

$$\|u_{\Delta t}(\,\cdot\,, t_1) - u_{\Delta t}(\,\cdot\,, t_2)\|_{L^1} \leq \text{const } \sqrt{|t_1 - t_2|}.$$

Hence we can conclude, using Theorem A.11, that a subsequence of $\{u_{\Delta t}\}$ converges strongly in L^1.
To show that the limit is a weak solution, we have to do a long and tortuous calculation. We give only an outline here. Firstly,

$$\iint \left(u_{\Delta t}\varphi_t + f(u_{\Delta t})\varphi_x + u_{\Delta t}\varphi_{xx}\right) dx\, dt \\ = \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \left(U_j^n \varphi_t + f_j^n \varphi_x + U_j^n \varphi_{xx}\right) dx\, dt.$$

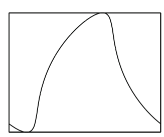
After a couple of partial summations this equals

$$-\sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi^{n+1}(x)\, dx \left[U_j^{n+1} - U_j^n\right] \tag{C.10}$$

$$-\sum_{n,j} \int_{t_n}^{t_{n+1}} \varphi_{j+1/2}(t)\, dt \left[f_j^n - f_{j-1}^n\right] \tag{C.11}$$

$$-\sum_{n,j} \int_{t_n}^{t_{n+1}} \varphi'_{j+1/2}(t)\, dt D_j^{n+1/2} \tag{C.12}$$

$$-\sum_{n,j} \int_{t_n}^{t_{n+1}} \left(\varphi'_{j+1/2}(t) - \varphi'_{j-1/2}(t)\right)\, dt \left[U_j^{n+1/2} - U_j^n\right]. \tag{C.13}$$



Now split (C.10) by writing the term in the square brackets as

$$\left[\mu\left(D_j^{n+1/2} - D_{j-1}^{n+1/2}\right) + \frac{1}{2}\left(D_j^n - D_{j-1}^n\right) - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right)\right]. \tag{C.14}$$

The trick is now to pair the term in (C.14) with μ with (C.12), and the term with λ with (C.11). The limits of the sums of these are zero, as is the limit of (C.13), and the remaining part of (C.14) also tends to zero as Δt approaches zero. If you carry out all of this, you will find that the limit is a weak solution.

4.6 **a.** To verify the left-hand side of the inequality, we proceed as before. When doing this, we have a right-hand side given by

$$\iint\iint \operatorname{sign}(u - v)\,(g(u) - g(v))\psi\omega_{\varepsilon_0}\omega_{\varepsilon_1}\,dx\,dt\,dy\,ds.$$

Now we can send ε_0 and ε_1 to zero and obtain the answer.

b. By choosing the test function as before, we find that the double integral on the left-hand side is less than zero. Therefore, we find that

$$h(0) - h(T) \geq \iint \operatorname{sign}(u - v)\,(g(u) - g(v))\psi(x,t)\,dxdt,$$

and therefore

$$\begin{aligned} h(T) &\leq h(0) + \int_0^T \int |g(u) - g(v)|\,\psi\,dx\,dt \\ &\leq h(0) + \gamma \int_0^T \int |u - v|\,\psi\,dx\,dt. \end{aligned}$$

c. By making M arbitrarily large we obtain the desired inequality.

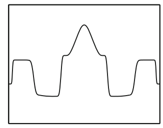
4.7 **a.** We set

$$u_{\Delta t}(x,0) = u_0(x),$$

$$u_{\Delta t}(x,t) = \begin{cases} S\,(2\,(t - t_n))\,u_n(x), & t \in (t_n, t_{n+1/2}], \\ R(x, 2(t - t_{n+1/2}) + t_n, t_{n+1/2})u_{n+1/2}(x), & t \in (t_{n+1/2}, t_n]. \end{cases}$$

b. A bound on $\|u_{\Delta t}\|_\infty$ is obtained as before. To obtain a bound on the total variation we first note that

$$\text{T.V.}\left(u^{n+1/2}\right) \leq \text{T.V.}\left(u^n\right).$$



Let $u(x,t)$ and $v(y,t)$ be two solutions of the ordinary differential equation in the interval $[t_n, t_{n+1}]$. Then as before, we find that

$$\begin{aligned}
|u(x,t) - v(x,t)|_t \leq\ & |g(x,t,u(x,t)) - g(x,t,v(y,t))| \\
& + |g(x,t,v(y,t)) - g(y,t,v(y,t))| \\
\leq\ & \gamma\, |u(x,t) - v(y,t)| \\
& + |g(x,t,v(y,t)) - g(y,t,v(y,t))| \,.
\end{aligned}$$

Setting $w = u - v$, we obtain

$$|w(t)| \leq e^{\gamma(t-t_{n+1})}\left(|w(0)| + \int\limits_{t_n}^{t_{n+1}} |g(x,t,v(y,t)) - g(y,t,v(y,t))|\ dt\right).$$

If now $u(0) = u^{n+1/2}(x)$ and $v(0) = u^{n+1/2}(y)$, this reads

$$\begin{aligned}
&\left|u^{n+1}(x) - u^{n+1}(y)\right| \\
&\quad \leq e^{\gamma\Delta t}\Bigg(\left|u^{n+1/2}(x) - u^{n+1/2}(y)\right| \\
&\qquad + \int\limits_{t_n}^{t_{n+1}} \left| g\left(x,t,u_{\Delta t}\left(y, \frac{t-t_n}{2} + t_{n+1/2}\right)\right) \right. \\
&\qquad\qquad \left. - g\left(y,t,u_{\Delta t}\left(y, \frac{t-t_n}{2} + t_{n+1/2}\right)\right)\right| dt\Bigg).
\end{aligned}$$

By the assumption on g this implies that

$$\text{T.V.}\left(u^{n+1}\right) \leq e^{\gamma\Delta t}\left(\text{T.V.}\left(u^{n+1/2}\right) + \int\limits_{t_n}^{t_{n+1}} b(t)\,dt\right),$$

and thus

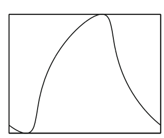
$$\text{T.V.}\left(u_{\Delta t}\right) \leq e^{\gamma T}\left(\text{T.V.}\left(u_0\right) + \|b\|_1\right).$$

We can now proceed as before to see that

$$\int\limits_I \left|u^{n+1}(x) - u^n(x)\right|\, dx \leq \text{const}\ \Delta t.$$

Hence we conclude, using Theorem A.11, that a subsequence of $\{u_{\Delta t}\}$ converges strongly in L^1 to a function u of bounded variation.

c. To show that u is an entropy solution, we can use the same argument as before.



4.8 Write $v = u_t$ and differentiate the heat equation $u_t = \varepsilon \Delta u$ with respect to t. Thus

$$v_t = \varepsilon \Delta v.$$

Let sign_η be a smooth sign function, namely,

$$\operatorname{sign}_\eta(x) = \begin{cases} 1 & \text{for } x \geq \eta, \\ x/\eta & \text{for } |x| < \eta, \\ -1 & \text{for } x \leq -\eta. \end{cases}$$

If we multiply

$$v_t = \varepsilon \Delta v$$

by $\operatorname{sign}_\eta(v)$ and integrate over $\mathbb{R}^m \times [0, t]$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^m} \int_0^t v_t \operatorname{sign}_\eta(v)\, dt\, dx &= \varepsilon \int_{\mathbb{R}^m} \int_0^t \Delta v \operatorname{sign}_\eta(v)\, dt\, dx \\ &= -\varepsilon \int_{\mathbb{R}^m} \int_0^t |\nabla v|^2 \operatorname{sign}'_\eta(v)\, dt\, dx \\ &\leq 0. \end{aligned}$$

For the left-hand side we obtain

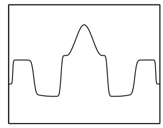
$$\begin{aligned} \int_{\mathbb{R}^m} \int_0^t v_t \operatorname{sign}_\eta(v)\, dt\, dx = &\int_{\mathbb{R}^m} \int_0^t (v \operatorname{sign}_\eta(v))_t\, dt\, dx \\ &- \int_{\mathbb{R}^m} \int_0^t v\, v_t \operatorname{sign}'_\eta(v)\, dt\, dx. \end{aligned}$$

The last term vanishes in the limit $\eta \to 0$; see Lemma B.5. Thus we conclude that as $\eta \to 0$,

$$\|v(t)\|_{L^1} - \|v(0)\|_{L^1} \leq 0.$$

From this we obtain

$$\|u(t) - u_0\|_{L^1} = \int_{\mathbb{R}^m} \left| \int_0^t v(\tilde{t})\, d\tilde{t} \right| dx \leq \int_0^t \|v(\tilde{t})\|_{L^1}\, d\tilde{t} \leq \|v(0)\|_1\, t.$$



Chapter 5, Sect. 5.8

5.1 Using the derivation of the shallow-water equations based on the Navier–Stokes equations, we obtain as before

$$\begin{aligned} v_t + vv_x + uv_y &= -p_x, \\ p_y &= -1. \end{aligned}$$

When we integrate the pressure in the vertical direction, we get

$$p = \delta b(x,t) + h(x,t) - y \tag{C.15}$$

when we as before normalize the pressure to vanish at the surface of the water. By inserting this in the first equation, we obtain

$$v_t + vv_x + (\delta b + h)_x = 0.$$

Apply Green's theorem to a region R of the fluid confined between $x = x_1$ and $x = x_2$:

$$\begin{aligned} 0 &= \iint_R \left(v_x + u_y\right) dx\, dy = \int_{\partial R} (-u\, dx + v\, dy) \\ &= \int_{x_1}^{x_2} \left(((\delta b + h)_x v + (\delta b + h)_t)\, dx - v(\delta b + h)_x\, dx\right) \\ &\quad - \int_{x_1}^{x_2} \left((\delta b_x v + \delta b_t)\, dx - v\delta b_x\, dx\right) \\ &\quad + v(x_2,t)h(x_2,t) - v(x_1,t)h(x_1,t) \\ &= \int_{x_1}^{x_2} \left(h_t + (vh)_x\right) dx, \end{aligned} \tag{C.16}$$

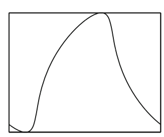
or $h_t + (vh)_x = 0$, where we used that $v\, dy = v(\delta b + h)_x\, dx$ along the curve $y = (\delta b + h)(x,t)$ and $v\, dy = v\delta b_x\, dx$ along the curve $y = \delta b(x,t)$.
By combining these equations and rewriting in conservative form, we conclude that

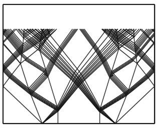
$$\begin{aligned} h_t + (vh)_x &= 0, \\ (vh)_t + (v^2 h + \frac{1}{2}h^2 + \delta hb)_x &= 0. \end{aligned} \tag{C.17}$$

5.2 The eigenvalues are $\lambda = \pm\sqrt{-p'(v)}$. Hence we need that p' is negative.

5.3 The shock curves are given by

$$\begin{aligned} S_1\colon u &= u_l + (v - v_l)/\sqrt{vv_l}, \quad v \le v_l, \\ S_2\colon u &= u_l - (v - v_l)/\sqrt{vv_l}, \quad v \ge v_l, \end{aligned}$$





and the rarefaction curves read

$$\begin{aligned} R_1\colon u &= u_l + \ln(v/v_l), \quad v \le v_l, \\ R_2\colon u &= u_l - \ln(v/v_l), \quad v \ge v_l. \end{aligned}$$

We see that for $v_l > 0$ the wave curves extend to infinity. Given a right state (v_r, u_r) and a left state (v_l, u_l), we see that the forward slow curves (i.e., the S_1 and the R_1 curves above) intersect the backward fast curves from (v_r, u_r) (i.e., the curves of left states that can be connected to the given right state (v_r, u_r)) at a unique point (v_m, u_m). Hence the Riemann problem has a unique solution for all initial data in the half-plane $v > 0$.

5.4 The shock curves are given by

$$\begin{aligned} S_1\colon u &= u_l - \sqrt{(v - v_l)(p(v_l) - p(v))}, \quad v \le v_l, \\ S_2\colon u &= u_l - \sqrt{(v - v_l)(p(v_l) - p(v))}, \quad v \ge v_l, \end{aligned}$$

and the rarefaction curves read

$$\begin{aligned} R_1\colon u &= u_l + \int_{v_l}^{v} \sqrt{-p'(y)}\, dy, \quad v \ge v_l, \\ R_2\colon u &= u_l - \int_{v_l}^{v} \sqrt{-p'(y)}\, dy, \quad v \le v_l. \end{aligned}$$

If $\int_{v_l}^{\infty} \sqrt{-p'(y)}\, dy < \infty$, then there are points (v_r, u_r) for which the Riemann problem does not have a solution. For further details, see [169, pp. 306 ff].

5.5 The solution consists of a slow rarefaction wave connecting the left state $(h_l, 0)$ and an intermediate state (h_m, v_m) with

$$v_m = -2\big(\sqrt{h_m} - \sqrt{h_l}\,\big),$$

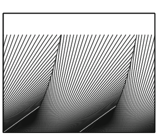
and there is a fast shock connecting (h_m, v_m) and $(h_r, 0)$, where

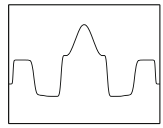
$$v_m = \frac{1}{\sqrt{2}}(h_r - h_m)\sqrt{\frac{1}{h_m} + \frac{1}{h_r}}.$$

The intermediate state is determined by the relation

$$2\sqrt{2}\big(\sqrt{h_m} - \sqrt{h_l}\big) = (h_r - h_m)\sqrt{\frac{1}{h_m} + \frac{1}{h_r}},$$

which has a unique solution (the left-hand side is increasing in h_m, ending up at zero for h_l, while the right-hand side is decreasing in h_m, starting at zero for h_r).





5.6 **a.** Set $f(w) = \varphi w$ and compute

$$df = \begin{pmatrix} \varphi_u + \varphi & \varphi_v u \\ \varphi_u v & \varphi_v v + \varphi \end{pmatrix} = \begin{pmatrix} \varphi_u u & \varphi_v u \\ \varphi_u v & \varphi_v v \end{pmatrix} + \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} = A + \varphi I.$$

Hence an eigenvector of A with eigenvalue μ is also an eigenvector of df with corresponding eigenvalue $\mu + \varphi$. Using this, we find that

$$\lambda_1 = \varphi, \quad r_1 = \begin{pmatrix} \varphi_v \\ -\varphi_u \end{pmatrix}, \quad \lambda_2 = \varphi + (\varphi_u u + \varphi_v v), \quad r_2 = \begin{pmatrix} u \\ v \end{pmatrix}.$$

b. In this case $\varphi_u = u$ and $\varphi_v = v$, and hence

$$\lambda_1 = \frac{1}{2}(u^2 + v^2), \quad r_1 = \begin{pmatrix} v \\ -u \end{pmatrix}, \quad \lambda_2 = \frac{3}{2}(u^2 + v^2), \quad r_2 = \begin{pmatrix} u \\ v \end{pmatrix}.$$

We find that the contours of φ, i.e., circles about the origin, are contact discontinuities with associated speed equal to half the radius squared. These correspond to the eigenvalue λ_1. The rarefaction curves of the eigenvalue λ_2 are half-lines starting at (u_l, v_l) given by

$$\frac{u}{u_l} = \frac{v}{v_l}$$

such that $u^2 + v^2 \geq u_l^2 + v_l^2$. For the shock part of the solution the Rankine–Hugoniot relation gives

$$(\varphi - \varphi_l)(v u_l - v_l u) = 0.$$

If $\varphi = \varphi_l$, we are on the contact discontinuity, so we must have

$$\frac{u}{v} = \frac{u_l}{v_l}.$$

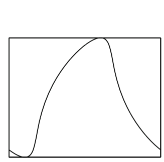
This is again a straight line through the origin and through u_l, v_l. What parts of this line can we use? The shock speed is given by

$$\begin{aligned} \sigma &= \frac{u_l^2 + v_l^2}{2}\left[\left(\frac{v}{v_l}\right)^2 + \left(\frac{v}{v_l}\right) + 1\right] \\ &= \frac{u_l^2 + v_l^2}{2}\left[\left(\frac{u}{u_l}\right)^2 + \left(\frac{u}{u_l}\right) + 1\right] \\ &= \frac{u_l^2 + v_l^2}{2 u_l v_l}\left[uv + \sqrt{u u_l v v_l} + u_l v_l\right]. \end{aligned}$$

Using polar coordinates $r^2 = 2\varphi$, $u = r\cos(\theta)$, and $v = r\sin(\theta)$, we see that

$$\sigma = \varphi_l\left[\left(\frac{r}{r_l}\right)^2 + \left(\frac{r}{r_l}\right) + 1\right].$$

We want (this is an extra condition!) to have σ decreasing along the shock path, so we define the admissible part of the Hugoniot locus to be the line segment bounded by (u_l, v_l) and the origin.



Now the solution of the Riemann problem is found by first using a contact discontinuity from (u_l, v_l) to some (u_m, v_m), where

$$\varphi(u_l, v_l) = \varphi(u_m, v_m) \quad \text{and} \quad \frac{u_m}{u_r} = \frac{v_m}{v_r},$$

and (u_m, v_m) is on the same side of the origin as (u_r, v_r). Finally, (u_m, v_m) is connected to (u_r, v_r) with a 2-wave.

c. In this case

$$\varphi_u = \varphi_v = \frac{-1}{(1+u+v)^2},$$

and therefore

$$\lambda_2 = \frac{1}{(1+u+v)^2}.$$

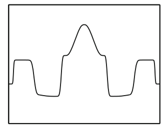
The calculation regarding the Hugoniot loci remains valid; hence the curves $\varphi = \text{const}$ are contact discontinuities. These are the lines given by $v = c - u$ and $u, v > 0$. The rarefaction curves of the second family remain straight lines through the origin, as are the shocks of the second family. On a Hugoniot curve of the second family, the speed is found to be

$$\sigma = \frac{1}{(1+u_l+v_l)(1+u+v)},$$

and if we want the Lax entropy condition to be satisfied, we must take the part of the Hugoniot locus pointing away from the origin. In this case both families of wave curves are straight lines. Such systems are often called *line fields*, and this system arises in chromatography. See [153, 154].
Observe that the family denoted by subscript 2 now has the smallest speed. Hence when we find the solution of the Riemann problem, we first use the second family, then the contact discontinuity.

5.7 Integrate the exact solution u_j^n over the rectangle $[(j-1)\Delta x, (j+1)\Delta x] \times [n\Delta t, (n+1)\Delta t]$. The CFL condition yields that no waves cross the lines $(j \pm 1)\Delta x$. Thus

$$\begin{aligned}
0 &= \int_{(j-1)\Delta x}^{(j+1)\Delta x} \int_{n\Delta t}^{(n+1)\Delta t} \big((u_j^n)_t + f(u_j^n)_x\big)\,dx\,dt \\
&= \int_{(j-1)\Delta x}^{(j+1)\Delta x} \Big|_{n\Delta t}^{(n+1)\Delta t} u_j^n \,dx + \int_{n\Delta t}^{(n+1)\Delta t} \Big|_{(j-1)\Delta x}^{(j+1)\Delta x} f(u_j^n)\,dt \\
&= \int_{(j-1)\Delta x}^{(j+1)\Delta x} u_j^n(x, \Delta t)\,dt - \Delta x\big(U_{j+1}^n + U_{j-1}^n\big) \\
&\quad + \Delta t\big(f(U_{j+1}^n) - f(U_{j-1}^n)\big) \\
&= \int_{(j-1)\Delta x}^{(j+1)\Delta x} u_j^n(x, \Delta t)\,dt - 2\Delta x U_j^{n+1}.
\end{aligned}$$



5.8 **a.** Choose a hypersurface M transverse to $r_k(u_0)$. Then the linear first-order equation $\nabla w(u) \cdot r_k(u)$ with w equal to some regular function w_0 on M has a unique solution locally, using, e.g., the method of characteristics. On M we may choose $n-1$ linearly independent functions, say $w_0^1, \dots, w_0^{n-1}$. The corresponding solutions w^j, $j = 1, \dots, n-1$, will have linearly independent gradients. See [167, p. 117] for more details, and [169, p. 321] for a different argument.

b. The kth rarefaction curve is the integral curve of the kth right eigenvector field, and hence every k-Riemann invariant is constant along that rarefaction curve.

c. The set of equations (5.185) yields $n(n-1)$ equations to determine n scalar functions w_j. See [56, pp. 127 ff] for more details.

d. For the shallow-water equations we have

$$w_1 = \frac{q}{h} - 2\sqrt{h}, \quad w_2 = \frac{q}{h} + 2\sqrt{h}.$$

5.9 **a.** Recall from Exercise 5.3 that the rarefaction curves are given by

$$\ln(v/v_l) = \mp (u - u_l).$$

Therefore, the Riemann invariants are given by

$$w_-(v, u) = \ln(v) + u, \quad w_+(v, u) = \ln(v) - u.$$

b.–e. Introducing $\mu = w_-$ and $\tau = w_+$, we find that

$$v = \exp\left(\frac{\mu + \tau}{2}\right), \quad u = \frac{\mu - \tau}{2}.$$

The Hugoniot loci are given by

$$u - u_l = \mp\left(\sqrt{\frac{v}{v_l}} - \sqrt{\frac{v_l}{v}}\right),$$

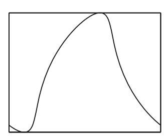
which in (μ, τ)-coordinates reads

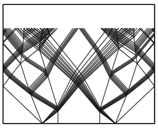
$$\begin{aligned} &\frac{1}{2}\left[(\mu - \mu_l) - (\tau - \tau_l)\right] \\ &\quad = \mp\left(\exp\left(\frac{\mu - \mu_l + \tau - \tau_l}{4}\right) - \exp\left(\frac{\mu_l - \mu + \tau_l - \tau}{4}\right)\right). \end{aligned}$$

Using $\Delta\mu$ and $\Delta\tau$, this relation says that

$$\Delta\mu - \Delta\tau = \mp 4 \sinh\left(\frac{\Delta\mu + \Delta\tau}{4}\right). \tag{C.18}$$

Hence the Hugoniot loci are translation-invariant. The rarefaction curves are coordinate lines, and trivially translation-invariant. To show that H_-





(the part of the Hugoniot locus with the minus sign) is the reflection of H_+, we note that such a reflection maps

$$\Delta\mu \mapsto \Delta\tau \quad \text{and} \quad \Delta\tau \mapsto \Delta\mu.$$

Hence a reflection leaves the right-hand side of (C.18) invariant, and changes the sign of the left-hand side. Thus the claim follows.
Recalling that the "–" wave corresponds to the eigenvalue $+1/u$, we must use the "+" waves first when solving the Riemann problem. Therefore, we now term the "+" waves 1-waves, and the "–" waves 2-waves. Hence the lines parallel to the μ-axis are the rarefaction curves of the first family. Then the 1-wave curve is given by

$$\begin{cases} u - u_l = \ln\left(\frac{v}{v_l}\right) & \text{for } v > v_l, \\ u - u_l = \sqrt{\frac{v}{v_l}} - \sqrt{\frac{v_l}{v}} & \text{for } u < u_l, \end{cases}$$

which in (μ, τ) coordinates reads

$$\begin{cases} \tau = \tau_l & \text{for } \mu > \mu_l, \\ \tau = \tau(\mu) & \text{for } \mu < \mu_l. \end{cases}$$

Here, the curve $\tau(\mu)$ is given implicitly by (C.18), and we find that

$$\tau'(\mu) = \frac{1 - \cosh(\cdots)}{1 + \cosh(\cdots)}.$$

From this we conclude that

$$0 \geq \frac{d\tau}{d\mu} \geq -1, \qquad \frac{d^2\tau}{d\mu^2} \geq 0,$$

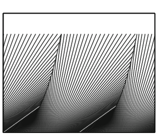
and $\tau(\mu_l) = \tau_l$ and $\lim_{\mu \to -\infty} \tau'(\mu) = -1$. By the reflection property we can also find the second wave curve in a similar manner.

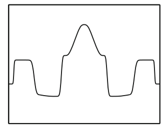
5.10 **a.** We obtain (cf. (2.15))

$$\begin{aligned} 0 &= \iint (u_t + f(u)_x)\, \nabla_u \eta(u) \phi \, dx\, dt \\ &= \iint \eta(u)_t \phi \, dx\, dt + \iint df(u) u_x \nabla_u \eta(u) \phi \, dx\, dt \\ &= -\iint \eta(u) \phi_t \, dx\, dt + \iint \nabla_u q(u) u_x \phi \, dx\, dt \\ &= -\iint \left(\eta(u)\phi_t + q(u)\phi_x\right) dx\, dt. \end{aligned}$$

b. The right-hand side of (5.186) is the gradient of a vector-valued function q if

$$d^2\eta(u) df(u) = df(u)^t d^2\eta(u), \tag{C.19}$$





where $df(u)^t$ denotes the transpose of $df(u)$, and $d^2\eta(u)$ denotes the Hessian of η. The relation (C.19) follows from the fact that mixed derivatives need to be equal, i.e.,

$$\frac{\partial^2 q}{\partial u_j \partial u_k} = \frac{\partial^2 q}{\partial u_k \partial u_j}.$$

Equation (C.19) imposes $n(n-1)/2$ conditions on the scalar function η.

c. We obtain

$$\eta(u) = \frac{1}{2}u^2 - \int^v p(y)\,dy, \quad q(u) = up(v).$$

d. Equation (C.19) reduces in the case of the shallow-water equations to one hyperbolic equation,

$$\Big(\frac{q^2}{2h} + h\Big)\eta_{qq} = \eta_{hh} + 2\frac{q}{h}\eta_{hq},$$

with solution

$$\eta(u) = \frac{q^2}{2h} + \frac{1}{2}h^2, \quad Q(u) = \frac{q^3}{2h^2} + hq$$

(where for obvious reasons we have written the entropy flux with capital Q rather than q).

5.11 See [40, Sect. 3.4].

5.12 **a.** The result follows directly from the definitions.

b. Given $u_t + f(u)_x = 0$, introduce new variables w such that $u = g(w)$ for some function g. We have $u_t = Gw_t$ and $u_x = Gw_x$, where $G = \partial g(w)/\partial w$, which implies

$$w_t + A(w)w_x = 0,$$

where $A = G^{-1}df(g(w))G$. In our case $w = (\rho, v, e)$, and we obtain by straightforward computations

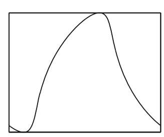
$$G = \begin{pmatrix} 1 & 0 & 0 \\ v & \rho & 0 \\ \frac{1}{2}v^2 + e & \rho v & 1 \end{pmatrix}, \quad G^{-1} = \frac{1}{\rho}\begin{pmatrix} \rho & 0 & 0 \\ -v & 1 & 0 \\ \frac{1}{2}v^2 - e & -v & 1 \end{pmatrix}.$$

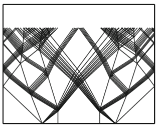
Thus

$$A = \begin{pmatrix} v & \rho & 0 \\ (\gamma-1)e/\rho & v & (\gamma-1) \\ 0 & (\gamma-1)e & v \end{pmatrix}.$$

We can also rewrite this as

$$A = \begin{pmatrix} v & \rho & 0 \\ p_\rho/\rho & v & p_e/\rho \\ 0 & p/\rho & v \end{pmatrix}.$$





c. Clearly, we obtain the same eigenvalues as for $df(u)$, that is, $v, v \pm c$. The corresponding eigenvectors read

$$\lambda_1 = v - c, \quad r_1 = \begin{pmatrix} \rho \\ -c \\ p/\rho \end{pmatrix},$$

$$\lambda_2 = v, \qquad r_2 = \begin{pmatrix} p_e \\ 0 \\ p_\rho \end{pmatrix},$$

$$\lambda_3 = v + c, \quad r_3 = \begin{pmatrix} \rho \\ c \\ p/\rho \end{pmatrix}.$$

Furthermore,

$$\nabla\lambda_1 \cdot r_1 = -\frac{1}{2}(1+\gamma)c, \quad \nabla\lambda_2 \cdot r_2 = 0, \quad \nabla\lambda_3 \cdot r_3 = \frac{1}{2}(1+\gamma)c.$$

Thus the first and third families are genuinely nonlinear, while the second family is linearly degenerate.

d. With $R = S = (\gamma - 1)\log(\rho) - \log(e) - \log(\gamma - 1)$, we obtain

$$R \cdot r_1 = R \cdot r_3 = 0,$$

while with $R = v + 2c/(\gamma - 1)$, we see that

$$R \cdot r_1 = 0.$$

(It is convenient to use that $\partial c/\partial e = c/(2e)$.) If $R = p$, we obtain

$$R \cdot r_2 = 0.$$

e. We obtain

$$\begin{aligned} \left(\frac{\partial \rho S}{\partial w}\right)^T &= \big(S + (\gamma - 1), 0, -\rho/e\big), \\ \left(\frac{\partial \rho v S}{\partial w}\right)^T &= \big(vS + (\gamma - 1)v, \rho S, -\rho v/e\big). \end{aligned}$$

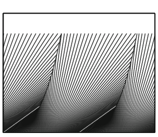
A straightforward computation yields the result.

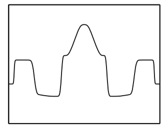
5.13 The third equation in (5.157) will give

$$\frac{zc^2}{\pi(\gamma - 1)} + \frac{1}{2}z^2w^2 = \frac{c^2}{\gamma - 1} + \frac{1}{2}w^2,$$

and solving for w/c gives

$$\left(\frac{w}{c}\right)^2 = \frac{2}{\gamma - 1}\frac{\pi - z}{\pi(1 - z^2)}. \tag{C.20}$$





The second equation in (5.157) now reads

$$\frac{c^2}{\gamma} + w^2 = \frac{c^2}{\gamma\pi} + zw^2.$$

Solving for w/c, we obtain

$$\left(\frac{w}{c}\right)^2 = \frac{1-\pi}{\gamma\pi(1-z)}.$$

Setting the two expressions for $(w/c)^2$ equal and solving for z, we get (5.161)! However, now we insert this into (C.20), and after several lines of computations, we get

$$\left(\frac{w}{c}\right)^2 = \frac{2}{\gamma-1}\frac{\beta+\pi}{\pi(\beta^2-1)}.$$

This implies (5.157).

Chapter 6, Sect. 6.4

6.1 **a.** Up to second order in ϵ, the wave curves are given by

$$S_i(\epsilon) = \exp(\epsilon r_i).$$

For an interaction between an i-wave and a j-wave ($j < i$) we have that (up to second order)

$$u_r = \exp(\epsilon_i r_i)\exp(\epsilon_j)u_l.$$

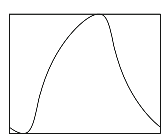
After the interaction,

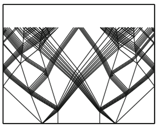
$$u_r = \exp(\epsilon'_n r_n)\cdots\exp(\epsilon'_j r_j)\cdots\exp(\epsilon'_i r_i)\cdots\exp(\epsilon'_1 r_1)u_l$$

up to second order. Comparing these two expressions and using part **c** and the fact that $\{r_i\}$ are linearly independent, we find that

$$\epsilon'_k = \delta_{kj}\epsilon_j + \delta_{ki}\epsilon_i + \text{second-order terms}.$$

6.2 **a.** The definition of the front-tracking algorithm follows from the general case and the grid in the (μ, τ)-plane. In this case we do not need to remove any front of high generation. To show that T_n is nonincreasing in n, we must study interactions. In this case we have a left wave ϵ_l separating states (μ_l, τ_l) and (μ_m, τ_m) colliding with a wave ϵ_r separating (μ_m, τ_m) and (μ_r, τ_r). Now, the family of ϵ_l must be greater than or equal to the family of ϵ_r. The claim follows by studying each case, and recalling that (consult Exercise 5.9) $0 \geq \tau'(\mu) > -1$.





b. If the total variation of μ_0 and τ_0 is finite, then we have enough to show that front tracking produces a compact sequence. Hence if

$$\text{T.V.}\,(\ln(u_0)) \quad \text{and} \quad \text{T.V.}\,(v_0)$$

are finite, then the sequence produced by front tracking is compact. A reasonable condition could then be that $u_0(x) \geq c > 0$ almost everywhere, $\text{T.V.}\,(u_0) < M$, and $\text{T.V.}\,(v_0) < M$.

c. To show that the limit is a weak solution, we proceed as in the general case.

Chapter 7, Sect. 7.4

7.1 We have already established **A** and **B** in Theorem 6.7. To prove **C**, let T be defined by (6.23) and Q by (6.22), and assume that $t = \gamma(x)$ is a curve with Lipschitz constant L that is smaller than the inverse of the largest characteristic speed, i.e.,

$$L < \frac{1}{\max_{u \in \mathcal{D}, k} |\lambda_k(u)|}.$$

For such a curve we have that all fronts in u_δ will cross γ from below. Let $\gamma_t(x)$ be defined by

$$\gamma_t(x) = \min\{t, \gamma(x)\}.$$

Since all fronts will cross γ, and hence also γ_t, from below, we can define $T|_{\gamma_t}$. Then we have that

$$T|_{\gamma_t} \leq (T + kQ)|_{\gamma_t} \leq (T + kQ)(t) \leq (T + kQ)(0).$$

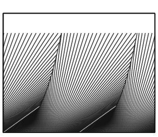
Since $\gamma = \lim_{t \to \infty} \gamma_t$, it follows that the total variation of u_δ on γ is finite, independent of δ. Hence the total variation on γ of the limit u is also finite.

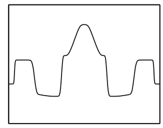
7.2 In the linearly degenerate case the Hugoniot loci coincide with the rarefaction curves. If you (meticulously) recapitulate all cases, you will find that some of the estimates are easier because of this, while others are identical. If you do this exercise, you have probably mastered the material in Sect. 7.1!

Chapter 8, Sect. 8.5

8.1 If a_l and a_r are both nonnegative, then the solution reads

$$u(x,t) = \begin{cases} u_l & x < 0, \\ \frac{a_l}{a_r} u_l & 0 \leq x \leq a_r t, \\ u_r & x > a_r t. \end{cases}$$





A similar description holds if both a_l and a_r are nonpositive. If $a_l < 0 < a_r$, then the solution reads

$$u(x,t) = \begin{cases} u_l & x < a_l t, \\ 0 & a_l t \le x \le a_r t, \\ u_r & x \ge a_r t. \end{cases}$$

If $a_l > 0 > a_r$, we have that $H_l(u_l) = \{u_l\}$ and $H_r(u_r) = \{u_r\}$, so unless $a_l u_l = a_r u_r$, there is no solution.

8.2 We have that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - f\frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial x} = s\frac{\partial}{\partial y}.$$

Using this, we get

$$s_\tau - f s_y + s f_y = 0.$$

Dividing by s^2, we get the first equation in (8.5). Combining this with the second equation in (8.4) will give $c_\tau = 0$.

Appendix A, Sect. A.2

A.1 Observe first that

$$\left|\int \phi(x) f(x)\, dx\right| \le \int |f(x)|\, dx,$$

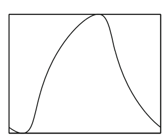
for every function $|\phi| \le 1$. Now let ω_ε be a standard mollifier, and define

$$\phi_\varepsilon = \omega_\varepsilon * \operatorname{sign}(f).$$

Clearly, $f\phi_\varepsilon \to |f|$ pointwise, and a simple application of dominated convergence implies that indeed,

$$\int \phi_\varepsilon(x) f(x)\, dx \to \int |f(x)|\, dx \text{ as } \varepsilon \to 0.$$

A.2 Write as usual $h^\delta = v^\delta - w^\delta$, where v^δ and w^δ both are nondecreasing functions. Both sequences $\{v^\delta\}$ and $\{w^\delta\}$ satisfy the conditions of the theorem, and hence we may pass to a subsequence such that both v^δ and w^δ converge in L^1. After possibly taking yet another subsequence, we conclude that we may obtain pointwise convergence almost everywhere. Let v be the limit of v^δ, which we may assume to be nondecreasing (by possibly redefining it on a set of measure zero). Write $v(x\pm)$ for the right, respectively left, limits of v at x. Fix



$x \in (a,b)$. Let $\epsilon > 0$ and $\eta > 0$ be such that $v(y) < v(x+) + \epsilon$ whenever $x < y < x + \eta$. If $v^\delta \geq v(x+) + 2\epsilon$, then

$$\left\| v^\delta - v \right\|_{L^1} \geq \int_x^{x+\eta} (v^\delta(y) - v(y))\, dy > \eta\epsilon,$$

and so, since $\left\| v^\delta - v \right\|_{L^1} \to 0$ as $\delta \to 0$, we must conclude that $v^\delta(x) < v(x+) + 2\epsilon$ for δ sufficiently small. Similarly, $v^\delta(x) > v(x-) - 2\epsilon$. In particular, $v^\delta(x) \to v(x)$ whenever v is continuous at x, thus at all but a countable set of points x. In the same way we show that $w^\delta(x) \to w(x)$ for all but at most a countable set of points x. A diagonal argument shows that we can pass to a subsequence such that both v^δ and w^δ converge pointwise for all x in $[a,b]$.

Appendix B, Sect. B.2

B.1 For more details on this exercise including several applications, as well as how to extend the result to hyperbolic systems in the $\mu \to 0$ limit, please consult [96].

a. Just follow the argument in Theorem B.1.
b. Mimic the argument starting with (B.15). (Let δ in (B.15) be negative.)
c. Redo the previous point, assuming that $u_{0,j} \geq u_*$.

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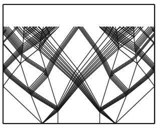

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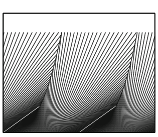

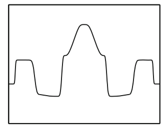

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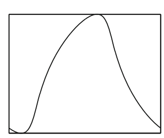

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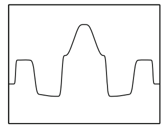

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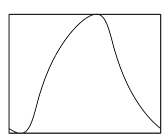

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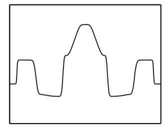

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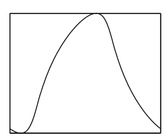

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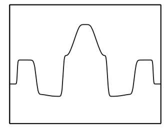

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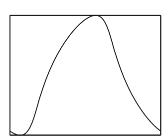

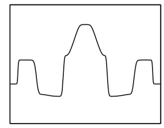

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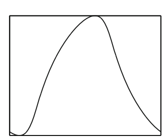

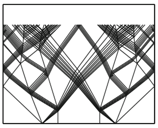

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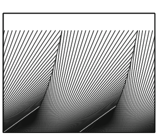

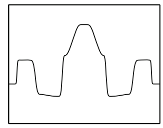

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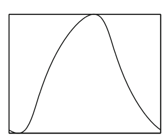

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— R.T. Davies, in foreword to D. Adams's
The Hitchhiker's Guide to the Galaxy