Chapter 11
Stabilization of Stochastic RNNs with Stochastic Delays

The research in Chaps. 4–10 is focused on the qualitative analysis of complex neural networks with delays. It is well known that the qualitative analysis of nonlinear dynamical systems is the foundation of controlling the systems. Therefore, in this chapter controller design problem will be studied for a class of stochastic Cohen-Grossberg neural networks with mode-dependent mixed time delays and Markovian switching, in which the neural dynamical networks will be stabilized. The contents in this chapter are from the research result in [1].

11.1 Introduction

In recent decades, neural networks have been successfully applied to various fields such as optimization, image processing, and associative memory design. In such application, it is important to know the stability properties of the designed neural network, these properties include asymptotic stability and exponential stability. However, time delays inevitably exist in neural networks due to various reasons [2]. The existence of *time delay* may lead to some complex dynamic behaviors such as oscillation, divergence, chaos, instability, or other poor performance of the neural networks. Since neural networks usually have a spatial extent, there is a distribution of propagation delays over a period of time. In these circumstances, the signal propagation is not instantaneous and cannot be modeled with discrete-time delays [3]. A more appropriate way is to incorporate discrete and continuously distributed time delays in the neural network model [2, 4]. Stability analysis for neural networks with delays has attracted more and more interests in recent years, for example, see [5–21] and references therein.

On the other hand, the stabilization issue has been an important focus of research in the control fields, and several feedback stabilizing control design approaches have been proposed (see [7, 22–25]). Some interesting results [6, 26–35] on the

Z. Wang et al., *Qualitative Analysis and Control of Complex Neural Networks with Delays*, Studies in Systems, Decision and Control 34,
DOI 10.1007/978-3-662-47484-6_11

stabilization of a wide range and different types of neural networks have been reported in the literature. For a class of discrete-time dynamic neural networks, reference [29] proposes two methods, namely, the gradient projection and the minimum distance projection to investigate the stabilization. For a class of dynamic neural network systems, a global robust stabilizing controller with unknown nonlinearities is developed in [6] via Lyapunov stability and inverse optimality. For a class of linearly coupled stochastic neural networks, some results are derived in [31] on the design of the minimum number of controllers for the pinning stabilization, which are expressed in terms of strict linear matrix inequality (LMI). For a class of neutral neural networks with varying delays, a novel criterion is obtained in [28] for the global stabilization using the Razumikhin's method. For a class of so-called standard neural network models with time delays, a few stabilization criteria are presented [30] which are based on the Lyapunov–Krasovskii stability theory and the LMI approach. For a class of impulsive high-order Hopfield-type neural networks with time-varying delays, some stabilization criteria are reported in [26] by employing the Lyapunov–Razumikhin technique. Very recently, for a class of neural networks with various activation functions and time-varying continuously distributed delays, LMI-based delay-dependent conditions are obtained in [27] for the global exponential stabilization. Despite some good progress on the stability analysis of delayed neural networks with various activation functions [36–38], the stabilization issue has not been fully explored in the existing studies.

Although the stabilization problem for some kinds of neural networks with or without time delays is investigated by some authors, there has been no literature reported on the stabilization of stochastic *Cohen-Grossberg neural networks* with both Markovian jumping parameters and mixed mode-dependent time delays. As well known, mode-dependent time delays are of practical significance since the signal may switch between different modes and also propagate in a distributed way during a certain time period with the presence of an amount of parallel pathways [24]. The purpose of this chapter is to make an attempt to deal with the control problem for a class of stochastic neural networks with mode-dependent delays [1]. By introducing a new Lyapunov–Krasovskii functional that accounts for the mode-dependent mixed delays, stochastic analysis is conducted in order to derive delay-dependent criteria for the exponential stabilization problem. The feedback stabilizing controller is designed to satisfy some exponential stability constraints on the closed-loop poles. The stabilization criteria are obtained in terms of LMI and hence the gain control matrix is easily determined by numerical MATLABs LMI Control Toolbox. Three numerical examples are carried out to demonstrate the feasibility of our delay-dependent stabilization criteria.

Throughout this chapter, the shorthand col$\{M_1, M_2, \ldots, M_l\}$ denotes a column matrix with the matrices $M_1, M_2, \ldots, M_l$. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P}\right)$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and contains all $\mathcal{P}$-null sets. $\mathcal{L}^p_{\mathcal{F}_0}([-h, 0], \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable $\mathbb{C}([-h, 0]; \mathbb{R}^n)$-valued random variables

$\xi = \{\xi(\theta) : -h \le \theta \le 0\}$ such that $\sup_{-h\le\theta\le 0} \mathbb{E}|\xi(\theta)|^p < \infty$, where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

11.2 Problem Formulation and Preliminaries

We consider the following stochastic neural network with both feedback control law and Markovian jumping parameters described by

$$\begin{aligned} \mathrm{d}x(t) = &- \alpha(x(t), \eta_t)\Bigg[\beta(x(t), \eta_t) - A(\eta_t) f(x(t)) \\ &- B(\eta_t) f(x(t - \tau(t, \eta_t))) \\ &- C(\eta_t) \int_{t-\upsilon(t,\eta_t)}^{t} g(x(s))\mathrm{d}s - D(\eta_t)u(t, \eta_t)\Bigg]\mathrm{d}t \\ &+ \Bigg[E_1(\eta_t)x(t) + E_2(\eta_t)x(t - \tau(t, \eta_t)) \\ &+ E_3(\eta_t) f(x(t)) + E_4(\eta_t) f(x(t - \tau(t, \eta_t))) \\ &+ E_5(\eta_t) \int_{t-\upsilon(t,\eta_t)}^{t} g(x(s))\mathrm{d}s\Bigg]\mathrm{d}\omega(t), \end{aligned} \tag{11.1}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$ denotes the neuron state at time t, $u(t) \in L_2([0, s), \mathbb{R}^m)$, $\forall s > 0$, is the control input vector of the neural networks, $\alpha(x(t), \eta_t) = \mathrm{diag}\{\alpha_j(x_j(t), \eta_t), \ldots, \alpha_n(x_n(t), \eta_t)\}$ denotes the amplification function, $\beta(x(t), \eta_t) = \mathrm{diag}\{\beta_j(x_j(t), \eta_t), \ldots, \beta_n(x_n(t), \eta_t)\}$ denotes the appropriately behaved function such that the solution of the model given in (11.1) remains bounded, and $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T$, $g(x(s)) = [g_1(x_1(s)), \ldots, g_n(x_n(s))]^T$ denote the activation functions. $f(x(t - \tau(t, \eta_t))) = [f_1(x_1(t - \tau(t, \eta_t))), \ldots, f_n(x_n(t - \tau(t, \eta_t)))]^T$. $0 \le \tau(t, \eta_t) \le \bar{\tau}(\eta_t) \le \bar{\tau}, 0 \le \upsilon(t, \eta_t) \le \bar{\upsilon}(\eta_t) \le \bar{\upsilon}$ $(j = 1, \ldots, n)$ are bounded and unknown delays. The matrices $A(\eta_t), B(\eta_t), C(\eta_t) \in \mathbb{R}^{n\times n}$, $D(\eta_t) \in \mathbb{R}^{n\times m}$ are the connection weight matrix, the discretely delayed connection weight matrix, the distributively delayed connection weight matrix and the control input weights, respectively. $E_j(\eta_t)(j = 1, 2, \ldots, 5)$ is known real constant matrix with appropriate dimension, $\omega(t)$ is a one-dimensional Brownian motion defined on complete probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P}\big)$ with $\mathbb{E}\{\mathrm{d}\omega(t)\} = 0$, $\mathbb{E}\{[\mathrm{d}\omega(t)]^2\} = \mathrm{d}t$. $\{\eta_t = \eta(t), t \ge 0\}$ is a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in finite set $\wp = \{1, 2, \ldots, N\}$ with given probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P}\big)$ and the initial model η_0. It is assumed that the initial condition of neural network (11.1) has the form $x(t) = \varphi(t)$ for $t \in [-\varpi, 0]$, where $\varphi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T$, function $\varphi_j(t)(j = 1, 2, \ldots, n)$ is continuous, $\varpi = \max\{\bar{\tau}, \bar{\upsilon}\}$. Let $\aleph = [\pi_{ij}]_{i,j\in\wp}$ denote the transition rate matrix with given probability:

$$P(\eta_{t+\delta}=j|\eta_t=i)=\begin{cases}\pi_{ij}\delta+o(\delta), & i\neq j,\\ \pi_{ii}\delta+o(\delta)+1, & i=j,\end{cases}$$

where $\delta>0$, $\lim_{\delta\to 0^+}\frac{o(\delta)}{\delta}=0$ and π_{ij} is the transition rate from mode i to mode j satisfying $\pi_{ij}\geq 0$ for $i\neq j$ with $\pi_{ii}=-\sum_{j=1,i\neq j}^{N}\pi_{ij}$, $i,j\in\wp$.

For convenience, each possible value of η_t is denoted by $i(i\in\wp)$ in the sequel. Then we have

$$\begin{aligned}&\alpha_i(x(t))=\alpha(x(t),\eta_t),\ \beta_i(x(t))=\beta(x(t),\eta_t),\\ &A_i=A(\eta_t),\ B_i=B(\eta_t),\ C_i=C(\eta_t),\\ &D_i=D(\eta_t),\ \tau_i(t)=\tau(t,\eta_t),\ \upsilon_i(t)=\upsilon(t,\eta_t),\\ &E_{li}=E_l(\eta_t),\ l=1,\ldots,5.\end{aligned}$$

In the following, we need the following definitions, assumptions, and lemmas.

Definition 11.1 ([24, 27]) Given $r>0$, and any initial condition $\varphi\in\mathcal{L}^2_{\mathcal{F}_0}([-\varpi,0],\mathbb{R}^n)$ with $u(t,\eta_t)=0$. The zero solution of system (11.1) is said to be r-*exponentially stable* in the mean square, if there exists a positive scalar M such that any solution $x(t,\varphi)$ of the system satisfies the following inequality,

$$\mathbb{E}||x(t,\phi)||^2\leq M\sup_{-\varpi\leq s\leq 0}\mathbb{E}||\phi(s)||^2e^{-2rt},\quad\forall\, t\geq 0.$$

Definition 11.2 ([24, 27]) Given $r>0$. The system (11.1) is said to be r-*exponentially stabilizable* in the mean square, if there is a feedback control law $u(t,\eta_t)=\overline{U}(\eta_t)x(t)$, such that the following closed-loop system

$$\begin{aligned}\mathrm{d}x(t)=&-\alpha(x(t),\eta_t)\Big[\beta(x(t),\eta_t)-A(\eta_t)f(x(t))\\ &-B(\eta_t)f(x(t-\tau(t,\eta_t)))\\ &-C(\eta_t)\int_{t-\upsilon(t,\eta_t)}^{t}g(x(s))\mathrm{d}s-D(\eta_t)\overline{U}(\eta_t)x(t)\Big]\mathrm{d}t\\ &+\Big[E_1(\eta_t)x(t)+E_2(\eta_t)x(t-\tau(t,\eta_t))\\ &+E_3(\eta_t)f(x(t))+E_4(\eta_t)f(x(t-\tau(t,\eta_t)))\\ &+E_5(\eta_t)\int_{t-\upsilon(t,\eta_t)}^{t}g(x(s))\mathrm{d}s\Big]\mathrm{d}\omega(t),\\ x(t)=&\varphi(t),\quad t\in[-\varpi,0],\end{aligned}$$

is r-exponentially stable.

Assumption 11.3 ([8]) Each $\alpha_{ji}(\cdot)$ is a continuous function and satisfies $\bar{\alpha}_{ji}\geq\alpha_{ji}(\cdot)\geq\underline{\alpha}_{ji}>0$, $j=1,2,\ldots,n, i=1,2,\ldots,N$.

Here, we denote $\underline{\alpha}_i = \min_{1\leq j\leq n}\{\underline{\alpha}_{ji}\}$, $\bar{\alpha}_i = \max_{1\leq j\leq n}\{\bar{\alpha}_{ji}\}$ for simplicity.

Assumption 11.4 Each function $\beta_{ji}(\cdot)$ is locally Lipschitz continuous, $\beta_{ji}(0) = 0$ and there exist constants $\bar{\beta}_{ji} > \underline{\beta}_{ji} \geq 0$ such that

$$\underline{\beta}_{ji} s^2 \leq \beta_{ji}(s)s \leq \bar{\beta}_{ji} s^2,$$

for any $s \in \mathbb{R}$, $j = 1, 2, \ldots, n, i = 1, 2, \ldots, N$.

For simplicity, we denote $\Pi_i = \mathrm{diag}\{\bar{\beta}_{1i}, \ldots, \bar{\beta}_{ni}\}$, $\Gamma_i = \mathrm{diag}\{\underline{\beta}_{1i}, \ldots, \underline{\beta}_{ni}\}$.

Assumption 11.5 For $j = 1, 2, \ldots, n$, $f_j(0) = g_j(0) = 0$. Furthermore, there exist constants $\varrho_j^-, \varrho_j^+, \psi_j^-, \psi_j^+$ such that $\varrho_j^- < \varrho_j^+, \psi_j^- < \psi_j^+$ and

$$\varrho_j^- \leq \frac{f_j(s)}{s} \leq \varrho_j^+, \quad \psi_j^- \leq \frac{g_j(s)}{s} \leq \psi_j^+,$$

for any $s \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

Remark 11.6 As pointed out in [24], the constants $\varrho_j^-, \varrho_j^+, \psi_j^-, \psi_j^+$ in Assumption 11.5 are allowed to be positive, negative, or zero. Then, those previously used Lipschitz conditions are just the special cases of Assumption 11.5. Hence, the activation functions can be of more general descriptions than those earlier forms.

For notational simplicity, we denote

$$\begin{aligned}
\bar{\Sigma} &= \mathrm{diag}\left\{\varrho_1^+, \varrho_2^+, \ldots, \varrho_n^+\right\},\\
\Sigma &= \mathrm{diag}\left\{\varrho_1^-, \varrho_2^-, \ldots, \varrho_n^-\right\},\\
F_1 &= \mathrm{diag}\left\{\varrho_1^-\varrho_1^+, \varrho_2^-\varrho_2^+, \ldots, \varrho_n^-\varrho_n^+\right\},\\
F_2 &= \mathrm{diag}\left\{\frac{\varrho_1^- + \varrho_1^+}{2}, \frac{\varrho_2^- + \varrho_2^+}{2}, \ldots, \frac{\varrho_n^- + \varrho_n^+}{2}\right\},\\
F_3 &= \mathrm{diag}\left\{\psi_1^-\psi_1^+, \psi_2^-\psi_2^+, \ldots, \psi_n^-\psi_n^+\right\},\\
F_4 &= \mathrm{diag}\left\{\frac{\psi_1^- + \psi_1^+}{2}, \frac{\psi_2^- + \psi_2^+}{2}, \ldots, \frac{\psi_n^- + \psi_n^+}{2}\right\}.
\end{aligned}$$

Lemma 11.7 (Jensen integral inequality, see [39]) *For any constant matrix $M > 0$, any scalars a and b with $a < b$, and a vector function $\chi(t) : [a, b] \to \mathbb{R}$ such that the integrals concerned are well defined, then the following inequality holds*

$$\left\langle \int_a^b \chi(s)\mathrm{d}s, M\int_a^b \chi(s)\mathrm{d}s \right\rangle \leq (b-a)\int_a^b \chi(s)^T M\chi(s)\mathrm{d}s,$$

where $\left\langle A, B \right\rangle = A^T B$ denotes the inner product.

Lemma 11.8 *Assume that $\nu, \mu, \underline{\vartheta}, \bar{\vartheta}$ are real scalars such that $\nu \leq 1, \nu + \mu \leq 4$, and $\underline{\vartheta} < \bar{\vartheta}$. Let $\vartheta : \mathbb{R} \to (\underline{\vartheta}, \bar{\vartheta})$ be a real function. Then for any nonnegative scalars a, b, the following inequality holds*

$$
\begin{aligned}
&- \frac{a}{\vartheta(t) - \underline{\vartheta}} - \frac{b}{\bar{\vartheta} - \vartheta(t)} \\
&\leq \frac{1}{\bar{\vartheta} - \underline{\vartheta}} \max\{-\nu a - \mu b, -\mu a - \nu b\}.
\end{aligned} \tag{11.2}
$$

Proof Without loss of generality, we assume that $\nu \leq \mu$. First consider the case that $a \leq b$. It is easy to see that $\max\{-\nu a - \mu b, -\mu a - \nu b\} = -\mu a - \nu b$. Therefore, we have

$$
\begin{aligned}
&\left(\vartheta(t) - \underline{\vartheta}\right)\left(\bar{\vartheta} - \vartheta(t)\right)(-\mu a - \nu b) \\
&+ \left(\bar{\vartheta} - \underline{\vartheta}\right)\left[\left(\bar{\vartheta} - \vartheta(t)\right) a + \left(\vartheta(t) - \underline{\vartheta}\right) b\right] \\
=& \left(\bar{\vartheta} - \vartheta(t)\right)\left[\bar{\vartheta} + (\mu - 1)\underline{\vartheta} - \mu\vartheta(t)\right] a \\
&+ \left(\vartheta(t) - \underline{\vartheta}\right)\left[(1 - \nu)\left(\bar{\vartheta} - \vartheta(t)\right) + \left(\vartheta(t) - \underline{\vartheta}\right)\right] b \\
\geq& \left\{\left(\bar{\vartheta} - \vartheta(t)\right)\left[\bar{\vartheta} + (\mu - 1)\underline{\vartheta} - \mu\vartheta(t)\right]\right. \\
&\left. + \left(\vartheta(t) - \underline{\vartheta}\right)\left[(1 - \nu)\left(\bar{\vartheta} - \vartheta(t)\right) + \left(\vartheta(t) - \underline{\vartheta}\right)\right]\right\} a \\
=& \frac{a}{4}\left[(\nu + \mu)\left(2\vartheta(t) - \underline{\vartheta} - \bar{\vartheta}\right)^2 + (4 - \nu - \mu)\left(\bar{\vartheta} - \underline{\vartheta}\right)^2\right] \\
\geq& 0.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\frac{1}{\bar{\vartheta} - \underline{\vartheta}} \max\{-\nu a - \mu b, -\mu a - \nu b\} \\
&= \frac{1}{\bar{\vartheta} - \underline{\vartheta}}(-\mu a - \nu b) \\
&\geq - \frac{a}{\vartheta(t) - \underline{\vartheta}} - \frac{b}{\bar{\vartheta} - \vartheta(t)}.
\end{aligned}
$$

Similarly, we can also conclude that the inequality (11.2) holds for $a > b$. Now, the proof of Lemma 11.8 is completed.

Remark 11.9 If we set $\nu = 1, \mu = 3$, then we get Lemma 3 of [40] from Lemma 11.8. Thus, based on Lemma 11.8, we can get some conditions of exponential stabilization problem with less conservativeness.

11.3 Stabilization Result

As is well known, for stochastic systems, Itô's formula plays an important role in the stability analysis of stochastic systems and we cite some related results here [41]. Consider a general stochastic system

$$\mathrm{d}x(t) = f(x(t), t, \eta_t)\mathrm{d}t + g(x(t), t, \eta_t)\mathrm{d}\omega(t) \tag{11.3}$$

on $t \geq t_0$ with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times \mathbb{R}^+ \times \wp \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^+ \times \wp \rightarrow \mathbb{R}^{n+m}$. Let $\mathbb{C}^{2,1}\big(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+\big)$ denote the family of all nonnegative functions $V(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}^+$ which are continuously differentiable in t and twice differentiable in x. Let £ be the weak infinitesimal generator of the random process $\{x(t), \eta(t)\}_{t\geq 0}$ along the system (11.3) (see [24, 42, 43]), i.e.,

$$\begin{aligned}\pounds V(x_t, t, i) := \lim_{\delta\rightarrow 0^+} \frac{1}{\delta} \sup\Big[\mathbb{E}\big\{V(x_{t+\delta}, t+\delta, \eta(t+\delta))\big|x(t),\\ \eta(t) = i\big\} - V(x_t, t, \eta(t) = i)\Big],\end{aligned}$$

then, by the generalized Itô's formula, one can get

$$\mathbb{E}V(x, t, i) = \mathbb{E}V(x_0, t_0, i) + \mathbb{E}\int_{t_0}^{t} \pounds V(x(s), s, i)\mathrm{d}s.$$

Theorem 11.10 *Given $r > 0$. For any given scalars $\bar{\tau}_i > 0$, $\bar{\upsilon}_i > 0$, $\upsilon_i' < 1$, considering the system (11.1) satisfying Assumptions 11.3–11.5 and $\dot{\tau}_i(t) \leq \tau_i'$, $\dot{\upsilon}_i(t) \leq \upsilon_i'$, the system (11.1) is globally r-exponentially stabilized if there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, symmetric nonnegative definite matrices $Q_{ji}, R_i, M_i, S_l, Z_i$ $(j = 1, \ldots, 4, l = 1, \ldots, 9)$, positive diagonal matrices G_i, U_i, T_i, W_i, H, K, and real matrices X_i satisfying the following inequalities $(i = 1, \ldots, N)$*

$$\sum_{j=1}^{N} \pi_{ij} Q_{lj} < S_l, \quad l = 1, 2, 3, 4, \tag{11.4}$$

$$\sum_{j=1}^{N} \pi_{ij} R_j < S_5, \tag{11.5}$$

$$\sum_{j=1}^{N} \pi_{ij} Z_j < S_6, \tag{11.6}$$

$$\sum_{j=1}^{N} \pi_{ij} \bar{\upsilon}_j R_j < S_7, \tag{11.7}$$

$$\sum_{j=1}^{N} \pi_{ij}\bar{\tau}_j Z_j < S_8, \tag{11.8}$$

$$\sum_{j=1}^{N} \pi_{ij}\bar{\tau}_j Q_{4j} < S_9, \tag{11.9}$$

$$\begin{bmatrix} \Omega_i + \widetilde{\Omega}_i & \mathcal{E}^T \\ \mathcal{E} & \overline{Z}_i \end{bmatrix} < 0, \tag{11.10}$$

$$\begin{bmatrix} \Omega_i + \widehat{\Omega}_i & \mathcal{E}^T \\ \mathcal{E} & \overline{Z}_i \end{bmatrix} < 0, \tag{11.11}$$

where

$$\Omega_i = \begin{bmatrix} \Omega_{1i} & \Omega_{2i} & \Omega_{4i} & \Omega_{7i} \\ * & \Omega_{3i} & \Omega_{5i} & 0 \\ * & * & \Omega_{6i} & \Omega_{8i} \\ * & * & * & \Omega_{9i} \end{bmatrix},$$

$$\widetilde{\Omega}_i = -\frac{2}{\bar{\bar{\tau}}_i}\mathbb{I}^T Q_{4i}\mathbb{I}, \quad \widehat{\Omega}_i = -\frac{2}{\bar{\bar{\tau}}_i}\Im^T Q_{4i}\Im,$$

$$\mathcal{E} = [\, E_{1i} \;\; E_{2i} \;\; E_{3i} \;\; E_{4i} \;\; 0 \;\; E_{5i} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,],$$

$$\overline{Z}_i = \frac{\bar{\tau}^2}{2}S_6 + \bar{\tau}S_8 + \bar{\tau}_i Z_i + \widetilde{Z}_i,$$

$$\widetilde{Z}_i = \underline{\alpha}_i^{-1}[P_i + H(\Pi_i - \Gamma_i) + K(\bar{\Sigma} - \Sigma)],$$

with

$$\Omega_{1i} = \begin{bmatrix} \Omega_{11i} & \Omega_{12i} & \Omega_{13i} & \Omega_{14i} & \Omega_{15i} & \Omega_{16i} \\ * & \Omega_{22i} & 0 & \Omega_{24i} & 0 & 0 \\ * & * & \Omega_{33i} & \Omega_{34i} & 0 & \Omega_{36i} \\ * & * & * & \Omega_{44i} & 0 & 0 \\ * & * & * & * & \Omega_{55i} & 0 \\ * & * & * & * & * & \Omega_{66i} \end{bmatrix},$$

$$\Omega_{2i} = [\, 0 \;\; 0 \;\; A_i \;\; B_i \;\; 0 \;\; C_i \,]^T G_i^T,$$

$$\Omega_{3i} = -2\bar{\alpha}_i^{-2}G_i + \underline{\alpha}_i^{-2}\Big[\frac{1}{2}\bar{\tau}^2 S_4 + \bar{\tau}_i Q_{4i} + \bar{\tau}S_9\Big],$$

$$\Omega_{4i} = \begin{bmatrix} \Omega_{18i} & 0 & \Omega_{1ai} \\ 0 & \Omega_{29i} & 0 \\ \Omega_{38i} & 0 & \Omega_{3ai} \\ 0 & 0 & \Omega_{4ai} \\ 0 & 0 & 0 \\ 0 & 0 & \Omega_{6ai} \end{bmatrix}, \quad \Omega_{7i} = \begin{bmatrix} \Omega_{1bi} & 0 \\ \Omega_{2bi} & \Omega_{2ci} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Omega_{5i} = G_i \begin{bmatrix} D_i D_i^T & 0 & -I \end{bmatrix},$$

$$\Omega_{6i} = \begin{bmatrix} \Omega_{88i} & 0 & \Omega_{8ai} \\ * & \Omega_{99i} & 0 \\ * & * & \Omega_{aai} \end{bmatrix}, \quad \Omega_{8i} = \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{9ci} \\ 0 & 0 \end{bmatrix},$$

$$\Omega_{9i} = \text{diag}\{\Omega_{bbi},\ \Omega_{cci}\},$$

$$\mathbb{I} = \begin{bmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & I \end{bmatrix},$$

$$\Im = \begin{bmatrix} -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix},$$

and

$$\begin{aligned} \Omega_{11i} = & -2P_i\Gamma_i + Q_{1i} + Q_{3i} - \frac{1}{\bar{\tau}_i}Q_{4i} \\ & + \sum_{j=1}^{N} \pi_{ij}\rho_{ij}^{-1}P_j + \bar{\tau}(S_1 + S_3) - U_iF_1 - W_iF_3 \\ & + \sum_{j=1}^{N} \bar{\pi}_{ij}\underline{\alpha}_i^{-1}\left[P_i + 2H(\Pi_i - \Gamma_i) + 2K(\bar{\Sigma} - \Sigma)\right], \end{aligned}$$

$$\Omega_{12i} = \frac{1}{\bar{\tau}_i}Q_{4i},$$

$$\Omega_{22i} = -(1 - \tau_i')Q_{1i} + \sum_{j=1}^{N} \bar{\pi}_{ij}\bar{\tau}_jQ_{1j} - \frac{2}{\bar{\tau}_i}Q_{4i} - T_iF_1,$$

$$\Omega_{13i} = P_iA_i + U_iF_2 - \Gamma_iHA_i - \Sigma KA_i,$$

$$\Omega_{33i} = Q_{2i} - U_i + KA_i + A_i^TK + \bar{\tau}S_2,$$

$$\Omega_{14i} = P_iB_i - \Gamma_iHB_i - \Sigma KB_i,$$

$$\Omega_{24i} = -T_iF_2, \quad \Omega_{34i} = KB_i,$$

$$\Omega_{44i} = -T_i - (1 - \tau_i')Q_{2i} + \sum_{j=1}^{N} \bar{\pi}_{ij}\bar{\tau}_jQ_{2j},$$

$$\Omega_{15i} = W_iF_4, \ \Omega_{55i} = -W_i + \bar{\upsilon}_iR_i + \frac{\bar{\upsilon}^2}{2}S_5 + \bar{\upsilon}S_7,$$

$$\Omega_{16i} = P_iC_i - \Gamma_iHC_i - \Sigma KC_i,$$

$$\Omega_{36i} = KC_i, \quad \Omega_{66i} = -\frac{1 - \upsilon_i'}{\bar{\upsilon}_i}R_i,$$

$$\Omega_{18i} = (P_i - \Sigma K + \Gamma_iH)D_iD_i^T + \bar{M}_i^T,$$

$$\Omega_{38i} = KD_iD_i^T, \ \Omega_{88i} = -2M_i, \ \Omega_{29i} = \frac{1}{\bar{\tau}_i}Q_{4i},$$

$$\Omega_{99i} = -\frac{1}{\bar{\tau}_i} Q_{4i} - Q_{3i} + \sum_{j=1}^{N} \bar{\pi}_{ij} \bar{\tau}_j Q_{3j},$$
$$\Omega_{1ai} = \Gamma_i H + \Sigma K, \quad \Omega_{3ai} = A_i^T H - K,$$
$$\Omega_{4ai} = B_i^T H, \ \Omega_{6ai} = C_i^T H, \ \Omega_{8ai} = D_i D_i^T H,$$
$$\Omega_{aai} = -2H, \quad \Omega_{1bi} = \frac{1}{\bar{\tau}_i} Q_{4i},$$
$$\Omega_{2bi} = -\frac{1}{\bar{\tau}_i} Q_{4i}, \quad \Omega_{bbi} = -\frac{1}{\bar{\tau}_i} Q_{4i} - Z_i,$$

$$\Omega_{2ci} = \frac{1}{\bar{\tau}_i} Q_{4i}, \quad \Omega_{9ci} = -\frac{1}{\bar{\tau}_i} Q_{4i}, \quad \Omega_{cci} = -\frac{1}{\bar{\tau}_i} Q_{4i} - Z_i,$$

and $\bar{\pi}_{ij} = \max\{\pi_{ij}, 0\}, \ \bar{M}_i = M_i X_i,$

$$\rho_{ij} = \begin{cases} \bar{\alpha}_i, & j = i \\ \underline{\alpha}_i, & j \neq i \end{cases}.$$

Furthermore, the feedback stabilizing control law is defined by $u_i(t) = D_i^T X_i x(t)$.

Proof From Assumption 11.3, we know that the amplification function $\alpha_i(x(t))$ is nonlinear and satisfies $\alpha_i(x(t))\alpha_i(x(t)) \leq \bar{\alpha}_i^2 I$. Following the way in [15], pre- and postmultiplying the left-hand sides of inequalities (11.10) and (11.11) by $\text{diag}\{I \ \ I \ \ I \ \ I \ \ I \ \ I \ \ \alpha_i(x(t)) \ \ I \ \ I \ \ I \ \ I \ \ I\}$, respectively, it follows that

$$\begin{bmatrix} \overline{\Omega}_i + \widetilde{\Omega}_i & \mathcal{E}^T \\ \mathcal{E} & \overline{Z}_i \end{bmatrix} < 0, \tag{11.12}$$
$$\begin{bmatrix} \overline{\Omega}_i + \widehat{\Omega}_i & \mathcal{E}^T \\ \mathcal{E} & \overline{Z}_i \end{bmatrix} < 0, \tag{11.13}$$

where

$$\overline{\Omega}_i \doteq \begin{bmatrix} \Omega_{1i} & \overline{\Omega}_{2i} & \Omega_{4i} \\ * & \overline{\Omega}_{3i} & \overline{\Omega}_{5i} \\ * & * & \Omega_{6i} \end{bmatrix},$$

with

$$\overline{\Omega}_{2i} = [\, 0 \ \ 0 \ \ A_i \ \ B_i \ \ 0 \ \ C_i \,]^T \alpha_i(x(t)) G_i^T,$$
$$\overline{\Omega}_{3i} = \bar{\tau}_i Q_{4i} - (G_i + G_i^T) + \frac{\bar{\tau}^2}{2} S_4 + \bar{\tau} S_9,$$
$$\overline{\Omega}_{5i} = G_i \alpha_i(x(t)) \begin{bmatrix} D_i D_i^T & 0 & -I & 0 & 0 \end{bmatrix}.$$

For any $j = 1, 2, \ldots, n$, from Assumption 11.5 we obtain that

$$\Big(f(x_j(t)) - \varrho_j^+ x_j(t)\Big)\Big(f(x_j(t)) - \varrho_j^- x_j(t)\Big) \le 0,$$
$$\Big(g(x_j(t)) - \psi_j^+ x_j(t)\Big)\Big(g(x_j(t)) - \psi_j^- x_j(t)\Big) \le 0.$$

Therefore, the following matrix inequalities hold for any positive diagonal matrices U_i, T_i, W_i,

$$\left\langle \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}, \begin{bmatrix} -U_iF_1 & U_iF_2 \\ U_iF_2 & -U_i \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \right\rangle \ge 0, \tag{11.14}$$
$$\left\langle \begin{bmatrix} x(t-\tau_i(t)) \\ f(x(t-\tau_i(t))) \end{bmatrix}, \begin{bmatrix} -T_iF_1 & T_iF_2 \\ T_iF_2 & -T_i \end{bmatrix} \begin{bmatrix} x(t-\tau_i(t)) \\ f(x(t-\tau_i(t))) \end{bmatrix} \right\rangle \ge 0, \tag{11.15}$$
$$\left\langle \begin{bmatrix} x(t) \\ g(x(t)) \end{bmatrix}, \begin{bmatrix} -W_iF_3 & W_iF_4 \\ W_iF_4 & -W_i \end{bmatrix} \begin{bmatrix} x(t) \\ g(x(t)) \end{bmatrix} \right\rangle \ge 0. \tag{11.16}$$

Denoting

$$\begin{aligned} \iota_i(t) = &- \beta_i(x(t)) + A_i f(x(t)) + B_i f(x(t-\tau_i(t))) \\ &+ C_i \int_{t-\upsilon_i(t)}^{t} g(x(s))\mathrm{d}s + D_i u_i(t), \end{aligned}$$

$$\begin{aligned} \vartheta_i(t) &= \alpha_i(x(t))\iota_i(t), \\ \sigma_i(t) &= E_{1i}x(t) + E_{2i}x(t-\tau_i(t)) + E_{3i}f(x(t)) \\ &\quad + E_{4i}f(x(t-\tau_i(t))) + E_{5i}\int_{t-\upsilon_i(t)}^{t} g(x(s))\mathrm{d}s, \end{aligned}$$

then system (11.1) can be rewritten as

$$\mathrm{d}x(t) = \vartheta_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}\omega(t). \tag{11.17}$$

Define the following Lyapunov–Krasovskii functional:

$$V(x_t, t, i) = \sum_{l=1}^{6} V_{li}(x_t, t), \tag{11.18}$$

where

$$V_{1i}(x_t, t) = \sum_{j=1}^{n} 2p_{ji} \int_0^{x_j(t)} \frac{s}{\alpha_{ji}(s)}\mathrm{d}s$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} 2h_j \int_0^{x_j(t)} \frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s \\
&+\sum_{j=1}^{n} 2k_j \int_0^{x_j(t)} \frac{f_j(s)-\varrho_j^{-}s}{\alpha_{ji}(s)}\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
V_{2i}(x_t,t) = &\int_{t-\tau_i(t)}^{t} \langle x(s),\, Q_{1i}x(s)\rangle \,\mathrm{d}s \\
&+\int_{t-\tau_i(t)}^{t} \langle f(x(s)),\, Q_{2i}f(x(s))\rangle \,\mathrm{d}s \\
&+\int_{t-\bar{\tau}_i}^{t} \langle x(s),\, Q_{3i}x(s)\rangle \,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
V_{3i}(x_t,t) = &\int_{-\bar{\tau}_i}^{0}\int_{t+\theta}^{t} \langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle \,\mathrm{d}s\mathrm{d}\theta \\
&+\int_{-\upsilon_i(t)}^{0}\int_{t+\theta}^{t} \langle g(x(s)),\, R_i g(x(s))\rangle \,\mathrm{d}s\mathrm{d}\theta \\
&+\int_{-\bar{\tau}_i}^{0}\int_{t+\theta}^{t} \langle \sigma_i(s),\, Z_i\sigma_i(s)\rangle \,\mathrm{d}s\mathrm{d}\theta,
\end{aligned}
$$

$$
\begin{aligned}
V_{4i}(x_t,t) = &\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} \big\{ \langle x(s),(S_1+S_3)x(s)\rangle \\
&\quad + \langle f(x(s)),\, S_2 f(x(s))\rangle \big\}\mathrm{d}s\mathrm{d}\theta, \\
V_{5i}(x_t,t) = &\int_{-\bar{\tau}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \langle \vartheta_i(s),\, S_4\vartheta_i(s)\rangle \,\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta \\
&+\int_{-\bar{\upsilon}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \langle g(x(s)),\, S_5 g(x(s))\rangle \,\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta \\
&+\int_{-\bar{\tau}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \langle \sigma_i(s),\, S_6\sigma_i(s)\rangle \,\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta, \\
V_{6i}(x_t,t) = &\int_{-\bar{\upsilon}}^{0}\int_{t+\theta}^{t} \langle g(x(s)),\, S_7 g(x(s))\rangle \,\mathrm{d}s\mathrm{d}\theta \\
&+\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} \big\{ \langle \sigma_i(s),\, S_8\sigma_i(s)\rangle \\
&\quad + \langle \vartheta_i(s),\, S_9\vartheta_i(s)\rangle \big\}\mathrm{d}s\mathrm{d}\theta,
\end{aligned}
$$

with $P_i = \mathrm{diag}\{p_{1i}, p_{2i}, \ldots, p_{ni}\}$, $H = \mathrm{diag}\{h_1, h_2, \ldots, h_n\}$, $K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$.

For any $\eta(t) = i \in \wp$, it can be shown that

$$
\begin{aligned}
&\pounds\left\{\sum_{j=1}^{n} 2p_{ji}\int_0^{x_j(t)} \frac{s}{\alpha_{ji}(s)}\mathrm{d}s\right\} \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\sum_{j=1}^{n} 2\left(\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]p_{jl} + p_{ji}\right)\right. \\
&\quad\times\int_0^{x_j(t+\Delta)} \frac{s}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s) + \alpha_{ji}(s)}\mathrm{d}s \\
&\quad\left.-\sum_{j=1}^{n} 2p_{ji}\int_0^{x_j(t)} \frac{s}{\alpha_{ji}(s)}\mathrm{d}s\right\} \\
&= \sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n} 2p_{jl}\int_0^{x_j(t)} \frac{s}{\alpha_{ji}(s)}\mathrm{d}s \\
&\quad+\lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\sum_{j=1}^{n} 2p_{ji}\left[-\int_0^{x_j(t)} \frac{s}{\alpha_{ji}(s)}\mathrm{d}s\right.\right. \\
&\quad\left.\left.+\int_0^{x_j(t+\Delta)} \frac{s}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s) + \alpha_{ji}(s)}\mathrm{d}s\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n} 2\int_0^{x_j(t)} \frac{s\left[p_{jl}\alpha_{ji}(s) - p_{ji}\alpha_{jl}(s)\right]}{\alpha_{ji}^2(s)}\mathrm{d}s \\
&\quad+2\langle\iota_i(t), P_i x(t)\rangle + \mathrm{trace}\left\langle\sigma_i(t), \alpha_i^{-1}(x(t))P_i\sigma_i(t)\right\rangle, \qquad (11.19)
\end{aligned}
$$

$$
\begin{aligned}
&\pounds\left\{\sum_{j=1}^{n} 2h_j\int_0^{x_j(t)} \frac{\beta_{ji}(s) - \underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s\right\} \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\sum_{j=1}^{n} 2h_j\left[-\int_0^{x_j(t)} \frac{\beta_{ji}(s) - \underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s\right.\right. \\
&\quad+\int_0^{x_j(t+\Delta)} \frac{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\beta_{jl}(s) + \beta_{ji}(s)}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s) + \alpha_{ji}(s)}\mathrm{d}s \\
&\quad\left.\left.-\int_0^{x_j(t+\Delta)} \frac{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\underline{\beta}_{jl}s + \underline{\beta}_{ji}s}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s) + \alpha_{ji}(s)}\mathrm{d}s\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\Delta\to 0^+} \frac{1}{\Delta}\mathbb{E}\left\{\sum_{j=1}^{n} 2h_j\Big[-\int_0^{x_j(t)} \frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s \right.\\
&\quad + \int_0^{x_j(t+\Delta)} \frac{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\beta_{jl}(s)}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s)+\alpha_{ji}(s)}\mathrm{d}s\\
&\quad - \int_0^{x_j(t+\Delta)} \frac{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\underline{\beta}_{jl}s}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s)+\alpha_{ji}(s)}\mathrm{d}s\\
&\quad \left. + \int_0^{x_j(t+\Delta)} \frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\sum_{l=1}^{N}[\pi_{il}\Delta + o(\Delta)]\alpha_{jl}(s)+\alpha_{ji}(s)}\mathrm{d}s\Big]\right\}\\
&= \sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2h_j\int_0^{x_j(t)} \frac{\big[\beta_{ji}(s)-\underline{\beta}_{ji}s\big]\big[\alpha_{ji}(s)-\alpha_{jl}(s)\big]}{\alpha_{ji}^2(s)}\mathrm{d}s\\
&\quad + 2\,\langle \iota_i(t), H\,(\beta_i(x(t))-\Gamma_i x(t))\rangle\\
&\quad + \mathrm{trace}\Big\langle \sigma_i(t), \alpha_i^{-1}(x(t))H(\Pi_i-\Gamma_i)\sigma_i(t)\Big\rangle,
\end{aligned}
\tag{11.20}
$$

$$
\begin{aligned}
&\pounds\left\{\sum_{j=1}^{n}2k_j\int_0^{x_j(t)}\frac{f_j(s)-\varrho_j^- s}{\alpha_{ji}(s)}\mathrm{d}s\right\}\\
&= \sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2k_j\int_0^{x_j(t)} \frac{\big[f_j(s)-\varrho_j^- s\big]\big[\alpha_{ji}(s)-\alpha_{jl}(s)\big]}{\alpha_{ji}^2(s)}\mathrm{d}s\\
&\quad + 2\,\langle \iota_i(t), K\,(f(x(t))-\Sigma x(t))\rangle\\
&\quad + \mathrm{trace}\Big\langle \sigma_i(t), \alpha_i^{-1}(x(t))K(\bar{\Sigma}-\Sigma)\sigma_i(t)\Big\rangle,
\end{aligned}
\tag{11.21}
$$

where $\alpha_i^{-1}(x(t)) = \mathrm{diag}\left\{\alpha_{1i}^{-1}(x_1(t)),\ \ldots,\ \alpha_{ni}^{-1}(x_n(t))\right\}$.

According to the definition of ρ_{il} and Assumptions 11.3–11.5 we have that

$$
\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2p_{jl}\int_0^{x_j(t)}\frac{s}{\alpha_{ji}(s)}\mathrm{d}s \le \Big\langle x(t), \sum_{l=1}^{N}\pi_{il}\rho_{il}^{-1}P_l x(t)\Big\rangle,
\tag{11.22}
$$

$$
\begin{aligned}
&-\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2p_{ji}\int_0^{x_j(t)}\frac{s\alpha_{jl}(s)}{\alpha_{ji}^2(s)}\mathrm{d}s\\
&\le -\pi_{ii}\sum_{j=1}^{n}2p_{ji}\int_0^{x_j(t)}\frac{s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le \Big\langle x(t), \sum_{l=1}^{N}\bar{\pi}_{il}\underline{\alpha}_i^{-1}P_i x(t)\Big\rangle,
\end{aligned}
\tag{11.23}
$$

$$
\begin{aligned}
&\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2h_j\int_0^{x_j(t)}\frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\sum_{l=1}^{N}\bar{\pi}_{il}\sum_{j=1}^{n}2h_j\int_0^{x_j(t)}\frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\Big\langle x(t),\,H\sum_{l=1}^{N}\bar{\pi}_{il}\underline{\alpha}_i^{-1}(\Pi_i-\Gamma_i)x(t)\Big\rangle, \qquad (11.24)\\
&-\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2h_j\int_0^{x_j(t)}\frac{\left[\beta_{ji}(s)-\underline{\beta}_{ji}s\right]\alpha_{jl}(s)}{\alpha_{ji}^2(s)}\mathrm{d}s\\
&\le-\pi_{ii}\sum_{j=1}^{n}2h_j\int_0^{x_j(t)}\frac{\beta_{ji}(s)-\underline{\beta}_{ji}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\Big\langle x(t),\,H\sum_{l=1}^{N}\bar{\pi}_{il}\underline{\alpha}_i^{-1}(\Pi_i-\Gamma_i)x(t)\Big\rangle, \qquad (11.25)\\
&\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2k_j\int_0^{x_j(t)}\frac{f_j(s)-\varrho_j^{-}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\sum_{l=1}^{N}\bar{\pi}_{il}\sum_{j=1}^{n}2k_j\int_0^{x_j(t)}\frac{f_j(s)-\varrho_j^{-}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\Big\langle x(t),\,K\sum_{l=1}^{N}\bar{\pi}_{il}\underline{\alpha}_i^{-1}(\bar{\Sigma}-\Sigma)x(t)\Big\rangle, \qquad (11.26)\\
&-\sum_{l=1}^{N}\pi_{il}\sum_{j=1}^{n}2k_j\int_0^{x_j(t)}\frac{\left[f_j(s)-\varrho_j^{-}s\right]\alpha_{jl}(s)}{\alpha_{ji}^2(s)}\mathrm{d}s\\
&\le-\pi_{ii}\sum_{j=1}^{n}2k_j\int_0^{x_j(t)}\frac{f_j(s)-\varrho_j^{-}s}{\alpha_{ji}(s)}\mathrm{d}s\\
&\le\Big\langle x(t),\,K\sum_{l=1}^{N}\bar{\pi}_{il}\underline{\alpha}_i^{-1}(\bar{\Sigma}-\Sigma)x(t)\Big\rangle. \qquad (11.27)
\end{aligned}
$$

Using the well-known Itô's differential formula [41, 44], we obtain

$$
\begin{aligned}
\pounds V_{1i}(x_t,t)\le\,&2\big\langle\iota_i(t),\;P_ix(t)+H\big[\beta_i(x(t))-\Gamma_ix(t)\big]+K\,(f(x(t))-\Sigma x(t))\big\rangle\\
&+\mathrm{trace}\big\langle\sigma_i(t),\;\alpha_i^{-1}(x(t))\big[P_i+H(\Pi_i-\Gamma_i)+K(\bar{\Sigma}-\Sigma)\big]\sigma_i(t)\big\rangle
\end{aligned}
$$

$$+\sum_{l=1}^{N}\pi_{il}\rho_{il}^{-1}\big\langle x(t),\, P_l x(t)\big\rangle$$
$$+\sum_{l=1}^{N}\bar{\pi}_{il}\Big\langle x(t),\ \underline{\alpha}_i^{-1}\big[P_i+2H\ \times(\Pi_i-\Gamma_i)+2K(\bar{\Sigma}-\Sigma)\big]x(t)\Big\rangle, \tag{11.28}$$

$$\begin{aligned}
\pounds V_{2i}(x_t,t) &= \langle x(t),\, Q_{1i}x(t)\rangle + \langle f(x(t)),\, Q_{2i}f(x(t))\rangle\\
&\quad -(1-\dot{\tau}_i(t))\big\{\,\langle x(t-\tau_i(t)),\, Q_{1i}x(t-\tau_i(t))\rangle\\
&\quad +\langle f(x(t-\tau_i(t))),\, Q_{2i}f(x(t-\tau_i(t)))\rangle\,\big\}\\
&\quad +\langle x(t),\, Q_{3i}x(t)\rangle - \langle x(t-\bar{\tau}_i),\, Q_{3i}x(t-\bar{\tau}_i)\rangle\\
&\quad +\sum_{j=1}^{N}\pi_{ij}\Big[\int_{t-\tau_i(t)}^{t}\big\langle x(s),\, Q_{1j}x(s)\big\rangle\,\mathrm{d}s\\
&\quad +\int_{t-\tau_i(t)}^{t}\big\langle f(x(s)),\, Q_{2j}f(x(s))\big\rangle\,\mathrm{d}s+\int_{t-\bar{\tau}_i}^{t}\big\langle x(s),\, Q_{3j}x(s)\big\rangle\,\mathrm{d}s\Big]\\
&\quad +\sum_{j=1}^{N}\pi_{ij}\tau_j(t)\big[\,\langle x(t-\tau_i(t)),\, Q_{1i}x(t-\tau_i(t))\rangle\\
&\quad +\langle f(x(t-\tau_i(t))),\, Q_{2i}f(x(t-\tau_i(t)))\rangle\,\big]\\
&\quad +\sum_{j=1}^{N}\pi_{ij}\bar{\tau}_j\,\langle x(t-\bar{\tau}_i),\, Q_{3i}x(t-\bar{\tau}_i)\rangle,
\end{aligned}\tag{11.29}$$

$$\begin{aligned}
\pounds V_{3i}(x_t,t) &= \bar{\tau}_i\,\langle \vartheta_i(t),\, Q_{4i}\vartheta_i(t)\rangle - \int_{t-\bar{\tau}_i}^{t}\langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle\,\mathrm{d}s\\
&\quad +\upsilon_i(t)\,\langle g(x(t)),\, R_i g(x(t))\rangle - \int_{t-\upsilon_i(t)}^{t}\langle g(x(t)),\, R_i g(x(t))\rangle\,\mathrm{d}s\\
&\quad +\bar{\tau}_i\,\langle \sigma_i(t),\, Z_i\sigma_i(t)\rangle - \int_{t-\bar{\tau}_i}^{t}\langle \sigma_i(t),\, Z_i\sigma_i(t)\rangle\,\mathrm{d}s\\
&\quad +\sum_{j=1}^{N}\pi_{ij}\Big[\int_{-\bar{\tau}_i}^{0}\int_{t+\theta}^{t}\big\langle \vartheta_i(s),\, Q_{4j}\vartheta_j(s)\big\rangle\,\mathrm{d}s\mathrm{d}\theta\\
&\quad +\int_{-\upsilon_i(t)}^{0}\int_{t+\theta}^{t}\big\langle g(x(s)),\, R_j g(x(s))\big\rangle\,\mathrm{d}s\mathrm{d}\theta\\
&\quad +\int_{-\bar{\tau}_i}^{0}\int_{t+\theta}^{t}\big\langle \sigma_i(s),\, Z_j\sigma_j(s)\big\rangle\,\mathrm{d}s\mathrm{d}\theta\Big]\\
&\quad +\sum_{j=1}^{N}\pi_{ij}\Big[\bar{\tau}_j\int_{t-\bar{\tau}_i}^{t}\langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle\,\mathrm{d}s
\end{aligned}$$

$$+\upsilon_j(t)\int_{t-\upsilon_i(t)}^{t}\Big\langle g(x(s)),\,R_j g(x(s))\Big\rangle\,\mathrm{d}s$$
$$+\bar{\tau}_j\int_{t-\bar{\tau}_i}^{t}\Big\langle \sigma_i(s),\,Z_j\sigma_i(s)\Big\rangle\,\mathrm{d}s\Bigg], \tag{11.30}$$

$$\begin{aligned}\pounds V_{4i}(x_t,t) &= \bar{\tau}\,\langle x(t),\,(S_1+S_3)x(t)\rangle + \bar{\tau}\,\langle f(x(t)),\,S_2 f(x(t))\rangle \\ &\quad - \int_{t-\bar{\tau}}^{t}\{\langle x(s),\,(S_1+S_3)x(s)\rangle\ + \langle f(x(s)),\,S_2 f(x(s))\rangle\}\,\mathrm{d}s,\end{aligned} \tag{11.31}$$

$$\begin{aligned}\pounds V_{5i}(x_t,t) &= \frac{\bar{\tau}^2}{2}\Big\{\,\langle \vartheta_i(t),\,S_4\vartheta_i(t)\rangle + \langle \sigma_i(t),\,S_6\sigma_i(t)\rangle\,\Big\} \\ &\quad - \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t}\langle \vartheta_i(s),\,S_4\vartheta_i(s)\rangle\,\mathrm{d}s\mathrm{d}\theta \\ &\quad + \frac{\bar{\upsilon}^2}{2}\,\langle g(x(t)),\,S_5 g(x(t))\rangle - \int_{-\bar{\upsilon}}^{0}\int_{t+\theta}^{t}\langle g(x(s)),\,S_5 g(x(s))\rangle\,\mathrm{d}s\mathrm{d}\theta \\ &\quad - \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t}\langle \sigma_i(s),\,S_6\sigma_i(s)\rangle\,\mathrm{d}s\mathrm{d}\theta,\end{aligned} \tag{11.32}$$

$$\begin{aligned}\pounds V_{6i}(x_t,t) &= \bar{\upsilon}\,\langle g(x(t)),\,S_7 g(x(t))\rangle - \int_{t-\bar{\upsilon}}^{t}\langle g(x(s)),\,S_7 g(x(s))\rangle\,\mathrm{d}s \\ &\quad + \bar{\tau}\,\langle \sigma_i(t),\,S_8\sigma_i(t)\rangle + \bar{\tau}\,\langle \vartheta_i(t),\,S_9\vartheta_i(t)\rangle \\ &\quad - \int_{t-\bar{\tau}}^{t}\langle \sigma_i(s),\,S_8\sigma_i(s)\rangle\,\mathrm{d}s - \int_{t-\bar{\tau}}^{t}\langle \vartheta_i(s),\,S_9\vartheta_i(s)\rangle\,\mathrm{d}s.\end{aligned} \tag{11.33}$$

Based on Assumption 11.4, we obtain that

$$-x^T(t)P_i\beta_i(x(t)) \le -x^T(t)P_i\Gamma_i x(t). \tag{11.34}$$

From Lemma 11.7, it follows that

$$\begin{aligned}&-\int_{t-\upsilon_i(t)}^{t}\langle g(x(s)),\,R_i g(x(s))\rangle\,\mathrm{d}s \\ &\le -\frac{1}{\bar{\upsilon}_i}\Big\langle \int_{t-\upsilon_i(t)}^{t} g(x(s))\mathrm{d}s,\,R_i\int_{t-\upsilon_i(t)}^{t} g(x(s))\mathrm{d}s\Big\rangle.\end{aligned} \tag{11.35}$$

For simplicity, we denote

$$\varsigma_{1i}(t) = \int_{t-\bar{\tau}}^{t-\tau_i(t)}\vartheta_i(s)\mathrm{d}s,\quad \varsigma_{2i}(t) = \int_{t-\tau_i(t)}^{t}\vartheta_i(s)\mathrm{d}s.$$

When $0 < \tau_i(t) < \bar{\tau}_i$, from Lemma 11.8 with $\nu = 1$, $\mu = 3$, one can obtain that

$$\begin{aligned}
&-\int_{t-\bar{\tau}_i}^{t} \langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle\, \mathrm{d}s \\
&= -\int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle\, \mathrm{d}s \\
&\quad -\int_{t-\tau_i(t)}^{t} \langle \vartheta_i(s),\, Q_{4i}\vartheta_i(s)\rangle\, \mathrm{d}s \\
&\le -\frac{1}{\bar{\tau}_i - \tau_i(t)} \langle \varsigma_{1i}(t),\, Q_{4i}\varsigma_{1i}(t)\rangle - \frac{1}{\tau_i(t)} \langle \varsigma_{2i}(t),\, Q_{4i}\varsigma_{2i}(t)\rangle \\
&\le \frac{1}{\bar{\tau}_i} \max\Big\{ -\langle \varsigma_{1i}(t),\, Q_{4i}\varsigma_{1i}(t)\rangle - 3\,\langle \varsigma_{2i}(t),\, Q_{4i}\varsigma_{2i}(t)\rangle\,, \\
&\qquad -3\,\langle \varsigma_{1i}(t),\, Q_{4i}\varsigma_{1i}(t)\rangle - \langle \varsigma_{2i}(t),\, Q_{4i}\varsigma_{2i}(t)\rangle \Big\}. \qquad (11.36)
\end{aligned}$$

Obviously, from Lemma 11.7, inequality (11.36) holds when $\tau_i(t) = 0$ or $\tau_i(t) = \bar{\tau}_i$. Therefore, inequality (11.36) holds for any t with $0 \le \tau_i(t) \le \bar{\tau}_i$.

On the other hand, by the *Leibniz-Newton formula*, we get

$$x(t) - x(t-\tau_i(t)) - \int_{t-\tau_i(t)}^{t} \vartheta_i(s)\mathrm{d}s - \int_{t-\tau_i(t)}^{t} \sigma_i(s)\mathrm{d}\omega(s) = 0, \qquad (11.37)$$

$$x(t-\tau_i(t)) - x(t-\bar{\tau}_i) - \int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \vartheta_i(s)\mathrm{d}s - \int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \sigma_i(s)\mathrm{d}\omega(s) = 0. \qquad (11.38)$$

It is easy to see that the following equality holds for any positive diagonal matrices G_i with compatible dimensions

$$0 = -2\,\langle G_i\vartheta_i(t), \vartheta_i(t) - \alpha_i(x(t))\iota_i(t)\rangle\,. \qquad (11.39)$$

Considering that the feedback stabilizing control law being defined by $u_i(t) = D_i^T X_i x(t)$, if we denote $y_i(t) = X_i x(t)$, then for any symmetric nonnegative definite matrices M_i, we have

$$0 = -2\,\langle M_i y_i(t), y_i(t) - X_i x(t)\rangle \;\; (i = 1, 2, \ldots, N). \qquad (11.40)$$

Noticing that the following equality holds

$$\begin{aligned}
-\int_{t-\bar{\tau}_i}^{t} \langle \sigma_i(s),\, Z_i\sigma_i(s)\rangle\, \mathrm{d}s &= -\int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \langle \sigma_i(s),\, Z_i\sigma_i(s)\rangle\, \mathrm{d}s \\
&\quad -\int_{t-\tau_i(t)}^{t} \langle \sigma_i(s),\, Z_i\sigma_i(s)\rangle\, \mathrm{d}s. \qquad (11.41)
\end{aligned}$$

From [14], we have

$$\mathbb{E}\left\{\int_{t-\tau_i(t)}^{t}\langle\sigma_i(s), Z_i\sigma_i(s)\rangle\,\mathrm{d}s\right\}$$
$$=\mathbb{E}\left\langle\int_{t-\tau_i(t)}^{t}\sigma_i(s)\mathrm{d}\omega(s),\, Z_i\int_{t-\tau_i(t)}^{t}\sigma_i(s)\mathrm{d}\omega(s)\right\rangle, \tag{11.42}$$
$$\mathbb{E}\left\{\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\langle\sigma_i(s), Z_i\sigma_i(s)\rangle\,\mathrm{d}s\right\}$$
$$=\mathbb{E}\left\langle\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\sigma_i(s)\mathrm{d}\omega(s),\, Z_i\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\sigma_i(s)\mathrm{d}\omega(s)\right\rangle. \tag{11.43}$$

By (11.4)–(11.9) and (11.14)–(11.43), we obtain

$$\frac{\mathrm{d}\mathbb{E}[V(x(t),t,i)]}{\mathrm{d}t}$$
$$\leq \mathbb{E}\max\left\{\left\langle\zeta_i(t), \left(\overline{\Omega}_i+\widetilde{\Omega}_i+\mathcal{E}^T\overline{Z}_i\mathcal{E}\right)\zeta_i(t)\right\rangle, \left\langle\zeta_i(t), \left(\overline{\Omega}_i+\widehat{\Omega}_i+\mathcal{E}^T\overline{Z}_i\mathcal{E}\right)\zeta_i(t)\right\rangle\right\}, \tag{11.44}$$

where

$$\zeta_i(t)=\mathrm{col}\Big\{x(t)\quad x(t-\tau_i(t))\quad f(x(t))$$
$$f(x(t-\tau_i(t)))\quad g(x(t))\quad \int_{t-\upsilon_i(t)}^{t}g(x(s))\mathrm{d}s$$
$$\vartheta_i(t)\quad y_i(t)\quad x(t-\bar{\tau}_i)\quad \beta_i(x(t))$$
$$\int_{t-\tau_i(t)}^{t}\sigma_i(s)\mathrm{d}\omega(s)\quad \int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\sigma_i(s)\mathrm{d}\omega(s)\Big\}.$$

Next, we prove that the *error system* is exponentially stable in mean square. For convenience, we define

$$\lambda_p=\min_{i\in\wp}\{\lambda_{\min}(P_i)\},$$

$$\lambda_M=\min_{i\in\wp}\{\lambda_{\min}(-\overline{\Omega}_i-\widetilde{\Omega}_i-\mathcal{E}^T\overline{Z}_i\mathcal{E}),\ \lambda_{\min}(-\overline{\Omega}_i-\widehat{\Omega}_i-\mathcal{E}^T\overline{Z}_i\mathcal{E})\}.$$

From (11.12) and (11.13) and the well-known *Schur complements*, it can be easily seen that $\lambda_M>0$. Furthermore, from (11.44) we have that

$$\frac{\mathrm{d}\mathbb{E}[V(x(t),t,i)]}{\mathrm{d}t}\leq-\lambda_M\mathbb{E}||\zeta_i(t)||^2\leq-\lambda_M\mathbb{E}||x(t)||^2. \tag{11.45}$$

Similar to [45], from (11.18) and the definition of $\vartheta_i(t)$, there exist positive scalars ε_1 and ε_2 such that

$$\mathbb{E}[V(x(t),t,i)] \le \varepsilon_1 \mathbb{E}||\zeta(t)||^2 + \varepsilon_2 \mathbb{E}\int_{t-\bar{\tau}_i}^{t} ||x(s)||^2 \mathrm{d}s.$$

To prove the mean square exponential stability, we modify the *Lyapunov function* candidate (11.18) as $\bar{V}(x(t),t,i) = e^{rt}V(x(t),t,i)$, where r is chosen such that $r(\varepsilon_1 + \bar{\tau}\varepsilon_2 e^{r\bar{\tau}}) \le \lambda_M$.

Then, we have

$$\mathbb{E}[\bar{V}(x(t),t,i)] \ge \lambda_p \mathbb{E}||x(t)||^2.$$

Furthermore, by the Dynkin's formula [14], for any $\eta(t) = i \in \wp, t > 0$, we obtain that

$$\begin{aligned}\mathbb{E}[\bar{V}(x(t),t,i)] &= \mathbb{E}[\bar{V}(x(0),0,\eta(0))] + \mathbb{E}\int_0^t e^{rs}\left[r\bar{V}(x(s),s,i) + \pounds\bar{V}(x(s),s,i)\right]\mathrm{d}s \\ &\le (\varepsilon_1 + \bar{\tau}\varepsilon_2) \sup_{-\varpi \le s \le 0} \mathbb{E}||x(t)||^2 + r\varepsilon_1 \int_0^t e^{rs}\mathbb{E}||x(s)||^2 \mathrm{d}s \\ &\quad + r\varepsilon_2 \mathbb{E}\int_0^t e^{rs}\int_{s-\bar{\tau}}^{s} ||x(\theta)||^2 \mathrm{d}\theta \mathrm{d}s - \lambda_M \int_0^t e^{rs}||x(s)||^2 \mathrm{d}s.\end{aligned}$$

By changing the integration sequence, we get

$$\begin{aligned}&\int_0^t e^{rs}\int_{s-\bar{\tau}}^{s} ||x(\theta)||^2 \mathrm{d}\theta \mathrm{d}s \\ \le &\int_{-\bar{\tau}}^{0} e^{rs}\int_0^{\theta+\bar{\tau}} ||x(\theta)||^2 \mathrm{d}s\mathrm{d}\theta + \int_0^t e^{rs}\int_0^{\theta+\bar{\tau}} ||x(\theta)||^2 \mathrm{d}s\mathrm{d}\theta \\ \le &\int_{-\bar{\tau}}^{0} (\theta+\bar{\tau})e^{r(\theta+\bar{\tau})}||x(\theta)||^2 \mathrm{d}\theta + \bar{\tau}\int_0^t e^{r(\theta+\bar{\tau})}||x(\theta)||^2 \mathrm{d}\theta \\ \le &\bar{\tau}e^{r\bar{\tau}}\left\{\sup_{-\varpi \le s \le 0} ||x(s)||^2 + \int_0^t e^{r\theta}||x(\theta)||^2 \mathrm{d}\theta\right\}.\end{aligned}$$

Therefore we have

$$\mathbb{E}||x(t)||^2 \le \epsilon e^{-rt} \sup_{-\varpi \le s \le 0} ||x(s)||^2,$$

or

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(\mathbb{E}||x(t)||^2) \le -r,$$

where $\epsilon = \lambda_p^{-1}(\varepsilon_1 + \bar{\tau}\varepsilon_2 + r\bar{\tau}^2\varepsilon_2 e^{r\bar{\tau}})$.

Consequently, we prove that the error system (11.1) is exponentially stable in mean square. So the system (11.1) is *r-exponentially stabilizable* in the mean square. This ends the proof.

Remark 11.11 The Lyapunov functional (11.18) of this chapter fully uses the information about the amplification function and the mode-dependent time-varying delays, but [15, 20] only use the information about delays when constructing their Lyapunov functionals. Therefore the Lyapunov functional is more general than those in [15, 20], and the stability criteria in this chapter may be less conservativeness.

Remark 11.12 When one of the time-varying delays $\dot{\tau}_i(t)$ is not differentiable or unknown, the result in Theorem 11.10 is no longer applicable. For this case, by setting $Q_{1i} = Q_{2i} = 0$ in Theorem 11.10, one can obtain a result of the mean square exponential stability of system (11.1).

If there are no stochastic disturbances, that is $E_j(\eta_t) = 0$ $(j = 1, \ldots, 5)$, then the neural network (11.1) is simplified to

$$\begin{aligned}\dot{x}(t) = &-\alpha(x(t), \eta_t)\bigg[\beta(x(t), \eta_t) - A(\eta_t) f(x(t)) - B(\eta_t) f(x(t - \tau(t, \eta_t)))\\ &- C(\eta_t)\int_{t-\upsilon(t,\eta_t)}^{t} g(x(s))\mathrm{d}s - D(\eta_t)u(t, \eta_t)\bigg].\end{aligned} \tag{11.46}$$

For system (11.46), by setting $Z_i = S_6 = S_8 = 0$ in Theorem 11.10 and deleting $\int_{t-\tau_i(t)}^{t} \sigma_i(s)\mathrm{d}\omega(s)$, $\int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \sigma_i(s)\mathrm{d}\omega(s)$ from $\zeta_i(t)$, we can get the following result of the mean square exponential stability.

Corollary 11.13 *Given $r > 0$. For any given scalars $\bar{\tau}_i > 0$, $\bar{\upsilon}_i > 0$, $\upsilon_i' < 1$, considering the system (11.46) satisfying Assumptions 11.3–11.5 and $\dot{\tau}_i(t) \leq \tau_i'$, $\dot{\upsilon}_i(t) \leq \upsilon_i'$, the system (11.46) is globally r-exponentially stabilizable if there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, symmetric nonnegative definite matrices Q_{li}, R_i, M_i, S_l $(l = 1, \ldots, 4, l = 1, \ldots, 5, 7, 9)$, positive diagonal matrices G_i, U_i, T_i, W_i, H, K, and real matrices X_i such that (11.4), (11.5), (11.7), (11.9) and the following inequalities hold,*

$$\begin{bmatrix} \underline{\Omega}_i + \check{\Omega}_i & \mathfrak{E}^T \\ \mathfrak{E} & \widetilde{Z}_i \end{bmatrix} < 0, \tag{11.47}$$

$$\begin{bmatrix} \underline{\Omega}_i + \grave{\Omega}_i & \mathfrak{E}^T \\ \mathfrak{E} & \widetilde{Z}_i \end{bmatrix} < 0, \tag{11.48}$$

where

$$\underline{\Omega}_i = \begin{bmatrix} \Omega_{1i} & \Omega_{2i} & \Omega_{4i} \\ * & \Omega_{3i} & \Omega_{5i} \\ * & * & \Omega_{6i} \end{bmatrix},$$

$$\check{\Omega}_i = -\frac{2}{\bar{\tau}_i}\mathcal{I}^T Q_{4i}\mathcal{I}, \quad \grave{\Omega}_i = -\frac{2}{\bar{\tau}_i}\mathbb{J}^T Q_{4i}\mathbb{J},$$
$$\mathcal{I} = \begin{bmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix},$$
$$\mathbb{J} = \begin{bmatrix} -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathfrak{E} = [\ E_{1i} \ \ E_{2i} \ \ E_{3i} \ \ E_{4i} \ \ 0 \ \ E_{5i} \ \ 0 \ \ 0 \ \ 0 \ \ 0\ ],$$

$i = 1, \ldots, N$, and other parameters are defined in Theorem 11.10. Furthermore, the feedback stabilizing control law is defined by $u_i(t) = D_i^T X_i x(t)$.

11.4 Illustrative Examples

In this section, we provide three numerical examples to demonstrate the feasibility of our delay-dependent stabilization criteria.

Example 11.14 Consider system (11.1) with $N = 2$,

$$\begin{aligned}
\alpha_{ji}(x_j(t)) &= 0.4\sin(x_j(t)) + 0.8,\\
\beta_{ji}(x_j(t)) &= 7.5x_j(t) + 0.5\sin(x_j(t)),\\
f_j(x_j(t)) &= g_j(x_j(t)) = \tanh(x_j(t)), \quad j = 1, 2,\\
\tau_i(t) &= 0.2\sin(t) + 0.2,\\
\upsilon_i(t) &= 0.3\sin(t) + 0.3, \quad i = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \begin{bmatrix} 1 & -0.01 \\ 0.1 & 1.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.1 & -0.01 \\ 0.1 & 1.2 \end{bmatrix},\\
B_1 &= \begin{bmatrix} 5.2 & 1.2 \\ 1.12 & 2.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5.3 & 1.1 \\ 1.11 & 2.3 \end{bmatrix},\\
C_1 &= \begin{bmatrix} 1.2 & 0.11 \\ 0.1 & 1.22 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.1 & 0.12 \\ 0.1 & 1.22 \end{bmatrix},\\
D_1 &= D_2 = 0, \quad E_{11} = E_{21} = 0.5I,\\
E_{12} &= E_{22} = 0.4I, \quad E_{l1} = E_{l2} = 0, \ l = 3, 4, 5;\\
\aleph &= \begin{bmatrix} -0.8 & 0.8 \\ 0.3 & -0.3 \end{bmatrix}.
\end{aligned}$$

For this system without external controller, Fig. 11.1a shows the results of time response of $x_1(t)$ and $x_2(t)$.

However, if we set

$$D_1 = \begin{bmatrix} 4 \\ 2.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

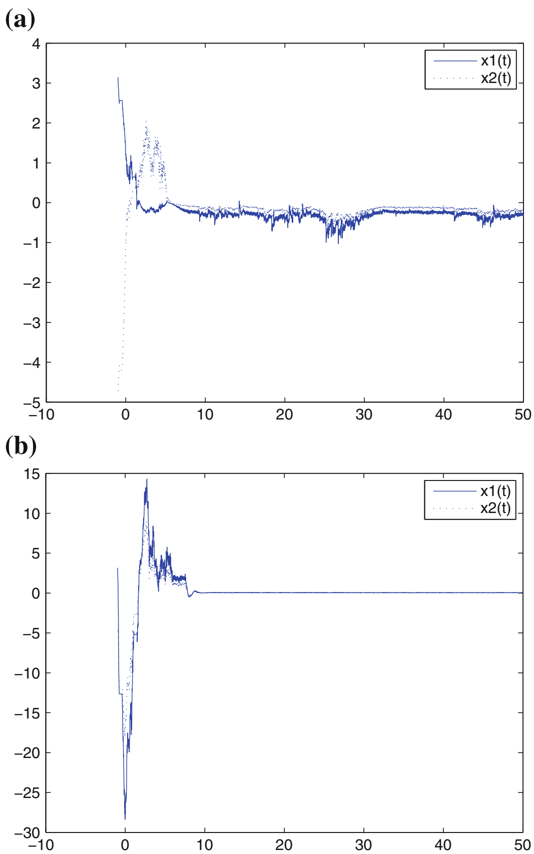

Fig. 11.1 **a** Time response of $x_1(t)$ and $x_2(t)$ without external controller in Example 11.14, **b** Time response of $x_1(t)$ and $x_2(t)$ with external controller $u_1(t), u_2(t)$ in Example 11.14

it is easy to see that Assumptions 11.3–11.5 are satisfied with $\underline{\alpha}_i = 0.4, \bar{\alpha}_i = 1.2, \Pi_i = 8I, \Gamma_i = 7I, \bar{\Sigma} = I, \Sigma = F_1 = F_3 = 0, F_2 = F_4 = 0.5I$, and $\bar{\tau} = \bar{\tau}_i = 0.4, \bar{\upsilon} = \bar{\upsilon}_i = 0.6,\ i = 1, 2$. Using the Matlab LMI Toolbox, the LMIs (11.4)–(11.11) are feasible and the feedback control is

$$u_1(t) = [\ -15.9876 \quad 28.4673\ ]x(t),$$
$$u_2(t) = [\ -9.8136 \quad 17.5622\ ]x(t).$$

The simulation of the solution is given in Fig. 11.1b for $t \in [-0.65, 200]$. It is clear that both $x_1(t)$ and $x_2(t)$ converge exponentially to zeros.

Example 11.15 Consider system (11.46) with $N = 2$,

$$B_1 = \begin{bmatrix} 6.2 & 1.2 \\ 1.12 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 6.3 & 1.1 \\ 1.11 & 0.3 \end{bmatrix},$$

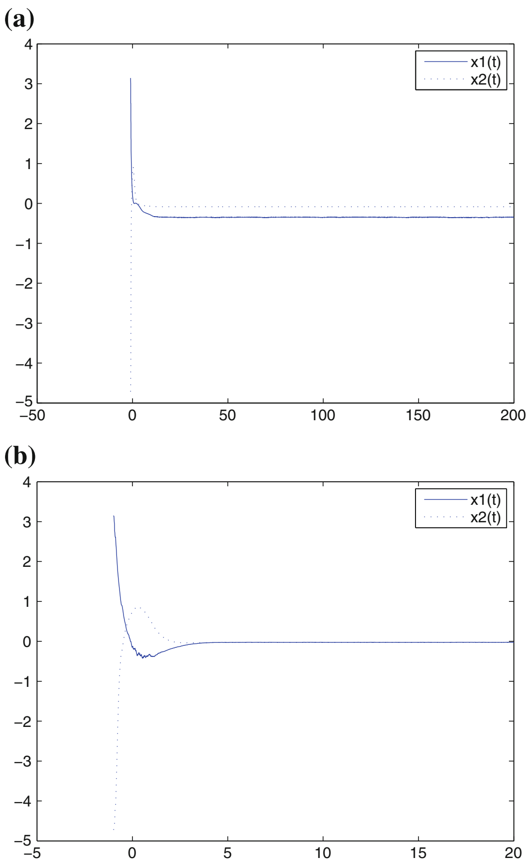

Fig. 11.2 **a** Time response of $x_1(t)$ and $x_2(t)$ without external controller in Example 11.15, **b** Time response of $x_1(t)$ and $x_2(t)$ with external controller $u_1(t)$, $u_2(t)$ in Example 11.15

and other parameters are defined in Example 11.14.

For this system without external controller, Fig. 11.2a shows the results of time response of $x_1(t)$ and $x_2(t)$.

However, if we set $D_1 = D_2 = [\ 4 \ \ 0\ ]^T$, it is easy to see that Assumptions 1-3 are satisfied. Using the Matlab LMI Toolbox, the LMIs (11.4), (11.5), (11.7), (11.9), (11.47) and (11.48) are feasible and the feedback control is

$$u_1(t) = [\ -0.9144 \quad -1.20177\ ]x(t),$$
$$u_2(t) = [\ -1.0149 \quad -0.1481\ ]x(t).$$

The simulation of the solution is given in Fig. 11.2b for $t \in [-0.65, 200]$. It is clear that both $x_1(t)$ and $x_2(t)$ converge exponentially to zeros.

Example 11.16 Consider system (11.46) with $N = 1$,

$$
\begin{aligned}
\alpha_{j1}(x_j(t)) &= 1,\ \beta_{j1}(x_j(t)) = 8x_j(t),\\
f_j(x_j(t)) &= g_j(x_j(t)) = \tanh(x_j(t)), \quad j = 1, 2,\\
\tau_1(t) &= 8.5, \quad \upsilon_1(t) = 2.5,
\end{aligned}
$$

and

$$
A_1 = \begin{bmatrix} 1 & -0.01 \\ 0.1 & 1.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5.2 & 1.2 \\ 1.12 & 2.3 \end{bmatrix},
$$
$$
C_1 = \begin{bmatrix} 1.2 & 0.11 \\ 0.1 & 1.22 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -1.2 \\ 0.2 \end{bmatrix}.
$$

For this system, Assumptions 11.3–11.5 are satisfied with $\underline{\alpha}_i = \bar{\alpha}_i = 1$, $\Pi_i = \Gamma_i = 8I$, $\bar{\Sigma} = I$, $\Sigma = F_1 = F_3 = 0$, $F_2 = F_4 = 0.5I$, and $\bar{\tau} = \bar{\tau}_1 = 8.5$, $\bar{\upsilon} = \bar{\upsilon}_1 = 2.5$. It is easy to verify that Theorem 1 of [27] admits no feasible solution. However, using the Matlab LMI Toolbox, the LMIs (11.4), (11.5), (11.7), (11.9), (11.47) and (11.48) are feasible with the following matrices:

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 72.4939 & -13.8747 \\ -13.8747 & 103.5930 \end{bmatrix},\\
Q_{11} &= \begin{bmatrix} 16.7250 & -26.0105 \\ -26.0105 & 88.6304 \end{bmatrix},\\
Q_{21} &= \begin{bmatrix} 644.0687 & 178.7808 \\ 178.7808 & 234.9008 \end{bmatrix},\\
Q_{31} &= \begin{bmatrix} 14.2369 & -26.9039 \\ -26.9039 & 81.3660 \end{bmatrix},\\
Q_{41} &= \begin{bmatrix} 0.3839 & -0.0816 \\ -0.0816 & 0.6447 \end{bmatrix},\\
R_1 &= \begin{bmatrix} 175.2573 & -2.4592 \\ -2.4592 & 269.3507 \end{bmatrix},\\
M_1 &= \begin{bmatrix} 95.6377 & -2.9849 \\ -2.9849 & 75.6384 \end{bmatrix},\\
\bar{M}_1 &= \begin{bmatrix} -334.3276 & 58.5190 \\ 58.5190 & -17.4669 \end{bmatrix},\\
T_1 &= \mathrm{diag}\{12.6571,\ 49.6002\},\\
U_1 &= \mathrm{diag}\{125.4140,\ 170.7878\},
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \mathrm{diag}\{10.2844,\ 37.0834\},\\
G_1 &= \mathrm{diag}\{9.2136,\ 15.5713\},\\
H &= \mathrm{diag}\{11.6382,\ 15.4633\},\\
K &= \mathrm{diag}\{5.1548,\ 8.9735\},
\end{aligned}
$$

and accordingly the feedback control is

$$u_1(t) = [\ 0.3810 \quad 0.2053\ ]x(t).$$

Based on Example 11.16, it is easy to see that the obtained results are better than those in [27]. Hence, the proposed method is an improvement over the existing ones.

11.5 Summary

In this chapter, the problem of designing a feedback control law to exponentially stabilize a class of stochastic Cohen-Grossberg neural networks with both Markovian jumping parameters and mixed mode-dependent time delays has been studied. The mixed time delays consist of both discrete and distributed delays. Using a new Lyapunov–Krasovskii functional that accounts for the mode-dependent mixed delays, a new delay-dependent condition for the global exponential stabilization has been established in terms of linear matrix inequalities. Upon the feasibility of the LMI, all the control parameters can be easily computed and the design of a stabilizing controller can be accomplished.

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